Probability and Random Process

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Outline

- 4. Random Process-II
  - Introduction to Markov Processes
  - Classifications of States and MCs
  - Computing State Probabilities
  - Continuous-time MC
  - Ergodicity Theorems
  - Series Expansions

Computing State Probabilities

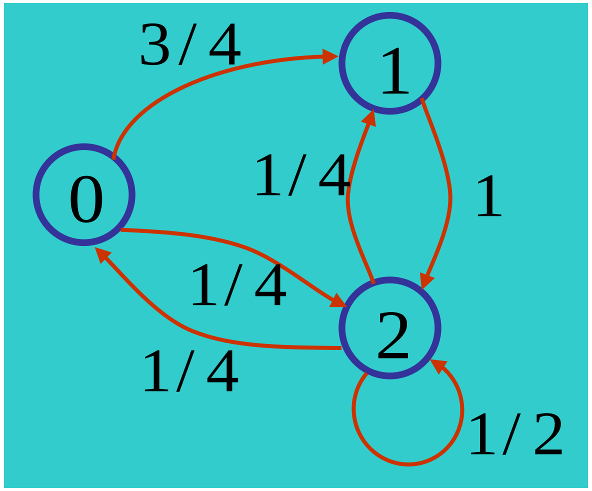
Computing State Probabilities

- When it exists, the steady state distribution (i.e. probability vector) can be found:
  1. Repeated propagation of the state vector using a computer program until it is sufficiently converged
  2. As the eigenvector of P^T corresponding to the unity eigenvalue, normalized by the sum of its elements
  3. Direct solving of the equations:

$$\begin{bmatrix}\pi_0 & \pi_2 & \cdots & \pi_{N-1}\end{bmatrix} = \begin{bmatrix}\pi_0 & \pi_2 & \cdots & \pi_{N-1}\end{bmatrix}\mathbf{\Pi}$$

and $$\sum_{i=0}^{N-1} \pi_i = \mathbf{1}$$

Example



$$\mathbf{\Pi} = \begin{bmatrix} P_{00} & P_{01} & P_{02} \\ P_{10} & P_{11} & P_{12} \\ P_{20} & P_{21} & P_{22} \end{bmatrix} = \begin{bmatrix} 0 & 3/4 & 1/4 \\ 0 & 0 & 1 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

- All pairs of states communicate, therefore this MC is irreducible.
- It is aperiodic (because $p_{22} > 0$).
- Because the state space is finite, the states must be recurrent non-null.

⇨ Therefore, the steady state probability vector exists.

Example, Cont'd

$$\begin{bmatrix}\pi_0 & \pi_1 & \pi_2\end{bmatrix} = \begin{bmatrix}\pi_0 & \pi_1 & \pi_2\end{bmatrix}\begin{bmatrix} 0 & 3/4 & 1/4 \\ 0 & 0 & 1 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$\pi_0 = \frac{\pi_2}{4} \qquad (1)$$

$$\pi_1 = \frac{3}{4}\pi_0 + \frac{\pi_2}{4} \qquad (2)$$

$$\pi_2 = \frac{\pi_0}{4} + \pi_1 + \frac{\pi_2}{2} \qquad (3)$$

$$\pi_0 + \pi_1 + \pi_2 = 1 \qquad (4)$$

Substitute (1) into (2) and (4):

$$\pi_1 = \frac{3}{4}\pi_0 + \pi_0 \qquad (5)$$

$$\pi_0 + \pi_1 + 4\pi_0 = 1 \qquad (6)$$

Substitute (5) into (6):

$$\frac{7}{4}\pi_0 + 5\pi_0 = 1 \Rightarrow \pi_0 = \frac{4}{27}$$

Example, Cont'd

Now, substitute back into (5): $$\pi_1 = \frac{7}{27}$$

Now use (4) to get: $$\pi_2 = 1 - \frac{(4+7)}{27} = \frac{16}{27}$$

$$\therefore \quad \boldsymbol{\pi}^{(\infty)} = \left[\frac{4}{27}, \frac{7}{27}, \frac{16}{27}\right]$$

Calculating Non-Steady State Distributions

- Suppose we want $p^{(n)}$ for some very large, but finite, n?
- We can calculate $p^{(n)}$ directly in closed form using Z-transforms.
- Take Z-transform of both sides of $\boldsymbol{\pi}^{(n+1)} = \boldsymbol{\pi}^{(n)}\boldsymbol{\Pi}$

$$Z\left\{\boldsymbol{\pi}^{(n+1)}\right\} = Z\left\{\boldsymbol{\pi}^{(n)}\boldsymbol{\Pi}\right\}$$

Let $G(z) = Z\left\{\boldsymbol{\pi}^{(n)}\right\}$

What is $Z\left\{p^{(n+1)}\right\}$?

