Static Electric Fields

1 Materials to Cover

- Vector Analysis <Vector Algebra, Orthogonal Coordinate Systems, Vector Calculus>
- Static Electric Fields Part I <Electrostatic Equations, Solving E, Conductors and Insulators>
- Static Electric Fields Part II <Boundary Condition, Capacitor, Electrostatic Energy and Force>

2 Vector Algebra

2.1 Vector Operation

2.1.1 Dot Product

- Definition: $\vec{A}\vec{B} = |A||B|cos\theta_{AB}$
- Commutative Law $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- Associative Law: **Not** Associative
- Distribution Law: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
- In XYZ Coordinate: $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$

2.1.2 Cross Product

- Definition: $\vec{A} \times \vec{B} = |A||B|sin\theta_{AB}\hat{c}$ where $\hat{c} = \hat{a} \times \hat{b}$
- Commutative Law **Not** commutative $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- Associative Law: **Not** Associative
- Distribution Law: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

- In XYZ Coordinate: $\vec{A}\times\vec{B} = (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$

Comments:

1 Triple Cross Product (bac-cab): $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

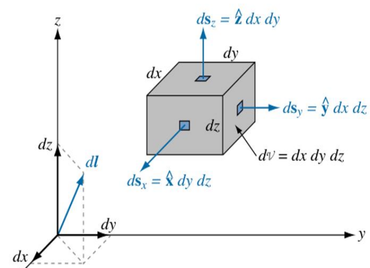
2. 3. Scalar Triple Product:

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = det(\begin{bmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{bmatrix})$$

2.2 Orthogonal Coordinate System

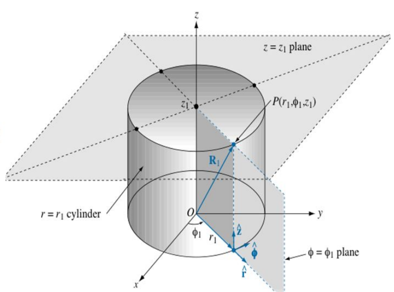
Definition: Three perpendicular coordinate systems can form a orthogonal coordinate system. And the unit vectors are defined as the **normal** direction of each coordinate surface.

2.2.1 basic intro



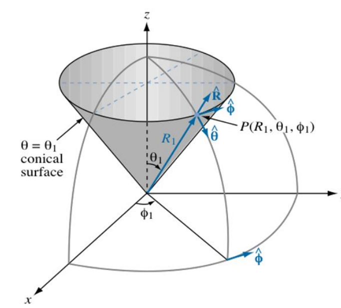
Cartesian Coordinates

(x, y, z)



Cylindrical Coordinates

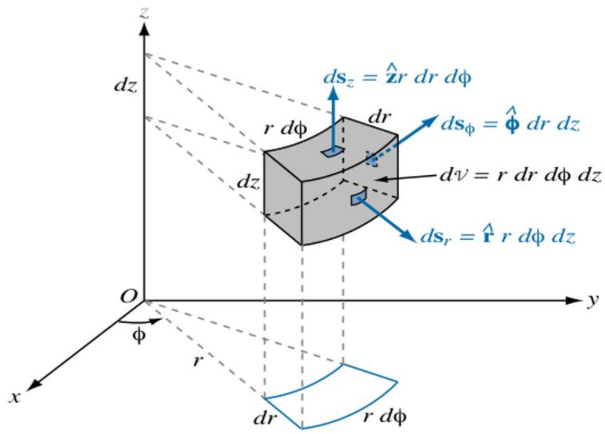
(r, ϕ, z)

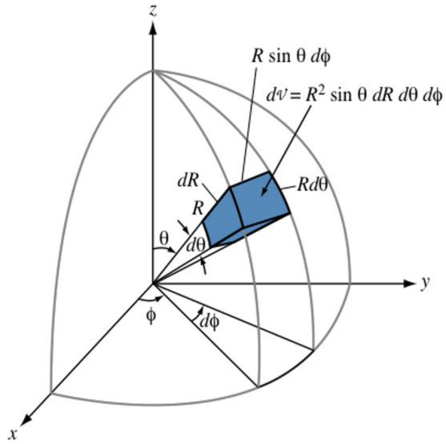


Spherical Coordinates

(R, θ, ϕ)

