Static Electric Fields

1 Basic Concept

Electrostatics:

i. electric charges are **at rest(not moving)**;

ii electric field **do not change with time**.

2 Electrostatics in Free Space

Static electric charges (source) in free space → electric field

2.1 Electric field intensity

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}}{q} \quad (\mathbf{V}/\mathrm{m})$$

2.2 Fundamental Postulates of Electrostatics

- Differential form:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \quad (divergence)$$
$$\nabla \times \mathbf{E} = 0 \quad (curl)$$

  , where ρ is the volume charge density of free charges (C/m^3), ϵ_0 is the permittivity of free space, a universal constant.

- Integral form:

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$
$$\oint_C \mathbf{E} \cdot d\ell = 0$$

  , where Q is the total charge contained in volume V bounded by surface S. Also, the scalar line integral of the static electric field intensity around any closed path vanishes.

E is **not solenoidal** (unless $\rho = 0$), but **irrotational (conservative)**

3 Coulomb's Law

3.1 Electric Field due to a System of Discrete Charges

- **a single point charge (charge on the origin):**

$$\mathbf{E} = \mathbf{a}_R E_R = \mathbf{a}_R \frac{q}{4\pi\epsilon_0 R^2} \quad (\mathbf{V}/\mathrm{m})$$

- **a single point charge (charge is not on the origin):**

$$\mathbf{E}_p = \frac{q\left(\mathbf{R}-\mathbf{R}'\right)}{4\pi\epsilon_0\left|\mathbf{R}-\mathbf{R}'\right|^3} \quad (\mathbf{V}/\mathrm{m})$$

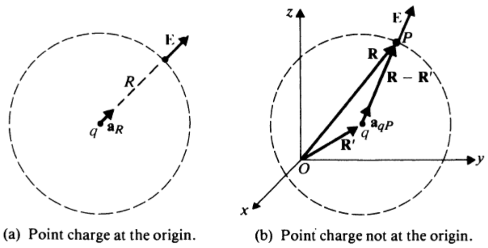


FIGURE 3–2
Electric field iFIGURE due to a point charge.

When a point charge q_2 is placed in the field of another point charge q_1 at the origin, a force F_{12} is experienced by q_2 due to the electric field intensity $\boldsymbol{E_{12}}$ of q_1 at q_2. Then we have:

$$\boldsymbol{F_{12}} = q_2\boldsymbol{E_{12}} = \boldsymbol{a_R}\frac{q_1q_2}{4\pi\epsilon_0R^2}$$

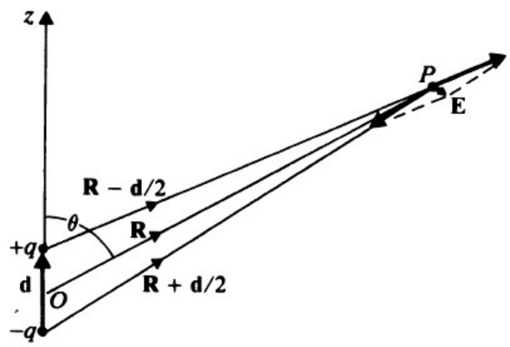
Example:
A total charge Q is put on a thin spherical shell of radius b. Determine the electric field intensity at an arbitrary point inside the shell

- **several point charges:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_{k=1}^{n}\frac{q_k\left(\mathbf{R}-\mathbf{R}'_k\right)}{\left|\mathbf{R}-\mathbf{R}'_k\right|^3}$$

3.2 Electric Dipole

- **Electric Field**



general expression:

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{R}-\frac{\mathbf{d}}{2}}{\left|\mathbf{R}-\frac{\mathbf{d}}{2}\right|^3}-\frac{\mathbf{R}+\frac{\mathbf{d}}{2}}{\left|\mathbf{R}+\frac{\mathbf{d}}{2}\right|^3}\right\}$$

if $d << R$:

$$\mathbf{E} \cong \frac{q}{4\pi\epsilon_0 R^3}\left[3\frac{\mathbf{R}\cdot\mathbf{d}}{R^2}\mathbf{R}-\mathbf{d}\right]$$

- **Electric Dipole Moment**
  Definition:

$$\mathbf{p} = q\mathbf{d}$$

,where q is the charge, vector $\mathbf{d}$ goes from $-q$ to $+q$.

$$\mathbf{p} = \mathbf{a}_z p = p\left(\mathbf{a}_R\cos\theta - \mathbf{a}_\theta\sin\theta\right)$$

$$\mathbf{R}\cdot\mathbf{p} = Rp\cos\theta$$

- **Electric Field:** (spherical coordinate)

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 R^3}\left(\mathbf{a}_R 2\cos\theta + \mathbf{a}_\theta\sin\theta\right) \quad (\mathrm{V/m})$$

3.3 Electric Field due to a Continuous Distribution of Charge

- **General Differential Element:**

$$d\mathbf{E} = \mathbf{a}_R\frac{\rho dv'}{4\pi\epsilon_0 R^2}$$

, where dv' is the differential volume element.

