Chapter 3: Static Electric Fields

Lecturer: Nana Liu

Summer 2020

JOINT INSTITUTE
交大密西根学院

Many slides courtesy of Sung-Liang Chen

3-10 Capacitance and Capacitors

- Deposit charges Q on a conductor ➔ V
- kQ ➔ kρ_s ➔ kV

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\int_{S'} \frac{\rho_s}{R}\, ds' \qquad \text{(V);}}$$

- The ratio Q/V unchanged

$$\boxed{Q = CV,}$$

C: capacitance (C/V, or Farad)

Capacitor

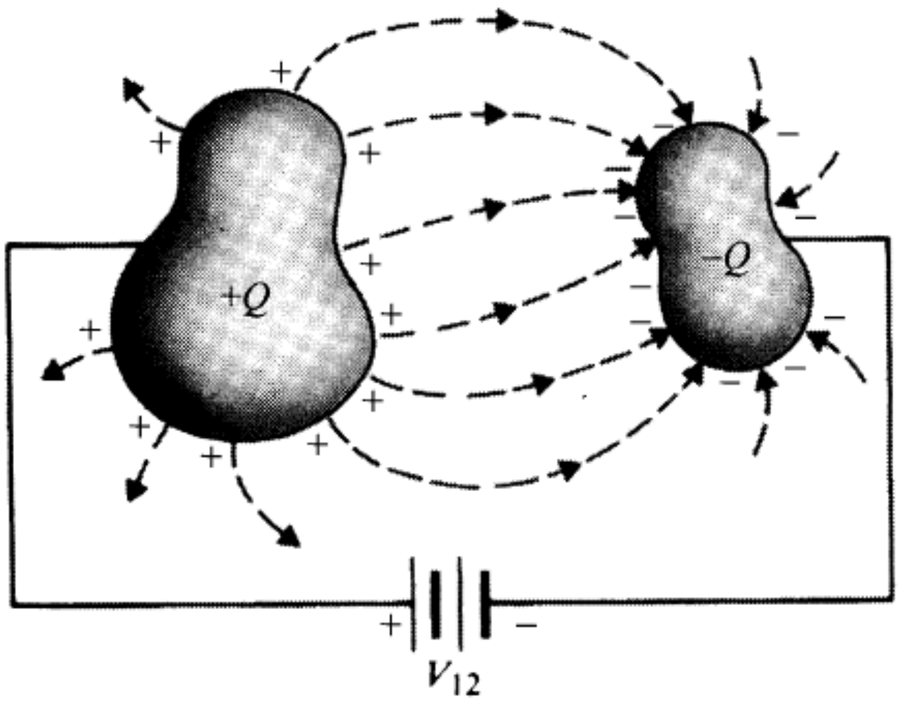


FIGURE 3–27
A two-conductor capacitor.

E $\perp$ conductor surfaces (equipotential surfaces)

$$\boxed{C = \frac{Q}{V_{12}} \qquad \text{(F).}}$$

Capacitance

- C depends on
  - the geometry of the conductors
  - the permittivity of the medium between conductors
  - **Independent of Q and V**
- Measurement of C
  - Method 1: V_{12} known, determine Q (Chap.4)
  - Method 2: Q known, determine V_{12}

Capacitance

- Method 2: Q known, determine V_{12}
  1. Choose a proper coordinate system
  2. Assume +Q, –Q on the conductors
  3. Find **E** from Q (Gauss's law, etc.)
  4. Find V_{12} by $V_{12} = -\int_2^1 \mathbf{E} \cdot d\boldsymbol{\ell}$
  5. $C=Q/V_{12}$

3-10.1 Series and Parallel Connections of Capacitors

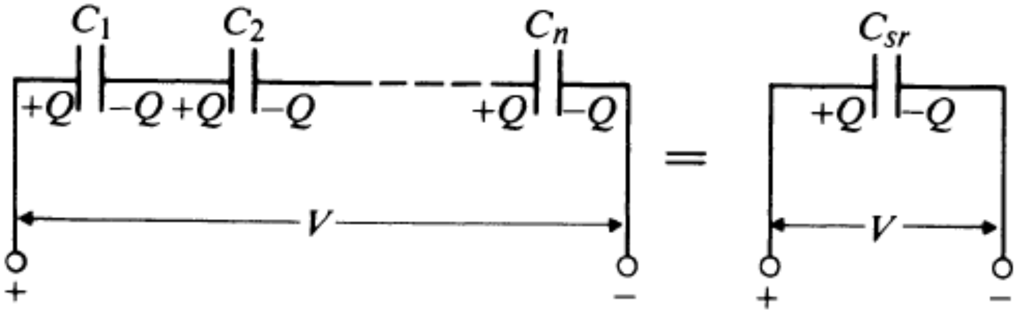


FIGURE 3–31
Series connection of capacitors.

