VE230 Chapter 3 Part 1

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1 Basic Concept

Electrostatics:

i. electric charges are **at rest(not moving)**;

ii. electric field **do not change with time**.

2 Electrostatics in Free Space

Static electric charges (source) in free space → electric field

2.1 Electric field intensity

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \quad (\mathbf{V}/\mathrm{m})$$

$\vec F = q\vec E$

2.2 Fundamental Postulates of Electrostatics

- Differential form:

$$\begin{cases} ① \ \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \\ ② \ \nabla \times \mathbf{E} = 0 \end{cases}$$

$\Rightarrow \begin{cases} \int_V \nabla \cdot \vec E \, dv = \int_V \frac{\rho}{\varepsilon_0} dv = \frac{1}{\varepsilon_0} \int_V \rho \, dv \\ \int_V \nabla \cdot \vec E \, dv = \oint_S \vec E \cdot d\vec S \quad = \frac{Q}{\varepsilon_0} \end{cases}$

- Integral form:

$$\begin{cases} \oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0} \\ \oint_C \mathbf{E} \cdot d\ell = 0 \end{cases}$$

(Gauss' Law)

$\begin{cases} \int_S \nabla \times \vec E \cdot d\vec S = 0 \\ \int_S \nabla \times \vec E \cdot d\vec S = \oint_L \vec E \cdot d\vec l \end{cases}$

E is **not solenoidal**, but **irrotational (conservative)**

3 Coulomb's Law

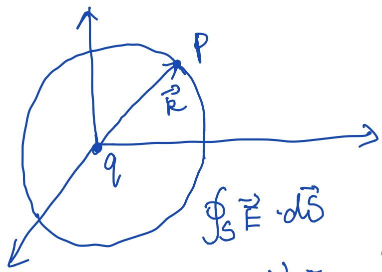
3.1 Electric Field due to a System of Discrete Charges

$\vec E = \hat a_R E_R$

- **a single point charge (charge on the origin):**

$\hat a_R = \frac{\vec R}{|\vec R|}$

$$\frac{\vec R q}{4\pi\varepsilon_0 R^3} = \mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \quad (\mathbf{V}/\mathrm{m})$$

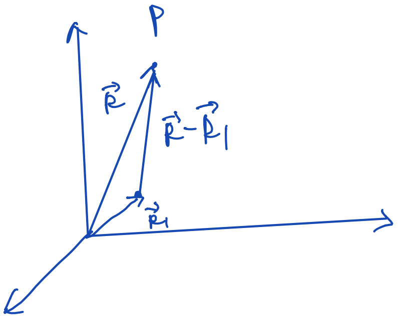


$\oint_S \vec E \cdot d\vec S = 4\pi R^2 \cdot E = \frac{q}{\varepsilon_0}$

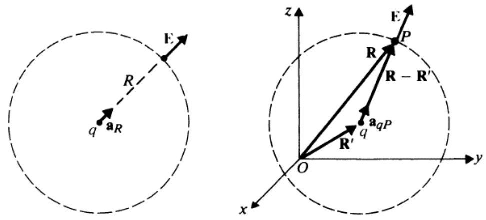
$E = \frac{q}{4\pi R^2 \varepsilon_0}$

- **a single point charge (charge is not on the origin):**

$$\mathbf{E}_p = \frac{q\,(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 \,|\mathbf{R} - \mathbf{R}'|^3} \quad (\mathbf{V}/\mathrm{m})$$



$\frac{\vec R}{|\vec R|} \rightarrow \vec R - \vec R_1$



(a) Point charge at the origin. (b) Point charge not at the origin.

FIGURE 3–2
Electric field iFIGURE due to a point charge.

Example:
Determine the electric field intensity at P(-0.2,0,-2.3) due to a point charge of +5(nC) at Q(0.2,0.1,-2.5) in air. (All dimensions are in meters).

$\vec{R}_1 = \hat{a}_x 0.2 + \hat{a}_y 0.1 - \hat{a}_z 2.5$

$\vec{R} = -\hat{a}_x 0.2 - \hat{a}_z 2.3$

$\vec{R} - \vec{R}_1 = -\hat{a}_x 0.4 - \hat{a}_y 0.1 + \hat{a}_z 0.2$

$|\vec{R} - \vec{R}_1| = \sqrt{0.4^2 + 0.1^2 + 0.2^2}$

$\vec{E} = \dfrac{(\vec{R} - \vec{R}_1) \cdot q}{4\pi \epsilon_0 |\vec{R} - \vec{R}_1|^3}$

