Probability and Random Process

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Nov. 5 2020

Ve501 2020-2021 Fall

Based on Lecture Notes by Prof. Chong

Outline

- 4. Random Process-I
  - Introduction to Random Processes
  - Brownian Motion//Wiener Process
  - Poisson Process
  - Complex RV and RP
  - Stationarity
  - PSD, QAM, White Noise
  - Response of Systems
  - LTI Systems and RPs

Introduction to Random Processes

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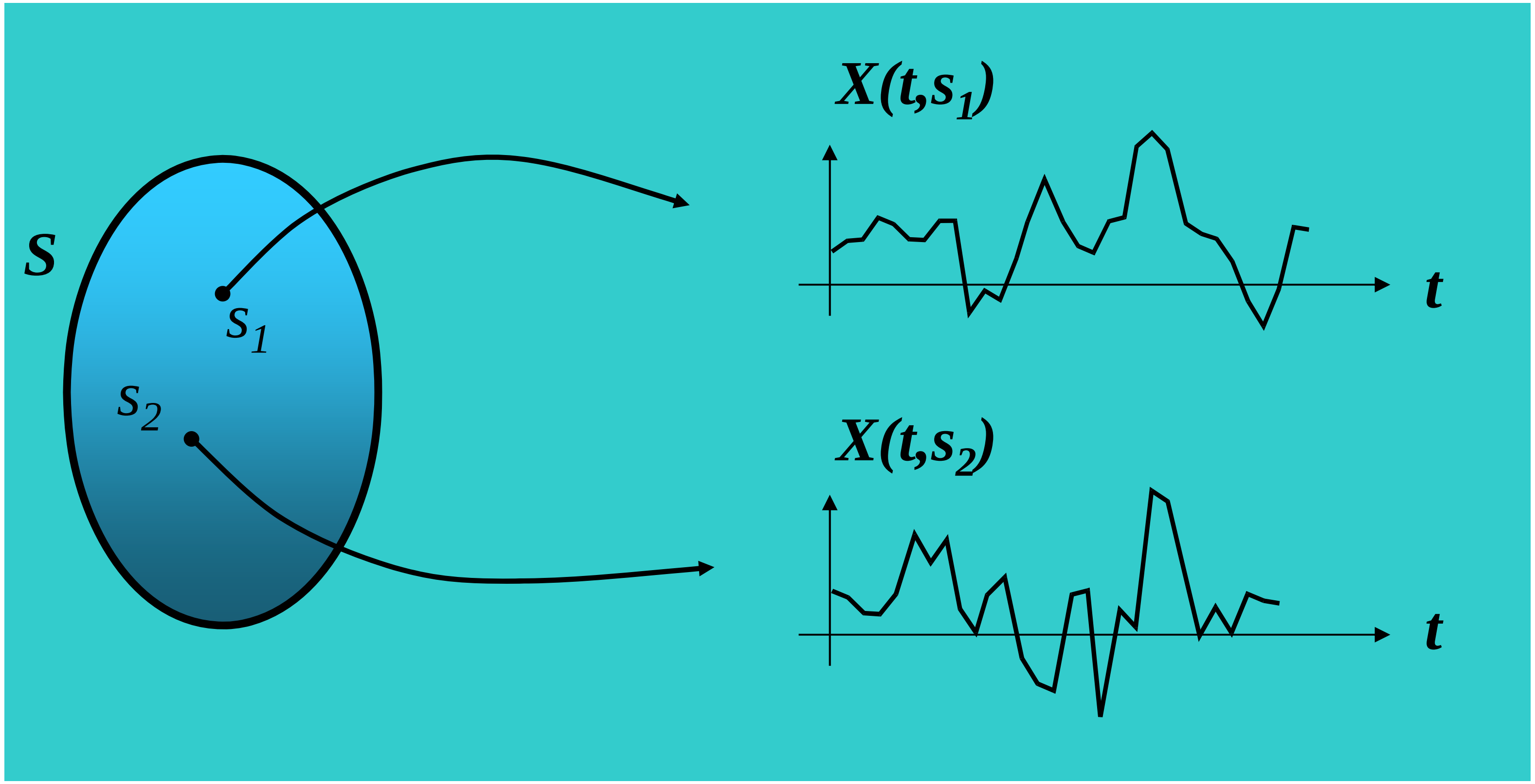
Random Processes

A random process (RP) is a function that maps each outcome from a sample space **S** to a function of time.



