Probability and Random Process

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Outline

- 1. Introduction to Probability
  - Application example
  - Review of set and functions
  - Models of random experiments
  - Axioms and properties of probability
  - Conditional probability
  - Independence of events
  - Combinatorics and probability

Application Areas

- Application areas of probability and random processes
  - Signal processing
  - Communications
  - Control
  - Industrial engineering
  - Economics
  - Aerospace
  - Information science
  - Computer science
  - …

Example: Signal Processing

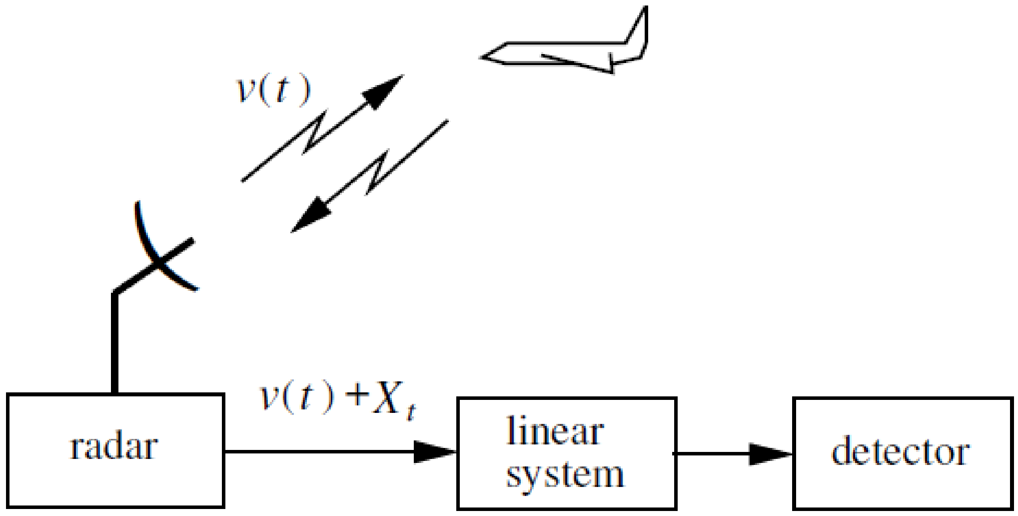


Figure **1.1.** Block diagram of radar detection system.

- To determine the presence of an aircraft, a known radar pulse $v(t)$ is sent out.
- The overall goal is to decide whether the received waveform is noise only or signal plus noise.
  - No object in range of radar, noise waveform only X_t.
  - An object in range, reflected radar pulse plus noise $v(t) + X_t$.

