
Static Electric Fields

1 Materials to Cover

- Vector Analysis <Vector Algebra, Orthogonal Coordinate Systems, Vector Calculus>
- Static Electric Fields Part I <Electrostatic Equations, Solving E, Conductors and Insulators>
- Static Electric Fields Part II <Boundary Condition, Capacitor, Electrostatic Energy and Force>

2 Vector Algebra

2.1 Vector Operation

2.1.1 Dot Product

- Definition: $\vec{A}\vec{B} = |A||B|\cos\theta_{AB}$
- Commutative Law $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$
- Associative Law: **Not** Associative
- Distribution Law: $\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$
- In XYZ Coordinate: $\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z$

2.1.2 Cross Product

- Definition: $\vec{A} \times \vec{B} = |A||B|\sin\theta_{AB}\hat{c}$ where $\hat{c} = \hat{a} \times \hat{b}$
- Commutative Law **Not** commutative $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$
- Associative Law: **Not** Associative
- Distribution Law: $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

- In XYZ Coordinate: $\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{x} + (A_z B_x - A_x B_z)\hat{y} + (A_x B_y - A_y B_x)\hat{z}$

Comments:

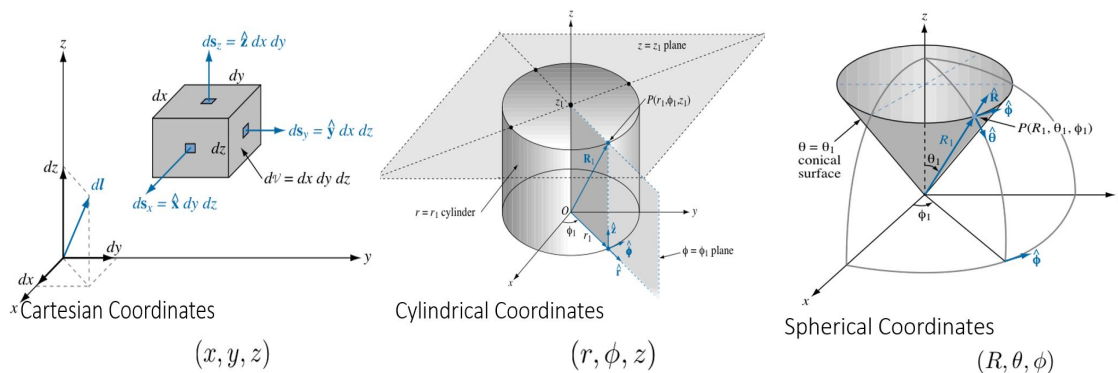
- 1 Triple Cross Product (bac-cab): $\vec{A} \times (\vec{B} \times \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$
2. 3. Scalar Triple Product:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \det \begin{bmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{bmatrix}$$

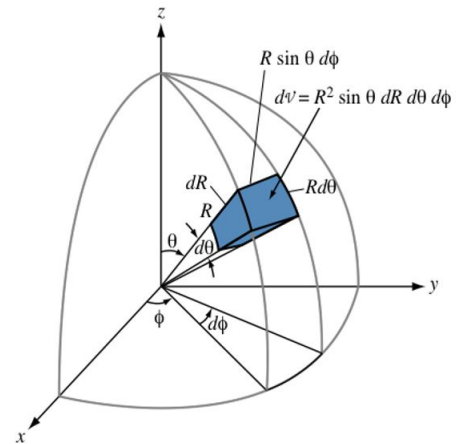
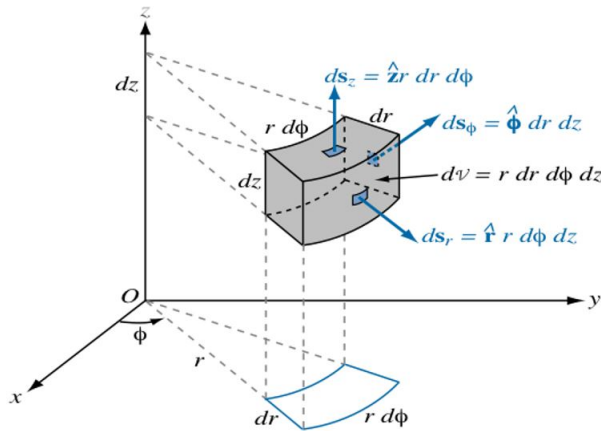
2.2 Orthogonal Coordinate System

Definition: Three perpendicular coordinate systems can form a orthogonal coordinate system. And the unit vectors are defined as the **normal** direction of each coordinate surface.