Index Set

In general, t belongs to an index set I.

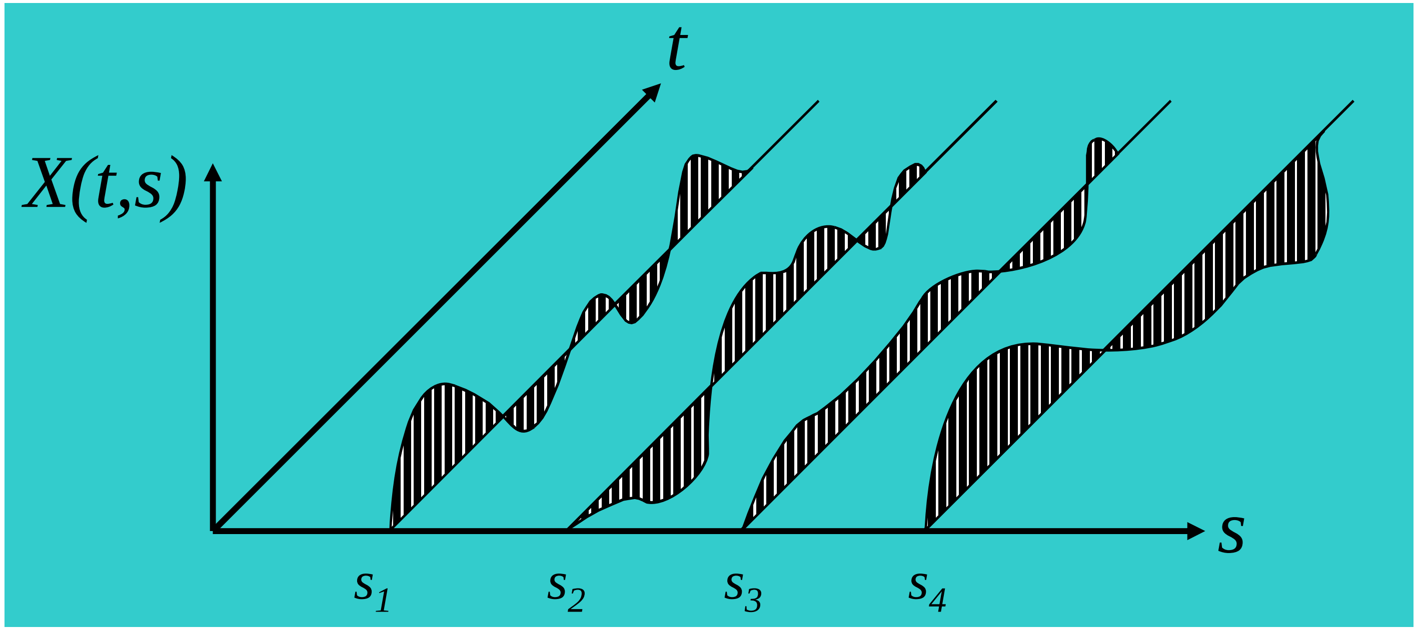
$I = \mathbf{R}$: $X(t,s)$ is a continuous-time random process.

$I = \mathbf{Z}$ (the integers): $X(n,s)$ is a discrete-time RP, also known as a random sequence.

$I = \{1,2,\ldots,N\}$: $X(n,s)$ is a random vector.

