Vector Analysis

1 Materials to cover

- Vectors related properties.
- Coordinates: Cartesian Coordinates, Cylindrical Coordinates, Spherical Coordinates, convertsion between these coordinates.
- Integrals: line integral, (double/tripple integral), review of some common integrals.
- Fields: scalar field, vector field, divergence and curl, Divergence Theorem, Stoke's Theorem.

2 Vectors

- dot product:
$$\vec{A} \cdot \vec{B} = ABcos\theta_{\vec{A}\vec{B}}$$
  - Commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
  - Distributive: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
  - Not associative: $\vec{A} \cdot (\vec{B} \cdot \vec{C}) \neq (\vec{A} \cdot \vec{B}) \cdot \vec{C}$
    e.g. $(\vec{a_x} \cdot \vec{a_y}) \cdot \vec{a_z} \neq \vec{a_x} \cdot (\vec{a_y} \cdot \vec{a_z})$
  - For the three edges A, B, C in a triangle, $C^2 = A^2 + B^2 - 2ABcos(\theta_{A,B})$
- cross product:
$$\vec{A} \times \vec{B} = \vec{a_n}|ABsin\theta_{\vec{A}\vec{B}}|$$
  - The cross product is always perpendicular to both $\vec{A}, \vec{B}$, the direction follows right hand rule.
  - Not Commutative: $\vec{A} \times \vec{B} \neq \vec{B} \times \vec{A}$ (have opposite directions).
  - Distributive: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$
  - Not associative: $\vec{A} \times (\vec{B} \times \vec{C}) \neq (\vec{A} \times \vec{B}) \times \vec{C}$
    e.g. $\vec{a_x} \times (\vec{a_x} \times \vec{a_y}) = \vec{a_x} \times \vec{a_z} = -\vec{a_y}$,
    $(\vec{a_x} \times \vec{a_x}) \times \vec{a_y} = 0 \neq -\vec{a_y}$
- $\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B}) = Volume$

- BAC-CAB: $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$

2.1 Exercise

- P.2-1 Given three vectos $\vec{A}, \vec{B}$ and $\vec{C}$ as follows,
  $\vec{A} = \vec{a_x} + \vec{a_y}2 - \vec{a_z}3$,
  $\vec{B} = -\vec{a_y}4 + \vec{a_z}$,
  $\vec{C} = \vec{a_x}5 - \vec{a_z}2$,
  find
  a) (Don't need to solve this problem) $\vec{a_A}$: note a_A represents the unit vector of $\vec{A}$.
  e) the component of $\vec{A}$ in the direction of $\vec{C}$

  f) $\vec{A} \times \vec{C}$ (1. use properties; 2. calculate by matrix)

g) (Don't need to solve this problem) $\vec{A} \cdot (\vec{B} \times \vec{C})$ and $(\vec{A} \times \vec{B}) \cdot \vec{C}$: note these two are equal to the volume.

h) $\vec{A} \times \vec{B}) \times \vec{C}$ and $\vec{A} \times (\vec{B} \times \vec{C})$ (convert the form to the standard BAC-CAB)

- P.2-2 Given

$$\vec{A} = \vec{a_x} - \vec{a_y}2 + \vec{a_x}3$$

,

$$\vec{B} = \vec{a_x} + \vec{a_y} - \vec{a_z}2$$

, find the expression for a unit vector $\vec{C}$ that is perpendicular to both $\vec{A}$ and $\vec{B}$.

- P.2-4 Show that, if $\vec{A} \cdot \vec{B} = \vec{A} \cdot \vec{C}$ and $\vec{A} \times \vec{B} = \vec{A} \times \vec{C}$, where $\vec{A}$ is not a null vector, then $\vec{B} = \vec{C}$.

3 Coordinates

Three basis (u_1, u_2, u_3): number of linearly independent basis = dimension of the space. For the three types of coordinates we discuss, u_i is orthogonal to each other.