2.2.1 basic intro



Cartesian Coordinate	Cylindrical Coordinate	Spherical Coordinate
$\vec{OP} = \mathbf{a}_x x_1 + \mathbf{a}_y y_1 + \mathbf{a}_z z_1$	$\vec{OP} = \mathbf{a}_r r_1 + \mathbf{a}_z z_1$	$\vec{OP} = \mathbf{a}_r r_1$
$\mathbf{A} = \mathbf{a}_x A_x + \mathbf{a}_y A_y + \mathbf{a}_z A_z$	$\mathbf{A} = \mathbf{a}_r A_r + \mathbf{a}_\phi A_\phi + \mathbf{a}_z A_z$	$\mathbf{A} = \mathbf{a}_R A_R + \mathbf{a}_\theta A_\theta + \mathbf{a}_\phi A_\phi$
$h_1 = h_2 = h_3 = 1$	$h_1 = h_3 = 1, h_2 = r$	$h_1 = 1, h_2 = R, h_3 = R \sin\theta$
$d\mathbf{l} = \mathbf{a}_x dx + \mathbf{a}_y dy + \mathbf{a}_z dz$	$d\mathbf{l} = \mathbf{a}_r dr + \mathbf{a}_\phi r d\phi + \mathbf{a}_z dz$	$d\mathbf{l} = \mathbf{a}_R dR + \mathbf{a}_\theta R d\theta + \mathbf{a}_\phi R \sin\theta d\phi$
$ds_x = dy dz$	$ds_r = r d\phi dz$	$ds_R = R^2 \sin\theta d\theta d\phi$
$ds_y = dx dz$	$ds_\phi = r dr dz$	$ds_\theta = R \sin\theta dR d\phi$
$ds_z = dx dy$	$ds_z = r dr d\phi$	$ds_\phi = R dR d\theta$
$dv = dx dy dz$	$dv = r dr d\phi dz$	$dv = R^2 \sin\theta dR d\theta d\phi$



2.2.2 Vector Transformation in Different Coordinate

Key Points: $\vec{A} = A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$

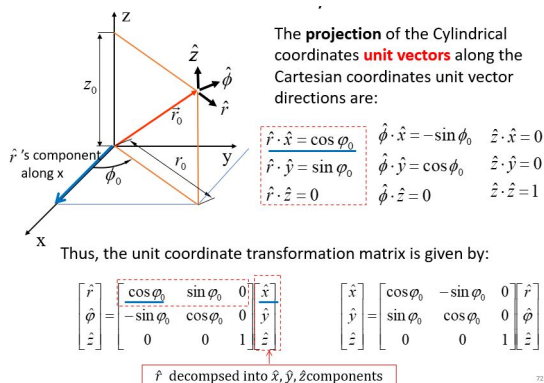
1. Unit vector transformation matrix
2. Corresponding magnitude of each direction
3. Calculate the needed elements' value like $r, \theta, \phi, R, x, y, z$ in it and replace them.

$\mathbf{A} \cdot \sim$ on both sides

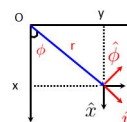
$$\mathbf{A} \cdot \mathbf{x} = A_x$$

$$\begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos \phi_0 & -\sin \phi_0 & 0 \\ \sin \phi_0 & \cos \phi_0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{r} \\ \hat{\phi} \\ \hat{z} \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix}$$

Application I: Cartesian to cylindrical



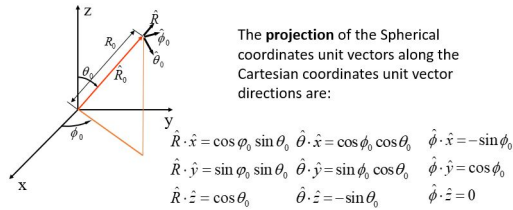
$$\vec{A} = \hat{x} A_x + \hat{y} A_y + \hat{z} A_z = \hat{r} A_r + \hat{\phi} A_\phi + \hat{z} A_z$$



$$\begin{aligned} x &= r \cos \phi, \\ y &= r \sin \phi, \\ z &= z. \end{aligned}$$

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, \\ \phi &= \tan^{-1} \frac{y}{x}, \\ z &= z. \end{aligned}$$

Application II: Cartesian to spherical



Thus, the unit coordinate transformation matrix is given by:

$$\begin{bmatrix} \hat{R} \\ \hat{\theta} \\ \hat{\phi} \end{bmatrix} = \begin{bmatrix} \cos \phi_0 \sin \theta_0 & \sin \phi_0 \sin \theta_0 & \cos \theta_0 \\ \cos \phi_0 \cos \theta_0 & \sin \phi_0 \cos \theta_0 & -\sin \theta_0 \\ -\sin \phi_0 & \cos \phi_0 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix}$$