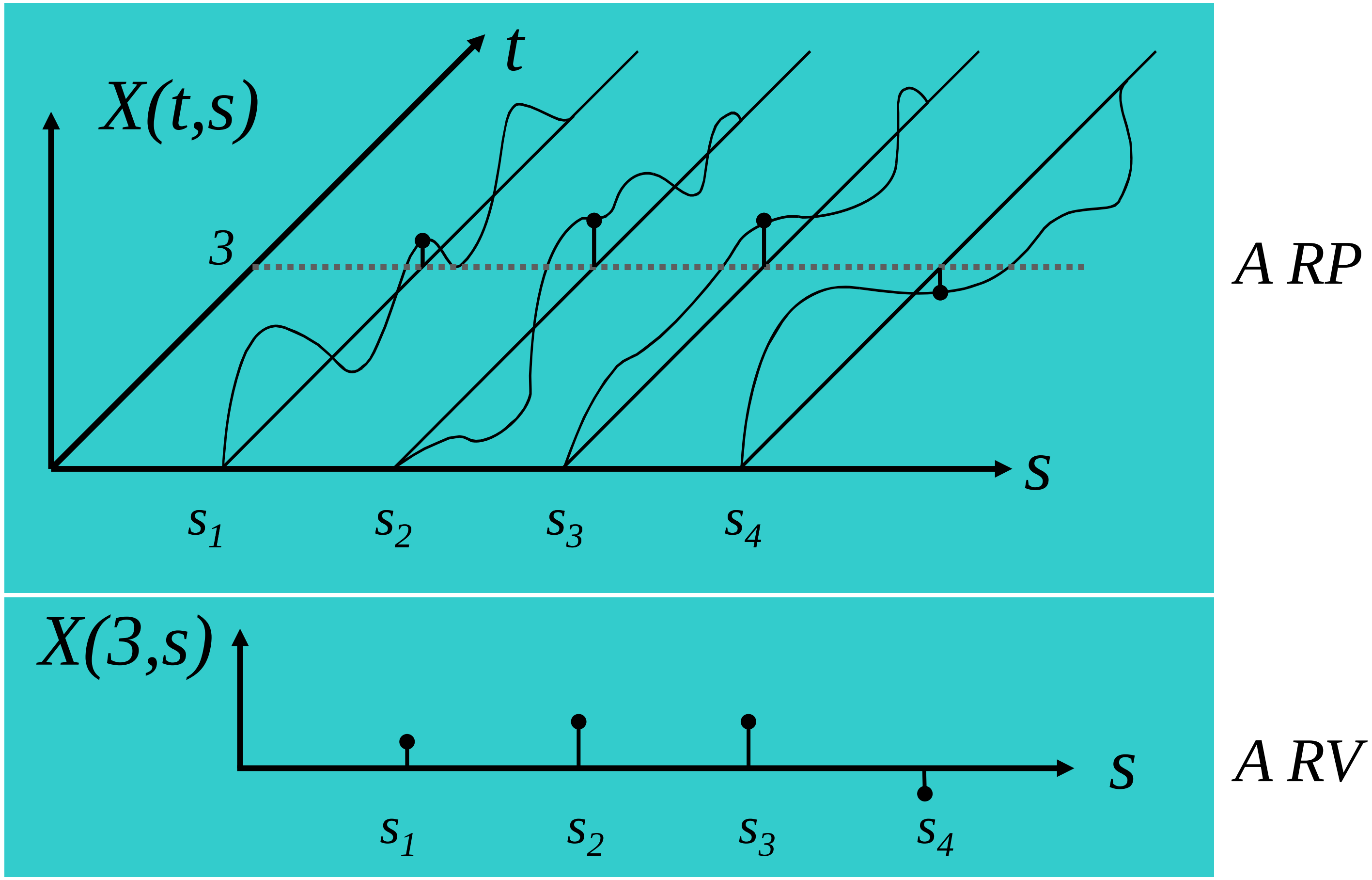
Random Processes - Second View

A RP is a function that maps from the cartesian product $S \times I$ to the real numbers.



Getting a RV from a RP

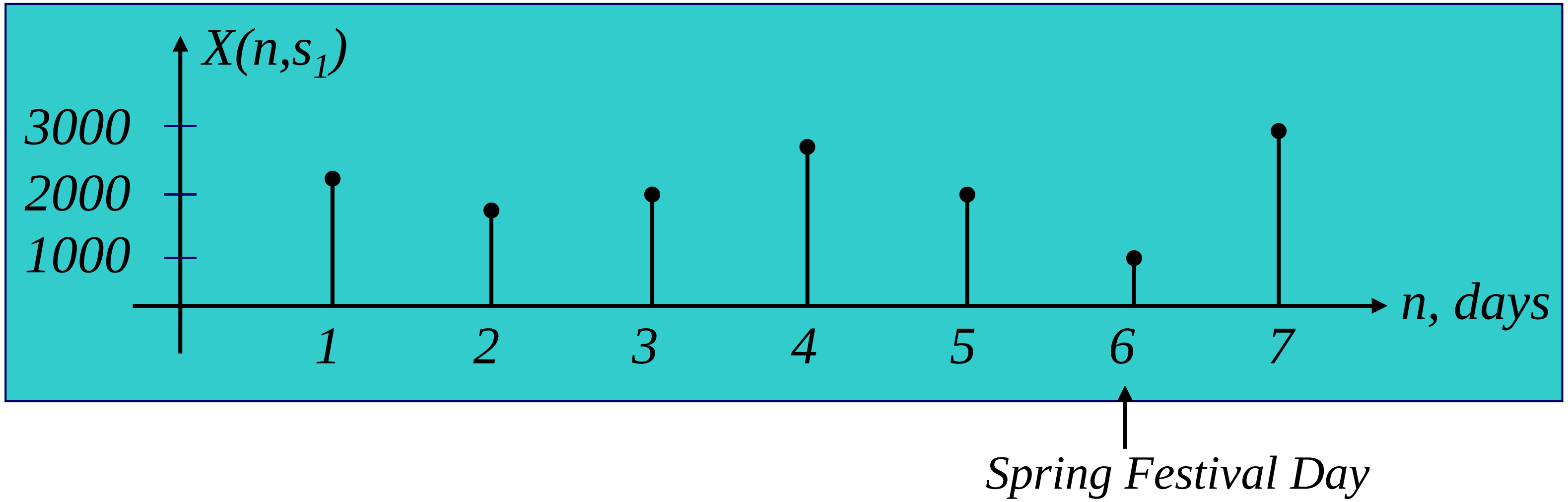
Specifying a particular point in time in a RP yields a RV.



