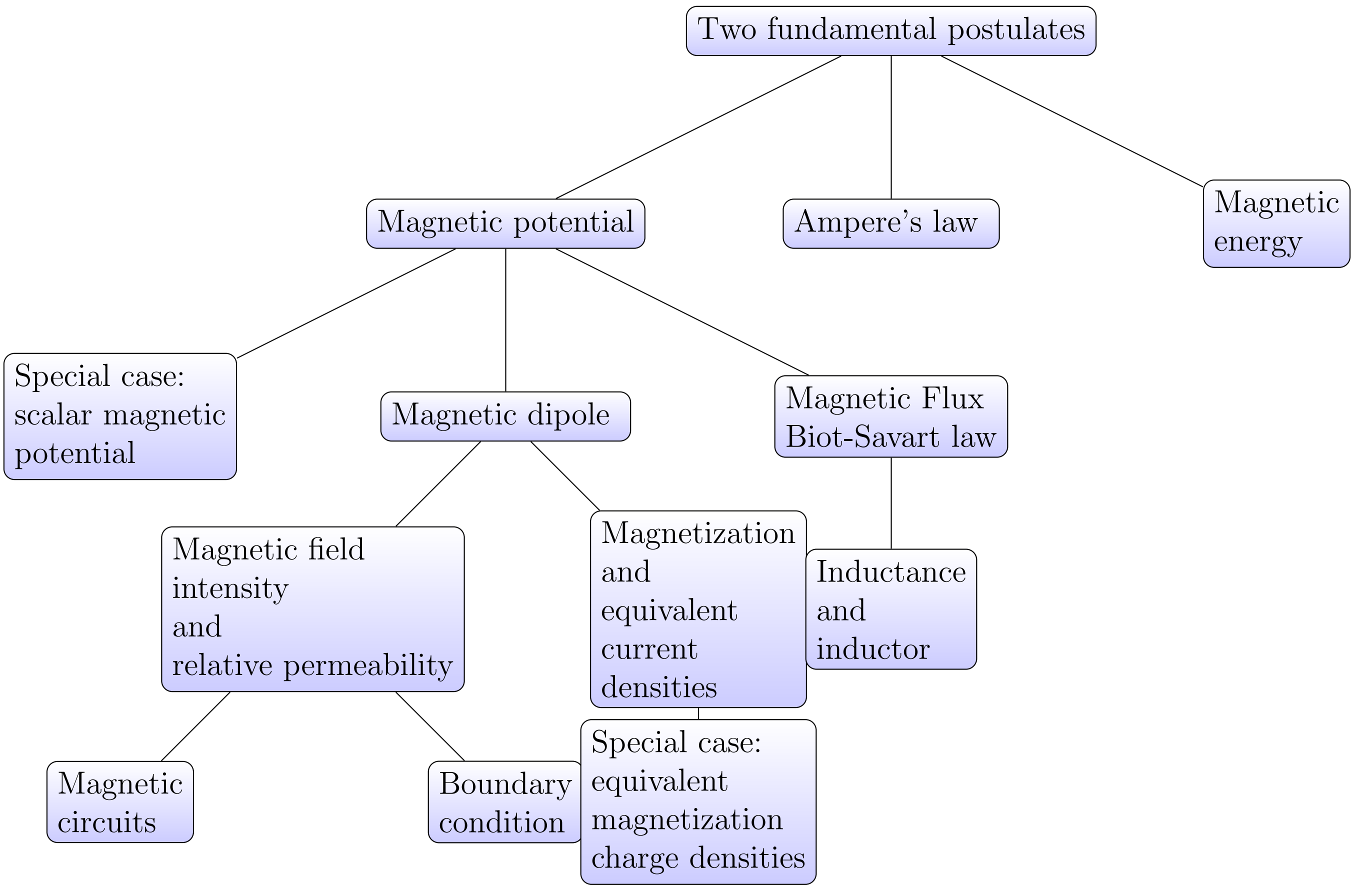
Steady Magnetic Fields

1 Chapter 6 Concept diagram



And also Magnetic forces and torques, including

1. Hall effect
2. Forces and torques on current-carrying conductors
3. Forces and torques in terms of stored magnetic energy
4. Forces and torques in terms of mutual inductance

2 Fundamental Postulates

differential form	integral form	Comment
$\nabla \cdot \boldsymbol{B} = 0$	$\oint_S \boldsymbol{B} \cdot d\boldsymbol{s} = 0$	$\boldsymbol{B}$ is solenoidal, Conservation of magnetic flux: no isolated magnetic charges, no magnetic flow source, flux lines always close upon themselves
$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}$	$\oint_C \boldsymbol{B} \cdot dl = \mu_0 I$	Ampere's circuital law

where μ_0 is the permeability of free space, $\mu_0 = 4\pi \times 10^{-7} \quad H/m$.
Because the divergence of the curl of any vector field is zero,

$$\nabla \cdot \boldsymbol{J} = \frac{\nabla \cdot (\nabla \times \boldsymbol{B})}{\mu_0} = 0$$

which is consistent with the formula

$$\nabla \cdot \boldsymbol{J} = \frac{\partial \rho}{\partial t} = 0$$

for steady current.

3 Vector Magnetic Potential

As $\nabla \cdot \boldsymbol{B} = 0$, $\boldsymbol{B}$ is solenoidal, thus could be expressed as:

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad (T) \tag{1}$$

, where $\boldsymbol{A}$ is called the vector magnetic potential.
Magnetic flux Φ:

$$\Phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{s} = \oint_C \boldsymbol{A} \cdot dl$$

For Eq 1, by doing Laplacian transformation and assume $\nabla \cdot \boldsymbol{A} = 0$, vector Poisson's equation:

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}$$

The solution is then

$$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\boldsymbol{J}}{\boldsymbol{R}} dv' \tag{2}$$

For a thin wire with cross-sectional area S, $dv' = Sdl'$, current flow is entirely along the wire, we then have

$$\boldsymbol{J}dv' = JSdl' = Idl'$$

Thus, Eq 2 becomes:

$$\boldsymbol{A} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{dl'}{R}$$

Based on this form and properties of differentiation, we can get Biot-Savart law:

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\oint_{C'}\frac{d\boldsymbol{l'}\times\boldsymbol{a_R}}{R^2}$$

The formula for Biot-Savart law could also be written as:

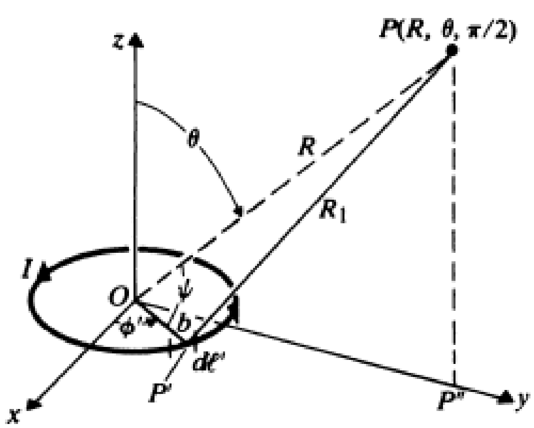
$$\boldsymbol{B} = \oint_{C'} d\boldsymbol{B}$$

and

$$d\boldsymbol{B} = \frac{\mu_0 I}{4\pi}\left(\frac{d\boldsymbol{l'}\times\boldsymbol{a_R}}{R^2}\right) = \frac{\mu_0 I}{4\pi}\left(\frac{d\boldsymbol{l'}\times\boldsymbol{R}}{R^3}\right)$$

Comment: Biot-Savart law is more difficult to apply than Ampere's circuital law, but Ampere's circuital law cannot be used to determine $\boldsymbol{B}$ from I in a circuit if a closed path cannot be found where $\boldsymbol{B}$ has a constant magnitude.

Example 6-7 Find the magnetic flux density at a distant point of a small circular loop of radius b that carries current I (a magnetic dipole)



$$\boldsymbol{A} = \frac{\mu_0\boldsymbol{m}\times\boldsymbol{a_R}}{4\pi R^2}$$

, where $\boldsymbol{m} = \boldsymbol{a_z}I\pi b^2 = \boldsymbol{a_z}IS = \boldsymbol{a_z}m$ is defined as the magnetic dipole moment. We call a small current-carrying loop a magnetic dipole.

3.1 Scalar magnetic potential

If a region is current free, i.e. $\boldsymbol{J} = 0$,

$$\nabla \times \boldsymbol{B} = 0$$

,

thus $\boldsymbol{B}$ can be expressed as the gradient of a scalar field.
Assume

$$\boldsymbol{B} = -\mu_0 \nabla V_m \tag{3}$$

, where V_m is called the scalar magnetic potential, the negative sign is conventional, μ_0 is the permeability of free space.

Thus, between two points P_1, P_2,

$$V_{m2} - V_{m1} = -\int_{P_1}^{P_2} \frac{1}{\mu_0} \boldsymbol{B} \cdot dl$$

If there were magnetic charges with a volume density ρ_m in a volume V', we could find V_m from:

$$V_m = \frac{1}{4\pi} \int_{V'} \frac{\rho_m}{R} dv'$$

Then we could obtain $\boldsymbol{B}$ by Eq 3. Note that this is only a mathematical model, isolated magnetic charges have never been found.