Coordinate Transformation - Cartesian to Spherical

$$\begin{aligned}\vec{A} &= \hat{x}A_x + \hat{y}A_y + \hat{z}A_z = \hat{R}A_R + \hat{\theta}A_\theta + \hat{\phi}A_\phi \\ R &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \tan^{-1} \sqrt{\frac{x^2 + y^2}{z^2}} \\ \phi &= \tan^{-1} \frac{y}{x} \\ x &= R \sin \theta \cos \phi \\ y &= R \sin \theta \sin \phi \\ z &= R \cos \theta\end{aligned}$$

2.3 vector calculus

2.3.1 Integrals

$$\begin{aligned}\int_V \mathbf{F} dv, \\ \int_C V d\ell, \\ \int_C \mathbf{F} \cdot d\ell, \\ \int_S \mathbf{A} \cdot d\mathbf{s},\end{aligned} \quad \oint_C V d\ell.$$

Key Points:

1. Determine it is a scalar or a vector
2. For vector, find the corresponding magnitude in each unit vector direction and unit vector.

2.3.2 Gradient of a scalar Field

- Physical meaning: A **vector** who shows along which direction scalar **increase** fastest, and its magnitude describes the maximum space rate of change of the scalar **per unit length**.

$$\text{grad } V = \nabla V = \mathbf{a}_n \frac{dV}{dn}$$

- Calculate the change of scalar in any direction \mathbf{a}_l , $\frac{dV}{dl} = (\nabla V) \cdot \mathbf{a}_l$
- Calculate gradient in orthogonal Coordinate: $\nabla V = \mathbf{a}_{u_1} \frac{\partial V}{\partial l_1} + \mathbf{a}_{u_2} \frac{\partial V}{\partial l_2} + \mathbf{a}_{u_3} \frac{\partial V}{\partial l_3} = \mathbf{a}_{u_1} \frac{\partial V}{h_1 \partial u_1} + \mathbf{a}_{u_2} \frac{\partial V}{h_2 \partial u_2} + \mathbf{a}_{u_3} \frac{\partial V}{h_3 \partial u_3}$
- Gradient Operator: $\nabla \equiv \mathbf{a}_{u_1} \frac{\partial}{h_1 \partial u_1} + \mathbf{a}_{u_2} \frac{\partial}{h_2 \partial u_2} + \mathbf{a}_{u_3} \frac{\partial}{h_3 \partial u_3}$

2.3.3 Divergence of a vector field

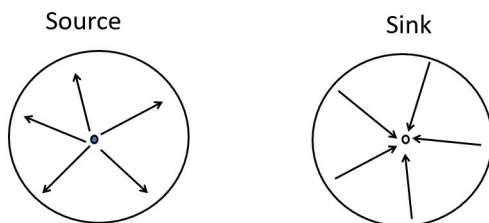
- Definition:** The net outward flux of \mathbf{A} per unit volume as the volume about the point tends to zero. **It is a scalar**

$$\text{div} \bar{\mathbf{A}} = \lim_{\Delta V} \frac{\oint \bar{\mathbf{A}} \cdot d\bar{\mathbf{S}}}{\Delta V}$$

Outward water **density**/time

- physical meaning:** flux density over a tiny volume of a closed surface. **rain**
- Application:** Whether the point is a sink or source in the vector field.

Source or Sink of A Vector Field



1. Divergence > 0 , source exists
2. Divergence < 0 , sink exists
3. Divergence $= 0$, **divergenceless** or **solenoidal** (pipe)

net outward flux $>$ inward flux \rightarrow source \rightarrow divergence > 0
 net outward flux $<$ inward flux \rightarrow sink \rightarrow divergence < 0

- Expression:** $\nabla \cdot \mathbf{A} \equiv \text{div} \mathbf{A}$

$$\nabla \cdot \mathbf{A} = \frac{1}{h_2 h_3 h_1} \left[\frac{\partial}{\partial u_1} (h_1 h_2 A_1) + \frac{\partial}{\partial u_2} (h_1 h_3 A_2) + \frac{\partial}{\partial u_3} (h_1 h_2 A_3) \right]$$

- Divergence Theorem:** The Volume integral of the divergence of a vector field equals the total outward flux of the vector through the surface that bounds the volume.

