Ve501 Probability and Random Processes

2021 Fall

Homework 4

Due: November 18, 2021, in the class

Submission Instructions

1. Follow the JI Honor Policies.
2. Write down the key intermediate steps, instead of simply giving the final answers.
3. Submit your homework in A4 papers. Neat and tidy handwriting is allowed.
4. No late homework submission is allowed.

1. Let X and Y be independent Gaussian random variables with mean zero and variance one. Let $Z = 3X + 4Y$.

(1) Find $P_r(Z \geq 5)$.

(2) Find the correlation coefficient ρ_{XZ}.

(3) Find $E[(X - Z)^3]$.

2. Suppose X and Y have the joint PDF

$$f_{XY}(x, y) = \begin{cases} 3e^{-3(x-y)}u(x-y), & 0 \leq y \leq 1, y \leq x \leq +\infty \\ 0, & \text{otherwise} \end{cases}$$

(a) Sketch the conditional PDF $f_{x|y}(x|1/4)$, in words, the PDF of X conditioned on $Y = 1/4$.

(b) What is the marginal PDF of Y? (Try to avoid integration here)

(c) Suppose the RV Z is defined as $Z = XY^4$. Give an expression for the conditional mean, $E\{Z|Y\}$; note that this is also a function of the RV Y. Try to use the fact from lecture that the mean of the exponential RV is $1/\alpha$ if the PDF is

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x}, & 0 \leq x \leq +\infty \\ 0, & \text{otherwise} \end{cases}.$$

(d) Use iterated expectation to give a single-integral expression for $E\{Z\}$.

3. The joint density of X and Y is

$$f_{XY}(x, y) = \begin{cases} 2xy, & (0 \leq x \leq 1, 0 \leq y \leq 1) \text{or} (-1 \leq x \leq 0, -1 \leq y \leq 0) \\ 0, & \text{otherwise} \end{cases}.$$

(a) Find the MMSE estimator for X based on Y and find the resulting MSE.

(b) Find the MMSE linear estimator for X based on Y and find the resulting MSE.

4. Suppose a 2-dimensional random vector $X = [X_1\ X_2]$ has the mean vector $[-2, 12]$ and correlation matrix $R = \begin{bmatrix} 13 & -6.3 \\ -6.3 & 180 \end{bmatrix}$.

(a) Let $Z = X_1 + X_2$. Give the mean and variance of Z.
(b) Give the optimal linear homogeneous estimator of X_2, given X_1.
(c) Give the MSE performance of the estimator of part (b).
(d) Give the angle between the estimator of part (b) and the space spanned by X_1.
(e) Give the optimal linear nonhomogeneous estimator of X_2, given X_1.
(f) Give the MSE performance of the estimator of part (e).

5. Suppose $X_1, X_2, X_3 \ldots$, is an iid random sequence with finite variance σ^2.
(a) Using the Chebyshev Inequality, find the minimum number of samples needed to ensure that the sample mean is within $\sigma/4$ of the true mean with a probability of at least 0.98.
(b) Recalculate the minimum number using the Central Limit Theorem.