For a bar magnet the fictitous magnetic charges $+q_m, -q_m$ assumed to be separated by d (magnetic dipole), the scalar magnetic potential V_m is given by:

$$V_m = \frac{\boldsymbol{m} \cdot \boldsymbol{a_R}}{4\pi R^2}$$

4 Magnetization and Equivalent Current Densitites

Define magnetization vector, $\boldsymbol{M}$, as

$$\boldsymbol{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{n\Delta v} \boldsymbol{m_k}}{\Delta v}$$

, which is the volume density of magnetic dipole moment,

1. The effect of magnetization is vector is equivalent to both
   (a) a volume current density:
   $$\boldsymbol{J_m} = \nabla \times \boldsymbol{M}$$
   (b) a surface current density:
   $$\boldsymbol{J_{ms}} = \boldsymbol{M} \times \boldsymbol{a_n}$$
2. Then we can determine A by:
   $$\boldsymbol{A} = \frac{\mu_0}{4\pi} \int_{V'} \frac{\nabla' \times \boldsymbol{M}}{R} dv' + \frac{\mu_0}{4\pi} \oint_{S'} \frac{\boldsymbol{M} \times \boldsymbol{a'_n}}{R} ds'$$
3. Then we could obtain $\boldsymbol{B}$ from $\boldsymbol{A}$.

4.1 Equivalent Magnetization Charge Densities

In current-free region, a manetized body may be replaced by an equivalent/fictitous magnetization surface charge density

$$\rho_{ms} = \boldsymbol{M} \cdot \boldsymbol{a_n}$$

and an equivalent/fictitous magnetization volume charge density

$$\rho_m = -\nabla \cdot \boldsymbol{M}$$

5 Magnetic Field Intensity, Relative Permeability and Magnetic circuit

5.1 Magnetic field intensity H

Considering the effect of both the internal dipole moment and the induced magnetic moment in a magnetic material, the magnetic field intensity $\boldsymbol{H}$ is defined as:

$$\boldsymbol{H} = \frac{\boldsymbol{B}}{\mu_0} - \boldsymbol{M}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J}$$

directly relates the magnetic field intensity with the density of free current density.
The integral form of which is then,

$$\oint_C \boldsymbol{H} \cdot dl = I$$

It is another form of Ampere's circuital law: the circulation of the magnetic field intensity around any closed path is equal to the free current flowing through the surface bounded by the path.

If the closed path C is chosen to enclose N turns of a winding carrying a current I that excites a magnetic circuit, we have

$$\oint_C \boldsymbol{H} \cdot dl = NI = V_m$$

,

V_m is analogous to electromotive force (emf) and is called magnetomotive force (mmf).

5.2 Relative Permeability μ_r

If we define:

$$\boldsymbol{M} = \chi_m H$$

,

where χ_m is magnetic susceptibility, Then

$$\boldsymbol{B} = \mu_0(1+\chi_m)\boldsymbol{H} = \mu_0\mu_r\boldsymbol{H} = \mu\boldsymbol{H}$$

$$\boldsymbol{H} = \frac{1}{\mu}\boldsymbol{B}$$

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_0}$$

where μ_r is the relative permeability of the medium. $\mu = \mu_r\mu_0$ is the absolute permeability/permeability.

5.3 magnetic Circuits

We have the analogous quantities:

Magnetic Circuits	Electric Circuits
mmf, $\mathscr{V}_m$ $(=NI)$	emf, $\mathscr{V}$
magnetic flux, Φ	electric current, I
reluctance, $\mathscr{R}$	resistance, R
permeability, μ	conductivity, σ

And the analysis of the magnetic circuit is similar to the electric circuits.

Similar to Kirchhoff's law,

$$\sum_j N_j I_j = \sum_k R_k \Phi_k$$

,

around a closed path in a magnetic circuit, the algebraic sum of ampere-turns is equal to the algebraic sum of the products of the reluctances and fluxes.

$$\sum_j \Phi_j = 0$$

,

the algebraic sum of all the magnetic fluxes flowing out of a junction in a magnetic circuit is zero.

But the condition for ideal transformer magnetic circuit is hard to achieve (leakage fluxes, fringing effect, permeability depends on the magnetic field intensity).

6 Behavior of Magnetic Materials

1. Diamagnetic, if $\mu_r \lesssim 1$ (χ_m is a very small negative number)
2. Paramagnetic, if $\mu_r \gtrsim 1$ (χ_m is a very small positive number)
3. Ferromagnetic, if $\mu_r >> 1$ (χ_m is a large positive number)

7 Boundary Conditions for Magnetostatic Fields

1. The normal component of $\boldsymbol{B}$ is continuous across an interface,

$$B_{1n} = B_{2n}$$

, which is

$$\mu_1 H_{1n} = \mu_2 H_{2n}$$

,

2. The tangential component of the $\boldsymbol{H}$ field is discontinuous across an interface where a free surface current exists.

$$a_{n2} \times (\boldsymbol{H_1} - \boldsymbol{H_2}) = \boldsymbol{J_s}$$

When the conductivities of both media are finite, currents are defined by volume current densities and free surface currents do not exist on the interface. Hence $\boldsymbol{J_s} = 0$, the tangential component of $\boldsymbol{H}$ is continuous across the boundary of almost all physical media; it is discontinuous only when an interface with an ideal perfect conductor or a superconductor is assumed.

3. **Analog of Boundary Condition Problems:**

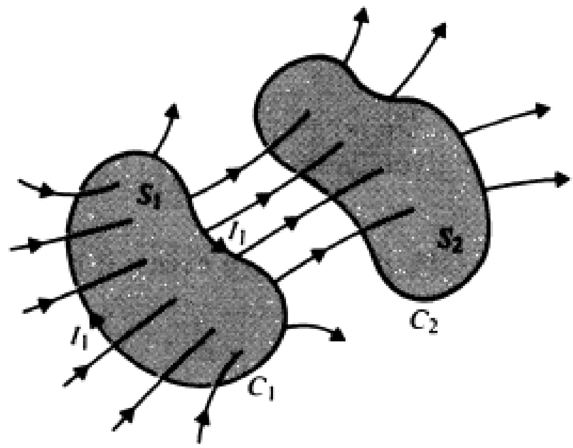
For current-free regions, $\boldsymbol{B} = -\mu \nabla V_m$, thus,

$$\nabla^2 V_m = 0$$

, similar to the Lapace's equation. Similar to what we have discussed in Chapter 4, we could expect the similar approach here to solve boundary-value problems.

8 Inductances and Inductors

For two loops C_1, C_2,



At first, we can get the mutual flux

$$\Phi_{12} = \int_{S_2} \mathbf{B_1} \times d\mathbf{S_2}$$

The flux linkage is

$$\Lambda_{12} = N_2 \Phi_{12}$$

For linear media where permeability does not change with I_1,

$$L_{12} = \frac{\Lambda_{12}}{I_1}$$

,

where Λ_{12} is the flux linkage, L_{12} is the mutual inductance between loops C_1 and C_2. A more general definition for L_{12} is then:

$$L_{12} = \frac{d\Lambda_1 2}{dI_1}$$

The total flux linkage with C_1 caused by I_1 is calculated as:

$$\Lambda_{11} = N_1 \Phi_{11} > N_1 \Phi_{12}$$

,

the self-inductance of loop C_1 could be calculated as:

$$L_{11} = \frac{\Lambda_{11}}{I_1}$$

for linear medium. In general:

$$L_{11} = \frac{d\Lambda_{11}}{dI_1}$$

A conductor arranged in an appropriate shape (e.g. a conducting wire wound as a coil) to supply a certain amount of self-inductance is called an inductor. It can store magnetic energy.

Method to determine the self-inductance of an inductor:

1. Choose an appropriate coordinate system for the given geometry.
2. Assume a current I in the conducting wire.
3. Find $\boldsymbol{B}$ from I by Ampere's circuital law,

$$\oint_C \boldsymbol{B} \cdot dl = \mu_0 I$$

   , if symmetry exists; if not, Biot-Savart law

$$\boldsymbol{B} = \frac{\mu_0 I}{4\pi} \oint_{C'} \frac{dl' \times \boldsymbol{a_R}}{R^2}$$

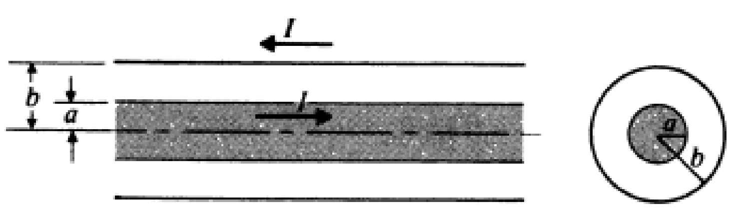
   must be used.
4. Find the flux linking with each turn, Φ, from $\boldsymbol{B}$ by integration:

$$\Phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{s}$$

   , where S is the area over which $\boldsymbol{B}$ exists and links with the assumed current.
5. Find the flux linkage Λ by mutiplying Φ by the number of turns.
6. Find L by taking ratio $L = \Lambda / I$.