- **Line Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{L'} \mathbf{a}_R \frac{\rho_\ell}{R^2} d\ell' \quad (\mathbf{V}/\mathbf{m})$$

- **Surface Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{\mathbf{S}'} \mathbf{a}_R \frac{\rho_s}{R^2} ds' \quad (\mathbf{V}/\mathrm{m})$$

- **Volume Charge:**

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{V'} \mathbf{a}_R \frac{\rho}{R^2} dv' \quad (\mathbf{V}/\mathbf{m})$$

Example:
Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_ℓ in air.

P.3-8:

A line charge of uniform density ρ_l in free space forms a semicircle of radius b. Determine the magnitude and direction of the electric field intensity at the center of the semicircle.

P.3-9:

Three uniform line charges - $\rho_{l_1}, \rho_{l_2}, \rho_{l_3}$, each of length L -form an equilateral triangle. Assuming that $\rho_{l_1} = 2\rho_{l_2} = 2\rho_{l_3}$, determine the electric field intensity at the center of the triangle.

4 Gauss's Law and Application

4.1 Definition

The total outward flux of the E-field over any closed surface in free space is equal to **the total charge enclosed in the surface** divided by ϵ_0. (Note that we can choose arbitrary surface S for our convenience.)

$$\oint_S \mathbf{E} \cdot d\mathbf{s} = \frac{Q}{\epsilon_0}$$

4.2 Application

- **Conditions for Maxwell's Integral Equations:**
  There is **a high degree of symmetry** in the charge distribution or in the electrical field (i.e., spherically symmetric, planar, line charge, etc.

Example:
Determine the electric field intensity of an infinitely long, straight, line charge of a uniform density ρ_s in air.

Example:

Determine the electric field intensity of an infinite planar charge with a uniform surface charge density ρ_s.

Example:

Determine the **E** field caused by a spherical cloud of electrons with a volume charge density $\rho = -\rho_o$ for $0 \leq R \leq B$ (both ρ_o and b are positive) and $\rho = 0$ for $R > b$.

P.3-12:
Two infinite long coaxial cylindrical surfaces, $r = a$ and $r = b(b > a)$, carry surface charge densities ρ_{sa}, ρ_{sb}, respectively.

a) Determine $\boldsymbol{E}$ everywhere.

b) What must be the relation between a and b in order that $\boldsymbol{E}$ vanishes for $r > b$?

4.3 Several Useful Models

Note: The charge distribution should be **uniform**.

different models	E(magnitude)
infinitely long, line charge	$E = \frac{\rho_\ell}{2\pi r\epsilon_0}$
infinite planar charge	$E = \frac{\rho_s}{2\epsilon_0}$
uniform spherical surface charge with radius R	$\begin{cases} E = 0(r < R) \\ E = \frac{Q}{4\pi r^2\epsilon_0}(r > R) \end{cases}$
uniform sphere charge with radius R	$\begin{cases} E = \frac{Qr}{4\pi R^3}(r < R) \\ E = \frac{Q}{4\pi r^2\epsilon_0}(r > R) \end{cases}$
infinitely long, cylindrical charge with radius R	$\begin{cases} E = \frac{\rho_v r}{2\epsilon_0}(r < R) \\ E = \frac{\rho_v R^2}{2r\epsilon_0}(r > R) \end{cases}$

5 Electric Potential

- **Expression:**

$$\mathbf{E} = -\nabla V$$

the reason for the negative sign: consistent with the convention that in going against the $\mathbf{E}$ field, the electric potential V increases.

- **Electric Potential Difference:**

$$V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\ell$$

- **Electric Potential due to a Charge Distribution**

$$V = \frac{q}{4\pi\epsilon_0 R}$$

  i. **Line Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{L'} \frac{\rho_\ell}{R} d\ell' \quad (V)$$

  ii. **Surface Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{\mathbf{S}'} \frac{\rho_s}{R} ds' \quad (V)$$

  – **Volume Charge:**

$$V = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho}{R} dv' \quad (V)$$

- **Example:**
Obtain a formula for the electric field intensity and potential on the axis of a circular disk of radius b that carries a uniform surface charge ρ_s.

P.3-13:

Determine the work done in carrying a $-2(\mu C)$ charge from $P_1(2,1,-1)$ to $P(8,2,-1)$ in the field $\boldsymbol{E} = \boldsymbol{a_x} y + \boldsymbol{a_y} x$.

a) along the parabola $x = 2y^2$.

b) along the straight line joining P_1 and P_2.

P.3-16:

A finite line charge of length L carrying uniform line charge density ρ_l is coincident with the x-axis.

a) Determine V in the plane bisecting the line charge.

b) Determine $\boldsymbol{E}$ on the bisecting plane from ρ_l directly by applying Coulomb's law.

c) Check the answer in part (b) with $-\nabla V$.

P.3-19:

A charge Q is distributed uniformly over the wall of a circular tube of radius b and height h. Determine V and $\boldsymbol{E}$ on its axis.

a) at a point outside the tube, Then

b) at a point inside the tube.