$\epsilon_0 = \dfrac{1}{c^2 \mu_0}$

- **several point charges:**

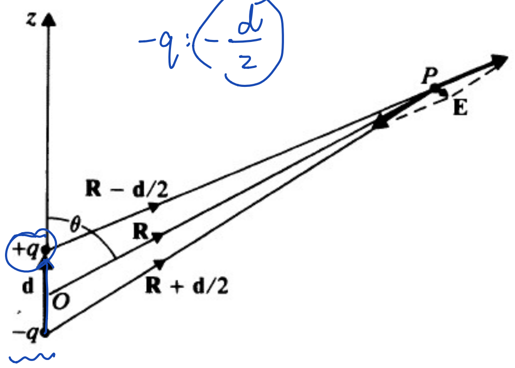
$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k \left(\mathbf{R} - \mathbf{R}_k'\right)}{\left|\mathbf{R} - \mathbf{R}_k'\right|^3}}$$

3.2 Electric Dipole

- **Electric Field**

$+q: \frac{\vec{d}}{2}$

$-q: -\frac{\vec{d}}{2}$



$\vec{d}$

general expression:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{\mathbf{R} - \frac{\mathbf{d}}{2}}{\left|\mathbf{R} - \frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R} + \frac{\mathbf{d}}{2}}{\left|\mathbf{R} + \frac{\mathbf{d}}{2}\right|^3} \right\}$$

if $d \ll R$:

$|\vec{A}| = \sqrt{\vec{A} \cdot \vec{A}}$

$$\left|\vec{R} - \frac{\vec{d}}{2}\right|^{-3} = \left[\left(\vec{R} - \frac{\vec{d}}{2}\right) \cdot \left(\vec{R} - \frac{\vec{d}}{2}\right)\right]^{-\frac{3}{2}}$$

$$= \left[R^2 - \vec{R} \cdot \vec{d} + \frac{d^2}{4}\right]^{-\frac{3}{2}}$$

$$= (R^2)^{-\frac{3}{2}} \left[1 - \frac{\vec{R} \cdot \vec{d}}{R^2} + \cancel{\frac{d^2}{4R^2}}\right]^{-\frac{3}{2}}$$

$$\simeq R^{-3} \underbrace{\left[1 - \frac{\vec{R} \cdot \vec{d}}{R^2}\right]^{-\frac{3}{2}}} \simeq 1 + \left(-\frac{3}{2}\right)\left(-\frac{\vec{R} \cdot \vec{d}}{R^2}\right) + \cdots$$

$$\simeq R^{-3} \left(1 + \frac{3}{2} \frac{\vec{R} \cdot \vec{d}}{R^2}\right)^{-\frac{3}{2}}$$

$$\left|\vec{R} + \frac{\vec{d}}{2}\right|^{-3} \simeq R^{-3} \left(1 - \frac{3}{2} \frac{\vec{R} \cdot \vec{d}}{R^2}\right)^{-\frac{3}{2}}$$

if $d << R$:

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right] = \frac{1}{4\pi\epsilon_0}\left[\frac{3\vec{R}\cdot \overset{\vec{p}}{q\vec{d}}}{R^5} - \overset{\vec{p}}{q\vec{d}}\right]$$

- **Electric Dipole Moment Definition:**

$\hat{a}_z = \hat{a}_R\cos\theta - \hat{a}_\theta \sin\theta$

$$\mathbf{p} = q\mathbf{d}$$

$$\mathbf{p} = \mathbf{a}_z p = p\left(\mathbf{a}_R\cos\theta - \mathbf{a}_\theta\sin\theta\right)$$

$$\underset{\hat{a}_R R}{\mathbf{R}}\cdot\mathbf{p} = Rp\cos\theta$$

- **Electric Field:**

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}\left(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta\sin\theta\right) \quad (\mathrm{V/m})$$

3.3 Electric Field due to a Continuous Distribution of Charge

- **General Differential Element:**

$$d\mathbf{E} = \mathbf{a}_R\frac{\rho dv'}{4\pi\epsilon_0 R^2}$$

dq

- **Line Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'}\mathbf{a}_R\frac{\rho_\ell}{R^2}d\ell' \quad (\mathbf{V/m})$$

- **Surface Charge:**

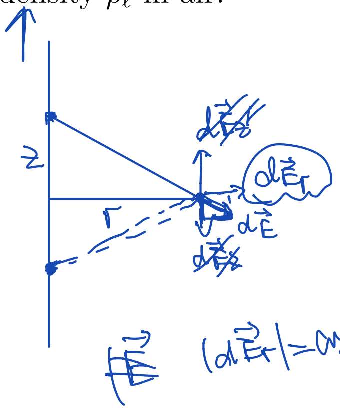
$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{S'}\mathbf{a}_R\frac{\rho_s}{R^2}ds' \quad (\mathbf{V}/\mathrm{m})$$

- **Volume Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'}\mathbf{a}_R\frac{\rho}{R^2}dv' \quad (\mathbf{V/m})$$

Example:
Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ in air.

