Probability and Random Process

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Outline

- 4. Random Process
  - Introduction to Random Processes
  - Brownian Motion
  - Poisson Process
  - Complex RV and RP
  - Stationarity
  - PSD, QAM, White Noise
  - Response of Systems
  - LTI Systems and RPs

PSD, QAM, White Noise

Power Spectral Density

Used to indicate frequency content of a RP

The power spectral density (PSD) of a WSS RP (real or complex) is the Fourier Transform of its autocorrelation function:

$R_X(\tau)$

the only arguement

$$S_X(\omega) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-j\omega\tau} d\tau$$

inverse Fourier function

$$R_X(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_X(\omega) e^{j\omega\tau} d\omega$$

Symmetry

frequency

$$S_X(\omega) = \int_{-\infty}^{+\infty} R_X(\tau) e^{-j\omega\tau}\, d\tau$$

$\omega = 2\pi f$

欧拉 ω

even, odd, $=0$

$$= \int_{-\infty}^{+\infty} R_X^e(\tau)\cos(\omega\tau)\, d\tau + \int_{-\infty}^{+\infty} R_X^o(\tau)\underbrace{\cos(\omega\tau)\, d\tau}_{=0}$$

odd, $=0$

$$- j\int_{-\infty}^{+\infty} R_X^e(\tau)\underbrace{\sin(\omega\tau)\, d\tau}_{=0} - j\int_{-\infty}^{+\infty} R_X^o(\tau)\sin(\omega\tau)\, d\tau$$

odd, odd, even

$$= \int_{-\infty}^{+\infty} R_X^e(\tau)\cos(\omega\tau)\, d\tau - j\int_{-\infty}^{+\infty} R_X^o(\tau)\sin(\omega\tau)\, d\tau$$

imaginary part

Since $R_X(\tau)$ has Hermitian symmetry, $S_X(\omega)$ is real.

$R_X(\tau) = R^*(-\tau)$ ✓

Real part real

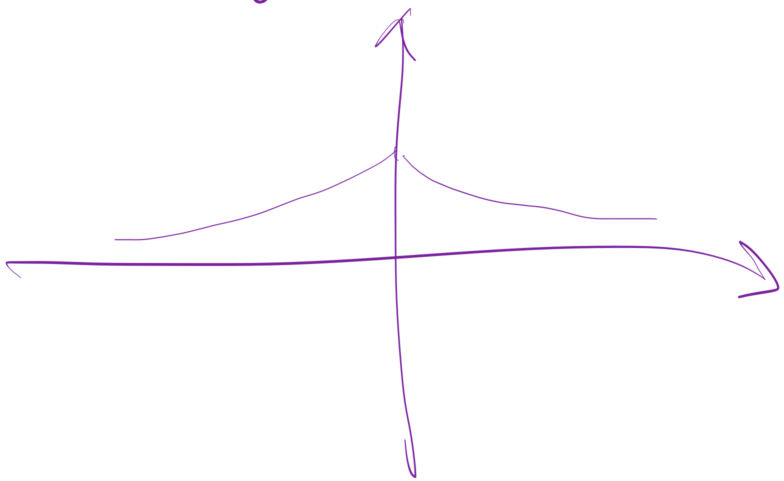
Image part odd

Symmetry

If X(t) is real, $R_X(\tau)$ is real and even

$S_X(\omega)$ is real and even

usually



EE Interpretation

If $X(t)$ is a WSS voltage waveform with PSD $S_X(\omega)$, then the average power (Watts) of $X(t)$ in the frequency band is

$$P_{avg}[\omega_a, \omega_b] = \frac{1}{2\pi}\int_{-\omega_b}^{-\omega_a} S_X(\omega)d\omega + \frac{1}{2\pi}\int_{\omega_a}^{\omega_b} S_X(\omega)d\omega$$

Total average power:

WSS, no matter t, E{x²} is the same

$$E\left\{|X(t)|^2\right\} = R_X(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_X(\omega)d\omega$$

$E[X(t)\cdot X^*(t)]$

important

Radians/sec and Hz

$S_X(\omega)$ has the unit of Watts/Hz, regardless of its expression as a function of ω or f.