For arbitrary vector $\vec{A}$:

$$\vec{A} = \vec{a_{u1}} A_{u1} + \vec{a_{u2}} A_{u2} + \vec{a_{u3}} A_{u3}$$

,

Norm of $\vec{A}$:

$$|\vec{A}| = \sqrt{A_{u1}^2 + A_{u2}^2 + A_{u3}^2}$$

For a differential length dl,

$$dl = \vec{a_{u1}}(h_1 du_1) + \vec{a_{u2}}(h_2 du_2) + \vec{a_{u3}}(h_3 du_3)$$

, h_i is called metric coefficient.

differential volume:

$$dv = h_1 h_2 h_3 du_1 du_2 du_3$$

differential area vector with a direction normal to the surface,

$$d\vec{s} = \vec{a_n} ds$$

differential area ds_1 normal to the unit vector $\vec{a_{u1}}$.

$$ds_1 = h_2 h_3 du_2 du_3 \tag{1}$$

Note that for ds_i, the foot indices on the right hand side are the ones that don't show up on the left hand side.

3.1 Cartesian Coordinates

- Figure for Cartesian coordinate.

- $$(u_1, u_2, u_3) = (x, y, z)$$

- Right hand rule:

$$\vec{a_x} \times \vec{a_y} = \vec{a_z}$$

- $$\vec{A} = \vec{a_x}A_x + \vec{a_y}A_y + \vec{a_z}A_z$$

  , where $\vec{a_i}$ is the basis for i-axis.

- dot product and cross product:

- differential length:

$$d\vec{l} = \vec{a_x}dx + \vec{a_y}dy + \vec{a_z}dz \tag{2}$$

- differential area:

$$ds_x = dydz$$

  , as $h_1 = h_2 = h_3 = 1$,

  (ds_x is the surface perpendicular to the x-axis, the forms for other surfaces follow the same pattern).

- differential volume:

$$dv = dxdydz$$

3.2 Cylindrical Coordinate

- Figure for cylindrical coordinate. Notice the position of θ.

- $$(u_1, u_2, u_3) = (r, \phi, z)$$

  Claim: as a_r can change its direction in the x-y plane, vectors in x-y plane could be represented simply by $\vec{a_r}$. Thus, all vectors in cylindrical coordinate could be represented by $\vec{a_r}$ and $\vec{a_z}$.

  Question: Do you agree with the claim? Why is $\vec{a_\phi}$ necessary?

- Right hand rule:
  $$\vec{a_r} \times \vec{a_\phi} = \vec{a_z}$$

- $$\vec{A} = \vec{a_r}Ar + \vec{a_\phi}A_\phi + \vec{a_z}A_z$$

- differential length:
  $$d\vec{l} = \vec{a_r}dr + \vec{a_\phi}rd\phi + \vec{a_z}dz \tag{3}$$
  , as $h_1 = 1, h_2 = r, h_3 = 1$

- differential area:
  $$ds_r = rd\phi dz$$
  , based on Eq 1.

- differential volume:
  $$dv = rdrd\phi dz$$

- From cylindrical coordinate to Cartesian coordinate: represent A_x by the quantities in cylindrical coordinate.

The same applies to A_y.

Write the formula in the form of matrix.

- conversion of quantities between Cartesian coordinate and Cylindrical coordinate:

a)

$$\begin{cases} x = rcos\phi \\ y = rsin\phi \\ z = z \end{cases}$$

b)

$$\begin{cases} r = \sqrt{x^2 + y^2} \\ \phi = arctan\frac{y}{x} \\ z = z \end{cases}$$

Question: what is the difference between these conversion equations and the conversion equations related to A_r, A_ϕ, A_z?

Question: When we calculate the norm of $\vec{A}$ in cylindrical coordinate, we can use

$$|\vec{A}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

. Is

$$|\vec{A}| = \sqrt{A_r^2 + A_\phi^2 + A_z^2} \tag{4}$$

also true? Can we prove Eq 4 is the valid equation to calculate norm in cylindrical coordinate?

3.3 Spherical Coordinate

- Figure for Spherical Coordinate. Notice the position of ϕ, θ.

- $$(u_1, u_2, u_3) = (R, \theta, \phi)$$

- Right hand rule:
$$\vec{a_R} \times \vec{\theta} = \vec{\phi}$$

- $$\vec{A} = \vec{a_R} A_R + \vec{a_\theta} A_\theta + \vec{a_\phi} A_\phi$$

- differential length:
$$d\vec{l} = \vec{a_R} dR + \vec{a_\theta} R d\theta + \vec{a_\phi} R sin\theta d\phi \tag{5}$$
, as $h_1 = 1, h_2 = R, h_3 = Rsin\theta$.

- differential area:
$$ds_R = R^2 sin\theta d\theta d\phi$$
, based on Eq 1.

