Probability and Random Process

Aimin Tang

The University of Michigan- Shanghai Jiao Tong University Joint Institute
Shanghai Jiao Tong University

Nov. 24 2020

Ve501 2020-2021 Fall

Based on Lecture Notes by Prof. Chong

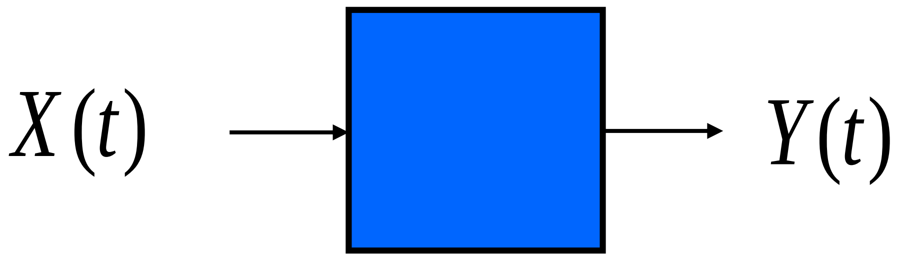
Outline

- 4. Random Process-I
  - Introduction to Random Processes
  - Brownian Motion
  - Poisson Process
  - Complex RV and RP
  - Stationarity
  - PSD, White Noise
  - Response of Systems
  - LTI Systems and RPs

Response of Systems

Response of Systems

- The objective is to determine the statistics of the output of a system, given the statistics of the input to the system



Nonlinear Memoryless

$$X(t) \longrightarrow \boxed{G(X)} \longrightarrow Y(t)$$

if nonlinear → Markov Process

- Can use "function of a RV" methods to get PDF for $Y(t)$ and joint PDFs for $Y(t_1)$, $Y(t_2)$, etc.

Linear Time-invariant

- Linear time-invariant (LTI) systems are described by their impulse response $h(t)$ or their frequency response $H(\omega)$

→ treated as filter

- This topic is treated in the next module

Systems Described by Differential Equations

- Let $L_t(X(t))$ be the linear operator operating on the function with t-dependence.

- **Ex:** Let $Y(t) = L_t(X(t))$, where the operator is represented by the differential equation:

$$a(t)\dot{Y}(t) + b(t)Y(t) = X(t)$$

if constant => time invariant

Initial conditions can be random vectors or variables

initial condition can be non-zero and random

$$Y(0) = A \qquad \dot{Y}(0) = B$$

X(t) is a white noise: $R(\tau) = q\delta(\tau)$

Non-zero Initial Conditions

- In a linear system, the output signal can be decomposed into the sum of these:
  - Zero-input solution
    - system response to non-zero initial conditions; these initial conditions can be RVs
  - Zero-state solution
    - assumes that the system is at rest prior to application of the stimulus (the input RP), i.e. the initial conditions are zero

initial condition = 0.

condition of decompose : initial conditions are non-random
or initial conditions independent of RP

Non-zero Initial Conditions, Concluded

- One approach: characterize these two responses separately. Convenient if the initial conditions are non-random or independent of the input RP.

- Alternatively, under certain conditions (e.g. jointly Gaussian initial conditions and input RP), the output RP can be analyzed as a Markov Process.

Zero Initial Conditions

- A differential equation with zero initial conditions is equivalent to an impulse response
- If the differential equation has constant coefficients and initial conditions are zero, the system is LTI
- Zero initial conditions are assumed in this course

Mean and Autocorrelation

The first and second order moments depend on the linear operator as follows

Linear operator can be all kind.

$$m_Y(t) = L_t[m_X(t)]$$

① ② → 两种都对

① ② → 都可以

$$R_{XY}(t_1,t_2) = L_{t_2}[R_{XX}(t_1,t_2)]$$

First, t_1 is considered a fixed parameter

Linear Operator

$$R_{YY}(t_1,t_2) = L_{t_1}[R_{XY}(t_1,t_2)]$$

Second, t_2 is considered a fixed parameter

Example: Derivative of Wiener Process

用前页方法。

Let X(t) be a Wiener process. Then,

$$R_X(t_1, t_2) = \sigma^2 \min(t_1, t_2), \quad \begin{matrix} t_1 > 0 \\ t_2 > 0 \end{matrix}$$

and

$$E[X(t)] = 0 \qquad X(0) = 0 \qquad \text{for } t \text{ £ } 0$$

and X(t) is Gaussian for t>0.