Cartesian Coordinate	Cylindrical Coordinate	Spherical Coordinate
$\bar{OP} = \boldsymbol{a_x}x_1 + \boldsymbol{a_y}y_1 + \boldsymbol{a_z}z_1$	$\bar{OP} = \boldsymbol{a_r}r_1 + \boldsymbol{a_z}z_1$	$\bar{OP} = \boldsymbol{a_r}r_1$
$\boldsymbol{A} = \boldsymbol{a_x}A_x + \boldsymbol{a_y}A_y + \boldsymbol{a_z}A_z$	$\boldsymbol{A} = \boldsymbol{a_r}A_r + \boldsymbol{a_\phi}A_\phi + \boldsymbol{a_z}A_z$	$\boldsymbol{A} = \boldsymbol{a_R}A_R + \boldsymbol{a_\theta}A_\theta + \boldsymbol{a_\phi}A_\phi$
$h_1 = h_2 = h_3 = 1$	$h_1 = h_3 = 1, h_2 = r$	$h_1 = 1, h_2 = R, h_3 = Rsin\theta$
$d\boldsymbol{l} = \boldsymbol{a_x}dx + \boldsymbol{a_y}dy + \boldsymbol{a_z}dz$	$d\boldsymbol{l} = \boldsymbol{a_r}dr + \boldsymbol{a_\phi}rd\phi + \boldsymbol{a_z}dz$	$d\boldsymbol{l} = \boldsymbol{a_R}dR + \boldsymbol{a_\theta}Rd\theta + \boldsymbol{a_\phi}Rsin\theta d\phi$
$ds_x = dydz$ $ds_y = dxdz$ $ds_z = dxdy$	$ds_r = rd\phi dz$ $ds_\phi = drdz$ $ds_z = rdrd\phi$	$ds_R = R^2sin\theta d\theta d\phi$ $ds_\theta = Rsin\theta dRd\phi$ $ds_\phi = RdRd\theta$
$dv = dxdydz$	$dv = rdrd\phi dz$	$dv = R^2sin\theta dRd\theta d\phi$





2.2.2 Vector Transformation in Different Coordinate

Key Points: $\vec{A} = A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z}$

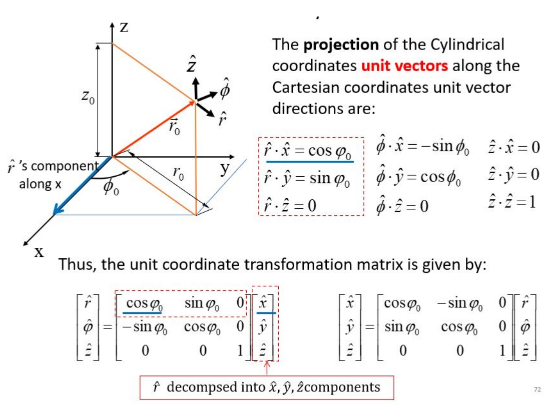
1. Unit vector transformation matrix
2. Corresponding magnitude of each direction
3. Calculate the needed elements' value like r,θ,ϕ, R,x,y,z in it and replace them.

A• ~ on both sides

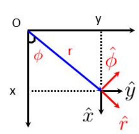
A•x=A_x

$$\begin{bmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{bmatrix} = \begin{bmatrix}\cos\varphi_0 & -\sin\varphi_0 & 0\\ \sin\varphi_0 & \cos\varphi_0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\hat{r}\\ \hat{\varphi}\\ \hat{z}\end{bmatrix} \Rightarrow \boxed{\begin{bmatrix}A_x\\ A_y\\ A_z\end{bmatrix} = \begin{bmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}A_r\\ A_\phi\\ A_z\end{bmatrix}.}$$

Application I: Cartesian to cylindrical

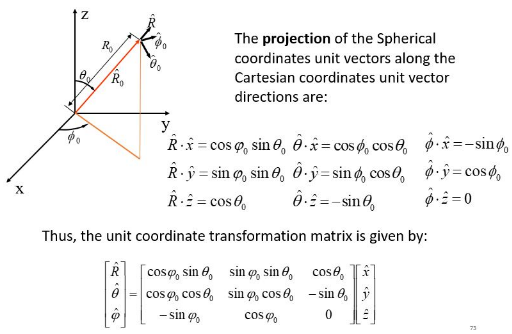


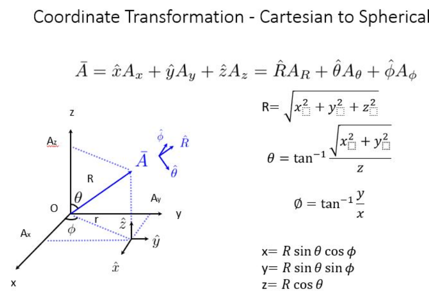
$$\bar{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$$



$$\boxed{\begin{aligned} x &= r\cos\phi,\\ y &= r\sin\phi,\\ z &= z.\end{aligned}} \qquad \boxed{\begin{aligned} r &= \sqrt{x^2+y^2},\\ \phi &= \tan^{-1}\frac{y}{x},\\ z &= z.\end{aligned}}$$

Application II: Cartesian to spherical





2.3 vector calculus

2.3.1 Integrals

$$\int_V \mathbf{F}\,dv,$$
$$\int_C V\,d\ell, \qquad \oint_C V\,d\ell.$$
$$\int_C \mathbf{F}\cdot d\ell,$$
$$\int_S \mathbf{A}\cdot d\mathbf{s}.$$

Key Points:

1. Determine it is a scalar or a vector
2. For vector, find the corresponding magnitude in each unit vector direction and unit vector.