- differential volume:
$$dv = R^2 sin\theta dR d\theta d\phi$$

- conversion of quantities between Cartesian coordinate and Spherical coordinate:

  a)
$$\begin{cases} x = Rsin\theta cos\phi \\ y = Rsin\theta sin\phi \\ z = Rcos\theta \end{cases}$$

  b)
$$\begin{cases} R = \sqrt{x^2 + y^2 + z^2} \\ \theta = arctan\frac{\sqrt{x^2+y^2}}{z} \\ \phi = arctan\frac{y}{x} \end{cases}$$

- From Spherical coordinate to Cartesian coordinate: represent A_x by the quantities in Spherical coordinate; write the formula in the form of matrix. (Similar to cylindrical coordinate).

P.2-17 A field is expressed in spherical coordinates by $\vec{E} = \vec{a_R}(25/R^2)$.

b) Find the angle that $\vec{E}$ makes with the vector $\vec{B} = \vec{a_x}2 - \vec{a_y}2 + \vec{a_z}$ at point $P(-3, 4, -5)$.

P.2-18 Express the base vectors $\vec{a_R}, \vec{a_\theta}, \vec{a_\phi}$ of a spherical coordinate system in Cartesian coordinates.

P.2-19 Determine the values of the following products of base vectors.

a) $\vec{a_x} \cdot \vec{a_\phi}$

c) $\vec{a_r} \times \vec{a_x}$

4 Integrals

4.1 Line integrals

When integrated along a certain differential length, use Eq 2, Eq 3, Eq 5 to convert differential length to integrable quantities with regard to different coordinates.

4.2 Some common integrals

- Integrals for sinusoidal functions: (from derivatives to integrals)

4.3 Exercise

EXAMPLE 2-14: Given $\vec{F} = \vec{a_x}xy - \vec{a_y}2x$, evaluate the scalar line integral along the quarter-circle shown in Fig. 2-21.

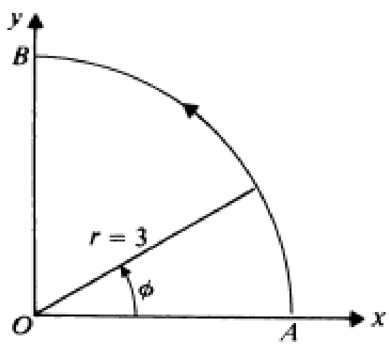

FIGURE 2–21
Path for line integral (Example 2–14).

- By Cartesian coordinate:

- By Cylindrical coordinate:

5 Fields

5.1 Scalar field and vector field

R.2-14: What is the difference between a scalar quantity and a scalar field? Between a vector quantity and a vector field?

5.2 Gradient of a scalar field

R.2-15: What is the physical definition of the gradient of a scalar field?

- $$\nabla V = \vec{a_n}\frac{dV}{dn}$$
- ∇V at certain point is a vector.
- $$dV = (\nabla V) \cdot d\vec{l}$$
  : the space rate of increase of V in the $\vec{a_l}$ direction is equal to the projection (the component) of the gradient of V in that direction. (The simple proof is on Page 43 of the text book).
- $$\nabla V = \vec{a_{u1}}\frac{\partial V}{h_1 \partial u_1} + \vec{a_{u2}}\frac{\partial V}{h_2 \partial u_2} + \vec{a_{u3}}\frac{\partial V}{h_3 \partial u_3}$$
  , when V is taken off:
  $$\nabla \equiv \vec{a_{u1}}\frac{\partial}{h_1 \partial u_1} + \vec{a_{u2}}\frac{\partial}{h_2 \partial u_2} + \vec{a_{u3}}\frac{\partial}{h_3 \partial u_3} gamma$$

5.3 Divergence of a vector field

- divergence of a vector field $\vec{A}$ at a point $div\vec{A}$ as the net outward flux of $\vec{A}$ per unit volume as the volume about the point tends to zero:
  $$div\vec{A} = \lim_{\Delta v \to 0} \frac{\oint_S \vec{A}\,\mathrm{d}s}{\Delta v} \tag{6}$$
  source: net positive divergence; sink: net negative divergence. zero divergence: no source/sink.
- $div\vec{A}$ at certain point is a scalar.

- For Cartesian coordinate,

$$div\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

, the proof is presented on page 48 in the text book.

