Probability and Random Process

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Oct. 26 2021

- ## 3. Multiple Random Variables
  - Two Random Variables
  - Marginal PDF
  - Functions of Two Random Variables
  - Conditional PDF
  - Joint Moments
  - Mean Square Error Estimation
  - Probability bound
  - Random Vectors
  - Sample Mean
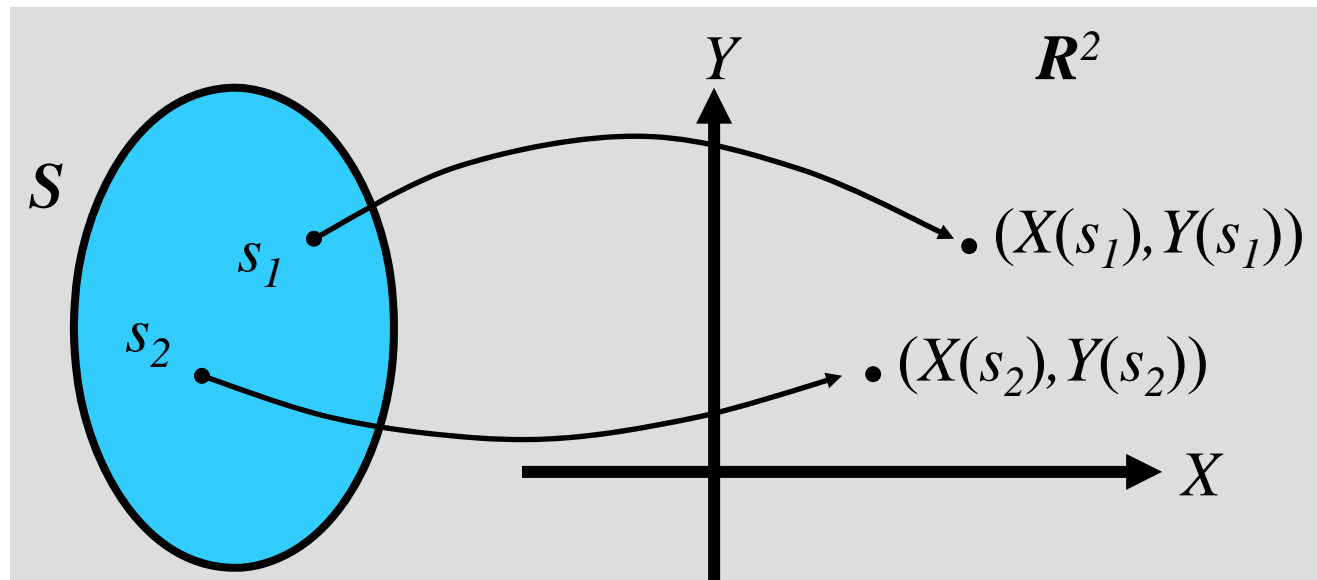 - Convergence of Random Sequences
  - Central Limit Theorem
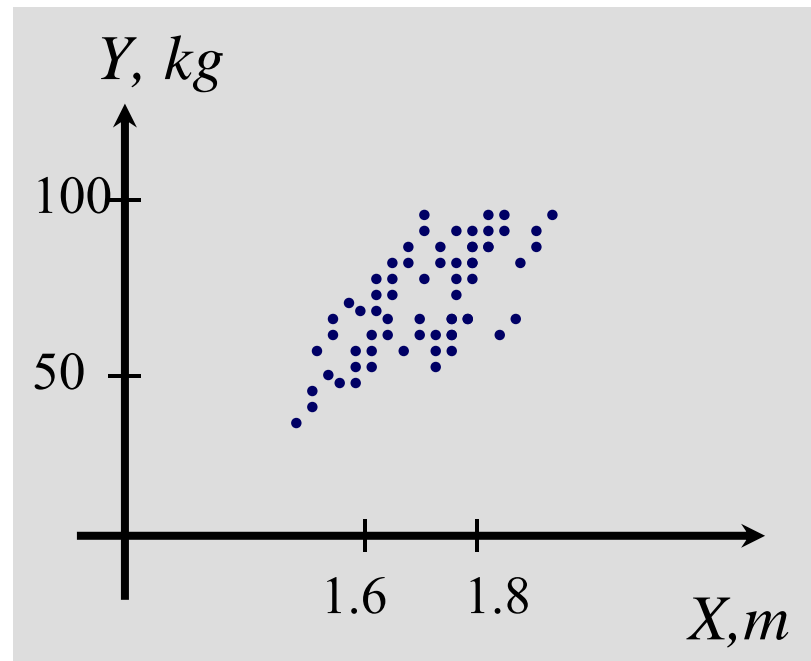
Two Random Variables

Two Random Variables

Also known as:

- Two-dimensional random variable
- Bivariate random variable

Let S be all the students at JI. Let $(X(s), Y(s))$ be the (*height, weight*) of students.
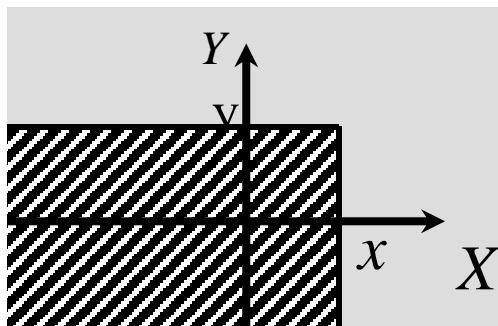

A scatter diagram is used to display measured data

Definition of Joint CDF -- I

Definition of the joint cumulative distribution function (CDF):

$$F_{XY}(x, y) = P(X \leq x \cap Y \leq y)$$

Properties:

1. $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0$

2. $F_{XY}(x, +\infty) = F_X(x), \quad F_{XY}(+\infty, y) = F_Y(y)$

$F_{XY}(x, y) =$ Probability that *(X,Y)* is in this region

How about $F_{XY}(x_1, y_1) - F_{XY}(x_2, y_2)$?

Definition of the joint probability density function (PDF):

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u,v)\,dv\,du$$
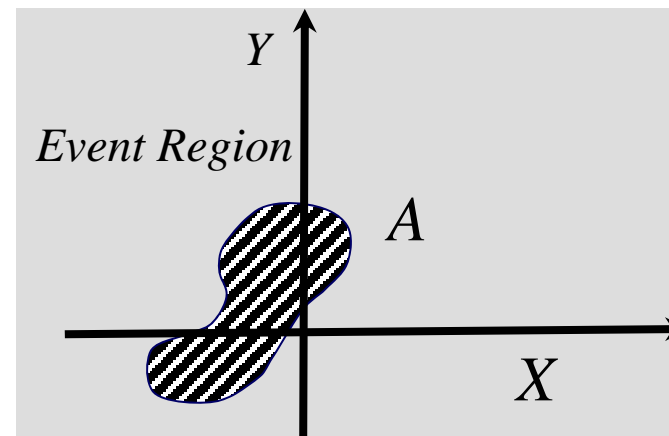
The joint PDF is a surface. The volume under $f_{XY}(x,y)$ is probability.

1. $f_{XY}(x, y) \geq 0$ for all $("\forall")$ $x, y \in R^2$

2. $\displaystyle\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x, y)\,dxdy = 1$

3. $P\big((X,Y) \in A\big) = \displaystyle\int\int_A f_{XY}(x, y)\,dxdy$

Event Region

Y

A

X

Suppose the joint PDF is flat ("uniform") over the triangle with vertices (0,0), (0,1), (1,0).

$$f_{XY}(x, y)$$

(0,1)

Y

(1,0)

X

the line

Y=-X+1
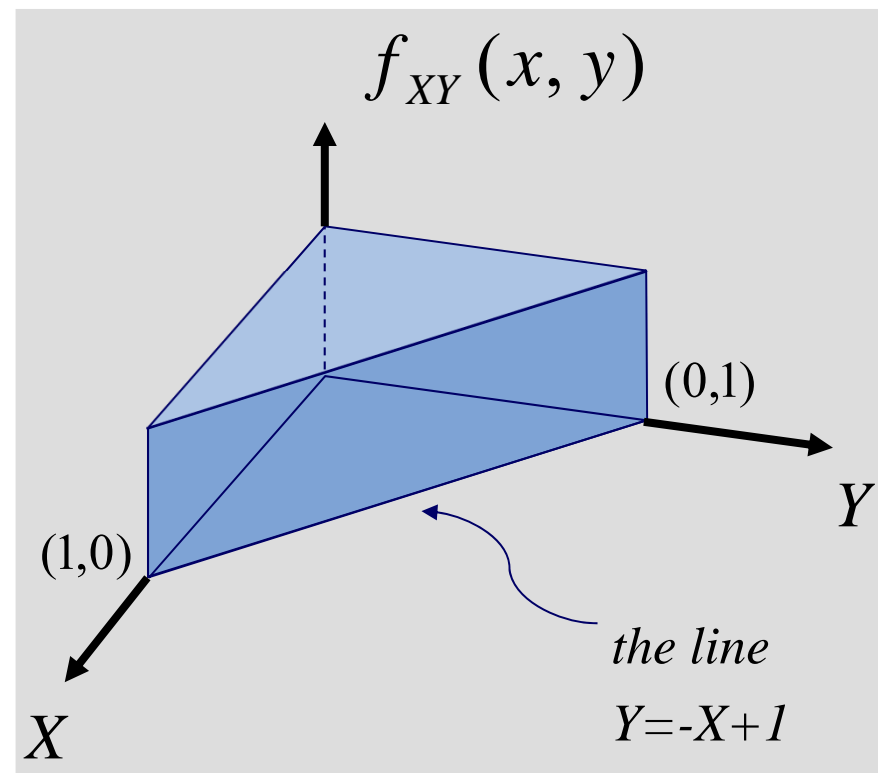
The region of support (ROS) is the set of all points (x,y) such that ("s.t.") $f_{XY}(x, y) > 0.$