Derivative of Wiener Process, Cont'd

- Let

这个例子里是求导

$$Y(t) = L_t[X(t)] = \frac{dX(t)}{dt}$$

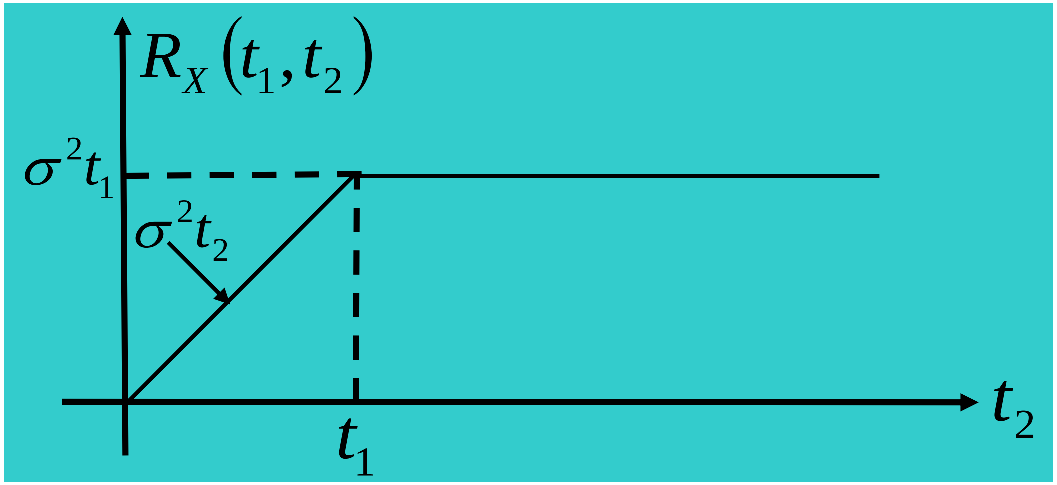
then

$$m_Y(t) = L_t[m_X(t)] = 0$$

$$R_{XY}(t_1, t_2) = L_{t_2}[R_X(t_1, t_2)] = \frac{dR_X(t_1, t_2)}{dt_2}$$

Derivative of Wiener Process, Cont'd

View $R_X(t_1,t_2)$ as a function of t_2 with t_1 fixed.

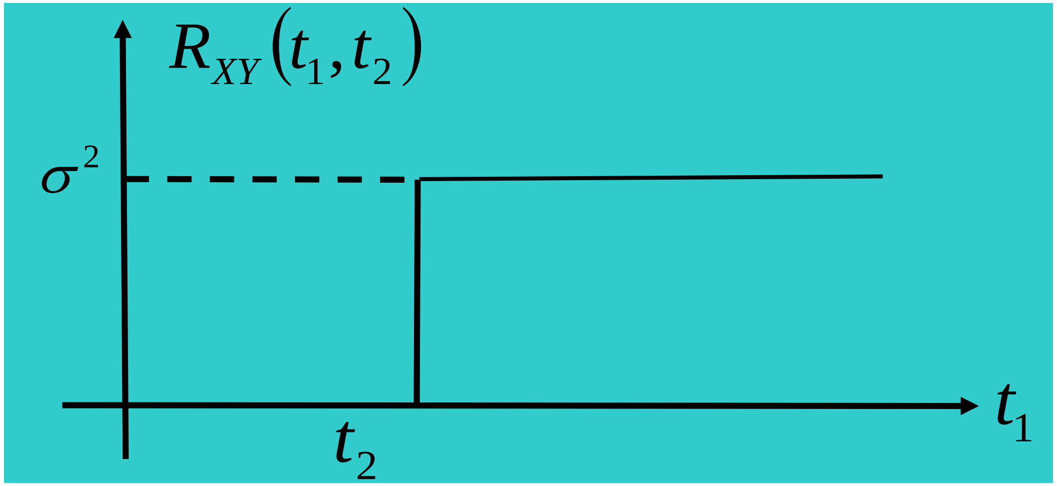


$$R_{XY}(t_1,t_2) = \frac{dR_X(t_1,t_2)}{dt_2} = \begin{cases} \sigma^2 & t_2 < t_1 \\ 0 & ow \end{cases}$$

Derivative of Wiener Process, Cont'd

$$R_{YY}(t_1,t_2) = \frac{d}{dt_1} R_{XY}(t_1,t_2)$$

Now view $R_{XY}(t_1,t_2)$ as a function of t_1 with t_2 fixed.



$$R_{YY}(t_1,t_2) = \sigma^2 \delta(t_1 - t_2)$$

Summary

- For nonlinear, memoryless systems, use "functions of RVs" approach
- For general linear systems, mean and autocorrelation can be derived using the same linear operator on mean, auto- and cross-correlation functions
- The derivative of the Wiener process is Gaussian white noise (GWN)

LTI Systems and RPs

Linear Time-Invariant Systems and RPs

- LTI systems are described by their
  - Impulse response, $h(t)$, or their
  - Frequency response, $H(f) = \Im\{h(t)\}$

- Our goal: Suppose $X(t)$ is an input RP with mean $m_X(t)$ and correlation $R_X(t_1, t_2)$. What are the mean and autocorrelation of the output $Y(t)$?