Z-Transform Details

$$Z\left[\boldsymbol{\pi}^{(n+1)}\right]=\sum_{n=0}^{\infty}\boldsymbol{\pi}^{(n+1)}z^{-n} = z\sum_{n=0}^{\infty}\boldsymbol{\pi}^{(n+1)}z^{-(n+1)}$$

Let $k = n + 1$, $\quad = z\sum_{k=1}^{\infty}\boldsymbol{\pi}^{(k)}z^{-(k)} = z\left[G(z) - \boldsymbol{\pi}^{(0)}\right]$

So $\quad Z\left[\boldsymbol{\pi}^{(n+1)}\right]= Z\left[\boldsymbol{\pi}^{(n)}\boldsymbol{\Pi}\right]$

$$z\left[G(z) - \boldsymbol{\pi}^{(0)}\right]= G(z)\boldsymbol{\Pi}$$

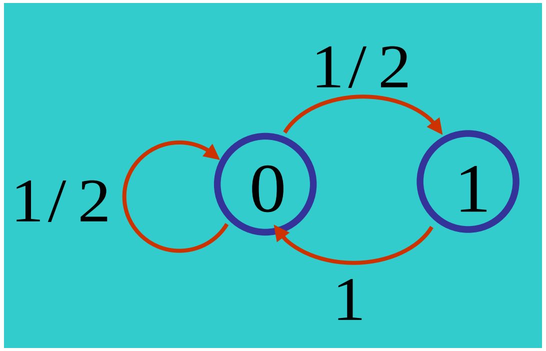
$$G(z)\left[z\mathbf{I} - \boldsymbol{\Pi}\right]= z\boldsymbol{\pi}^{(0)}$$

$$G(z)= z\boldsymbol{\pi}^{(0)}\left[z\mathbf{I} - \boldsymbol{\Pi}\right]^{-1}$$

Then inverse Z-transform is performed on G(z) for \pi$^{(n)}$

Example

$$\mathbf{\Pi} = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$



Suppose $\boldsymbol{\pi}^{(0)} = \begin{bmatrix} 1/4 & 3/4 \end{bmatrix}$

We want an expression for $p^{(n)}$

$$G(z) = z\begin{bmatrix} 1/4 & 3/4 \end{bmatrix}\left[z\mathbf{I} - \mathbf{\Pi}\right]^{-1}$$

$$\left[z\mathbf{I} - \mathbf{\Pi}\right]^{-1} = \left[\begin{bmatrix} z & 0 \\ 0 & z \end{bmatrix} - \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}\right]^{-1} = \begin{bmatrix} z-1/2 & -1/2 \\ -1 & z \end{bmatrix}^{-1}$$

$$= \frac{\begin{bmatrix} z & 1/2 \\ 1 & z-1/2 \end{bmatrix}}{z(z-1/2)-1/2} = \frac{\begin{bmatrix} z & 1/2 \\ 1 & z-1/2 \end{bmatrix}}{(z-1)(z+1/2)}$$

Example, Cont'd

Partial Fraction Expansion on each matrix term yields:

$$[z\mathbf{I}-\mathbf{\Pi}]^{-1}=\begin{bmatrix}\left(\dfrac{2/3}{z-1}+\dfrac{1/3}{z+1/2}\right) & \left(\dfrac{1/3}{z-1}-\dfrac{1/3}{z+1/2}\right)\\ \left(\dfrac{2/3}{z-1}-\dfrac{2/3}{z+1/2}\right) & \left(\dfrac{1/3}{z-1}+\dfrac{2/3}{z+1/2}\right)\end{bmatrix}$$

$$G(z)=z[1/4 \quad 3/4][z\mathbf{I}-\mathbf{\Pi}]^{-1}$$

$$=\left[\left(\frac{2z/3}{z-1}-\frac{5z/12}{z+1/2}\right)\quad\left(\frac{z/3}{z-1}+\frac{5z/2}{z+1/2}\right)\right]$$

$$\boldsymbol{\pi}^{(n)}=\left[\frac{2}{3}-\left(-\frac{1}{2}\right)^{n}\frac{5}{12},\quad\frac{1}{3}+\left(-\frac{1}{2}\right)^{n}\frac{5}{12}\right]$$