$\uparrow$

strictly

In the previous example, the ROS is the shaded area:
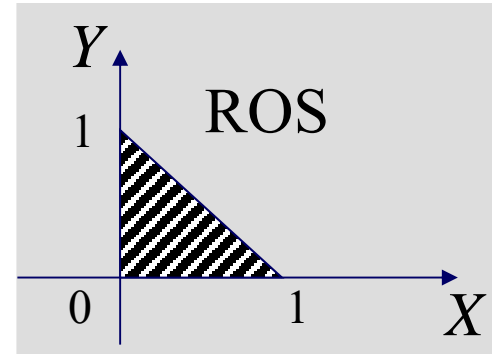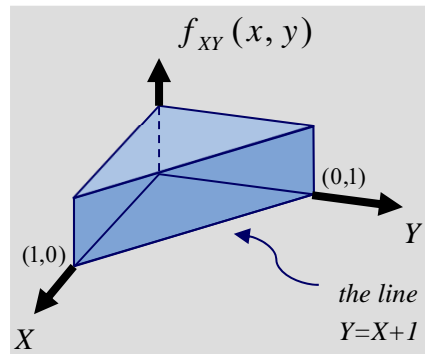
The area of this ROS is ½. What is the PDF?

For a uniform joint PDF, the total volume is the height times the area of the ROS.

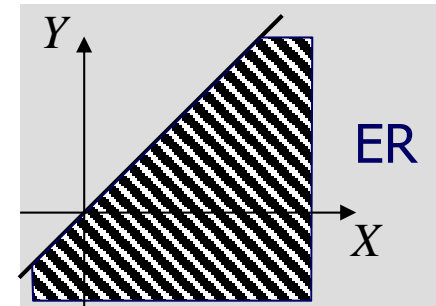The total volume must be 1, so the height must be 2.

Symbolic description of the joint PDF:

$$f_{XY}(x, y) = \begin{cases} 2 & x \geq 0, \, y \geq 0, \, y \leq -x + 1 \\ 0 & \text{otherwise} \end{cases}$$
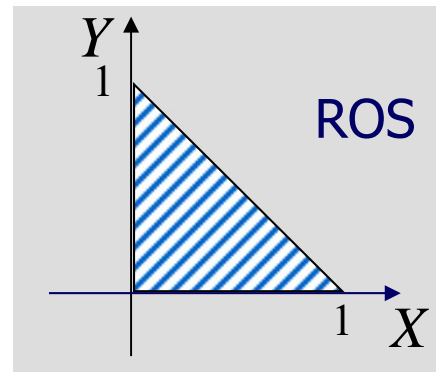
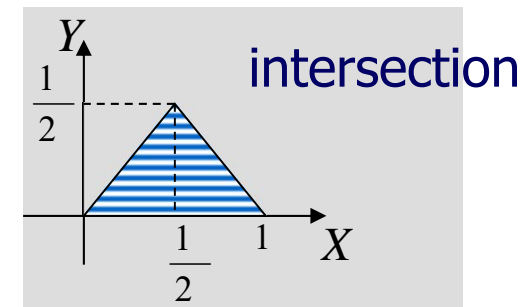The "0 otherwise" is necessary for completeness.

- Calculate P(X>Y)
- General Procedures:

  1. Identify the event region (ER).

  2. Identify the ROS.

  3. Integrate the joint PDF over the intersection of the ER and ROS.

Because the joint PDF is uniform and the line $X=Y$ bisects the ROS, we observe that $\Pr(X > Y) = 1/2$.

In general, double integration must be performed to compute probability.

For this example:

$$P(X > Y) = \int_{y=0}^{1/2} \int_{x=y}^{1-y} 2\, dx\, dy = 1/2$$
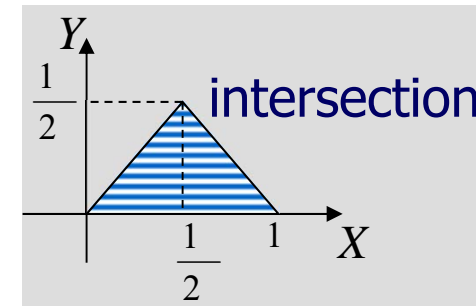
How to determine the integral boundaries?

