Probability and Random Process

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Outline

- 3. Multiple Random Variables
  - Two Random Variables
  - Marginal PDF
  - Functions of Two Random Variables
  - Conditional PDF
  - Joint Moments
  - Mean Square Error Estimation
  - Probability bound
  - Random Vectors
  - Sample Mean
  - Convergence of Random Sequences
  - Central Limit Theorem

Conditional PDF

Conditional PDF

- The "PDF of X conditioned on Y" is used to model the uncertainty in X when the joint PDF of X and Y is known and a value of Y is given.

$$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

- Related forms: $f_{X|A}(x|A),\ P(A|Y = y), \Pr(X \in A|Y \in B)$
- Recall Conditioning a RV on an Event Involving Only that RV

$$F_{X|B}(x \mid B) = P(X \leq x \mid B) = \frac{P(\{X \leq x\} \cap B)}{P(B)}$$

Derivation of Conditional PDF -- I

Begin with the conditional CDF:

$$F_{X|Y}(x \mid y) = P(X \leq x \mid Y = y)$$

This has the form: $$P(A \mid Y = y) = \frac{f_{Y|A}(y \mid A)P(A)}{f_Y(y)}$$?

$$F_{X|Y}(x \mid y) = \frac{f_{Y|X \leq x}(y \mid X \leq x)P(X \leq x)}{f_Y(y)}$$

$$= \frac{\frac{d}{dy} F_{Y|X \leq x}(y \mid X \leq x)P(X \leq x)}{f_Y(y)} = \frac{\frac{d}{dy} P(Y \leq y \cap X \leq x)}{f_Y(y)}$$

Derivation of Conditional PDF -- II

$$F_{X|Y}(x \mid y) = \frac{\frac{\partial}{\partial y} F_{XY}(x, y)}{f_Y(y)}$$

$$f_{X|Y}(x \mid y) = \frac{d}{dx} F_{X|Y}(x \mid y)$$

$$= \frac{\frac{\partial^2}{\partial x \partial y} F_{XY}(x, y)}{f_Y(y)} = \frac{f_{XY}(x, y)}{f_Y(y)}$$

Conditional PMF

- The **conditional probability mass function** of X given Y is

$$p_{X|Y}(x|y) = \begin{cases} \Pr(X = x|Y = y), & \Pr(Y = y) > 0 \\ 0, & \text{otherwise} \end{cases}$$

Property

- $\mathrm{P_r}(X \in B | Y = y) = \int_B f_{X|Y}(x|y)\, dx$
- $f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$
- $f_{XY}(x, y) = f_{X|Y}(x|y) f_Y(y)$
- **Conditional CDF**

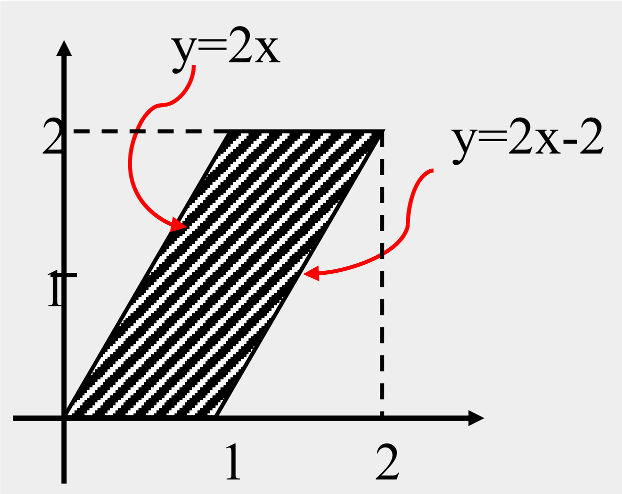
$$\boxed{F_{Y|X}(y|x) = \int_{-\infty}^{y} f_{Y|X}(z|x) dz}$$

- **Bayes' rule**

$$\boxed{f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}}$$

Example of Conditional PDF -- I

Let X and Y be uniform over the shaded area:



