Question 1

1. Suppose the functions in Fig 1 are $\boldsymbol{v}_a = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$, $\boldsymbol{v}_b = \hat{\boldsymbol{z}}$, and $\boldsymbol{v}_c = z\hat{\boldsymbol{z}}$. Calculate their divergence.

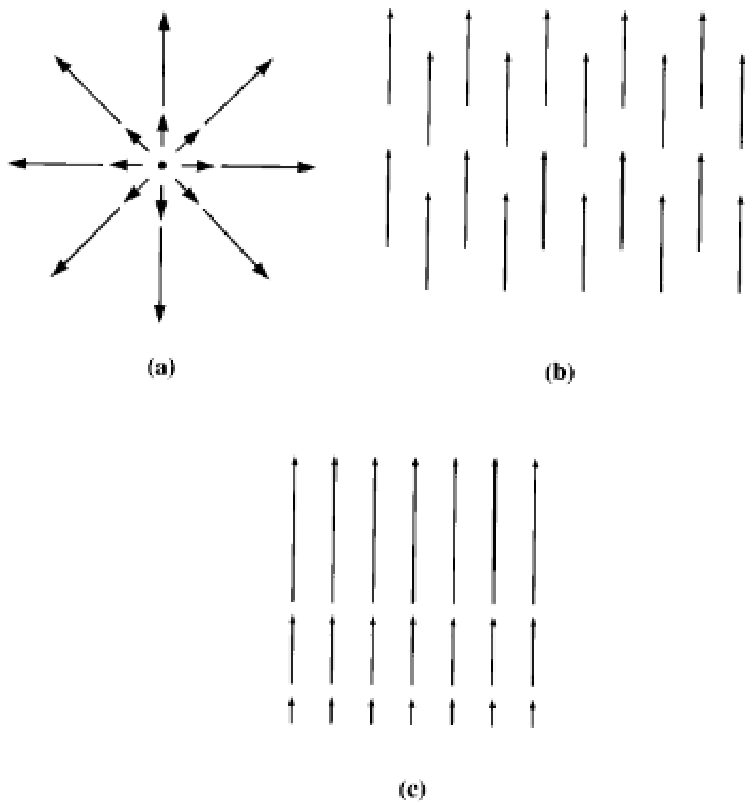

Figure 1: Figure for Question 1.1

2. Suppose the function sketched in Fig 2a is $\boldsymbol{v}_a = -y\hat{\boldsymbol{x}} + x\hat{\boldsymbol{y}}$, and that in Fig 2$b$ is $\boldsymbol{v}_b = x\hat{\boldsymbol{y}}$. Calculate their curls.

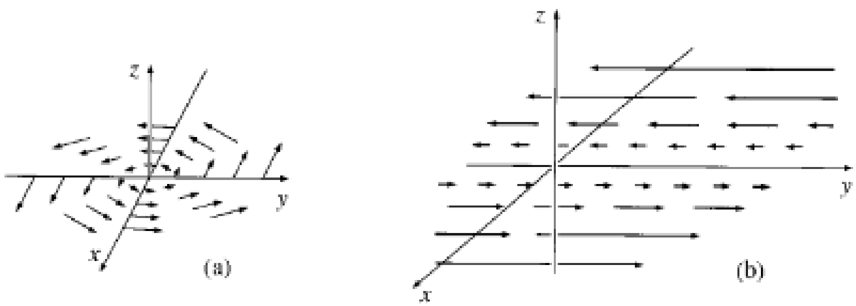

Figure 2: Figure for Question 1.2

3. Construct a vector function that has zero divergence and zero curl everywhere. (A *constant will do the job, of course, but make it something a little more interesting than that!*)

Question 2

The expression of the electric field for a point charge (Fig 3) is:

$$\boldsymbol{E}(\boldsymbol{r}) \equiv \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i}{r_i^2} \hat{\boldsymbol{r}}_{\boldsymbol{i}} \tag{1}$$

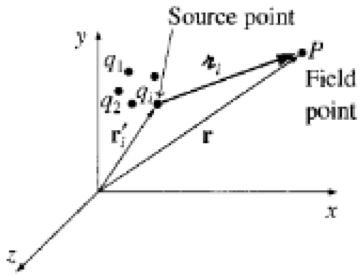

Figure 3: Figure for Eq 1

1. Find the electric field (magnitude and direction) a distance z above the midpoint between two equal charges, q, a distance d apart (Fig 4). Check that your result is consistent with what you'd expect when $z >> d$.