Recall the avg power expression:

$$P_{avg}[\omega_a,\omega_b] = \frac{1}{2\pi}\int_{-\omega_b}^{-\omega_a} S_X(\omega)d\omega + \frac{1}{2\pi}\int_{\omega_a}^{\omega_b} S_X(\omega)d\omega$$

Change variables $f = \dfrac{\omega}{2\pi}$ and let $S_X^{(Hz)}(f) = S_X^{(Rad)}(2\pi f)$

$$P_{avg}[\omega_a,\omega_b] = \int_{-f_b}^{-f_a} S_X(f)df + \int_{f_a}^{f_b} S_X(f)df$$

EXCEPTION: impulses: $\delta^{(Rad)}(\omega) = \dfrac{1}{2\pi}\delta^{(Hz)}(f)$

期末 cttp 要 Fourier table
几何公式也要用 例 $\cos(A+B) = \cdots$

Random Phase Sinusoid

this is WSS

$$X(t) = \sin(\omega_0 t + \theta) \quad \theta \sim U[-\pi, \pi]$$

$R_X(\tau) = E[X(t)\, X(t-\tau)]$

$= E[\sin(\omega_0 t + \theta)\sin(\omega_0(t-\tau)+\theta)]$

$= \dfrac{e^{j(\omega_0 t+\theta)} - e^{-j(\omega_0 t+\theta)}}{2j} \cdot \dfrac{e^{j(\omega_0 t+\theta-\omega_0\tau)} - e^{-j(\omega_0 t+\theta-\omega_0\tau)}}{2j}$

$= \dfrac{e^{j2(\omega_0 t+\theta)-j\omega_0\tau} - e^{-j2(\omega_0 t+\theta)+j\omega_0\tau}}{-4}$

$= -\dfrac{\cos(2\omega_0 t+2\theta-\omega_0\tau)}{2}$

$= \frac{1}{2}\cos(2\omega_0 t+2\theta-\omega_0\tau)$

expectation ω

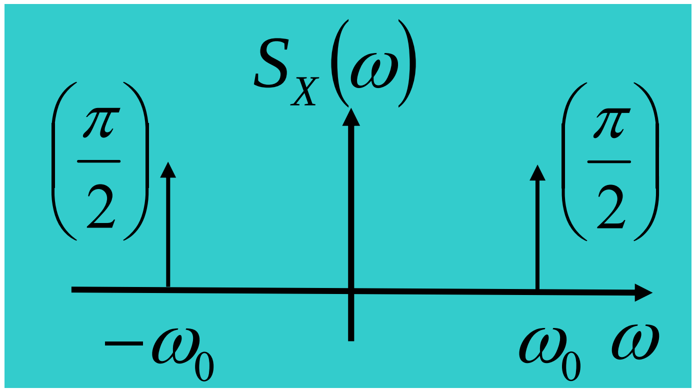
① calculate this

$$R_X(\tau) = \frac{1}{2}\cos(\omega_0\tau) \rightarrow ② \; S_X(\omega) = \Im\{R_X(\tau)\}$$

$$\cos(\omega_0\tau) \leftrightarrow \pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)$$

③

$$\therefore S_X(\omega) = \frac{\pi}{2}\delta(\omega-\omega_0) + \frac{\pi}{2}\delta(\omega+\omega_0)$$