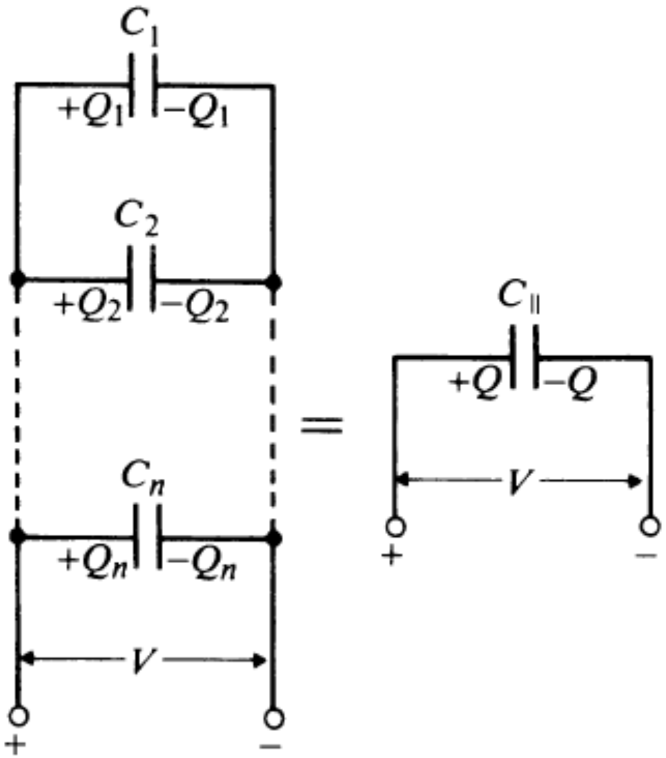


FIGURE 3–32
Parallel connection of capacitors.

Series

- V
  - ➔ +Q and –Q on two external terminals
  - ➔ +Q and –Q also induced internally

$$V = \frac{Q}{C_{sr}} = \frac{Q}{C_1} + \frac{Q}{C_2} + \cdots + \frac{Q}{C_n},$$

$$\boxed{\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots + \frac{1}{C_n}.}$$

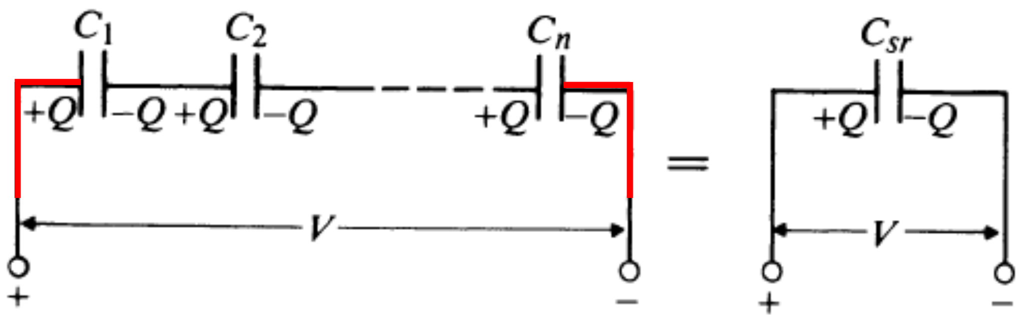
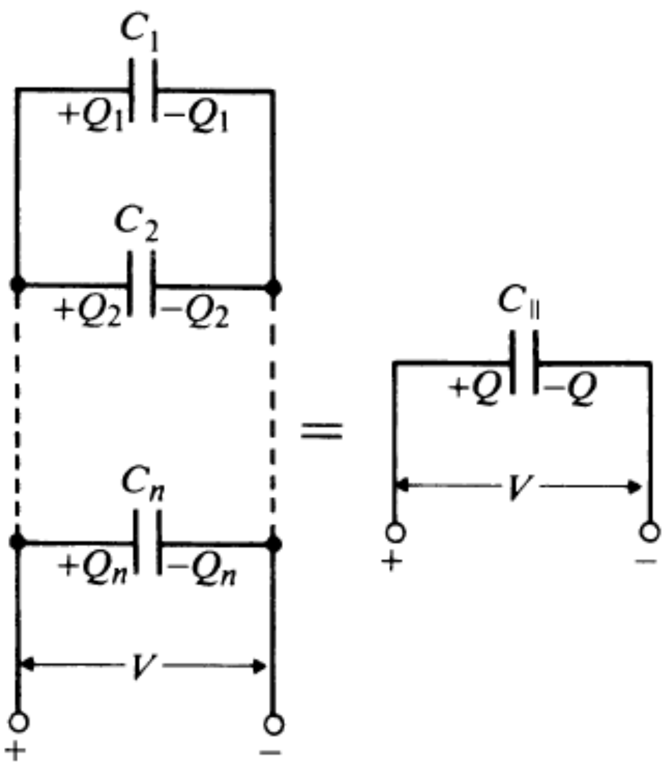