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

$m_X(t)$
$R_X(t_1, t_2)$

$H(f) = \mathcal{F}[h(t)]$

$m_Y(t)$
$R_Y(t_1, t_2)$

The Mean of the Output

$$Y(t) = \int_{-\infty}^{+\infty} h(s) X(t-s) ds$$

$$E\{Y(t)\} = \int_{-\infty}^{+\infty} h(s) E\{X(t-s)\} ds$$

convolution

$$m_Y(t) = \int_{-\infty}^{+\infty} h(s) m_X(t-s) ds = h(t) * m_X(t)$$

The Autocorrelation of the Output

$$R_Y(t_1,t_2) = E\{Y(t_1)Y^*(t_2)\} = E\left\{\int_{-\infty}^{+\infty} h(s)X(t_1-s)ds\,Y^*(t_2)\right\}$$

$$= \int_{-\infty}^{+\infty} h(s)E\{X(t_1-s)Y^*(t_2)\}ds$$

$$= \int_{-\infty}^{+\infty} h(s)R_{XY}(t_1-s,t_2)ds = \underbrace{h(t_1)}_{\text{known}} * \underbrace{R_{XY}(t_1,t_2)}_{\text{unknown}}$$

Cross-correlation of Input and Output

$E[X(t_1)Y^*(t_2)]$

$$R_{XY}(t_1,t_2) = E\left\{ X(t_1)\int_{-\infty}^{+\infty} h^*(\alpha)X^*(t_2-\alpha)d\alpha \right\}$$

$$= \int_{-\infty}^{+\infty} h^*(\alpha)R_X(t_1,t_2-\alpha)d\alpha$$

convolution

$$= h^*(t_2) * R_X(t_1,t_2)$$

then briefly

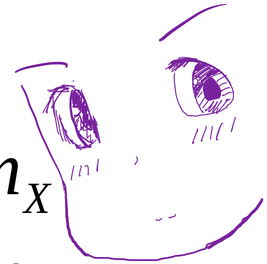
$$R_Y(t_1,t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(s)h^*(\alpha)R_X(t_1-s,t_2-\alpha)dsd\alpha$$

$$= h(t_1) * h^*(t_2) * R_X(t_1,t_2)$$

WSS Input - Mean

Mean:

$$m_X(t) = m_X$$



$$m_Y(t) = \int_{-\infty}^{+\infty} h(s) m_X(t-s)\,ds = m_X \int_{-\infty}^{+\infty} h(s)\,ds$$

Recall the DC response of the system:

$$H(0) = \left[\int_{-\infty}^{+\infty} h(t) e^{-jwt}\,dt \right] \Bigg|_{w=0}$$

Then,

$$m_Y = m_X H(0)$$

WSS Case - Autocorrelation

$$R_Y(t_1,t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(t_1-s,t_2-\alpha)h(s)h^*(\alpha)dsd\alpha$$

$$= \underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(t_1-s-t_2+\alpha)h(s)h^*(\alpha)dsd\alpha}_{\text{Whole thing is a function of } t_1-t_2}$$

So far …

$$m_Y(t) = m_Y \quad \text{(constant)}$$

$$R_Y(t_1,t_2) = R_Y(t_1-t_2)$$

For LTI:
if input WSS,
output also WSS

Y(t) is WSS.

