1 Poisson's Equation and Laplace's Equation

1.1 Poisson's Equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

$$\begin{cases} \nabla \cdot \vec{D} = \rho \\ \nabla \times \vec{E} = 0 \end{cases}$$

$\vec{E} = -\nabla V$

$\vec{D} = \epsilon \vec{E} = -\nabla \epsilon V$

$\nabla \cdot (-\nabla \epsilon V) = -\epsilon \nabla \cdot \nabla V$
$= -\epsilon \nabla^2 V = \rho$

- In Cartesian System:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

- In Cylindrical System:

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial^2 \varphi} + \frac{\partial^2 V}{\partial z^2}$$

- In Spherical System:

$$\nabla^2 V = \tfrac{1}{r^2}\tfrac{\partial}{\partial r}\left(r^2\tfrac{\partial V}{\partial r}\right) + \tfrac{1}{r^2 \sin\theta}\tfrac{\partial}{\partial \theta}\left(\sin\theta\tfrac{\partial V}{\partial \theta}\right) + \tfrac{1}{r^2 \sin^2\theta}\tfrac{\partial^2 V}{\partial \varphi^2}$$

1.2 Laplace's Equation:

For a simple medium where there is no free charge:

$$\nabla^2 V = 0$$

For problem involving conductors:

- Use Laplace's Equation to obtain electric potential V.
- Use $\mathbf{E} = -\nabla V$ to work out E.
- Use $\rho_s = \epsilon E$ to get charge density on the conductor surface.

1.3 Uniqueness of Electrostatic Solutions

A solution of Poisson's Equation or Laplace's Equation that satisfies the given boundary conditions is a unique solution.

1.4 Examples

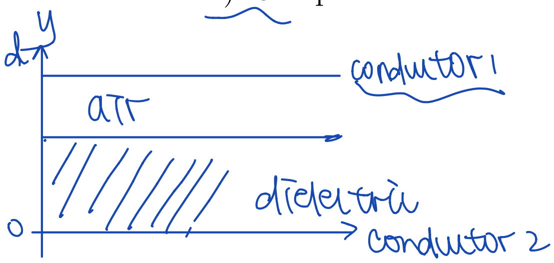
1. The upper and lower conducting plates of a large parallel-plate capacitor are separated by a distance d and maintained at potentials V_0 and 0, respectively. A dielectric slab of dielectric constant 6.0 and uniform thickness of $0.8d$ is placed over the lower plate. Assuming negligible fringing effect, determine

   a) the potential and electric field distribution in the dielectric slab,

   b) the potential and electric field distribution in the air space between the dielectric slab and the upper plate,

c) the surface charge densities on the upper and lower plates.

d) Compare the results in part (b) with those without the dielectric slab.



In air: $\nabla^2 V_a = 0$; $V_a = a_1 y + a_2$

In dielectric $\nabla^2 V_{di} = 0$; $V_{di} = a_3 y + a_4$

$\vec{E} = -\nabla V \Rightarrow \begin{cases} \vec{E}_a = -\hat{a}_y a_1 \\ \vec{E}_{di} = -\hat{a}_y a_3 \end{cases}$

$D_{1n} = D_{2n} \to \epsilon_1 E_1 = \epsilon_2 E_2$

$\begin{cases} V(0) = a_4 = 0 \\ V(d) = a_1 d + a_2 = V_0 \\ V(0.8d) = a_1 \cdot 0.8d + a_2 = a_3 \cdot 0.8d + a_4 \\ \epsilon_0 a_1 = \epsilon_0 \cdot a_3 \cdot 6 \end{cases}$

$\begin{cases} a_1 = \frac{3V_0}{d} \\ a_2 = -2V_0 \\ a_3 = \frac{V_0}{2d} \\ a_4 = 0 \end{cases} \Rightarrow \begin{cases} \vec{E}_a = -\hat{a}_y \frac{3V_0}{d} \\ \vec{E}_d = -\hat{a}_y \frac{V_0}{2d} \end{cases}$

$\rho_s = \epsilon E.$

$\rho_s = \epsilon_0 \cdot \frac{3V_0}{d}$

$\rho_{s\_l} = -\epsilon_0 \frac{3V_0}{d}$

2. Prove the scalar potential V in:

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R} dv'$$

satisfies Poisson's Equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

Note: For a volume charge, the potential function inside the charge is different from the potential function outside the charge.

