Probability and Random Process

Aimin Tang

The University of Michigan- Shanghai Jiao Tong University Joint Institute
Shanghai Jiao Tong University

Oct. 12 2021

Ve501 2021-2022 Fall

Based on Lecture Notes by Prof. Chong Han

Outline

- 2. Random Variables
  - Introduction to Random Variables
  - PMF and Discrete Random Variables
  - PDF and Continuous Random Variables
  - Gaussian CDF
  - Conditional Probability
  - Function of a RV
  - Expectation of a RV
  - Transform Methods and Probability Generating Function

PDF and Continuous Random Variables

The Probability Density Function

- The probability density function (PDF) is the derivative of the CDF, and denoted:

$$f_X(x) = \frac{d}{dx} F_X(x)$$

$$F_X(x) = \int_{-\infty}^{x} f_X(y)dy$$

- Some properties of $f_X(x)$ inherited from $F_X(x)$:

1. $\lim\limits_{x \to +\infty} F_X(x) = 1 \Rightarrow \int_{-\infty}^{+\infty} f_X(y)dy = 1$

★REMEMBER PDFs have unit area.

More Properties of PDFs

2. $F_X(x)$ non-decreasing $\Rightarrow f_X(x) \geq 0$

3. $P(a < X \leq b) = F_X(b) - F_X(a) = \int_a^b f_X(x)dx$

PDFs must be integrated to get probability, hence the name "density".

Question

- Is $P(a \leq X \leq b) = F_X(b) - F_X(a)$?
  - If X is continuous, it is true, because $\Pr(X = a) = 0$.
  - If X is discrete, it may not be true.

Question

- Does pdf $f_X\,(x)$ of a continuous r.v. need to be a continuous function?
  - No. e.g.. Exponential density is not continuous.(see later)

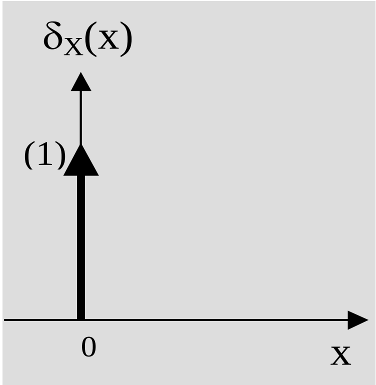
4. If $F_X(x)$ has a discontinuity at x_0, then $f_X(x)$ has a delta function at x_0 with area equal to the height of the jump.

Dirac Delta Function Definition

- The function, denoted $\delta(x)$, has undefined (infinite) height and unit area. It is defined implicitly by the following equation:

$$U(x) = \int_{-\infty}^{x} \delta(y)dy,$$

where U(x) is the unit step function.



It is OK to say:

$$\delta(x) = \frac{d}{dx}U(x)$$

Dirac Delta Function Properties

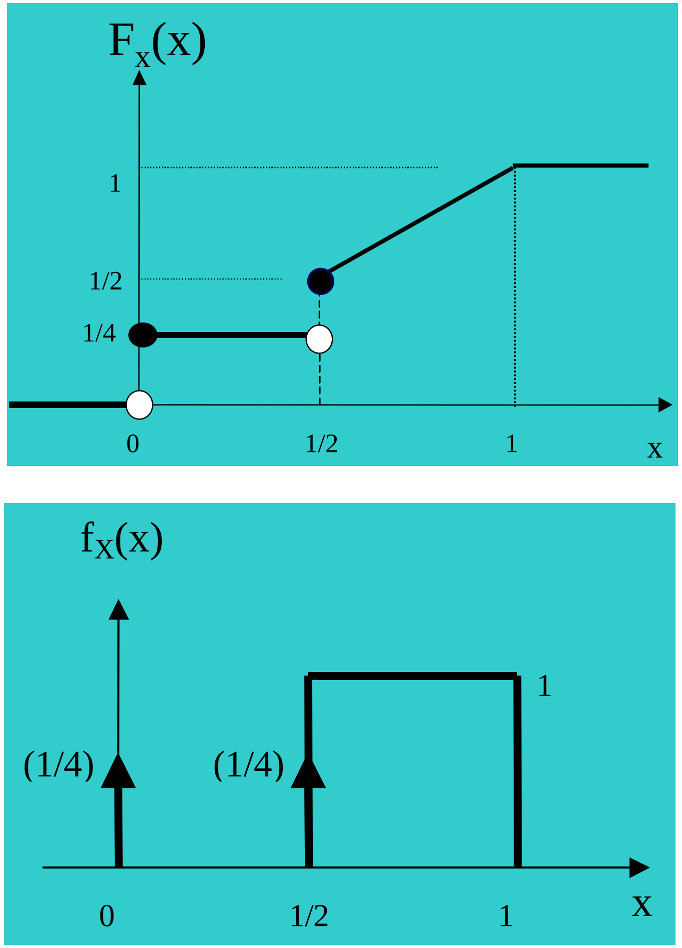
- $\delta(x)$ is also called the "shifting function" because it "shifts out" one value from another function that multiplies it:

$$\int_{-\infty}^{+\infty} g(x)\delta(x-a)dx = g(a)$$

★REMEMBER

A Dirac delta function only has meaning under an integral sign

PDF Example



$$P(0 < X < 1/2) = \text{area under } f_X(x)$$

between 0 and 1/2 , but not including end points, so deltas are not included

$$= 0$$

$$P(1/2 \le X \le 3/4) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Area of impulse at x=1/2

Half the area of the rectangle

PDFs for Discrete RV's

- PDFs can be used for discrete RV's - these PDFs comprise only impulses.

$$\text{Binomial PMF}: \quad p_X(x) = \binom{n}{x} p^x q^{n-x} \qquad x = 0, 1, \ldots, n$$

$$\text{Binomial PDF}: \quad f_X(x) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k} \delta(x-k)$$