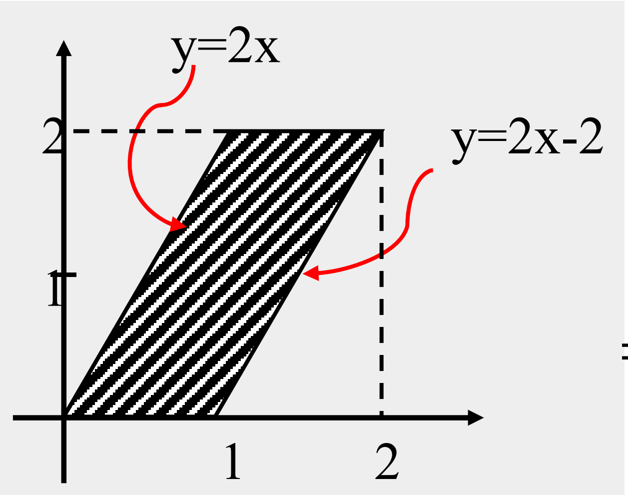
$$f_{XY}(x,y) = \begin{cases} \dfrac{1}{2} & \dfrac{y}{2} \le x \le \dfrac{y}{2}+1,\ \ 0 \le y \le 2 \\ 0 & \text{otherwise} \end{cases}$$

What is $f_{X|Y}(x|y)$?

We obtain marginal pdf on Y:

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)dx = \begin{cases} 1/2 & 0 \le y \le 2 \\ 0 & o.w. \end{cases}$$

Example of Conditional PDF -- II



$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

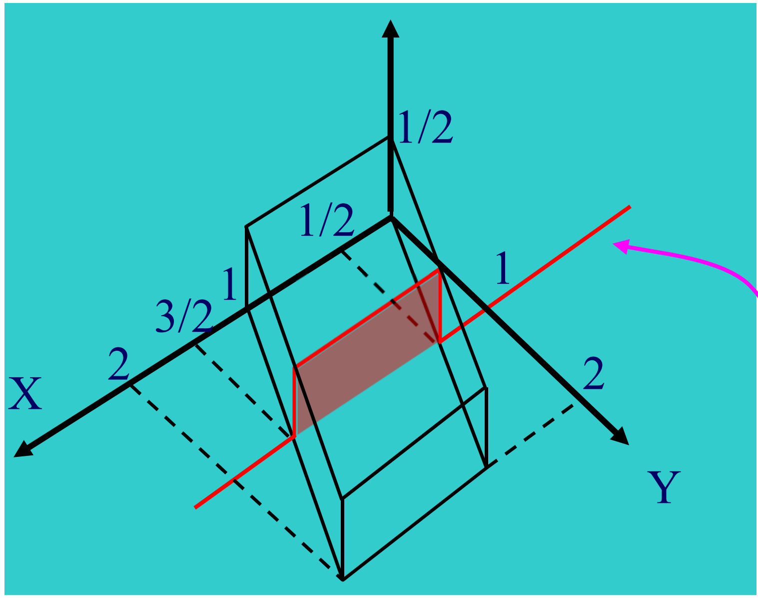
$$= \begin{cases} \dfrac{1/2}{1/2} & \dfrac{y}{2} \le x \le \dfrac{y}{2}+1, \quad 0 \le y \le 2 \\ 0 & o.w. \end{cases}$$

$$= \begin{cases} 1 & \dfrac{y}{2} \le x \le \dfrac{y}{2}+1, \quad 0 \le y \le 2 \\ 0 & o.w. \end{cases}$$

The ROS of the conditional PDF is the same as the ROS of the joint PDF.

"Slice and Normalize"

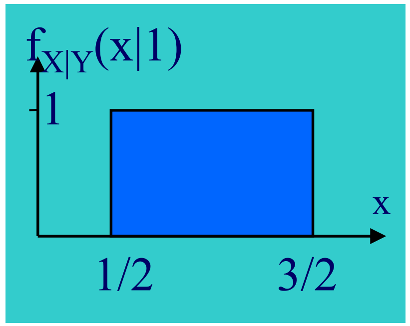
- $f_{XY}(x,y)$ with y fixed is like slicing the joint PDF along the line $Y=y$. For example, $Y=1$.
- The conditional PDF is just this slice, normalized by its own area to yield a valid PDF.



$f_{XY}(x,1)$ is not a valid PDF of X because its area$=1/2$.