In Hz domain,

$f = \frac{\omega}{2\pi}$

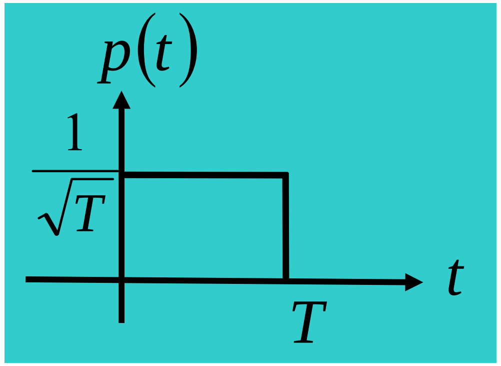
④

$$S_X(f) = \frac{1}{4}\delta(f-f_0) + \frac{1}{4}\delta(f+f_0)$$

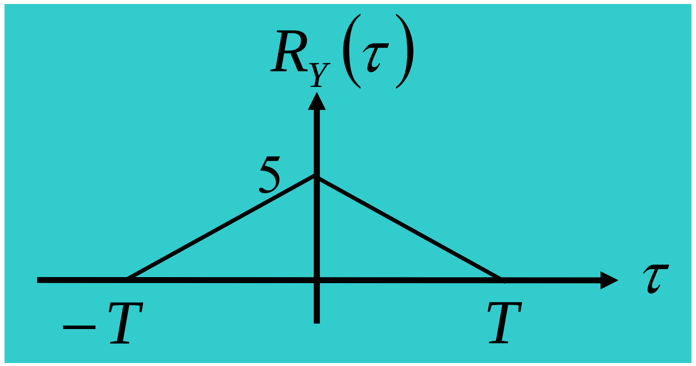
Baseband 4-level PAM Example

$$Y(t)=\sum_{i=-\infty}^{+\infty} A_i\, p(t-iT+\theta)$$

RV. pulse



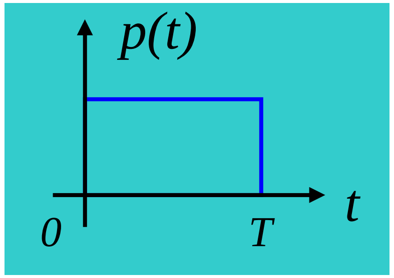
Unit energy pulse



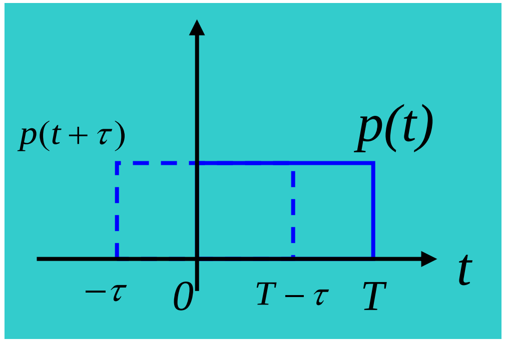
$$R_Y(\tau)=\begin{cases}\frac{5}{T}\left(T-|\tau|\right) & |\tau|\le T\\ 0 & \text{o.w.}\end{cases}$$

PAM Example

If $p(t)$ is as shown on the right, then only the $m=0$ term contributes to the integral.



For $\tau > 0$



$$C_Y(\tau) = \frac{5}{T}\int_0^T p(t)p(t+\tau)dt = \frac{5}{T}(T-\tau)$$

Since $C_Y(\tau)$ is even,
$$C_Y(\tau) = \begin{cases} \frac{5}{T}(T-|\tau|) & |\tau| \le T \\ 0 & o.w. \end{cases}$$

4-level PAM, Concluded

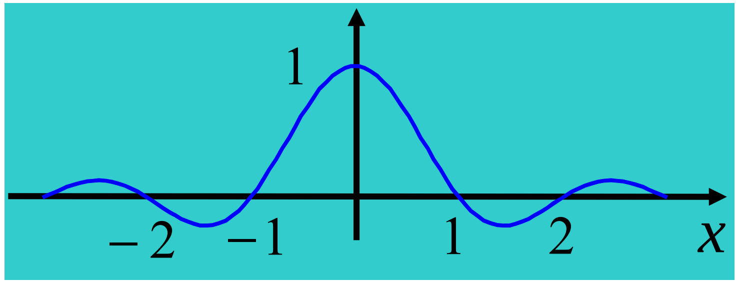
Mathmatics Book Definition:
$\mathrm{sinc}(x) = \frac{\sin x}{x}$

Engineering Book:
$\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$

From F.T. table, $S_Y(\omega) = 5T\left[\frac{\sin(\omega T/2)}{\omega T/2}\right]^2$