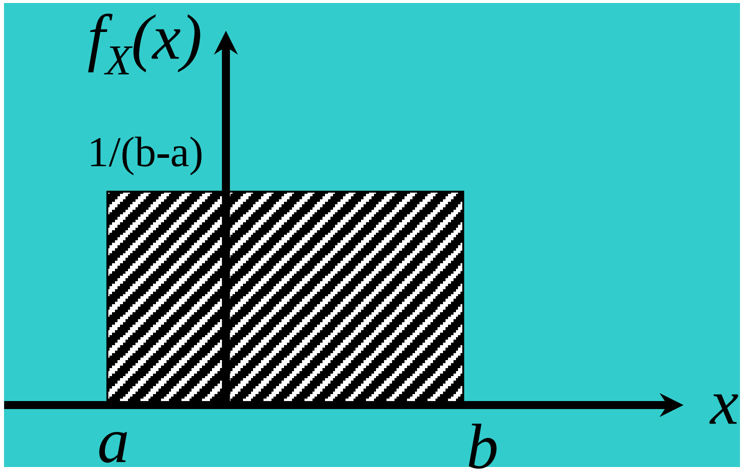
Some Special Continuous Random Variables

- Uniform
- Exponential
- Gaussian
- Rician
- Rayleigh
- Cauchy

Uniform

X ~ **uniform[a, b]** if a < b and

$$f_X(x) = \begin{cases} \dfrac{1}{b - a} & a \le x \le b \\ 0 & o.w. \end{cases}$$



Shorthand: $X \sim U[a, b]$

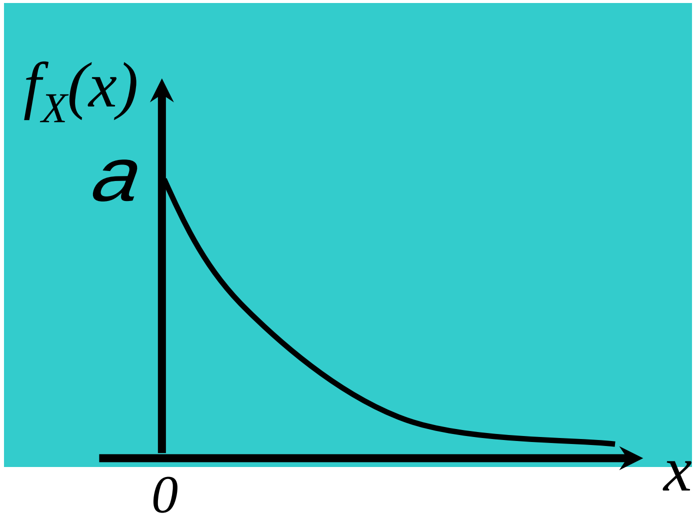
Used to describe random quantities that we can bound, but otherwise know nothing about.

Ex: Phase of a radio frequency (RF) carrier at a receiver: $U[-\pi, \pi]$

Exponential

$$X \sim \exp(\alpha)$$

$$f_X(x) = \begin{cases} \alpha e^{-\alpha x} & x \geq 0, \alpha > 0 \\ 0 & o.w. \end{cases}$$



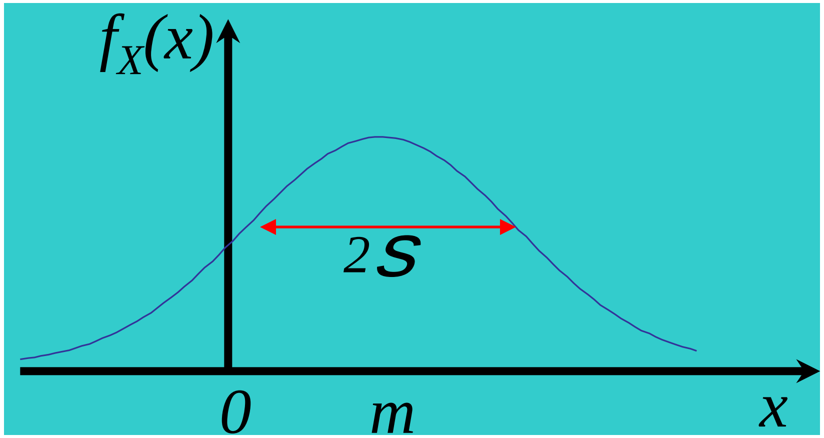
- Used to describe times between randomly occurring events.
  - ➢Time-to-failure
  - ➢Inter-arrival times
- α is also called "rate parameter"
  - Relationship with Poisson r.v ?

CDF of Exponential R.V.

- The CDF of exponential R.V. is given by

$$F_X(x) = \begin{cases} 1 - e^{\alpha x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Gaussian (Normal)



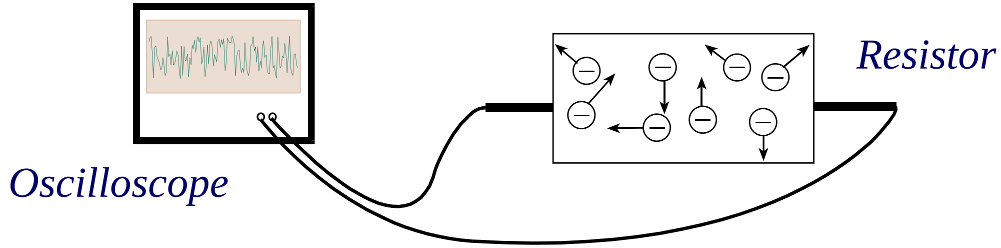
$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-m)^2}{2\sigma^2}}$$

Shorthand: $X \sim N(m, \sigma^2)$

$X \sim N(0,1)$ called standard normal.

Used to model effects that are accumulations of large numbers of independent effects.