$\nabla^2 V = \nabla \cdot \nabla V = \nabla \cdot \nabla \frac{1}{4\pi\epsilon} \int_{v'} \frac{\rho}{R} dv'$

$= \nabla \cdot \frac{1}{4\pi\epsilon} \int_{v'} \nabla \frac{\rho}{R} dv$

$= \nabla \cdot \frac{1}{4\pi\epsilon} \left(-\int_{v'} \frac{\rho \vec{R}}{R^3} dv \right)$

$= \frac{1}{4\pi\epsilon} \left(-\int_{v'} \nabla \cdot \frac{\rho \vec{R}}{R^3} dv \right)$

$= \frac{\rho}{4\pi\epsilon} \cdot (-4\pi)$

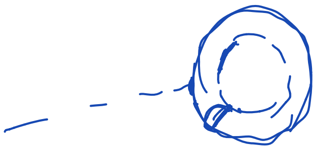
$= -\frac{\rho}{\epsilon}$

$\nabla \cdot \frac{\vec{R}}{R^3} = \begin{cases} 0, & R \neq 0 \\ 4\pi \end{cases}$

outside:

$V = \frac{Q}{4\pi\epsilon_0 R}$

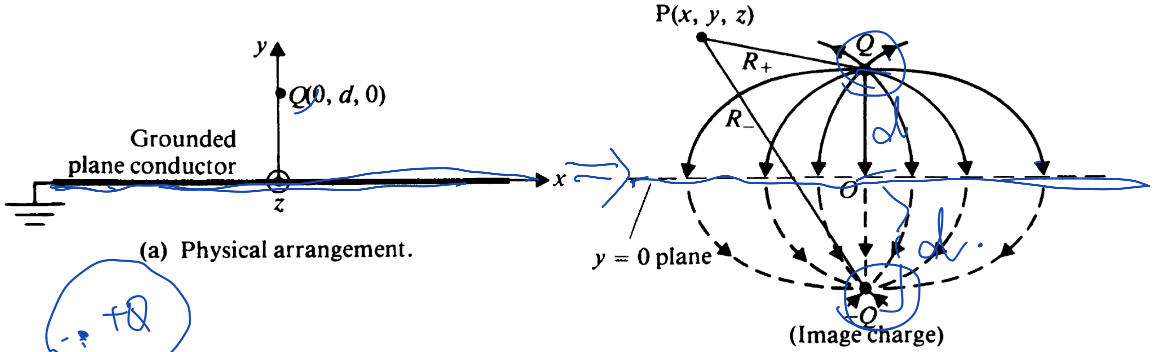
inside:



$V = \frac{3Q}{8\pi\epsilon_0 R} - \frac{Q r^2}{8\pi\epsilon_0 {}^3}$

2 Method of Images

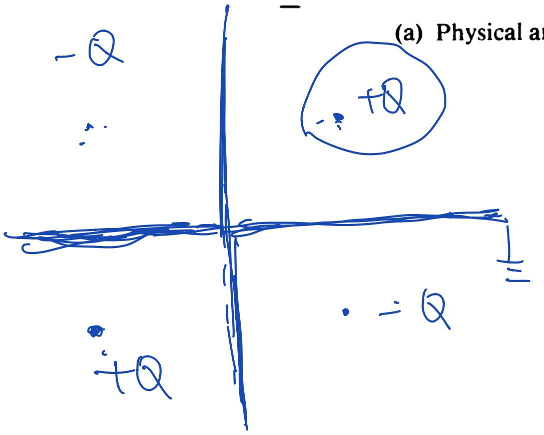
2.1 Point Charge and Conducting Planes



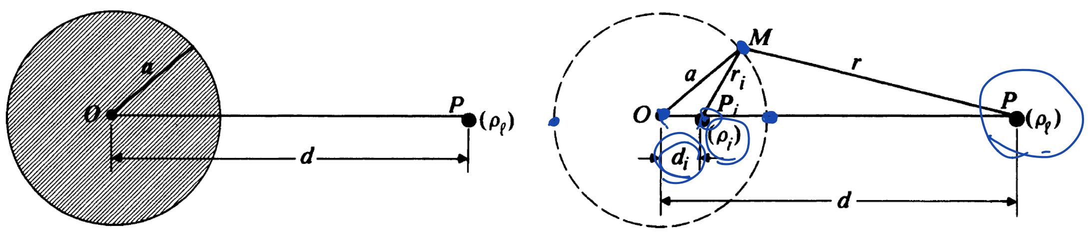
(a) Physical arrangement.