Partial Fraction Details

$$\frac{H(z)}{(z-1)(z+1/2)} = \frac{A}{z-1} - \frac{B}{z+1/2} = \frac{zA + A/2 + Bz - B}{(z-1)(z+1/2)}$$

For $H(z) = z \Rightarrow \left.\begin{matrix} A+B=1 \\ A/2-B=0 \end{matrix}\right\}$ $A=\frac{2}{3},\quad B=\frac{1}{3}$

For $H(z) = 1/2 \Rightarrow \left.\begin{matrix} A+B=0 \\ A/2-B=1/2 \end{matrix}\right\}$ $A=\frac{1}{3},\quad B=-\frac{1}{3}$

For $H(z) = 1 \Rightarrow \left.\begin{matrix} A+B=0 \\ A/2-B=1 \end{matrix}\right\}$ $A=\frac{2}{3},\quad B=-\frac{2}{3}$

For $H(z) = z-1/2 \Rightarrow \left.\begin{matrix} A+B=1 \\ A/2-B=-1/2 \end{matrix}\right\}$ $A=\frac{1}{3},\quad B=\frac{2}{3}$

Example, Cont'd

$$\boldsymbol{\pi}^{(n)} = \left[\frac{2}{3} - \left(-\frac{1}{2}\right)^n \frac{5}{12}, \quad \frac{1}{3} + \left(-\frac{1}{2}\right)^n \frac{5}{12} \right]$$

Check: $\boldsymbol{\pi}^{(0)} = \left[\frac{2}{3} - \frac{5}{12}, \quad \frac{1}{3} + \frac{5}{12} \right] = \left[\frac{3}{12}, \quad \frac{9}{12} \right]$ ✓

$$\boldsymbol{\pi}^{(1)} = \left[\frac{2}{3} + \frac{5}{24}, \quad \frac{1}{3} - \frac{5}{24} \right] = \left[\frac{7}{8}, \quad \frac{1}{8} \right]$$

$$= \boldsymbol{\pi}^{(0)} \boldsymbol{\Pi} = \left[\frac{1}{4}, \quad \frac{3}{4} \right] \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix} = \left[\frac{7}{8}, \quad \frac{1}{8} \right]$$ ✓

Summary

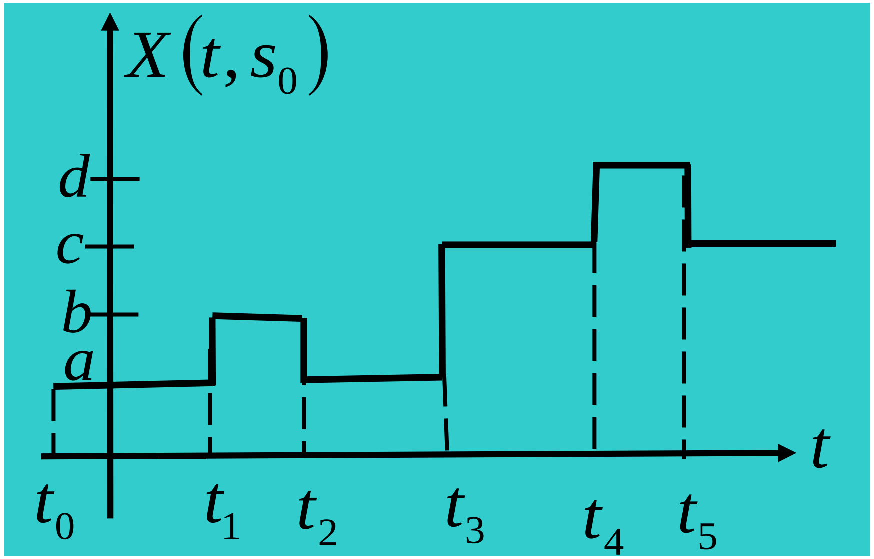
- Compute steady state distribution
  - Check if steady state probability vector exists
- Calculate non-steady state distribution
  - By using the Z-transform

