

Ve215 Electric Circuits

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Chapter 6

Capacitors and Inductors

6.1 Introduction

- In this chapter, we study two passive circuit elements: the capacitor and the inductor.
- Capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. For this reason, capacitors and inductors are called *storage* elements.

6.2 Capacitors

- A capacitor is a passive element designed to store energy in its electric field.
- A capacitor is typically constructed as depicted in Fig. 6.1. It consists of two conducting plates separated by an insulator (or dielectric).

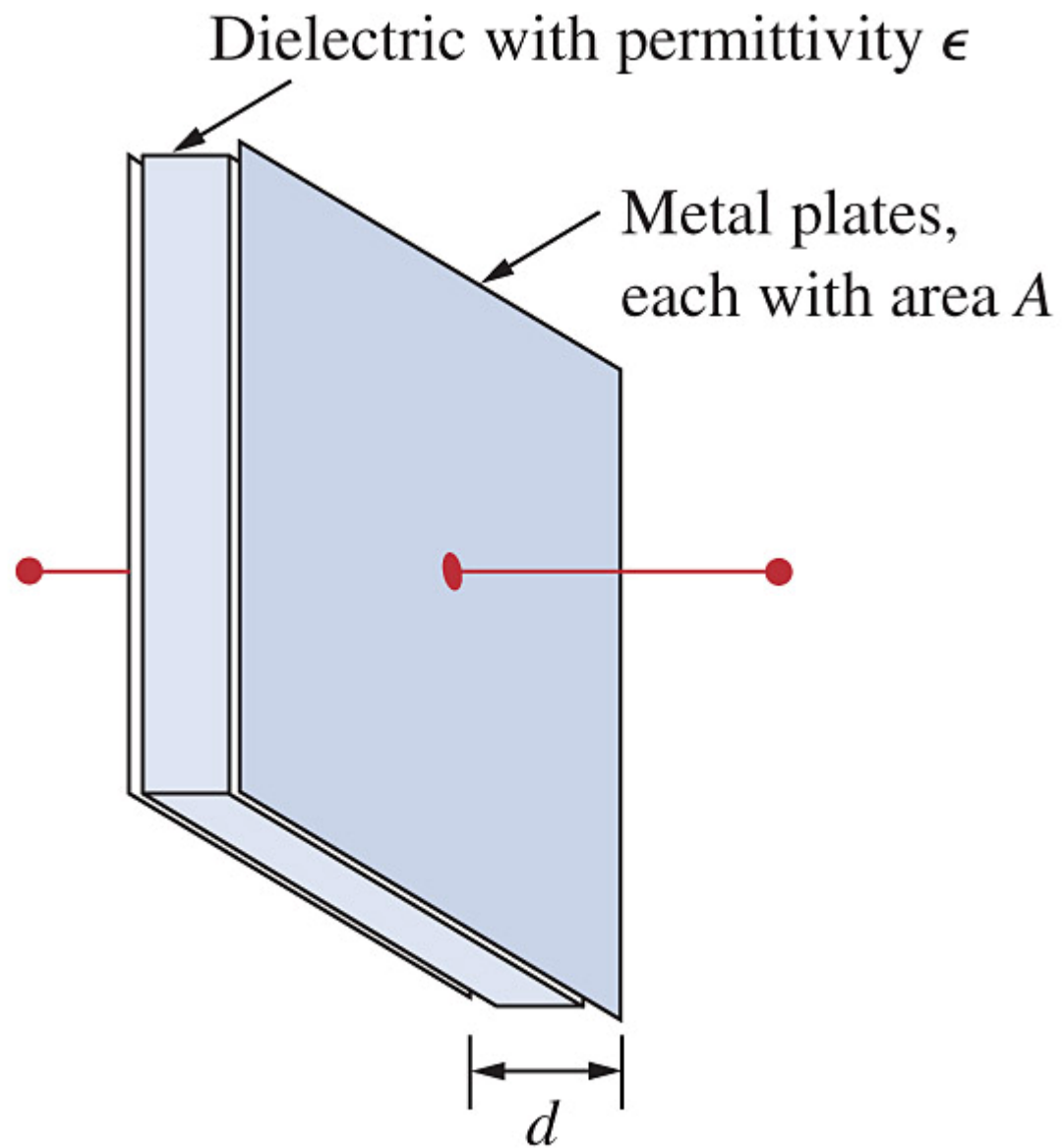


Figure 6.1 A typical capacitor.

When a voltage source v is connected to the capacitor, as in Fig. 6.2, the source deposits a positive charge q on one plate and a negative charge $-q$ on the other. The capacitor is said to store the electric charge.

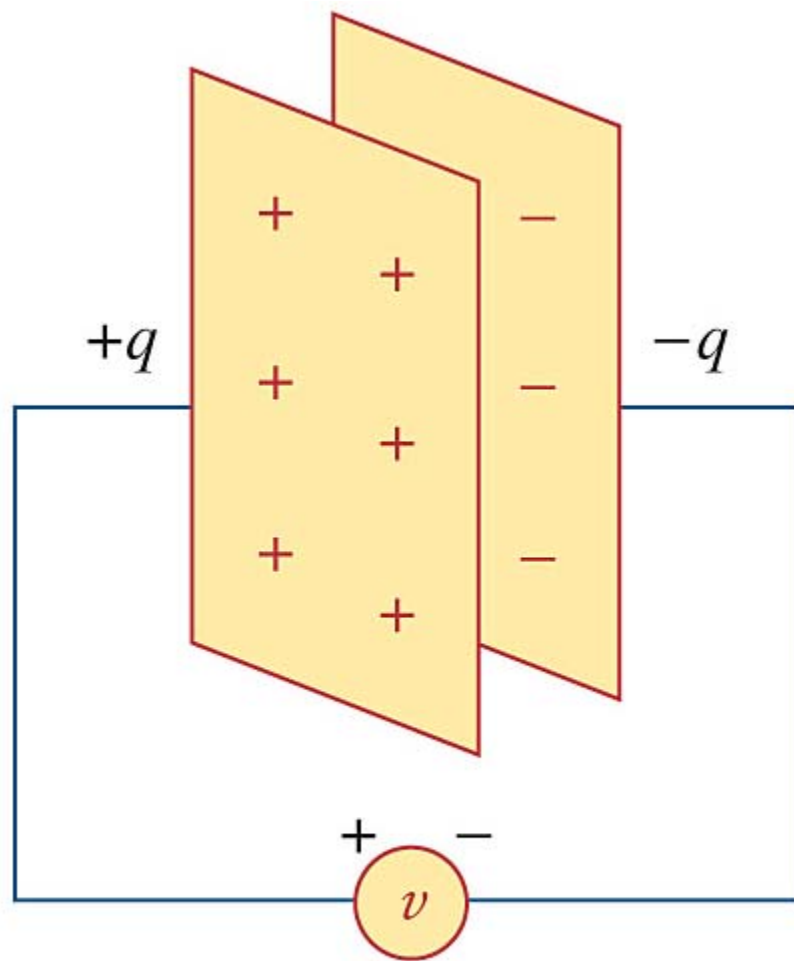


Figure 6.2 A capacitor with applied voltage v .

The amount of charge stored, represented by q , is given by

$$q = Cv$$

where C is known as the capacitance of the capacitor, measured in farads (F). If C is independent of v , the capacitor is said to be *linear*.

- **Capacitance** is the ability of a body to store an electrical charge.
- Capacitance can **also** be calculated if the geometry of the conductors and the dielectric properties of the insulator between the conductors are known.

$$C = \epsilon_r \epsilon_0 \frac{A}{d}$$

where

- C is the capacitance, in Farads;
- A is the area of overlap of the two plates, in square meters;
- ϵ_r is the relative static permittivity (sometimes called the dielectric constant) of the material between the plates (for a vacuum, $\epsilon_r = 1$);
- ϵ_0 is the electric constant ($\epsilon_0 \approx 8.854 \times 10^{-12} \text{ F m}^{-1}$)
- d is the separation between the plates, in meters;

Similar to linear resistors, the capacitance of linear capacitors is dependent on materials and geometry and **independent of v**

Figure 6.3 shows the circuit symbols for fixed and variable capacitors. Note that according to the passive sign convention, if $v \cdot i > 0$, the capacitor is being charged, and if $v \cdot i < 0$, the capacitor is discharging.

Being charged = absorbing power ($p > 0$)
Discharging = supplying power ($p < 0$)

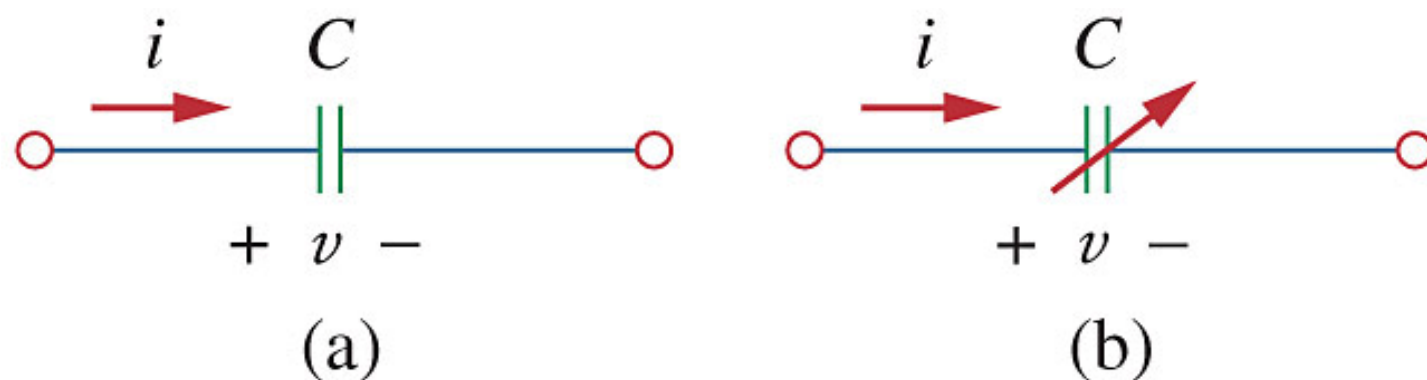


Figure 6.3 Circuit symbols for capacitors: (a) fixed capacitor, (b) variable capacitor.

The current-voltage relationship for the capacitor is given by

$$i = C \frac{dv}{dt}$$

as illustrated in Fig. 6.6.

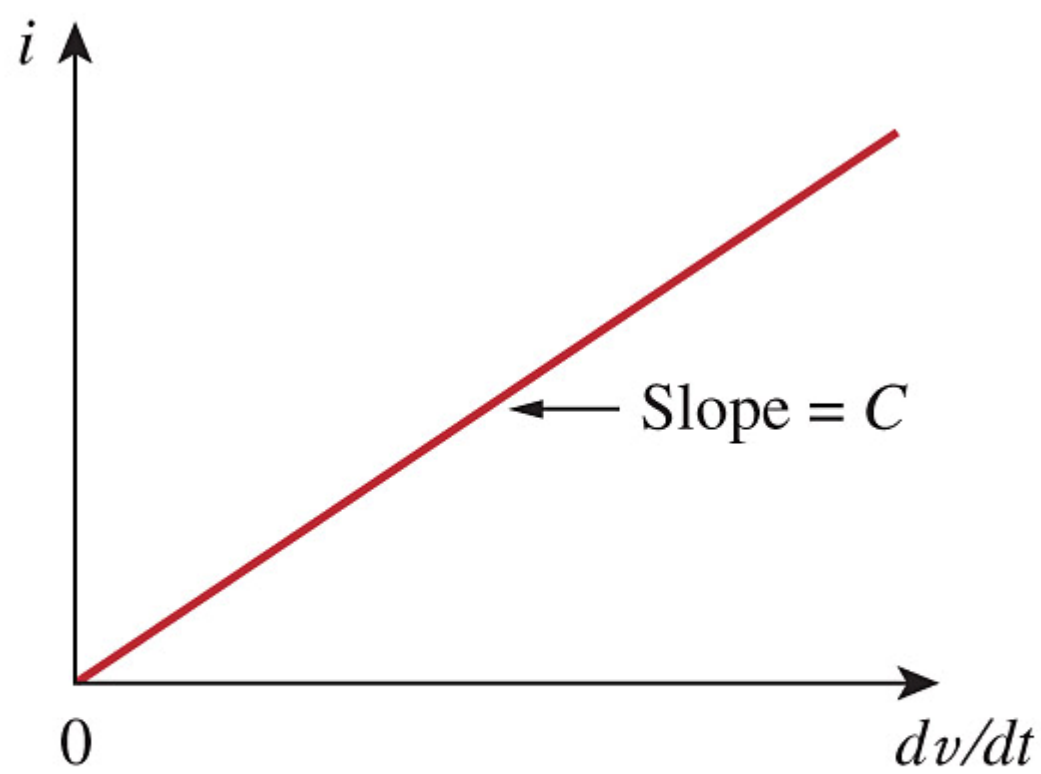


Figure 6.6 Current-voltage relationship of a capacitor.

The voltage-current relation of the capacitor is

$$v = \frac{1}{C} \int_{-\infty}^t i dt = \frac{1}{C} \int_{t_0}^t i dt + v(t_0)$$

where $v(t_0) = \frac{1}{C} \int_{-\infty}^{t_0} i dt = \frac{q(t_0)}{C}$ is the voltage across the capacitor at time t_0 .

The instantaneous power delivered to the capacitor is

$$p = vi = v \left(C \frac{dv}{dt} \right)$$

$$p(t) = v(t)i(t)$$

The energy stored in the capacitor is
therefore

For a linear capacitor

$$w = \int_{-\infty}^t p dt = \int_{-\infty}^t v \left(C \frac{dv}{dt} \right) dt \downarrow = C \int_{v(-\infty)}^v v dv$$

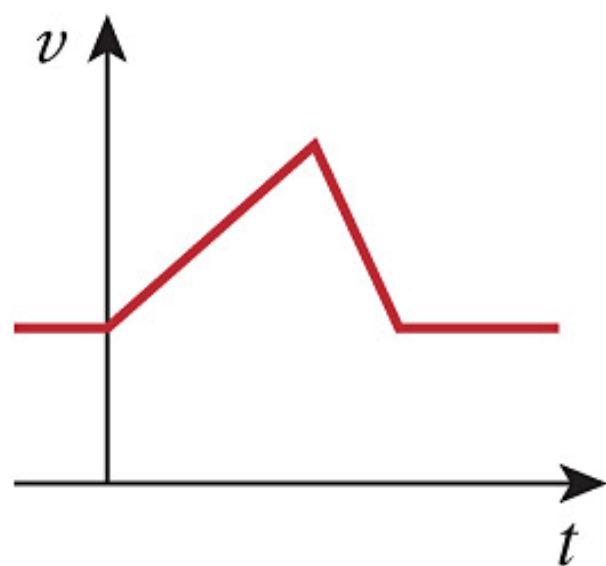
$$= \frac{1}{2} C v^2 \Big|_{v(-\infty)}^v = \frac{1}{2} C v^2$$

$$v(-\infty) = 0$$

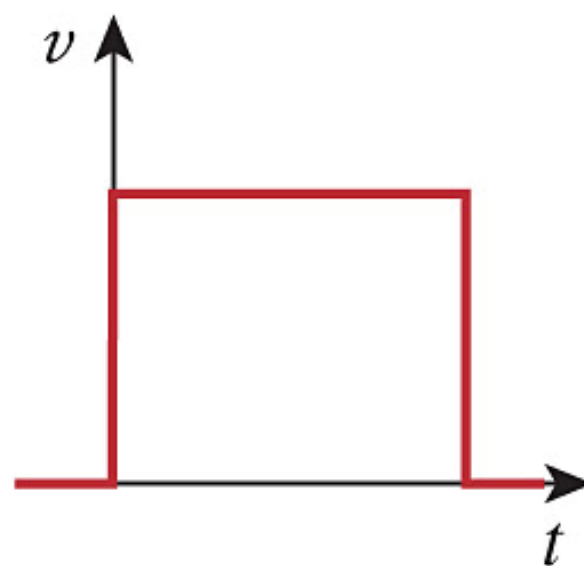
Capacitors have the following important properties:

1. When the voltage across a capacitor is not changing with time (i.e., dc voltage), the current through the capacitor is zero. Thus, a capacitor is an open circuit to dc.

2. The voltage on a capacitor must be continuous. In other words, the voltage on a capacitor cannot change abruptly. (A discontinuous change in voltage requires an infinite current, which is physically impossible.)



(a)



(b)

Figure 6.7 Voltage across a capacitor:
(a) possible, (b) impossible.

3. The ideal capacitor does not dissipate energy. It takes power from the circuit when storing energy in its electric field and returns previously stored energy when delivering power to the circuit.

