Homework 6 Solutions

1.

We will show that $p_{N_{k+1}|N_k}(n_{k+1}|n_k) = p_{N_{k+1}|N_k,\cdots,N_1}(n_{k+1}|n_k,\cdots,n_1)$ and, thus, that $\{N_l\}_{l\geq 0}$ a Markov process. Using substitution law, we get that

$$\begin{aligned}
p_{N_{k+1}|N_k,\cdots,N_1}(n_{k+1}|n_k,\cdots,n_1) &= \mathsf{P}_\mathrm{r}(N_{k+1} = n_{k+1}|N_i = n_i,\ i = k,\cdots,1) \\
&= \mathsf{P}_\mathrm{r}\left(\sum_{j=1}^{k+1} X_j = n_{k+1}|N_i = n_i,\ i = k,\cdots,1\right) \\
&= \mathsf{P}_\mathrm{r}\left(X_{k+1} = n_{k+1} - \sum_{j=1}^{k} X_j|N_i = n_i,\ i = k,\cdots,1\right) \\
&= \mathsf{P}_\mathrm{r}(X_{k+1} = n_{k+1} - N_k|N_i = n_i,\ i = k,\cdots,1) \\
&= \mathsf{P}_\mathrm{r}(X_{k+1} = n_{k+1} - n_k|N_i = n_i,\ i = k,\cdots,1) \\
&= \mathsf{P}_\mathrm{r}(X_{k+1} = n_{k+1} - n_k) \quad (X_{k+1} \text{ is independent of } N_i,\ i = 1,\ldots,k) \\
&= p_X(n_{k+1} - n_k).
\end{aligned}$$

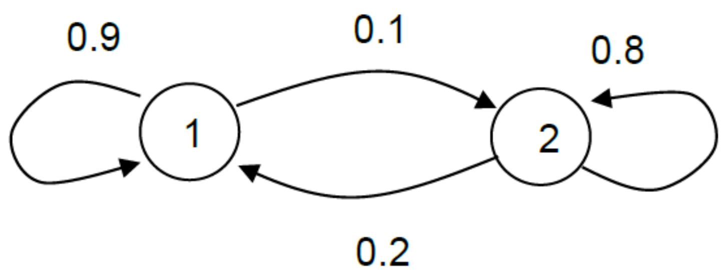
Similarly,

$$\begin{aligned}
p_{N_{k+1}|N_k}(n_{k+1}|n_k) &= \mathsf{P}_\mathrm{r}(N_{k+1} = n_{k+1}|N_k = n_k) \\
&= \mathsf{P}_\mathrm{r}\left(\sum_{j=1}^{k+1} X_j = n_{k+1}|N_k = n_k\right) \\
&= \mathsf{P}_\mathrm{r}\left(X_{k+1} = n_{k+1} - \sum_{j=1}^{k} X_j|N_k = n_k\right) \\
&= \mathsf{P}_\mathrm{r}(X_{k+1} = n_{k+1} - N_k|N_k = n_k) \\
&= \mathsf{P}_\mathrm{r}(X_{k+1} = n_{k+1} - n_k|N_k = n_k) \\
&= \mathsf{P}_\mathrm{r}(X_{k+1} = n_{k+1} - n_k) \quad (X_{k+1} \text{ is independent of } N_k) \\
&= p_X(n_{k+1} - n_k).
\end{aligned}$$

2. (a)

Prove by induction: $\pi^1 = \pi^0\Pi$. Suppose that $\pi^{n-1} = \pi^0\Pi^{n-1}$. Then we left multiply by both sides to get $\pi^{n-1}\Pi = \pi^0\Pi^{n-1}\Pi = \pi^0\Pi^n = \pi^n$.

(b)



(c)

Directly from the state transition diagram, we start at state $X[0] = 1$ with probability 0.9 to stay in this state and probability 0.1 to transit to state 2. Thus, the probability that the first transition to state 2 occurs at time n is:

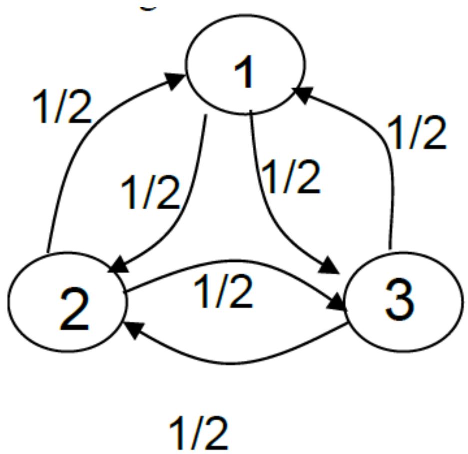
$$P\{X[k] = 1 | k \leq n-1 \text{ and } X[n] = 2\} = 0.9^{n-1} \cdot 0.1$$

3.

(a)

$$\begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

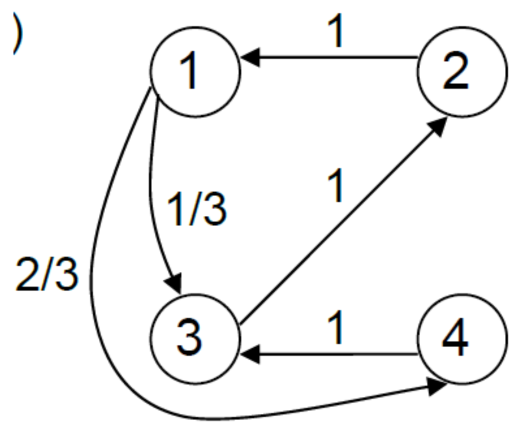
The MC is finite, irreducible and aperiodic. Therefore, all states are ergodic (recurrent non-null and aperiodic).