So divide $f_{XY}(x,1)$ by $1/2$:

$$f_{X|Y}(x\,|\,1) = \frac{f_{XY}(x,1)}{1/2}$$



Independence in terms of conditional pdf's

- Independence in terms of conditional pdf's

$$f_{X|Y}(x|y) = f_X(x), \qquad \forall y: f_Y(y) \neq 0$$

- Can you prove that?

Law of total probability

- **Law of total probability**

$$\Pr(X \in A) = \int_{-\infty}^{\infty} \Pr(X \in \mathrm{A} | Y = y) f_Y(y)\, dy$$

- Discrete case

$$\Pr(X \in A) = \sum_{Y} p_Y(y) \sum_{X \in \mathrm{A}} p_{X|Y}(x|y)$$

Proof

$$\int_{-\infty}^{\infty} \mathrm{P_r}(X \in A | Y = y) f_Y(y)\, dy = \int_{-\infty}^{\infty} \int_A f_{X|Y}(x|y)\, dx\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \int_A \underbrace{f_{X|Y}(x|y) f_Y(y)}_{f_{XY}(x,y)}\, dx\, dy$$

interchange the order of integration

$$= \int_A \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy\, dx = \int_A f_X(x)\, dx = \mathrm{P_r}(X \in A)$$

$$\boxed{\mathrm{P_r}(X \in A) = \int_{-\infty}^{\infty} \mathrm{P_r}(X \in A | Y = y)\, f_Y(y)\, dy}$$

Substitution law

- $Z = g(X, Y)$ is a function of continuous random variable X and Y, then

$$\Pr(g(X,Y) \in A | Y = y) = \Pr(g(X,y) \in A | Y = y)$$

- This is known as the **substitution law** of conditional probability

Law of total probability

- **Law of total probability** for a function of random variables

$$\Pr((X,Y) \in A) = \int_{-\infty}^{\infty} \Pr((X,Y) \in A \mid Y = y)\, f_Y(y)\,dy$$

Proof

$$\int_{-\infty}^{\infty} P_r((X,Y) \in A | Y = y) f_Y(y)\, dy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_A(x,y) f_{X|Y}(x|y)\, dx\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_A(x,y) \underbrace{f_{X|Y}(x|y) f_Y(y)}_{f_{XY}(x,y)}\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_A(x,y) f_{XY}(x,y)\, dx\, dy = \iint_A f_{XY}(x,y)\, dx\, dy = P_r((X,Y) \in A)$$

$$\boxed{P_r((X,Y) \in A) = \int_{-\infty}^{\infty} P_r((X,Y) \in A | Y = y) f_Y(y)\, dy}$$

- $I_A(x,y)$ is the indicator function of A

Example

- Let X and Y be absolutely continuous random variables with joint pdf $f_{XY}(x, y)$. Let $Z = X + Y$. Compute the pdf and cdf of Z.

Solution

$$F_Z(z) = \mathrm{P_r}(Z \le z) = \mathrm{P_r}(X + Y \le z)$$

$$= \int_{-\infty}^{\infty} \mathrm{P_r}(X + Y \le z | Y = y)\, f_Y(y)\, dy \quad (\textit{law of total probability})$$

$$= \int_{-\infty}^{\infty} \mathrm{P_r}(X + y \le z | Y = y)\, f_Y(y)\, dy \quad (\textit{substitution law})$$

$$= \int_{-\infty}^{\infty} \underbrace{\mathrm{P_r}(X \le z - y | Y = y)}\, f_Y(y)\, dy$$

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{z-y} \underbrace{f_{X|Y}(x|y)}\, dx \right] \underbrace{f_Y(y)}\, dy$$

$$\boxed{= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x, y)\, dx\, dy}$$

Solution

$$f_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z} F_Z(z) = \frac{\mathrm{d}}{\mathrm{d}z} \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \left[\frac{\mathrm{d}}{\mathrm{d}z} \int_{-\infty}^{z-y} f_{XY}(x, y)\, dx \right] dy$$

$$= \boxed{\int_{-\infty}^{\infty} f_{XY}(z - y, y)\, dy} \quad (\textit{Leibniz's rule})$$

Conditional Expectation

The conditional expected value or mean of Y given X is the mean of the conditional PDF of Y given X

$$E(Y \mid X = x) = \int_{-\infty}^{+\infty} y \, f_{Y|X}(y \mid x) dy$$

Compare with unconditional expected value

$$E(Y) = \int_{-\infty}^{+\infty} y \, f_Y(y) dy$$

Law of total expectation

- Law of total expectation

$$E[g(X,Y)] = \int_{-\infty}^{\infty} E[g(X,Y)|Y = y] f_Y(y)\, dy$$

for continuous random variable X and Y.