4. A real capacitor has a large leakage resistance, as shown in Fig. 6.8.

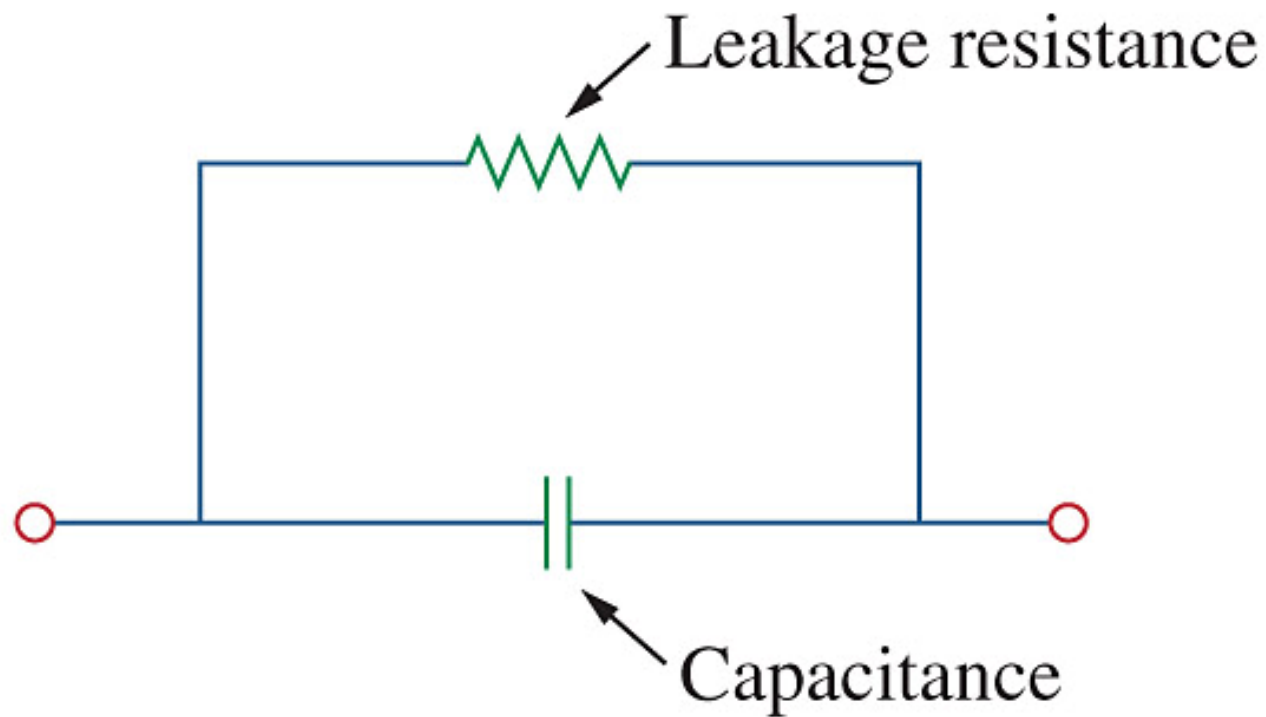


Figure 6.8 Circuit model of a nonideal capacitor.

Example 6.3 Determine the voltage across a $2\text{-}\mu\text{F}$ capacitor if the current through it is

$$i(t) = 6e^{-3000t} \text{ mA}, \quad t \geq 0$$

Assume the initial capacitor voltage is zero.

Solution :

$$\begin{aligned} v &= \frac{1}{C} \int_0^t i dt + v(0) = \frac{1}{C} \int_0^t i dt \\ &= \frac{1}{2 \times 10^{-6}} \int_0^t 6e^{-3000t} \times 10^{-3} dt \\ &= \frac{3 \times 10^3}{-3000} e^{-3000t} \Big|_0^t = 1 - e^{-3000t} \text{ (V)} \end{aligned}$$

Example 6.5 Obtain the energy stored in each capacitor in Fig. 6.12(a) under dc conditions.

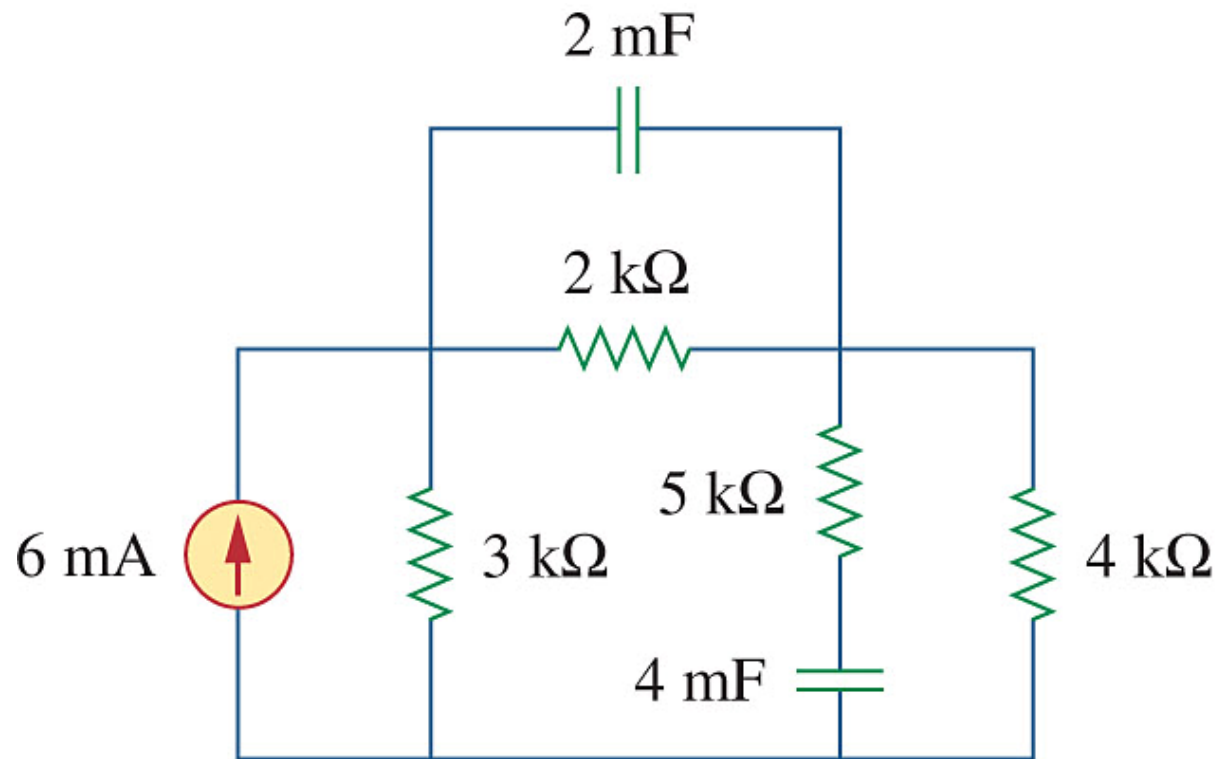


Figure 6.12(a)

Solution : We replace each capacitor with an open circuit, as shown in Fig. 6.12(b).

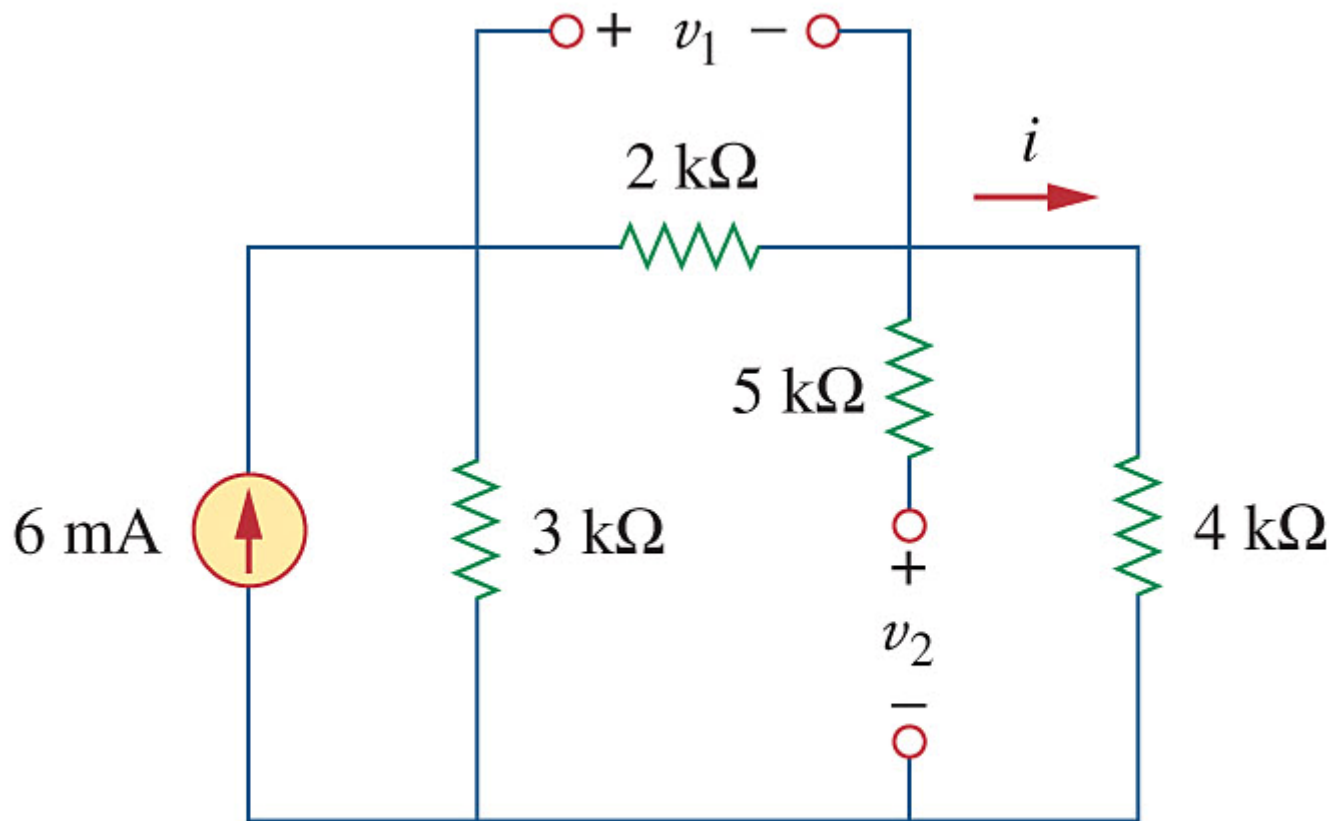


Figure 6.12(b)

$$i = \frac{3}{3 + 2 + 4} \times 6 = 2 \text{ (mA)}$$

$$v_1 = 2i = 2 \times 2 = 4 \text{ (V)}$$

$$v_2 = 4i = 4 \times 2 = 8 \text{ (V)}$$

$$w_1 = \frac{1}{2} C_1 v_1^2 = \frac{1}{2} \times 2 \times 10^{-3} \times 4^2 = 0.016 \text{ (J)}$$

$$w_2 = \frac{1}{2} C_2 v_2^2 = \frac{1}{2} \times 4 \times 10^{-3} \times 8^2 = 0.128 \text{ (J)}$$

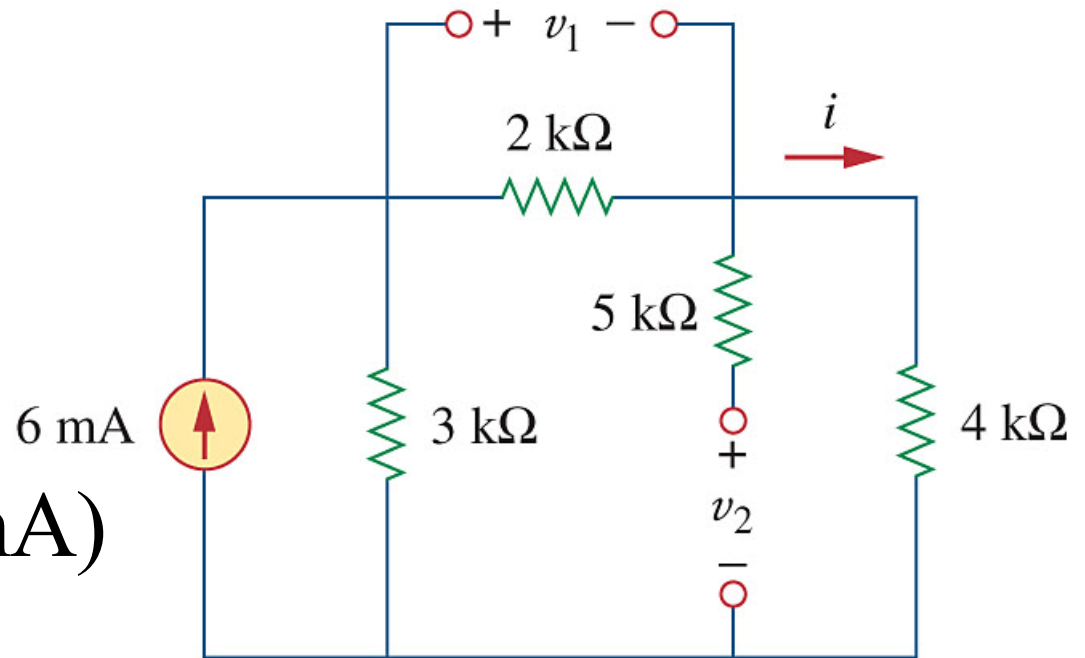


Figure 6.12(b)

6.3 Series and Parallel Capacitor

- The series-parallel combination technique can be extended to series-parallel connections of capacitors.

The equivalent capacitance of N parallel-connected capacitors is the sum of the individual capacitances:

$$C_{eq} = C_1 + C_2 + \dots + C_N$$

Proof :

$$i = \sum_{k=1}^N i_k = \sum_{k=1}^N C_k \frac{dv}{dt} = \left(\sum_{k=1}^N C_k \right) \frac{dv}{dt} = C_{eq} \frac{dv}{dt}$$

$$C_{eq} = \sum_{k=1}^N C_k$$

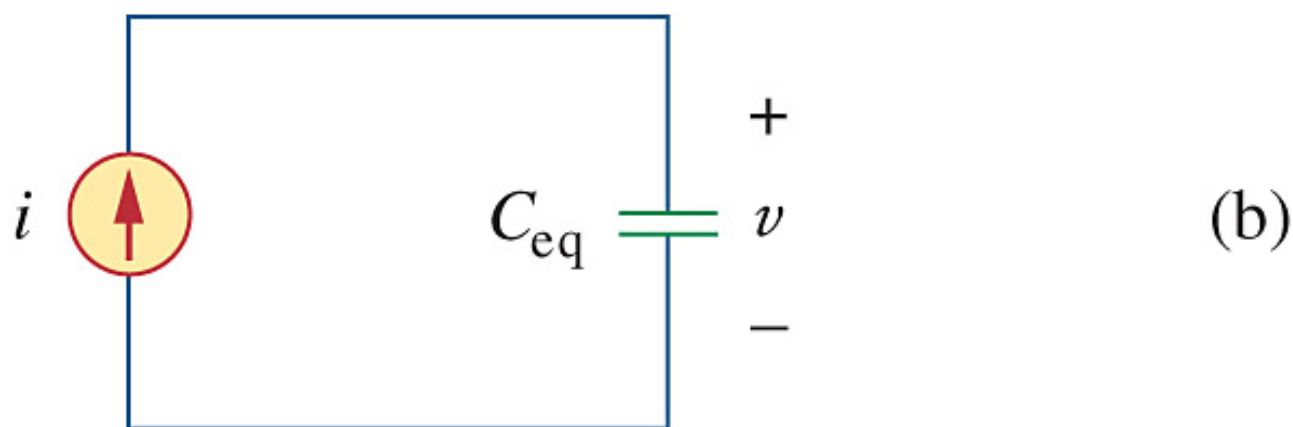
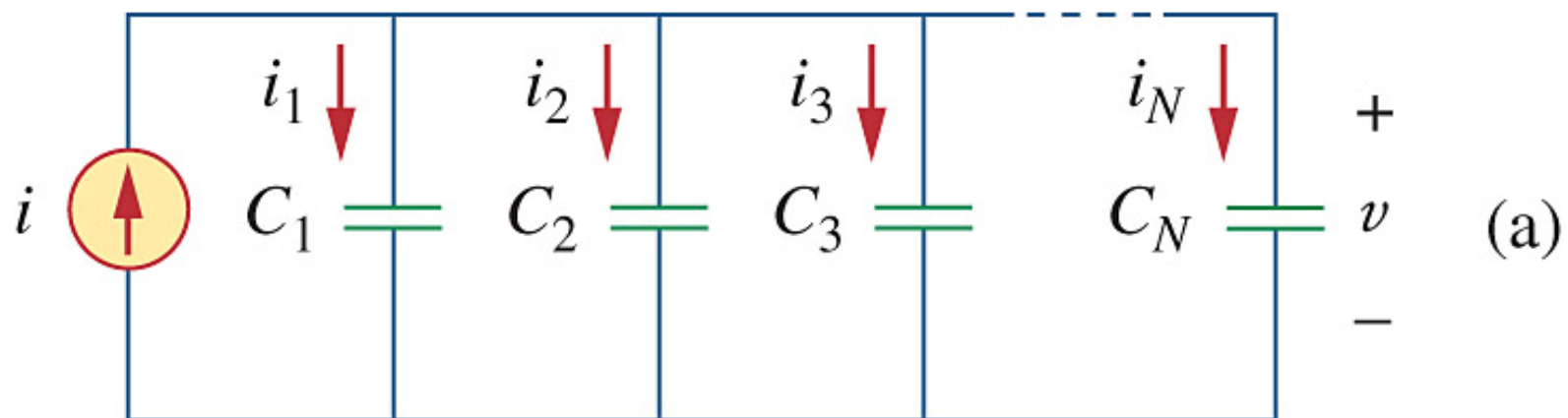


Figure 6.14 (a) Parallel-connected N capacitors, (b) equivalent circuit for the parallel capacitors.