Ex: Thermal Noise is the voltage caused by the independent motions of thermally excited electrons in a resistance.

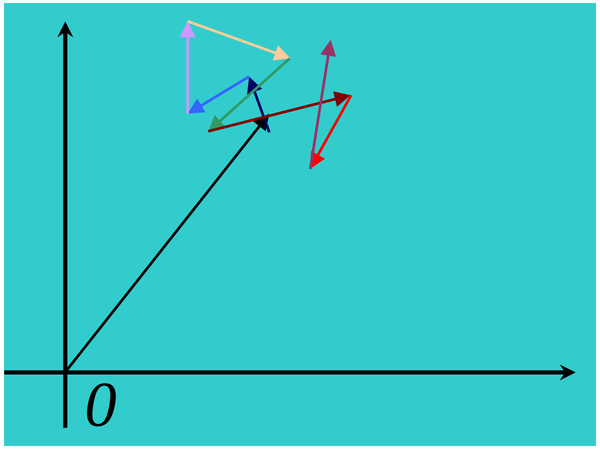


Rician

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2} \exp\left\{ -\dfrac{x^2 + A^2}{2\sigma^2} \right\} \cdot \mathrm{I}_0\left(\dfrac{\mathrm{Ax}}{\sigma^2} \right) & A \geq 0, x \geq 0 \\ 0 & x < 0 \end{cases}$$

Bessel function of the first kind, order 0.

X models the magnitude of the sum of one large known vector and lots of small random vectors.

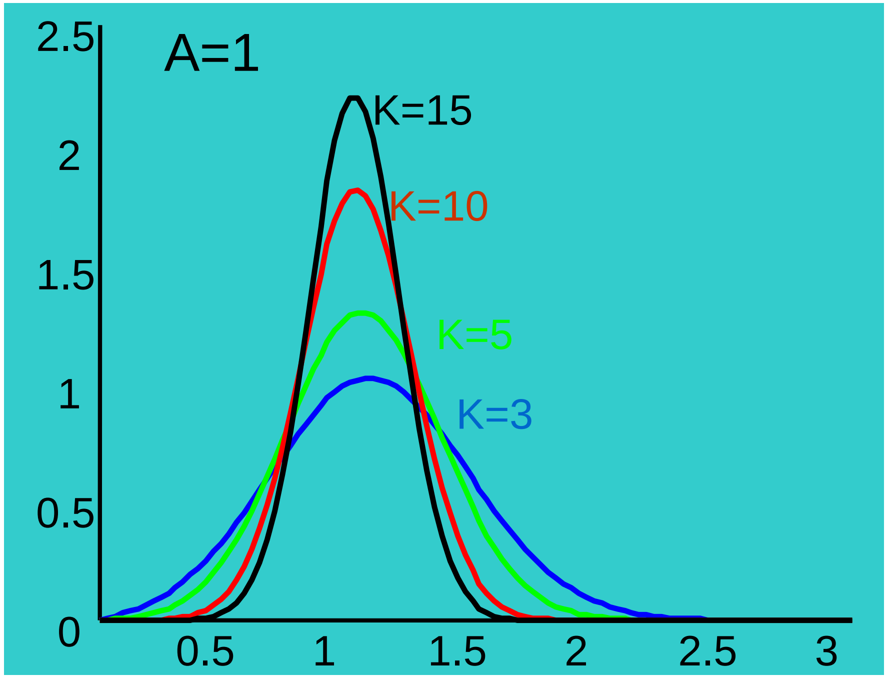


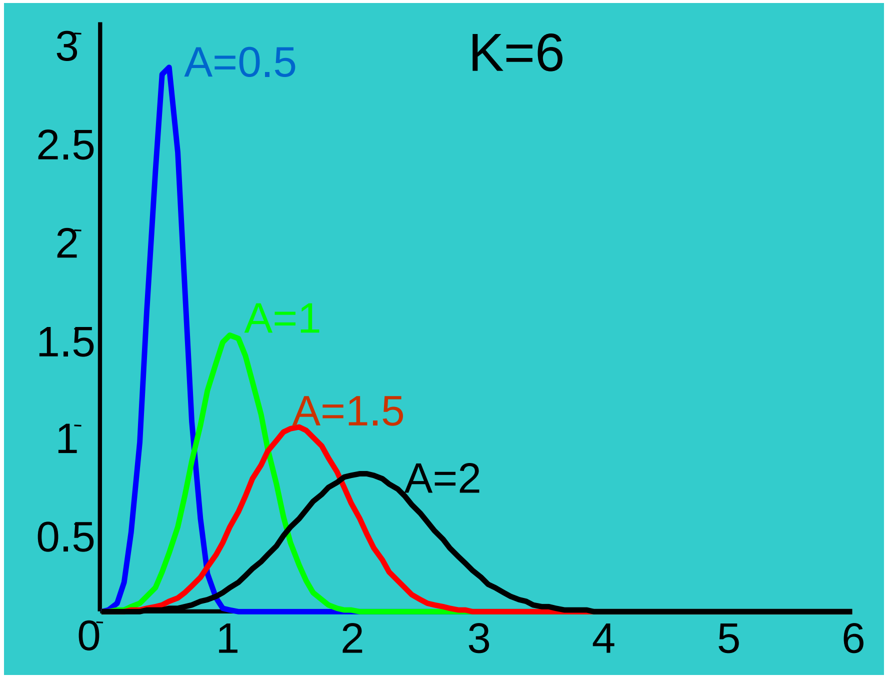
A is the length of the large vector.

In radio links with line-of-sight (LOS) and scattered paths, the signal envelope is Rician-distributed.

Rician, Cont'd

$K = A^2/2\sigma^2$ is called the "K-factor" or "Rician factor," and represents the dominant component power over the average power of the sum of the other components.