$\int_{-\infty}^{\infty}$



$|d\vec{E}_r| = \cos\theta\,|d\vec{E}|$

$$\int d\vec{E} = \int d\vec{E}_r$$

$$d\vec{E}_r = \frac{\rho_l r\, dl}{4\pi\epsilon_0 r^3} \quad dz$$

$$= \frac{\rho_l}{4\pi\epsilon_0}\cdot\frac{\vec{r}}{(r^2+z^2)^{\frac{3}{2}}}dz$$

$$\int d\vec{E}_r = \int_{-\infty}^{\infty}\frac{\rho_l}{4\pi\epsilon_0}\frac{r}{(r^2+z^2)^{\frac{3}{2}}}dz$$

$$= \frac{\hat{a}_r\rho_l r}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{dz}{(r^2+z^2)^{\frac{3}{2}}}$$

$$\vec{E} = \hat{a}_r\frac{\rho_l}{2\pi\epsilon_0 r}$$

4 Gauss's Law and Application

4.1 Definition

The total outward flux of the E-field over any closed surface in free space is equal to **the total charge enclosed in the surface** divided by ϵ_0.

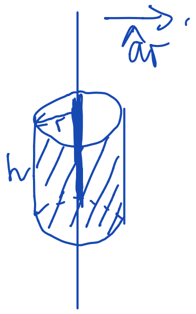
$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

4.2 Application

- **Conditions for Maxwell's Integral Equations:**
  There is **a high degree of symmetry** in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc.

Example:
Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ in air.

$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$



$\oint_S \vec{E} \cdot d\vec{S} = 2\pi r h \cdot E$

$Q_{total} = \rho_l \cdot h$

$2\pi r h \cdot E = \frac{Q}{\epsilon_0} = \frac{\rho_l \cdot h}{\epsilon_0}$

$E = \frac{\rho_l}{2\pi r \epsilon_0}$

$\vec{E} = \hat{a}_r \frac{\rho_l}{2\pi\epsilon_0 r}$

Determine the **E** field caused by a spherical cloud of electrons with a volume charge density $\rho = -\rho_o$ for $0 \leq R \leq \cancel{B}$ b (both ρ_o and b are positive) and $\rho = 0$ for $R > b$.

① $0 \leq R \leq b$.

$\oint_S \vec{E} \cdot d\vec{S} = \frac{Q}{\epsilon_0}$

$\oint_S \vec{E} \cdot d\vec{S} = 4\pi R^2 \cdot E$

$Q_{total} = \rho V = -\rho_0 \cdot \frac{4\pi R^3}{3}$

$4\pi R^2 \cdot E = -\frac{\rho_0 4\pi R^3}{3\epsilon_0}$

$\vec{E} = -\hat{a}_R \frac{\rho_0 R}{3\epsilon_0}$

② $R > b$.

$\oint_S \vec{E} \cdot d\vec{S} = 4\pi R^2 \cdot E$

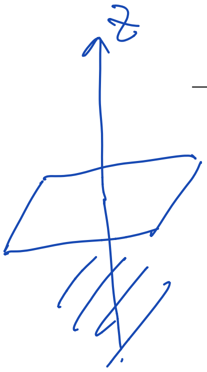
$Q_{total} = -\rho_0 \frac{4\pi b^3}{3}$

$4\pi R^2 \cdot E = -\rho_0 \frac{4\pi b^3}{3\epsilon_0}$

$\vec{E} = -\hat{a}_R \frac{\rho_0 b^3}{3R^2\epsilon_0}$

4.3 Several Useful Models

Note: The charge distribution should be **uniform**.



different models	E(magnitude)
infinitely long, line charge $\hat{a}_r$	$E = \frac{\rho_\ell}{2\pi r\epsilon_0}$
infinite planar charge $\hat{a}_z$ / $-\hat{a}_z$	$E = \frac{\rho_s}{2\epsilon_0}$
$\hat{a}_R$ uniform spherical surface charge with radius R	$\begin{cases} E = 0 (r < R) \\ E = \frac{Q}{4\pi r^2\epsilon_0}(r > R) \end{cases}$
uniform sphere charge with radius R	$\begin{cases} E = \frac{Qr}{4\pi R^3}(r < R) \\ E = \frac{Q}{4\pi r^2\epsilon_0}(r > R) \end{cases}$
$\hat{a}_r$ infinitely long, cylindrical charge with radius R	$\begin{cases} E = \frac{\rho_v r}{2\epsilon_0}(r < R) \\ E = \frac{\rho_v R^2}{2r\epsilon_0}(r > R) \end{cases}$