The equivalent capacitance of N series-connected capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_N}$$

The initial voltage across C_{eq} is the sum of the individual initial voltages:

$$v(t_0) = v_1(t_0) + v_2(t_0) + \dots + v_N(t_0)$$

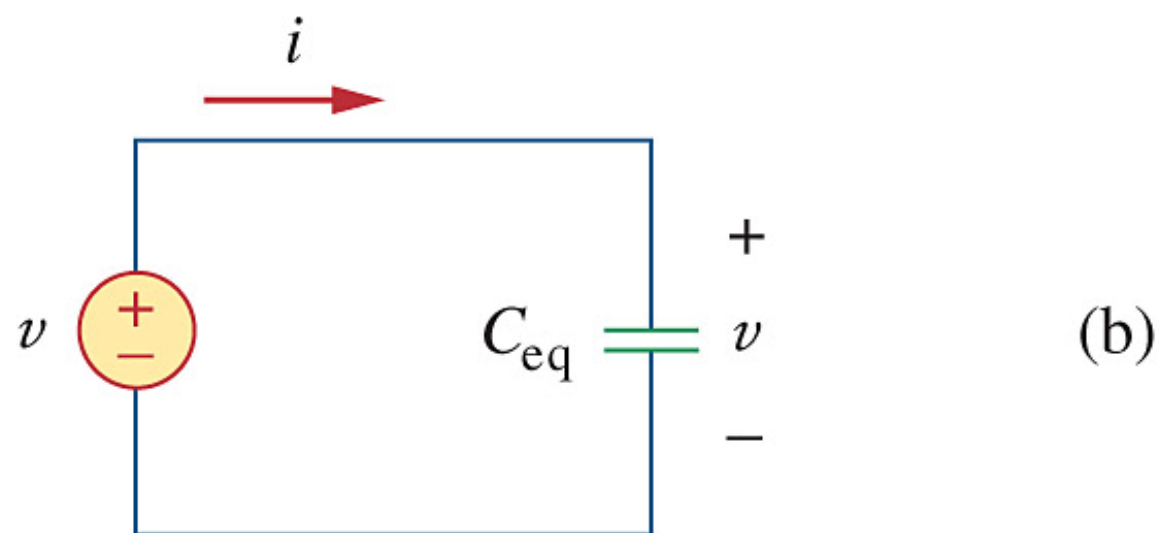
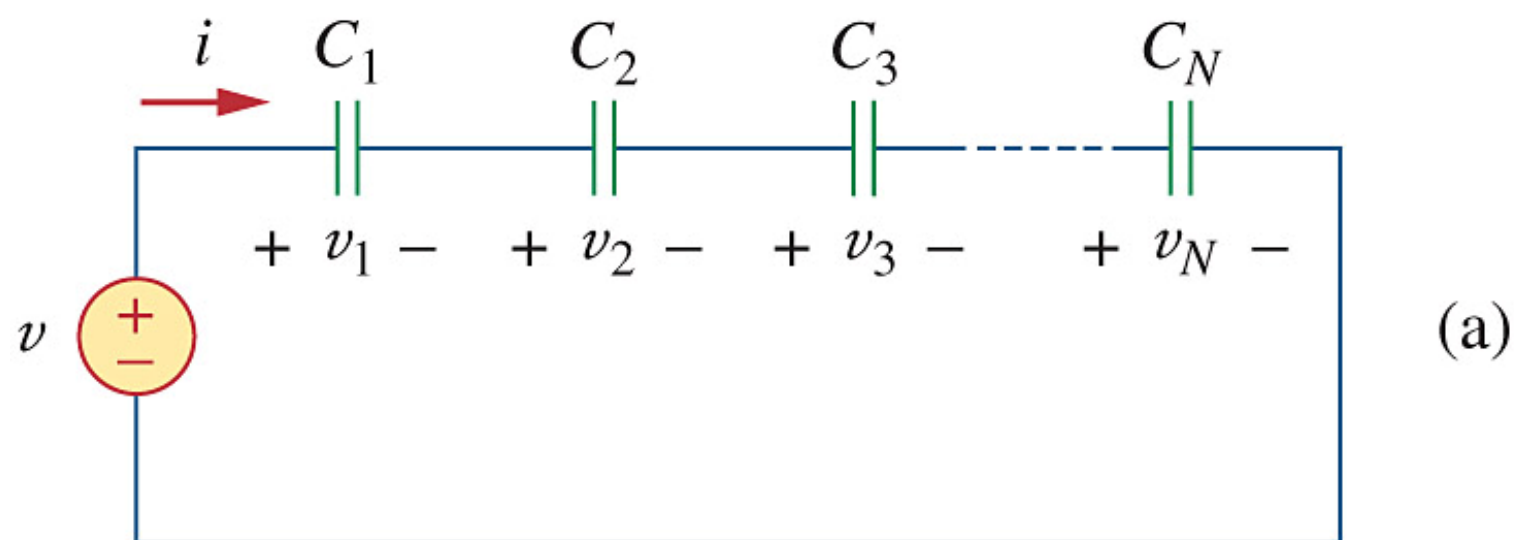


Figure 6.15 (a) Series-connected N capacitors, (b) equivalent circuit for the series capacitors.

Proof :

$$v = \sum_{k=1}^N v_k = \sum_{k=1}^N \left(\frac{1}{C_k} \int_{t_0}^t i dt + v_k(t_0) \right)$$

$$= \left(\sum_{k=1}^N \frac{1}{C_k} \right) \int_{t_0}^t i dt + \sum_{k=1}^N v_k(t_0)$$

$$= \frac{1}{C_{eq}} \int_{t_0}^t i dt + v(t_0)$$

$$\frac{1}{C_{eq}} = \sum_{k=1}^N \frac{1}{C_k}, \quad v(t_0) = \sum_{k=1}^N v_k(t_0)$$

Example 6.6 Find the equivalent capacitance seen between terminals a and b of the circuit in Fig. 6.16.

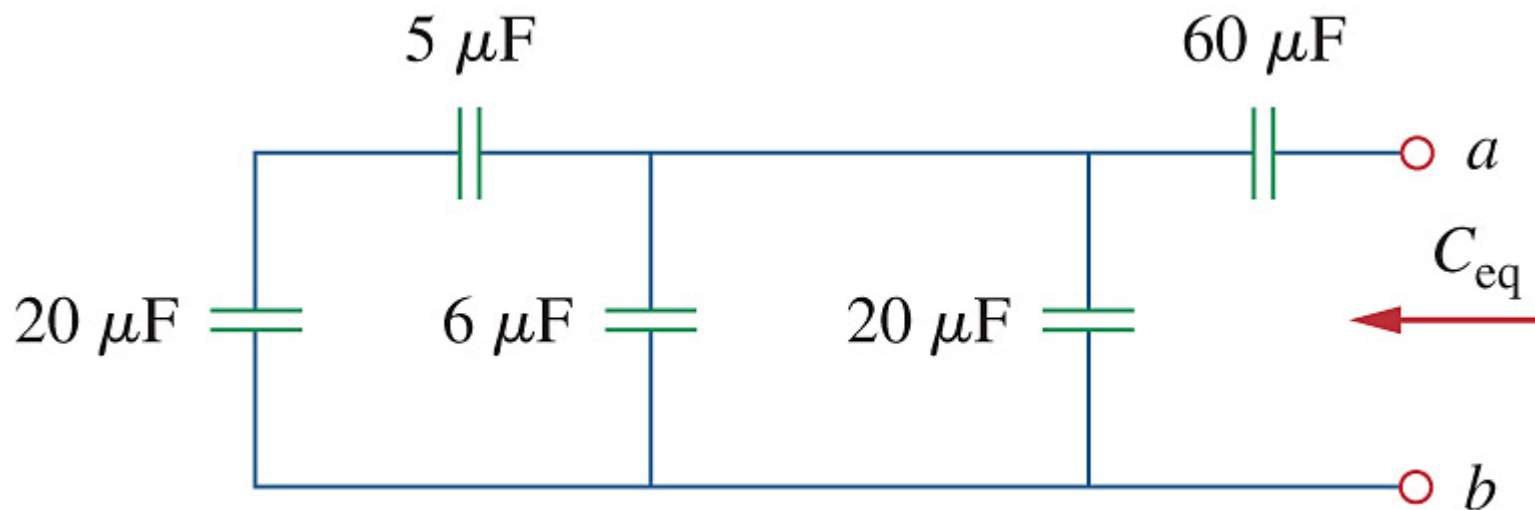


Figure 6.16

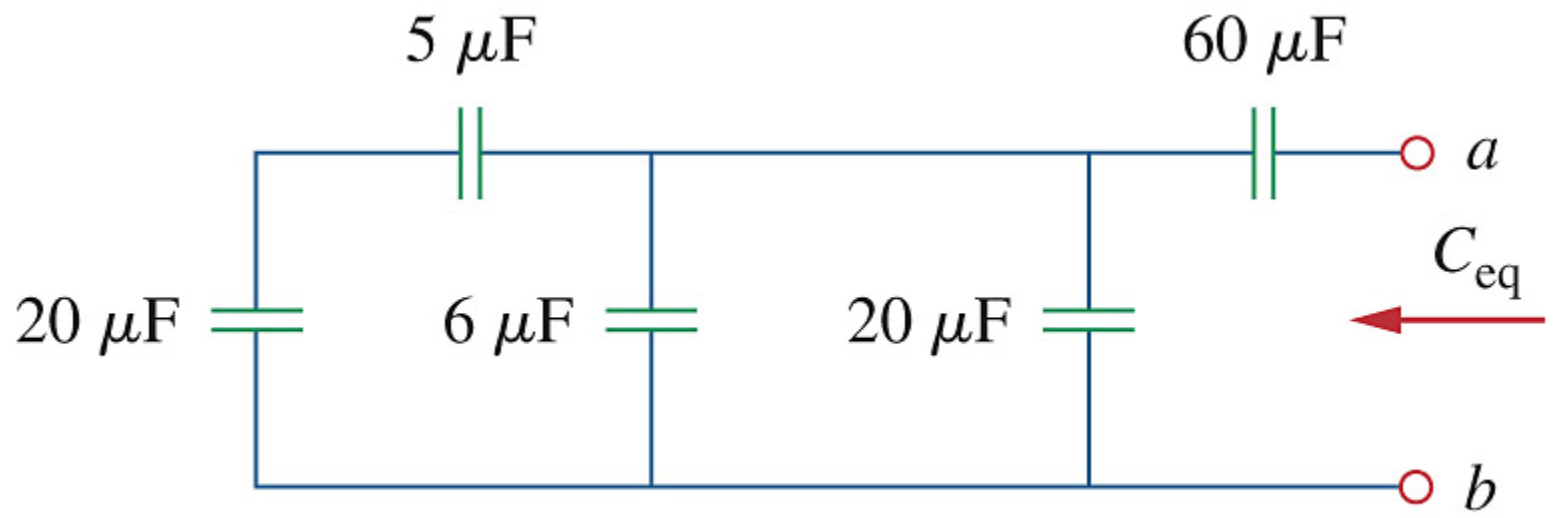


Figure 6.16

Solution :

$$\frac{1}{C_{eq}} = \frac{1}{60} + \frac{1}{6 + 20 + \frac{1}{\frac{1}{5} + \frac{1}{20}}} = \frac{1}{60} + \frac{1}{30} = \frac{1}{20}$$

$$C_{eq} = 20\ (\mu\text{F})$$

6.4 Inductors

- An inductor is a passive element designed to store energy in its magnetic field.
- An inductor is usually formed into a cylindrical coil with many turns of conducting wire, as shown in Fig. 6.21.

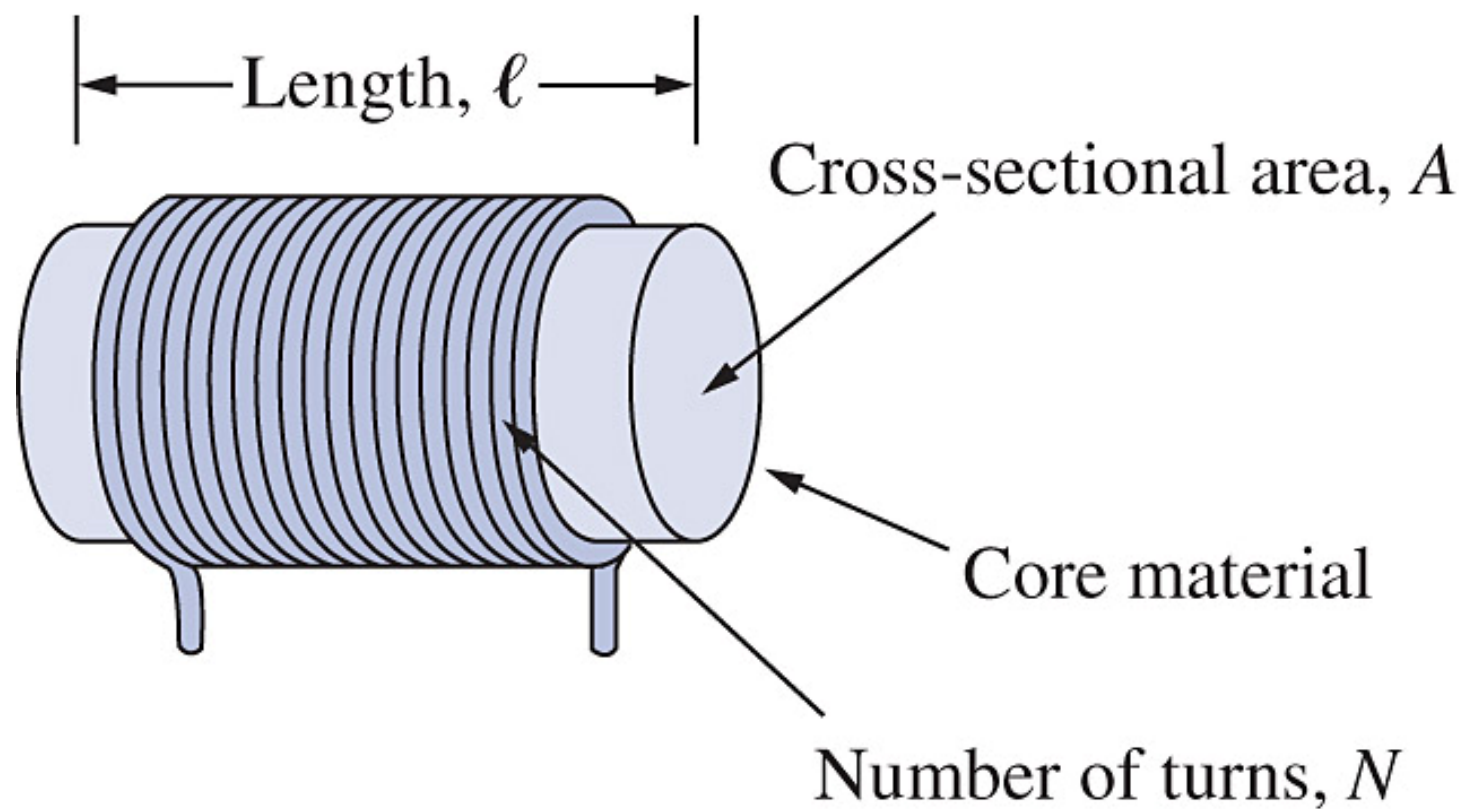


Figure 6.21 Typical form of an inductor.