Continuous-time MC

Continuous-Time Discrete-State

- In a continuous-time discrete-state MC, or CTMC, a transition can occur *at any instant of time*.
- Instead of a probability transition matrix, as in discrete-time MCs, there is a transition rate matrix.
- A CTMC can be specified by its underlying point process (the list of transition times) and the embedded MC (the list of values corresponding to the transitions).

Example



For this realization, the point process is:

$$\{\ldots, t_0, t_1, t_2, t_3, t_4, \ldots\}$$

And the embedded MC is:

$$\{\ldots, a, b, a, c, d, \ldots\}$$

A Stepping Stone to Rates: Transition Probabilities

- A CTMC, X(t), may also be described by its initial state probabilities

$$p_i(t_0) = P(X(t_0) = a_i)$$

Where a_i is the value associated with state i, and its transition probabilities

$$\pi_{ij}(t_1, t_2) = P(X(t_2) = a_j \mid X(t_1) = a_i)$$

- As in the DTMC case, these quantities satisfy:

$$\sum_j \pi_{ij}(t_1, t_2) = 1, \quad \sum_i p_i(t_1)\pi_{ij}(t_1, t_2) = p_j(t_2)$$

and the Chapman-Kolmogorov equations

$$\pi_{ij}(t_1, t_3) = \sum_r \pi_{ir}(t_1, t_2)\pi_{rj}(t_2, t_3), \quad t_1 < t_2 < t_3$$

Homogenous CTMCs

The CTMC is homogenous if its transition probabilities depend only on the difference $\tau = t_2 - t_1$:

$$\pi_{ij}(\tau) = P\left(X(t+\tau) = a_j \mid X(t) = a_i\right)$$

The Chapman-Kolmogorov equations become

$$\pi_{ij}(\tau+\alpha) = \sum_r \pi_{ir}(\tau)\pi_{rj}(\alpha)$$

or in matrix form:

$$\mathbf{\Pi}(\tau+\alpha) = \mathbf{\Pi}(\tau)\mathbf{\Pi}(\alpha)$$

And the state probability vector, $\mathbf{p}(t)$, satisfies:

$$p(\tau+t) = p(t)\mathbf{\Pi}(\tau)$$

Transition Rates

- The derivative-from-the-right of the transition probability matrix

$$\boldsymbol{\Lambda} = \frac{d}{d\tau}\boldsymbol{\Pi}(\tau)\Big|_{\tau=0^+} = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{N1} \\ \lambda_{11} & \ddots & & \vdots \\ \vdots & & & \\ \lambda_{N1} & \cdots & & \lambda_{NN} \end{bmatrix}$$

  is the transition rate matrix.
- λ_{ij} indicates the rate, in terms of probability mass per unit time, that probability mass moves out of state i and into state j (λ_{ij} can be < 0).

Conservation of Mass

- If probability mass is not flowing out of a state, then it must be staying in
- Recall $\sum_j \pi_{ij}(\tau) = 1$
- Differentiation of this yields $\sum_j \lambda_{ij} = 0$
- $-\lambda_{ii}$ is the rate that mass moves out of state i. It does not include mass coming in from other states.

$$-\lambda_{ii} = \sum_{j \neq i} \lambda_{ij}$$

Inter-transition Times

- All Markov processes share the interesting property that the time it takes for a change of state is an exponentially distributed random variable

- All CTMCs stay in given state for an exponentially distributed period of time, with parameter $- /_{ii}$