$$\text{WSS in} \longrightarrow \boxed{h(t)} \longrightarrow \text{WSS out}$$

WSS Autocorrelation Simplification

$$R_Y(\tau) = E\{Y(t+\tau)Y^*(t)\} = E\left\{\int_{-\infty}^{+\infty} h(s)X(t+\tau-s)ds\,Y^*(t)\right\}$$

$$= \int_{-\infty}^{+\infty} h(s)R_{XY}(\tau-s)ds$$

$$= h(\tau) * R_{XY}(\tau)$$

WSS Cross-correlation Simplification

$$R_{XY}(\tau) = E\{X(t+\tau)Y^*(t)\}$$

$$= E\left\{ X(t+\tau)\int_{-\infty}^{+\infty} h^*(\alpha)X^*(t-\alpha)d\alpha \right\}$$

$$= \int_{-\infty}^{+\infty} h^*(\alpha)R_X(\tau+\alpha)d\alpha$$

$$= h^*(-\tau) * R_X(\tau)$$

WSS Output Autocorrelation, Concluded

Putting the results

$$R_Y(\tau) = h(\tau) * R_{XY}(\tau)$$

$$R_{XY}(\tau) = h^*(-\tau) * R_X(\tau)$$

together, yields

$$R_Y(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(\tau - s + \alpha) h(s) h^*(\alpha) ds d\alpha$$

$$= h(t) * h^*(-t) * R_X(t)$$

PSD of the Output

- We know Y(t) is WSS. The PSD is

$$
\begin{aligned}
S_Y(\omega) &= \Im\{R_Y(\tau)\} \\
&= \Im\{h(\tau) * h^*(-\tau) * R_X(\tau)\} \\
&= \Im\{h(\tau)\}\Im\{h^*(-\tau)\}\Im\{R_X(\tau)\} \\
&= H(\omega)\Im\{h^*(-\tau)\}S_X(\omega)
\end{aligned}
$$

convolution
$\Downarrow$
multiplication

Final Results

- Change of variables

$$\Im\left\{h^*(-\tau)\right\} = \int_{-\infty}^{+\infty} h^*(-\tau) e^{-j\omega\tau} d\tau$$

Let $s = -\tau$

$$\Im\left\{h^*(-\tau)\right\} = \left[\int_{-\infty}^{+\infty} h(s) e^{-j\omega s} ds\right]^* = H^*(\omega)$$

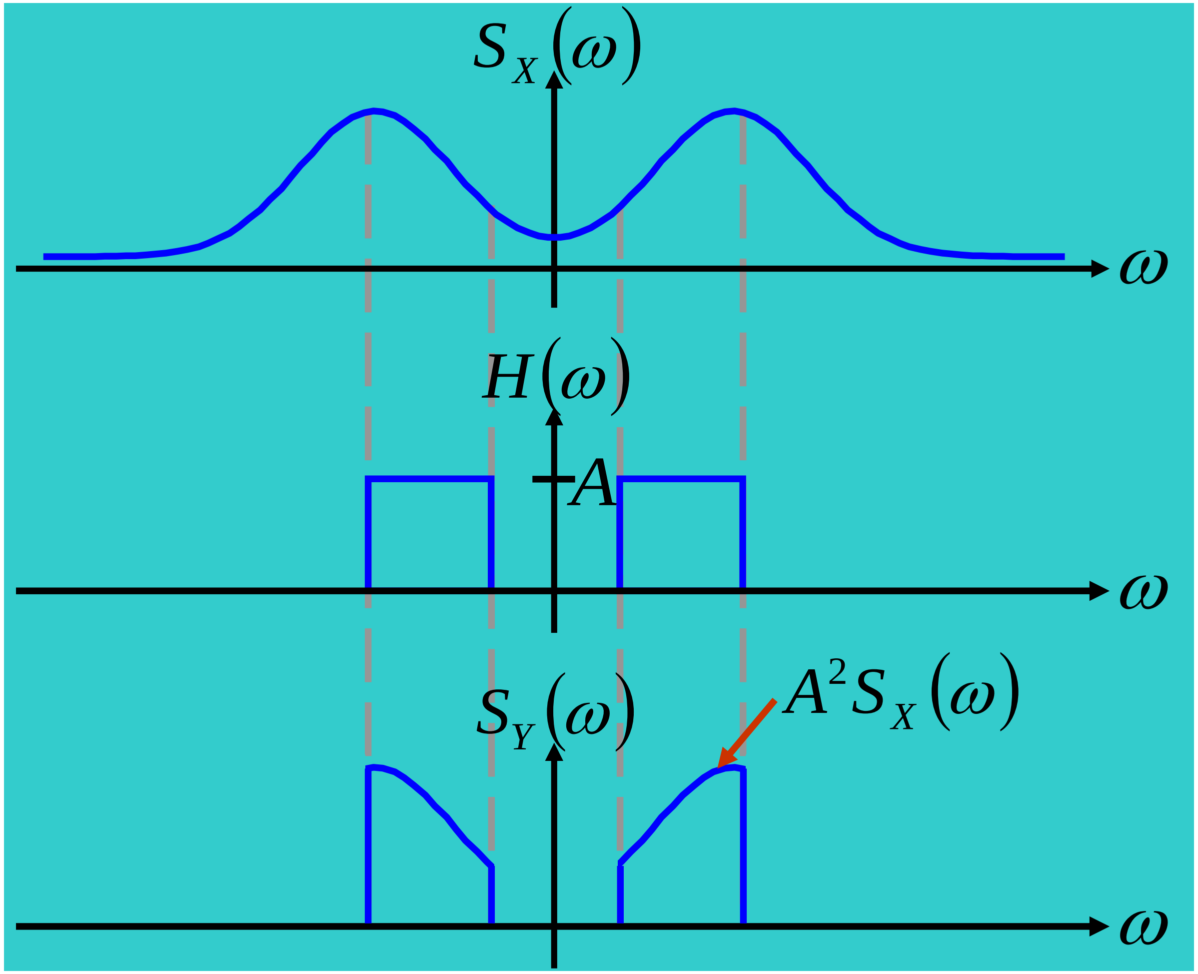
$[(1+j)(1-2j)]^*$
$= [1-2j+j+2]^* = 3+j$
$(1+j)^*(1-2j)^*$
$= (1-j)(1+2j)$
$= 1+2j-j+2$