(Image charge)

(b) Image charge and field lines.



2.2 Line Charge and Parallel Conducting Cylinder



(a) Line charge and parallel conducting cylinder.

(b) Line charge and its image.

$\rho_i = -\rho_l$

$$V_M = \frac{\rho_l}{2\pi\epsilon_0}\ln\frac{r_0}{r} + \frac{\rho_i}{2\pi\epsilon_i}\ln\frac{r_0}{r_i}$$

$$= \frac{\rho_i}{2\pi\epsilon_0}\ln\frac{r_i}{r}$$

$$\frac{r_i}{r} = constant$$

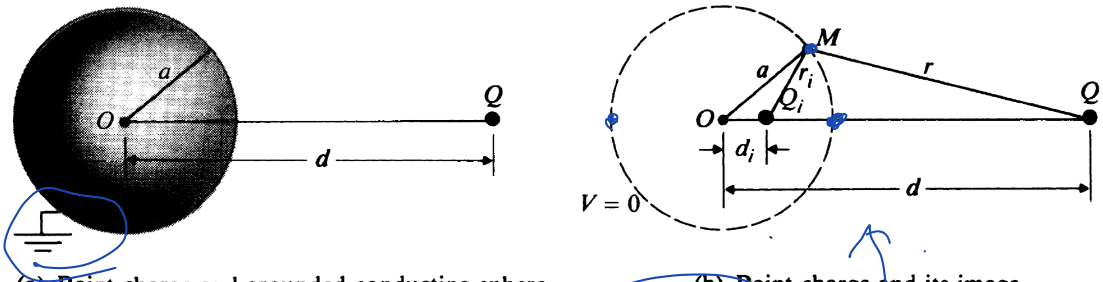
$$\frac{a-d_i}{d-a} = \frac{a+d_i}{a+d} = constant$$

→ reference zero potential

$$d_i = \frac{a^2}{d}$$

$$\frac{r_i}{r} = \frac{a-d_i}{d-a} = \frac{a}{d}.$$

2.3 Point Charge and a Conducting Sphere



(a) Point charge and grounded conducting sphere. (b) Point charge and its image.

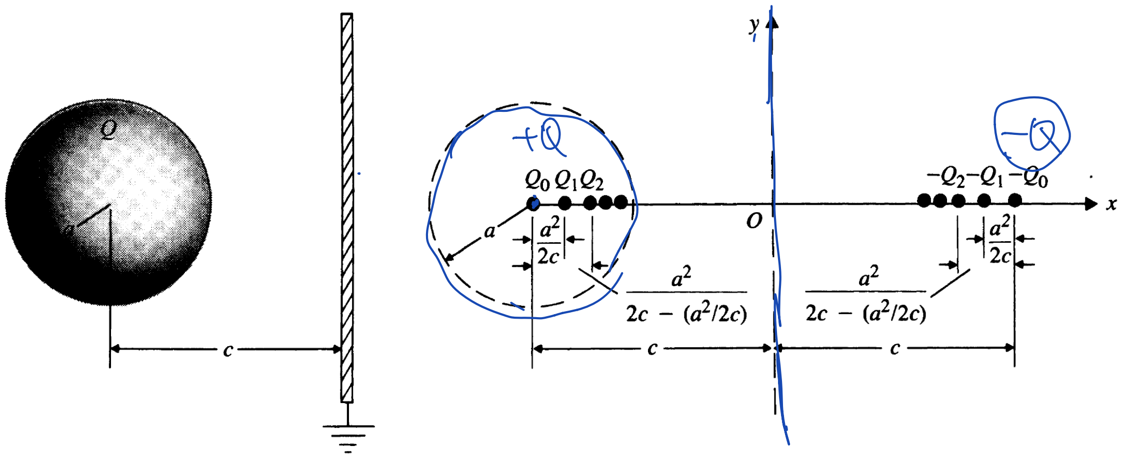
$$V_M = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r} + \frac{Q_i}{r_i}\right) = 0$$