(b)

$$\begin{bmatrix} 0 & 0 & 1/3 & 2/3 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

All states of the MS are recurrent non-null and aperiodic, i.e., ergodic.



For example, there are only two ways to return to state 1 after leaving it: in 3 steps (path: $1 \to 3 \to 2 \to 1$, with probability $f_1^{(3)} = 1/3$) and in 4 steps (path: $1 \to 4 \to 3 \to 2 \to 1$, with probability $f_1^{(4)} = 2/3$). This is not periodic. Similar results can be checked for states 2 and 3.

Strictly speaking, state 4 is not periodic because in order to be periodic, the transition probabilities from state i to state j in n steps $p_{ij}(n)$ should be zero for all n except for any multiple of some integer number (the period). Since this is not satisfied for any integer, state 4 is not periodic.

4.

Ans: There are two ways to work on this problem. The first way is to find the true mean and show that the time average of any particular outcome will or will not converge to the true mean. The second way is to use the "limit of the integral of covariance" approach.

First way: $E\{X(t)\} = 0$. $X(t)$ is mean ergodic, because if you time-averaged a sample function over an ever-longer interval, your average would approach zero, because of the cosine term.

Second way: $C_X(\tau) = \frac{E\{Y^2\}}{2}\cos(\omega_0\tau)$, and $\frac{1}{T}\int_0^T C(\tau)\ \mathrm{d}\tau \to 0$ as $T \to \infty$ implies mean ergodicity.

Ans: There are three ways to work this problem.

First way: $Var\{X(t)\} = \sigma_Y^2 R_Z(0)$, where $Z(t) = \cos(\omega_0 t + \theta)$. Therefore, $Var\{X(t)\} = \frac{\sigma_Y^2}{2}$. The time estimate of variance is $V_T = \frac{1}{T}\int_0^T X^2(t)\ \mathrm{d}t$. Substituting, $V_T = \frac{1}{T}\int_0^T Y^2\cos^2(\omega_0 t+\theta)\ \mathrm{d}t$. In the limit as T goes to infinity, this approaches $\frac{Y^2}{2}$, which is dependent on the outcome of Y, and therefore $X(t)$ cannot be variance ergodic.

Second way:

$$\sigma_{V_T}^2 = E\{V_V^2\} = \frac{1}{T^2}\int_0^T\int_0^T E\{X^2(t)X^2(s)\}\ \mathrm{d}t\mathrm{d}s$$

$$E\{X^2(t)X^2(s)\} = E\{Y^4\}E\{Z^2(t)Z^2(s)\}$$

$$\begin{aligned}E\{Z^2(t)Z^2(s)\} =& \frac{1}{4}E\{[1+\cos(2\omega_0 t+2\theta)][1+\cos(2\omega_0 s+2\theta)]\}\\ =& \frac{1}{4}E\{1+\cos(2\omega_0 t+2\theta)+\cos(2\omega_0 s+2\theta)+\cos(2\omega_0 t+2\theta)\cos(2\omega_0 s+2\theta)\}\end{aligned}$$

The last term is

$$\cos(2\omega_0 t+2\theta)\cos(2\omega_0 s+2\theta) = \frac{1}{2}[\cos(2\omega_0[t-s]) + \cos(2\omega_0[t+s]+4\theta)]$$

After this is substituted, $Z^2(t)Z^2(s) = \frac{1}{4} + \frac{1}{8}\cos(2\omega_0[t-s]) + f(\theta)$. When the expectation with respect to θ is taken, the $f(\theta)$ term goes to zero. Therefore, $\sigma_{V_T}^2 = E\{V_T^2\} = \frac{1}{T^2}\int_0^T\int_0^T \frac{1}{4} + \frac{1}{8}\cos(2\omega_0[t-s])\ \mathrm{d}t\mathrm{d}s$. This quantity will not approach zero as T goes to infinity, so $X(t)$ is not variance ergodic.

Third way: The ergodicity conditions of a RP can be expressed in terms of the properties of the covariance spectrum.
$X(t)$ is WSS because:

1. Its mean is zero regardless of the mean of Y because the mean of $Z(t) = \cos(\omega_0 t+\theta)$ is also zero, i.e., $E\{X(t)\} = E\{YZ(t)\} = E\{Y\}E\{Z(t)\} = 0$
2. Its autocorrelation is $R_X(\tau) = E\{Y^2\}R_Z(\tau) = E\{Y^2\}\frac{1}{2}\cos(\omega_0\tau)$

Therefore, its autocovariance is $C_X(\tau) = R_X(\tau) - (E\{X(t)\})^2 = R_X(\tau)$, whose Fourier transform (the covariance spectrum) comprises a pair of impulses:

$$S_X^C(\omega) = \frac{E\{Y^2\}}{2}(\pi\delta(\omega-\omega_0) + \pi\delta(\omega+\omega_0))$$

- $X(t)$ is mean ergodic if and only if $S_X^C(\omega)$ has no impulses at the origin, $w = 0$. Therefore, $X(t)$ is mean ergodic.
- $X(t)$ is variance ergodic if and only if $S_X^C(\omega)$ has no impulses anywhere. Therefore, $X(t)$ is not variance ergodic.