When a current source i is connected to the inductor, the source sets up a magnetic field. The magnetic flux linkage in the inductor, represented by ψ , is given by

$$\psi = Li$$

ψ : magnetic flux (Wb)

B: magnetic flux density or magnetic field (Wb/m²)

$$\psi = \mathbf{B} \bullet \mathbf{S}$$

- **Inductance** results from the magnetic field around a current-carrying conductor
- Inductance can **also** be calculated if the geometry of the current path and the magnetic permeability of nearby materials

E.g., Cylindrical air-core coil

$$L = \frac{1}{l} \mu_0 K N^2 A$$

where

- L = inductance in Henrys (H)
- μ_0 = permeability of free space = 4×10^{-7} H/m
- K = Nagaoka coefficient
- N = number of turns
- A = area of cross-section of the coil in square meters (m²)
- l = length of coil in meters (m)

A = area of cross-section
of the coil

l = length of coil

permeability

less inductance



more inductance



less inductance



more inductance



less inductance



air core
(permeability = 1)

more inductance



soft iron core
(permeability = 600)

where L is known as the inductance of the inductor, measured in henrys (H). If L is independent of i , the inductor is said to be linear.

The circuit symbols for inductors are shown in Figure 6.23, following the passive sign convention.

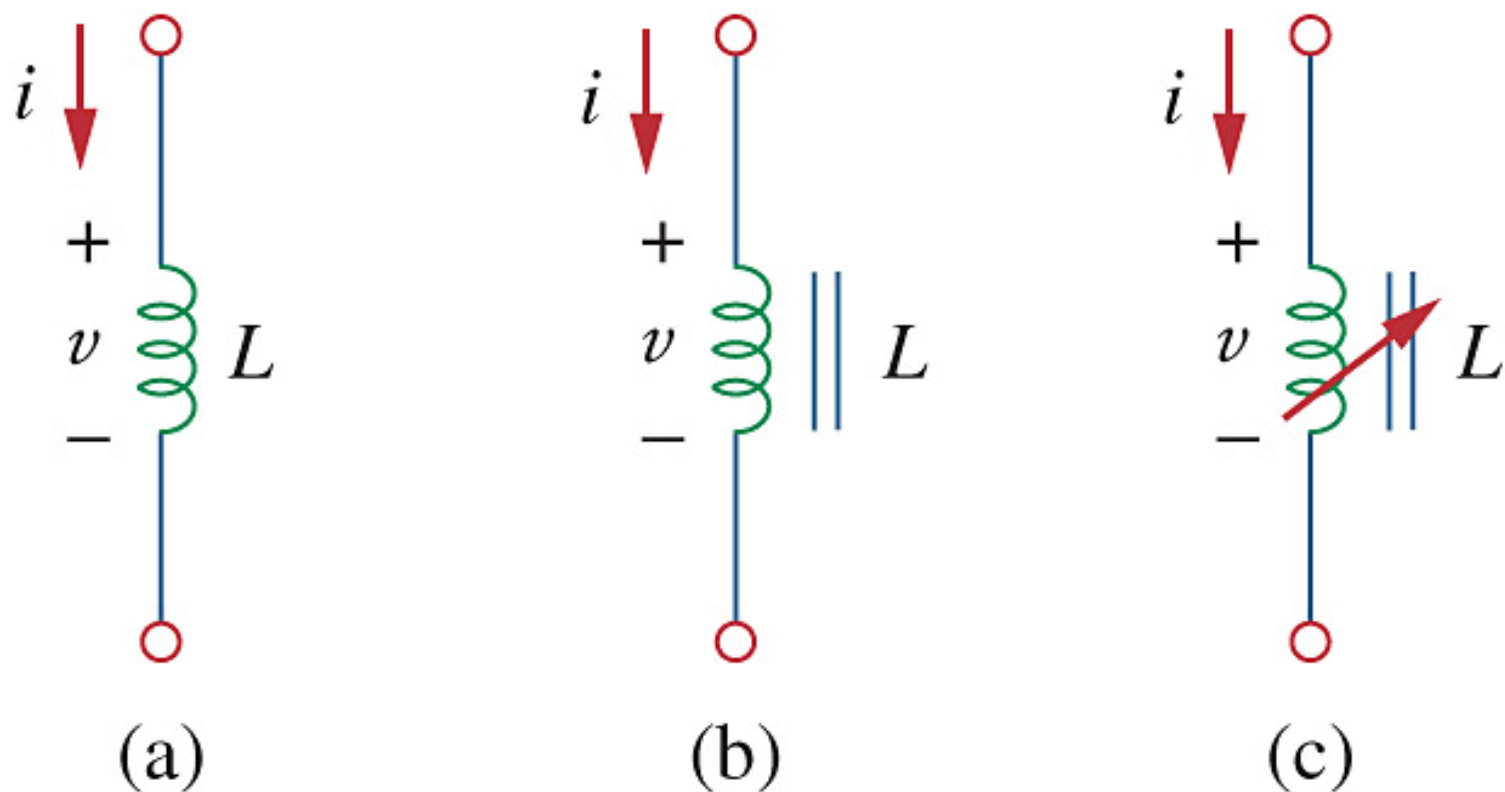


Figure 6.23 Circuit symbols for inductors: air-core, (b) iron-core, (c) variable iron-core.

The voltage-current relationship for the inductor is given by

$$v = L \frac{di}{dt}$$

as illustrated in Fig. 6.24.

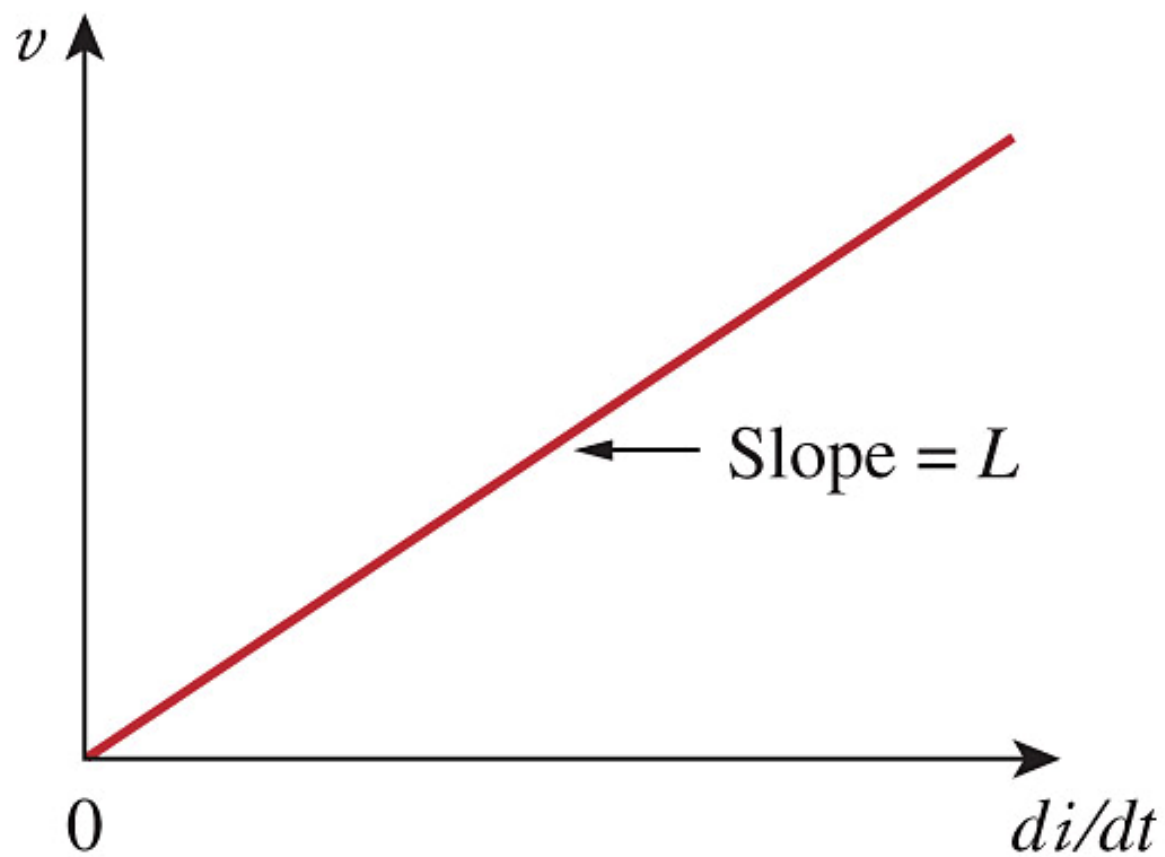


Figure 6.24 Voltage-current relationship of an inductor.

The current-voltage relation of the inductor is

$$i = \frac{1}{L} \int_{-\infty}^t v dt = \frac{1}{L} \int_{t_0}^t v dt + i(t_0)$$

where $i(t_0) = \frac{1}{L} \int_{-\infty}^{t_0} v dt = \frac{\psi(t_0)}{L}$ is the

current through the inductor at time t_0 .

The instantaneous power delivered to the inductor is

$$p = vi = \left(L \frac{di}{dt} \right) i$$

The energy stored in the inductor is
therefore

$$w = \int_{-\infty}^t p dt = \int_{-\infty}^t \left(L \frac{di}{dt} \right) i dt = L \int_{i(-\infty)}^i i di$$
$$= \frac{1}{2} L i^2 \Big|_{i(-\infty)}^i = \frac{1}{2} L i^2$$

Analogy between capacitors and inductors

	Capacitor	Inductor
Electric/magnetic	q	ψ
	$q = Cv$	$\psi = Li$
i-v (or v-i) relation	$i = C \times dv/dt$	$v = L \times di/dt$
energy	$1/2 Cv^2$	$1/2 Li^2$

C: the capacity to store charges

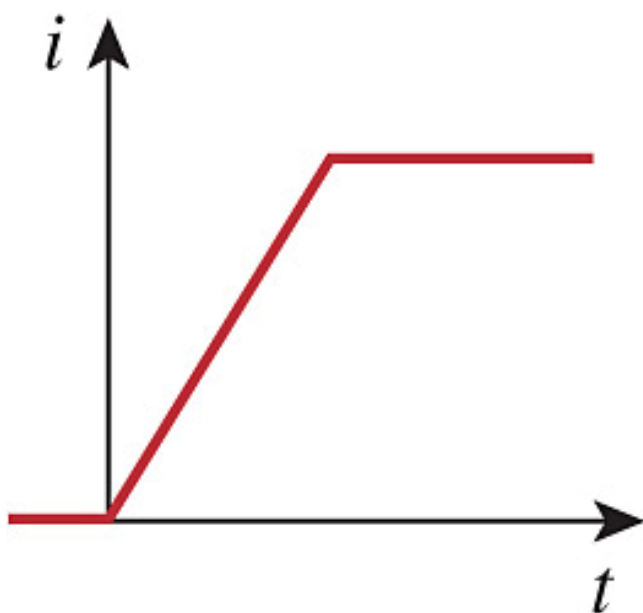
L: the capability to induce magnetic flux

Inductor have the following important properties:

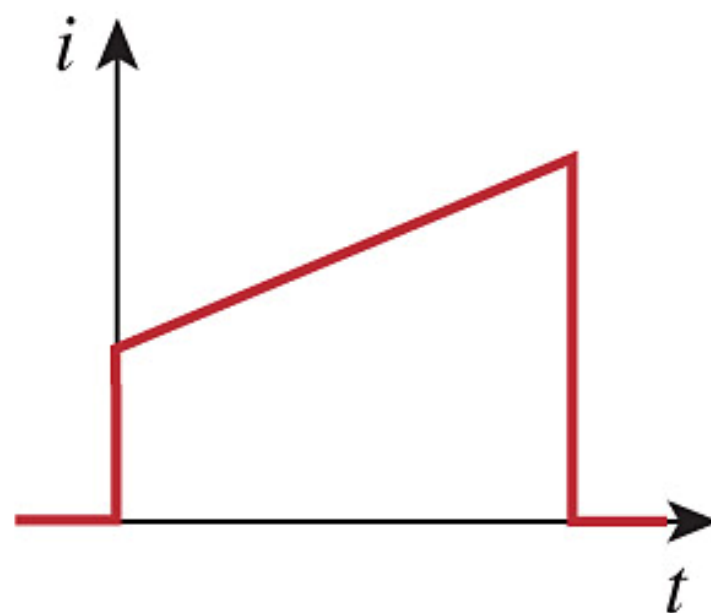
1. When the current through an inductor is not changing with time (i.e., dc current), the voltage across the inductor is zero.

Thus, an inductor is a short circuit to dc.

2. The current in an inductor must be continuous. In other words, the current in an inductor cannot change abruptly. (A discontinuous change in current requires an infinite voltage, which is physically impossible.)



(a)



(b)

Figure 6.25 Current through an inductor:
(a) possible, (b) impossible.

3. The ideal inductor does not dissipate energy. It takes power from the circuit when storing energy in its magnetic field and returns previously stored energy when delivering power to the circuit.
4. A real inductor has a significant *winding resistance* and a small *winding capacitance*, as shown in Fig. 6.26.

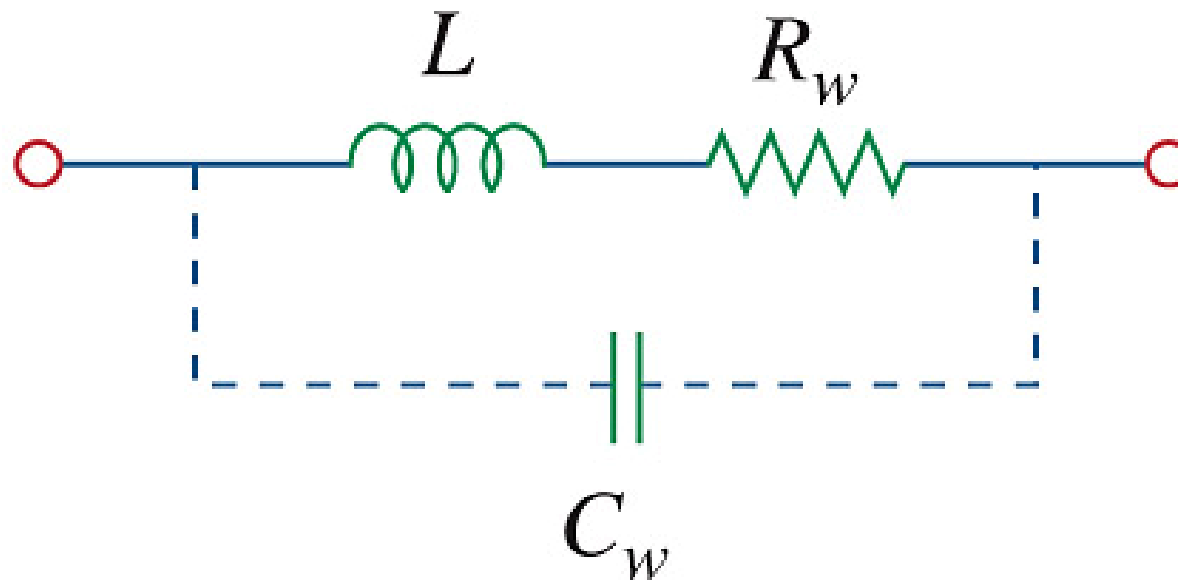


Figure 6.26 Circuit model for a practical inductor.

Example 6.8 The current through a 0.1-H inductor is

$$i(t) = 10te^{-5t} \text{ A}$$

Find the voltage across the inductor and the energy stored in it.

Solution :

$$\begin{aligned} v &= L \frac{di}{dt} = 0.1 \frac{d}{dt} (10te^{-5t}) = \frac{d}{dt} (te^{-5t}) \\ &= e^{-5t} + t(-5e^{-5t}) = e^{-5t} (1 - 5t) \text{ (V)} \\ w &= \frac{1}{2} Li^2 = \frac{1}{2} \times 0.1 \times (10te^{-5t})^2 \\ &= 5t^2 e^{-10t} \text{ (J)} \end{aligned}$$

Example 6.10 Consider the circuit in Fig. 6.27. Under dc conditions, find (a) i , v_C , and i_L , (b) the energy stored in the capacitor and inductor.