FIGURE 3–31
Series connection of capacitors.

Parallel

- V

➔ Q_1, Q_2, Q_3, … on each capacitor



$$Q = Q_1 + Q_2 + \cdots + Q_n$$
$$= C_1 V + C_2 V + \cdots + C_n V = C_{||} V$$

$$\boxed{C_{||} = C_1 + C_2 + \cdots + C_n.}$$

FIGURE 3–32
Parallel connection of capacitors.

3-10.2 Capacitances in Multiconductor Systems

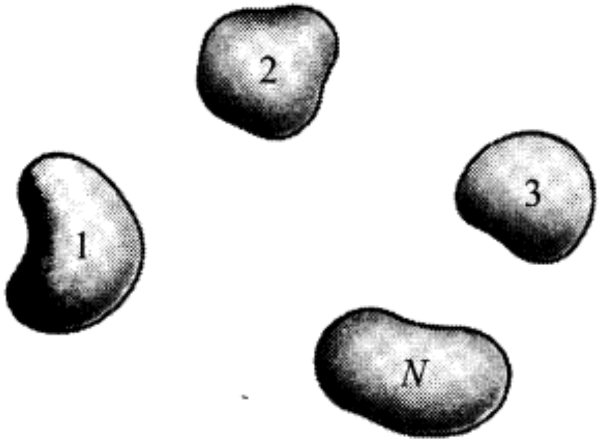


FIGURE 3–34
A multiconductor system.

Presence of a charge on any one of the conductors
➔ Affect potential of all the other conductors

$$\begin{aligned}
V_1 &= p_{11}Q_1 + p_{12}Q_2 + \cdots + p_{1N}Q_N, \\
V_2 &= p_{21}Q_1 + p_{22}Q_2 + \cdots + p_{2N}Q_N, \\
&\vdots \\
V_N &= p_{N1}Q_1 + p_{N2}Q_2 + \cdots + p_{NN}Q_N.
\end{aligned}$$

p_{ij}: coefficients of potential; depends on
1. Shape and position of the conductor
2. Permittivity of surroundings

For an isolated system $Q_1 + Q_2 + Q_3 + \cdots + Q_N = 0.$

$$
\begin{aligned}
Q_1 &= c_{11}V_1 + c_{12}V_2 + \cdots + c_{1N}V_N, \\
Q_2 &= c_{21}V_1 + c_{22}V_2 + \cdots + c_{2N}V_N, \\
&\vdots \\
Q_N &= c_{N1}V_1 + c_{N2}V_2 + \cdots + c_{NN}V_N,
\end{aligned}
$$

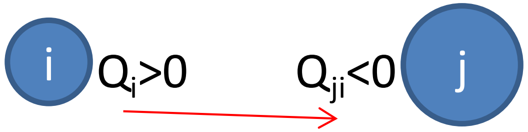
c_{ii}: coefficients of capacitance
c_{ij}: coefficients of induction ($i \neq j$)

c_{ii}: ground all other conductors, then $c_{ii}=Q_i/V_i$

c_{ji}: Induced charge $Q_{ji}=c_{ji}V_i$

If Q_i on ith conductor, $V_i>0$ ➔ induced $Q_{ji}<0$

Thus, $c_{ii}>0$; $c_{ji}<0$



By reciprocity, $p_{ij} = p_{ji}$ and $c_{ij} = c_{ji}$

A Four-conductor System

$$\begin{aligned}
Q_1 &= c_{11}V_1 + c_{12}V_2 + \cdots + c_{1N}V_N, \\
Q_2 &= c_{21}V_1 + c_{22}V_2 + \cdots + c_{2N}V_N, \\
&\vdots \\
Q_N &= c_{N1}V_1 + c_{N2}V_2 + \cdots + c_{NN}V_N,
\end{aligned}$$