- Determine y first, since dy is the outer integral
- Fix y, obtain the boundaries of x

$$P(X > Y) = \int_{y=0}^{1/2} \int_{x=y}^{1-y} 2 \, dx \, dy = 1/2$$
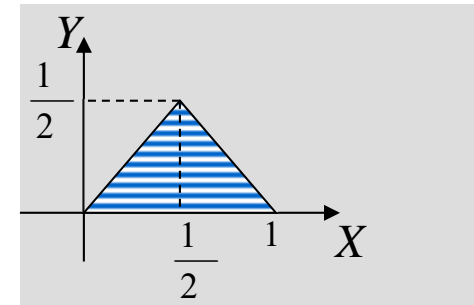
- **Joint probability mass function (pmf)** of X and Y is defined by

$$p_{XY}(x, y) = \Pr(X = x, Y = y)$$

- Properties

1. $p_{XY}(x, y) \geq 0$

2. $\sum_X \sum_Y p_{XY}(x, y) = 1$

3. $\Pr(X \in A, Y \in B) = \sum_{X \in A} \sum_{Y \in B} p_{XY}(x, y)$

4. $\Pr((X, Y) \in G) = \sum_{(X,Y) \in G} p_{XY}(x, y)$

Example

- X and Y are integer valued.

$$\Pr(1 \leq X \leq 3, 2 \leq Y \leq 4) = \sum_{x=1}^{3} \sum_{y=2}^{4} p_{XY}(x, y)$$

Marginal PDF

Marginal PDFs

A marginal PDF is obtained by "integrating out" the unwanted variable in a joint PDF:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY(x,y)}(x, y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY(x,y)}(x, y)\,dx$$