- Final answers

$$S_Y(w) = H(w) H^*(w) S_X(w)$$

$$S_Y(w) = \left|H(w)\right|^2 S_X(w)$$

Graphical Example



Example: White Noise

Ex: Input WSS white noise $$S_X(\omega) = \frac{N_0}{2}$$

$$S_Y(\omega) = \frac{N_0}{2}\left|H(\omega)\right|^2$$

Average power of the output:

$$R_Y(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_Y(\omega)d\omega = \frac{N_0}{2}\underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|H(\omega)\right|^2 d\omega\right]}_{\text{Energy of the system impulse response}}$$

Summary

- WSS in, WSS out, for LTI systems

- $S_Y(\omega) = \left|\underbrace{H(\omega)}_{\mathcal{F}[h(t)]}\right|^2 S_X(\omega)$

White noise with power spectral density $N_0/2$ is passed through a linear, time-invariant (LTI) system with impulse response $h(t) = 1/(1+t^2)$. If Y_t denotes the filter output, find $E[Y_{t+\frac{1}{2}}Y_t]$.

Method 1:

$$E[Y_{t+\frac{1}{2}}Y_t]$$

$$= R_Y(t+\tfrac{1}{2}, t)$$

$$Y(t) = \int_{-\infty}^{+\infty} h(s)\, X(t-s)\, ds$$

$$\int_{-\infty}^{+\infty} h(s) E[Y(t+\tfrac{1}{2})\, X(t-s)]\, ds$$

$$= \int_{-\infty}^{+\infty} h(s)\, E\left[\int_{-\infty}^{+\infty} h(v) X(t+\tfrac{1}{2}-v)\, dv\, X(t-s)\right] ds$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(s)h(v)\, E[X(t+\tfrac{1}{2}-v)\, X(t-s)]\, dv\, ds$$

$$= \iint_{-\infty}^{+\infty} h(s)h(v)\, R_X(t+\tfrac{1}{2}-v,\, t-s)\, dv\, ds$$

$$= \iint h(s)h(v)\, q\,\delta(s+\tfrac{1}{2}-v)\, dv\, ds$$

$$=$$

Thank You!

Method 2:

We need to find $E[Y_{t+\frac{1}{2}}Y_t]$ which is $R_Y(\frac{1}{2})$. We find this by finding the power spectral density and then taking the inverse Fourier transfrom. The power spectral density is

$$S_Y(f) = S_X(f)|H(f)|^2 = \frac{N_0}{2}\pi^2 e^{-4\pi|f|},$$

where $H(f) = \pi e^{-2\pi|f|}$. Then

$$R_Y(\tau) = \mathcal{F}^{-1}(S_Y(f)) = \mathcal{F}^{-1}(\tfrac{N_0}{2}\pi^2 e^{-4\pi|f|}) = \frac{N_0\pi}{2}\frac{2}{4+t^2} = \frac{N_0\pi}{4+t^2}$$

and

$$E[Y_{t+\frac{1}{2}}Y_t] = R_Y(\tfrac{1}{2}) = \frac{N_0\pi}{4+\frac{1}{4}} = \boxed{\frac{4N_0\pi}{17}}.$$