To determine the mutual inductance L_{12} between two circuits, after choosing an appropriate coordinate system, assume $I_1 \rightarrow$ Find $\boldsymbol{B_1} \rightarrow$ Find Φ_{12} by integrating $\boldsymbol{B_1}$ over surface $S_2 \rightarrow$ Find flux linkage $\Lambda_{12} = N_2 \Phi_{12} \rightarrow$ Find $L_{12} = \Lambda_{12}/I_1$.

Example 6-16 An air coaxial transmission line has a solid inner conductor of radius a and a very thin outer conductor of inner radius b. Determine the inductance per unit length of the line.



In hight frequency applications, current tends to concentrate in the ”skin” of the inner conductor as a surface current, internal self-inductance tends to zero.

Neumann formula for mutual inductance:

$$L_{12} = L_{21} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{dl_1 \cdot dl_2}{R}$$

9 Magnetic Energy

For a system of N loops carrying currents,

$$W_m = \frac{1}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} L_{jk} I_j I_k$$

,

for a single inductor,

$$W_m = \frac{1}{2} L I^2$$

,

$$W_m = \frac{1}{2} \sum_{k=1}^{N} I_k \Phi_k$$

,

9.1 Magnetic Energy In Terms of Field Quantities

Generally,

$$\begin{aligned} W_m &= \frac{1}{2} \int_{V'} \boldsymbol{A} \cdot \boldsymbol{J} dv' \\ &= \frac{1}{2} \int_{V'} \boldsymbol{H} \cdot \boldsymbol{B} dv' \\ &= \frac{1}{2} \int_{V'} \frac{B^2}{\mu} dv' \\ &= \frac{1}{2} \int_{V'} \mu H^2 dv' \end{aligned}$$

,

if define a magnetic energy density w_m such that

$$W_m = \int_{V'} w_m dv'$$

,

then

$$\begin{aligned} w_m &= \frac{1}{2} \boldsymbol{H} \cdot \boldsymbol{B} \\ &= \frac{B^2}{2\mu} \\ &= \frac{1}{2} \mu H^2 \end{aligned}$$

And we have

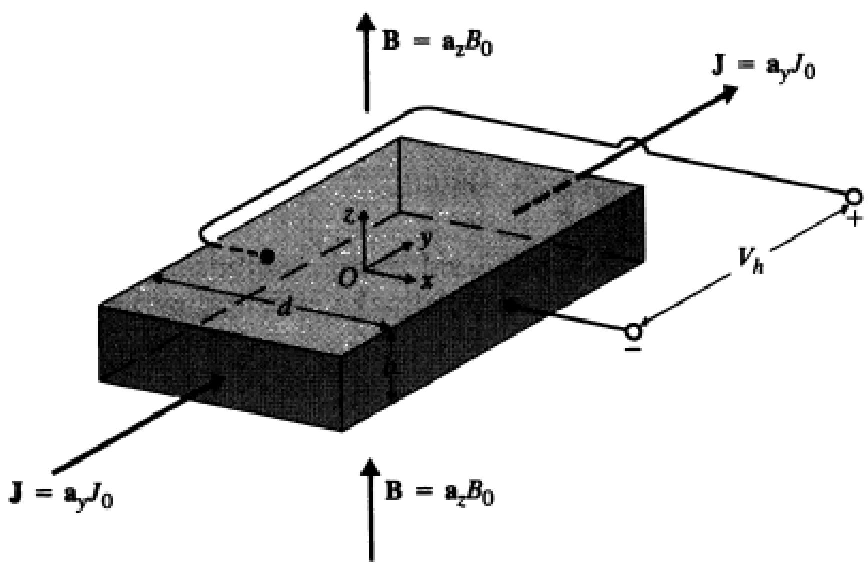
$$L = \frac{2W_m}{I^2}$$

10 Magnetic Forces and Torques

a charge q moving with a velocity $\boldsymbol{u}$ in a magnetic field with flux density $\boldsymbol{B}$ experiences a magnetic forces F_m,

$$F_m = q\boldsymbol{u} \times \boldsymbol{B}$$

10.1 Hall Effect



Before the steady state, the magnetic force tends to move the electrons in the positive x-direction, creating a transverse electric field.

For the steady state, the net force on the charge carriers is zero:

$$\boldsymbol{E_h} + \boldsymbol{u} \times \boldsymbol{B} = 0$$

This is known as the Hall effect. $\boldsymbol{E_h}$ is called the Hall field.

$$V_h = -\int_0^d E_h dx = u_0 B_0 d$$

,

where V_h is called the Hall voltage. $E_x/J_y B_z = 1/Nq$ is called Hall coefficient.

10.2 Forces and torques on current-carrying conductors

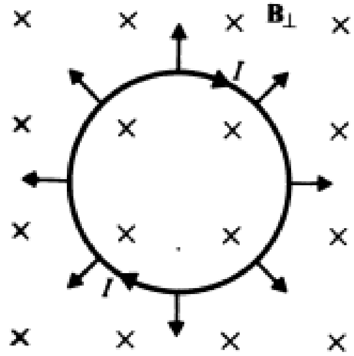
$$d\boldsymbol{F_m} = I dl \times \boldsymbol{B}$$

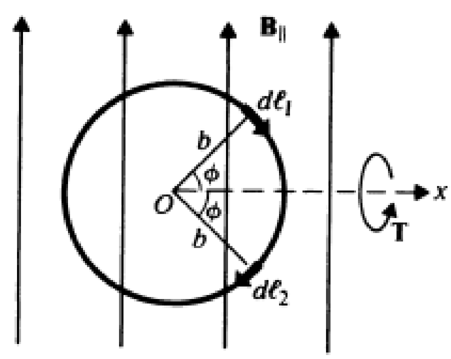
$$\boldsymbol{F_m} = I \oint_C dl \times \boldsymbol{B}$$

$$F_{21} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C'} \oint_{C'} \frac{dl_1 \times (dl_2 \times \boldsymbol{a_{R_{21}}})}{R_{21}^2}$$

,

which is the Ampere's law of force.
Two wires with the same current flowing direction will attract each other.





For total torque acting on the loop is calculated as:

$$\boldsymbol{T} = \boldsymbol{m} \times \boldsymbol{B}$$

10.3 Forces and Torques In Terms of Stored Magnetic Energy

1. System of circuits with constant flux linkage:

   we obtain

$$\boldsymbol{F_\Phi} = -\nabla W_m$$

,

$$(T_\Phi)_z = -\frac{\partial W_m}{\partial \Phi}$$

,

2. System of Circuits with Constant Currents:

$$F_I = \nabla W_m$$

,

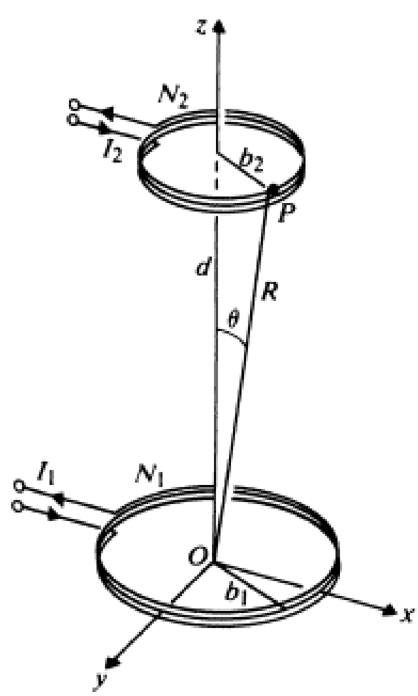
$$(T_I)_z = \frac{\partial W_m}{\partial \phi}$$

10.4 Forces and Torques In Terms of Mutual Inductance

$$\boldsymbol{F_I} = I_1 I_2 (\nabla L_{12})$$

$$(T_I)_z = I_1 I_2 \frac{\partial L_{12}}{\partial \phi}$$

Example 6-24 Determine the force between two coaxial circular coils of radii b_1 and b_2 separated by a distance d that is much larger than the radii ($d >> b_1, b_2$). The coils consist of N_1 and N_2 closely wound turns and carry currents I_1 and I_2, respectively.



11 Topics to be covered in Chapter 7:

I Faraday's Law;

II Revised Maxwell Equation;

III Boundary Condition between two media;

IV Scalar and Vector potential Wave function and Its solution;

IV Time-Harmonic field with phasor;

12 Review of Static Case:

Fundamental Relations	Electrostatic Model	Magnetostatic Model
Governing equations	$\nabla \times \mathbf{E} = 0$ $\nabla \cdot \mathbf{D} = \rho$	$\nabla \cdot \mathbf{B} = 0$ $\nabla \times \mathbf{H} = \mathbf{J}$
Constitutive relations (linear and isotropic media)	$\mathbf{D} = \epsilon \mathbf{E}$	$\mathbf{H} = \frac{1}{\mu}\mathbf{B}$

In static case, $\mathbf{E}, \mathbf{B}$ can exist together(In a conducting medium), but they won't influence each other.