Web Page Hits Example

Define $X(n,s)$ to be the number of visits to the JI home page on day n.

Example sample function



One possible question: What is the expected value of the number of visits on Spring Festival day?

State Space

The set of all possible values (or states) that a RP may take is its state space

A discrete state space RP (chain) occupies only a finite or countable number of states

Ex: Number of objects in a queue

A continuous state space RP can take any real value

Ex: Temperature

Discrete vs. Continuous Valued r.p.'s

	Discrete-valued	Continuous-valued
Discrete -time	seq of stock prices	seq of temp's in time or distance
Continuous -time	number of customers in line at time	waveform from microphone

It is essential that you keep separate track of time and value axes and properties.

Lessons

A random process maps from $S \times I$ to the real numbers

Random processes are classified as being
- discrete or continuous state
- discrete or continuous time

A Random process at a fixed time is a random variable

Characterization of a RP

Most general: joint CDF for any subset of the index set

Ex: For a continuous time RP X(t), given the subset {6, 10, 110, 200}, produce the function:

$$F_{X(6)X(10)X(110)X(200)}(x_1, x_2, x_3, x_4)$$

The joint PDF may also be used

As with RVs, the s-dependence is often dropped from the notation $X(t, s) = X(t)$, or simply use X_t

Distribution

1. **1st-order distribution**
   This is the distribution of X_t for all $t \in \mathcal{T}$, *e.g.*

$$F_{X_t}(x), \quad \forall t \in \mathcal{T}, \forall x$$

   It tells us nothing about dependence among variables.
2. **2nd-order distribution**
   The joint distribution of X_t, X_s for all $t, s \in \mathcal{T}$.
3. **nth-order distribution**
   The joint distribution of $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ for all $t_1, \ldots, t_n \in \mathcal{T}$.
   This can become overwhelming as n increases.

Moments of a RP

Moments provide partial characterization:

Mean (first moment):

$$m_X(t) = E\{X(t)\} = \int_{-\infty}^{+\infty} x f_{X(t)}(x)dx$$

Auto-correlation:

$$R_X(t_1,t_2) = E\{X(t_1)X(t_2)\}$$
$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy f_{X(t_1)X(t_2)}(x,y)dxdy$$

Auto-correlation properties

Property

1. symmetric function of t and s.

$$R_X(t,s) = \mathrm{E}[X_t X_s] = \mathrm{E}[X_s X_t] = R_X(s,t)$$

2. $R_X(t,t) = \mathrm{E}\left[X_t^2\right] \geq 0$

$$|R_X(t,s)| \leq \sqrt{\mathrm{E}\left[X_t^2\right] \mathrm{E}\left[X_s^2\right]}$$

Cauchy-Schwarz inequality

$$|R_X(t,s)| = |\mathrm{E}[X_t X_s]| \leq \sqrt{\mathrm{E}\left[X_t^2\right] \mathrm{E}\left[X_s^2\right]}$$

Auto-covariance

$$C_X(t_1, t_2) = E\{(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (x - m_X(t_1))(y - m_X(t_2)) f_{X(t_1)X(t_2)}(x, y)\,dx\,dy$$