$$\int_V \nabla \cdot \bar{\mathbf{A}} dV = \oint_S \bar{\mathbf{A}} \cdot d\bar{\mathbf{S}}$$

flux outward density/volume* volume= total flux

2.3.4 Curl of a Vector field

- Circulation** The net circulation around a closed path $\oint_c \bar{\mathbf{A}} \cdot d\bar{\mathbf{l}}$
- Curl** The net circulation **per unit area**. **It is a vector, and we can use right-hand rule.**

$$\nabla \times \bar{\mathbf{A}} = \lim_{\Delta s} \frac{1}{\Delta s} \left(\hat{n} \oint_c \bar{\mathbf{A}} \cdot d\bar{\mathbf{l}} \right)$$

- **Physical meaning:** circulation density over a surface of a closed path.
- **Application:** If curl free, the vector field is a irrotational field, which is also called conservative field.
- **Expression:**

$$\nabla \times \mathbf{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} \mathbf{a}_1 h_1 & \mathbf{a}_2 h_2 & \mathbf{a}_3 h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

- **Stokes' Theorem** The surface integral of the curl of a vector field over an open surface is equal to the closed line integral of the vector along the contour bounding the surface.

$$\int_s (\nabla \times \bar{\mathbf{A}}) \cdot d\bar{\mathbf{S}} = \oint_c \bar{\mathbf{A}} \cdot d\bar{\mathbf{l}}$$

Circulation density per unit surface* surface=total circulations

2.4 Other Operators

- $\nabla \times (\nabla V) = 0$ If a vector field is curl free, it can be expressed by the gradient of a scalar.
- $\nabla \cdot (\nabla \times \mathbf{A}) = 0$. If a vector field is divergence-less, it can be expressed by the curl of a vector field.

- **Laplace:** $\nabla^2 V = \nabla \cdot \nabla V$

In Cartesian Coordinate: $\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$

In Cylindrical Coordinate: $\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} (r \frac{\partial V}{\partial r}) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

In Spherical Coordinate: $\nabla^2 V = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \frac{\partial V}{\partial r}) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta \frac{\partial V}{\partial \theta}) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$

- **Chain Rule**

$$\nabla(f(\vec{r})g(\vec{r})) = f\nabla g + g\nabla f$$

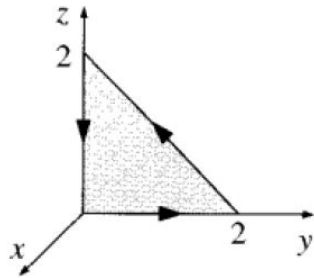
$$\nabla \cdot (f(\vec{r})\vec{G}(\vec{r})) = f\nabla \cdot \vec{G} + \vec{G} \cdot \nabla f$$

$$\nabla \times (f(\vec{r})\vec{G}(\vec{r})) = f\nabla \times \vec{G} + \nabla f \times \vec{G}$$

$$\nabla \cdot (\vec{F}(\vec{r}) \times \vec{G}(\vec{r})) = \vec{G} \cdot \nabla \times \vec{F} - \vec{F} \cdot \nabla \times \vec{G}$$

P 2.32 A vector field $\mathbf{D} = \mathbf{a}_R(\cos^2 \phi)/R^3$ exists in the region between two spherical shells defined by $R=1$ and $R=2$. Evaluate $\int \mathbf{D} \cdot d\mathbf{s}$, and $\int \nabla \cdot \mathbf{D} dv$.

. **quiz 1.2.2** Test Stokes's Theorem for the function $\mathbf{v} = (xy)\hat{x} + (2yz)\hat{y} + (3zx)\hat{z}$ using the triangular shaded area of below figure.



3 Electric Statics I

3.1 Basic Concept

Electrostatics:

- i. electric charges are **at rest(not moving)**;
- ii. electric field **do not change with time**.

3.2 Electric Field Intensity

Static electric charges (source) in free space \rightarrow electric field

$$\mathbf{E} = \lim_{q \rightarrow 0} \frac{\mathbf{F}}{q} \quad (\text{V/m})$$

IF q is small enough not to disturb the charge distribution of the source, $\mathbf{F} = q\mathbf{E}$ (N).