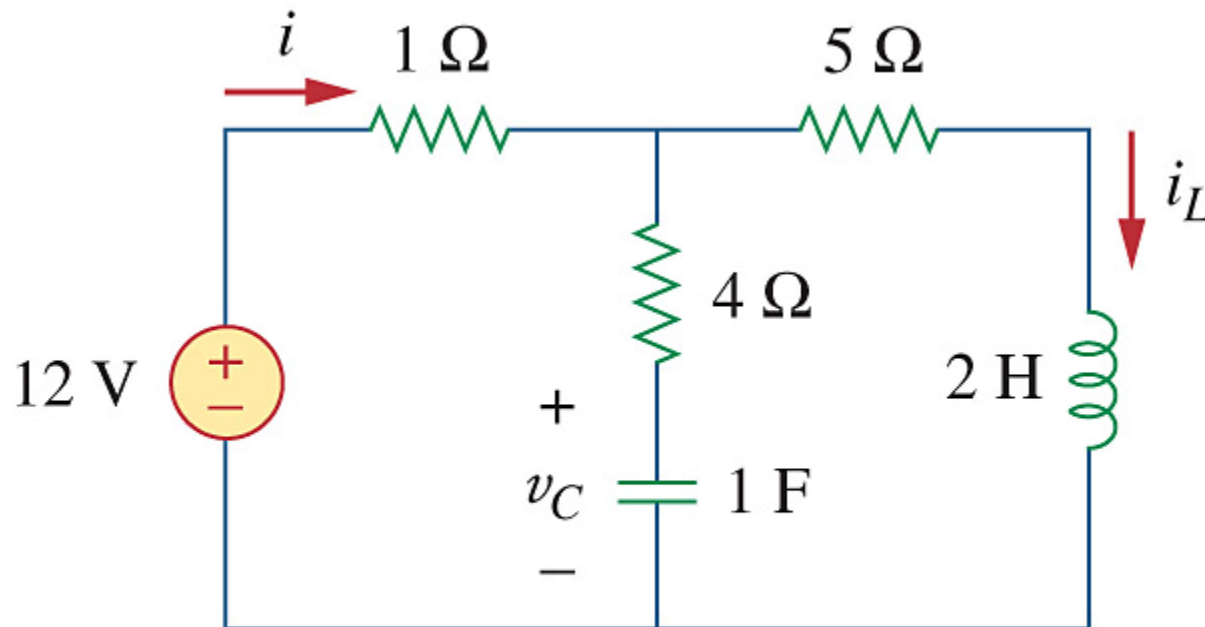


Figure 6.27(a)

Solution : We replace the capacitor with an open circuit and the inductor with a short circuit, as shown in Fig. 6.27(b).

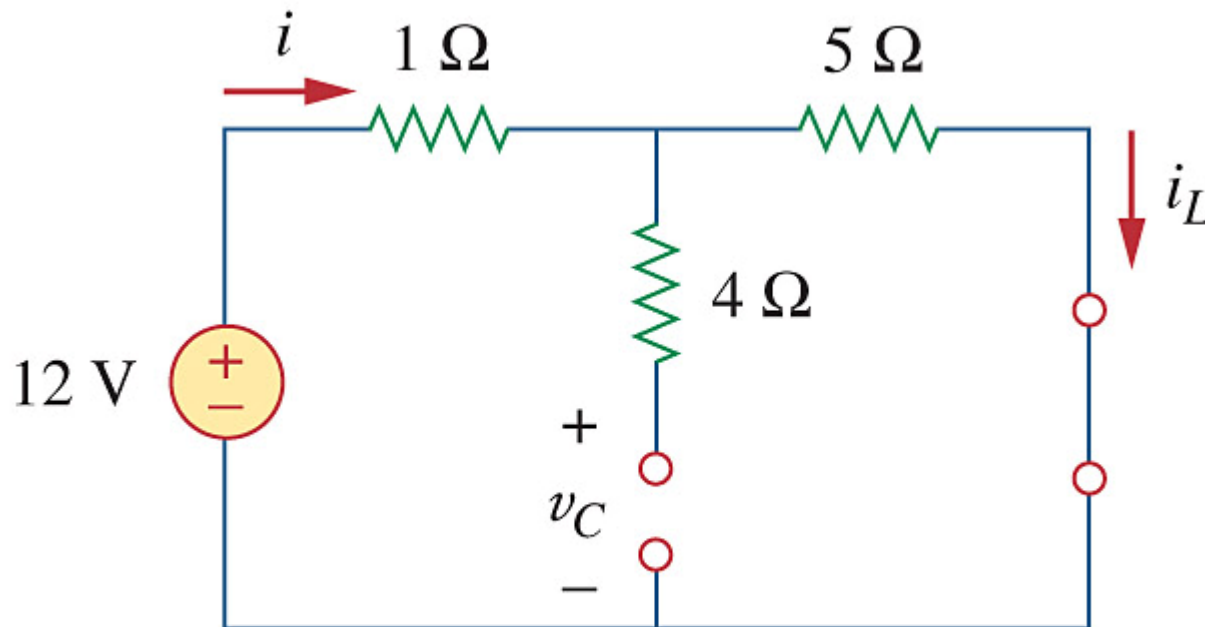


Figure 6.27(b)

(a)

$$i = i_L = \frac{12}{1+5} = 2 \text{ (A)}$$

$$v_c = 5i_L = 5 \times 2 = 10 \text{ (V)}$$

(b)

$$w_C = \frac{1}{2} C v_C^2 = \frac{1}{2} \times 1 \times 10^2 = 50 \text{ (J)}$$

$$w_L = \frac{1}{2} L i_L^2 = \frac{1}{2} \times 2 \times 2^2 = 4 \text{ (J)}$$

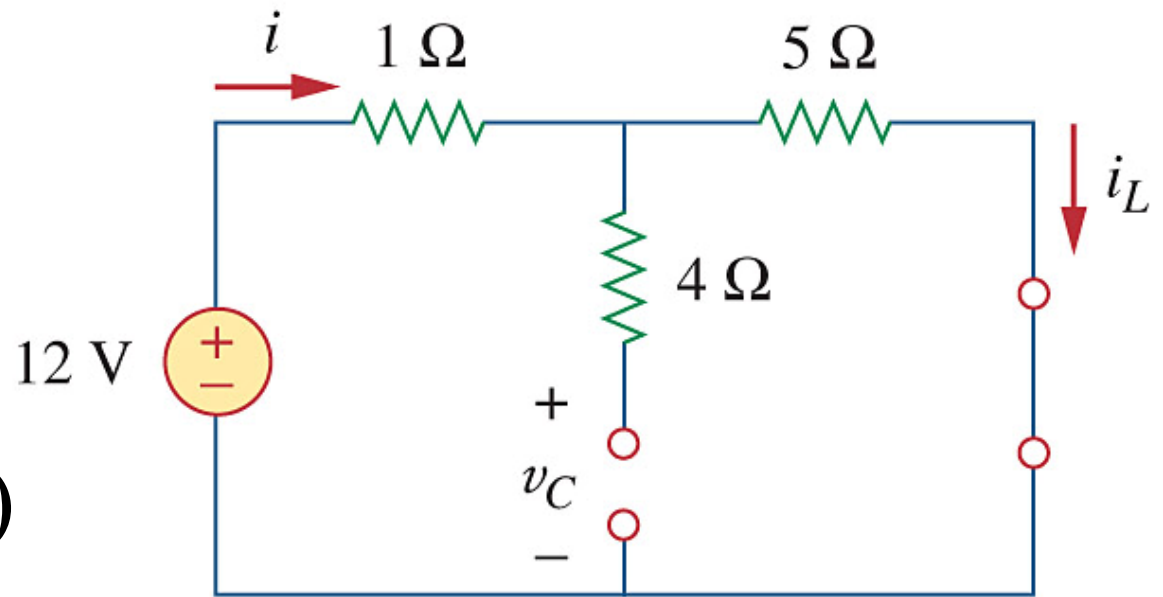


Figure 6.27(b)

6.5 Series and Parallel Inductor

- The series-parallel combination technique can be extended to series-parallel connections of inductors.

The equivalent inductance of N series-connected inductors is the sum of the individual inductances:

$$L_{eq} = L_1 + L_2 + L + L_N$$

Proof :

$$v = \sum_{k=1}^N v_k = \sum_{k=1}^N L_k \frac{di}{dt} = \left(\sum_{k=1}^N L_k \right) \frac{di}{dt} = L_{eq} \frac{di}{dt}$$

$$L_{eq} = \sum_{k=1}^N L_k$$

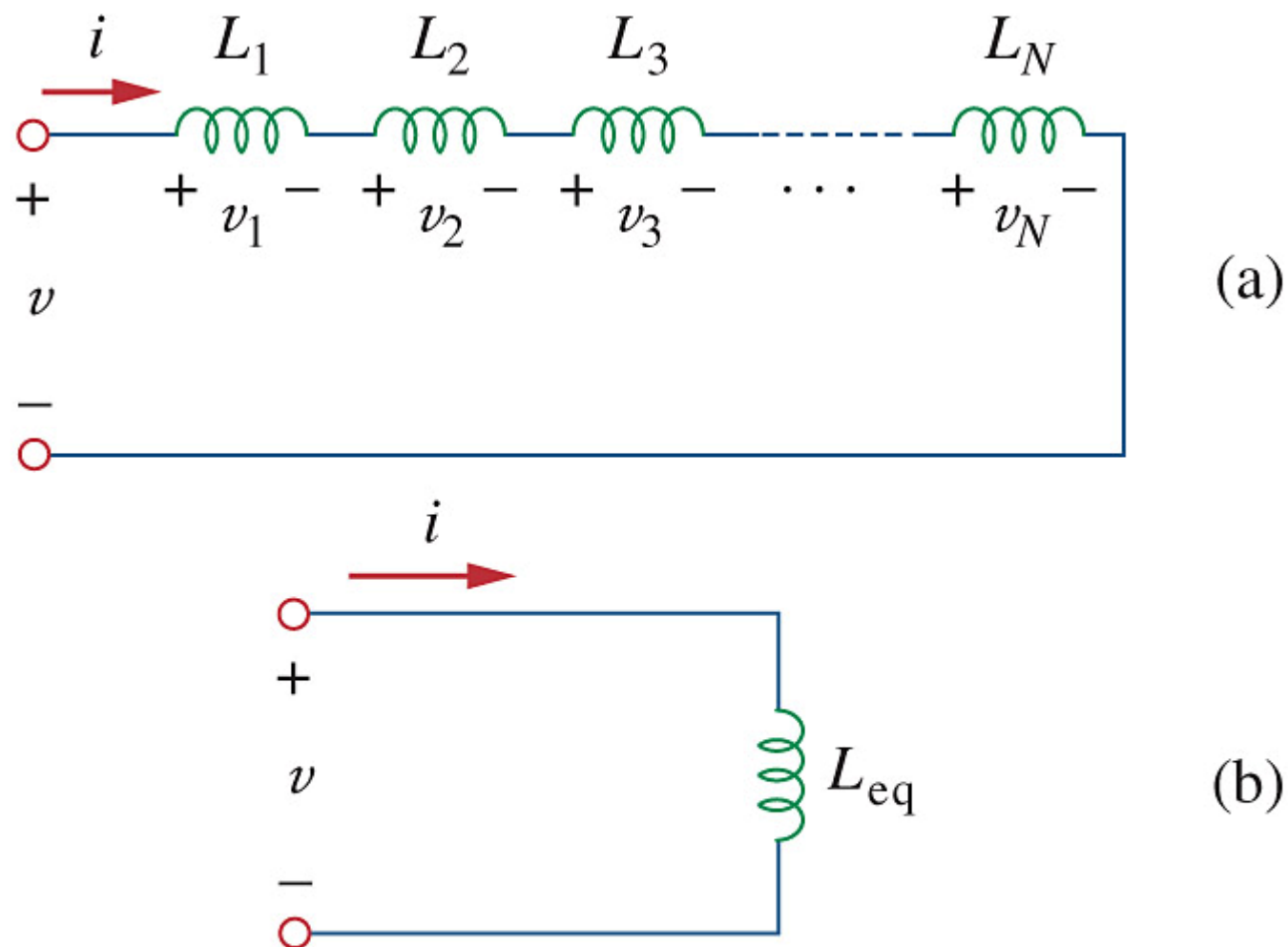


Figure 6.29 (a) A series connection of N inductors, (b) equivalent circuit for the series inductors.

The equivalent inductance of N parallel-connected inductors is the reciprocal of the sum of the reciprocals of the individual inductances:

$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \dots + \frac{1}{L_N}$$

The initial current through L_{eq} is the sum of the individual initial currents:

$$i(t_0) = i_1(t_0) + i_2(t_0) + \dots + i_N(t_0)$$

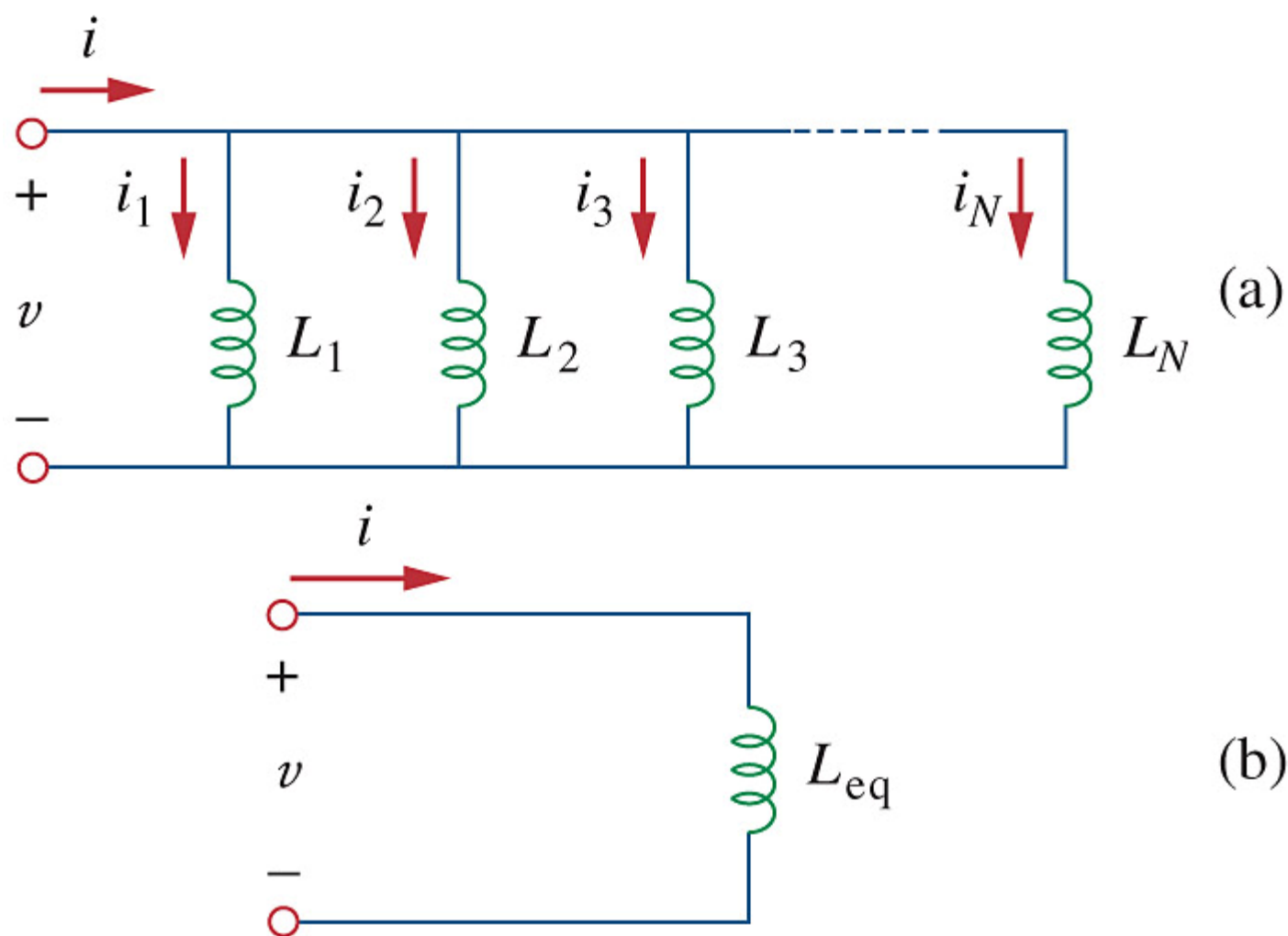


Figure 6.30 A parallel connection of N inductors,
(b) equivalent circuit for the parallel inductors.

Proof :

$$i = \sum_{k=1}^N i_k = \sum_{k=1}^N \left(\frac{1}{L_k} \int_{t_0}^t v dt + i_k(t_0) \right)$$

$$= \left(\sum_{k=1}^N \frac{1}{L_k} \right) \int_{t_0}^t v dt + \sum_{k=1}^N i_k(t_0)$$

$$= \frac{1}{L_{eq}} \int_{t_0}^t v dt + i(t_0)$$

$$\frac{1}{L_{eq}} = \sum_{k=1}^N \frac{1}{L_k}, \quad i(t_0) = \sum_{k=1}^N i_k(t_0)$$

Practice Problem 6.11 Calculate the equivalent inductance for the inductive ladder network in Fig. 6.32.

