# Probability and Random Process

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# Outline

- 4. Random Process
  - Introduction to Random Processes
  - Brownian Motion
  - Poisson Process
  - Complex RV and RP
  - Stationarity
  - PSD, QAM, White Noise
  - Response of Systems
  - LTI Systems and RPs

# Complex RV and RP

# Motivation

Complex RVs are used to describe samples of complex envelopes (or complex amplitudes) of modulated signals

$$W(t) = \mathrm{Re}\{Z(t)e^{j\omega t}\} = X(t)\cos\omega t - Y(t)\sin\omega t$$

# One Sample

Consider the sample of the complex amplitude at $t=t_1$

$$Z = Z(t_1) = X(t_1) - jY(t_1) = X + jY$$

dropping the $t_1$ only for notational simplicity

Z is a complex RV

# The "PDF" of a Complex RV

The statistics of a complex RV are described by the joint PDF of its real and imaginary parts

$$f_Z(z) = f_{XY}(x, y)$$

# Independent Gaussian Example

If the real and imaginary parts are iid zero mean Gaussian, you may see

$$f_Z(z) = \frac{1}{2\pi\sigma^2} \exp\left\{ -\frac{|z|^2}{2\sigma^2} \right\} \quad \text{for all } z$$

only as shorthand for

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left\{ -\left( \frac{x^2}{2\sigma_X{}^2} + \frac{y^2}{2\sigma_Y{}^2} \right) \right\}$$

for all $x$ and $y$. ($\sigma_X$ and $\sigma_Y$ are equal for iid)

# Complex Random Processes

The real and imaginary parts are random processes defined on the same sample space (i.e., they are jointly distributed)

$$Z(t,s) = X(t,s) + jY(t,s)$$

# Autocorrelation for Real RPs

Order of times doesn't matter for real RPs

$$\begin{aligned} R_X(t_1,t_2) &= E\left\{ X(t_1)X(t_2) \right\} \\ &= E\left\{ X(t_2)X(t_1) \right\} \\ &= R_X(t_2,t_1) \end{aligned}$$

# Autocorrelation for Complex RPs

For complex RPs, order of times matters, because the second factor (associated with the second time argument) is conjugate by definition

$$
\begin{aligned}
R_Z(t_1, t_2) &= E\left\{Z(t_1)Z^*(t_2)\right\} \\
&= E\left\{Z(t_2)^* Z(t_1)\right\} \\
&= \left[E\left\{Z(t_2)Z^*(t_1)\right\}\right]^* \\
&= R_Z^*(t_2, t_1)
\end{aligned}
$$

# Autocovariance for Complex RPs

Straightforward, with same conjugate symmetry as $R_Z$:

$$C_Z(t_1,t_2)=E\left\{\left[Z(t_1)-m_Z(t_1)\right]\left[Z(t_2)-m_Z(t_2)\right]^*\right\}$$

$$=R_Z(t_1,t_2)-m_Z(t_1)m_Z^*(t_2)$$

$$=C_Z^*(t_2,t_1)$$

# Short Summary

A complex random process is defined by the joint statistics of its real and imaginary processes

Order of times matters

# Stationarity

# Stationarity

Stationary RPs have time-invariant statistics.

Every statistic of a Strict-Sense Stationary (SSS) RP is time-invariant.

The mean and autocorrelation functions of a Wide-Sense Stationary (WSS) RP are time-invariant.

# Strict-Sense Stationarity

A RP is (strict-sense) stationary if, for any positive integer $k$ and any subset of time indices in $I$ (the index set for the RP), $t_1, t_2, \ldots, t_k$, the joint PDF of