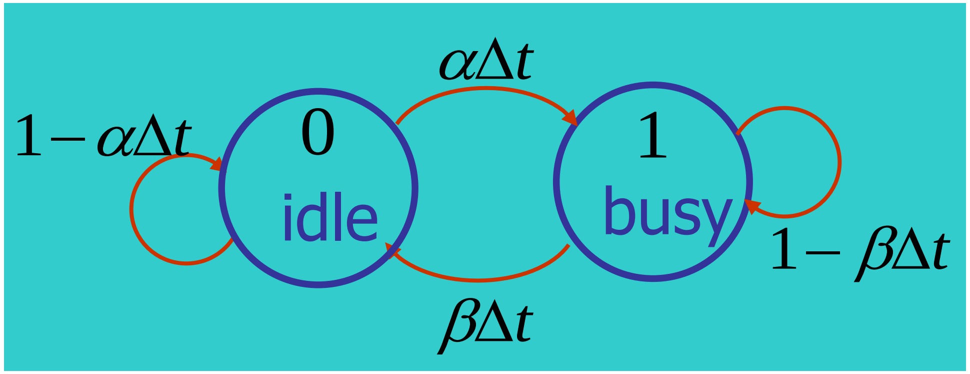
Global Balance Equations

- In steady state, rate of flow into a state must equal the rate of flow out of a state.
- Consider state j over a very small interval Dt :

$$\left.\begin{array}{ll}\text{Into j:} & \\ & \displaystyle\sum_{\substack{i\\ j\neq i}} \Delta t\lambda_{ij}\rho_i \\ \text{Out of j:} & \displaystyle\sum_{\substack{i\\ j\neq i}} \Delta t\lambda_{ji}\rho_j \end{array}\right\} \quad \sum_{\substack{i\\ j\neq i}} \Delta t\lambda_{ij}\rho_i = \rho_j \sum_{\substack{i\\ j\neq i}} \Delta t\lambda_{ji}$$

CTMC Example

Simple Queuing System [Ex. 8.15 from Kleinrock]



Global Balance Equations

$$\begin{bmatrix} \lambda_{00} & \lambda_{01} \\ \lambda_{10} & \lambda_{11} \end{bmatrix} = \begin{bmatrix} -\alpha & \alpha \\ \beta & -\beta \end{bmatrix}$$

GBE, repeated:

$$\sum_{\substack{i \\ j \neq i}} \Delta t \lambda_{ij} \rho_i = \rho_j \sum_{\substack{i \\ j \neq i}} \Delta t \lambda_{ji}$$

$$j = 0: \quad \lambda_{10}\rho_1 = \rho_0 \lambda_{01} \quad \Rightarrow \beta\rho_1 = \rho_0\alpha \quad (1)$$

$$j = 1: \quad \lambda_{01}\rho_0 = \rho_1 \lambda_{10} \quad \Rightarrow$$ Same (no help)

Can use: $\rho_0 + \rho_1 = 1 \quad (2)$

Eliminate r_1 by substituting $r_1 = 1 - r_0$

$$\beta(1 - \rho_0) = \rho_0\alpha \quad \Rightarrow \beta = (\alpha + \beta)\rho_0 \quad \Rightarrow \begin{cases} \rho_0 = \dfrac{\beta}{\alpha + \beta} \\ \rho_1 = \dfrac{\alpha}{\alpha + \beta} \end{cases}$$

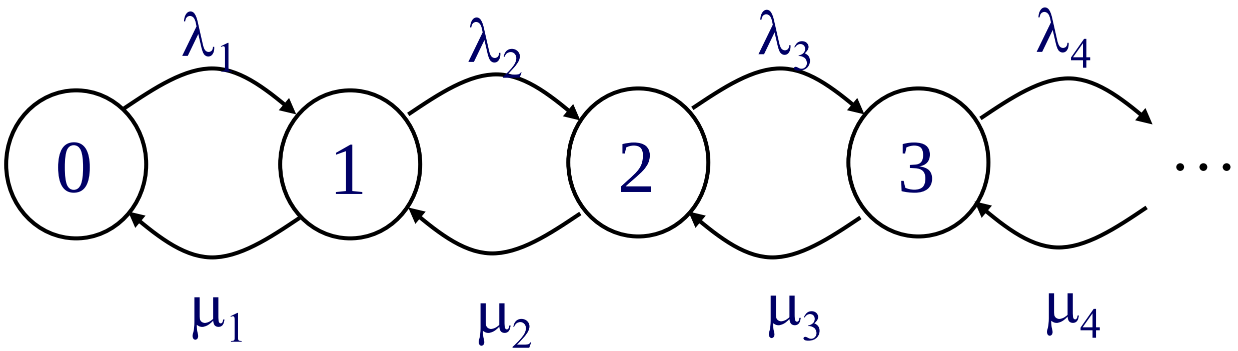
Continuous-Time Discrete-State Examples

- Birth and Death Process
- M/M/1 Single-Server Queue
- M/M/c Multi-Server Queue

Birth and Death Process

- The birth–death process can go only from state n to state n-1 or n+1 in one transition
  - The state transitions are of only two types: "birth", which increases the state variable by one and "death", which decreases the state by one