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$$\nabla \cdot \vec{A} \equiv div\vec{A}$$

-

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3}[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_1 h_3 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)]$$

Example 2-17: Find the divergence of the position vector to an arbitrary point. (For spherical cooridnate only)

5.4 Divergence Theorem

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$$\int_V \nabla \cdot \vec{A} dv = \oint_S \vec{A} \cdot d\vec{s}$$

, the volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

- Proof is presented on page 50-51 in the textbook. The proof is based on the Eq 6.

5.5 Curl of a vector field

-

$$curl\vec{A} \equiv \nabla \times \vec{A} = \lim_{\Delta s \to 0} \frac{1}{\Delta s}[\vec{a_n} \oint_C \vec{A} \cdot d\vec{l}]_{max}$$

: the curl of a vector field $\vec{A}$, denoted by $curl\vec{A}$ or $\nabla \times \vec{A}$, is a vector whose magnitude is the maximum net circulation of $\vec{A}$ per unit area as the area tends to zero and whose direction is the normal direction of the area when the area is oriented to make the net circulation maximum. (Right hand rule defines the positive normal to an area).

- The proof for the form in Cartesian is presented on page 55-56 in the textbook.

- $\nabla \times \vec{A}$ in a general coordinate:

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \vec{a_{u1}} h_1 & \vec{a_{u2}} h_2 & \vec{a_{u3}} h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

- curl-free vector field ($\nabla \times \vec{A} = 0$): **irrotational** or **conservative field**

P.2-21 Given a vector function $\vec{E} = \vec{a_x} y + \vec{a_y} x$, evaluate the scalar line integral $\int \vec{E} \cdot d\vec{l}$ from $P_1(2, 1, -2)$ to $P_2(8, 2, -1)$.

a) along the parabola $x = 2y^2$,

b) along the straight line joining two points.

Is this $\vec{E}$ a conservative field?

P.2-22 For the $\vec{E}$ of P.2-21, evaluate $\int \vec{E} d\vec{l}$ from $P_3(3, 4, -1)$ to $P_4(4, -3, 1)$ by converting both $\vec{E}$ and the position of P_3 and P_4 into cylindrical coordinate. (Review of the inverse of a matrix)

5.6 Stokes's Theorem

- $$\int_S (\nabla \times \vec{A}) d\vec{s} = \oint_C \vec{A} \cdot d\vec{l}$$

  : the surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

5.7 Other identities

I
- $$\nabla \times (\nabla V) \equiv 0$$

  : the curl of the gradient of any scalar field is identically zero.
- Another interpretation: If a vector field is curl-free, it can be expressed as the gradient of a scalar field.
- Since a curl-free vector field is irrotational or conservative, an irrotational/conservative vector field can always be expressed as the gradient of a scalar field.

II
- $$\nabla \cdot (\nabla \times \vec{A}) \equiv 0$$

  : the divergence of the curl of any vector field is identically zero.
- Another interpretation: if a vector field is divergenceless, it can be expressed as the curl of another vector field.
- Divergenceless field is called solenoidal field, which will be further discussed in later classes.

Helmholtz's Theorem: A vector field (vector point function) is determined to within an additive constant if both its divergence and its curl are specified everywhere.

Example 2-23: Given a vector function

$$\vec{F} = \vec{a_x}(3y - c_1 z) + \vec{a_y}(c_2 x - 2z) - \vec{a_x}(c_3 y + z)$$

b) Determine the scalar potential function V whose negative gradient equals $\vec{F}$.

6 Other useful vector properties

$$\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$$

$$\nabla\cdot(\psi\mathbf{A}) = \psi\nabla\cdot\mathbf{A} + \mathbf{A}\cdot\nabla\psi$$

$$\nabla\times(\psi\mathbf{A}) = \psi(\nabla\times\mathbf{A}) + \nabla\psi\times\mathbf{A}$$

$$\nabla\cdot(\nabla\times\mathbf{A}) = 0$$

$$\nabla\times(\nabla\psi) = \mathbf{0}$$

$$\nabla\cdot(\nabla\psi) = \nabla^2\psi \text{ (scalar Laplacian)}$$

$$\nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) = \nabla^2\mathbf{A} \text{ (vector Laplacian)}$$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\nabla\times(\mathbf{A}\times\mathbf{B}) = \mathbf{A}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{A}) + (\mathbf{B}\cdot\nabla)\mathbf{A} - (\mathbf{A}\cdot\nabla)\mathbf{B}$$

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$$