2.3.2 Gradient of a scalar Field

- Physical meaning: A **vector** who shows along which direction scalar **increase** fastest, and its magnitude describes the maximum space rate of change of the scalar **per unit length**.

$$grad\ V = \bigtriangledown V = \boldsymbol{a_n}\frac{dV}{dn}$$

- Calculate the change of scalar in any direction $\boldsymbol{a_l}$, $\frac{dV}{dl} = (\bigtriangledown V)\cdot\boldsymbol{a_l}$

- Calculate gradient in orthogonal Coordinate: $\bigtriangledown V = \boldsymbol{a_{u1}}\frac{\partial V}{\partial l_1} + \boldsymbol{a_{u2}}\frac{\partial V}{\partial l_2} + \boldsymbol{a_{u3}}\frac{\partial V}{\partial l_3} = \boldsymbol{a_{u1}}\frac{\partial V}{h_1\partial u_1} + \boldsymbol{a_{u2}}\frac{\partial V}{h_2\partial u_2} + \boldsymbol{a_{u3}}\frac{\partial V}{h_3\partial u_3}$

- Gradient Operator: $\bigtriangledown \equiv \boldsymbol{a_{u1}}\frac{\partial}{h_1\partial u_1} + \boldsymbol{a_{u2}}\frac{\partial}{h_2\partial u_2} + \boldsymbol{a_{u3}}\frac{\partial}{h_3\partial u_3}$

2.3.3 Divergence of a vector field

- **Definition:** The net outward flux of **A** per unit volume as the volume about the point tends to zero. It is a scalar

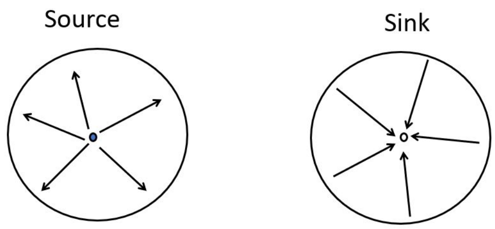
$$div\bar{A} = \lim \frac{\oint \bar{A} \cdot d\bar{S}}{\Delta V}$$

Outward water **density**/time

- **physical meaning:** flux density over a tiny volume of a closed surface.rain

- **Application:** Whether the point is a sink or source in the vector field.

Source or Sink of A Vector Field



1. Divergence >0, source exists
2. Divergence <0, sink exists
3. Divergence = 0, divergenceless or solenoidal (pipe)

net outward flux > inward flux → source → divergence > 0
net outward flux < inward flux → sink → divergence < 0

- **Expression:** $\bigtriangledown \cdot \boldsymbol{A} \equiv div\boldsymbol{A}$

$$\bigtriangledown \cdot \boldsymbol{A} = \frac{1}{h_2 h_3 h_3}\left[\frac{\partial}{\partial u_1}(h_1 h_2 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

- **Divergence Theorem:** The Volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

$$\int_v \nabla \cdot \bar{A} dV = \oint_s \bar{A} \cdot d\bar{S}$$

flux outward density/volume* volume= total flux

2.3.4 Curl of a Vector field

- **Circulation** The net circulation around a closed path $\oint_c \bar{A} \cdot d\bar{l}$

- **Curl** The net circulation per unit area. It is a vector, and we can use right-hand rule.

$$\nabla \times \bar{A} = \lim \frac{1}{\Delta s}\left(\hat{n} \oint_c \bar{A} \cdot d\bar{l}\right)$$

- **Physical meaning:** circulation density over a surface of a closed path.

- **Application:** If curl free, the vector field is a irrational field, which is also called conservative field.

- **Expression:**

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{a}_{u_1} h_1 & \mathbf{a}_{u_2} h_2 & \mathbf{a}_{u_3} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}.$$

- **Stokes' Theorem** The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

$$\int_s (\nabla \times \bar{A}) \cdot d\bar{S} = \oint_c \bar{A} \cdot d\bar{l}$$

Circulation desity per unit surface* surface=total circulations

2.4 Other Operators

- $\bigtriangledown \times (\bigtriangledown V) = 0$ If a vector field is curl free, it can be expressed by the gradient of a scalar.

- $\bigtriangledown \cdot (\bigtriangledown \times \boldsymbol{A}) = 0$. If a vector field is divergence-less, it can be expressed by the curl of a vector field.