- Graphical interpretation: any process with the following state diagram



GBE for Birth-Death

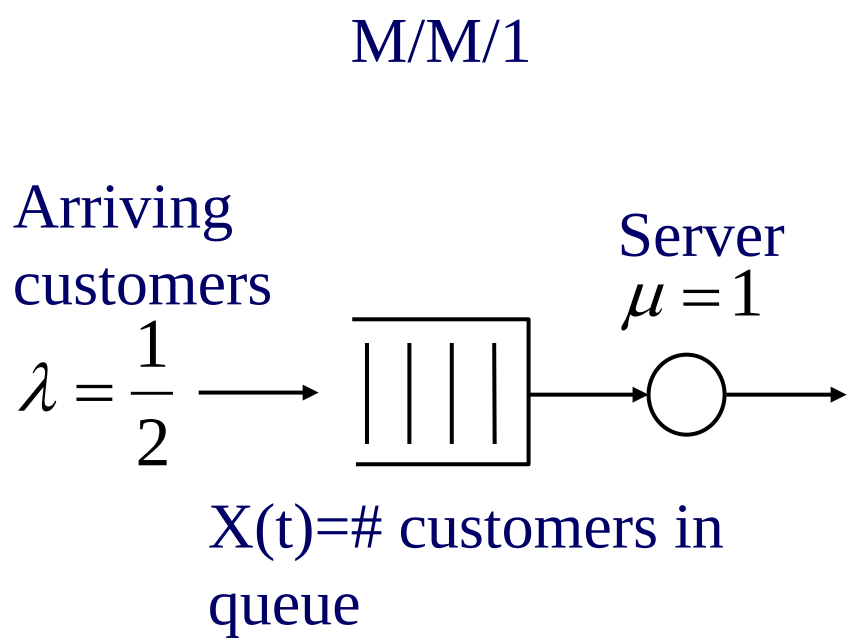
$$p_j = r_j p_{j-1} \quad \text{and} \quad p_j = \prod_{k=1}^{j} r_k p_0, \quad \text{with} \quad r_k = \frac{\lambda_k}{\mu_k}$$

$$\text{Let } \mathrm{R}_{\mathrm{j}} = \prod_{\mathrm{k}=1}^{\mathrm{j}} r_k \text{ then the condition } \sum_{\mathrm{j}=0}^{\infty} p_j = 1 \text{ yields}$$

$$p_0 = \frac{1}{\sum_{\mathrm{j}=0}^{\infty} R_j} \quad \text{and} \quad p_j = \frac{R_j}{\sum_{\mathrm{j}=0}^{\infty} R_j}$$