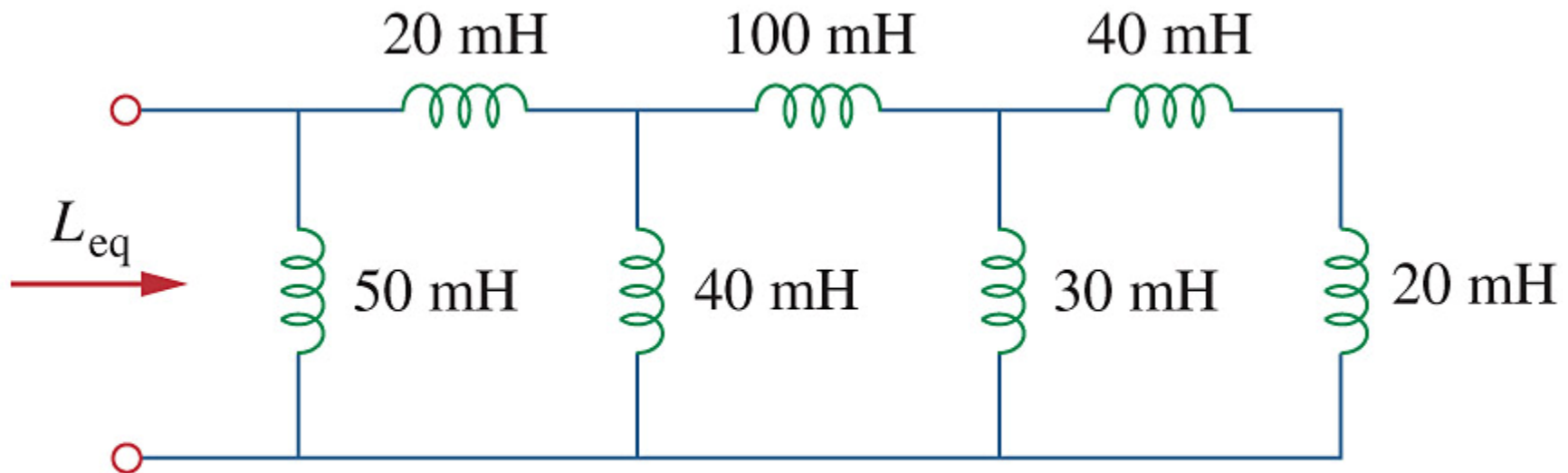


Figure 6.32

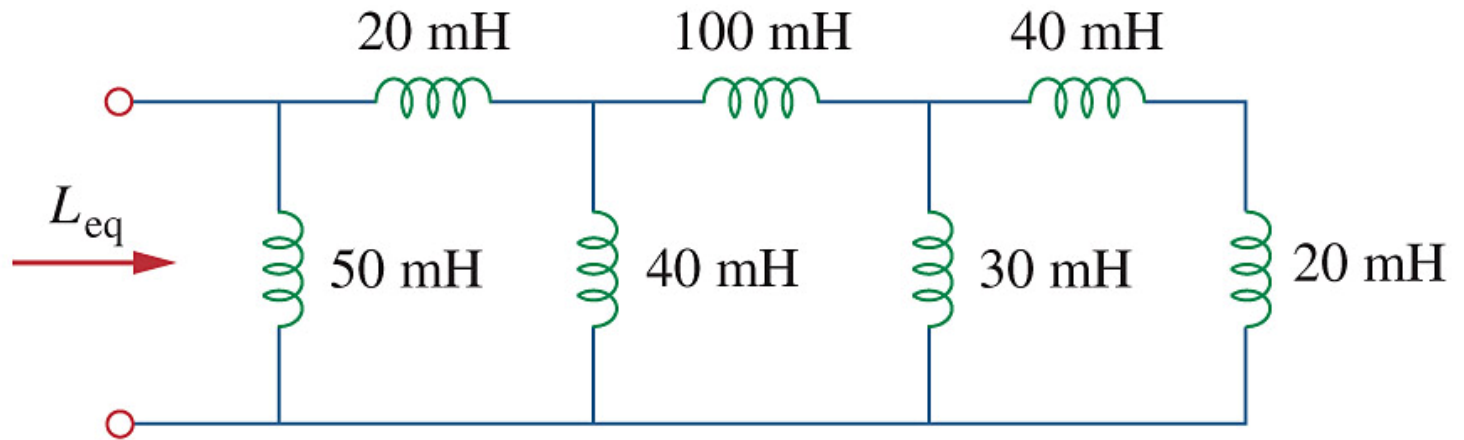


Figure 6.32

Solution :

$$\frac{1}{L_{eq}} = \frac{1}{50} + \frac{1}{20 + \frac{1}{\frac{1}{40} + \frac{1}{100 + \frac{1}{\frac{1}{30} + \frac{1}{40 + 20}}}}}$$

$$L_{eq} = 25 \text{ (mH)}$$

6.6 Applications

- Circuit elements such as resistors and capacitors are commercially available in either discrete form or integrated-circuit form. Inductors usually come in discrete form and tend to be more bulky and expensive.

- There are several applications in which inductors have no practical substitute. They are routinely used in relays, delays, sensing devices, pick-up heads, telephone circuits, radio and TV receivers, power supplies, electric motors, microphones, and loudspeakers, to mention a few.

- The capacity to store energy makes capacitors and inductors useful as **temporary voltage sources and current sources**, respectively.
- Capacitors and inductors are frequency sensitive. This property makes them useful for **frequency discrimination**.

- The property of opposing any abrupt change in voltage makes capacitors useful for **glitch (or spike) elimination**.
- The property of opposing any abrupt change in current makes inductors useful for **spark (or arc) suppression**.
- This property also makes capacitors and inductors useful for converting pulsating dc supply into relatively smooth dc supply.

An *integrator* is an op amp circuit whose output is proportional to the integral of the input. For the circuit in Fig. 6.35,

$$v_o = -\frac{1}{RC} \int_0^t v_i dt + v_o(0)$$

If $v_o(0) = 0$, then

$$v_o = -\frac{1}{RC} \int_0^t v_i dt$$

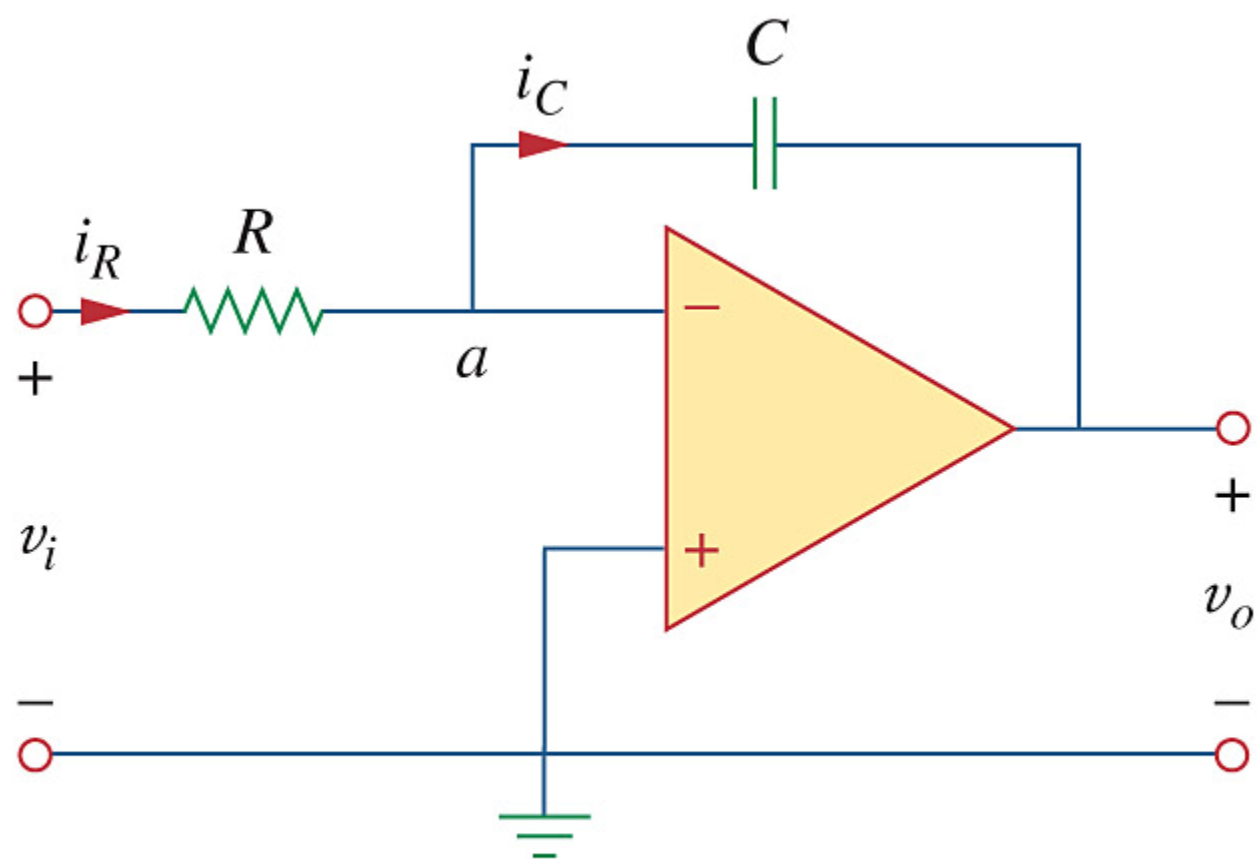


Figure 6.35(b) An integrator circuit.

Proof :

$$\begin{cases} i_R = i_C \\ i_R = \frac{v_i}{R}, i_C = -C \frac{dv_o}{dt} \end{cases} \Rightarrow \frac{dv_o}{dt} = -\frac{v_i}{RC}$$

$$v_o - v_o(0) = -\frac{1}{RC} \int_0^t v_i dt$$

$$v_o = -\frac{1}{RC} \int_0^t v_i dt + v_o(0)$$

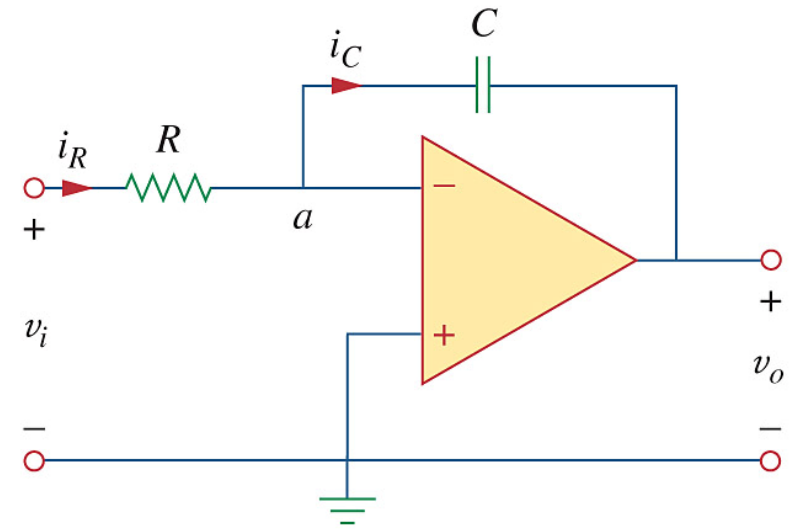


Figure 6.35(b) An integrator circuit.

A differentiator is an op amp circuit whose output is proportional to the derivative of the input. For the circuit in Fig. 6.37,

$$v_o = -RC \frac{dv_i}{dt}$$

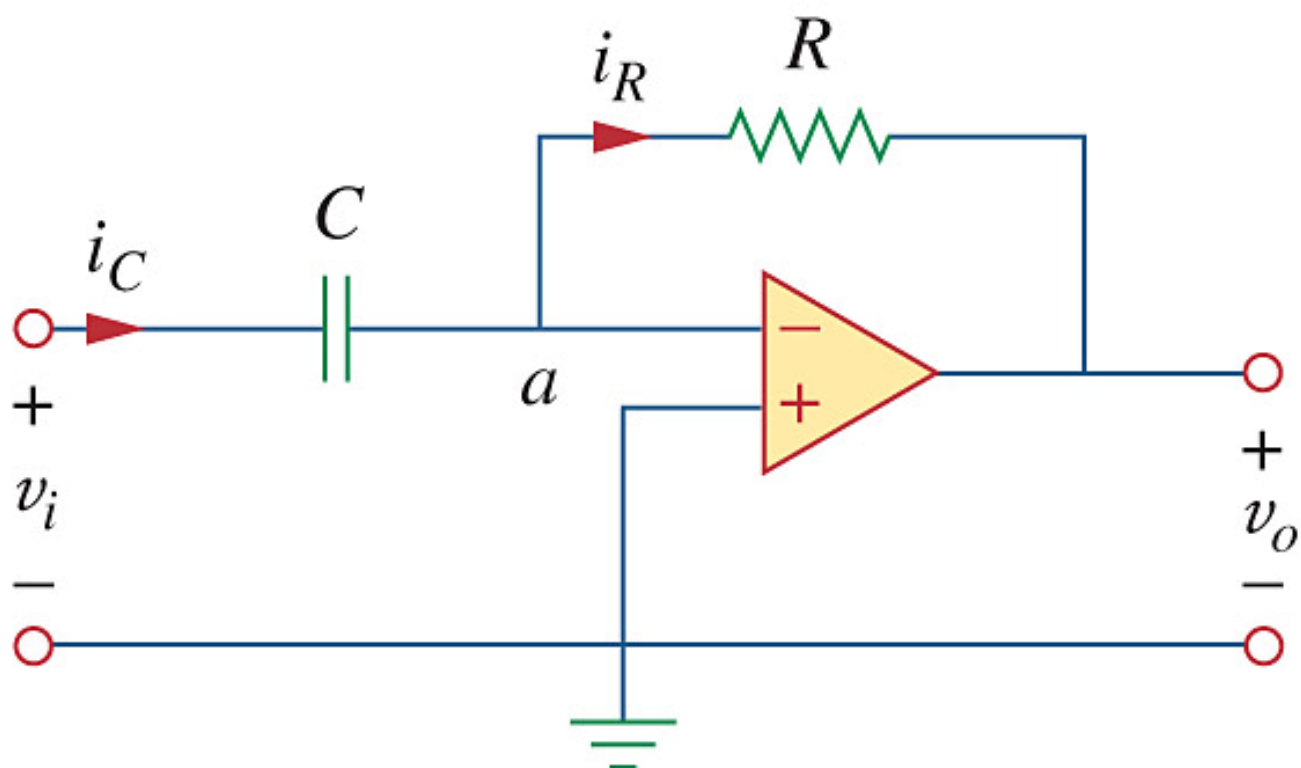


Figure 6.37 A differentiator circuit.

Proof :

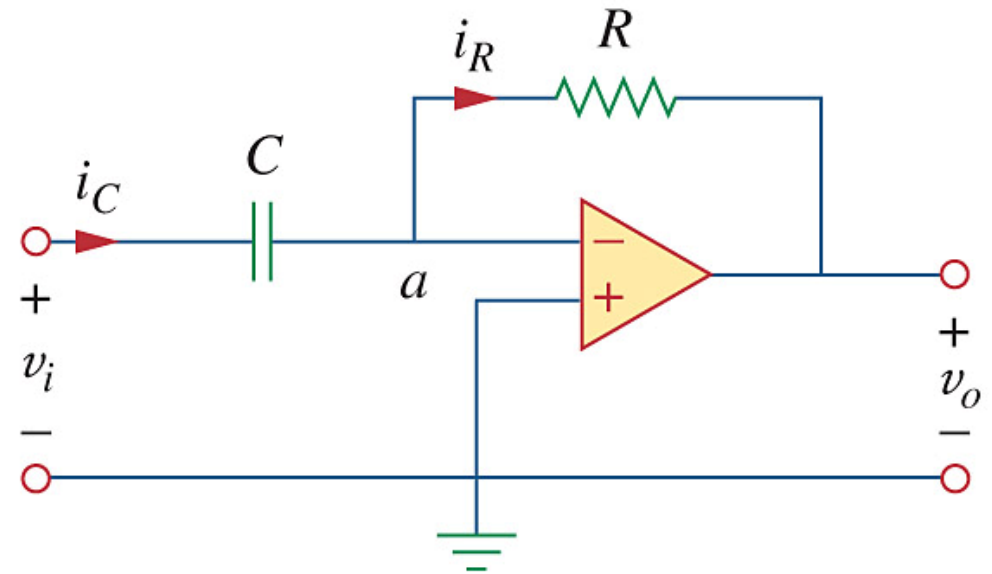


Figure 6.37 A differentiator circuit.

$$\begin{cases} i_C = i_R \\ i_C = C \frac{dv_i}{dt}, i_R = -\frac{v_o}{R} \end{cases} \Rightarrow v_o = -RC \frac{dv_i}{dt}$$

It is better not to use differentiator. If the input signal is contaminated by noise, taking derivative will magnify the noise.

Op amps were initially developed for electronic *analog computers*. Analog computers can be programmed to solve differential equations.