$$\Rightarrow \frac{Q_i}{Q} = -\frac{r_i}{r}$$

$$\frac{r_i}{r} = \frac{a-d_i}{d-a} = \frac{a+d_i}{d+a}$$

$$\begin{cases} d_i = \dfrac{a^2}{d} \\ Q_i = -\dfrac{a}{d}Q \end{cases}$$

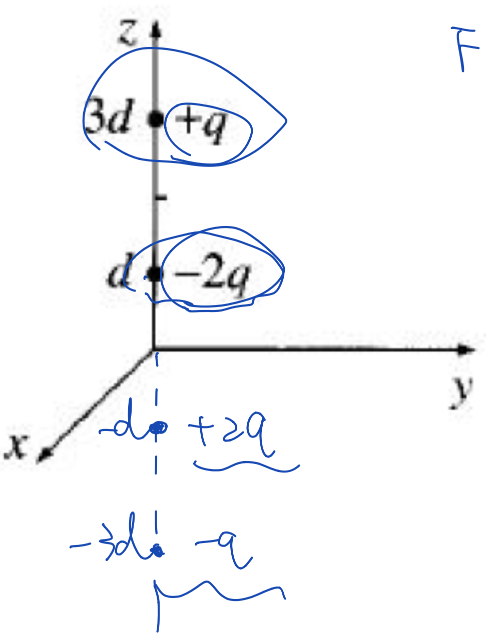
2.4 Charge Sphere and Grounded Plane



(a) Physical arrangement. (b) Two groups of image point charges.

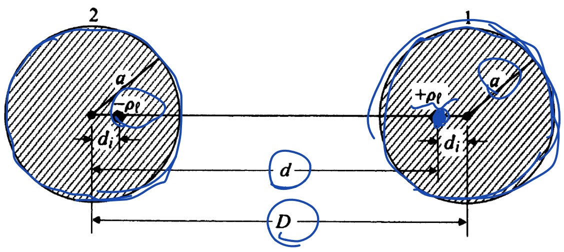
2.5 Examples

1. Find the force on the charge +q in Fig.1. (The xy plane is an grounded conductor.)



$$F = \hat{a}_z \frac{q\cdot(-2q)}{4\pi\varepsilon_0\cdot(2d)^2} + \hat{a}_z \frac{q\cdot 2q}{4\pi\varepsilon_0\cdot(4d)^2} + \hat{a}_z \frac{q\cdot(-q)}{4\pi\varepsilon_0\cdot(6d)^2}$$

2. Determine the capacitance per unit length between two long, parallel, circular conducting wires of radius a. The axes of the wires are separated by a distance D



$$C = \frac{Q}{\Delta V}$$

$$d_i = \frac{a^2}{d}$$

$$d = D - d_i = D - \frac{a^2}{d}$$

$$\Rightarrow d = D - \frac{a^2}{d}$$

$$\Rightarrow d = \frac{1}{2}\left(D + \sqrt{D^2 - 4a^2}\right)$$

$$V_2 = \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{a}{d}\right)$$

$$V_1 = \frac{-\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{a}{d}\right)$$

$$C = \frac{Q}{V_1 - V_2}$$

$$= \frac{\rho_l}{-\frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{a}{d}\right) - \frac{\rho_l}{2\pi\varepsilon_0}\ln\left(\frac{a}{d}\right)}$$

$$= \frac{\pi\varepsilon_0}{\ln\left(\frac{a}{d}\right)}$$

3 Boundary Value problem in Cartesian Coordinates

3.1 Boundary Condition Problem

- In order to find specific voltage on conductor systems without isolated free charge.
- **General idea:** Use boundary condition to find coefficients for general solution form from Laplace equation.
- **Types of boundary condition:** (1) Dirichlet: V is specific; (2) Neumann: $\frac{dV}{dn}$ is specified on boundaries (3) Mixed: V is specific on some boundaries; $\frac{dV}{dn}$ is specified on some boundaries.
- **Solution Form:** Separation of variables, which means $V(x,y,z) = X(x)Y(y)Z(z)$. When the potential or normal derivative is specified, and it coincide with coordinate surfaces of an orthogonal, curvilinear coordinate system.

3.2 Boundary condition value in Cartesian Coordinate

(1) Laplace's Equation for V in Cartesian coordinates is

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

(2) To use the Separation of variables and take it into Laplace's Equation.