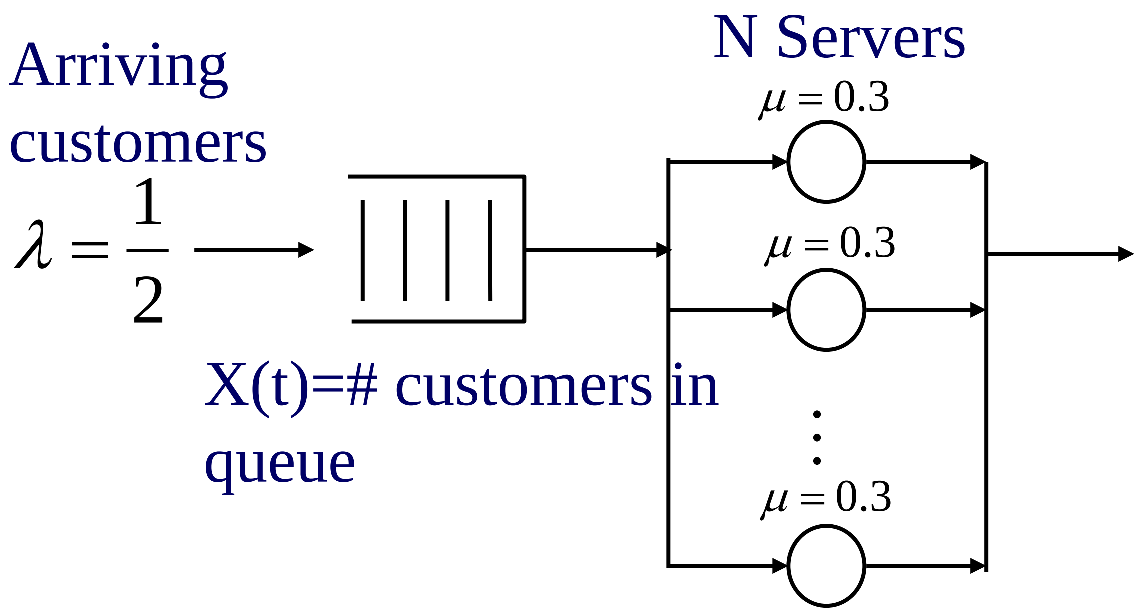
Queue Notation

- a/b/c
  - a indicates customer arrival distribution. M=Poisson
  - b indicates distribution of service times. M=Exponential
  - c=Number of servers



Multiple Servers

M/M/N



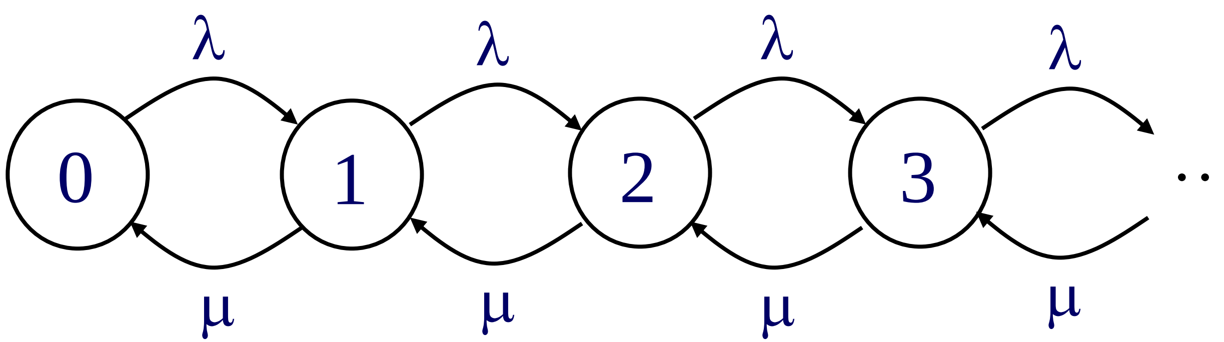
M/M/1 Queue

- Arrivals occur according to a Poisson process with average rate λ.
- Inter-arrivals are iid exponential with mean $1/\lambda$.
- Customers are served, one at a time, with iid exponentially distributed service times with parameter μ

$$f_{\tau_s}(t) = \begin{cases} \mu e^{-\mu t} & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad E\{\tau_s\} = \frac{1}{\mu}$$

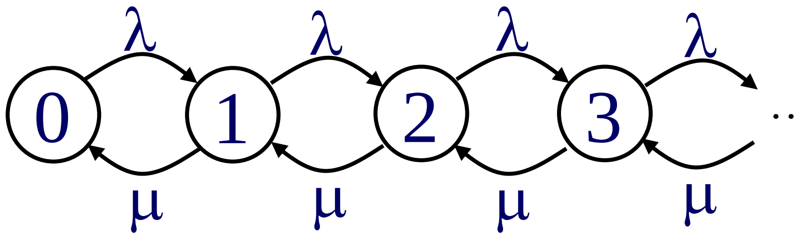
M/M/1 State Diagram

- State space: the number of customers in the system
  - Including any currently in service



M/M/1 is a special case of birth-death process, with the identical arriving and serving rates