13 Faraday's Law of Electromagnetic Induction

13.1 Faraday's law

- **Content:** The relationship between induced emf and the negative rate of change of flux linkage.

- **Expressions:**

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint_c \mathbf{E} \cdot d\ell = -\int_s \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

13.2 Static Circuit in a Time-varying Magnetic Field

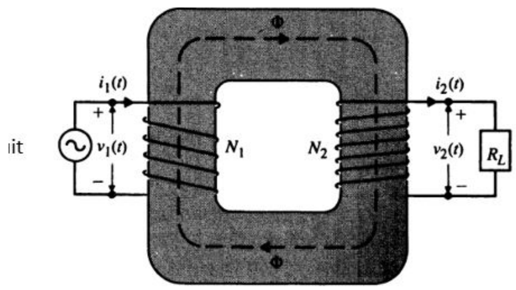
If we assign emf $V == \oint_c \mathbf{E} \cdot d\ell$ and magnetic flux$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s}$, we can get the Faraday's Law

$$V = -\frac{d\Phi}{dt}$$

If there are N turns wires, the total magnetic flux is $N\Phi$, $V = -N\frac{d\Phi}{dt}$.

Lenz's Law: The induced emf will cause a current to flow in the closed loop in such a direction as to oppose the change in the linking magnetic flux.

Transformer:two or more coils coupled magnetically through a common ferromagnetic core.



The general equation is

$$N_1 i_1 - N_2 i_2 = \frac{\ell}{\mu S}\Phi$$

- **Ideal Transformer**: $\mu \to \infty$, and then we can retake equation (1) to get

  $$\frac{i_1}{i_2} = \frac{N_2}{N_1}$$

  Given Faraday's Law, We get ratio of emf as

  $$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$

  If we have RL in secondary circuit, we can see source Resistor $R_1 = \frac{V_1}{i_1}$ as

  $$(R_1)_{eff} = (\frac{N_1}{N_2})^2 R_L$$

* **Real Transformer**: Constrains: leakage flux(k¡1). non-infinite inductances, nonzero winding resistance, and presence of hysteresis and eddy-current losses.
  We can also get v_1, v_2 by

  $$v_1 = L_1\frac{di_1}{dt} - L_{12}\frac{di_2}{dt}$$

  $$v_2 = L_{12}\frac{di_1}{dt} - L_2\frac{di_2}{dt}$$

where L1, L2, can be calculated given what we learn before,

$$L_1 = \frac{\mu S}{\ell} N_1^2$$

$$L_2 = \frac{\mu S}{\ell} N_2^2$$

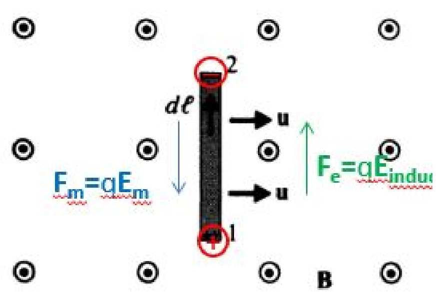
And L_{12} is related to flux leakage

$$L_{12} = k\sqrt{L_1 L_2}, \quad k < 1$$

13.3 Moving Conductor in Static Magnetic field

- **Working process**: $F_m = q\mathbf{u} \times \mathbf{B}$ → positive and negative charge move to opposite direction → built induced Electric field $E_{induced} = -\mathbf{u} \times \mathbf{B}$ → Other charges in equilibrium (they won't move along the bar)
- **Motional EMF**

$$\mathscr{V}' = \oint_c (\mathbf{u} \times \mathbf{B}) \cdot d\ell$$



13.4 Moving Circuit in a Time-varying Magnetic Field

- **Lorentz's force equation**: $\mathbf{F} = q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$
- **Effective electric field E′**: If an observer have the same movement with q, Lorentz's force on q can be seen as effective electric field

$$\mathbf{E}' = \mathbf{E} + \mathbf{u} \times \mathbf{B}$$

- **General form of Faraday's law**:

$$\oint_C \mathbf{E}' \cdot d\ell = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} + \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\ell$$

Where left side talks about emf induced in **a moving frame of reference**, and on right side, **transformer emf** equals to

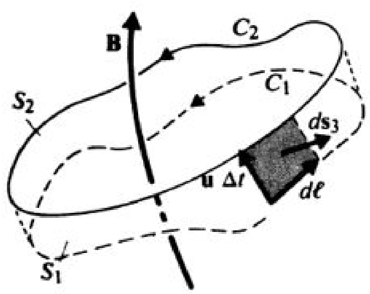
$$V = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

and **motional emf** equals to

$$V = \oint_C (\mathbf{u} \times \mathbf{B}) \cdot dl$$

- Faraday's Law also works a moving circuit. If we use $V = \oint_C \mathbf{E}' \cdot d\mathbf{l}$, we can get $V = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s} = -\frac{d\Phi}{dt}$

derive process for $\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{s} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s} - \oint_C (\mathbf{u} \times \mathbf{B}) \cdot d\ell$



14 Maxwell's Equation

Differential Form	Integral Form	Significance
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_c \mathbf{E} \cdot d\ell = -\frac{d\Phi}{dt}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_c \mathbf{H} \cdot d\ell = I + \int_s \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{s}$	Ampère's circuital law
$\nabla \cdot \mathbf{D} = \rho$	$\oint_s \mathbf{D} \cdot d\mathbf{s} = Q$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_s \mathbf{B} \cdot d\mathbf{s} = 0$	No isolated magnetic charge

Other useful equations:

$$\nabla \cdot \mathbf{J} = \frac{\partial \rho}{\partial t}$$

$$\mathbf{H} = \mathbf{B}/\mu$$

$$\mathbf{D} = \epsilon \mathbf{E}$$

15 Potential Function

- **Electric Field time-varying field**

$$E = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

where $-\nabla V$ comes from charge distribution, and $-\frac{\partial \mathbf{A}}{\partial t}$ comes from time-varying current. Strictly speaking, V and **A** are calculated by the Poisson's Equation in time varying field (Which will be discussed later)

- **Quasi-static fields**:
  If ρ and **J** vary slowly with time and the range of R is small in comparison with the wavelength(low frequency, long wavelength), We can use below 2 equations to find quasi-static fields.

$$V = \frac{1}{4\pi\epsilon_0}\int_{V'}\frac{\rho}{R}dv'$$

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_{V'}\frac{\mathbf{J}}{R}dv'$$

  Note:(1) These above equations are solutions of Poisson's equation in static case.
  (2) if there is high-frequency sources, we need to consider time-retardation effects, which means as source changes in time, it take time to change the potential at a certain distance from the source.

- **Non-homogeneous wave equation for vector potential**:
  if we choose divergence and curl of **A** as

$$\nabla \cdot \mathbf{A} + \mu\epsilon\frac{\partial V}{\partial t} = 0$$

$$B = \nabla \times \mathbf{A}$$

  which is also called Lorentz condition, we can find the non-homogeneous wave equations as

$$\nabla^2\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J}$$

- **Non-homogeneous wave equation for Scalar Potential V**:

$$\nabla^2 V - \mu\epsilon\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon}$$

16 Electromagnetic Boundary Condition

16.1 General Boundary Condition Equations

$$E_{1t} = E_{2t} \quad (\mathrm{V/m}) \tag{4}$$

$$\mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \quad (\mathrm{A/m}) \tag{5}$$

$$\mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad \left(\mathrm{C/m^2}\right) \tag{6}$$

$$B_{1n} = B_{2n} \quad (\mathrm{T}) \tag{7}$$

Note that (1)(4) are equivalent and (2)(3) are equivalent. Since divergence equation can be derived from curl equations with continuity equation.

16.2 Interface between two lossless Linear Media

- lossless media: $\sigma = 0$, then we can get $J = 0$
- Usually, no free charge and no surface currents at the interface of two lossless.($\rho_s = 0, \mathbf{J_s} = 0$)
- Boundary condition:

$$E_{1t} = E_{2t} \to \frac{D_{1t}}{D_{2t}} = \frac{\epsilon_1}{\epsilon_2}$$

$$H_{1t} = H_{2t} \to \frac{B_{1t}}{B_{2t}} = \frac{\mu_1}{\mu_2}$$

$$D_{1n} = D_{2n} \to \epsilon_1 E_{1n} = \epsilon_2 E_{2n}$$

$$B_{1n} = B_{2n} \to \mu_1 H_{1n} = \mu_2 H_{2n}$$

16.3 Interface between a Dielectric and a perfect conductor

- perfect conductor: $\sigma \to \infty$, then we know $\mathbf{E_{inside}} = 0$, the charge only exists on the surface.
- $\mathbf{D}, \mathbf{B}, \mathbf{H} = 0$ for point inside a conductor.
- Boundary condition equation (2 is perfect)

On the Side of Medium 1	On the Side of Medium 2
$E_{1t} = 0$	$E_{2t} = 0$
$\mathbf{a}_{n2} \times \mathbf{H}_1 = \mathbf{J}_s$	$H_{2t} = 0$
$\mathbf{a}_{n2} \cdot \mathbf{D}_1 = \rho_s$	$D_{2n} = 0$
$B_{1n} = 0$	$B_{2n} = 0$

17 Wave Equations and Solutions

17.1 solutions for wave equations for potentials

The calculation process is on the textbook.