Fundamental Postulates of Electrostatics

- Differential form:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon_0} \\ \nabla \times \mathbf{E} &= 0\end{aligned}$$

- Integral form: Gauss' Law and KVL

$$\begin{aligned}\oint_S \mathbf{E} \cdot d\mathbf{s} &= \frac{Q}{\epsilon_0} \\ \oint_C \mathbf{E} \cdot d\boldsymbol{\ell} &= 0\end{aligned}$$

\mathbf{E} is **not solenoidal**, but **irrotational (conservative)**

3.3 Coulomb's Law

3.3.1 Electric Field due to a System of Discrete Charges

- a single point charge (charge on the origin):

$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \quad (\text{V/m})$$

- a single point charge (charge is not on the origin):

$$\mathbf{E}_p = \frac{q(\mathbf{R} - \mathbf{R}')}{4\pi\epsilon_0 |\mathbf{R} - \mathbf{R}'|^3} \quad (\text{V/m})$$

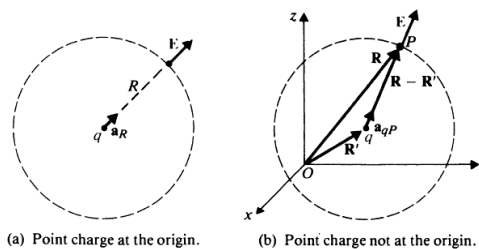


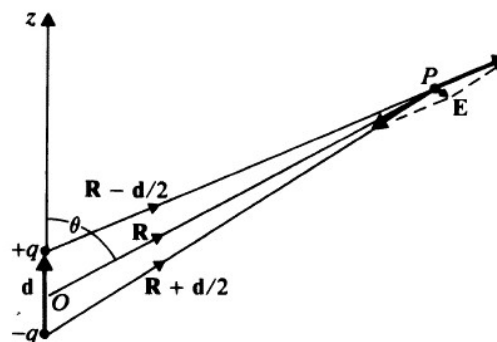
FIGURE 3-2
Electric field due to a point charge.

- several point charges:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^n \frac{q_k (\mathbf{R} - \mathbf{R}'_k)}{|\mathbf{R} - \mathbf{R}'_k|^3}$$

3.3.2 Electric Dipole

- Electric Field



general expression:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \left\{ \frac{\mathbf{R} - \frac{\mathbf{d}}{2}}{|\mathbf{R} - \frac{\mathbf{d}}{2}|^3} - \frac{\mathbf{R} + \frac{\mathbf{d}}{2}}{|\mathbf{R} + \frac{\mathbf{d}}{2}|^3} \right\}$$

if $d \ll R$:

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3} \left[3 \frac{\mathbf{R} \cdot \mathbf{d}}{R^2} \mathbf{R} - \mathbf{d} \right]$$

- **Electric Dipole Moment**

Definition:

$$\mathbf{p} = q\mathbf{d}$$

$$\mathbf{p} = \mathbf{a}_z p = p (\mathbf{a}_R \cos \theta - \mathbf{a}_\theta \sin \theta)$$

$$\mathbf{R} \cdot \mathbf{p} = R p \cos \theta$$

- **Electric Field:**

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3} (\mathbf{a}_R 2 \cos \theta + \mathbf{a}_\theta \sin \theta) \quad (\text{V/m})$$

3.3.3 Electric Field due to a Continuous Distribution of Charge

- **General Differential Element:**

$$d\mathbf{E} = \mathbf{a}_R \frac{\rho dv'}{4\pi\epsilon_0 R^2}$$

- **Line Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2} d\ell' \quad (\text{V/m})$$

- **Surface Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{S'} \mathbf{a}_R \frac{\rho_s}{R^2} ds' \quad (\text{V/m})$$

- **Volume Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int_{V'} \mathbf{a}_R \frac{\rho}{R^2} dv' \quad (\text{V/m})$$

3.4 Gauss's Law and Application

3.4.1 Definition

The total outward flux of the E-field over any closed surface in free space is equal to the total charge enclosed in the surface divided by ϵ_0 .

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

3.4.2 Application

- **Conditions for Maxwell's Integral Equations:**

There is **a high degree of symmetry** in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc).

3.4.3 Several Useful Models

Note: The charge distribution should be **uniform**.

different models	E(magnitude)
infinitely long, line charge	$E = \frac{\rho_\ell}{2\pi r \epsilon_0}$
infinite planar charge	$E = \frac{\rho_s}{2\epsilon_0}$
uniform spherical surface charge with radius R	$\begin{cases} E = 0 (r < R) \\ E = \frac{Q}{4\pi r^2 \epsilon_0} (r > R) \end{cases}$
uniform sphere charge with radius R	$\begin{cases} E = \frac{Qr}{4\pi R^3} (r < R) \\ E = \frac{Q}{4\pi r^2 \epsilon_0} (r > R) \end{cases}$
infinitely long, cylindrical charge with radius R	$\begin{cases} E = \frac{\rho_v r}{2\epsilon_0} (r < R) \\ E = \frac{\rho_v R^2}{2r \epsilon_0} (r > R) \end{cases}$