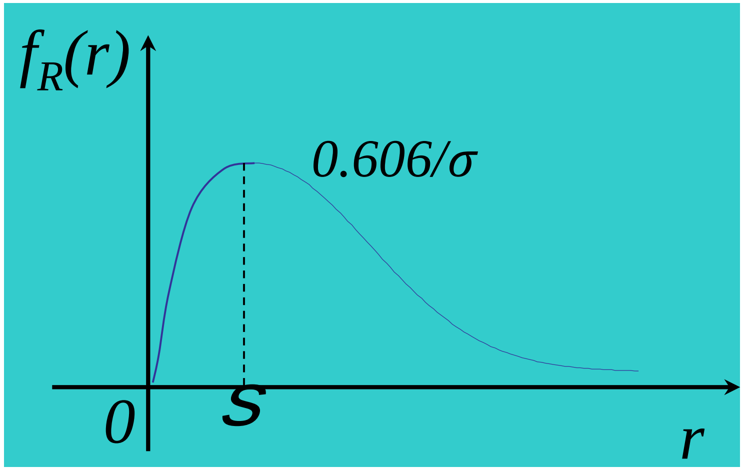
Rayleigh

$$f_X(x) = \begin{cases} \dfrac{x}{\sigma^2}\exp\left\{-\dfrac{x^2}{2\sigma^2}\right\} & x \ge 0 \\ 0 & x < 0 \end{cases}$$

Special case of Rician when $A=0$.

Models envelopes in radio signals with no LOS path

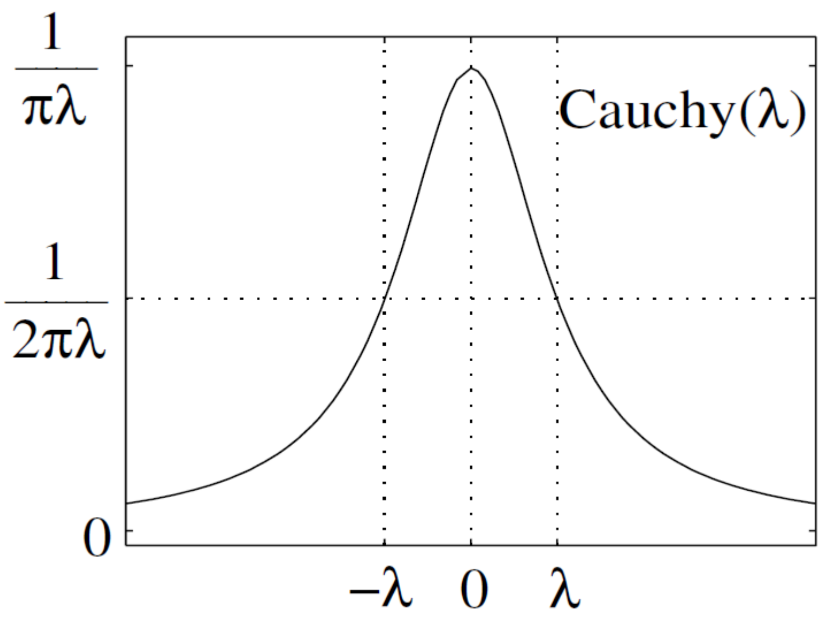
If $Z=X^2$, Z is an exponential RV.



Cauchy

- The Cauchy random variable with parameter $\lambda > 0$

$$f_X(x) = \frac{\lambda / \pi}{\lambda^2 + x^2}$$



- X~ Cauchy(λ)

- The Cauchy random variable arises as the **tangent of a uniform random variable** and also as the **quotient of independent Gaussian random variables.**

Short Summary

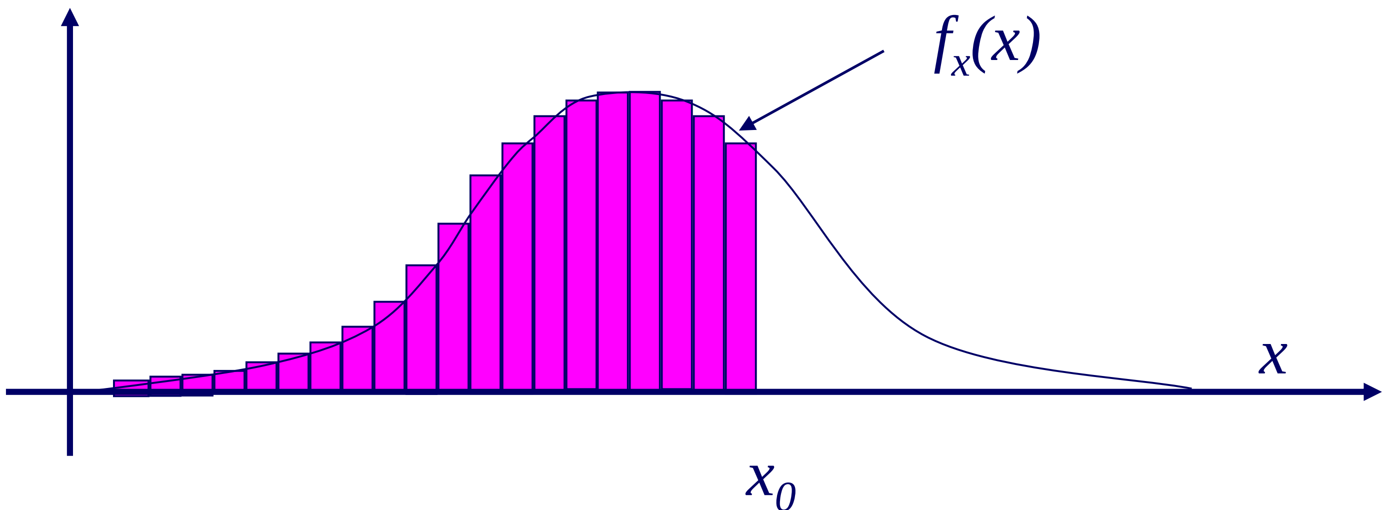
- PDFs are derivatives of CDFs

- Some continuous random variables
  - Uniform
  - Exponential
  - Gaussian
  - Rician
  - Rayleigh
  - Cauchy

Gaussian CDF

A Problem with the Gaussian PDF

REMEMBER The Gaussian PDF has no anti-derivative. In practice, it must be integrated numerically.



$$F_x(x_o) = \int_{-\infty}^{x_o} f_x(x)dx \approx \text{sum of rectangular areas}$$

Some Common Functions

- Several functions are commonly available in software packages, on calculators, and tabulated in books:

$\Phi(x)$ = CDF of the "standard normal" RV. The standard normal has zero mean and unit variance.