Autocovariance can also be expressed:

$$C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1) m_X(t_2)$$

Variance (second moment):

$$\sigma_X^2(t_1) = C_X(t_1, t_1)$$

Cross-correlation & covariance

Let $\{X_t, t \in \mathcal{T}\}$ and $\{Y_t, t \in \mathcal{T}\}$ be random processes. Their cross-correlation is defined as

$$R_{XY}(t,s) = E[X_t Y_s], t, s \in \mathcal{T}$$

Let $\{X_t, t \in \mathcal{T}\}$ and $\{Y_t, t \in \mathcal{T}\}$ be random processes. Their cross-covariance is defined as

$$C_{XY}(t,s) = Cov\{X_t Y_s\} = E\big[\big(X_t - m_X(t)\big)\big(Y_s - m_Y(s)\big)\big]$$
$$= R_{XY}(t,s) - m_X(t) m_Y(s)$$

Random Line Example - I

Let A and B be independent random variables. Then, let

$$X(t) = A + Bt$$

Find $m_X(t)$, $C_X(t_1,t_2)$, and $f_{X(t)}(x)$.

$$m_X(t) = E\{X(t)\} = m_A + m_B t$$

$$C_X(t_1,t_2) = E\{[A + Bt_1 - m_A - m_B t_1][A + Bt_2 - m_A - m_B t_2]\}$$

$$= E\{[(A - m_A) + (B - m_B)t_1][(A - m_A) + (B - m_B)t_2]\}$$

$$= \sigma_A^2 + \sigma_B^2 t_1 t_2$$

Why did the cross terms drop out?

Random Line Example - II

X(t) is a function of two RVs.

Let $D = Bt$, then $f_D(d) = \dfrac{f_B(\frac{d}{t})}{|t|}$

Since $X(t) = A + D$ and A and D are independent,

$$f_{X(t)}(x) = f_A(x) * f_D(x) = f_A(x) * \frac{f_B(\frac{x}{t})}{|t|}$$

convolution

Discrete-time Random Walk

Let B[n] be a sequence of iid RVs, such that:

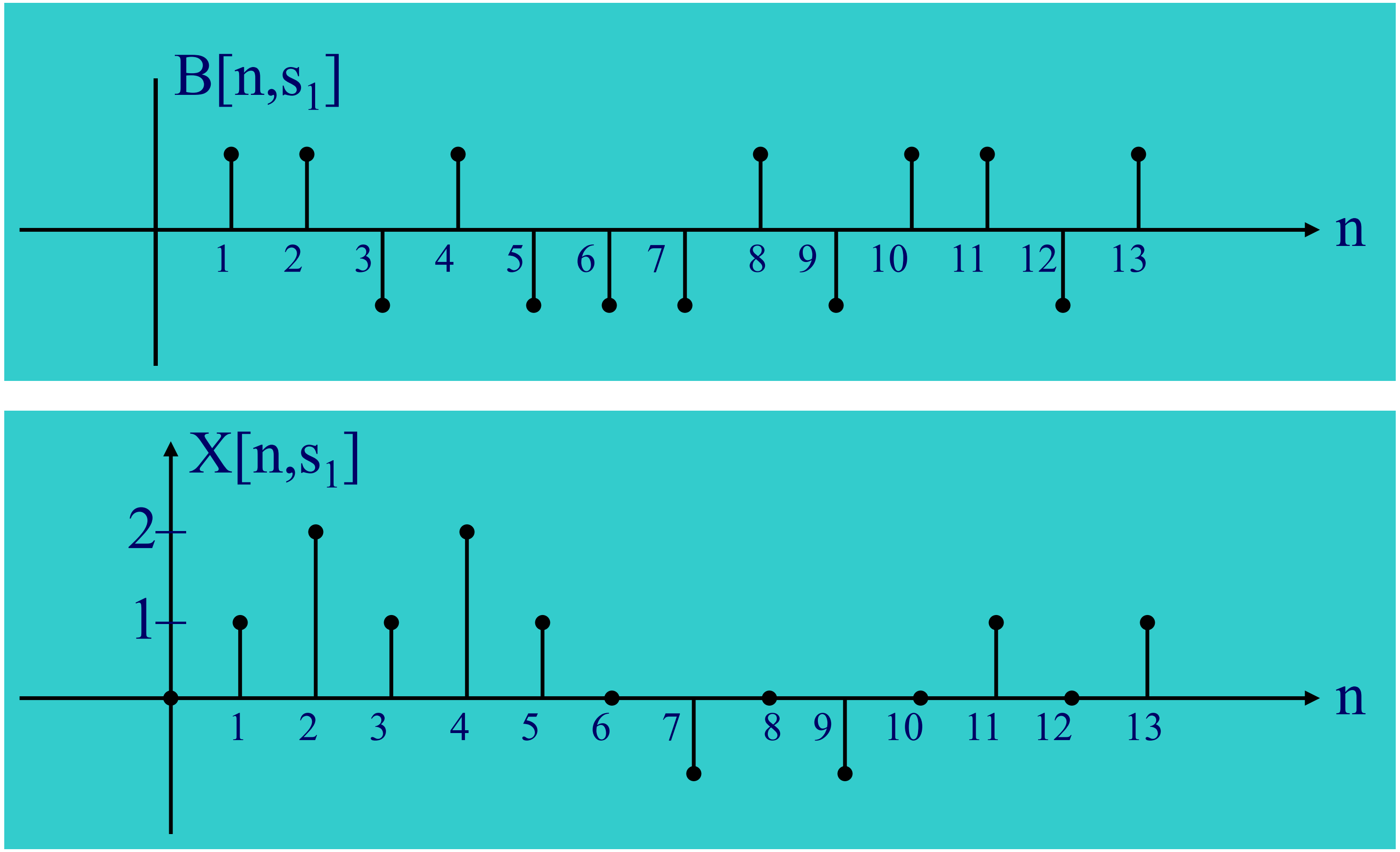
$$P(B[n] = 1) = P(B[n] = -1) = 1/2$$

Let $X[n] = \sum_{k=1}^{n} B[k]$, and set X[0]=0

X[n] is an example of a sum process

X[n] is the discrete time random walk

Discrete-time Random Walk - II



DT Random Walk Moments - I

$$m_X[n] = E\{x[n]\} = \sum_{k=1}^{n} E\{B[k]\} = \sum_{k=1}^{n} m_B[k] = 0$$

$$\begin{aligned}
C_X[n_1, n_2] &= E\{(X[n_1] - m_X[n_1])(X[n_2] - m_X[n_2])\} \\
&= E\left\{\sum_{k=1}^{n_1} (B[k] - m_B[k]) \sum_{j=1}^{n_2} (B[j] - m_B[j])\right\} \\
&= E\left\{\sum_{k=1}^{n_1} \sum_{j=1}^{n_2} (B[k] - m_B[k])(B[j] - m_B[j])\right\} \\
&= \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} C_B[k, j]
\end{aligned}$$

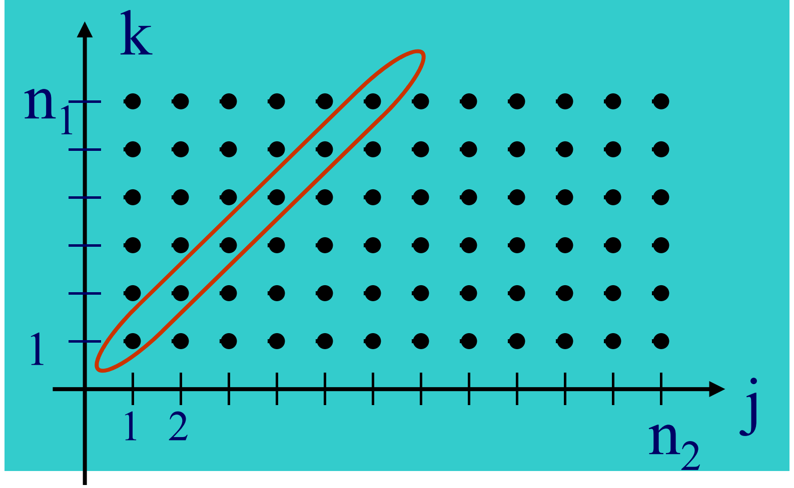
DT Random Walk Moments - II

Recall $C_X[n_1,n_2] = \sum_{k=1}^{n_1} \sum_{j=1}^{n_2} C_B[k, j]$

This is the general form for a sum process:

B[n] iid $\Rightarrow$ $C_B[k, j] = \begin{cases} 0 & \text{for } k \neq j \\ \sigma_B^2 & \text{for } k = j \end{cases}$

Example: $n_1 < n_2$.