- Proof:

$$E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y) dx dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X|Y}(x|y) f_Y(y) dx dy$$

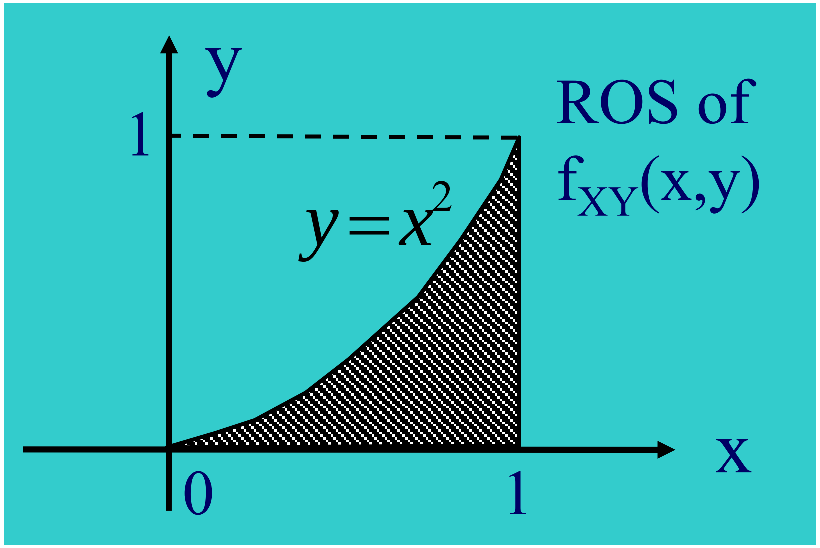
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X|Y}(x|Y = y) dx f_Y(y) dy$$

$$= \int_{-\infty}^{\infty} E[g(X,Y)|Y = y] f_Y(y)\, dy$$

Example

Suppose $f_{XY}(x, y)$ is uniform over the shaded area

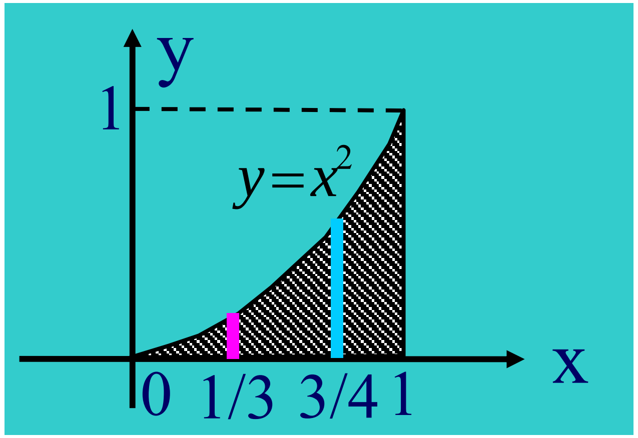
The cross sections parallel to the y axis are all rectangles, so all $f_{Y|X}(y|x)$'s are uniform



Example

The conditional PDF is

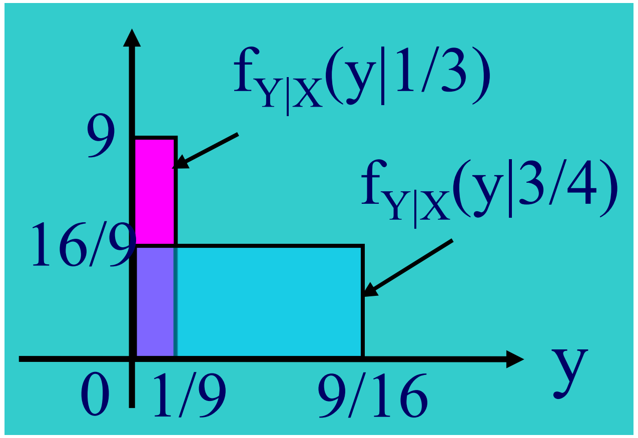
$$f_{Y|X}(y \mid x) = \begin{cases} \dfrac{1}{x^2} & 0 \le y \le x^2 \\ 0 & \text{else} \end{cases}$$