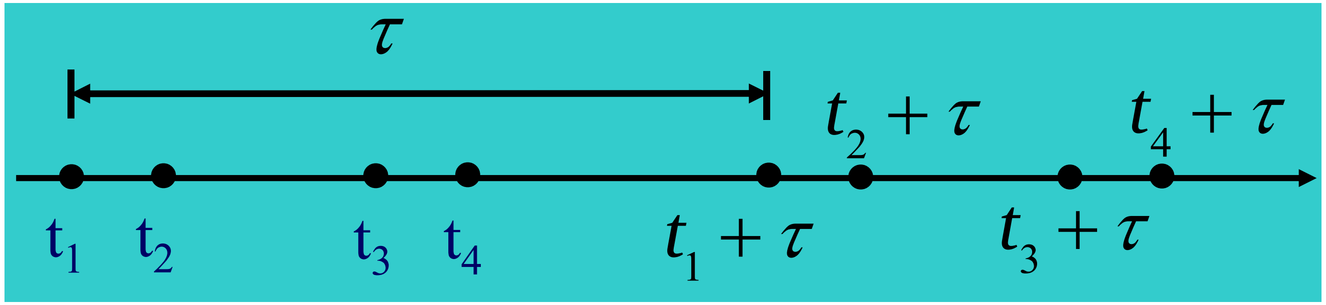
$$\mathbf{X} = \left[X(t_1), X(t_2), \ldots, X(t_k)\right]$$

is the same as the joint PDF of

$$\mathbf{Y} = \left[X(t_1 + \tau), X(t_2 + \tau), \ldots, X(t_k + \tau)\right]$$

All arguments shifted by $\tau$ for any $\tau$.

# Time Shift




- For a strict-sense stationary r.p., the probability distributions of random variables (and vectors) do not change with time shifts.
- The probability of something happening at time is the same as the probability of it happening at any other time.

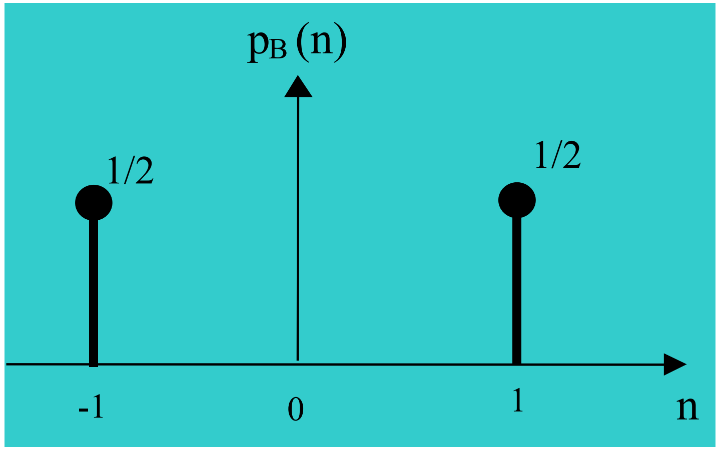
# SSS Example

Recall the B(n) process that was used to define the Discrete-time random walk.

Consider the joint PMF of B(1) and B(5) and the joint PMF of B(101) and B(105) .

$$p_{B(1)B(5)}(n,k) = p_{B(1)}(n)p_{B(5)}(k) = p_B(n)p_B(k)$$

where




Similarly, $p_{B(101)B(105)}(n,k) = p_B(n)p_B(k)$

# Strict-sense stationary: properties (1)

- All singles $X_t$'s are identical.
- All pairs $(X_{t_1}, X_{t_2})$ and $(X_{t_1+\tau}, X_{t_2+\tau})$ are identical.
- All triples $(X_{t_1}, X_{t_2}, X_{t_3})$ and $(X_{t_1+\tau}, X_{t_2+\tau}, X_{t_3+\tau})$ are identical.
- ...
- All n-tuples $(X_{t_1}, X_{t_2}, X_{t_3}, \cdots, X_{t_n})$ and $(X_{t_1+\tau}, X_{t_2+\tau}, X_{t_3+\tau}, \cdots, X_{t_n+\tau})$ are identical.

# Strict-sense stationary: properties (2)

For any function $g$, every $n$ and $t_1, \ldots, t_n \in \mathcal{T}$, and $\tau$ such that $t_1 + \tau, \ldots, t_n + \tau \in \mathcal{T}$

$$E\left[g\left(X_{t_1}, \ldots, X_{t_n}\right)\right] = E[g(X_{t_1+\tau}, \ldots, X_{t_n+\tau})]$$

This property can be used to show a r.p. is stationary

$$E[I_B(X)] = \Pr(X \in B)$$

# Strict-sense stationary: properties (3)

Mean function $m_X(t)$ is the same for all t

Average power in $X_t$: $E[X_t^2]$ is the same for all t

Auto-correlation $R_X(t+\tau, t)$ does not depend on t

When $X_t$ is stationary, we consider the auto-correlation function to have just one argument

$$R_X(\tau) \triangleq E[X_{t+\tau}X_t], R_X(t_1 - t_2) \triangleq E[X_{t_1}X_{t_2}]$$

# Strict-sense stationary

In general, it is quite difficult to show that a random process is strict sense stationary since to do so, one needs to be able to express the general n-th order pdf/pdf/pmf.