Each dot represents a term in the sum.

$$\boxed{C_X[n_1,n_2] = \min[n_1,n_2]\sigma_B^2}$$

DT Random Walk Moments - III

$$\sigma_B^2 = E\left\{B^2[n]\right\} - \left(\overset{0}{\cancel{E\left\{B[n]\right\}}}\right)^2 = 1$$

$$\boxed{\therefore C_X[n_1, n_2] = \min[n_1, n_2]; \quad \sigma_X^2[n] = n}$$

The variance grows linearly with time: “a clock”

DT Random Walk Moments - IV

To get the PMF of X[n], observe that it can be expressed as a function of a Binomial RV

Let D[n] be a Bernoulli sequence with p=1/2. Then B[k]=2D[k]-1, and

$$X[n] = \sum_{k=1}^{n} B[k] = 2\underbrace{\left(\sum_{k=1}^{n} D[k]\right)}_{\text{Binomial} = Z[n]} - n$$

DT Random Walk Moments - V

$$p_{X[n]}[l] = P\left(X[n] = l\right) = P\left(2Z[n] - n = l\right)$$

$$= P\left[Z[n] = \frac{l+n}{2}\right] = \begin{pmatrix} n \\ \frac{l+n}{2} \end{pmatrix} \frac{1}{2^n} \qquad ,\text{with } p = \frac{1}{2}$$

Recall that $E(X[n]) = 0$ and $\sigma_X^2[n] = n$

By the Central Limit Theorem,

$$\boxed{F_{X[n]}(l) \to \Phi\left(\frac{l}{\sqrt{n}}\right)}$$

Lessons

Some ways to characterize a RP:

- mean
- autocorrelation or autocovariance
- joint PMF (discrete space) or joint PDF

Discrete-time random walk

- zero mean
- variance=n
- CDF approaches Gaussian

Increments

An increment of a RP is

$$X(b) - X(a)$$

where $a < b$

Two types of increments that are useful in analysis are

Independent increments

Stationary increments

Independent Increments

A RP X(t) has independent increments if, for any $a < b \leq c < d$,

$$\left[X(b) - X(a)\right] \text{ is independent from } \left[X(d) - X(c)\right]$$

All sum processes that sum over sequences of independent RVs are ind. inc.

Example: Let $U_1, U_2, U_3, \ldots$ be a sequence of independent RVs, such that $U_i \sim U[0,1]$. Let

$$X(n) = \sum_{i=1}^{n} U_i$$

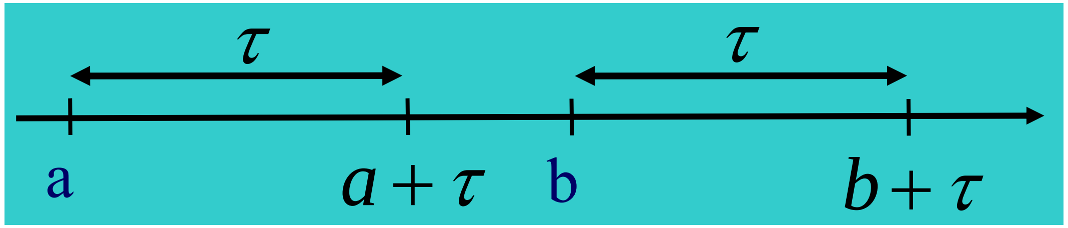
X(n) is an ind. inc. discrete-time RP

Discrete-time Random Walk is another example

Stationary Increments

A RP X(t) has stationary increments if, for any a, b and τ:

$$\left[X(a+\tau)-X(a)\right]$$ has the same PDF as $$\left[X(b+\tau)-X(b)\right]$$