Example: Signal Processing

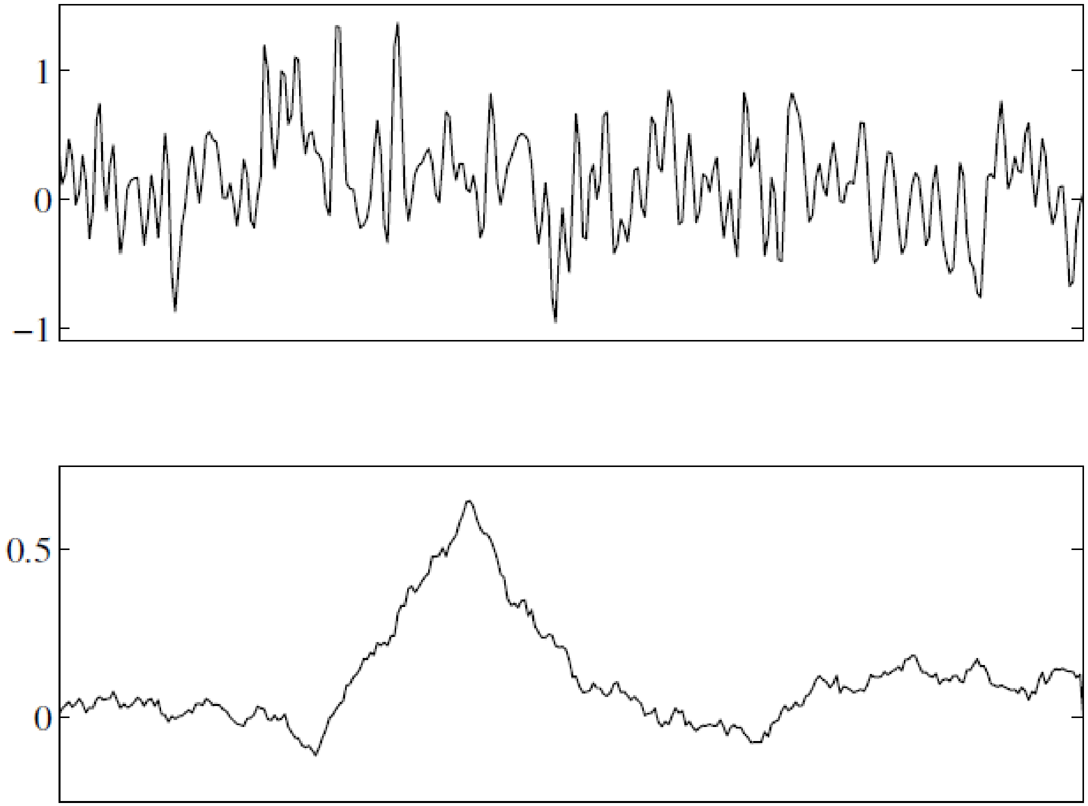


Figure 1.2. Matched filter input (top) in which the signal is hidden by noise. Matched filter output (bottom) in which the signal presence is obvious.

Describe Uncertainty

- How to describe/capture uncertainty in the behavior of engineering systems?
- What type of calculus does one develop to quantify uncertainty and show how uncertainty propagates through time?
- One way is through probability theory, random variables and random processes.

Review of set and functions

Set definition and representations

- A set is a collection of objects called elements or members of the set.
- Methods of specifying a set
  1. list them in curly brackets separately by commas$\{a, b, t, \ldots\}$
  2. predicate: $\{\text{real number } X : 0 \le X \le 1\}$ (colon means such that)
  3. intervals of the real line

$$[a, b) = \{\text{real number } x : a \le x < b\}$$

$$(a, b), (a, b], [a, b], (a < b)$$

  4. in terms of other sets: $A = B \cup C$
  5. Venn diagrams (Picture)

Venn diagram

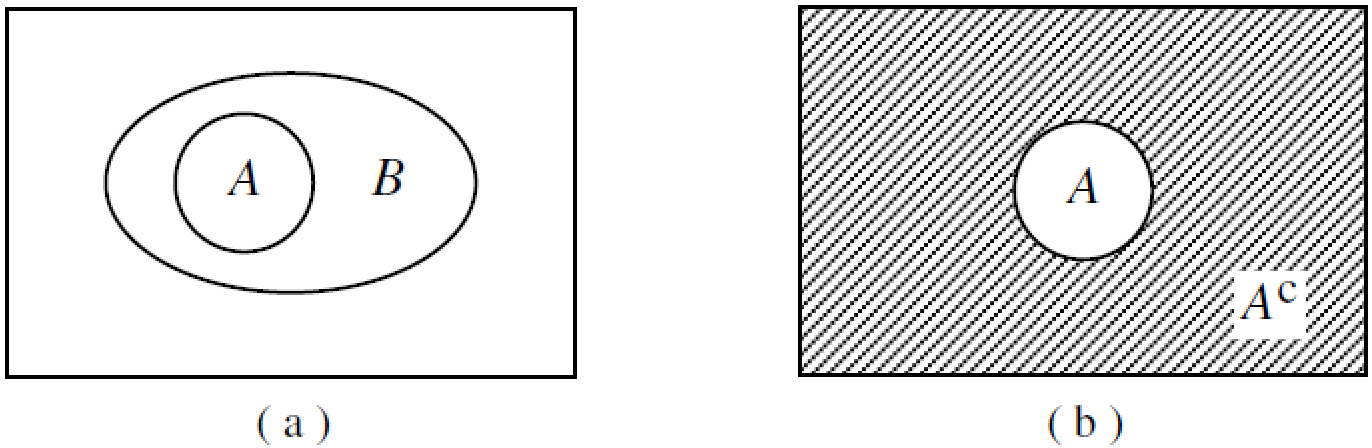


Figure **1.7.** (a) Venn diagram of $A \subset B$. (b) The complement of the disk A, denoted by A^c, is the shaded part of the diagram.