3.5 Electric Potential

- **Expression:**

$$\mathbf{E} = -\nabla V$$

- **Electric Potential Difference:**

$$V_2 - V_1 = - \int_{P_1}^{P_2} \mathbf{E} \cdot d\ell$$

- **Work**

In moving a unit charge from point P_1 to point P_2 in an electric field, work must be done against the field and is equal to

$$\frac{W}{q} = V_2 - V_1 = - \int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l} \text{ (J/C or V)}$$

- **Electric Potential due to a Charge Distribution**

- Point Charge & Several Point Charge**

$$V = \frac{q}{4\pi\epsilon_0 R}$$

$$V = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^n \frac{q_k}{|\mathbf{R} - \mathbf{R}'|}$$

ii. **dipole Charge**

$$V = \frac{q d \cos \theta}{4\pi\epsilon_0 R^2} = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2}$$

iii. **Line Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{L'} \frac{\rho_\ell}{R} d\ell' \quad (V)$$

iv. **Surface Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{S'} \frac{\rho_s}{R} ds' \quad (V)$$

v. **Volume Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R} dv' \quad (V)$$

3.6 Conductors and dielectrics in static electric field

- conductors:

- electrons migrate easily.
- charges reach the surface and conductor redistribute themselves in a way that both the charge and the field vanish.

- **static state conditions:**

- * inside the conductor:

$$\rho = 0, \mathbf{E} = 0$$

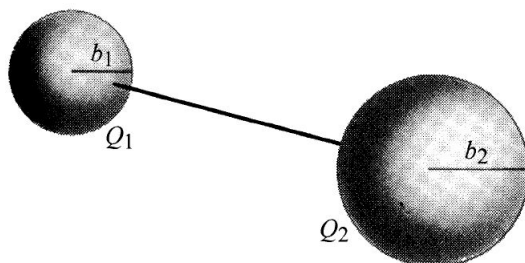
, where $\rho = 0$ represents no charge in the interior

- * on the conductor surface (boundary conditions)

$$E_t = 0, E_n = \frac{\rho_s}{\epsilon_0}$$

It is an equal-potential body.

- Electric field intensity tends to be higher at a point near the surface of a charged conductor with a larger curvature



- semiconductos:

- relatively small number of freely movable charges.
- insulators(dielectrics):
 - electrons are confined to their orbits.
 - external electric field polarizes a dielectric material and create electric dipoles. The induced electric dipoles will modify the electric field both inside and outside the dielectric material, as shown in Fig 1.

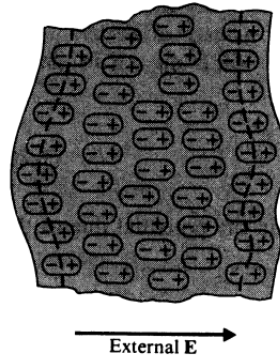


Figure 1: A cross section of a polarized dielectric medium

- **polarization charge densities/ bound-charge densities:**
 - * **polarization vector, \mathbf{P} :**

$$\mathbf{P} = \lim_{\Delta v \rightarrow 0} \frac{\sum_{k=1}^{n\Delta v} \mathbf{p}_k}{\Delta v}$$

, where the numerator represents the vector sum of the induced dipole moment contained in a very small volume Δv .

- * charge distribution on surface density:

$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n$$

,

- * volume charge distribution density:

$$\rho_p = -\nabla \cdot \mathbf{P}$$

.

p.3-13 Determine the work done in carrying a $-2(\mu C)$ charge from $P_1(2,1,-1)$ to $P_2(8,2,-1)$ in the field $\mathbf{E} = \mathbf{a}_x y + \mathbf{a}_y x$.

a) along the parabola $x = 2y^2$

b) along the straight line joining P_1 and P_2 .

P.3-23 Determine the electric field intensity at the center of a small spherical cavity cut out of a large block of dielectric in which a polarization \mathbf{P} exists.

4 Electric Statics II

4.1 Electric Flux Density and Dielectric Constant

- electric flux density/electric displacement, \mathbf{D} :

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (C/m^2)$$

•

$$\nabla \cdot \mathbf{D} = \rho \quad (C/m^3)$$

, where ρ is the volume density of free charges.

- Another form of **Gauss's law**:

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \quad (C)$$

, the total outward flux of the electric displacement (the total outward electric flux) over any closed surface is equal to the total free charge enclosed in the surface.