5 Electric Potential

$\vec{E}$ increasing of V

- **Expression:**

$$\mathbf{E} = -\nabla V$$

- **Electric Potential Difference:**

P_1, P_2

$$V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell$$

$V = \frac{a}{4\pi\epsilon_0 R}\ (V)$

a single point charge:

$\vec{E} = \hat{a}_R \frac{a}{4\pi R^2}$

$\Rightarrow V = -\int_{-\infty}^{R} \vec{E}\, d\vec{l}$

$= -\int_{-\infty}^{R} \left(\hat{a}_R \frac{a}{4\pi R^2}\right) \cdot \hat{a}_R dR$

$= \frac{a}{4\pi\epsilon_0 R}\ (V)$

- **Electric Potential due to a Charge Distribution**

  i. **Line Charge:**

$$V = \frac{1}{4\pi\epsilon_0}\int_{L'} \frac{\rho_\ell}{R} d\ell' \quad (V)$$

  ii. **Surface Charge:**

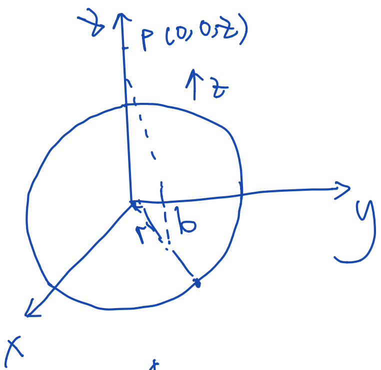
$$V = \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{\rho_s}{R} ds' \quad (V)$$

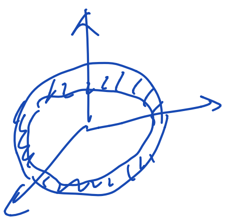
  – **Volume Charge:**

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'} \frac{\rho}{R} dv' \quad (V)$$

- **Example:**
  Obtain a formula for the electric field intensity and potential on the axis of a circular disk of radius b that carries a uniform surface charge ρ_s.

Example 3-9





① find V
② $\vec{E} = -\nabla V$

② $\vec{E} = -\nabla V$

$ds = r\,dr\,d\phi$

$V = \int_S \frac{\rho_s}{4\pi\epsilon_0} ds$

$= \int_0^{2\pi}\int_0^b \frac{\rho_s}{4\pi\epsilon_0} \cdot r\,dr\,d\phi$

$= \frac{\rho_s}{2\epsilon_0}\left[(z^2+b^2)^{1/2} - |z|\right]$

$$\nabla^2 \frac{1}{r} = \begin{cases} 0 & r \neq 0 \\ -4\pi\delta(\vec{r}) & r = 0. \end{cases}$$

$$\nabla^2 \frac{1}{r} = \nabla \cdot \left(\nabla \frac{1}{r}\right) \longrightarrow \qquad \nabla \frac{1}{r} = \hat{a}_r \frac{\partial}{\partial r}\left(\frac{1}{r}\right) = -\frac{\hat{a}_r}{r^2}$$

$$= \nabla \cdot \left(-\frac{\vec{r}}{r^3}\right) = -\nabla \cdot \frac{\vec{r}}{r^3} \qquad = -\frac{\vec{r}}{r^3}$$

$$= -\left(\frac{1}{r^3} \nabla \cdot \vec{r} + \vec{r}\, \nabla \frac{1}{r^3}\right). \qquad \nabla \cdot (\psi \vec{A}) = \psi \nabla \cdot \vec{A} + \vec{A} \cdot \nabla \psi$$

Scalar vector

$r \neq 0$

$$= \frac{3}{r^3} - 3\vec{r} \cdot \frac{\vec{r}}{r^5}$$

$$= 0$$

$$\frac{1}{r}$$

$$\hat{r}\, dS$$

$$\int \left(\nabla \cdot \frac{\vec{r}}{r^3}\right) dV \quad \lim_{R \to \infty} \oint_S \frac{\vec{r}}{r^3}\, d\vec{S}. = \lim_{R \to \infty} \oint \frac{1}{r^2}\, dS.$$

$$= 4\pi$$

$$r = 0 \Rightarrow \nabla \cdot \frac{\vec{r}}{r^3} = 4\pi \qquad \Rightarrow \qquad \nabla^2 \frac{1}{r} = -4\pi$$

$$\Rightarrow \quad \nabla^2 \frac{1}{r} = -4\pi\, \delta(\vec{r}).$$