The conditional expectations are:

$$E(Y \mid X = 1/3) = 1/18$$

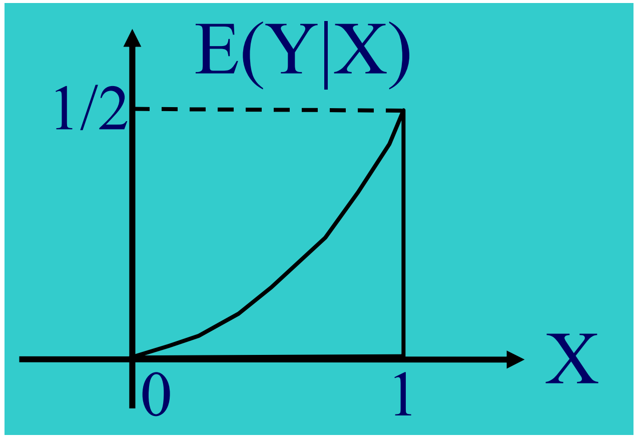
$$E(Y \mid X = 3/4) = 9/32$$



Example

The conditional expectation, without the X value specified, becomes function of X

$$\therefore \quad E(Y \mid X) = \frac{X^2}{2}$$



Iterated Expectation

- $E(Y|X)$ is just a function of X, denoted as $G(X)$.
- If we do not specify the point value of X, the "conditional expectation" becomes another RV

$$G(X) = E\left(Y \mid X\right)$$

What is the expected value of G(X) ?

Iterated Expectation

What is the expected value of G(X) ?

$$E[G(X)] = \int_{-\infty}^{+\infty} G(x) f_X(x) dx$$

$$= \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} y f_{Y|X}(y \mid x) dy \right] f_X(x) dx$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} y \underbrace{f_{Y|X}(y \mid x) f_X(x)}_{f_{XY}(x,y)} dy dx$$

Iterated Expectation

$$E\left[G(x)\right]=\int_{-\infty}^{+\infty} y\left[\int_{-\infty}^{+\infty} f_{XY}(x,y)dx\right]dy=\int_{-\infty}^{+\infty} yf_Y(y)dy$$

$$\therefore \quad E\left[E(Y\mid X)\right]=E(Y)$$

Very useful result: Iterated Expectation

Note:

- The inner expectation is w.r.t. Y
- The outer expectation is w.r.t. X

Example of Iterated Expectation

Suppose a wireless data user receives a packet of 400 bits

Because of multipath, the user's signal-to-noise ratio (SNR), γ, is exponentially distributed. The probability of bit error is given by

$$P(\text{bit error} \mid \gamma) = \mathrm{Q}\left(\sqrt{2\gamma}\right)$$

Proakis, Digital Communications, 4th ed.

Find the expected number of bits detected in error in the packet. Assume the error of each bit is independent and that all bits share the same SNR

Example

Since the error of each bit is independent and that all bits share the same SNR

$$
\begin{aligned}
E(\#\ \text{error bits} \mid \gamma) &= 400 * P(\text{bit error} \mid \gamma) \\
&= 400 * \mathrm{Q}\left(\sqrt{2\gamma}\right)
\end{aligned}
$$

Example

Using iterated expectation,

$$E(\#\,\text{error bits}) = E\left(E\left[\#\,\text{error bits} \mid \gamma\right] \right)$$

$$= E\left(400 * Q\left(\sqrt{2\gamma}\right)\right)$$

This problem is being solved in two stages, first assuming we know γ, and then we average over the possible values for γ

Example

Since γ is exponentially distributed,

$$E(\#\ \text{error bits}) = E\left(400 * Q\left(\sqrt{2\gamma}\right)\right)$$

$$= \int_0^{+\infty} 400 * Q\left(\sqrt{2\gamma}\right) \frac{1}{\bar{\gamma}} e^{-\frac{\gamma}{\bar{\gamma}}} d\gamma$$

$$= 200\left(1 - \sqrt{\frac{\bar{\gamma}}{1+\bar{\gamma}}}\right)$$

Proakis, Digital Communications, 4th ed.

where $\bar{\gamma}$ is the mean of the exponential distribution (mean SNR)

Example

- A gambler brings X dollars to a casino where X is a random variable with density

$$f_X(x) = \begin{cases} \dfrac{x}{80000}, & 0 \le x \le 400 \\ 0, & \text{otherwise} \end{cases}$$

- After a night of gambling the gambler leaves the casino with Y dollars, where Y is uniformly distributed between 0 and X.
    1. Given the gambler leaves the casino with less than 200 dollars, find the probability that he brought less than 200 dollars.
    2. Find the probability that his loss was less than 100 dollars.
    3. Find the probability that his loss was exactly 75 dollars.
    4. Find the density of Y.