- If the dielectric of the medium is **linear and isotropic**,

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

$$\mathbf{D} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon_0 \epsilon_r \mathbf{E} = \epsilon \mathbf{E}$$

, where χ_e is a dimensionless quantity called electric susceptibility,

ϵ_r is a dimensionless quantity called as relative permittivity/ electric constant of the medium,

ϵ is the absolute permittivity/permittivity of the medium (F/m).

- For anisotropic,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{31} & \epsilon_{32} & \epsilon_{33} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

, For biaxial,

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \epsilon_1 & 0 & 0 \\ 0 & \epsilon_2 & 0 \\ 0 & 0 & \epsilon_3 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

For uniaxial, $\epsilon_1 = \epsilon_2$, For isotropic, $\epsilon_1 = \epsilon_2 = \epsilon_3$ (the only kind of media we deal with in this course).

- dielectric breakdown: electric field is very strong, causes permanent dislocations and damage in the material.

dielectric strength: the maximum electric field intensity that a dielectric material can withstand without breakdown.

4.2 Boundary Conditions for Electrostatic Fields

- the tangential component of an \mathbf{E} field is continuous across an interface.

$$E_{1t} = E_{2t} \quad (V/m)$$

, or

$$\frac{D_{1t}}{\epsilon_1} = \frac{D_{2t}}{\epsilon_2}$$

- The normal component of \mathbf{D} field is discontinuous across an interface where a surface charge exists - the amount of discontinuity being equal to the surface charge density.

$$\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s$$

, or

$$D_{1n} - D_{2n} = \rho_s \quad (C/m^2)$$

P.3-28 Dielectric lenses can be used to collimate electromagnetic fields. In Fig.1 the left surface of the lens is that of a circular cylinder, and the right surface is a plane. If \mathbf{E}_1 at point $P(r_0, 45^\circ, z)$ in region 1 is $\mathbf{a}_r 5 - \mathbf{a}_\phi 3$, what must be the dielectric constant of the lens in order that \mathbf{E}_3 in region 3 is parallel to the x-axis?

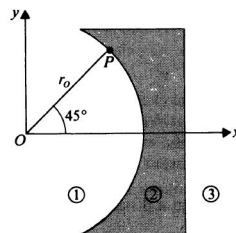


Figure 2: A dielectric lens

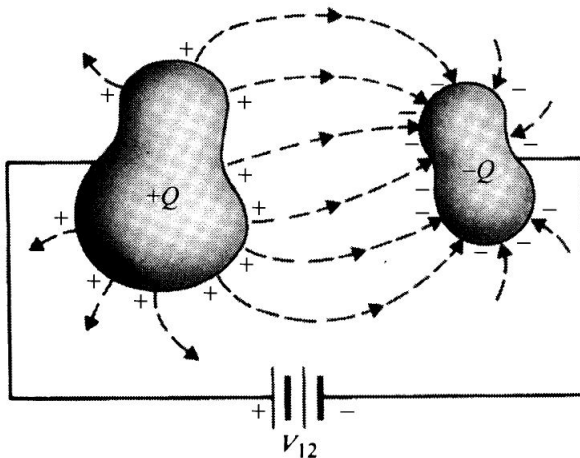
4.3 Capacitance and Capacitors

4.3.1 Capacitance

- Definition: The capacitance of isolated conducting body is the electric charge that must be added to the body per unit increase in its electric potential.
- $C = \frac{Q}{V}$ ($F = C/V$)

4.3.2 Capacitor

- **Components:** two conductors with arbitrary shapes are separated by free space or dielectric medium. $C = \frac{Q}{V_{12}}$



- **Capacitance:**
Its Capacitance is **independent of V and Q**, which means a capacitor has a capacitance even no voltage is applied to it and no free charges exist on its conductors.

- **How to calculate its capacitance:**

1. Choose a proper coordinate system
2. Assume $+Q, -Q$ on the conductors
3. Find \mathbf{E} from Q (like, Gauss's law, $D_n = \epsilon E_n = \rho_s$)
4. Find $V_{12} = -\int_2^1 \mathbf{E} \cdot d\mathbf{l}$
5. $C = Q/V_{12}$