$f_{xy}(x,y)$

Diamond Slab
Example

The sand represents
"probability mass"
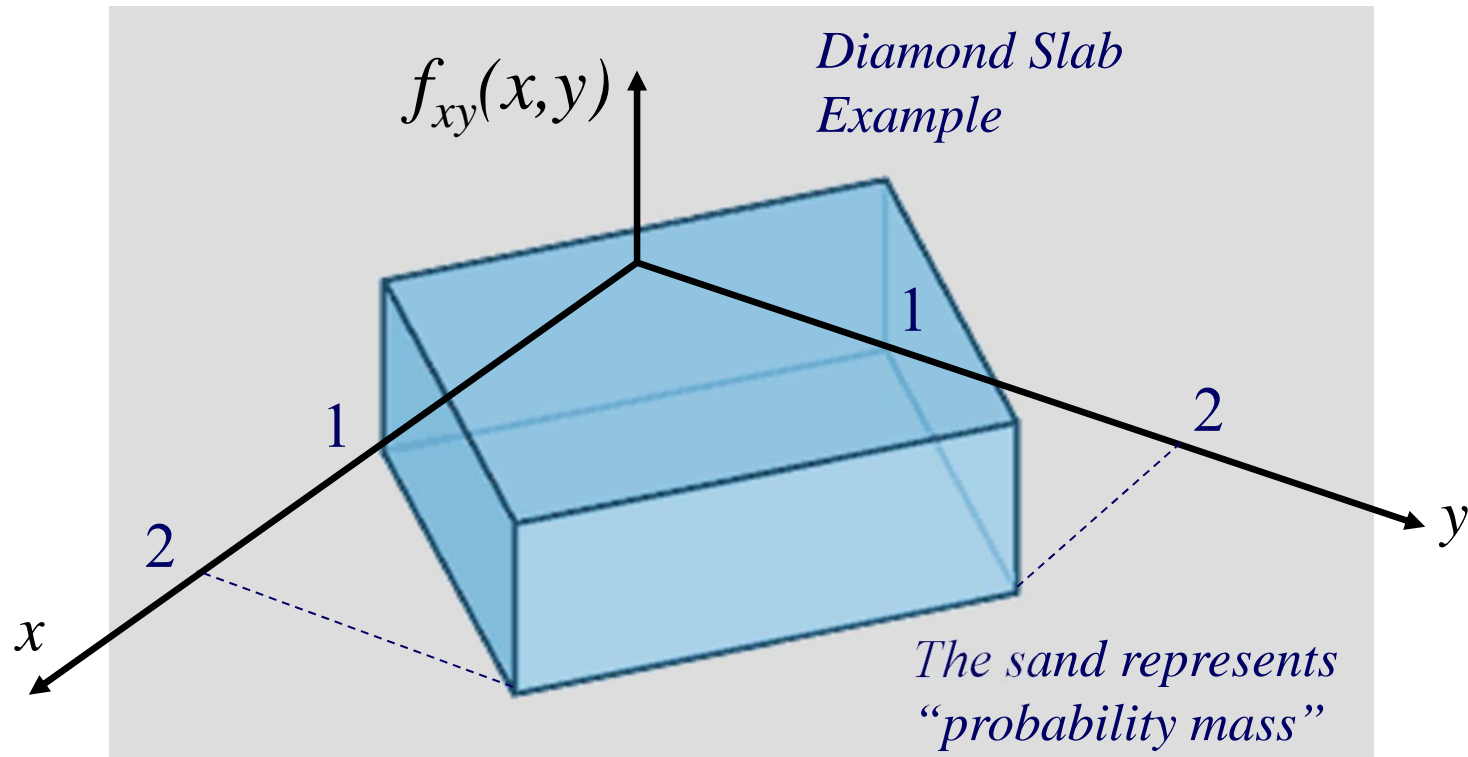
To get the marginal for X, imagine using boards that are parallel to the X-axis to push the sand **onto** the X-axis :
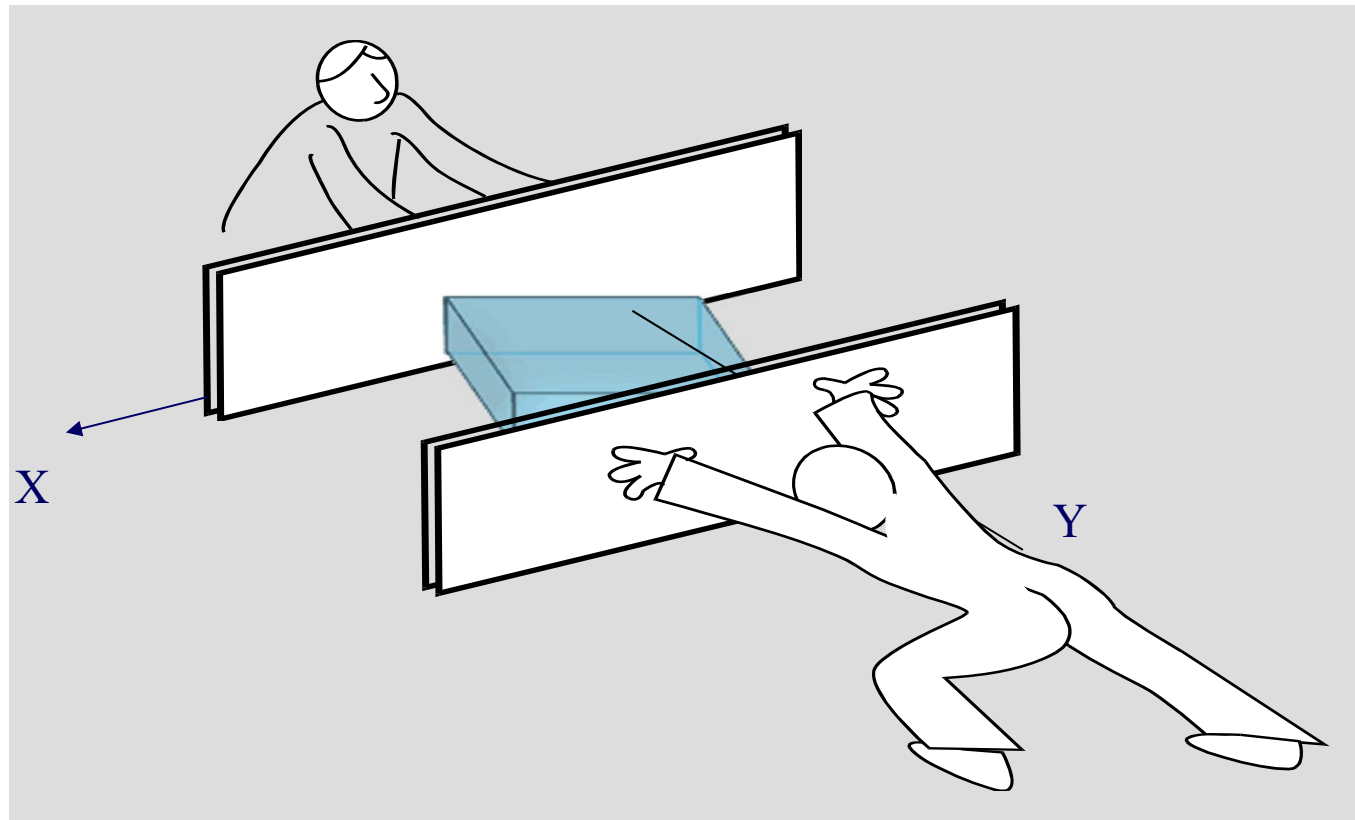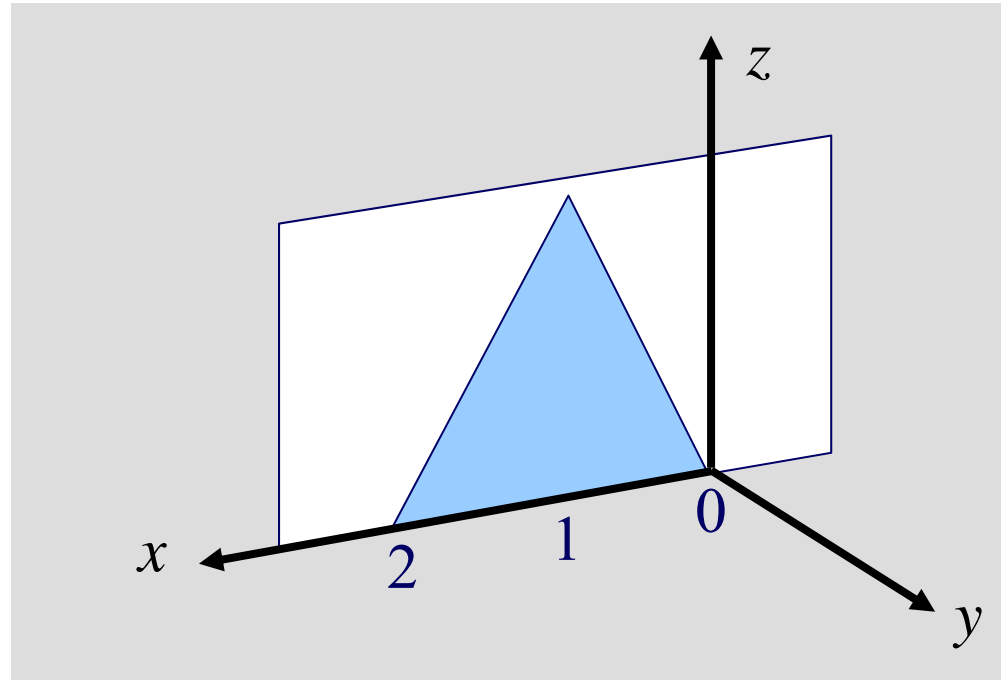
Use this idea to:

- Predict the ROS of $f_X(x)$
- Predict the profile of $f_X(x)$

The tough part is to set up the integral correctly.

It is very helpful to sketch ROS of the joint PDF to see the limits of integration.
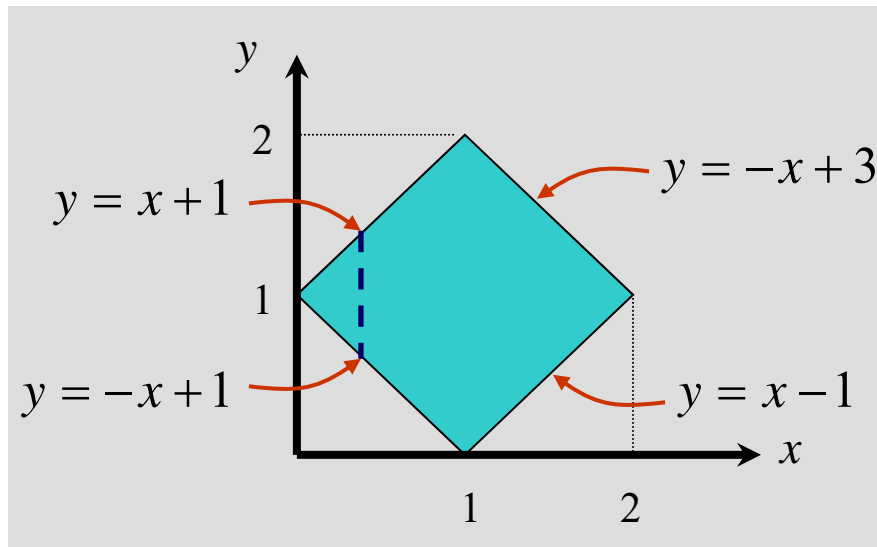
$$for \quad 0 < x \leq 1:$$

$$f_X(x) = \int_{-x+1}^{x+1} \frac{1}{2} dy = x$$

$$for \quad 1 < x \leq 2:$$

$$f_X(x) = \int_{x-1}^{-x+3} \frac{1}{2} dy = 2 - x$$

$$f_X(x) = \begin{cases} x & 0 < x \le 1 \\ 2 - x & 1 < x \le 2 \\ 0 & \text{ow} \end{cases}$$
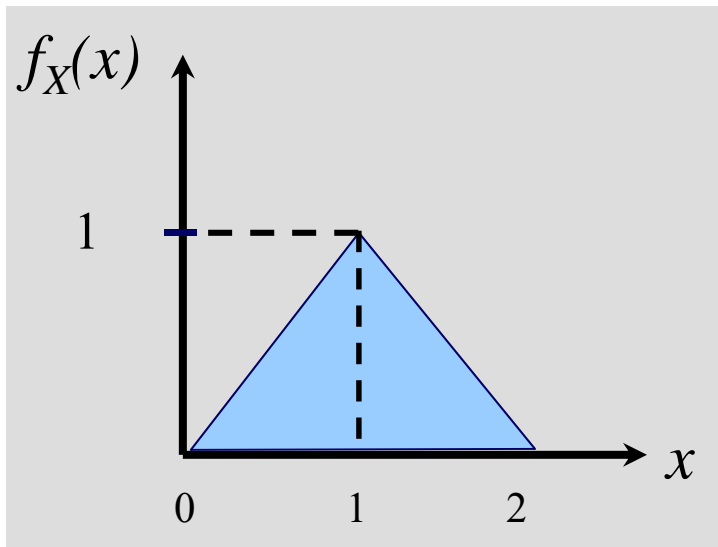
Two check-points:

Does $f_X(x)$ integrate to 1?