Solution

We must find

$$P_r(X \leq 200 | Y \leq 200) = \frac{P_r(X \leq 200, Y \leq 200)}{P_r(Y \leq 200)}$$

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x}, & 0 \leq y \leq x \\ 0, & \text{otherwise} \end{cases}$$

$$f_{XY}(x,y) = f_{Y|X}(y|x) f_X(x) = \begin{cases} \frac{1}{80{,}000}, & 0 \leq x \leq 400, 0 \leq y \leq x \\ 0, & \text{otherwise} \end{cases}$$

Solution

The numerator

$$
\begin{aligned}
& P_r(X \le 200, Y \le 200) \\
= \quad & \int_{-\infty}^{200} \int_{-\infty}^{200} f_{XY}(x, y)\, dy\, dx = \int_0^{200} \int_0^{x} \frac{1}{80,000}\, dy\, dx \\
= \quad & \int_0^{200} \frac{x}{80,000}\, dx = \frac{1}{80,000} \frac{1}{2} x^2 \Big|_0^{200} = \frac{1}{4}
\end{aligned}
$$

Solution

The denominator

$$
\begin{aligned}
\mathrm{P_r}(Y \le 200) &= \int_{-\infty}^{200} f_Y(y)\,dy = \int_{-\infty}^{200}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy \\
&= \int_{0}^{200}\int_{y}^{400} \frac{1}{80,000}\,dx\,dy = \int_{0}^{200} \frac{400-y}{80,000}\,dy \\
&= 1 - \frac{1}{80,000}\frac{1}{2}\,y^2\Big|_0^{200} = \frac{3}{4}.
\end{aligned}
$$

Solution

Substituting for the numerator and denominator gives

$$\begin{aligned} P_r(X \le 200 | Y \le 200) &= \frac{P_r(X \le 200, Y \le 200)}{P_r(Y \le 200)} \\ &= \frac{\frac{1}{4}}{\frac{3}{4}} \\ &= \boxed{\frac{1}{3}} \end{aligned}$$

Solution

Find the probability that his loss was less than 100 dollars.

The gambler's loss is $X - Y$, so we need to find

$$\mathrm{P_r}(X - Y < 100) = \int_{-\infty}^{\infty} \int_{x-100}^{x} f_{XY}(x, y)\, dy\, dx$$

$$= \int_{0}^{400} \int_{\max(0, x-100)}^{x} \frac{1}{80,000}\, dy\, dx$$

$$= \int_{0}^{100} \int_{0}^{x} \frac{1}{80,000}\, dy\, dx + \int_{100}^{400} \int_{x-100}^{x} \frac{1}{80,000}\, dy\, dx$$

$$= \int_{0}^{100} \frac{x}{80,000}\, dx + \int_{100}^{400} \frac{1}{800}\, dx = \frac{1}{80,000} \frac{1}{2} x^2 \Big|_0^{100} + \frac{300}{800}$$

$$= \frac{1}{16} + \frac{3}{8} = \boxed{\frac{7}{16}}$$

Solution

Find the probability that his loss was exactly 75 dollars.

$$P_r(X - Y = 75) = \int_{-\infty}^{\infty} \int_{x-75}^{x-75} f_{XY}(x, y)\, dy\, dx = \boxed{0}$$

because area of integration is zero.

Solution

Find the density of Y

$$
\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx \\
&= \begin{cases} \int_y^{400} \frac{1}{80{,}000}\,dx, & 0 \le y \le 400, \\ 0, & \text{otherwise} \end{cases} \\
&= \boxed{\begin{cases} \frac{400-y}{80{,}000}, & 0 \le y \le 400, \\ 0, & \text{otherwise} \end{cases}}
\end{aligned}
$$

Thank You!