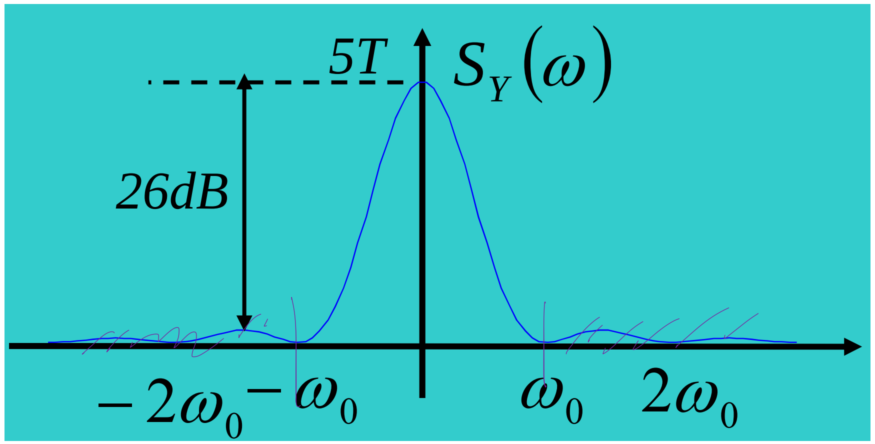
Definition of sinc function:

$$\sin c(x) = \frac{\sin(\pi x)}{\pi x}$$



$$\therefore S_Y(\omega) = 5T\mathrm{sinc}^2\left(\frac{\omega T}{2\pi}\right)$$

Let $\omega_0 = \frac{2\pi}{T}$,



Summary (1)

The PSD indicates frequency content of a RP

The PSD is the Fourier transform of the autocorrelation function of a WSS RP

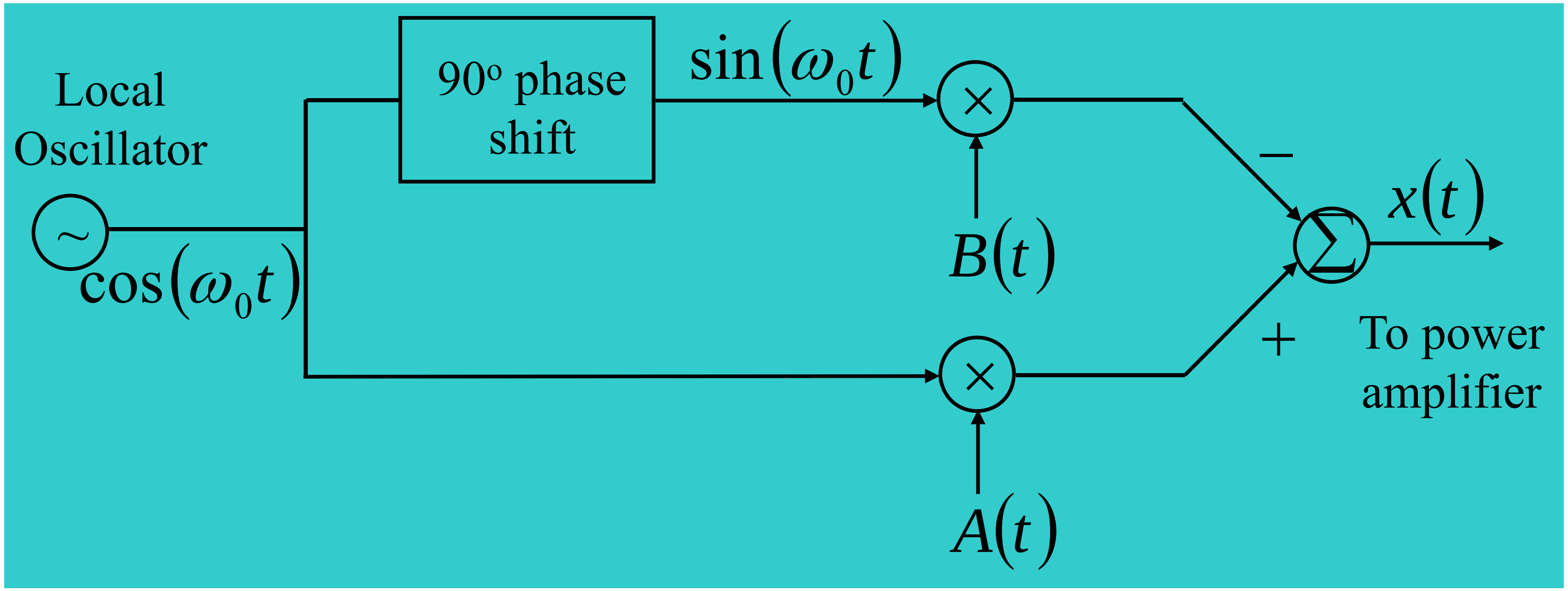
The PSD is integrated to get average power

If cyclo : Take average
then Fourier Transformation

Quadrature Amplitude Modulation (QAM)

Extremely popular in digital communication systems because of its high bandwidth efficiency (the number of bits that can be expressed in a band-limited waveform).