Global Balance Equations for M/M/1



$$\text{State } 0: \quad p_0\lambda = p_1\mu$$

$$\text{Can rewrite as}: \quad p_0\lambda - p_1\mu = 0$$

$$\text{State } 1: \quad p_1\lambda + p_1\mu = p_0\lambda + p_2\mu$$

$$\text{Can rewrite as}: \quad p_1\lambda - p_2\mu = p_0\lambda - p_1\mu = 0$$

$$\text{State j}: \quad p_j\lambda + p_j\mu = p_{j-1}\lambda + p_{j+1}\mu$$

$$\text{Can rewrite as}: \quad p_j\lambda - p_{j+1}\mu = p_{j-1}\lambda - p_j\mu = 0$$

GBE Concluded

$$p_{j-1}\lambda - p_j\mu = 0 \qquad \text{implies} \qquad p_j = \frac{\lambda}{\mu} p_{j-1}$$

$$\text{or} \qquad p_j = \left(\frac{\lambda}{\mu}\right)^j p_0$$

$$\text{Thus, requiring} \qquad \sum_{j=0}^{\infty} p_j = \sum_{j=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^j p_0 = 1$$

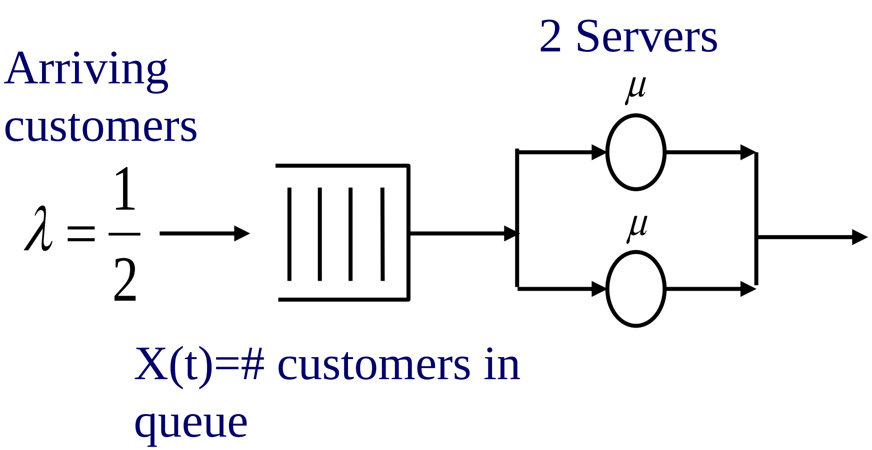
$$\text{yields} \qquad p_0 = 1 - \frac{\lambda}{\mu} \qquad \text{only if} \qquad \frac{\lambda}{\mu} < 1$$

Must be able to serve them at a higher rate than they arrive

M/M/c: Multi-server Queue

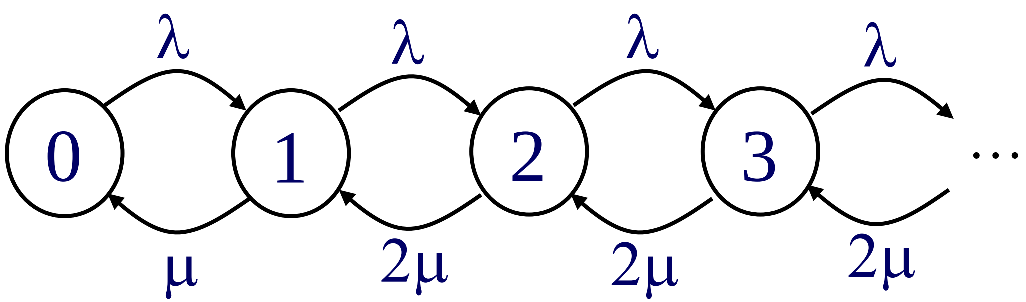
- Customer is served by first available server
- Let τ_i=time until server i is available
- Then time until next server is available is X=min(τ_1 ,τ_2,…, τ_n)

M/M/2



M/M/c State Diagram

- Since each of the servers works at rate μ, the total departure rate will be 2μ
- The state diagram becomes



Summary

- Continuous-time Markov Chains (CTMCs) can change state at any time.
  - The time between entering state i and leaving it is exponentially distributed with parameter $- /_{ii}$

- Global Balance Equations equalize mass flow into and out of a state, assuming steady state conditions.

- Birth-death processes
  - M/M/c queues

Thank You!