Conductors 0,1,2,3. Let conductor 0 be grounded (i.e., $V_0=0$).

$$\begin{aligned}
Q_1 &= c_{11}V_1 + c_{12}V_2 + c_{13}V_3, \\
Q_2 &= c_{12}V_1 + c_{22}V_2 + c_{23}V_3, \\
Q_3 &= c_{13}V_1 + c_{23}V_2 + c_{33}V_3,
\end{aligned}$$

A Four-conductor System

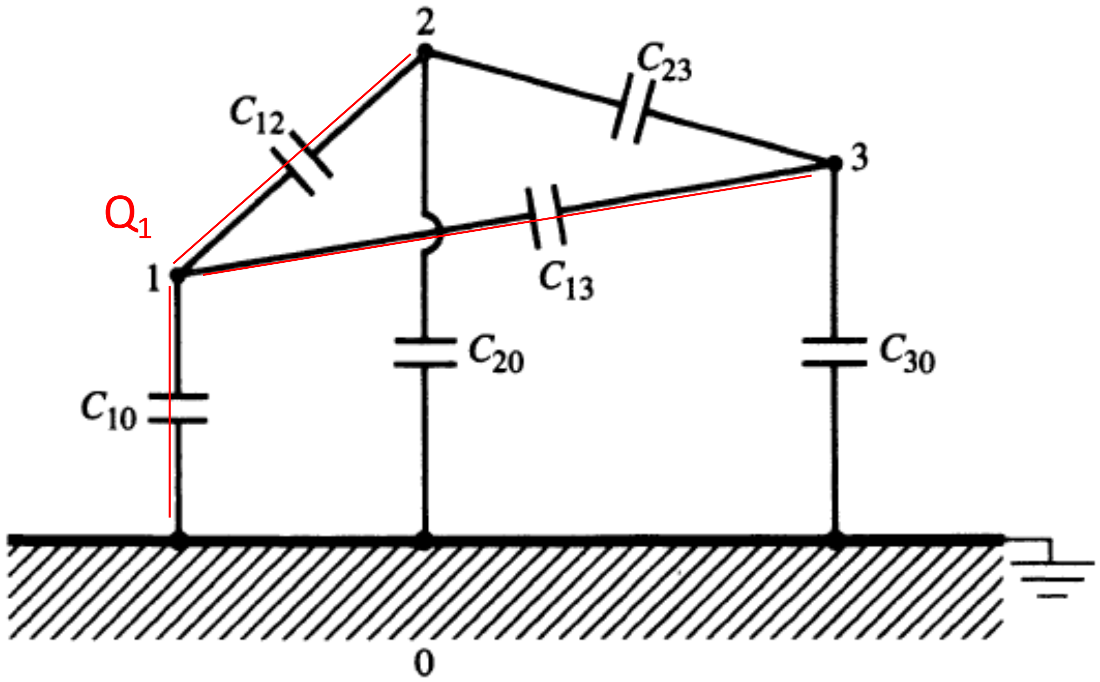


FIGURE 3–35
Schematic diagram of three conductors and the ground.

c: Coefficient of capacitance
C: Capacitance

Rewrite the Q ~ V relation

$$Q_1 = C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3),$$
$$Q_2 = C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3),$$
$$Q_3 = C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2),$$

C_{10},C_{20},C_{30}: self-partial capacitance
C_{ij} ($i \neq j$): mutual partial capacitance

$$Q_1 = (C_{10} + C_{12} + C_{13})V_1 - C_{12}V_2 - C_{13}V_3,$$
$$Q_2 = -C_{12}V_1 + (C_{20} + C_{12} + C_{23})V_2 - C_{23}V_3,$$
$$Q_3 = -C_{13}V_1 - C_{23}V_2 + (C_{30} + C_{13} + C_{23})V_3.$$

Compare with

$$\boxed{\begin{aligned} Q_1 &= c_{11}V_1 + c_{12}V_2 + c_{13}V_3, \\ Q_2 &= c_{12}V_1 + c_{22}V_2 + c_{23}V_3, \\ Q_3 &= c_{13}V_1 + c_{23}V_2 + c_{33}V_3, \end{aligned}}$$