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} + \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} + \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = 0$$

(3) In order to satisfied all x,y,z values, these three parts should be constant. Then we can get

$$\frac{1}{X(x)}\frac{d^2X(x)}{dx^2} = -k_x^2, \frac{1}{Y(y)}\frac{d^2Y(y)}{dy^2} = -k_y^2, \frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2} = -k_z^2$$

$$k_x^2 + k_y^2 + k_z^2 = 0$$

(4) List the boundary conditions we got.

(5) The general solution formats for above differential equation $\frac{d^2X(x)}{dx^2} + k_x^2X(x) = 0$ are:

k_x^2	k_x	$X(x)$	Exponential forms † of $X(x)$
0	0	$A_0x + B_0$	
+	k	$A_1 \sin kx + B_1 \cos kx$	$C_1e^{jkx} + D_1e^{-jkx}$
−	jk	$A_2 \sinh kx + B_2 \cosh kx$	$C_2e^{kx} + D_2e^{-kx}$

We need to choose the proper form of solution given boundary condition.
If V is independent of x, We can see X(x)=0;
If V goes to infinity or 0 as x goes to infinity, we choose k_x^2 is negative.

(6) Find the coefficients through boundary condition.

Quiz 2

Two infinite grounded metal plates lie parallel to the xz plane, one at y=0, the other at y=a(Fig.2). The left end, at x=0, is closed off with an infinite strip insulated form the two plates and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."

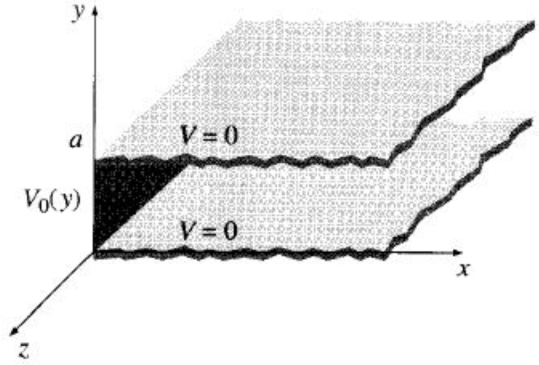


Figure 1: Question 2

Hint:

1. After using separation of variables you can get

$$\frac{1}{X}\frac{d^2X}{dx^2} = C_1 \quad and \quad \frac{1}{Y}\frac{d^2Y}{dy^2} = C_2, \qquad with \quad C_1 + C_2 = 0$$

This implies

$$\frac{d^2X}{dx^2} = k^2X, \qquad \frac{d^2Y}{dy^2} = -k^2Y$$

2. Mathematical equations you may use:

$$\sum_{n=1}^{\infty} C_n \int_0^a sin(n\pi y/a)sin(n'\pi y/a)dy = \int_0^a V_0(y)sin(n'\pi y/a)dy$$

With the left side:

$$\int_0^a \sin(n\pi y/a)\sin(n'\pi y/a)\,dy = \begin{cases} 0, & \text{if } n' \neq n \\ \frac{a}{2}, & \text{if } n' = n \end{cases}$$

And then:

$$C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)dy$$

4 Boundary-value Problems in Cylindrical Coordinates

(1) Laplace Equation:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}+\frac{\partial^2 V}{\partial z^2}=0$$

(2) General solution: Assuming V is independent of Z.

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2}=0$$

(3) Separation of variables: $V(r,\phi)=R(r)\Phi(\phi)$

(4) Equations for $\Phi(\phi)$

$$\frac{d^2\Phi(\phi)}{d\phi^2}+k^2\Phi(\phi)=0$$

Since the solution should be periodic among phi, we can get k=n and we should use $k^2>0$ be like

$$\Phi(\phi)=A_\phi\sin n\phi+B_\phi\cos n\phi$$

(5) Equations for R(r): After using separation of variables, we get

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]=k^2$$

Which is a second order differential Equation

$$r^2\frac{d^2R(r)}{dr^2}+r\frac{dR(r)}{dr}-n^2R(r)=0$$

And the general solution is

$$R(r)=A_r r^n+B_r r^{-n}$$

If we study the area including r=0, B_r=0, Otherwise, V goes to infinity at r=0
If we studay the area including r= ∞, S_r=0

