1 Steady Electric Currents

Types of electric currents caused by the motion of free charges:

1. **conduction currents**: drift motion of conduction electrons and/or holes in conductors/semiconductors.
2. electrolytic currents: migration of positive and negative ions.
3. convection currents: motion of electrons and/or ions in a vacuum.

2 Current Density and Ohm's Law

$$I = \int_S \vec{J} \cdot ds \quad (A)$$

where J is the volume current density or current density, defined by

$$J = Nqu \quad (A/m^2) \qquad \vec{J} = Nq\vec{u}$$

,

$\Delta Q = Nq\vec{u} \cdot \hat{a}_n \Delta s \Delta t$

$\Rightarrow I = \frac{\Delta Q}{\Delta t} = Nq\vec{u} \cdot \hat{a}_n \Delta s = Nq\vec{u} \cdot \Delta \vec{s}$

$\int I = \int \vec{J} \cdot \Delta \vec{s}$

where N is the number of charge carriers per unit volume, each of charges q moves with a velocity u.

Since Nq is the free charge per unit volume, by $\rho = Nq$, we have:

$$J = \rho u \quad (A/m^2)$$

Ex.5-1 In vaccum-tube diodes, electrons are emitted from a hot cathode at zero potential and collected by an anode maintained at a potential V_0, resulting in a convection current flow. Assuming that the cathode and the anode are parallel conducting plates and that the electrons leave the cathode with a zero initial velocity (space-charge limited condition), find the relation between the current density J and V_0.

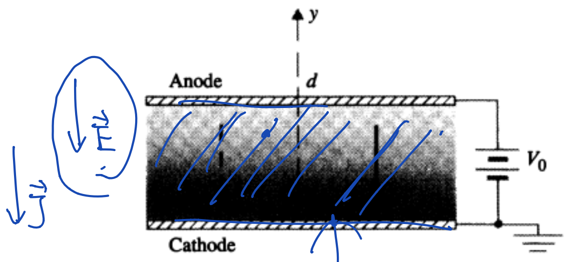


FIGURE 5–2
Space-charge-limited vacuum diode (Example 5–1).

$\vec{J} = -\hat{a}_y J = \hat{a}_y \rho(y) u(y)$

$\vec{F} = -e\vec{E} = ma = m\frac{du}{dt}$

$\vec{E} = -\nabla V = -\frac{dV(y)}{dy}$

$e\frac{dV(y)}{dy} = m\frac{du}{dt} = m\frac{du}{dy}\underbrace{\frac{dy}{dt}}_{=u} = mu\frac{du}{dy}$

$\Rightarrow e\frac{dV}{dy} = \frac{d(\frac{1}{2}mu^2)}{dy}$

$\Rightarrow eV = \frac{1}{2}mu^2$

$u = \left(\frac{2e}{m}V\right)^{\frac{1}{2}}$

$\Rightarrow \rho = -\frac{J}{u} = -J\sqrt{\frac{m}{2eV}}$

$\frac{d^2V}{dy^2} = -\frac{\rho}{\varepsilon_0}$

$\Rightarrow \int \frac{d^2V}{dy^2} = \int \frac{J}{\varepsilon_0}\sqrt{\frac{m}{2eV}}$

$\frac{4}{3}V_0^{\frac{3}{4}} = 2\sqrt{\frac{J}{\varepsilon_0}}\left(\frac{m}{2e}\right)^{\frac{1}{4}} d.$

$\vec{u} = -\mu_e \vec{E}$

For conduction currents, $\vec{J} = \rho \vec{u} = (-\rho_e \mu_e)\vec{E}$, $= \sigma$

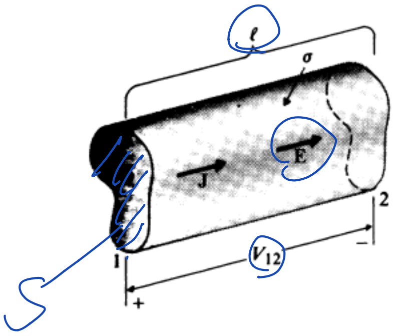
$V = RI$

$$\boxed{J = \sigma E \quad (A/m^2)} \Rightarrow \vec{J} = \sigma \vec{E}$$

where $\sigma = \rho_e \mu_e$ is conductivity, a macroscopic constitutive parameter of the medium. $\rho_e = -Ne$ is the charge density of the drifting electrons and is negative. $u = -\mu_e E \quad (m/s)$ where μ_e is the electron mobility measured in $(m^2/V \cdot s)$.

Materials where $J = \sigma E$ (A/m^2) holds are called ohmic media. The form can be referred as the point form of Ohm's law.

Derivation of voltage-current relationship of a piece of homogeneous material by the point form of Ohm's law.