$f_X(x)$ should have no y dependence.

- Relationship between joint pmf and marginal (individual) pmf's.

$$p_X(x) = \sum_y p_{XY}(x, y), \quad p_Y(y) = \sum_x p_{XY}(x, y)$$

- $p_{XY}(x, y) \Rightarrow p_X(x)$ and $p_Y(y)$?
- $p_X(x)$ and $p_Y(y) \Rightarrow p_{XY}(x, y)$?
  - Yes
  - No. If independent, yes.

- Marginal PDF
  - Obtained by integrating out the unwanted variable in a joint PDF

- Remarks
  - Be careful about the limits of integral
  - Two check-points

Independence

- Two random variables X and Y are said to be **equal (almost surely)**, written as $X = Y$, if the event $\{X = Y\}$ occurs with probability one, i.e.

$$\Pr(X = Y) = P(\{s : X(s) = Y(s)\}) = 1$$

- The random variables X and Y are **identically distributed** if

$$\Pr(X \in B) = \Pr(Y \in B), \forall B \in \mathcal{B}(\boldsymbol{R})$$

- equal $\Rightarrow$ identical ?
- identical $\Rightarrow$ equal ?
  - Yes
  - No

Consider the experiment of tossing a fair coin three times.
Define the random variables

$X \triangleq$ number of heads observed $Y \triangleq$ number of tails observed

The distributions of X and Y are:

ω	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
$X(\omega)$	3	2	2	2	1	1	1	0
$Y(\omega)$	0	1	1	1	2	2	2	3

- If the events $\{X \in B\}$ and $\{Y \in C\}$ are independent for all sets $B \in \mathcal{B}(\boldsymbol{R})$ and $C \in \mathcal{B}(\boldsymbol{R})$, we say that X and Y are **independent random variables**. In light of this definition, we see that X and Y are independent random variables if and only if

$$\Pr(X \in B, Y \in C) \;=\; \Pr(X \in B)\,\Pr(Y \in C)$$

for all Borel sets B and C .

- Given any finite number of random variables, say $X_1, \ldots, X_n$, we say they are **independent** if

$$\Pr\left(\bigcap_{j=1}^{n} \{X_j \in B_j\}\right) = \prod_{j=1}^{n} \Pr(X_j \in B_j)$$

- for all choices of the Borel sets $B_1, \ldots, B_n$. If $X_1, \ldots, X_n$ are independent, then so is any subset of them, e.g., X_1, X_3 and X_4.

- If $X_1, X_2, \dots$ are independent and every X_i and X_j are identically distributed for $i \neq j$, $i, j = 1, 2, \dots$, we say that $X_1, X_2, \dots$ are **independently and identically distributed (i.i.d.)**.

Independent RVs

Two RVs, X and Y are independent iff their joint PDF factors into a product of the marginal PDFs

Mathematically:
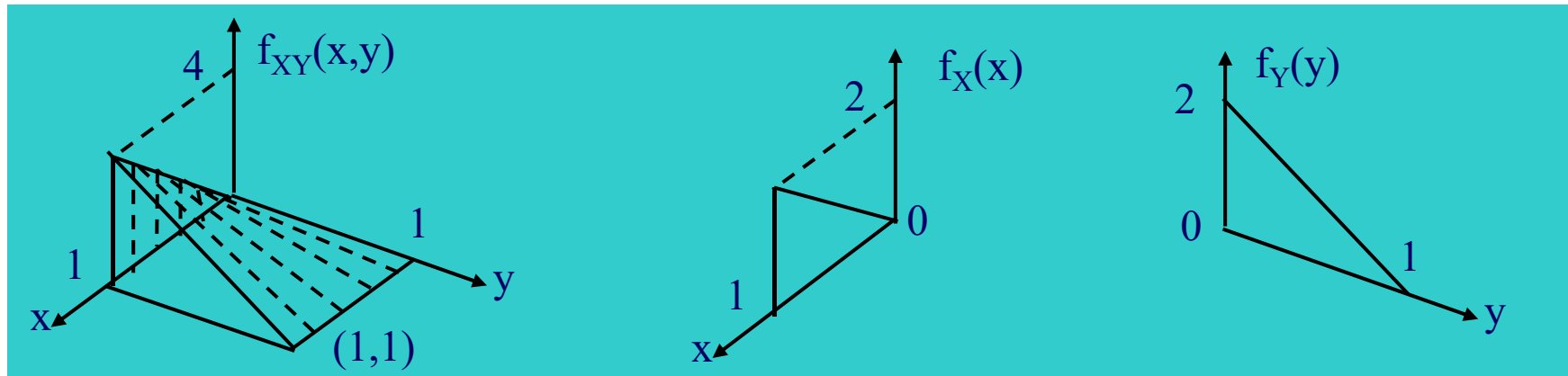$$f_{XY}(x, y) = f_X(x)f_Y(y)$$
Or
$$p_{XY}(x, y) = p_X(x)p_Y(y)$$