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$Q(x) = 1 - \Phi(x) = \int_{x}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$ = complement of $\Phi(x)$

$\mathrm{erf}(x)$ = error function

$\mathrm{erfc}(x)$ = complementary error function

Using $\Phi(x)$

- Use change of variables in a CDF integral to make the integrand look like a standard normal pdf.

$$\text{Ex: Let } X \sim N(5,9). \quad \text{Compute } P(X < 6.05).$$

Using $\Phi(x)$

Ex : Let $X \sim N(5,9)$. Compute $P(X < 6.05)$.

$$P(X < 6.05) = \int_{-\infty}^{6.05} \frac{1}{\sqrt{2\pi 9}} e^{-\frac{(x-5)^2}{2\cdot 9}} dx$$

Let $y = \dfrac{x-5}{3}$. Then, $P(X < 6.05) =$

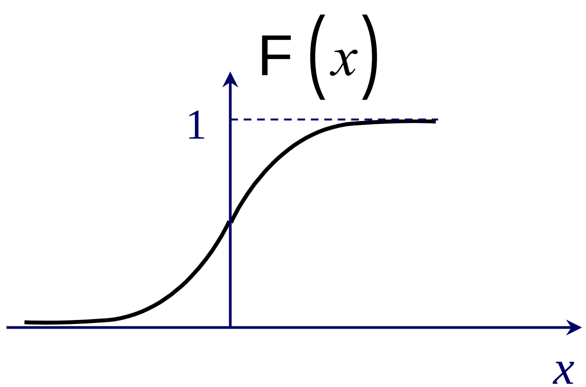
$$\int_{-\infty}^{\frac{6.05-5}{3}} \frac{1}{\sqrt{2\pi 9}} e^{-\frac{y^2}{2}} (3dy) = \int_{-\infty}^{0.35} \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}_{\text{std normal pdf}} dy = \Phi(0.35)$$

Use of $\Phi(x)$ Table

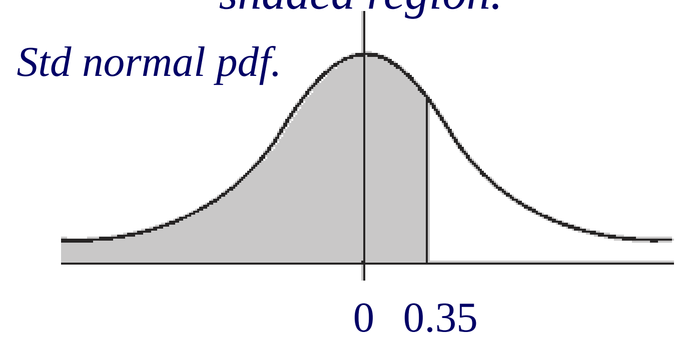
Part of numeric table for the Normal Distribution Function:

X	.00	0.01	0.02	.03	0.04	0.05
0.0	.5000	.5040	.5080	.5120	.5160	.5199
0.1	.5398	.5438	.5478	.5517	.5557	.5596
0.2	.5793	.5832	.5871	.5910	.5948	.5987
0.3	.6179	.6217	.6255	.6293	.6331	.6368
0.4	.6554	.6591	.6628	.6664	..6700	.6736
0.5	.6915	.6950	.6985	.7019	.7054	.7088
0.6	.7257	.7291	.7324	.7357	.7389	.7422
0.7	.7580	.7611	.7642	.7673	.7704	.7734
0.8	.7881	.7910	.7939	.7957	.7995	.8023
0.9	.8159	.8186	.8212	.8238	.8264	.8289

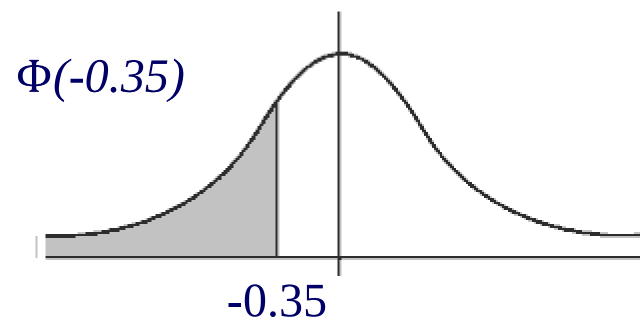
• • •



Φ(0.35) is the area of the shaded region.



$$F(0.35) = 1 - F(-0.35) = 0.6368$$



Using Q(x)

- The CDF of a Gaussian random variable is given by

$$
\begin{aligned}
F(x) &= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-m)^2}{2\sigma^2}}\, dt \\
&= 1 - \int_{x}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-m)^2}{2\sigma^2}}\, dt \\
&= 1 - \int_{\frac{x-m}{\sigma}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du \\
&= 1 - Q\left(\frac{x-m}{\sigma}\right)
\end{aligned}
$$