QAM Modulator:



Definition

Let A(t) and B(t) be real, jointly WSS RPs with zero mean, and let ω_0 be the carrier frequency in radians/sec. Then the QAM-modulated signal X(t) is defined as

$$X(t) = \underset{\text{RV}}{A(t)}\cos(\omega_0 t) - \underset{\text{RV}}{B(t)}\sin(\omega_0 t)$$

This RP is WSS as long as

Here assump: A, B joint WSS

$$R_A(\tau) = R_B(\tau) \quad \text{and} \quad R_{AB}(\tau) = -R_{BA}(\tau)$$

where $R_{AB}(\tau) = E\left\{A(t+\tau)B^*(t)\right\}$

Verify WSS

- Mean is constant
- Autocorrelation/autocovariance is constant
  $R_X(t_1, t_2) = R_X(\tau)$. where $\tau = t_1 - t_2$.

Proof that X(t) is WSS

$$R_X(\tau) = E\{X(t+\tau)X^*(t)\}$$
$$= E\{\,[A(t+\tau)\cos(\omega_0(t+\tau)) - B(t+\tau)\sin(\omega_0(t+\tau))]$$
$$\cdot [A(t)\cos(\omega_0 t) - B(t)\sin(\omega_0 t)]\,\}$$

$$= R_A(\tau)\cos(\omega_0(t+\tau))\cos(\omega_0 t) - R_{AB}(\tau)\cos(\omega_0(t+\tau))\sin(\omega_0 t)$$
$$- R_{BA}(\tau)\sin(\omega_0(t+\tau))\cos(\omega_0 t) + R_B(\tau)\sin(\omega_0(t+\tau))\sin(\omega_0 t)$$

since $R_A(\tau) = R_B(\tau)$

$$= R_A(\tau)\cos(\omega_0\tau) + R_{AB}(\tau)\sin(\omega_0\tau)$$

$R_{AB}(\tau) = -R_{BA}(\tau)$

PSD

$$S_X(\omega) = \Im\{R_A(\tau)\cos(\omega_0\tau) + R_{AB}(\tau)\sin(\omega_0\tau)\}$$

$$= \frac{1}{2\pi}\begin{bmatrix} S_A(\omega) * [\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0)] \\ +S_{AB}(\omega) * \left[\frac{\pi}{j}\delta(\omega-\omega_0) - \frac{\pi}{j}\delta(\omega+\omega_0)\right] \end{bmatrix}$$

where $S_{AB}(\omega) = \int_{-\infty}^{+\infty} R_{AB}(\tau) e^{-j\omega\tau} d\tau$

OK, real.

$$S_X(\omega) = \frac{1}{2\pi}\begin{bmatrix} \underline{\pi S_A(\omega-\omega_0) + \pi S_A(\omega+\omega_0)} \\ +\frac{\pi}{j} S_{AB}(\omega-\omega_0) - \frac{\pi}{j} S_{AB}(\omega+\omega_0) \end{bmatrix}$$

must be real

具体看 next page.

Imaginary Part

We have terms of the form $\frac{\pi}{j} S_{AB}(\omega-\omega_0) = -j\pi S_{AB}(\omega-\omega_0)$

Recall the assumption $R_{AB}(\tau) = -R_{BA}(\tau)$

Combine this with

$$R_{AB}(\tau) = E\{A(t+\tau)B(t)\} = E\{B(t)A(t+\tau)\} = R_{BA}(-\tau)$$

to get $R_{AB}(\tau) = -R_{AB}(-\tau)$ ⟵ ODD SYMMETRY

PSD, Cont'd

The odd symmetry of $R_{AB}(\tau)$ implies that $S_{AB}(\omega)$ is purely imaginary and has odd symmetry,

so $-j\pi S_{AB}(\omega)$ is real

$$\boxed{\begin{aligned} S_X(\omega) &= \frac{1}{2}\left[S_A(\omega-\omega_0) - jS_{AB}(\omega-\omega_0)\right] \\ &+ \frac{1}{2}\left[S_A(\omega+\omega_0) + jS_{AB}(\omega+\omega_0)\right] \end{aligned}}$$