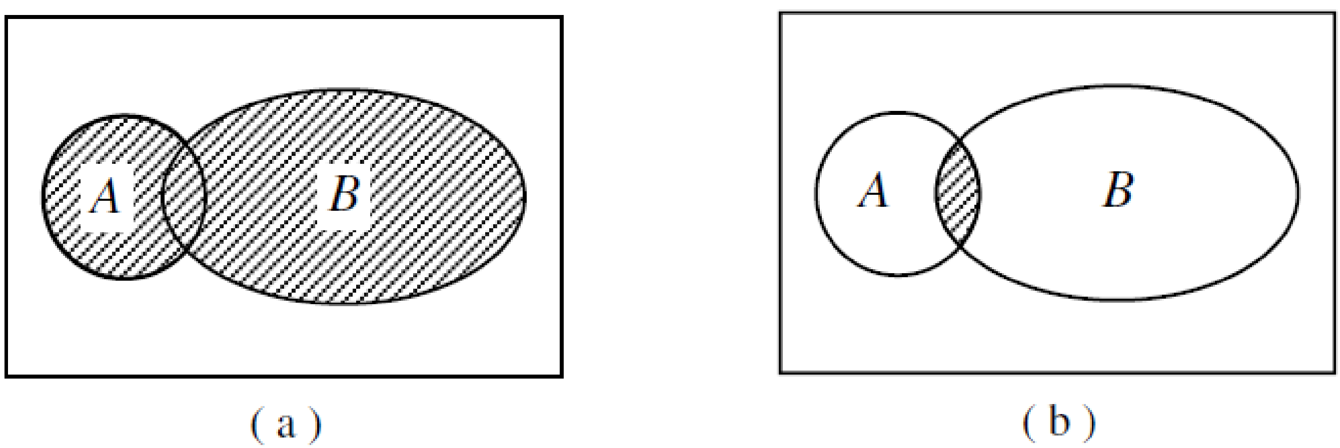


Figure **1.8.** (a) The shaded region is $A \cup B$. (b) The shaded region is $A \cap B$.

Set notations

• convention:
whole set: Ω element: ω

- Let Ω be a set of points. Let A and B be two collections of points in Ω.
  - element/member/point of a set

    $\omega \in \Omega$, ω is an element of Ω, Ω contains ω
  - Subset
    - $A \subset B$. A is contained in B (A is a subset of B). Every element of A is also in B.
    - $A \supset B$ (superset). B is a subset of A (A contains B)
  - Equality

    $A = B$. A equals B. A and B have the same elements.

    $A \subset B$ and $A \supset B$ is a way to prove $A = B$

    $A \subset A$
  - proper subset
    - If $A \subset B$ but $A \neq B$, we say that A is a proper subset of B

Set operations: complement

- If $A \subset \Omega$ and $\omega \in \Omega$ does not belong to A, we write $\omega \notin A$. The set of all such ω is called the **complement** of A in , i.e. $A^c = \{\omega \in \Omega : \omega \notin A\}$
- The empty set or null set contains no points in Ω. It is denoted $\emptyset$
  - for any $A \subset \Omega$, $\emptyset \subset A$
  - $\Omega^c = \emptyset$

Set operations: union

- The **union** of two subsets A and B is

$$A \cup B \triangleq \{\omega \in \Omega\colon \omega \in A \text{ or } \omega \in B\}$$

It is a set contains all elements of A and all elements of B.

  - Here "or" is inclusive; i.e., if $\omega \in A \cup B$, we permit ω to belong either to A or to B or to both.

Set operations: infinite union

- Suppose $A_i \subset \Omega, i = 1,2, \ldots$. Then the **infinite union** is

$$\bigcup_{i=1}^{\infty} A_i \triangleq \{\omega \in \Omega, \omega \in A_i \text{ for some } 1 \leq i < \infty\}$$

$\omega \in \bigcup_{i=1}^{\infty} A_i$ iff for at least one integer i satisfying $1 \leq i < \infty$, $\omega \in A_i$.

  – This definition admits the possibility that $\omega \in A_i$ for more than one value of i.