- for given charge and current distribution ρ and $\mathbf{J}$,in order to get the $\mathbf{E}, \mathbf{B}$, we firstly need to find solutions for $\mathbf{A}, V$ in nonhomogeneous wave equation.

- **solution for scalar potential**:

$$V(R,t) = \frac{1}{4\pi\epsilon}\int_{V'}\frac{\rho(t-R/u)}{R}dv'$$

It takes time R/u for the effect of ρ to be felted at the distance R, which means there is time Retardation $\triangle t = R/u$ from ρ to V.

- **solution for vector potential**:

$$\mathbf{A}(R,t) = \frac{\mu}{4\pi}\int_{V'}\frac{\mathbf{J}(t-R/u)}{R}dv' \quad (\mathbf{W}b/\mathrm{m})$$

17.2 Source-Free Wave Equations in simple nonconducting media

- **Source-free**; $\rho = 0$, $\mathbf{J} = 0$
- **In a simple nonconducting media**: ϵ, μ are constant, $sigma = 0$
- **Rewrite the Maxwell Equations**:

$$\nabla \times \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t}$$
$$\nabla \times \mathbf{H} = \epsilon\frac{\partial \mathbf{E}}{\partial t}$$
$$\nabla \cdot \mathbf{E} = 0$$
$$\nabla \cdot \mathbf{H} = 0$$

- **Wave Equations** for $\mathbf{E}, \mathbf{H}$ can be found directly.

$$\nabla^2\mathbf{E} - \frac{1}{u^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0$$
$$\nabla^2\mathbf{H} - \frac{1}{u^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = 0$$

18 Time Harmonic Fields

18.1 Time Harmonic Electromagnetic

- **Time harmonic Vector Phasor**: $\mathbf{E}(x,y,z,t) = Re[\mathbf{E}(x,y,z)e^{jwt}]$

we different or integrate it, we can find

$$\partial\mathbf{E}(x,y,z,t)/\partial t = j\omega\mathbf{E}(x,y,z)$$

$$\int \mathbf{E}(x,y,z,t)dt = \mathbf{E}(x,y,z)/j\omega$$

- **Revised Maxwell Equation**

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + j\omega\epsilon\mathbf{E}$$

$$\nabla \cdot \mathbf{E} = \rho/\epsilon$$

$$\nabla \cdot \mathbf{H} = 0$$

- **Time harmonic wave equations**

$$\nabla^2 V + k^2 V = -\tfrac{\rho}{\epsilon}$$

$$\nabla^2 \mathbf{A} + k^2 \mathbf{A} = -\mu\mathbf{J}$$

$$\text{where} \quad k = \omega\sqrt{\mu\epsilon} = \tfrac{\omega}{u} = 2\pi/\lambda$$

Since we need to follow the Lorentz condition, which is $\nabla \cdot \mathbf{A} + j\omega\mu\epsilon V = 0$ in phasor form.

- **Phasor solutions**

$$V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho e^{-jkR}}{R} dv' \quad (V)$$

$$\mathbf{A}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J} e^{-jkR}}{R} dv' \quad (\mathbf{Wb}/\mathrm{m})$$

Note that $kR = 2\pi\frac{R}{\lambda} << 1$ when $R << \lambda$.

18.2 Procedure for determining E and H

1 Find V and **A** by

$$V(R) = \frac{1}{4\pi\epsilon}\int_{V'} \frac{\rho e^{-jkR}}{R} dv' \quad \mathbf{A}(R) = \frac{\mu}{4\pi}\int_{V'} \frac{\mathbf{J} e^{-jkR}}{R} dv'$$

2 Find **B** and **E** by

$$\mathbf{E}(R) = -\nabla V - j\omega\mathbf{A} \quad \mathbf{B}(R) = \nabla \times \mathbf{A}$$

3 Find Instantaneous $\mathbf{E}(t)$ and $\mathbf{B}(t)$ by

$$\mathbf{E}(R,t) = \mathscr{R}e\left[\mathbf{E}(R)e^{j\omega t}\right] \quad \mathbf{B}(R,t) = \mathscr{R}e\left[\mathbf{B}(R)e^{j\omega t}\right]$$

18.3 Source-Free Fields in non-conducting Simple Media

1 Find Wave function given previous section

$$\nabla^2\mathbf{E} - \frac{1}{u^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0 \quad \nabla^2\mathbf{H} - \frac{1}{u^2}\frac{\partial^2\mathbf{H}}{\partial t^2} = 0$$

2 Find homogeneous equations in phasor form:

$$\nabla^2\mathbf{E} + k^2\mathbf{E} = 0 \quad \nabla^2\mathbf{H} + k^2\mathbf{H} = 0$$

- If medium is conducting, We use $\nabla \times \mathbf{H} = j\omega\epsilon_c\mathbf{E}$ where complex permitivity $\epsilon_c = \epsilon - j\frac{\sigma}{\omega}\ (F/m)$

problem 7.27 It is known that the electric field intensity of a spherical wave in free space is

$$\mathbf{E} = \mathbf{a}_\theta \frac{E_0}{R}\sin\theta\cos(\omega t - kR)$$

. Determine the magnetic field intensity **H** and the value of k.

19 Chapter 8:Plane Wave in LossLess Media

- **Uniform Plane Wave: $\mathbf{E}, \mathbf{H}$** with same direction, same magnitude, and same phase in infinite planes perpendicular to the direction of propagation.

- **Wave equation for source free, in free space**

$$\nabla^2 \mathbf{E} + k_0^2 \mathbf{E} = 0$$

where wave number

$$k_0 = \frac{2\pi}{\lambda} = \omega\sqrt{\mu_0 \epsilon_0} = \frac{\omega}{c} \quad \text{(rad/m)}$$

If we assume E is wave is uniform in xy plane (wave will propragate in z direction), we can get the solution in space domain

$$\begin{aligned} E_x(z) &= E_x^+(z) + E_x^-(z) \\ &= E_0^+ e^{-jk_0 z} + E_0^- e^{jk_0 z} \end{aligned}$$

If we check the wave equation in time domain, and we use phasor here

$$\begin{aligned} E_x(z,t) &= \mathscr{R}[E_x^+(z)e^{j\omega t} + E_x^-(z)e^{j\omega t}] \\ &= \mathscr{R}[E_0^+ e^{j(\omega t - k_0 z)} + E_0^- e^{j(\omega t + k_0 z)}] \\ &= E_0^+ \cos(\omega t - k_0 z) + E_0^- \cos(\omega t + k_0 z) \end{aligned}$$

If we just consider the wave propagate in +z direction, we can get [**phase velocity**, which means the propagation velocity of a point with particular phase on wave, is equal to

$$\mu_p = \frac{dz}{dt} = \frac{\omega}{k_o} = \frac{1}{\sqrt{u_0 \epsilon_0}} = c$$

- **Associative Magnetic Field**:

In chapter 7, by using Maxwell Equation's in phaor form, we get

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

If we consider $E_x(z) = E_x^+(Z)$, We can get

$$H_y^+ = \frac{k_0}{\omega\mu_0} E_x^+(z)$$

Here we introduce **intrinsic impedance**:

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 120\pi$$

Then, we get magnetic field intensity quickly by

$$\begin{aligned} \mathbf{H}(z,t) &= \mathbf{a}_y H_y^+(z,t) = \mathbf{a}_y \mathscr{R}\varepsilon \left[H_y^+(z) e^{j\omega t} \right] \\ &= \mathbf{a}_y \frac{E_0^+}{\eta_0} \cos(\omega t - k_0 z) \end{aligned}$$

Example 8.1 A uniform plane wave with $\mathbf{E} = \mathbf{a_x} E_x$ propagates in a lossless simple media $(\epsilon_r = 4, \mu_r = 1, \sigma = 0)$ in the +z direction. Assume that E_x is sinusoidal with a frequency 100MHz and has a maximun value of $+10^{-4}(V/m)$ at t=0 and z=1/8 m,
a) Write the instantaneous expressions for $\mathbf{E}$ for any t and z.
b) Write the instantaneous expression for $\mathbf{H}$.
c) Determine the locations where E_x is a positive maximum when $t = 10^{-8}s$

19.1 Doppler Effect

If the source moves close to the receiver, the received wave shifts to higher frequency. If source moves away the receiver, the received wave shift to lower frequency. The equation look like this

$$f^{‘} = \frac{f}{1 - \frac{mu}{c} cos\theta} \approx f(1 + \frac{\mu}{c} cos\theta$$