- : **Laplace:**$\bigtriangledown^2 V = \bigtriangledown \cdot \bigtriangledown V$
  In Cartesian Coordinate: $\bigtriangledown^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$
  In Cylindrical Coordinate: $\bigtriangledown^2 V = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial V}{\partial r}) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$
  In Spherical Coordinate: $\bigtriangledown^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial V}{\partial r}) + \frac{1}{r^2 sin\theta}\frac{\partial}{\partial \theta}(sin\theta\frac{\partial V}{\partial \theta}) + \frac{1}{r^2 sin^2\theta}\frac{\partial^2 V}{\partial \phi^2}$

- **Chain Rule**

$$\nabla(f(\vec{r})g(\vec{r})) = f\nabla g + g\nabla f$$

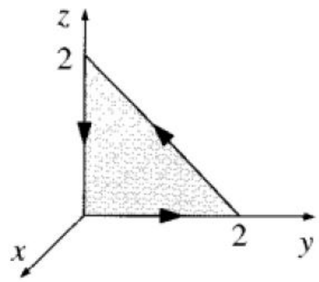
$$\nabla \cdot \left(f(\vec{r})\vec{G}(\vec{r})\right) = f\,\nabla \cdot \vec{G} + \vec{G} \cdot \nabla f$$

$$\nabla \times \left(f(\vec{r})\vec{G}(\vec{r})\right) = f\,\nabla \times \vec{G} + \nabla f \times \vec{G}$$

$$\nabla \cdot \left(\vec{F}(\vec{r}) \times \vec{G}(\vec{r})\right) = \vec{G} \cdot \nabla \times \vec{F} - \vec{F} \cdot \nabla \times \vec{G}$$

P 2.32 A vector field $\mathbf{D} = \mathbf{a_R}(cos^2\phi)/R^3$ exists in the region between two spherical shells defined by R=1 and R=2. Evaluate $\int \mathbf{D} \cdot d\mathbf{s}$, and $\int \nabla \cdot \mathbf{D} dv$.

. **quiz 1.2.2** Test stokes's Theorem for the function $\mathbf{v} = (xy)\hat{x} + (2yz)\hat{y} + (3zx)hatz$ using the triangular shaded area of below figure.



3 Electric Statics I

3.1 Basic Concept

Electrostatics:

i. electric charges are **at rest(not moving)**;

ii. electric field **do not change with time**.

3.2 Electric Field Intensity

Static electric charges (source) in free space → electric field

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \quad (\mathbf{V}/\mathrm{m})$$

IF q is small enough not to disturb the charge distribution of the source, $\mathbf{F} = q\mathbf{E}$ (N).
Fundamental Postulates of Electrostatics

- Differential form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$
$$\nabla \times \mathbf{E} = 0$$

- Integral form: Gauss' Law and KVL

$$\oint_S \mathbf{E} \cdot ds = \frac{Q}{\epsilon_0}$$ for a closed surface
$$\oint_C \mathbf{E} \cdot d\ell = 0$$ for a closed contour

E is **not solenoidal**, but **irrotational (conservative)**

3.3 Coulomb's Law

3.3.1 Electric Field due to a System of Discrete Charges

- **a single point charge (charge on the origin):**

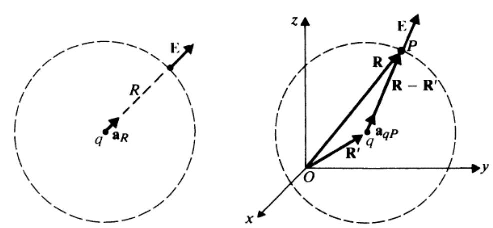
$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \quad (\mathbf{V}/\mathrm{m})$$

- **a single point charge (charge is not on the origin):**

$$\mathbf{E}_p = \frac{q\left(\mathbf{R} - \mathbf{R}'\right)}{4\pi\epsilon_0 \left|\mathbf{R} - \mathbf{R}'\right|^3} \quad (\mathbf{V}/\mathrm{m})$$

$\bar{R}'$: vector that represents the "q"

$\bar{R}$: vector that represents the point P.



(a) Point charge at the origin. (b) Point charge not at the origin.

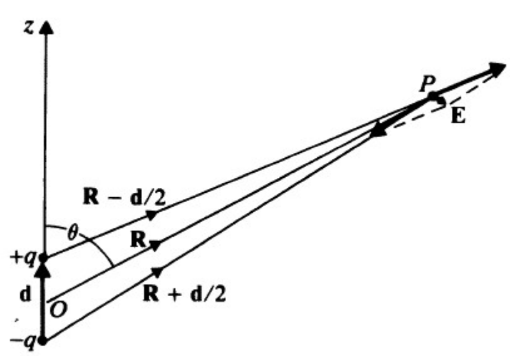
FIGURE 3-2
Electric field iFIGUREdue to a point charge.

- **several point charges:** (Superposition)

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^{n} \frac{q_k\left(\mathbf{R} - \mathbf{R}'_k\right)}{\left|\mathbf{R} - \mathbf{R}'_k\right|^3}$$

3.3.2 Electric Dipole

- **Electric Field**



general expression:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3} - \frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right\}$$

if $d << R$:

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R} - \mathbf{d}\right]$$

- **Electric Dipole Moment**
  Definition:

$$\mathbf{p} = q\mathbf{d}$$

$$\mathbf{p} = \mathbf{a}_z p = p\left(\mathbf{a}_R\cos\theta - \mathbf{a}_\theta\sin\theta\right)$$

$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta$$

q_e: magnitude of the positive/negative charge.
$\vec{d}$: vector from $-q_e$ to $+q_e$.