Intervals are same length and may be overlapping

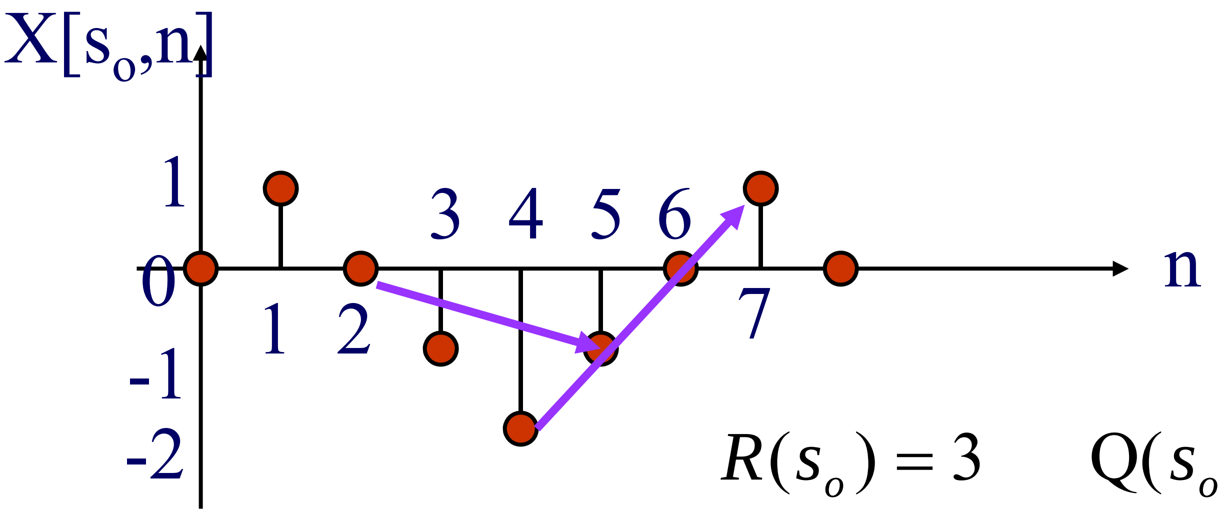
Caution: The values of the increments are not equal (in general); only their statistics are.

Stationary Increment Example

Discrete-time Random Walk:

Consider two intervals of length 3.

$$R = [X(7) - X(4)] \text{ and } Q = [X(5) - X(2)]$$



$$R(s_o) = 3 \qquad Q(s_o) = -1$$

Overlap

The overlap means that the increments are dependent.

For example, R = -3, Q = 3 is not possible.

To get joint PMF, apply Total Probability, conditioning on the overlap value.

$$p_{RQ}(r,q) = P(R = r \cap Q = q)$$

$$= \sum_{i=-1,1} P(R = r \cap Q = q \mid B_5 = i) P(B_5 = i)$$

These are conditionally independent

$$= \sum_{i=-1,1} P(R = r \mid B_5 = i) P(Q = q \mid B_5 = i) P(B_5 = i)$$

PDFs of the Increments

$$R = [X(7) - X(4)] \quad \text{and} \quad Q = [X(5) - X(2)]$$

$$R = \sum_{i=5}^{7} B(i) \quad f_R(r) = \sum_{k=0}^{3} \frac{\binom{3}{k}}{2^3} \delta(r - [2k-3])$$

$$Q = \sum_{i=3}^{5} B(i) \quad f_Q(q) = \sum_{k=0}^{3} \frac{\binom{3}{k}}{2^3} \delta(q - [2k-3])$$

Same PDFs

What does overlap imply about R and Q?

Stationary but dependent increments

Lessons

Useful features are

- Independent increments
- Stationary increments

Do not confuse with stationary RPs

Quiz

- In a communication system, the carrier signal at the receiver is modeled by $X_t = \cos(2\pi f t + \Theta)$, where $\Theta \sim \text{uniform}[-\pi,\pi]$. Find the mean function and the correlation function of X_t.

Solution. For the mean, write

$$\begin{aligned}
\mathsf{E}[X_t] &= \mathsf{E}[\cos(2\pi f t + \Theta)] \\
&= \int_{-\infty}^{\infty} \cos(2\pi f t + \theta)\, f_\Theta(\theta)\, d\theta \\
&= \int_{-\pi}^{\pi} \cos(2\pi f t + \theta)\, \frac{d\theta}{2\pi}.
\end{aligned}$$

Be careful to observe that this last integral is with respect to θ, *not* t. Hence, this integral evaluates to zero.

For the correlation, first write

$$R_X(t_1, t_2) = \mathsf{E}[X_{t_1} X_{t_2}] = \mathsf{E}\big[\cos(2\pi f t_1 + \Theta)\cos(2\pi f t_2 + \Theta)\big].$$

Then use the trigonometric identity

$$\cos A \cos B = \tfrac{1}{2}[\cos(A+B) + \cos(A-B)] \tag{10.3}$$

to write

$$R_X(t_1, t_2) = \tfrac{1}{2}\mathsf{E}\big[\cos(2\pi f[t_1 + t_2] + 2\Theta) + \cos(2\pi f[t_1 - t_2])\big].$$

The first cosine has expected value zero just as the mean did. The second cosine is nonrandom, and therefore equal to its expected value. Thus, $R_X(t_1, t_2) = \cos(2\pi f[t_1 - t_2])/2$.

Thank You!