19.2 Transverse Electromagnetic Waves

In this subsection, we consider a uniform plane wave along an arbitrary direction. Firstly, we get the electric field expression

$$\mathbf{E}(x, y, z) = \mathbf{E_0} e^{-jk_x x - jk_y y - jk_z z}$$

In order to satisfy non homogeneous equation for $\mathbf{E}$. the condition is

$$k_x^2 + k_y^2 + k_z^2 = \omega^2 \mu \epsilon$$

If we define **wave number vector** as

$$\mathbf{k} = \mathbf{a_x} k_x + \mathbf{a_y} k_y + \mathbf{a_z} k_z$$

Then we get the Electric field intensity at a position vector $\mathbf{R}$ is equal to

$$\mathbf{E}(\mathbf{R}) = \mathbf{E_0} e^{-j\mathbf{k} \cdot \mathbf{R}} = \mathbf{E_0} e^{-jk\mathbf{a_n} \cdot \mathbf{R}}$$

Since this is a source free region, we need to satisfy $\nabla \cdot \mathbf{E} = 0$, and then we find that

$$\mathbf{a_n} \cdot \mathbf{E_0} = 0$$

Which means propagation direction is perpendicular to electric intensity direction, and plus, We can find magnetic field intensity

$$\mathbf{H}(\mathbf{R}) = \frac{1}{\eta}\left(a_n \times \mathbf{E}_0\right) e^{-jka_n \cdot \mathbf{R}}$$

summary: A uniform plane wave propagating in an arbitrary direction $\mathbf{a_n}$ is TEM wave ($\mathbf{E}, \mathbf{H}, \mathbf{a_n}$ are perpendicular to each other.)

19.3 Polarization of Plane Waves

Polarization of a uniform plane wave is to find time-varying behavior of E vector at a given point in space.

linearity polarization:E always in a diection (can be reversed), like $\mathbf{E} = \mathbf{a_x}E$

find the polarization at a specific point: at first find the Electric field expression in time and space domain; $\mathbf{E}(\mathbf{R}, t)$

Find the position vector $\mathbf{R_0}$ and take it into above equation

If the solution looks like this:

$$\left[\frac{E_2(R_0,t)}{E_{20}}\right]^2 + \left[\frac{E_1(R_0,t)}{E_{10}}\right]^2 = 1$$

This is an elliptical or circular polarization. For more details, please check the lecture slides.

20 Plane Waves in Lossy Media

In a source free but lossy media, we can still use the homogeneous equation for electric field intensity as

$$\nabla^2 \mathbf{E} - \gamma^2 \mathbf{E} = 0$$

Where **propagation constant** is

$$\gamma = jk_c = j\omega\sqrt{\mu\epsilon_c}$$

Note wave number becomes to $k_c = \omega\sqrt{\mu\epsilon_c} = \omega\sqrt{u(\epsilon' - j\epsilon'')}$ Then we prefer to write propagation constant as

$$\gamma = \alpha + j\beta = j\omega\sqrt{\mu\epsilon'}\left(1 - j\frac{\epsilon''}{\epsilon'}\right)^{1/2} = j\omega\sqrt{\mu\epsilon}\left(1 + \frac{\sigma}{j\omega\epsilon}\right)^{1/2}$$

where we call α as attenuation constant, β phase constant. Assume the wave propagation along +z, and E is in +x direction, the solution for homogeneous wave equation is

$$E_X = E_0 e^{-\alpha z} e^{-j\beta z}$$

And the intrinsic impedance becomes to

$$\eta_c = (\mu/\varepsilon)^{1/2} = \left(\mu/\left(\varepsilon' - j\varepsilon''\right)\right)^{1/2}$$

20.1 Low Loss Dielectric

- $\epsilon'' << \epsilon'$
- $\alpha \cong \frac{\omega\epsilon''}{2}\sqrt{\frac{\mu}{\epsilon'}}$ (Np/m)
- $\beta \cong \omega\sqrt{\mu\epsilon'}\left[1 + \frac{1}{8}\left(\frac{\epsilon''}{\epsilon'}\right)^2\right]$
- $\eta_c =\cong \sqrt{\frac{\mu}{\epsilon'}}\left(1 + j\frac{\epsilon''}{2\epsilon'}\right)$

20.2 Good Conductor

- $\frac{\sigma}{\omega\epsilon} >> 1$
- $\alpha = \beta = \sqrt{\pi f \mu\sigma}$
- $\eta_c = \sqrt{\frac{\mu}{\epsilon}} \cong \sqrt{\frac{j\omega\mu}{\sigma}} = (1+j)\sqrt{\frac{\pi f\mu}{\sigma}} = (1+j)\frac{\alpha}{\sigma}$
- Skin depth $$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\pi f\mu\sigma}}$$ (m)

20.3 Ionized Gas

- Plasma frequency $f_p = \frac{\omega_p}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{Ne^2}{m\epsilon_0}}$ (Hz)
- equivalent permitivity $\epsilon_p = \epsilon_0\left(1 - \frac{\omega_p^2}{\omega^2}\right) = \epsilon_0\left(1 - \frac{f_p^2}{f^2}\right)$
- $\gamma = j\omega\sqrt{\mu\epsilon_0}\sqrt{1 - \left(\frac{f_p}{f}\right)^2}$
- $\eta_p = \frac{\eta_0}{\sqrt{1-\left(\frac{f_p}{f}\right)^2}}$
- cutoff frequency: $f_p \approx 9\sqrt{N}$

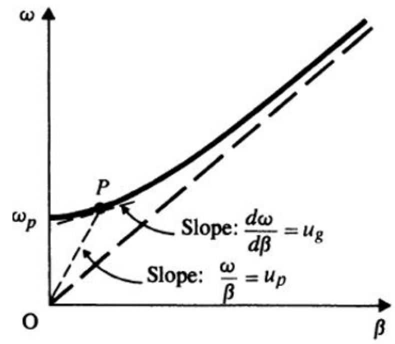
21 Group Speed

In a lossless media, the phase velocity $\mu_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}$ is independent of the frequency.

However, in a loss media, wave in different frequency will propagate in different phase velocity, which will cause a distortion on wave shape.

The group velocity is used to represent propagation velocity of the envelop of such a packet of signals with small different frequency df around fc.

$$u_g = \frac{dz}{dt} = \frac{\Delta\omega}{\Delta\beta} = \frac{1}{\Delta\beta/\Delta\omega}$$



I put the figure for w and β above, since we can get the relationship of ω and β:

$$\beta = \omega\sqrt{\mu\epsilon_0}\sqrt{1-\left(\frac{f_p}{f}\right)^2}$$
$$= \frac{\omega}{c}\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}$$

We can find that for $\omega > \omega_p$, wave can be propagated with

$$u_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}}$$

$$u_g = c\sqrt{1-\left(\frac{\omega_p}{\omega}\right)^2}$$

Note we can also find relationship among μ_p and μ_g are

$$u_g = \frac{u_p}{1-\frac{\omega}{u_p}\frac{du_p}{d\omega}}$$

22 Flow of Electromagnetics Power and the Poynting Vector

Power flow per unit area, Poynting vector, $\mathscr{P}$ is defined as:

$$\mathscr{P} = \boldsymbol{E}\times\boldsymbol{H}$$

Poynting vector is a power density vector associated with an electromagnetics field.

Poynting's theorem: the surface integral of $\mathscr{P}$ over a closed surface, equals the power leaving the enclosed volume, that is,

$$-\oint_S \mathscr{P}\cdot d\boldsymbol{s} = -\oint_S (\boldsymbol{E}\times\boldsymbol{H})\cdot d\boldsymbol{s}$$
$$= \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\epsilon E^2 + \frac{1}{2}\mu H^2\right)dv + \int_V \sigma E^2 dv$$
$$= \frac{\partial}{\partial t}\int_V (w_e + w_m)dv + \int_V p_\sigma dv$$

,

where

$$w_e = \frac{1}{2}\epsilon E^2 = \frac{1}{2}\boldsymbol{E}\cdot\boldsymbol{E}^* = \text{Electric energy density}$$

$$w_m = \frac{1}{2}\mu H^2 = \frac{1}{2}\mu\boldsymbol{H}\cdot\boldsymbol{H}^* = \text{Magnetic energy density}$$

$$p_\sigma = \sigma E^2 = J^2/\sigma = \sigma\boldsymbol{E}\cdot\boldsymbol{E}^* = \boldsymbol{J}\cdot\boldsymbol{J}^*/\sigma = \text{Ohmic power density}$$

If the region is loseless ($\sigma = 0$),

total power flow in = rate of increase of the stored electric and magnetic energies

,

moreover, for static situation, w_e and w_m vanish, the total power flowing into the closed surface equals to the ohmic power (usually heat) dissipated in the enclosed volume.

