Homework 5 Solutions

1. (a)

Ans: $E\{N(8)\} = 5$ implies that $\lambda = \frac{5}{8}$. $N(10)$ is Poisson distributed with mean $\lambda 10 = 50/8 = 25/4$. The variance of a Poisson RV is the same as its mean.

(b) Ans: $N(3)$ is Poisson distributed with mean $15/8$.

$$\begin{aligned} P\{N(3) \leq 2\} &= P\{N(3) = 0\} + P\{N(3) = 1\} + P\{N(3) = 2\} \\ &= e^{-3\lambda} + \frac{3\lambda}{1!}e^{-3\lambda} + \frac{(3\lambda)^2}{2!}e^{-3\lambda} \\ &= e^{-15/8}\left[1 + \frac{15/8}{1} + \frac{(15/8)^2}{2}\right] \\ &= 0.7105 \end{aligned}$$

(c) Ans:
$$P\{N(12) = 6|N(4) = 3\} = \frac{P\{N(12) = 6 \cap N(4) = 3\}}{P\{N(4) = 3\}}$$

where
$$P\{N(12) = 6 \cap N(4) = 3\} = P\{N(12) - N(4) = 3 \cap N(4) = 3\}$$

Because of independent and stationary increments,

$$P\{N(12)-N(4) = 3 \cap N(4) = 3\} = P\{N(12)-N(4) = 3\}P\{N(4) = 3\} = P\{N(8) = 3\}P\{N(4) = 3\}$$

Therefore,

$$P\{N(12) = 6|N(4) = 3\} = \frac{P\{N(8) = 3\}P\{N(4) = 3\}}{P\{N(4) = 3\}} = P\{N(8) = 3\}$$

The mean of $N(8)$ is $8 \cdot 5/8 = 5$. Hence,

$$P\{N(12) = 6|N(4) = 3\} = \frac{5^3}{3!}e^{-5} = 0.1404$$

2. Ans: Because of stationary increments, we can change the time interval to 0 to 3.

$$\begin{aligned} P\{X(1) = 2|X(3) = 14\} &= \frac{P\{X(1) = 2 \cap X(3) = 14\}}{P\{X(3) = 14\}} \\ &= \frac{P\{X(1) = 2 \cap X(3) - X(1) = 12\}}{P\{X(3) = 14\}} \\ &= \frac{P\{X(1) = 2\}P\{X(3) - X(1) = 12\}}{P\{X(3) = 14\}} \\ &= \frac{P\{X(1) = 2\}P\{X(2) = 12\}}{P\{X(3) = 14\}} \\ &= \frac{\left(\frac{5^2}{2!}e^{-5}\right)\left(\frac{10^{12}}{12!}e^{-10}\right)}{\frac{15^{14}}{14!}e^{-15}} \\ &= 0.078 \end{aligned}$$

To solve using uniform RVs, let $U_1, U_2, \ldots, U_{14}$ be iid uniform RVs that are distributed over $[0, 3]$. One possible scenario is as follow:

$$P\{X(1) = 2|X(2) = 12\} = P\{U_1 < 1 \cap U_2 < 1 \cap U_3 > 1 \cap U_4 > 1 \cap \ldots \cap U_{14} > 1 | U_1 \leq U_2 \leq \ldots \leq U_{14}\}$$

Following this idea, we can calculate the unconditional version of the above probability, and multiply it by all possible ways we can assign two of the 14 RVs to fall into the region $[0, 1]$ and the other 12 RVs to be in the region $(1, 3]$. For one uniform RV, there is a 1/3 probability that the RV is less than 1, and a 2/3 probability that the RV is greater than 1. There are "14 select 2" ways to choose the pair of RVs that are less than 1. Therefore,

$$P\{X(1) = 2|X(2) = 12\} = \frac{14!}{2!12!}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{12} = 0.078$$

3.

(a) $$E\{X(t)\} = E\{Y\cos(2\pi f_0 t + \theta)\} = E\{Y\}E\{\cos(2\pi f_0 t + \theta)\}$$

The second factor is the expectation of the random phase sinusoid, which is zero. Therefore, $E\{X(t)\} = 0$.

(b) $$E\{X(t+\tau)X(t)\} = E\{Y^2\}E\{\cos(2\pi f_0[t+\tau] + \theta)\cos(2\pi f_0 t + \theta)\}$$

The second expectation is the autocorrelation of the random phase sinusoid, which is $\frac{1}{2}\cos(2\pi f_0 \tau)$. $E\{Y^2\} = \sigma^2 + \mu^2$. Therefore, $E\{X(t+\tau)X(t)\} = (\sigma^2 + \mu^2)\frac{1}{2}\cos(2\pi f_0 \tau)$.

(c)

Ans: Because the mean of $X(t)$ is constant and its autocorrelation depends on τ and not t, $X(t)$ is WSS.