$$f_{XY}(x,y) = \begin{cases} 4x(1-y) & \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{array} \\ 0 & otherwise \end{cases}$$

is the product of

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & o.w. \end{cases} \quad \text{and} \quad f_Y(y) = \begin{cases} 2(1-y) & 0 \leq y \leq 1 \\ 0 & o.w. \end{cases}$$

therefore, X and Y are independent.

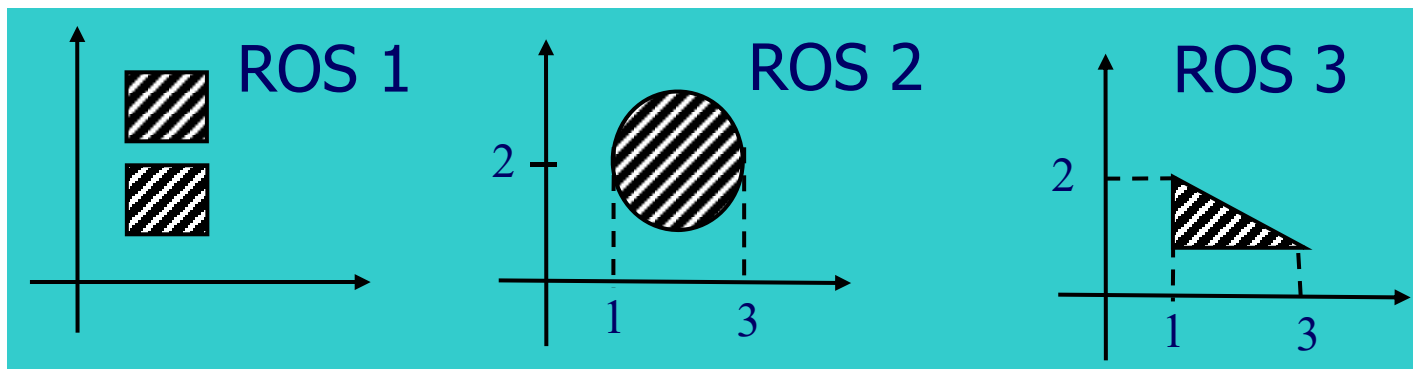Observe that the factorization of $f_{XY}(x, y)$ occurs in two places:

Product of expressions

Cartesian Product of sets

$$f_{XY}(x, y) = \begin{cases} [2x][2(1-y)] & \{0 \le x \le 1\} \times \{0 \le y \le 1\} \\ 0 & o.w. \end{cases}$$

If either the expression or the ROS for $f_{XY}(x, y)$ is NOT factorable, then the RVs are dependent

A joint PDF ROS that is not a Cartesian product is easy to identify:

In ROS 2, $X=1$ tells us $Y=2$. Dependent!

In ROS 3, $X=3$ tells us $Y=1$. Dependent!

★REMEMBER

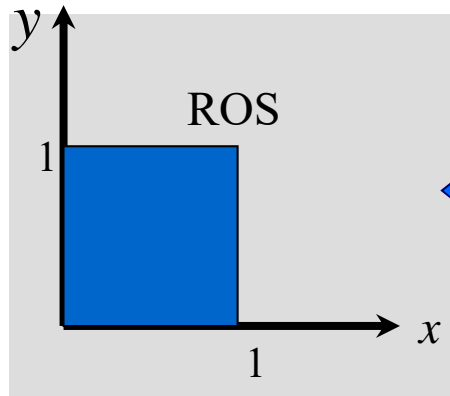If a joint PDF ROS has a sloped or curved boundary, the RVs are dependent.

A Cartesian product ROS does not imply independence.

Example:

$$f_{XY}(x, y) = \begin{cases} Ke^{xy} & \begin{array}{l} 0 \leq x \leq 1 \\ 0 \leq y \leq 1 \end{array} \\ 0 & o.w. \end{cases}$$

where K is a proper constant of normalization.

$$f_{XY}(x, y) = \begin{cases} Ke^{xy} & \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1 \end{array} \\ 0 & o.w. \end{cases}$$

$$f_X(x) = \int_0^1 Ke^{xy}\, dy = \frac{K}{x}(e^x - 1) \quad 0 < x \le 1$$

Thank You!