22.1 Instantaneous and average power densities

Note that $\Re$ represents the real part of a complex form.
for Instantaneous expression for the Poynting vector or the power density vector,

$$\mathscr{P}(z,t) = \boldsymbol{E}(z,t)\times\boldsymbol{H}(z,t) = \Re[\boldsymbol{E}(z)e^{j\omega t}]\times\Re[\boldsymbol{H}(z)e^{j\omega t}]$$

for the time-average Poynting vector,

$$\mathscr{P} - av(z) = \frac{1}{T}\int_0^T \mathscr{P}(z,t)dt = \boldsymbol{a_z}\frac{E_0^2}{2|\eta|}e^{-2\alpha z}cos\theta_\eta \quad (W/m^2)$$

More generally, if the wave does not propagate in z-direction,

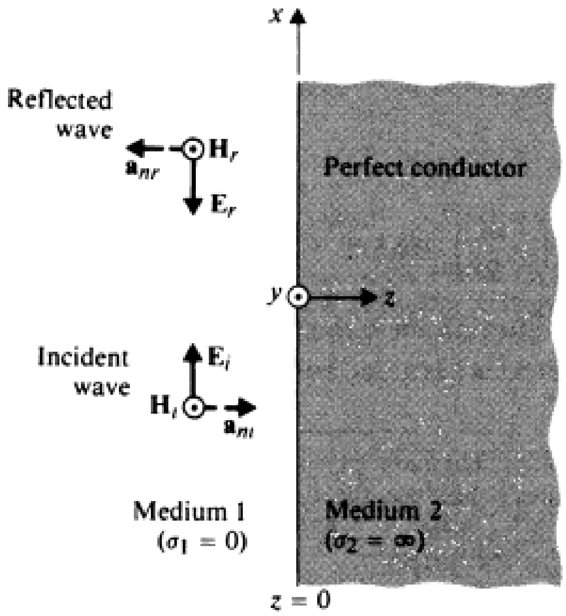
$$\mathscr{P}_{av} = \frac{1}{2}\Re(\boldsymbol{E}\times\boldsymbol{H}^*)$$

23 Incidence at a Plane Conducting Boundary

Consider an incident uniform plane wave ($\boldsymbol{E}_i, \boldsymbol{H}_i$) travels in a loseless medium 1 ($\sigma_1 = 0$). The boundary is an interface with perfect conductor, that is, medium 2, $\sigma_2 = \infty$.

23.1 Normal incidence at a plane conducting boundary

Assume the boundary exists at $z = 0$



1. Medium 1:

$$\boldsymbol{E}_i(z) = \boldsymbol{a_x} E_{i0} e^{-j\beta_1} z$$

, where E_{i0} is the magnitude of $\boldsymbol{E}_i$ at $z = 0$, β_1, η_1 are the phase constant and the instrinsic ipedance of medium 1.

$$\boldsymbol{H}_i(z) = \boldsymbol{a_y} \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}$$

2. Medium 2: since it is a perfect conductor, $\boldsymbol{E}_2 = 0, \boldsymbol{H}_2 = 0$.

The reflected electric field intensity can be written as

$$\boldsymbol{E}_r(z) = \boldsymbol{a_x} E_{r0} e^{+j\beta_1 z}$$

,

the change of sign for the exponent is becasuse the reflected wave travels in the opposite direction as the incident wave.

Thus, the total electric field intensity at medium 1 is that:

$$\boldsymbol{E}_1(z) = \boldsymbol{E}_i(z) + \boldsymbol{E}_r(z)$$

,

Continuity of the tangential component (boundary conditions for $\boldsymbol{E}$) of $\boldsymbol{E}$-field at the boundary requires:

$$\boldsymbol{E_1}(0) = \boldsymbol{a_x}(E_{i0} + E_{r0}) = \boldsymbol{E}_2(0) = 0$$

,

thus, $E_{r0} = -E_{i0}$, then we could represent $E_1(z)$ with only E_{i0} as:

$$\boldsymbol{E}_1(z) = -\boldsymbol{a_x} j2E_{i0} sin\beta_1 z$$

,

we could calculate the related $\boldsymbol{H}_r(z)$ by:

$$\boldsymbol{H}_r(z) = \frac{1}{\eta_1}\boldsymbol{a_{nr}} \times \boldsymbol{E}_r(z)$$

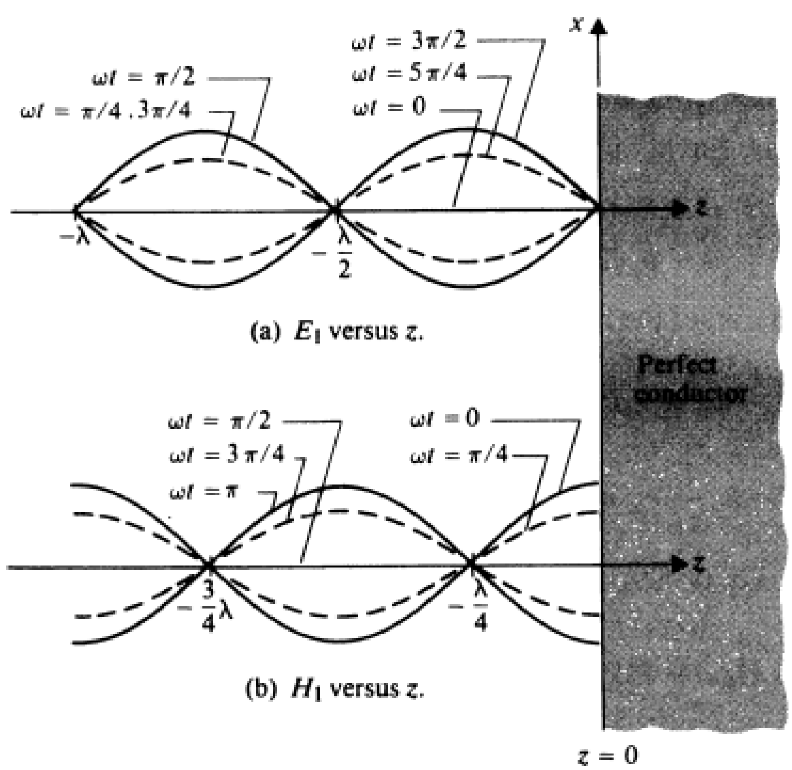
,

the same procedure could be applied to $\boldsymbol{H}_1$.
Thus,

$$\boldsymbol{E}_1(z,t) = \boldsymbol{a_x} 2E_{i0} sin\beta_1 z sin\omega t,$$
$$\boldsymbol{H}_1(z,t) = \boldsymbol{a_y} 2\frac{E_{i0}}{\eta_1} cos\beta_1 z cos\omega t$$

1. Zeros of $\boldsymbol{E}_1(z,t)$ and Maxima of $\boldsymbol{H}_1(z,t)$ occur at $\beta_1 z = -n\pi$, or $z = -n\frac{\lambda}{2}$, where $n = 0, 1, 2....$
2. Zeros of $\boldsymbol{H}_1(z,t)$ and Maxima of $\boldsymbol{E}_1(z,t)$ occur at $\beta_1 z = -(2n+1)\frac{\pi}{2}$, or $z = -(2n+1)\frac{\lambda}{4}$, where $n = 0, 1, 2....$

The resulted wave is a standing wave from the superposition of the two waves (incident and the reflected)



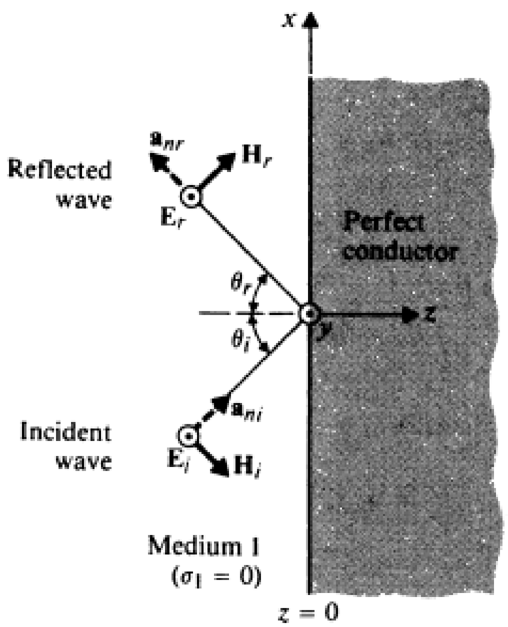
Note that

1. $\boldsymbol{E}_1$ vanishes on the conducting boundary as well as those zero points.
2. $\boldsymbol{H}_1$ is at its maximum on the conducting boundary
3. the standing wave $\boldsymbol{E}_1$ and $\boldsymbol{H}_1$ are in time quadrature (90° phase difference), and are shifted in space by a quarter wavelength.

23.2 Oblique incidence at a plane conducting boundary

Plane of incidence: the plane containing the vector indicating the direction of propagation of the incident wave and normal to the boundary surface.

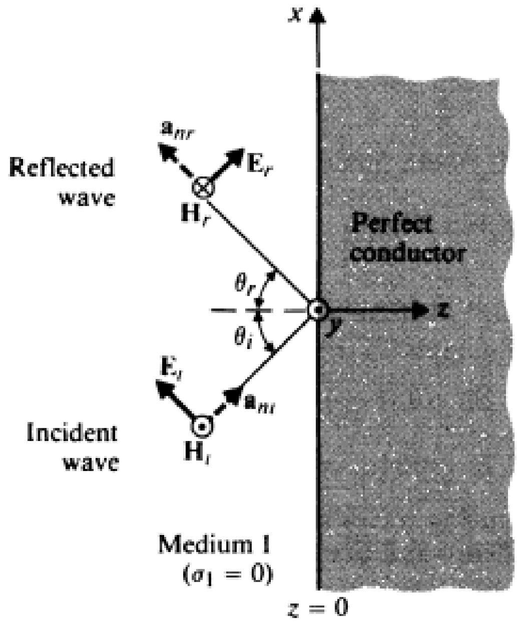
23.2.1 Perpendicular polarization



$\boldsymbol{E}_i$ is perpendicular to the plane of incidence,

Snell's law of reflection: the angle of reflection equals the angle of incidence.