- **Electric Field:**

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}\left(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta\sin\theta\right) \quad (\mathrm{V/m})$$

3.3.3 Electric Field due to a Continuous Distribution of Charge

- **General Differential Element:**

$$d\mathbf{E} = \mathbf{a}_R\frac{\rho dv'}{4\pi\epsilon_0 R^2}$$

- **Line Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'}\mathbf{a}_R\frac{\rho_\ell}{R^2}d\ell' \quad (\mathbf{V/m})$$

- **Surface Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{S}'}\mathbf{a}_R\frac{\rho_s}{R^2}ds' \quad (\mathbf{V}/\mathrm{m})$$

- **Volume Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'}\mathbf{a}_R\frac{\rho}{R^2}dv' \quad (\mathbf{V/m})$$

3.4 Gauss's Law and Application

3.4.1 Definition

The total outward flux of the E-field over any closed surface in free space is equal to **the total charge enclosed in the surface** divided by ϵ_0.

$$\oint_S \mathbf{E}\cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

3.4.2 Application

- **Conditions for Maxwell's Integral Equations:**
  There is **a high degree of symmetry** in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc.)

3.4.3 Several Useful Models

Note: The charge distribution should be **uniform**.

different models	E(magnitude)
infinitely long, line charge	$E = \frac{\rho_\ell}{2\pi r\epsilon_0}$
infinite planar charge	$E = \frac{\rho_s}{2\epsilon_0}$
uniform spherical surface charge with radius R	$\begin{cases} E = 0(r < R) \\ E = \frac{Q}{4\pi r^2\epsilon_0}(r > R) \end{cases}$
uniform sphere charge with radius R	$\begin{cases} E = \frac{Qr}{4\pi R^3}(r < R) \\ E = \frac{Q}{4\pi r^2\epsilon_0}(r > R) \end{cases}$
infinitely long, cylindrical charge with radius R	$\begin{cases} E = \frac{\rho_v r}{2\epsilon_0}(r < R) \\ E = \frac{\rho_v R^2}{2r\epsilon_0}(r > R) \end{cases}$

3.5 Electric Potential

- **Expression:**

$$\mathbf{E} = -\nabla V$$

- **Electric Potential Difference:**

$$V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell$$

- **Work**
  In moving a unit charge from point P_1 to point P_2 in an electric field, work must be done against the field and is equal to

$$\frac{W}{q} = V_2 - V_1 = -\int_{p_1}^{p_2} \mathbf{E} \cdot d\mathbf{l} \ (J/CorV)$$

- **Electric Potential due to a Charge Distribution**

  i. **Point Charge & Several Point Charge**

$$V = \frac{q}{4\pi\epsilon_0 R}$$

$$V = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{q_k}{|\mathbf{R} - \mathbf{R}'|}$$

superposition.

ii. **dipole Charge**

$$V = \frac{qdcos\theta}{4\pi\epsilon_0 R^2} = \frac{\mathbf{p} \cdot \mathbf{a_R}}{4\pi\epsilon_0 R^2}$$

iii. **Line Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{L'} \frac{\rho_\ell}{R} d\ell' \quad (V)$$

iv. **Surface Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{\mathbf{S}'} \frac{\rho_s}{R} ds' \quad (V)$$

v. **Volume Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R} dv' \quad (V)$$

3.6 Conductors and dielectrics in static electric field

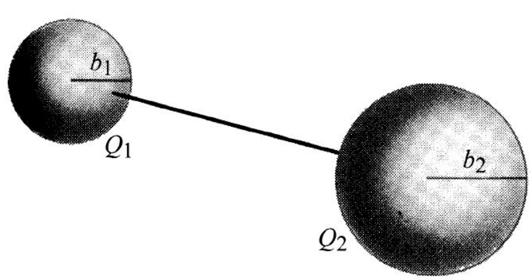
- conductors:
  - electrons migrate easily.
  - charges reach the surface and conductor redistribute themselves in a way that both the charge and the field vanish.
  - **static state conditions**:
    * inside the conductor:

      $$\rho = 0, \boldsymbol{E} = 0$$

      , where $\rho = 0$ represents no charge in the interior
    * on the conductor surface (boundary conditions)

      $$E_t = 0, E_n = \frac{\rho_s}{\epsilon_0}$$

      It is an equal-potential body.
  - Electric field intensity tends to be higher at a point near the surface of a charged conductor with a larger curvature



conductor: connected by a conducting wire ⇒ equal-potenti-al for the two surfaces

① $\frac{Q_1}{Q_2} = \frac{b_1}{b_2}$ ③ V: equal

② $E \sim \frac{1}{r}$

- semiconductos:

- relatively small number of freely movable charges.