Using Q(x) Table

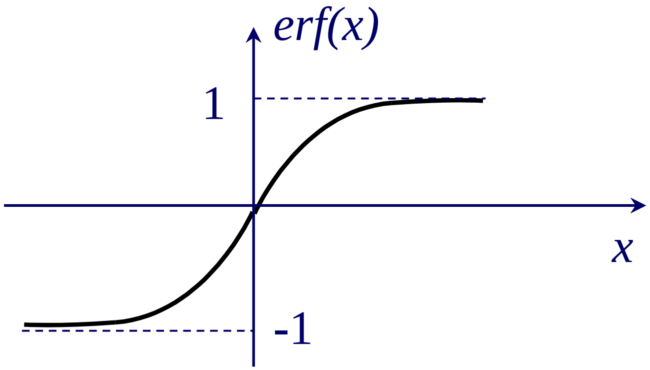
Table of Q Function Values

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0	0.500000	1.8	0.035930	3.6	0.000159	5.4	3.3320×10^{-8}
0.1	0.460170	1.9	0.028717	3.7	0.000108	5.5	1.8990×10^{-8}
0.2	0.420740	2	0.022750	3.8	7.2348×10^{-5}	5.6	1.0718×10^{-8}
0.3	0.382090	2.1	0.017864	3.9	4.8096×10^{-5}	5.7	5.9904×10^{-9}
0.4	0.344580	2.2	0.013903	4	3.1671×10^{-5}	5.8	3.3157×10^{-9}
0.5	0.308540	2.3	0.010724	4.1	2.0658×10^{-5}	5.9	1.8175×10^{-9}
0.6	0.274250	2.4	0.008198	4.2	1.3346×10^{-5}	6	9.8659×10^{-10}
0.7	0.241960	2.5	0.006210	4.3	8.5399×10^{-6}	6.1	5.3034×10^{-10}
0.8	0.211860	2.6	0.004661	4.4	5.4125×10^{-6}	6.2	2.8232×10^{-10}
0.9	0.184060	2.7	0.003467	4.5	3.3977×10^{-6}	6.3	1.4882×10^{-10}
1	0.158660	2.8	0.002555	4.6	2.1125×10^{-6}	6.4	7.7689×10^{-11}
1.1	0.135670	2.9	0.001866	4.7	1.3008×10^{-6}	6.5	4.0160×10^{-11}
1.2	0.115070	3	0.001350	4.8	7.9333×10^{-7}	6.6	2.0558×10^{-11}
1.3	0.096800	3.1	0.000968	4.9	4.7918×10^{-7}	6.7	1.0421×10^{-11}
1.4	0.080757	3.2	0.000687	5	2.8665×10^{-7}	6.8	5.2309×10^{-12}
1.5	0.066807	3.3	0.000483	5.1	1.6983×10^{-7}	6.9	2.6001×10^{-12}
1.6	0.054799	3.4	0.000337	5.2	9.9644×10^{-8}	7	1.2799×10^{-12}
1.7	0.044565	3.5	0.000233	5.3	5.7901×10^{-8}	7.1	6.2378×10^{-13}

Using erf(x)

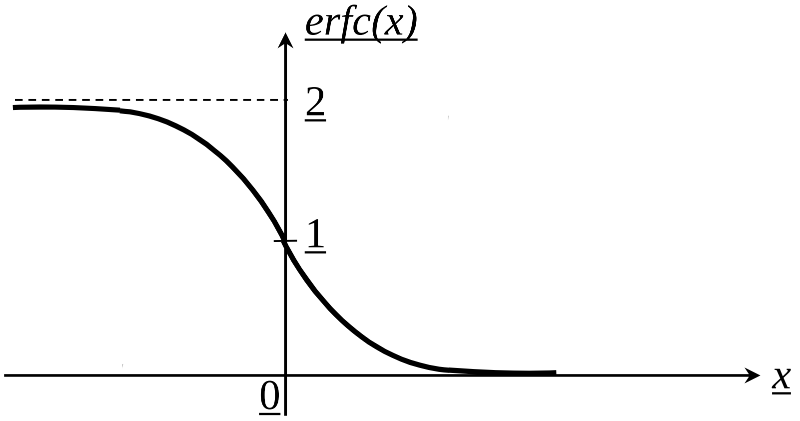
- erf(x) is a version of $\Phi(x)$ with odd symmetry, extreme values +1 and -1.

$$\mathrm{erf}(x) = 2\Phi\left(\sqrt{2}x\right) - 1$$
$$= 2\int_{-\infty}^{x} \frac{1}{\sqrt{\pi}} e^{-z^2}\, dz - 1$$
$$= \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-z^2}\, dz$$



erfc(x)

$$\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-z^2}\, dz$$



Gaussian Integration

- Integrands that are scaled Gaussian pdfs must be numerically integrated. Learn to recognize the form:

$$Ke^{-ax^2+bx+c}$$

  - K, b and c are any real numbers. "a" must be a positive real number.

Gaussian Integration Example

- Typical problem: Let a RV X have the following PDF:

$$f_X(x) = \begin{cases} Ke^{-4x^2+3x} & x > 2 \\ 0 & \text{ow} \end{cases}$$

What is K? Use the fact that $\int_{-\infty}^{\infty} f_X(x)dx = 1$

$$K = \frac{1}{\int_2^{\infty} e^{-4x^2+3x} dx} = \frac{1}{e^{\frac{9}{16}} \sqrt{\frac{\pi}{4}} \cdot Q\left(\frac{13}{\sqrt{8}}\right)}$$

Can you derive it?