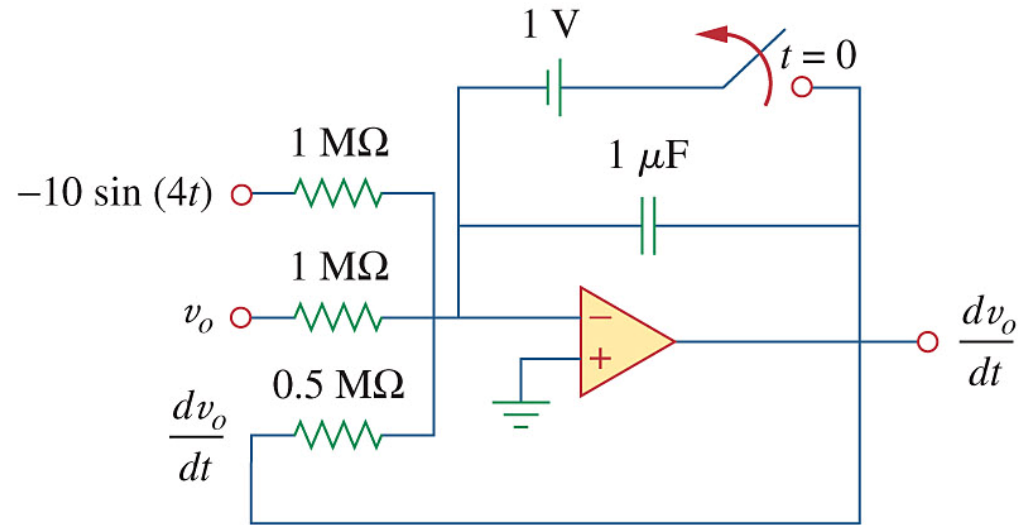
The analog computer consists of three types of op amp circuits: integrator circuits, summing amplifiers, and inverting or noninverting amplifiers. Example 6.15 illustrates how to build an analog computer.

Example 6.15 Design an analog computer circuit to solve the differential equation

$$\frac{d^2 v_o}{dt^2} + 2 \frac{dv_o}{dt} + v_o = 10 \sin 4t, \quad t > 0$$

subject to $v_o(0) = -4$, $v_o'(0) = 1$, where the prime refers to the time derivative.

$$v_o = -\frac{1}{RC} \int_0^t v_i dt + v_o(0)$$



Solution : For $t > 0$,

$$\frac{d^2 v_o}{dt^2} = 10 \sin 4t - 2 \frac{dv_o}{dt} - v_o$$

Figure 6.40(a) The "summing integrator" circuit.

- Use proper resistances to get corresponding coefficients
- The sign should be changed

$$\frac{dv_o}{dt} = \int_0^t \left(10 \sin 4t - 2 \frac{dv_o}{dt} - v_o \right) dt + v_o'(0)$$

We implement this equation using the "summing integrator" in Fig. 6.40(a).

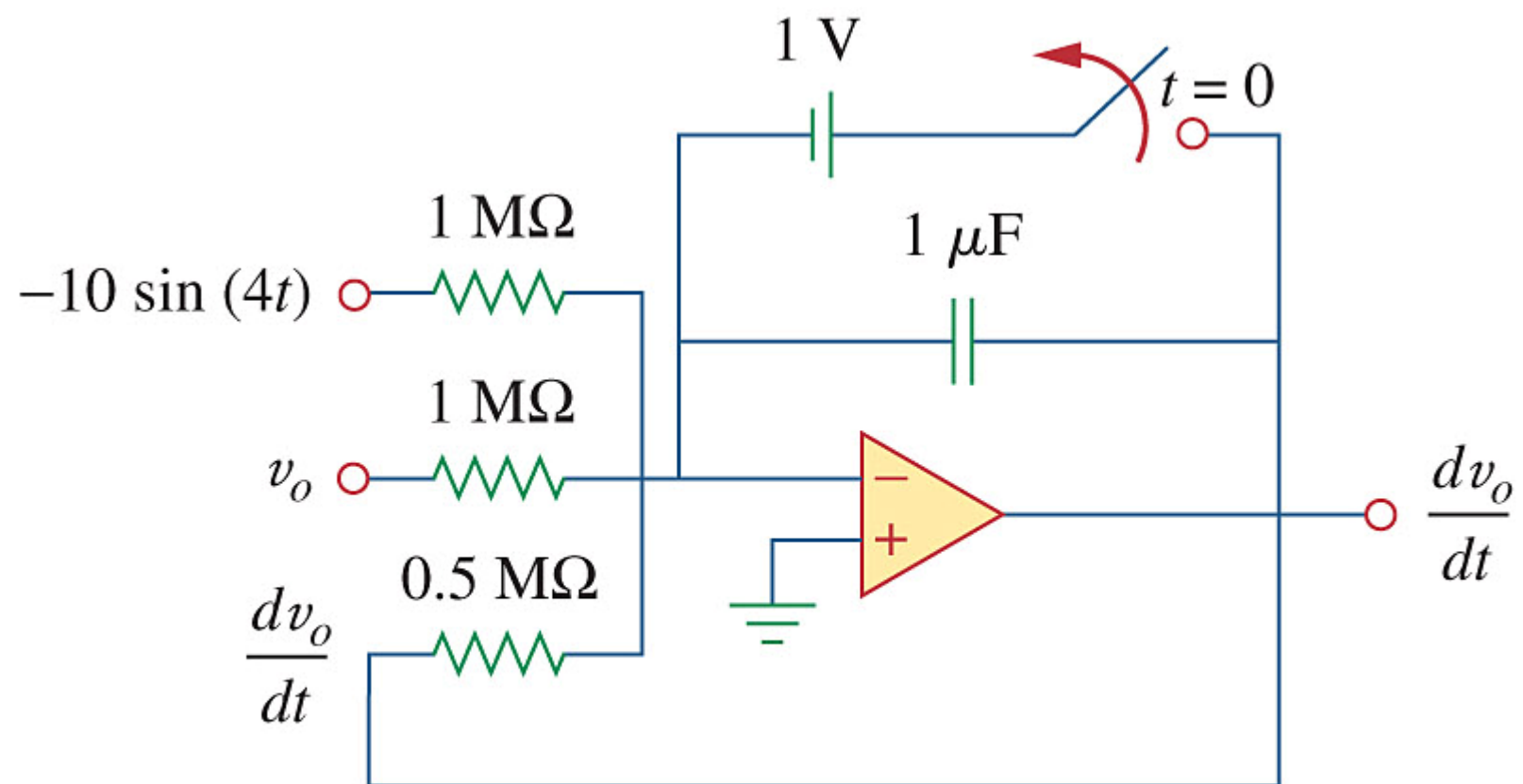
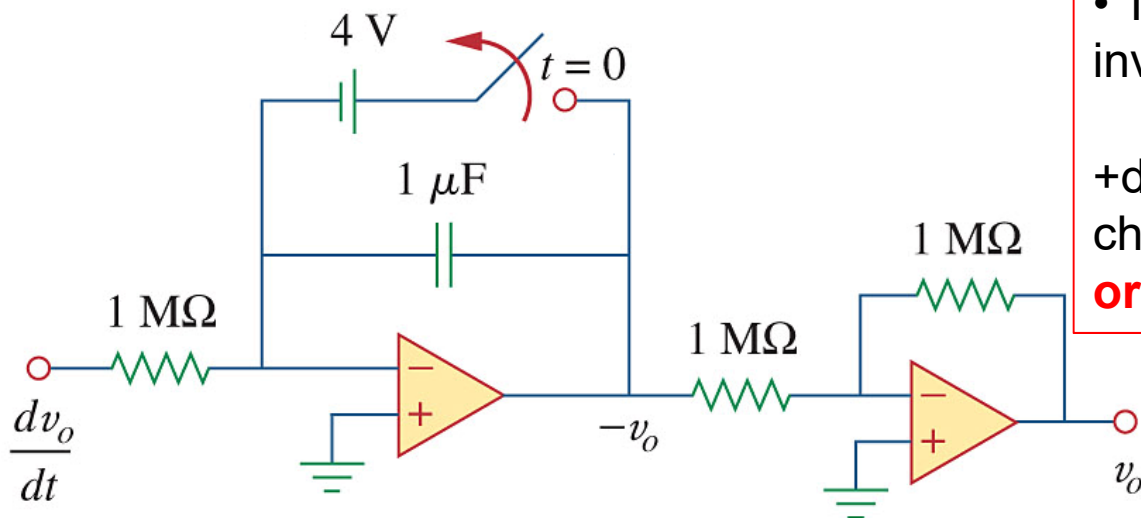


Figure 6.40(a) The "summing integrator" circuit.

$$v_o = \int_0^t \left(\frac{dv_o}{dt} \right) dt + v_o(0)$$

$$v_o = -\frac{1}{RC} \int_0^t v_i dt + v_o(0)$$

This is implemented with the circuit in Fig. 6.40(b). The complete circuit is shown in Fig. 6.40(c).



- Use proper resistances to get corresponding coefficients
- The sign should be changed (an inverter used here)

$+dv_o/dt$ in previous step \rightarrow also choose $+ dv_o/dt$ in this step **in order to cascade them.**

Figure 6.40(b) The cascading of an integrator and an inverting amplifier.

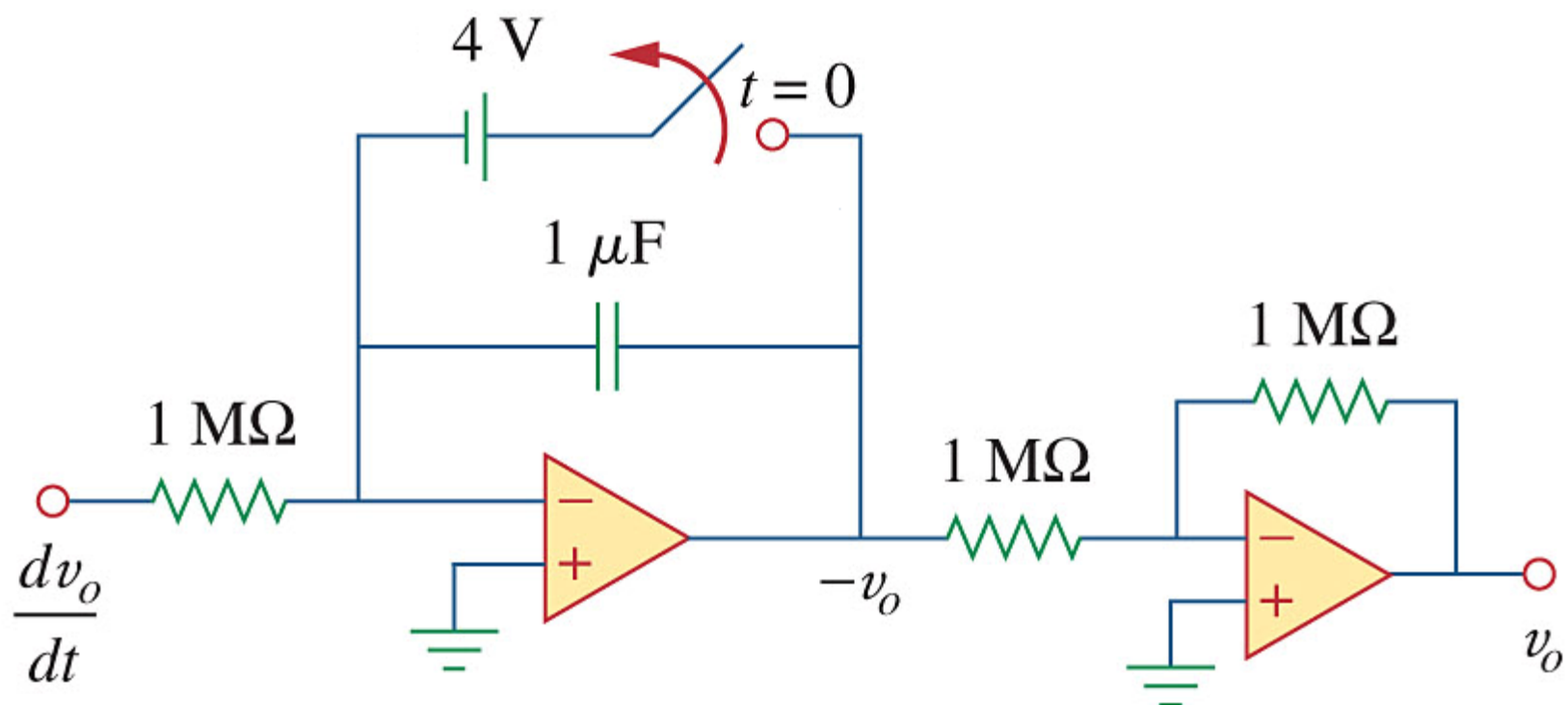


Figure 6.40(b) The cascading of an integrator and an inverting amplifier.

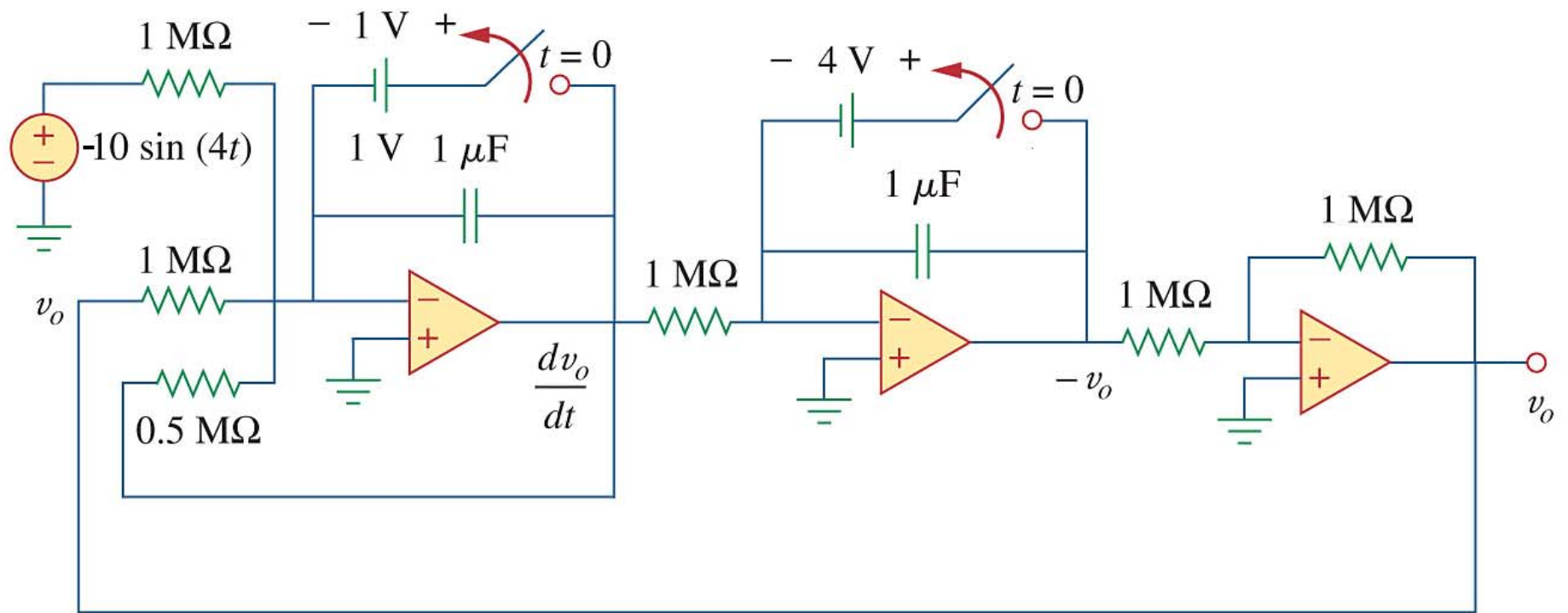
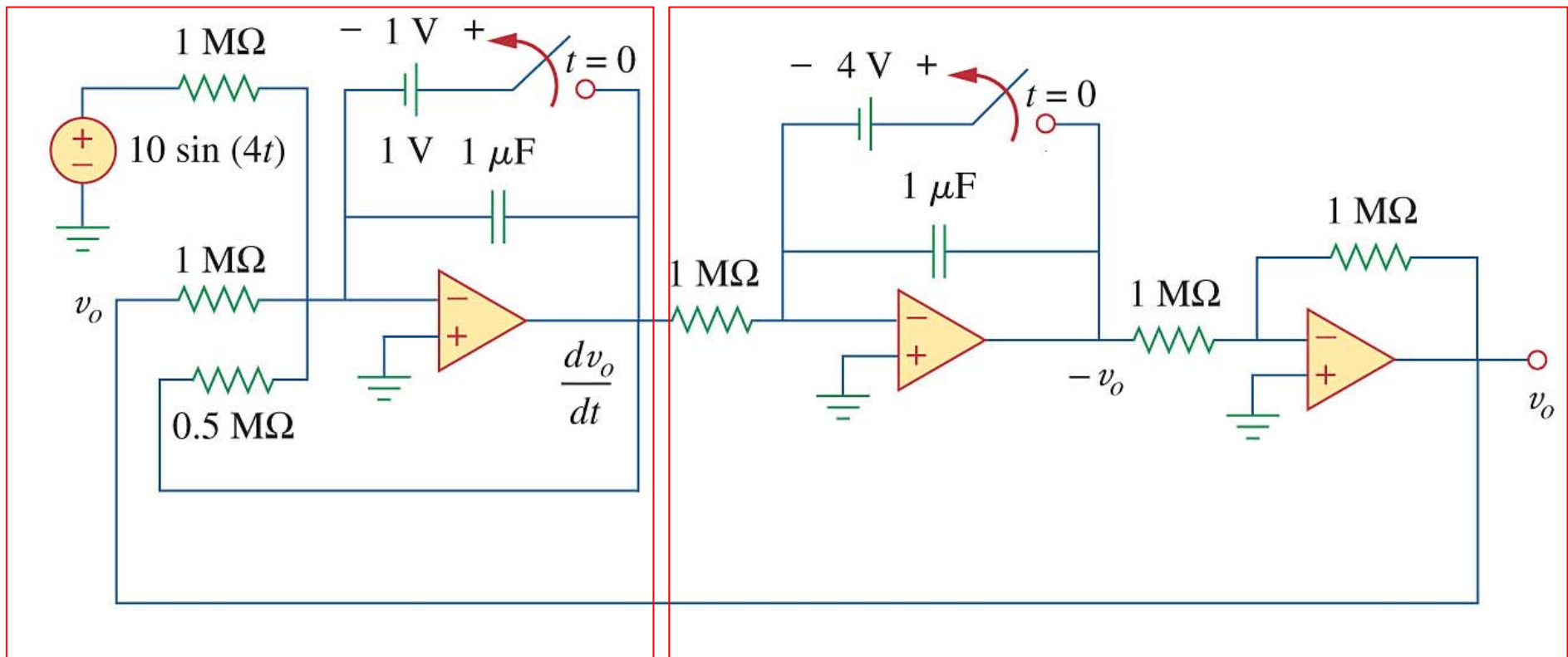