- insulators(dielectrics):
  - electrons are confined to their orbits.
  - external electric field polarizes a dielectric material and create electric dipoles. The induced electric dipoles will modify the electric field both inside and outside the dielectric material, as shown in Fig 1.

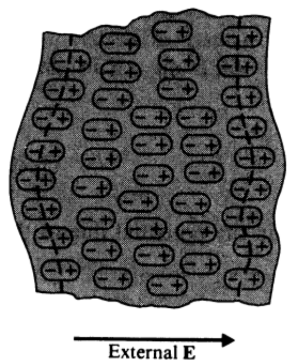


Figure 1: A cross section of a polarized dielectric medium

  - **polarization charge densities/ bound-charge densities:**
    * **polarization vector, $\boldsymbol{P}$**:

$$\boldsymbol{P} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \boldsymbol{p_k}}{\Delta v}$$

, where the numerator represents the vector sum of the induced dipole moment contained in a very small volume Δv.

    * charge distribution on surface density:

$$\rho_{ps} = \boldsymbol{P} \cdot \boldsymbol{a_n}$$

,

    * volume charge distribution density:

$$\rho_p = -\nabla \cdot \boldsymbol{P}$$

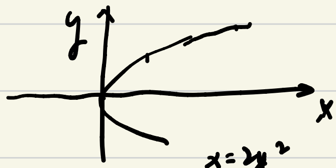
.

p.3-13 Determine the work done in carrying a $-2(\mu C)$ charge from P_1(2,1,-1) to $P_2(8, 2, -1)$ in the field $\boldsymbol{E} = \boldsymbol{a_x} y + \boldsymbol{a_y} x$.
a) along the parabola $x = 2y^2$
b) along the straight line joining P_1 and P_2.

<1> $W = -\int_L q\,\bar{E}\, d\bar{l}$

$\bar{E} = \hat{a}_x y + \hat{a}_y x, \quad d\bar{l} = dx \cdot \hat{a}_x + dy \cdot \hat{a}_y$

$\Rightarrow \bar{E} \cdot d\bar{l} = y\,dx + x\,dy$



$x = 2y^2$

and $x=2y^2$

from $P_1(2,1,-1)$ to $P_2(8,2,-1)$

$W = 2\times10^{-6} \int_L y\,dx + x\,dy$

$= 2\times10^{-6} \left(\int_2^8 \sqrt{\tfrac{1}{2}x}\, dx + \int_1^2 2y^2\, dy \right)$

$= 2\times10^{-6} \left(\left[\sqrt{\tfrac{1}{2}}\, x^{\frac{3}{2}} \cdot \tfrac{2}{3} \right]_2^8 + \left[\tfrac{2}{3} y^3 \right]_1^2 \right)$

$= 2\times10^{-6} \left(\frac{\sqrt{2}}{\cancel{2}} \cdot \frac{\cancel{2}}{3} \cdot (16\sqrt{2} - 2\sqrt{2}) + \frac{2}{3} \cdot 7 \right)$

$= 2\times10^{-6} \left(\frac{28}{3} + \frac{14}{3} \right)$

$= 28\ \mu J$

<2>: $P_1 P_2$: $y = \frac{1}{6}x + \frac{2}{3} \Rightarrow 6y - 4 = x$

$W = 2\times10^{-6} \left[\int_2^8 \left(\tfrac{1}{6}x + \tfrac{2}{3}\right) dx + \int_1^2 (6y-4)\, dy \right]$

$= 2\times10^{-6} \left[\left(\tfrac{1}{12}x^2 + \tfrac{2}{3}x\right)\Big|_2^8 + (3y^2 - 4y)\Big|_1^2 \right]$

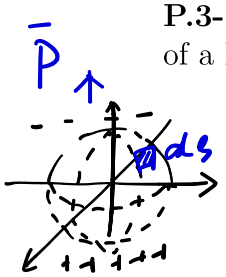
$= 2\times10^{-6} [5 + 4 + 9 - 4]$

$= 2\times10^{-6} \cdot 14$

$= 28\ \mu J$

assume $\bar{P}$ pointing upwards. to $\hat{a}_z$

P.3-23 Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization **P** exists.



cavity: empty inside:

only surface bound-charge has the influence.

$\rho_{ps} = \bar{P}\cdot\hat{a}_n = -P\cdot\cos\theta$. (pointing outward)

$$|\bar{E}| = \frac{1}{4\pi\epsilon_0}\oint_s \frac{-\rho_{ps}\cos\theta}{R^2}\,ds = -\frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\int_0^{\pi}\frac{\rho_{ps}\cdot\cos\theta}{R^2}\cdot R^2\cdot\sin\theta\, d\theta\, d\phi$$

$$= +\frac{1}{4\pi\epsilon_0}\cdot\int_0^{2\pi}\int_0^{\pi} P\cos^2\theta\cdot\sin\theta\cdot d\theta\, d\phi$$