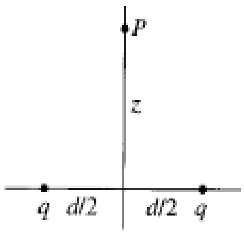

Figure 4: Figure for Question 2.1

2. Repeat part 1 in this question, only this time make the right-hand charge $-q$ instead of $+q$.

Question 3

The expression for the work required to bring together n charges is given by

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V(\boldsymbol{r}_{\boldsymbol{i}})$$

, where $V(\boldsymbol{r}_{\boldsymbol{i}})$ is the potential.

1. Three charges are situated at the corners of a square (side a), as shown in Fig 5. How much work does it take to bring in another charge, $+q$, from far away and place it in the fourth corners?

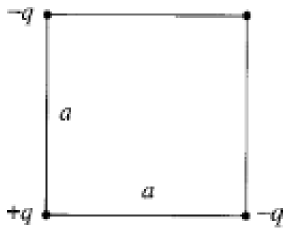

Figure 5: Figure for Question 3.1

2. How much work does it take to assemble the whole configuration of four charges?

Question 4

The original form of the Gauss's law is

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{1}{\epsilon_0} Q_{enc} \tag{2}$$

, where Q_{enc} is the total charge enclosed within the surface.

The electric displacement could be calculated by

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E} + \boldsymbol{P} \tag{3}$$

The revised form for the Gauss's law which considers the electric displacement is then

$$\oint \boldsymbol{D} \cdot d\boldsymbol{a} = Q_{f_{enc}} \tag{4}$$

, where $Q_{f_{enc}}$ indicates the total free charge enclosed in the volume.

1. Show that the energy of an ideal dipole $\boldsymbol{p}$ in an electric field $\boldsymbol{E}$ is given by

$$U = -\boldsymbol{p} \cdot \boldsymbol{E}$$

2. A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization

$$\boldsymbol{P}(\boldsymbol{r}) = \frac{k}{r} \hat{\boldsymbol{r}}$$

, where k is a constant and r is the distance from the center (Fig 6). (There is no *free* charge in the problem). Find the electric field in all three regions by two different methods:

   (a) Locate all the bound charge, and use Gauss's law (Eq 2) to calculate the field it produces.

   (b) Use Eq 4 to find $\boldsymbol{D}$, and then get $\boldsymbol{E}$ from Eq 3. [Notice that the second method is much faster, and avoids any explicit referencer to the bound charges.]

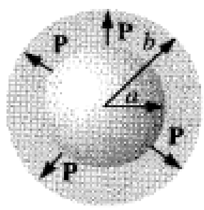

Figure 6: Figure for Question 4.2

Question 5

Choose either question A or question B to answer. If both questions are answered, only question A would be graded.

A

The Ampere's law with Maxwell's correction is given by:

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{5}$$

Suppose $\boldsymbol{J}(\boldsymbol{r})$ is constant in time but $\rho(\boldsymbol{r}, t)$ is *not* — conditions that might prevail, for instance, during the charging of a capacitor.