White Noise

General definition:

$X(t)$ is a white-noise process such that

$$C(t_1,t_2)=q(t_1)\delta(t_1-t_2)$$

$t_1=t_2 \Leftrightarrow C\neq 0.$

Any two samples of a white noise are uncorrelated, regardless of how close they are in time.

independent $\Leftrightarrow$ strict white Noise

Usually, white noise is assumed to have zero mean, so

$$R(t_1,t_2)=q(t_1)\delta(t_1-t_2)$$

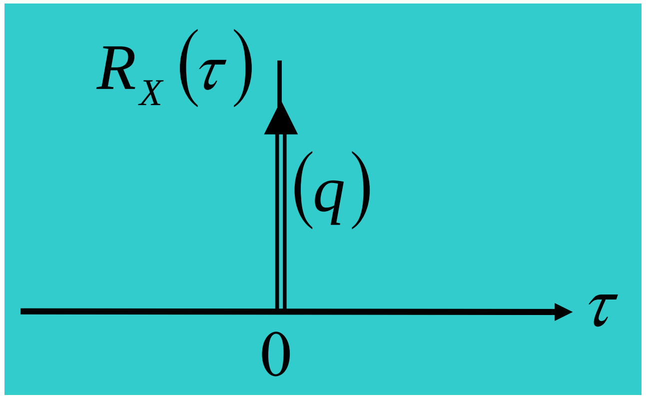
WSS white noise: $R(\tau)=q\delta(\tau)$

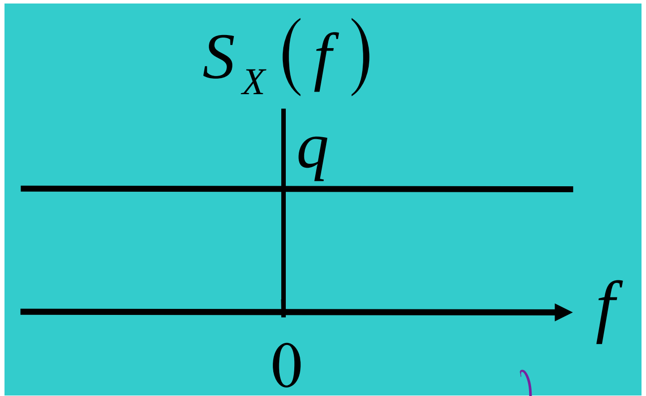
WSS White Noise

A white noise RP has a flat power spectral density

→equal energy in all bands of the same bandwidth.

Equivalently, its autocorrelation is a single impulse.





infinite energy

(power = ∫ power density)

How Much Power?

If is $\mathrm{X}(\mathrm{t})$ a WSS white noise RP, then the total average power is

$$E\left\{\left|X(t)\right|^2\right\} = \int_{-\infty}^{+\infty} S_X(f)\,df$$

$$= R_X(0)$$

$$= +\infty$$

How can this be?

since It's impossible.

It's an ideal model

White Noise is an Idealization

White noise doesn't exist outside of a convolution integral (similar to a delta function)

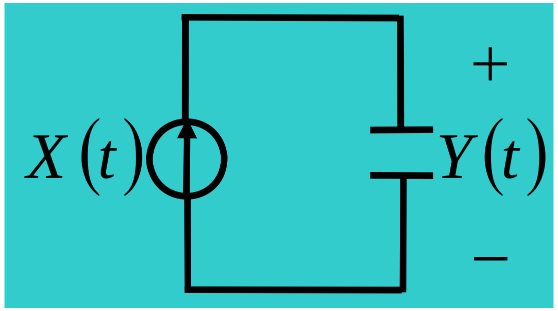
Any real system has finite bandwidth

As long as the actual input PSD is flat over the bandwidth of the system, it might as well be flat everywhere to simplify analysis

Integral of White Noise

Let $Y(t) = \int_0^t X(s)ds$

$Y(t) = X(t) * u(t)$, where u(t) is the unit step.