(6) Equations for $V_n(r,\phi)$,

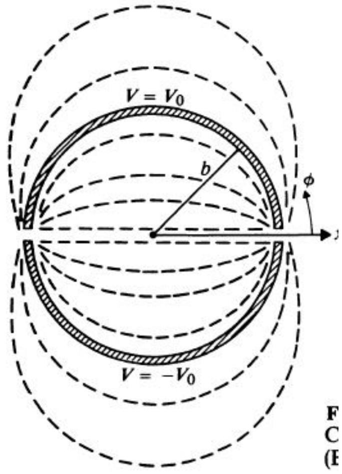
$$V_n(r,\phi)=r^n\left(A_n\sin n\phi+B_n\cos n\phi\right)+r^{-n}\left(A_n'\sin n\phi+B_n'\cos n\phi\right),\quad n\neq 0$$

(7) Special case: if V is independent of ϕ, k=0. Then we get

$$\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]=0$$

$$V(r)=C_1\ln r+C_2$$

Example 4-9 An infinity long, thin, conducting circular tube of radius b is split in two halves. The upper half is kept at a potential V=$V-0$ and the lower half at $V=-V_0$.Determine the potential distribution both inside and outside the tube.



5 Boundary-value Problem in Spherical Coordinates

(1) Since we only consider the situation that V is independent of ϕ, the Laplace Equation in Spherical coordinates is simplified to

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right)+\frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)=0$$

(2) By using separation of variables, we assign $V(R,\theta)=\Gamma(R)\Theta(\theta)$. Then it looks like

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2\frac{d\Gamma(R)}{dR}\right]+\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right]=0$$

(3) General solutions for $\Gamma(R)$. Firstly,we assume the part for $\Gamma(R)$ equals to k^2:

$$\frac{1}{\Gamma(R)}\frac{d}{dR}\left[R^2\frac{d\Gamma(R)}{dR}\right]=k^2$$

It is actually second differential Equation:

$$R^2\frac{d^2\Gamma(R)}{dR^2}+2R\frac{d\Gamma(R)}{dR}-k^2\Gamma(R)=0$$

The solution form is

$$\Gamma_n(R)=A_nR^n+B_nR^{-(n+1)},\ where\ k=n(n+1),n>0$$

(4) General Solutions for θ. Similarly, we can get

$$\frac{1}{\Theta(\theta)\sin\theta}\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right] = -k^2$$

Since we already know n(n+1)=k^2, we can get the second differential equation:

$$\frac{d}{d\theta}\left[\sin\theta\frac{d\Theta(\theta)}{d\theta}\right] + n(n+1)\Theta(\theta)\sin\theta = 0$$

. It is called Legendre's equation and for *theta* $\in [0, \pi]$, the solution has special forms called Legendre's polynomials:

$$\Theta_n(\theta) = P_n(\cos\theta)$$

There are some solutions forms for usual n.

n	$P_n(\cos\theta)$
0	1
1	$\cos\theta$
2	$\frac{1}{2}(3\cos^2\theta - 1)$
3	$\frac{1}{2}(5\cos^3\theta - 3\cos\theta)$

(5) By Combing them together,

$$V_n(R,\theta) = \left[A_n R^n + B_n R^{-(n+1)}\right] P_n(\cos\theta)$$

Example 4-10 An uncharged conducting sphere of radius b is placed in an initially uniform Electric Field $\mathbf{E}_0 = \mathbf{a}_x E_0$. Determine (a) the potential distribution V(R,θ) and (b) the electric field intensity $\mathbf{E}(R,\theta)$ after the introduction of sphere.

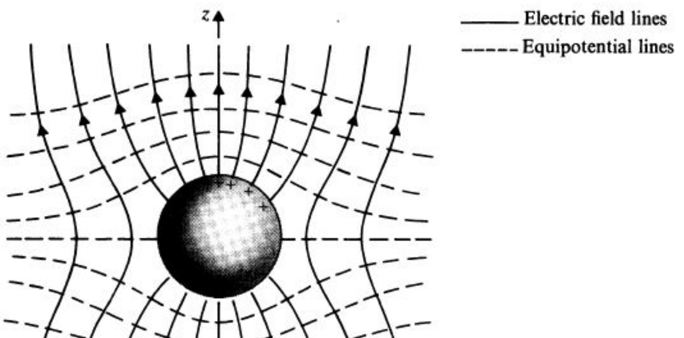


FIGURE 4–21
Conducting sphere in a un
(Example 4–10).

6 Steady Electric Currents

Types of electric currents caused by the motion of free charges:

1. **conduction currents**: drift motion of conduction electrons and/or holes in conductors/semiconductors.