Coefficient of capacitance:
c_{11} is the total capacitance between conductor 1 and all the other conductors connected

$$c_{11} = C_{10} + C_{12} + C_{13},$$
$$c_{22} = C_{20} + C_{12} + C_{23},$$
$$c_{33} = C_{30} + C_{13} + C_{23},$$

Coefficient of inductance:
c_{12} is negative of the C_{12} (mutual partial capacitance)

$$c_{12} = -C_{12},$$
$$c_{23} = -C_{23},$$
$$c_{13} = -C_{13}.$$

$$C_{10} = c_{11} + c_{12} + c_{13},$$
$$C_{20} = c_{22} + c_{12} + c_{23},$$
$$C_{30} = c_{33} + c_{13} + c_{23}.$$

3-10.3 Electrostatic Shielding

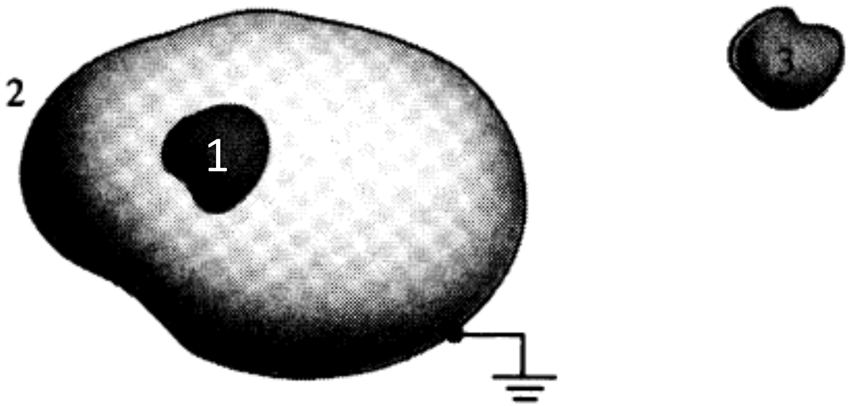


FIGURE 3–37
Illustrating electrostatic shielding.

$$\boxed{\begin{aligned} Q_1 &= C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3), \\ Q_2 &= C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3), \\ Q_3 &= C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2), \end{aligned}}$$

A three-conductor system

Setting $V_2=0$

$\Rightarrow$ $Q_1 = C_{10}V_1 + C_{12}V_1 + C_{13}(V_1 - V_3).$

When $Q_1=0$ $\Rightarrow$ **E** inside 2=0 $\Rightarrow$ $V_1=V_2=0$ $\Rightarrow$ $0=-C_{13}V_3$ $\Rightarrow$ $C_{13}=0$

Gauss's law

$$V = -\int \boldsymbol{E}\cdot dl$$

A change of V_3 will not affect Q_1

3-11 Electrostatic Energy and Forces

- From Eq. 3-44 $\frac{W}{q} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell$ $=V_{21}$
  - Work required to bring a charge *q* from P_1 to P_2 $W=qV_{21}$

- A charge Q_1 in free space. Work required to bring a **second** charge Q_2 from infinity to a distance R_{12} (position 2): $W=Q_2 \underline{V_{2\infty}}=Q_2V_2$

$$W_2 = Q_2V_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 R_{12}}$$

Against E field of charge Q_1
(V_2 is due to charge Q_1)

Rewrite $$W_2 = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 R_{12}}$$

$$W_2 = Q_1 \frac{Q_2}{4\pi\epsilon_0 R_{12}} = Q_1 V_1.$$

$Q_1V_1=Q_2V_2$

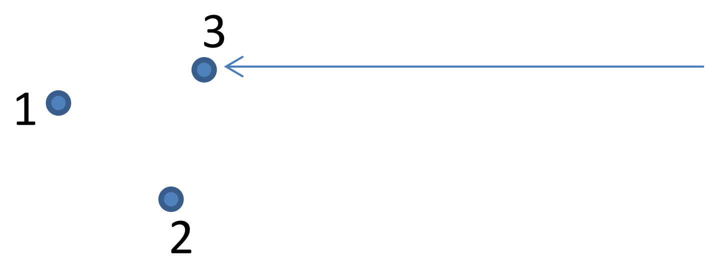
$\Rightarrow$ $Q_1V_1+Q_2V_2= 2Q_1V_1=2W_2$

$$W_2 = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2).$$

- Another charge Q_3. Work required to bring a **third** charge Q_3 from infinity to a distance R_{13} from Q_1 and R_{23} from Q_2: $\Delta W = Q_3 V_{3\infty}$