Gaussian Integration Example - I

To find *K*, use the property of PDFs: $\int_{-\infty}^{\infty} f_X(x)dx = 1$

$$\int_{2}^{\infty} Ke^{-4x^2+3x}dx = 1$$

Complete the square in the exponent:

$$-4x^2+3x = -4(x^2-\frac{3}{4}x) = -4\left[x^2-\frac{3}{4}x+\left(\frac{3}{8}\right)^2-\left(\frac{3}{8}\right)^2\right] = -4\left(x-\frac{3}{8}\right)^2+4\left(\frac{3}{8}\right)^2$$

Substitute into integral and use $e^{a+b} = e^a e^b$:

$$\int_{2}^{\infty} Ke^{-4\left(x-\frac{3}{8}\right)^2+4\left(\frac{3}{8}\right)^2}dx = Ke^{4\left(\frac{3}{8}\right)^2}\int_{2}^{\infty} e^{-4\left(x-\frac{3}{8}\right)^2}dx$$

Gaussian Integration Example - II

Now look at the exponent and match items with $-\frac{(x-m)^2}{2\sigma^2}$

$$-4\left(x-\frac{3}{8}\right)^2 = -\frac{(x-m)^2}{2\sigma^2}$$

$$\Rightarrow m = \frac{3}{8}, \quad \sigma^2 = \frac{1}{8}$$

Gaussian Integration Example - III

Construct proper coefficient to make integrand a Gaussian PDF (i.e. the $\frac{1}{\sqrt{2\pi\sigma}}$ factor)

$$\int_{2}^{\infty} K e^{-4x^2+3x} dx$$

$$= K e^{4\left(\frac{3}{8}\right)^2} \sqrt{2\pi\frac{1}{8}} \cdot \int_{2}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\frac{1}{8}}} e^{\frac{-4\left(x-\frac{3}{8}\right)^2}{2\cdot\frac{1}{8}}}}_{\text{A Gaussian pdf for } X\sim N\left(\frac{3}{8},\frac{1}{8}\right)} dx$$

Gaussian Integration Example - IV

The change of variables $y = \frac{x-m}{\sigma} = \frac{x-\frac{3}{8}}{\sqrt{\frac{1}{8}}}$

gives the $Q(x)$ form:

$$\int_{2}^{\infty} Ke^{-4x^2+3x}dx = Ke^{4\left(\frac{3}{8}\right)^2}\sqrt{2\pi\frac{1}{8}}\cdot Q\left(\frac{2-3/8}{\sqrt{1/8}}\right) = 1$$

$$\therefore K = \frac{1}{e^{\frac{9}{16}}\sqrt{\frac{\pi}{4}}\cdot Q\left(\frac{13}{\sqrt{8}}\right)}$$

Short Summary

- Gaussian CDF
- Some common functions
  - $\Phi(x)$ CDF of the standard Gaussian RV
  - Q function
  - Error function
  - Complementary error function
- Gaussian integration

Conditional Probability

Conditional Probability

- Let B be an event with P(B)>0. For any event A, we define the conditional probability of A given B as:

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(AB)}{P(B)}$$

$P(AB)$ is shorthand for $P(A \cap B)$

Conditioning a RV on an Event Involving Only that RV

- Here, we consider events of non-zero probability.
- Define the conditional CDF of X given the event B, assuming $\mathrm{P}(B) > 0$, as:

$$F_{X|B}(x \mid B) = P(X \leq x \mid B) = \frac{P(\{X \leq x\} \cap B)}{P(B)}$$

- $F_{X|B}(x|B)$ is a valid CDF:
  1. $0 \leq F_{X|B}(x|B) \leq 1$
  2. $\lim_{x \to +\infty} F_{X|B}(x|B) = 1$
  3. $F_{X|B}(x|B)$ is non-decreasing

Proof of Property 2

$\{X \to \infty\}$ is the certain event Ω

$$\therefore \{X \le +\infty\} \cap B = B$$

$$F_{X|B}(+\infty \mid B) = \frac{P(B)}{P(B)} = 1$$

Proof of Property 3

Let $x_1 \le x_2$. Then,

$$\{X \le x_2\} = \{X \le x_1\} \cup \{x_1 < X \le x_2\}$$

disjoint

Next, intersect both sides with the event B.

$$\{X \le x_2\} \cap B = (\{X \le x_1\} \cap B) \cup (\{x_1 < X \le x_2\} \cap B)$$

still disjoint

Take probability of both sides,

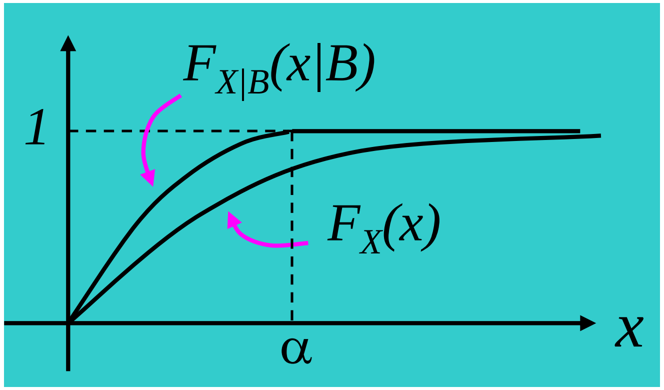
$$P(\{X \le x_2\} \cap B) = P(\{X \le x_1\} \cap B) + \underbrace{P(\{x_1 < X \le x_2\} \cap B)}_{\text{This term is } \ge 0}$$

$$\Rightarrow P(\{X \le x_1\} \cap B) \le P(\{X \le x_2\} \cap B)$$

$$\Rightarrow F_{X|B}(x_1) \le F_{X|B}(x_2)$$

Example

Suppose X is an exponential RV and let $B = \{ X \le \alpha \}$



$$\{X \le x\} \cap B = \begin{cases} X \le x & x < \alpha \\ X \le \alpha & x \ge \alpha \end{cases}$$

$$F_{X|B}(x \mid B) = \frac{P(\{X \le x\} \cap B)}{P(B)} = \begin{cases} \dfrac{F_X(x)}{F_X(\alpha)} & x < \alpha \\ 1 & x \ge \alpha \end{cases}$$