Set operations: intersection

- The **intersection** of two subsets A and B is

$$A \cap B \triangleq \{\omega \in \Omega: \ \omega \in A \text{ and } \omega \in B\}$$

$\omega \in A \cap B$ iff ω belongs to both A and B.

- Suppose $A_i \subset \Omega, i = 1,2, \ldots$. Then the **infinite intersection** is

$$\bigcap_{i=1}^{\infty} A_i \triangleq \{\omega \in \Omega, \omega \in A_i \text{ for all } 1 \le i < \infty\}$$

$\omega \in \bigcap_{i=1}^{\infty} A_i$ iff for every integer i satisfying $1 \le i < \infty$, $\omega \in A_i$.

Example

- $\bigcap_{n=1}^{\infty}(0,\frac{1}{n}) =?$ ϕ
- $\bigcup_{n=1}^{\infty}\left(\frac{1}{n},2\right] =?$ $(0,2]$

Set operations: difference

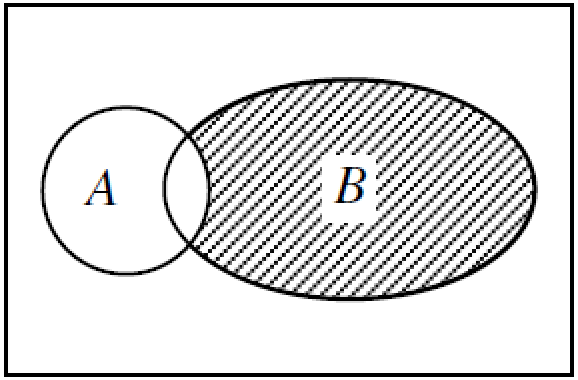
- The **difference** of two subsets A and B is

$$B\backslash \mathrm{A} \triangleq B - A \triangleq B \cap A^c = \{\omega \in \Omega \colon \omega \in B \text{ and } \omega \notin A\}$$

backslash

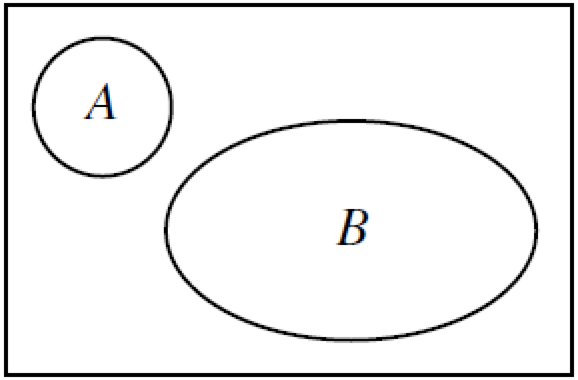
$B \cap A^c$ is a set ω ∈ B that do not A.

- $B\backslash\mathrm{A}$ is found by starting with all the points in B and then removing those that belong to A.



Set operations: disjoint

- Two subsets A and B are **disjoint** or **mutually exclusive** if $A \cap B = \emptyset$, i.e., there is not point in Ω that belongs to both A and B.



- Subsets $A_i \subset \Omega, i = 1,2, \ldots$ are **pairwise disjoint** if

$A_i \cap A_j = \emptyset$, for all $i \neq j$.

Set identities

Let A, B and C be subsets of Ω.

- communicative law

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A$$

- associative law

$$A \cap (B \cap C) = (A \cap B) \cap C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$

- distributive law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set identities

- generalized distributive law

$$B \cap \left(\bigcup_{i=1}^{\infty} A_i \right) = \bigcup_{i=1}^{\infty} (B \cap A_i), \quad B \cap \left(\bigcap_{i=1}^{\infty} A_i \right) = \bigcap_{i=1}^{\infty} (B \cap A_i)$$

- De Morgan's law

$$(A \cup B)^c = (A^c \cap B^c), \qquad (A \cap B)^c = (A^c \cup B^c)$$

- generalized De Morgan's law

$$\left(\bigcap_{i=1}^{\infty} A_i \right)^c = \bigcup_{i=1}^{\infty} A_i^c, \qquad \left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

Countable sets

- **Set size/cardinality** is the number of elements in a set A, denoted by $|A|$.
  - finite: countably finite
  - infinite: countably/uncountably infinite
- A set A is said to be **countable** iff it is either **finite**, or its elements can be **enumerated** or listed in a sequence: $a_1, a_2, \ldots$, i.e., A can be written in the form

$$A = \bigcup_{k=1}^{\infty} \{a_k\}$$

  - In other words, there is a **one-to-one** correspondence between elements of the set and **positive integers**

Uncountable sets

- A set is uncountably infinite if its cardinality is infinite but not countably.