White noise current generator

Properties of Integrated White Noise

1. $$E\left\{\left|Y(t)\right|^2\right\} = E\left\{\int_0^t X(s)ds \int_0^{t} X^*(v)dv\right\}$$

   $u(t)$

   $$= \int_0^t \int_0^t E\left\{X(s)X^*(v)\right\}dsdv$$

   $$= \int_0^t \int_0^t q(s)\delta(s-v)dsdv = \int_0^t q(v)dv$$

2. Y(t) has uncorrelated increments.

if said sample independent.

Y(t) increment also independent

if gaussian → independence

Gaussian White Noise

- If each sample is independent and has a normal distribution with zero mean, the signal is said to be Gaussian white noise
- Any integral of Gaussian White noise (GWN) is Gaussian.

If X(t) is WSS GWN, then

$Y(t) = \int_0^t X(s)ds$ has independent increments.

Y(t) is a Wiener Process

X(t) is the derivative of a Wiener Process.

Derivative of Wiener Process is Gaussian White Noise

Therefore, if $Y(t)$ is a Wiener process, then

$X(t) = \dfrac{dY(t)}{dt}$ is white noise with

$$E\left[X(t)\right] = 0$$

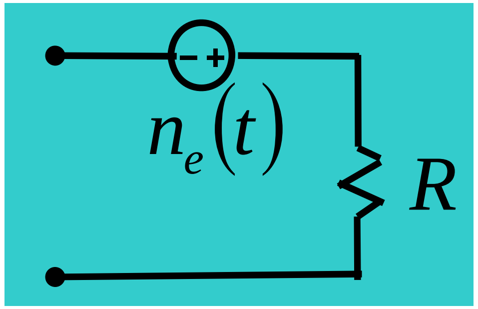
$$R_{XX}(t_1, t_2) = \sigma^2 \delta(t_1 - t_2), \quad 0 < t_1, \quad 0 < t_2$$

For $0<t_1$ and $0<t_2$, the RP is WSS. We often assume that the process "began" in the infinite past and assume it is WSS for all time.

It can be shown that any integral of $X(t)$ is Gaussian, therefore we call $X(t)$ Gaussian White Noise (GWN).

Thermal Noise

GWN is used to model thermal noise in communication receivers.



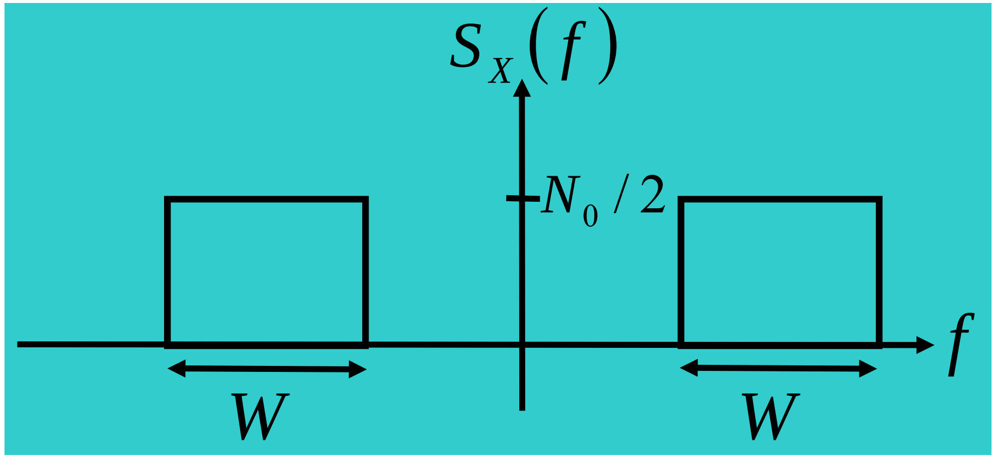
$$S_{n_e}(f) = 2kTR$$

Boltzman Constant

Temperature

Resistance

Band-limited White Noise



$$E\left\{\left|X(t)\right|^2\right\} = N_0 W$$

Summary (2)

- QAM is WSS
- White noise is an idealized RP that is simple to analyze
- Gaussian White Noise (GWN)
  - is the derivative of the Wiener Process
  - models thermal noise
- WSS white noise has a flat power density spectrum

Thank You!