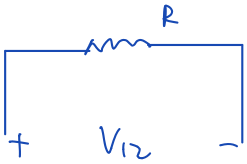


FIGURE 5–3
Homogeneous conductor with a constant cross section.

$V_{12} = El \Rightarrow E = \frac{V_{12}}{l}$

$J = \sigma E = \frac{\sigma V_{12}}{l}$

$I = JS = \frac{\sigma V_{12} S}{l}$

$R = \frac{V_{12}}{I} = \frac{V_{12}}{\frac{\sigma V_{12} S}{l}} = \frac{l}{\sigma S}$

Thus, the resistance is defined as

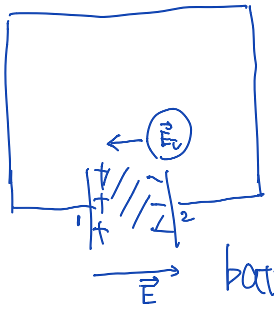
$$R = \frac{l}{\sigma S} \quad (\Omega)$$

where l is the length of the homogeneous conductor, S is the area of the uniform cross section. The conductance G (reciprocal of resistance), is defined by

$$G = \frac{1}{R} = \sigma \frac{S}{l} \quad (S)$$

1. Resistance in series:

$$R_{sr} = R_1 + R_2$$



battery

electromotive force $\Rightarrow$ emf

$$V = \int_2^1 \vec{E}_i \cdot d\vec{l} = -\int_2^1 \vec{E} \cdot d\vec{l}$$

2. Resistance in parallel:

$$\frac{1}{R_{||}} = \frac{1}{R_1} + \frac{1}{R_2}$$

, or

$$G_{||} = G_1 + G_2$$

3 Electromotive Force and Kirchhoff's Voltage Law

A steady current cannot be maintained in the same direction in a closed circuit by an electrostatic field, which is:

$$\oint_C \frac{1}{\sigma} J \cdot dl = 0$$

a

Kirchhoff's voltage law: around a closed path in an electric circuit, the algebraic sum of the emf's (voltage rises) is equal to the algebraic sum of the voltage drops across the resistance, which is:

$$\sum_j V_j = \sum_k R_k I_k \quad (V)$$

4 Equation of Continuity and Kirchhoff's Current Law

Equation of continuity:

$$I = \oint_S \vec{J} \cdot d\vec{S} = - \frac{dQ}{dt} = - \frac{d}{dt} \int_v \rho \, dv$$

$$= \int \nabla \cdot \vec{J} \, dv = \int_v - \frac{\partial \rho}{\partial t} \, dv$$

$$\nabla \cdot J = -\frac{\partial \rho}{\partial t} \quad (A/m^3)$$

,

where ρ is the volume charge density.

For steady currents, as $\partial\rho/\partial t = 0$, $\nabla \cdot J = 0$. By integral, we have Kirchhoff's current law, stating that the algebraic sum of all the currents flowing out of a junction in an electric circuit is zero:

$$\sum_j I_j = 0$$

For a simple medium conductor, the volume charge density ρ can be expressed as:

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \Rightarrow \sigma \nabla \cdot \vec{E} = -\frac{\partial \rho}{\partial t}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon}$$

$$\rho = \rho_0 e^{-(\rho/\epsilon)t} \quad (C/m^3)$$

where ρ_0 is the initial charge density at $t = 0$. The equation implies that the charge density at a given location will decrease with time exponentially.

Relaxation time: an initial charge density ρ_0 will decay to $1/e$ or 36.8% of its original value:

$$\Rightarrow -\frac{\partial \rho}{\partial t} = \frac{\rho}{\varepsilon}$$

$$\tau = \frac{\epsilon}{\sigma} \quad (s)$$

5 Power Dissipation and Joule's Law

For a given volume V that the total electric power converted to heat is:

$$P = \int_V \mathbf{E} \cdot \mathbf{J} dv \quad (= ds\,dl)$$

$$P = \int_L E d\ell \int_S J ds = VI = I^2 R$$

6 Boundary Conditions

6.1 Governing Equations for Steady Current Density

- Differential form:

$$\nabla \cdot \mathbf{J} = 0$$
$$\nabla \times \left(\frac{\mathbf{J}}{\sigma}\right) = 0$$

$\nabla \times \vec{E} = 0$, $\vec{E} = \frac{\vec{J}}{\sigma}$

- Integral form:

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = 0$$
$$\oint_C \frac{1}{\sigma}\mathbf{J} \cdot d\ell = 0$$