Figure 6.40(c)

- 0. identify v_o'' (move to one side!)
- 1. take integration to get $v_o' = \dots$
 use a summing integrator
 add a DC source and a switch to implement the IC condition
- 2. write down the integration relation between v_o and v_o'
 add a DC source and a switch to implement the IC condition
 (use an inverting amplifier if sign change is needed)
- 3. cascade them (connect dv_o/dt) and connect the v_o

Summing
integrator realized
with dv_o/dt as the
output

Method 1



Step 1
Output: dv_o/dt

Figure 6.40(c)

Step 2
Output: v_o

Practice Problem 6.15 Design an analog computer circuit to solve the differential equation

$$\frac{d^2 v_o}{dt^2} + 3 \frac{dv_o}{dt} + 2v_o = 4 \cos 105t, \quad t > 0$$

subject to $v_o(0) = 2$, $v_o'(0) = 0$.

Solution :

$$\frac{d^2 v_o}{dt^2} = -3 \frac{dv_o}{dt} - 2v_o + 4 \cos 105t$$

$$\frac{dv_o}{dt} = \int_0^t \left(\frac{d^2 v_o}{dt^2} \right) dt + v'(0)$$

$$v_o = \int_0^t \left(\frac{dv_o}{dt} \right) dt + v(0)$$

The circuit is shown in Fig. 6.41, where
 $RC = 1$ s.

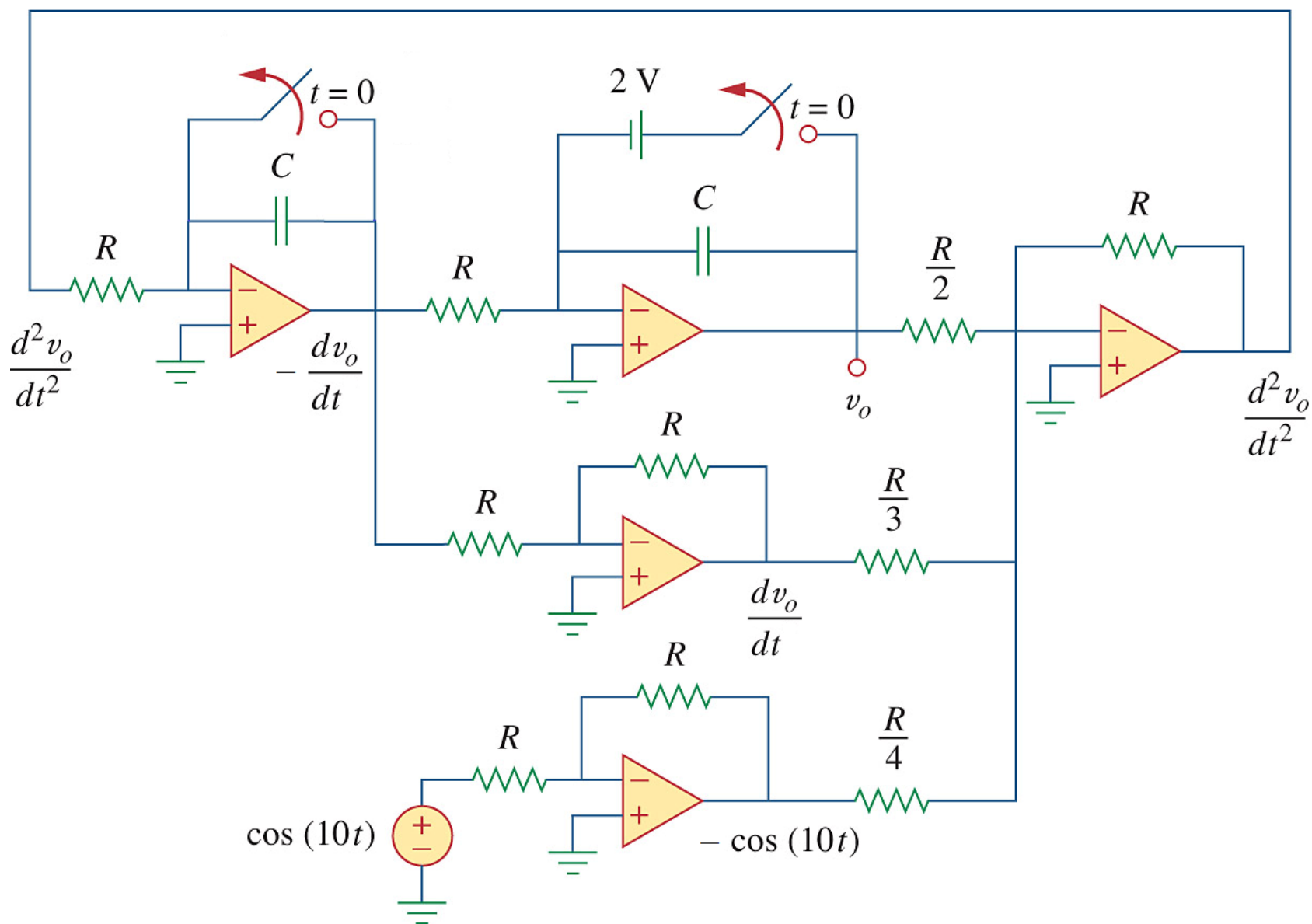


Figure 6.41

Method 2

- 0. identify v_o'' (move to one side!)
- 1. ~~take integration to get $v_o' = \dots$~~
use a summing **amplifier**
~~add a DC source and a switch to implement the IC condition~~
- 2. **write down the integration relation between v_o' and v_o''**
add a DC source and a switch to implement the IC condition
(use an inverting amplifier if sign change is needed)
- 3. write down the integration relation between v_o and v_o'
add a DC and a switch to implement the IC condition
(use an inverting amplifier if sign change is needed)
- 4. **connect v_o''**

Summing
amplifier realized
with d^2v_o/dt^2 as
output

$$v_o = \int_0^t \left(\frac{dv_o}{dt} \right) dt + v(0)$$

step3

step2

$$\frac{dv_o}{dt} = \int_0^t \left(\frac{d^2v_o}{dt^2} \right) dt + v'(0)$$

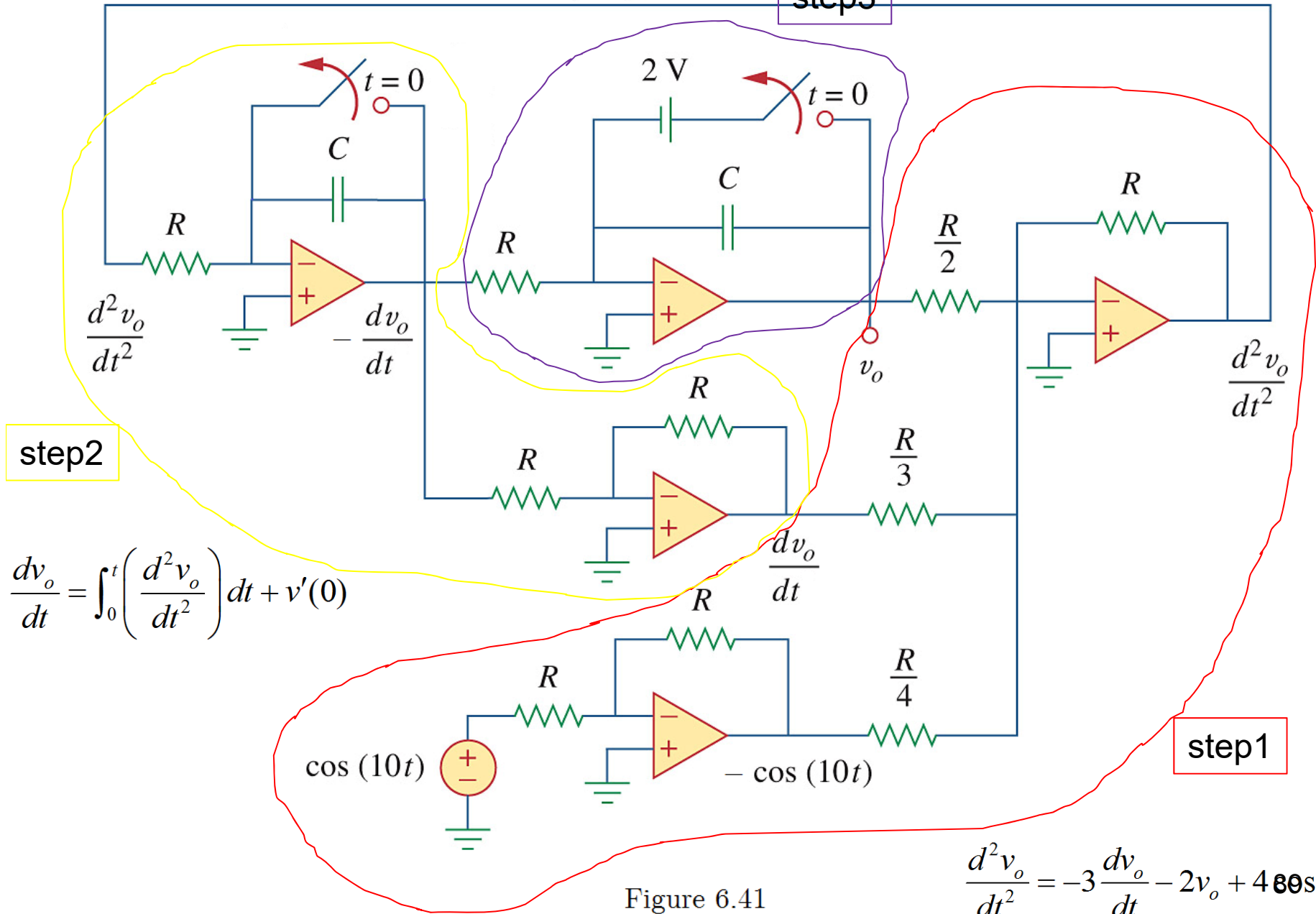
step1

$\cos(10t)$

$-\cos(10t)$

Figure 6.41

$$\frac{d^2v_o}{dt^2} = -3 \frac{dv_o}{dt} - 2v_o + 4 \cos 10t$$



Method 1

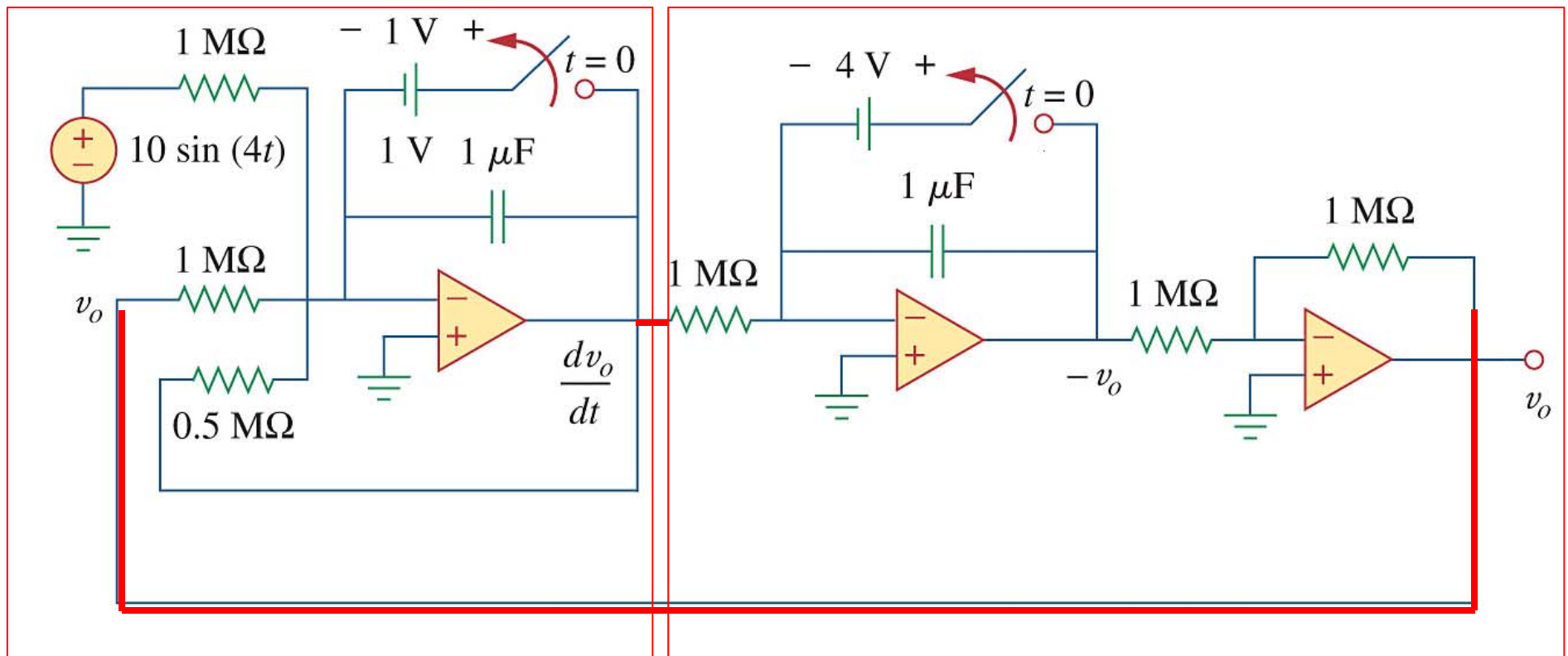
1. *Summing integrator* for v_o'
2. Integrator for v_o'
3. Cascade v_o' and connect v_o

Method 2

1. *Summing amplifier* for v_o''
2. Integrator for v_o'
3. Integrator for v_o
4. Cascade v_o , v_o' and connect v_o''

Method 1

1. Summing integrator for v_o'
2. Integrator for v_o'
3. Cascade v_o' and connect v_o



Step 1
Output: dv_o/dt

Figure 6.40(c)

Step 2
Output: v_o

Method 2

$$v_o = \int_0^t \left(\frac{dv_o}{dt} \right) dt + v(0)$$

1. Summing amplifier for $\frac{d^2 v_o}{dt^2}$
2. Integrator for $\frac{dv_o}{dt}$
3. Integrator for v_o
4. Cascade v_o , $\frac{dv_o}{dt}$ and connect v_o

step3

step2

$$\frac{dv_o}{dt} = \int_0^t \left(\frac{d^2 v_o}{dt^2} \right) dt + v'(0)$$

step1



Figure 6.41

$$\frac{d^2 v_o}{dt^2} = -3 \frac{dv_o}{dt} - 2v_o + 4 \cos 10t$$

00s105