- **Series Connections of Capacitors:**

$$\frac{1}{C_{sr}} = \frac{1}{C_1} + \frac{1}{C_2} + \dots + \frac{1}{C_n}$$

- **Parallel Connections of Capacitors:**

$$C_{||} = C_1 + C_2 + \dots + C_n$$

4.3.3 Capacitance in Multi-conductor System

- Isolated Conductor System $Q_0 + Q_1 + \dots + Q_N = 0$

- Four Conductor System Q-V Relationship I ($V_0 = 0$)

$$Q_1 = c_{11}V_1 + c_{12}V_2 + \dots + c_{13}V_N$$

$$Q_2 = c_{12}V_1 + c_{22}V_2 + \dots + c_{23}V_N$$

$$Q_3 = c_{13}V_1 + c_{23}V_2 + \dots + c_{33}V_N$$

where coefficients of capacitance $c_{ii} = Q_i/V_i$, $c_{ii} > 0$

coefficients of induction ($i \neq j$), $c_{ji} = Q_{ji}/V_i$, $c_{ji} = c_{ji} < 0$

- Four Conductor System Q-V Relationship II (Conductor 0 is grounded as well)

$$Q_1 = C_{10}V_1 + C_{12}(V_1 - V_2) + C_{13}(V_1 - V_3)$$

$$Q_2 = C_{20}V_2 + C_{12}(V_2 - V_1) + C_{23}(V_2 - V_3)$$

$$Q_3 = C_{30}V_3 + C_{13}(V_3 - V_1) + C_{23}(V_3 - V_2)$$

where self-partial capacitance: C_{i0}

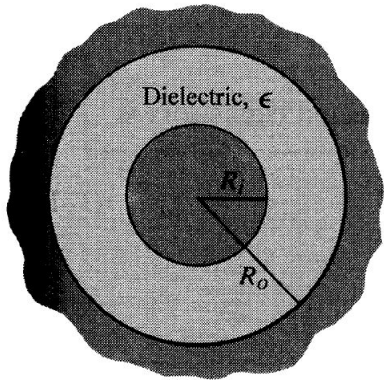
mutual partial capacitance: $C_{ij} = C_{ji}$

- Relationship between c and C

$$C_{i0} = c_{i1} + c_{i2} + c_{i3}$$

$$C_{ij} = -c_{ij}$$

E. 3-19 A spherical capacitor consists of an inner conducting sphere of radius R_i and an outer conductor with a sphere inner wall of radius R_o . The space in between is filled with a dielectric of permittivity ϵ . Determine the capacitance.



4.4 Electrostatic Energy and Forces

- Work done to bring a charge q from P_1 to P_2

$$\frac{W}{q} = V_{21} = V_2 - V_1 = - \int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}$$

- **Self Energy:** Work done to bring a charge Q_2 from infinity to distance R_{12} with Q_1 (initially, Q_1 is in space)

$$W = Q_2 V_2 = Q_2 \frac{Q_1}{4\pi\epsilon_0 R_{12}}$$

- **Mutual Energy:** Potential energy of a group of N discrete point charges at rest

$$W_e = \frac{1}{2} \sum_{k=1}^N Q_k V_k$$

where $V_k = \frac{1}{4\pi\epsilon_0} \sum_{j=1 \& j \neq k}^N \frac{Q_j}{R_{jk}}$ Note the W_e can be negative, for example, there are 2-point charge systems, and one charge is positive, the other is negative.

- **Electrostatic Energy density** w_e : $W_e = \int_{v'} w_e dv$

4.4.1 Electrostatic Energy in terms of Field Quantities

- v' can be all space.
- **A continuous Charge Distribution of Density ρ**

$$W_e = \frac{1}{2} \int_v \rho V dv = \frac{1}{2} \int_{v'} (\nabla \cdot \mathbf{D}) V dv$$

Another expression:

$$W_e = \frac{1}{2} \int_{v'} \mathbf{D} \cdot \mathbf{E} dv$$

- If it is a simple dielectric, it should be

$$W_e = \frac{1}{2} \int_{v'} \epsilon E^2 dv = \frac{1}{2} \int_{v'} \frac{D^2}{\epsilon} dv$$

4.4.2 Electrostatic Forces

Here we use **Principle of virtual displacement** to calculate Force in two situations.

- **System of bodies with fixed charges**

1. Mechanical work is from the reduced stored electrostatic energy

$$F_Q = -\nabla W_e(N)$$

2. Electric torque rotates one of the bodies by $d\phi$ (a virtual rotation) about an axis

$$T_Q = -\frac{\partial W_e}{\partial \phi}(N \cdot m)$$

- **System of conducting bodies with Fixed Potentials**

1. The fixed potential can be retained by connecting with an external source.
2. $F_v = \nabla W_e$
3. $T_v = \frac{\partial W_e}{\partial \phi}$

Example 3-22 Find the energy required to assemble a uniform sphere of charges of radius b and volume charge ρ .