2. electrolytic currents: migration of positive and negative ions.

3. convenction currents: motion of electrons and/or ions in a vacuum.

6.1 Current Density and Ohm's Law

$$I = \int_S J \cdot ds \quad (A)$$

where J is the volume current density or current density, defined by

$$J = Nqu \quad (A/m^2)$$

,

where N is the number of charge carriers per unit volume, each of charges q moves with a velocity u.

Since Nq is the free charge per unit volume, by $\rho = Nq$, we have:

$$J = \rho u \quad (A/m^2)$$

Ex.5-1 In vaccum-tube diodes, electrons are emitted from a hot cathode at zero potential and collected by an anode maintained at a potential V_0, resulting in a convection current flow. Assuming that the cathode and the anode are parallel conducting plates and that the electrons leave the cathode with a zero initial velocity (space-charge limited condition), find the relation between the current density J and V_0.

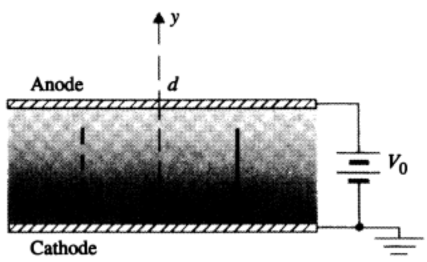


FIGURE 5–2
Space-charge-limited vacuum diode (Example 5–1).

For conduction currents,

$$J = \sigma E \quad (A/m^2)$$

where $\sigma = \rho_e \mu_e$ is conductivity, a macroscopic constitutive parameter of the medium. $\rho_e = -Ne$ is the charge density of the drifting electrons and is negative. $u = -\mu_e E \quad (m/s)$ where μ_e is the electron mobility measured in $(m^2/V \cdot s)$.

Materials where $J = \sigma E \quad (A/m^2)$ holds are called ohmic media. The form can be referred as the point form of Ohm's law.

Derivation of voltage-current relationship of a piece of homogeneous material by the point form of Ohm's law.

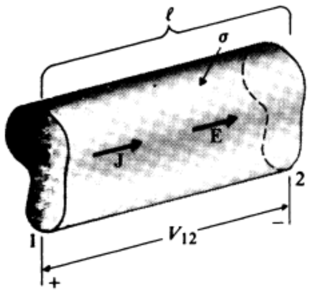


FIGURE 5–3
Homogeneous conductor with a constant cross section.

Thus, the resistance is defined as

$$R = \frac{l}{\sigma S} \quad (\Omega)$$

where l is the length of the homogeneous conductor, S is the area of the uniform cross section. The conductance G (reciprocal of resistance), is defined by

$$G = \frac{1}{R} = \sigma \frac{S}{l} \quad (S)$$

1. Resistance in series:

$$R_{sr} = R_1 + R_2$$

2. Resistance in parallel:

$$\frac{1}{R_{||}} = \frac{1}{R_1} + \frac{1}{R_2}$$

, or

$$G_{||} = G_1 + G_2$$

6.2 Electromotive Force and Kirchhoff's Voltage Law

A steady current cannot be maintained in the same direction in a closed circuit by an electrostatic field, which is:

$$\oint_C \frac{1}{\sigma} J \cdot dl = 0$$

Kirchhoff's voltage law: around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistance, which is:

$$\sum_j V_j = \sum_k R_k I_k \quad (V)$$

6.3 Equation of Continuity and Kirchhoff's Current Law

Equation of continuity:

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t} \quad (A/m^3)$$

,

where ρ is the volume charge density.

For steady currents, as $\partial\rho/\partial t = 0$, $\nabla \cdot J = 0$. By integral, we have Kirchhoff's current law, stating that the algebraic sum of all the currents flowing out of a junction in an electric circuit is zero:

$$\sum_j I_j = 0$$

For a simple medium conductor, the volume charge density ρ can be expressed as:

$$\rho = \rho_0 e^{-(\rho/\epsilon)t} \quad (C/m^3)$$

where ρ_0 is the initial charge density at $t = 0$. The equation implies that the charge density at a given location will decrease with time exponentially.

Relaxation time: an initial charge density ρ_0 will decay to $1/e$ or 36.8% of its original value:

$$\tau = \frac{\epsilon}{\sigma} \quad (s)$$