$$= +\frac{1}{2\epsilon_0}P\cdot\int_0^{\pi}\cdot\cos^2\theta\,\sin\theta\, d\theta$$

$$= +\frac{1}{2\epsilon_0}\cdot P\left[\left(\frac{1}{3}\cos^3\theta\right)\cdot(-1)\right]_0^{\pi}$$

$$= +\frac{P}{2\epsilon_0}\cdot\left(\frac{1}{3}+\frac{1}{3}\right)$$

$$= \frac{P}{3\epsilon_0}$$

4 Electric Statics II

4.1 Electric Flux Density and Dielectric Constant

- **electric flux density/electric displacement, $\boldsymbol{D}$**:

$$\boldsymbol{D} = \epsilon_0\boldsymbol{E} + \boldsymbol{P} \quad (C/m^2)$$

- $$\nabla\cdot\boldsymbol{D} = \rho \quad (C/m^3)$$

  , where ρ is the volume density of free charges.

- Another form of **Gauss's law**:

$$\oint_S \boldsymbol{D}\cdot d\boldsymbol{s} = Q \quad (\boldsymbol{C})$$

  , the total outward flux of the electric displacement (the total outward electric flux) over any closed surface is equal to the total free charge enclosed in the surface.

- If the dielectric of the medium is **linear and isotropic**,

$$\boldsymbol{P} = \epsilon_0\chi_e\boldsymbol{E}$$

$$\boldsymbol{D} = \epsilon_0(1+\chi_e)\boldsymbol{E} = \epsilon_0\epsilon_r\boldsymbol{E} = \epsilon\boldsymbol{E}$$

  , where χ_e is a dimensionless quantity called electric susceptibility,

  ϵ_r is a dimensionless quantity called as relative permittivity/ electric constant of the medium,

  ϵ is the absolute permittivity/permittivity of the medium (F/m).

- For anisotropic,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

, For biaxial,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

For uniaxial, $\epsilon_1 = \epsilon_2$, For isotropic, $\epsilon_1 = \epsilon_2 = \epsilon_3$ (the only kind of media we deal with in this course).

- dielectric breakdown: electric field is very strong, causes permanent dislocations and damage in the material.

  dielectric strength: the maximum electric field intensity that a dielectric material can withstand without breakdown.

4.2 Boundary Conditions for Electrostatic Fields

- the tangential component of an $\boldsymbol{E}$ field is continuous across an interface.

$$E_{1t} = E_{2t} \quad (V/m)$$

, or

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

- The normal component of $\boldsymbol{D}$ field is discontinous across an interface where a surface charge exists - the amount of discontinuity being equal to the surface charge density.

$$\boldsymbol{a_{n2}} \cdot (\boldsymbol{D_1} - \boldsymbol{D_2}) = \rho_s$$

, or

$$D_{1n} - D_{2n} = \rho_s \quad (C/m^2)$$

P.3-28 Dielectric lenses can be used to collimate electromagnetic fields. In Fig.1 the left surface of the lens is that of a circular cylinder, and the right surface is a plane. If $\boldsymbol{E_1}$ at point $P(r_0, 45°, z)$ in region 1 is $\boldsymbol{a_r}5 - \boldsymbol{a_\phi}3$, what must be the dielectric constant of the lens in order that $\boldsymbol{E_3}$ in region 3 is parallel to the x-axis?

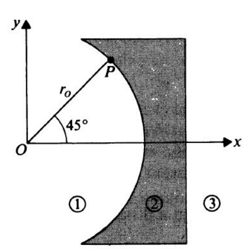


Figure 2: A dielectric lens

4.3 Capacitance and Capacitors

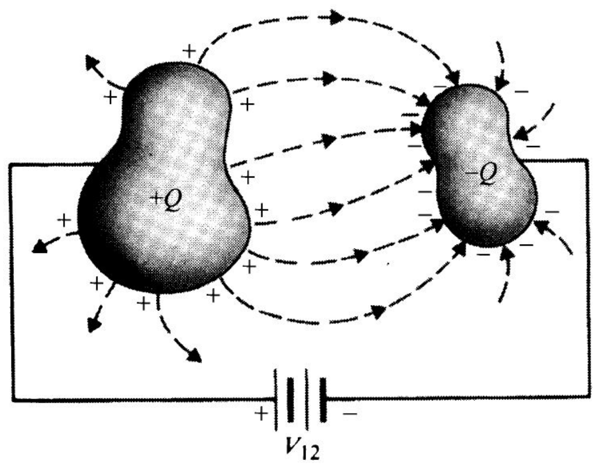
4.3.1 Capacitance

- Definition: The capacitance of isolated conducting body is the electric charge that must be added to the body per unit increase in its electric potential.

- $C = \frac{Q}{V}$ $(F = C/V)$

4.3.2 Capacitor

- **Components:** two conductors with arbitrary shapes are separated by free space or dielectric medium. $C = \frac{Q}{V12}$



- **Capacitance:**
  Its Capacitance is independent of V and Q, which means a capacitor has a capacitance even no voltage is applied to it and no free charges exist on its conductors.