6.2 Boundary Conditions:

- Normal Component:

$$J_{1n} = J_{2n}$$

- Tangential Component:

$$\frac{J_{1t}}{J_{2t}} = \frac{\sigma_1}{\sigma_2}$$

Combining with boundary conditions of electric field:

$$J_{1n} = J_{2n} \quad \rightarrow \sigma_1 E_{1n} = \sigma_2 E_{2n}$$
$$D_{1n} - D_{2n} = \rho_s \quad \rightarrow \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

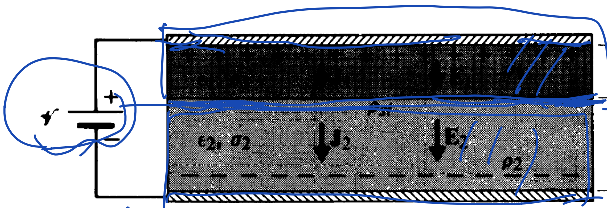
Surface charge density on the interface:

$$\rho_s = \left(\epsilon_1 \frac{\sigma_2}{\sigma_1} - \epsilon_2\right) E_{2n} = \boxed{\left(\epsilon_1 - \epsilon_2 \frac{\sigma_1}{\sigma_2}\right) E_{1n}}$$

If medium 2 is a much better conductor than medium 1:

$$\rho_s = \epsilon_1 E_{1n} = D_{1n}$$

EXAMPLE **5 − 4** An emf $\mathcal{V}$ is applied across a parallel-plate capacitor of area S. The space between the conducting plates is filled with two different lossy dielectrics of thicknesses d_1 and d_2, permittivities ϵ_1 and ϵ_2, and conductivities σ_1 and σ_2, respectively. Determine
(a) the current density between the plates,
(b) the electric field intensities in both dielectrics, and (c) the surface charge densities on the plates and at the interface.



(b) $\vec{J} = \sigma \vec{E} \Rightarrow \vec{E} = \frac{\vec{J}}{\sigma}$

$$\vec{E}_1 = -\hat{a}_y \frac{\sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2}$$
$$\vec{E}_2 = -\hat{a}_y \frac{\sigma_1 V}{\sigma_2 d_1 + \sigma_1 d_2}$$

$J_{1n} = J_{2n} \Rightarrow J_1 = J_2$

$$\sigma_1 E_1 = \sigma_2 E_2$$
$$E_1 d_1 + E_2 d_2 = V$$

(a) $R_1 = \frac{d_1}{\sigma_1 S}$, $R_2 = \frac{d_2}{\sigma_2 S}$

$$I = \frac{V}{R} = \frac{V}{\frac{d_1}{\sigma_1 S} + \frac{d_2}{\sigma_2 S}}$$

$$\Rightarrow J = \frac{I}{S} = \frac{V}{\frac{d_1}{\sigma_1} + \frac{d_2}{\sigma_2}} = \boxed{\frac{\sigma_1 \sigma_2 V}{\sigma_2 d_1 + \sigma_1 d_2}}$$

(c) $$\rho_S = \frac{(\epsilon_2 \sigma_1 - \epsilon_1 \sigma_2) V}{\sigma_2 d_1 + \sigma_1 d_2}$$

7 Resistance Calculation

$$C = \frac{Q}{V} = \frac{\oint_S \mathbf{D}\cdot d\mathbf{s}}{-\int_L \mathbf{E}\cdot d\ell} = \boxed{\frac{\oint_S \epsilon\mathbf{E}\cdot d\mathbf{s}}{-\int_L \mathbf{E}\cdot de}}$$

$$R = \frac{V}{I} = \frac{-\int_L \mathbf{E}\cdot d\ell}{\oint_S \mathbf{J}\cdot d\mathbf{s}} = \frac{-\int_L \mathbf{E}\cdot d\boldsymbol{\ell}}{\oint_S \sigma\mathbf{E}\cdot d\mathbf{s}}$$

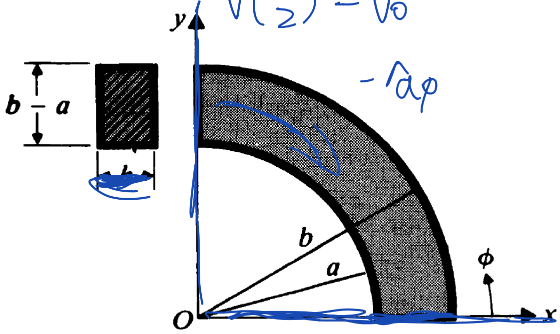
If σ and ϵ have the same space dependence or the medium is homogeneous:

$$RC = \frac{C}{G} = \frac{\epsilon}{\sigma}$$

Method of resistance calculation:

1. Choose a proper coordinate for the system.
2. Assume a voltage different V_0 between the two conductors.
3. Find the electric field E within the conductors (for homogenenous medium, solve Laplace's Equation ~~$\Delta^2 V = 0$~~ and get ~~$E = -\Delta V$~~).
   $\nabla^2 V = 0$ $\quad E = -\nabla V$
4. Find the total current
$$I = \int_S \mathbf{J}\cdot d\mathbf{s} = \int_S \sigma\mathbf{E}\cdot d\mathbf{s}$$
5. The resistance is $R = \frac{V_0}{I}$.