1. Show that the charge density at any particular point is a linear function of time:

   $$\rho(\boldsymbol{r}, t) = \rho(\boldsymbol{r}, 0) + \dot{\rho}(\boldsymbol{r}, 0)t,$$

   , where $\dot{\rho}(\boldsymbol{r}, 0)$ is the time derivative of ρ at $t = 0$. This is *not* an electrostatic or magnetostatic configuration; nevertheless — rather superisingly — both Coloumb's law and the Biot-Savart law hold, as you can confirm by showing that they satisfy Maxwell's equations. In particular:

2. Show that

   $$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{J}(\boldsymbol{r'}) \times \hat{\boldsymbol{r}}}{r^2} d\tau'$$

   obeys Ampere's law with Maxwell's displacement current term (Eq 5).

B

The Maxwell's equations with magnetic charge is given by:

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\epsilon_0} \rho_e, \tag{6a}$$

$$\nabla \cdot \boldsymbol{B} = \mu_0 \rho_m \tag{6b}$$

$$\nabla \times \boldsymbol{E} = -\mu_0 \boldsymbol{J_m} - \frac{\partial \boldsymbol{B}}{\partial t} \tag{6c}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J_e} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{E}}{\partial t} \tag{6d}$$

1. Show that Maxwell's equations with magnetic charge (Eq 6) are invariant under the duality transformation

   $$\left.\begin{aligned} \boldsymbol{E}' &= \boldsymbol{E} cos\alpha + c\boldsymbol{B} sin\alpha, \\ c\boldsymbol{B}' &= c\boldsymbol{B} cos\alpha - \boldsymbol{E} sin\alpha, \\ cq_e' &= cq_e cos\alpha + q_m sin\alpha, \\ q_m' &= q_m cos\alpha - cq_e sin\alpha, \end{aligned}\right\},$$

   where $c \equiv 1/\sqrt{\epsilon_0 \mu_0}$ and α is an arbitrary rotation angle in "$\boldsymbol{E}/\boldsymbol{B}$-space." Charge and current densities transform in the same way as q_e and q_m. [This means, in particular, that if you knows the fields produced by a configuration of electric charge, you can immediately (using $\alpha = 90°$) write down the fields produced by the corresponding arrangement of magnetic charge.]

2. Show that the force law

   $$\boldsymbol{F} = q_e(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) + q_m(\boldsymbol{B} - \frac{1}{c^2} \boldsymbol{v} \times \boldsymbol{E})$$

   is also invariant under the duality transformation.

Question 6

Choose either question A or question B to answer. If both questions are answered, only question A would be graded.

The magnetic potential $\boldsymbol{A}$ is introduced by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \quad (T) \tag{7}$$

Also, we have

$$\boldsymbol{E} = -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} \quad (V/m) \tag{8}$$

The nonhomogeneous wave equation for vector potential $\boldsymbol{A}$ is:

$$\nabla^2 \boldsymbol{A} - \mu\epsilon \frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu \boldsymbol{J} \tag{9}$$

The nonhomogeneous wave equation for scalar potential V is :

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{10}$$

A

The vector magnetic potential $\boldsymbol{A}$ and scalar electric potential V are not unique in that it is possible to add to $\boldsymbol{A}$ the gradient of a scalar ψ, $\nabla\psi$, with no change in $\boldsymbol{B}$ from Eq 7.

$$\boldsymbol{A}' = \boldsymbol{A} + \nabla\psi$$

In order not to change $\boldsymbol{E}$ in using Eq 8, V must be modified to V'.

1. Find the relation between V' and V.
2. Discuss the condition that ψ must satisfy so that the new potential $\boldsymbol{A}'$ and V' remain governed by the uncoupled wave equations Eq 9 and Eq 10

B

Substitute Eq 7 and Eq 8 in Maxwell's equations to obtain wave equations for scalar potential V and vector potential $\boldsymbol{A}$ for a linear, isotropic but inhomogeneous medium. Show that these wave equations reduce to Eq 9 and Eq 10 for simple media. (*Hint: Use the following gauge condition for potentials in an inhomogeneous medium:*

$$\nabla \cdot (\epsilon \boldsymbol{A}) + \mu\epsilon^2 \frac{\partial V}{\partial t} = 0$$

)

Congratulations on finishing VE230!

Figure 7: Figure for finishing VE230