1. (a) $V_a = x^2\hat{x} + 3xz^2\hat{y} - 2xz\hat{z}$

$\nabla \cdot V_a = \frac{\partial}{\partial x} A_x + \frac{\partial}{\partial y} A_y + \frac{\partial}{\partial z} A_z$

$= \frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(3xz^2) - \frac{\partial}{\partial z}(2xz)$

$= 2x - 2x = 0$

(b) $V_b = xy\,\hat{x} + 2yz\,\hat{y} + 3zx\,\hat{z}$

$\nabla \cdot V_b = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(2yz) + \frac{\partial}{\partial z}(3zx)$

$= y + 2z + 3x$

(c). $\nabla \cdot V_c = \frac{\partial}{\partial x}(y^2) + \frac{\partial}{\partial y}(2xy + z^2) + \frac{\partial}{\partial z}(2yz)$

$= 0 + 2x + 2y$

$= 2x + 2y$

2. (a) 1° calculate the divergence of $\vec{V_1}$

$\vec{V_1} = V_r \cdot \hat{r} \Rightarrow \nabla \cdot V_1 = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot \hat{r}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \cdot r^2)$

$= \frac{1}{r^2} \cdot 4r^4 = 4r$

2° Integrate $\nabla \cdot \vec{V_1}$ in spherical coordinate:

$\int_V \nabla \cdot \vec{V_1} dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (4r)(r^2 \sin\theta)\, dr\, d\theta\, d\phi$

$= \int_0^{2\pi}\int_0^{\pi}\int_0^{R} (4r^3 \sin\theta)\, dr\, d\theta\, d\phi$

$$= \int_0^{2\pi}\int_0^{\pi} R^4 \sin\theta \, d\theta \, d\phi = \int_0^{2\pi} 2R^4 d\phi = 2R^4 \phi \Big|_0^{2\pi}$$

$\because \int_0^{\pi} \sin\theta \, d\theta = -\cos\theta \Big|_0^{\pi} = 1$

$$= 4\pi R^4$$

The right hand side: $\oint_S A \cdot ds$

$$= \oint_S r^2 \hat{r} \, ds_R \qquad ds_R = r^2 \sin\theta \, d\theta \, d\phi \, \hat{r}$$

$$\Rightarrow \int (r^2 \hat{r}) \cdot (r^2 \sin\theta \, d\theta \, d\phi) \hat{r}$$

$$= R^2 \cdot \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} 1 \cdot d\phi \; \hat{r} \cdot \hat{r}$$

$$= R^2 \cdot 4\pi R^2 = 4\pi R^4$$

Therefore, $\int_V \nabla \cdot \vec{V}_1 dv = \oint_S A \cdot ds = 4\pi R^4$

the divergence theorem applies for V_1

(b) Take the gradient:

$$\nabla \cdot V_2 = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \cdot \frac{1}{r^2} \right) = \frac{1}{r^3} \cdot 0 = 0$$

Therefore $\int_V \nabla \cdot V_2 dV = \int_V 0 \cdot dv = 0$

$$\oint_S V_2 \cdot ds = \oint_S \left(\frac{1}{r^2} \hat{r} \right) ds = \oint_S \left(\frac{1}{r^2} \hat{r} \right) (r^2 \sin\theta \, d\theta \, d\phi \, \hat{r})$$

$$= \frac{1}{R^2} \cdot 4\pi R^2 = 4\pi$$

$\int_V \nabla \cdot V_2 \, dv \neq \oint_S V_2 \cdot ds$

V_2 doesn't satisfy the divergence theorem

(II) $\nabla \times \vec{V} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & 2yz & 3zx \end{vmatrix} = -2y\hat{x} - 3z\hat{y} - x\hat{z}$

$$\int_S (\nabla \times A) \, ds = \int_0^2 \int_0^2 -2y \, dy \, dz = -\frac{1}{3}z^3 + 2x^2 - 4z \Big|_0^2 = -\frac{8}{3}$$

$$\oint_C A \cdot dl = \int_0^2 2yz \, dy \qquad \because z = -y + 2$$

$$= \int_0^2 2y(-y+2) \, dy = \frac{16}{3} - 8 = -\frac{8}{3}$$

satisfied