EXAMPLE 5-6 A conducting material of uniform thickness h and conductivity σ has the shape of a quarter of a flat circular washer, with inner radius a and outer radius b, as shown in Fig. $5-8$. Determine the resistance between the end faces.



$V(\frac{\pi}{2}) = V_0$

$-\hat{a}_\phi$

$V(0) = 0$

$I = \int_S \vec{J}\cdot d\vec{S}$

$dz\,dr$

$= \int \left(-\hat{a}_\phi \frac{2\sigma V_0}{\pi r}\right)\cdot\left(-\hat{a}_\phi\, h\,dr\right)$

$= \frac{2\sigma h V_0}{\pi}\ln\left(\frac{b}{a}\right)$

$R = \frac{V_0}{I} = \frac{\pi}{2\sigma h \ln(b/a)}$

$\nabla^2 V = 0 \Rightarrow \frac{d^2V}{d\phi^2} = 0 \Rightarrow V = C_1\phi + C_2$

$\begin{cases} V(0) = C_2 = 0 \\ V(\frac{\pi}{2}) = C_1 = V_0 \end{cases} \Rightarrow \begin{cases} C_1 = \frac{2V_0}{\pi} \\ C_2 = 0 \end{cases}$

$V = \frac{2V_0}{\pi}\phi \qquad \nabla = \frac{1}{r}\frac{\partial V}{\partial \phi}$

$\vec{E} = -\nabla V$

$\vec{J} = \sigma\vec{E} = \sigma(-\nabla V) = -\hat{a}_\phi\,\sigma\frac{\partial V}{r\,\partial\phi}$

$= -\hat{a}_\phi \frac{2\sigma V_0}{\pi r}$

P.5-10 The space between two parallel conducting plates each having an area S is filled with an inhomogeneous ohmic medium whose conductivity varies linearly from σ_1 at one plate ($y = 0$) to σ_2 at the other plate ($y = d$). A d-c voltage V_0 is applied across the plates as in Fig.5-11. Determine
a) the total resistance between the plates,
b) the surface charge densities on the plates,
c) the volume charge density and the total amount of charge between the plates.

$\rho_s = \epsilon E$

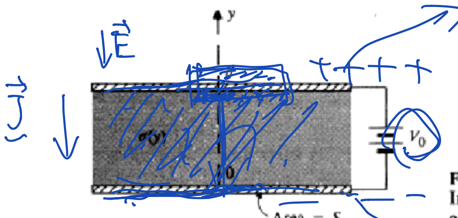


FIGURE 5–11
Inhomogeneous ohmic medium with conductivity $\sigma(y)$ (Problem P.5–10).

$\rho_s = -\epsilon E$

$\sigma = C_1 y + C_2 \Rightarrow \begin{cases} C_2 = \sigma_1 \\ C_1 d + C_2 = \sigma_2 \end{cases}$

$\Rightarrow \sigma(y) = \dfrac{\sigma_2 - \sigma_1}{d} y + \sigma_1$

$\vec{J} = -\hat{a}_y J$

$\vec{E} = \dfrac{\vec{J}}{\sigma} = -\dfrac{\hat{a}_y J}{\sigma(y)}$

$V_0 = -\int_0^d \vec{E} \cdot d\vec{l}$

$= -\int_0^d \dfrac{-\hat{a}_y J}{\sigma(y)} \cdot d\vec{l}$

$= \dfrac{J d}{\sigma_2 - \sigma_1} \ln\left(\dfrac{\sigma_2}{\sigma_1}\right)$

$R = \dfrac{V_0}{I} = \dfrac{V_0}{J \cdot S}$

$= \dfrac{d}{(\sigma_2 - \sigma_1) S} \ln\left(\dfrac{\sigma_2}{\sigma_1}\right)$

$\boxed{J = \dfrac{V_0 \sigma_1 \sigma_2}{\ln\left(\frac{\sigma_2}{\sigma_1}\right)}}$ ✓

upper: $\rho_s = \epsilon E$
$= \epsilon_0 \dfrac{J}{\sigma_2}$

lower: $\rho_s = -\epsilon E$
$= -\epsilon_0 \dfrac{J}{\sigma_1}$

(c) $\Rightarrow \rho = \nabla \cdot \vec{D}$

$\vec{E} = -\dfrac{\hat{a}_y J}{\sigma(y)}$

$\vec{D} = \epsilon \vec{E}$