Against E field of charge Q_1 and E field of charge Q_2
V_3 is due to charges Q_1 and Q_2

$$\Delta W = Q_3 V_3 = Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right).$$



- Total work to assemble the 3 charges Q_1, Q_2, and Q_3: $W_3 = W_2 + \Delta W$

$$W_3 = W_2 + \Delta W = \frac{1}{4\pi\epsilon_0}\left(\frac{Q_1Q_2}{R_{12}} + \frac{Q_1Q_3}{R_{13}} + \frac{Q_2Q_3}{R_{23}}\right).$$

Rewrite: 3 terms divided into 6 terms

$$\begin{aligned} W_3 &= \frac{1}{2}\left[\underline{Q_1\left(\frac{Q_2}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{13}}\right)} + Q_2\left(\frac{Q_1}{4\pi\epsilon_0 R_{12}} + \frac{Q_3}{4\pi\epsilon_0 R_{23}}\right)\right. \\ &\quad \left. + Q_3\left(\frac{Q_1}{4\pi\epsilon_0 R_{13}} + \frac{Q_2}{4\pi\epsilon_0 R_{23}}\right)\right] \\ &= \tfrac{1}{2}(Q_1\underline{V_1} + Q_2V_2 + Q_3V_3). \end{aligned}$$

Potential V_1 is caused by charges Q_2 and Q_3

Different from the previous V_1 due to Q_2 only $W_2 = Q_1 \dfrac{Q_2}{4\pi\epsilon_0 R_{12}} = Q_1V_1.$

General expression

Self energy: Work required to assemble the individual point charges

Mutual energy: the interacting energy

Initially, Q_1 in space

Introduce Q_2 $\Delta W = Q_2 V_{2\infty}$

Introduce Q_3 $\Delta W = Q_3 V_{3\infty}$

+

$$W_2 = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2).$$

$$W_3 = \tfrac{1}{2}(Q_1 V_1 + Q_2 V_2 + Q_3 V_3).$$

$$\vdots$$

$$\boxed{W_e = \frac{1}{2}\sum_{k=1}^{N} Q_k V_k \qquad \text{(J)},}$$

Potential V_k is caused by all the other charges

$$V_k = \frac{1}{4\pi\epsilon_0}\sum_{\substack{j=1\\(j\neq k)}}^{N} \frac{Q_j}{R_{jk}}.$$

3-11.1 Electrostatic Energy in terms of Field Quantities

- For a continuous charge distribution of density ρ

$$\boxed{W_e = \frac{1}{2}\int_{\underline{V'}} \rho \underline{V}\, dv \qquad \text{(J)}.}$$

Volume

Electrical potential

$$W_e = \frac{1}{2}\int_{V'} (\nabla \cdot \mathbf{D}) V\, dv.$$

$$\nabla \cdot (V\mathbf{D}) = V\nabla \cdot \mathbf{D} + \mathbf{D} \cdot \nabla V,$$

$$W_e = \frac{1}{2}\int_{V'} \nabla \cdot (V\mathbf{D})\,dv - \frac{1}{2}\int_{V'} \mathbf{D} \cdot \nabla V\,dv$$
$$= \frac{1}{2}\oint_{S'} V\mathbf{D} \cdot \mathbf{a}_n\,ds + \frac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E}\,dv,$$

- *V'* can be any volume
- Choose its radius R→∞ ➔ 1st term disappears

$$\boxed{W_e = \frac{1}{2}\int_{V'} \mathbf{D} \cdot \mathbf{E}\,dv \qquad \text{(J).}}$$

$\mathbf{D} = \epsilon\mathbf{E}$ For a linear medium

$$\boxed{W_e = \frac{1}{2}\int_{V'} \epsilon E^2\,dv \qquad \text{(J)}}$$

$$\boxed{W_e = \frac{1}{2}\int_{V'} \frac{D^2}{\epsilon}\,dv \qquad \text{(J).}}$$