- Example:
  - Real number **R**
  - The interval of real number [0, 1)

Example

- Which of the following sets are countable? Enumerate the countable sets.

① {1,3,5,7,9,…}

② {…,-2,-1,0,1,2,…}

③ {positive rational numbers X: $X=m/n$, m, n are integers, $n \neq 0$}

④ $B \cup C$, where $B = \{b_1, b_2, b_3, \ldots\}$, $C = \{c_1, c_2, c_3, \ldots\}$

b_{mn}

double index

$$\begin{matrix} b_{11} & b_{12} & b_{13} & \cdots \\ b_{21} & b_{22} & b_{23} & \\ b_{31} & b_{32} & & \end{matrix}$$

① if B and C are disjoint

$A = \{a_1, b_1, a_2, b_2, \ldots\}$ countable

Set operations

- **empty set** $\emptyset$
  - $\emptyset \subset A, A \cup \emptyset = A, A \cap \emptyset = \emptyset$
  - $A \cap B = \emptyset$ iff A, B are disjoint.
- **Singleton**
  - $\{X\}$ = singleton set containing only X

    element x.

- **Power set 2^A** of a set A is a set of all subsets of A.
- Example: $2^{\{1,2,3\}} = ?$
  - $\{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\emptyset\}$
- cardinality: $|2^A| = 2^{|A|}$

Partition and Cartesian product

- **Partition** of a set A is a set of sets (called cells or atoms of the partition) $\{B_1, B_2, \dots\}$ s.t. (such that) B_i 's are disjoint and their union is A: $\bigcup_{i=1}^{\infty} B_i = A$
- **Cartesian product**: $A \times B = \{(X, Y) : X \in A, Y \in B\}$
- Example:
  $[0,1] \times [2,3] = \{(X, Y) : 0 \le X \le 1, 2 \le Y \le 3\}$

Function

- A function consists of a set X of inputs called the domain and a rule or mapping f that associates to each $x \in X$ a value $f(x)$ that belongs to a set Y called the co-domain.

$\cup$
range

We write

$$f : X \to Y$$

and say that f maps X into Y.

Range and co-domain

- The set of all possible values of $f(x)$ is called the **range**. It is the set $\{f(x) : x \in X\}$.

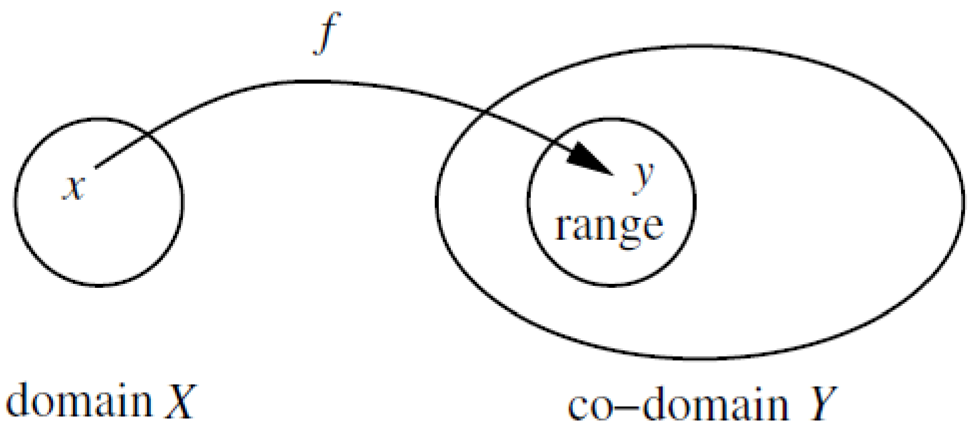
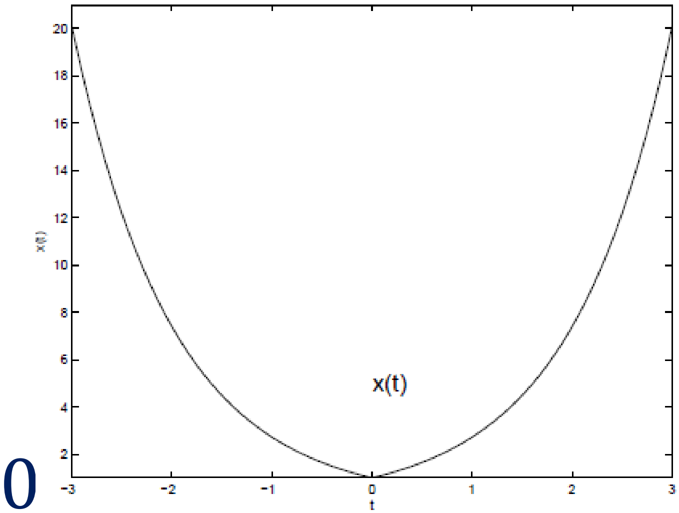