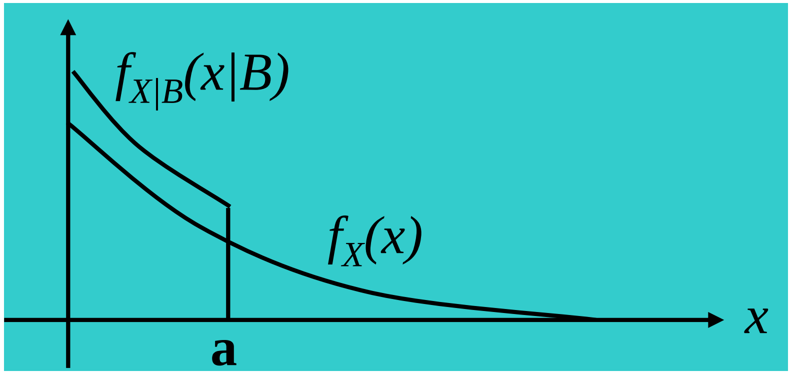
PDF Conditioned on an Event

Continuing the example,

$$f_{X|B}(x\,|\,B) = \frac{d}{dx} F_{X|B}(x\,|\,B)$$

$$f_{X|B}(x\,|\,B) = \begin{cases} \dfrac{d}{dx}\dfrac{F_X(x)}{F_X(\alpha)} & x < \alpha \\ 0 & x \ge \alpha \end{cases}$$

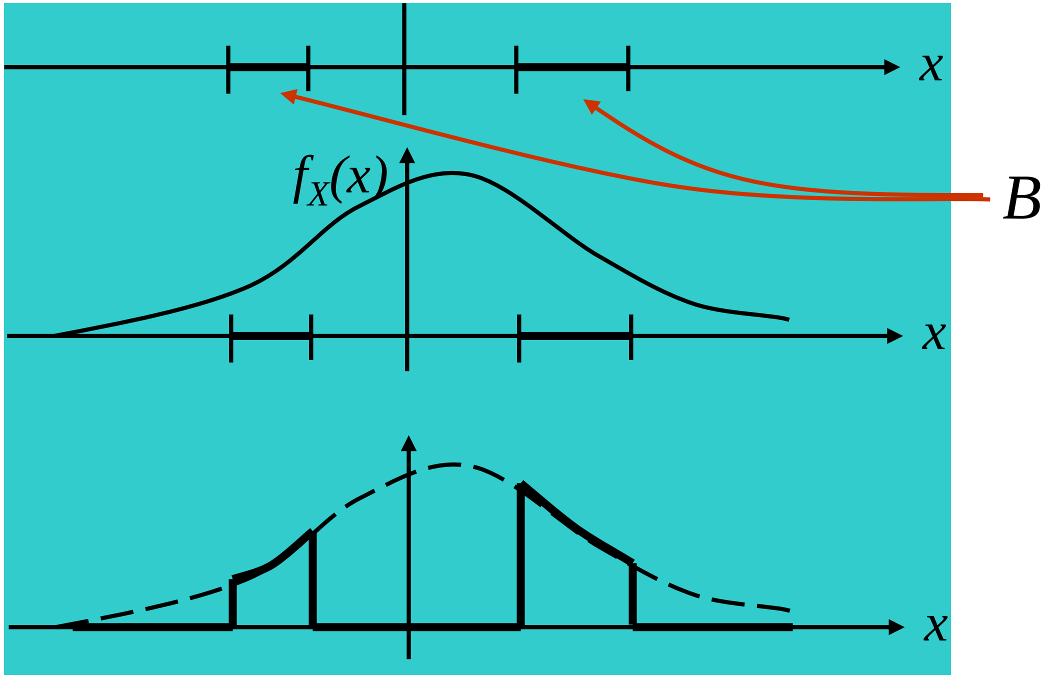
$$= \begin{cases} \dfrac{f_X(x)}{F_X(\alpha)} & x < \alpha \\ 0 & x \ge \alpha \end{cases}$$



Graphical Interpretation

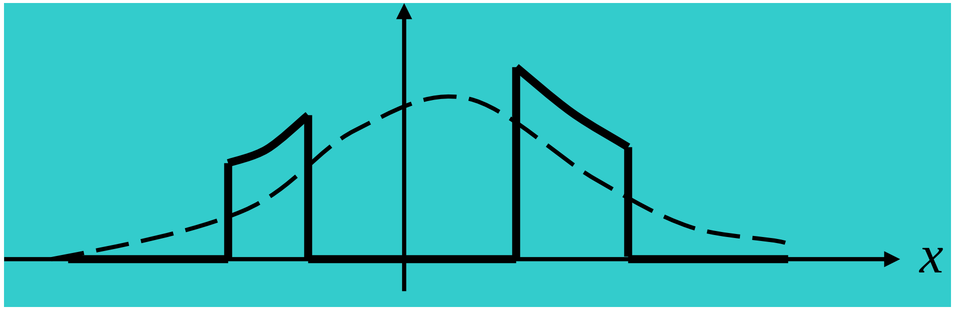
When B is a union of intervals on X, two steps are taken:

1. "zero-out" $f_X(x)$ everywhere but on these intervals.



Graphical Interpretation – Cont'd

2. Normalize (i.e. scale) zeroed-out version by its own area to get a unit-area function.



Short Summary

- Conditional probability
- Conditioning a RV on an event involving only that RV

$$F_{X|B}(x|B) = P(X \le x|B) = \frac{P(\{X \le x\} \cap B)}{P(B)}$$

$$f_{X|B}(x|B) = \frac{d}{dx} F_{X|B}(x|B)$$

- Graphical interpretation
  - Zero-out
  - Normalization

Thank You!