- **How to calculate its capacitance**:

  1. Choose a proper coordinate system
  2. Assum $+Q, -Q$ on the conductors
  3. Find $\mathbf{E}$ from Q (like, Gauss's law, $D_n = \epsilon E_n = \rho_s$)
  4. Find $V_{12} = -\int_2^1 \mathbf{E} \cdot d\mathbf{l}$
  5. $C = Q/V_{12}$

- **Series Connections of Capacitors:**

$$\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + ... + \frac{1}{C_n}$$

- **Parallel Connections of Capacitors:**

$$C_{||} = C_1 + C_2 + ... + C_n$$

4.3.3 Capacitance in Multi-conductor System

- Isolated Conductor System $Q_0 + Q_1 + .. + Q_N = 0$

- Four Conductor System Q-V Relationship I($V_0 = 0$)

$$\begin{aligned} Q_1 &= c_{11}V1 + c_{12}V2 + .. + c_{13}V_N \\ Q_2 &= c_{12}V1 + c_{22}V2 + .. + c_{23}V_N \\ Q_3 &= c_{13}V1 + c_{23}V2 + .. + c_{33}V_N \end{aligned}$$

  where coefficients of capacitance $c_{ii} = Q_i/V_i$, $c_{ii} > 0$
  coefficients of induction $(i \neq j)$, $c_{ji} = Q_{ji}/V_i$, $c_{ji} = cji < 0$

- Four Conductor System Q-V Relationship II (Conductor 0 is grounded as well)

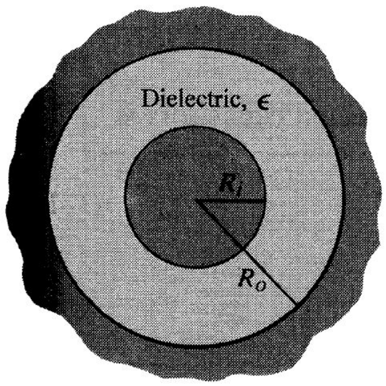
$$\begin{aligned} Q_1 &= C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3) \\ Q_2 &= C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3) \\ Q_3 &= C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2) \end{aligned}$$

  where self-partial capacitance: C_{i0}
  mutual partial capacitance: $C_{ij} = C_{ji}$

- Relationship between c and C

$$C_{i0} = c_{i1} + c_{i2} + c_{i3}$$
$$C_{ij} = -c_{ij}$$

E. 3-19 A spherical capacitor consists of an inner conducting sphere of radius R_i and an outer conductor with a sphere inner wall of radius R_o. The space in between is filled with a dielectric of permittvity ϵ. Determine the capacitance.



4.4 Electrostatic Energy and Forces

- Work done to bring a charge q from P_1 to P_2

$$\frac{W}{q} = V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$

- **Self Energy:**Work done to bring a charge Q_2 from infinity to distance R_{12} with Q_1 (initially, Q_1 is in space)

$$W = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 R_{12}}$$

- **Mutual Energy:** Potential energy of a group of N discrete point charges at rest

$$W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k$$

where $V_k = \frac{1}{4\pi\epsilon_0}\sum_{j=1\&j\neq k}^{N}\frac{Q_j}{R_{jk}}$ Note the W_e can be negative, for example, there are 2-point charge systems, and one charge is positive,the other is negative.

- **Electrostatic Energy density**w_e**:** $W_e = \int_{v'} w_e dv$

4.4.1 Electrostatic Energy in terms of Field Quantities

- v' can be all space.

- **A continuous Charge Distribution of Density** ρ

$$W_e = \frac{1}{2}\int_{v}\rho V dv = \frac{1}{2}\int_{v'}(\nabla\cdot\mathbf{D})V dv$$

Another expression:

$$W_e = \frac{1}{2}\int_{v'}\mathbf{D}\cdot\mathbf{E}dv$$

- If it is a simple dielectric, it should be

$$W_e = \frac{1}{2}\int_{v'}\epsilon E^2 dv = \frac{1}{2}\int_{v'}\frac{D^2}{\epsilon}dv$$

4.4.2 Electrostatic Forces

Here we use **Principle of virtual displacement** to calculate Force in two situations.

- **System of bodies with fixed charges**

1. Mechanical work is from the reduced stored electrostatic energy

$$F_Q = -\nabla W_e(N)$$

2. Electric torque rotates one of the bodies by $d\phi$(a virtual rotation) about an axis

$$T_Q = -\frac{\partial W_e}{\partial\phi}(N\cdot m)$$

- **System of conducting bodies with Fixed Potentials**

1. The fixed potential can be retained by connecting with an external source.
2. $F_v = \nabla W_e$
3. $T_v = \frac{\partial W_e}{\partial \phi}$

Example 3-22 Find the energy required to assemble a uniform sphere of charges of radius b and volume charge ρ.