Figure **1.11.** The mapping f associates each x in the domain X to a point y in the co-domain Y. The range is the subset of Y consisting of those y that are associated by f to at least one $x \in X$. In general, the range is a proper subset of the co-domain.

Describing a function

- Graphically :



- Braces or piecewise notation:

$$x(t) = \begin{cases} e^{-t}, t \geq 0 \\ e^{t}, t < 0 \end{cases}$$

- Formula: $x(t) = e^{-|t|}$
- In terms of other functions: $x(t) = s(t) + s(-t)$ where

$$s(t) = \begin{cases} e^{-t}, t > 0 \\ 1/2, t = 0 \\ 0, t < 0 \end{cases}$$

Function classes

- $f(X)$ is **one-to-one** if $f(X_1) \neq f(X_2)$ where $X_1, X_2 \in A$ and $X_1 \neq X_2$

range = co-domain

- $f(X)$ is **onto** if its range equal to its co-domain $f(X) = Y$

- $f(X)$ is **invertible** if it is one-to-one and onto, *i.e.*, for every $y \in Y$ there is a unique $x \in X$ with $f(x) = y$

Image and inverse image

- If $f: X \to Y$ and if $A \subset X$, then the **image** of A is
$$f(A) = \{f(x) : x \in A\}$$

- If $f: X \to Y$ and if $B \subset Y$, then the **inverse image** of B is
could be ϕ. if not invertible
$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

- This concept applies to any function whether or not it is invertible

Monotonicity

- non-decreasing: $f(X_2) \geq f(X_1)$ whenever $X_2 > X_1$
- strictly increasing: $f(X_2) > f(X_1)$whenever $X_2 > X_1$
- non-increasing: $f(X_2) \leq f(X_1)$ whenever $X_2 > X_1$
- strictly decreasing: $f(X_2) < f(X_1)$ whenever $X_2 > X_1$

Continuity

- $f: X \to Y$ (X, Y are intervals of the real line)

- f is **continuous** if

$$x_n \to x \Rightarrow f(x_n) \to f(x)$$

- Equivalently, for all $\epsilon > 0$, there is a $\delta > 0$ $s.t.$

$|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$

Example

for one value in domain only map to one value to co-domain

- For each of the following cases, determine if f is a **valid function** with domain A and co-domain B. For those that are valid functions, determine if they are **one-to-one, onto, continuous, monotonic** (if so, state the type of monotonicity), and find the **inverse image** of the set $(-0.1, 0.2)$

Not valid.

a) $A = [0,1], B = [-1,1], f(x)=\{y \in B: y^2 = x\}, \forall x \in A$

b) $A = [-1,1], B = [-\pi,\pi], f(x)=\{y \in B: \sin y = x\}, \forall x \in A$

not valid $\quad \sin^{-1}(0) = -\pi, \pi$

c) $A = [0,1], B = [-1,1], f(x) = \begin{cases} 1, x \in [\frac{1}{4}, 1] \\ 0, \text{otherwise} \end{cases}, \forall x \in A$

non-decreasing $\quad$ co-domain.

$[0, \frac{1}{4})$

Thank You!