4. (a)

Ans: $W(t)$ is Gaussian with zero mean and variance αt. Therefore,

$$P\{|W(t)| < 3\alpha\} = P\{W(t) < 3\alpha\} - P\{W(t) < -3\alpha\}$$

Recall that if Z is Gaussian with mean m and standard deviation s, then

$$P(Z < b) = \Phi\left(\frac{b-m}{s}\right)$$

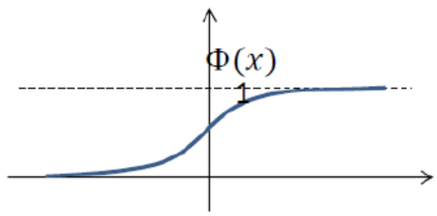
Applying this, we have:

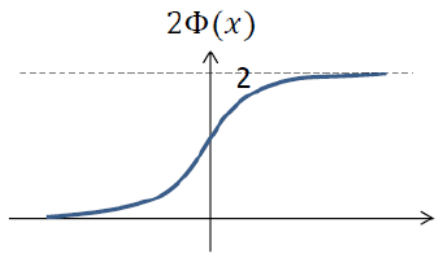
$$P\{|W(t)| < 3\alpha\} = \Phi\left(\frac{3\alpha - 0}{\sqrt{\alpha t}}\right) - \Phi\left(\frac{-3\alpha - 0}{\sqrt{\alpha t}}\right) = \Phi\left(3\sqrt{\frac{\alpha}{t}}\right) - \Phi\left(-3\sqrt{\frac{\alpha}{t}}\right)$$

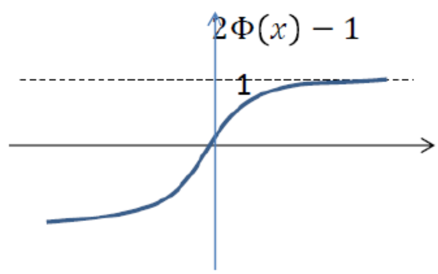
Using $\Phi(x) = 1 - \Phi(-x)$,

$$P\{|W(t)| < 3\alpha\} = 2\Phi\left(3\sqrt{\frac{\alpha}{t}}\right) - 1$$

The following graphs give an idea of the shape of $2\Phi(x) - 1$. As a function of t, $2\Phi\left(3\sqrt{\frac{\alpha}{t}}\right) - 1$ would be a version of $2\Phi(x) - 1$ that is distorted in time, and would apply only for the positive axis.







(b) Ans:

$$
\begin{aligned}
f_{W(t)|B}(w|B) =& \frac{\mathrm{d}}{\mathrm{d}w} F_{W(t)|B}(w|B) \\
F_{W(t)|B}(w|B) =& P(W(t) \le w | W(4) = 1) \\
=& \frac{P(W(t) \le w \cap W(4) = 1)}{P(W(4) = 1)} \\
=& \frac{P(W(t) - W(4) \le w - 1 \cap W(4) = 1)}{P(W(4) = 1)} \\
=& \frac{P(W(t) - W(4) \le w - 1) P(W(4) = 1)}{P(W(4) = 1)} \\
=& P(W(t) - W(4) \le w - 1) \\
=& P(W(t - 4) \le w - 1) \\
=& P(Z \le w - 1)
\end{aligned}
$$

where Z is zero mean Gaussian, with variance $\alpha(t - 4)$. Therefore,

$$F_{W(t)|B}(w|B) = F_Z(w - 1)$$

$$f_{W(t)|B}(w|B) = \frac{\mathrm{d}}{\mathrm{d}w} F_Z(w - 1) = f_Z(w - 1) = \frac{1}{\sqrt{2\pi\alpha(t - 4)}} \exp\left(-\frac{[w - 1]^2}{2\alpha(t - 4)}\right)$$

6.

Ans: Let $Y(t) = \frac{1}{2}X^2(t)$. Then $\frac{\mathrm{d}}{\mathrm{d}t}Y(t) = X(t) \cdot \frac{\mathrm{d}X(t)}{\mathrm{d}t}$. We know $E\{\frac{\mathrm{d}}{\mathrm{d}t}Y(t)\} = \frac{\mathrm{d}}{\mathrm{d}t}E\{Y(t)\}$. And $E\{Y(t)\} = E\{\frac{1}{2}X^2(t)\} = \frac{1}{2}R_X(0)$. Since this is not dependent on t, $\frac{\mathrm{d}}{\mathrm{d}t}E\{Y(t)\} = 0$. Thus, $E\{X(t)\frac{\mathrm{d}X(t)}{\mathrm{d}t}\} = 0$.

7.

Ans: Applying the above result to $R_Y(t_1, t_2)$, we arrive at

$$R_Y(t_1, t_2) = E\{X^2(t_1)\}E\{X^2(t_2)\} + 2(E\{X(t_1)X(t_2)\})^2 = R_X^2(0) + 2R_X(t_1, t_2)$$

Using the fact that $X(t)$ is WSS, we conclude that $Y(t)$ is also WSS, with autocorrelation

$$R_Y(\tau) = R_X^2(0) + 2R_X^2(\tau)$$