Electrostatic Energy Density w_e

$$\boxed{W_e = \frac{1}{2}\int_{V'} \mathbf{D}\cdot\mathbf{E}\,dv \qquad \text{(J).}}$$

$$\boxed{W_e = \frac{1}{2}\int_{V'} \epsilon E^2\,dv \qquad \text{(J)}}$$

$$\boxed{W_e = \frac{1}{2}\int_{V'} \frac{D^2}{\epsilon}\,dv \qquad \text{(J).}}$$

$$W_e = \int_{V'} w_e\,dv.$$

$$w_e = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E} \qquad (\text{J/m}^3)$$

$$w_e = \tfrac{1}{2}\epsilon E^2 \qquad (\text{J/m}^3)$$

$$w_e = \frac{D^2}{2\epsilon} \qquad (\text{J/m}^3).$$

Definition of density form is artificial;
Volume integral form can be verified.

3-11.2 Electrostatic Forces

- Using Coulomb's law to determine the force on one body that is caused by the charges on other bodies would be very tedious.
- Thus, a simple method of **principle of virtual displacement** is introduced.
  - System of bodies with fixed charges
  - System of conducting bodies with fixed potentials

System of Bodies with Fixed Charges

- An isolated system consisting of charged conductor and dielectric bodies.
- Condition: Charges are constant.
- Electric force displaces one of the bodies by dl (a virtual displacement)
  - Mechanical **work** done **by the system**:

$$dW = \mathbf{F}_Q \cdot d\ell,$$

$\mathbf{F}_Q$: total electric force acting on the body

– In other words, **reduced stored electrostatic energy** produces the mechanical **work**

$$dW = \underline{-dW_e} = \mathbf{F}_Q \cdot d\boldsymbol{\ell}.$$

Reduced stored electrostatic energy

$$dW_e = (\nabla W_e) \cdot d\boldsymbol{\ell}$$

$$\boxed{\mathbf{F}_Q = -\nabla W_e \qquad \text{(N)}.}$$

A very simple formula for the calculation of $\mathbf{F}_\mathbf{Q}$ from the electrostatic energy of the system

- Electric torque rotates one of the bodies by dϕ (a virtual rotation) about an axis (e.g., z axis)
  - Work done by the system:

$$dW = (T_Q)_z \, d\phi$$

$$\vdots$$

$$\boxed{(T_Q)_z = -\frac{\partial W_e}{\partial \phi} \qquad (\mathrm{N \cdot m}).}$$

System of Conducting Bodies with Fixed Potentials

- Condition: potentials are fixed.
- System connected to **external sources** to maintain fixed potentials
- A displacement dl ➔ dW_e, dQ_k
  to maintain fixed potentials V_k
  - 1. Work done by the external sources:

$$dW_s = \sum_k V_k \, dQ_k$$

  - 2. Produced mechanical work:

$$dW = \mathbf{F}_V \cdot d\ell$$

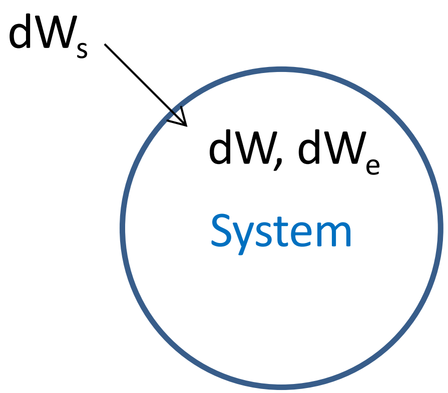
$\mathbf{F}_V$: electric force acting on the body

– 3. Change of electrostatic energy due to dQ_k:

$$dW_e = \frac{1}{2}\sum_k V_k\, dQ_k = \frac{1}{2}\, dW_s$$

- Thus, $dW + dW_e = dW_s.$

$$\boxed{dW_e = \frac{1}{2}\sum_k V_k\, dQ_k = \frac{1}{2}\, dW_s}$$



$dW = dW_e$

$$\mathbf{F}_V \cdot d\boldsymbol{\ell} = dW_e = (\nabla W_e)\cdot d\boldsymbol{\ell}$$

$$\boxed{\mathbf{F}_V = \nabla W_e \qquad \text{(N)}.}$$

- Similarly, a displacement $d\phi \rightarrow dW_e, dQ_k$

.
.
.

$$\boxed{(T_V)_z = \frac{\partial W_e}{\partial \phi} \qquad (\mathrm{N \cdot m}),}$$

The difference in formulas for fixed potentials and for fixed charges is only a sign change.