Derive $\boldsymbol{E}_1$ and $\boldsymbol{H}_1$ in terms of E_{i0} and the constants that describe the incident curve



23.3 Parallel polarization

$\boldsymbol{E_i}$ lies in the plane of incidence while a uniform plane wave impinges obliquely on a perfectly conducting plane boundary.

$$\boldsymbol{E}_i(x,z) = E_{i0}(\boldsymbol{a_x} cos\theta_i - \boldsymbol{a_z} sin\theta_i)e^{-j\beta_1(xsin\theta_i + zcos\theta_i)}$$

,

$$\boldsymbol{H}_i(x,z) = \boldsymbol{a_y}\frac{E_{i0}}{\eta_1}e^{-j\beta_1(xsin\theta_i + zcos\theta_i)}$$

$$\boldsymbol{E}_r(x,z) = \boldsymbol{E}_{r0}(\boldsymbol{a_x} cos\theta_r + \boldsymbol{a_z} sin\theta_r)e^{-j\beta_1(xsin\theta_r - zcos\theta_r)}$$

$$\boldsymbol{H}_r(x,z) = -\boldsymbol{a_y}\frac{E_{r0}}{\eta_1}e^{-j\beta_1}(xsin\theta_r - zcos\theta_r)$$

,

as at $z = 0$, the surface of a perfect conductor, the tangential component must vanish for electric field.

$$E_{ix}(x,0) + E_{rx}(x,0) = 0$$

,

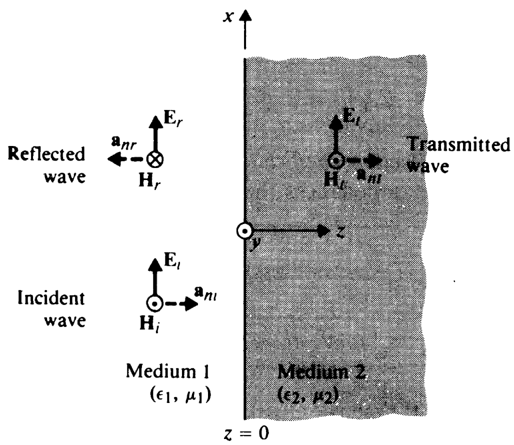
essentially $\boldsymbol{E_1}$ and $\boldsymbol{H_1}$ could be represented as:

$$\boldsymbol{E_1}(x,z) = -2E_{i0}[\boldsymbol{a_x} j cos\theta_i sin(\beta_1 z cos\theta_i) + \boldsymbol{a_z} sin\theta_i cos(\beta_1 z cos\theta_i)]e^{-j\beta_1 x sin\theta_i}$$

$$\boldsymbol{H}_1(x,z) = \boldsymbol{a_y} 2\frac{E_{i0}}{\eta_1}cos(\beta_1 z cos\theta_i)e^{-j\beta_1 x sin\theta_i}$$

24 Normal Incidence at a Plane Dielectric Boundary

- Both media are assumed that $\sigma_1 = \sigma_2 = 0$.
- There exists transmitted wave in medium 2.



1. Incident wave:

$$\mathbf{E}_i(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z}$$

$$\mathbf{H}_i(z) = \mathbf{a}_y \frac{E_{i0}}{\eta_1} e^{-j\beta_1 z}$$

2. Reflected wave:

$$\mathbf{E}_r(z) = \mathbf{a}_x E_{r0} e^{j\beta_1 z}$$

$$\mathbf{H}_r(z) = (-\mathbf{a}_z) \times \frac{1}{\eta_1}\mathbf{E}_r(z) = -\mathbf{a}_y \frac{E_{r0}}{\eta_1} e^{j\beta_1 z}$$

3. Transmitted wave:

$$\mathbf{E}_t(z) = \mathbf{a}_x E_{t0} e^{-j\beta_2 z}$$

$$\mathbf{H}_t(z) = \mathbf{a}_z \times \frac{1}{\eta_2}\mathbf{E}_t(z) = \mathbf{a}_y \frac{E_{t0}}{\eta_2} e^{-j\beta_2 z}$$

Boundary condition:

$$E_{i0} + E_{r0} = E_{t0}$$

$$\frac{1}{\eta_1}(E_{i0} - E_{r0}) = \frac{E_{t0}}{\eta_2}$$

Reflection and transmission coefficients:

- Reflection coefficient:

$$\Gamma = \frac{E_{r0}}{E_{i0}} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1}$$

- Transmission coefficient:

$$\tau = \frac{E_{t0}}{E_{i0}} = \frac{2\eta_2}{\eta_2 + \eta_1}$$

$$1 + \Gamma = \tau$$

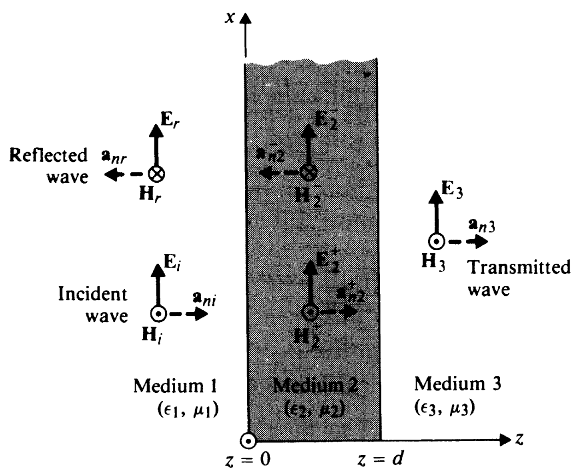
In medium 1:

$$\mathbf{E}_1(z) = \mathbf{a}_x E_{i0} e^{-j\beta_1 z}\left(1 + \Gamma e^{j2\beta_1 z}\right)$$

Standing wave ration:

$$S = \frac{|E|_{\max}}{|E|_{\min}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

25 Normal Incidence at Multiple Dielectric Interfaces



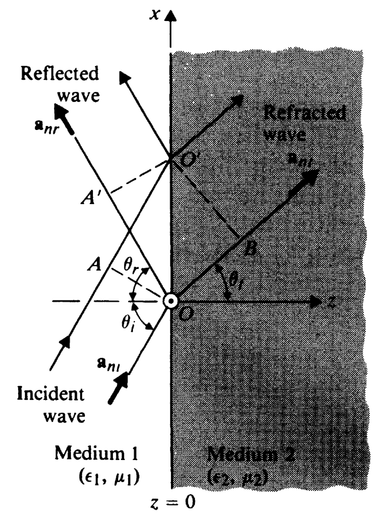
$$\mathbf{E}_2 = \mathbf{a}_x\left(E_2^+ e^{-j\beta_2 z} + E_2^- e^{j\beta_2 z}\right)$$

$$\mathbf{H}_2 = \mathbf{a}_y \tfrac{1}{\eta_2}\left(E_2^+ e^{-j\beta_2 z} - E_2^- e^{j\beta_2 z}\right)$$

$$\mathbf{E}_3 = \mathbf{a}_x E_3^+ e^{-j\beta_3 z}$$

$$\mathbf{H}_3 = \mathbf{a}_y \tfrac{E_3^+}{\eta_3} e^{-j\beta_3 z}$$

26 Oblique Incidence at a Plane Dielectric Boundary



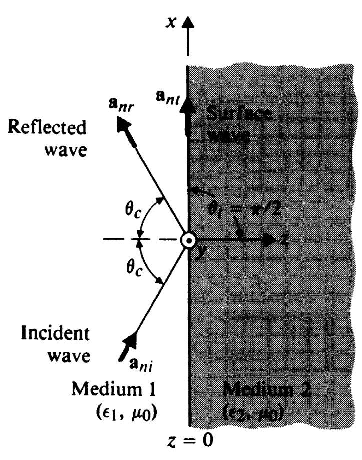
- Snell's law of reflection:

$$\frac{\sin\theta_t}{\sin\theta_i} = \frac{u_{p2}}{u_{p1}} = \frac{\beta_1}{\beta_2} = \frac{n_1}{n_2}$$

- Snell's law of refraction:

$$\frac{\sin\theta_t}{\sin\theta_i} = \sqrt{\frac{\epsilon_1}{\epsilon_2}} = \sqrt{\frac{\epsilon_{r1}}{\epsilon_{r2}}} = \frac{n_1}{n_2} = \frac{\eta_2}{\eta_1}$$

Total Reflection:



$$\theta_c = \sin^{-1}\sqrt{\frac{\epsilon_2}{\epsilon_1}} = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$