On the other hand, to show that a process is not strict sense stationary, one needs to show only that one cdf/pdf/pmf of any order is not invariant to a time shift.

# Wide Sense Stationary

In this case, we are only concerned with first and second order moments

- Mean
- Autocorrelation

$$m_X(t) = E[X(t)] = m_X$$

$$R_X(t_1, t_2) = E[X(t_1)X^*(t_2)] = \underbrace{R_X(t_1 - t_2)}$$

A function of only the difference between time arguments.

# Strict- and wide-sense stationary

(strict-sense) stationary ⇒ wide-sense stationary ?

wide-sense stationary ⇒ (strict-sense) stationary ?

Wide-sense stationarity is a weak kind of stationarity that is easier to check and work with, since it only depends on the mean and auto-correlation functions.

# WSS is Useful

The WSS assumption is often sufficient because

Many RPs are Gaussian.

All that is needed for Linear Estimation

Enough to define Power Spectral Density (PSD), which describes frequency content, for WSS RPs.

# Joint WSS

Jointly WSS Processes Two processes $X(t)$ and $Y(t)$ are said to be jointly WSS if:

(a) $X(t)$ is WSS;

(b) $Y(t)$ is WSS;

(c) $R_{XY}(t+\tau, t) = R_{XY}(\tau)$.

Note that all three conditions must be satisfied.

# Cyclostationary Definition

A random process is cyclostationary if the joint CDF of

$$x(t_1), x(t_2), \ldots x(t_k)$$

is the same as joint CDF for

$$x(t_1 + mT), x(t_2 + mT), \ldots x(t_k + mT)$$

for any integer $m$ and some $T$.

REMEMBER

Strict stationarity holds for any time shift, not just integer multiples of some period.

# Cyclostationarity

Characterizes digital communication waveforms

Enables a certain kind of blind equalizer

# Wide Sense Cyclostationarity

Mean and autocovariance functions are invariant to shifts in time by an integer multiple of some period $T$

$$m_X(t + mT) = m_X(t)$$
$$C_X(t_1 + mT, t_2 + mT) = C_X(t_1, t_2)$$

The mean and the autocovariance are periodic

# Alternative Definition

Since

$$C_X(t_1, t_2) = R_X(t_1, t_2) - m_X(t_1) m_X(t_2)$$

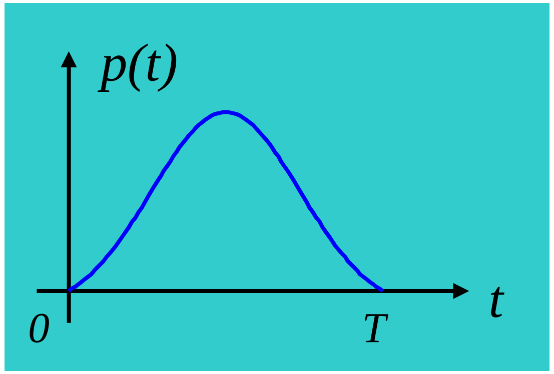
The condition for WSS cyclostationarity may also be expressed as

$$m_X(t + mT) = m_X(t)$$

$$R_X(t_1 + mT, t_2 + mT) = R_X(t_1, t_2)$$

The mean and autocorrelation are periodic

# Example: Pulse - Amplitude Modulated (PAM) Signal




$$X(t) = \sum_{n=-\infty}^{\infty} A_n \, p(t - nT)$$

$$A_n \in \{-3, -1, 1, 3\}$$

Values of $A_n$ are equally likely and independent.

$$E\{X(t)\} = \sum_{n=-\infty}^{\infty} \cancelto{0}{E\{A_n\}} \, p(t - nT) = 0$$

Therefore, $C_X(t_1, t_2) = R_X(t_1, t_2)$

# Example – Cont'd

$$C_X(t_1,t_2) = E\{X(t_1)X^*(t_2)\}$$

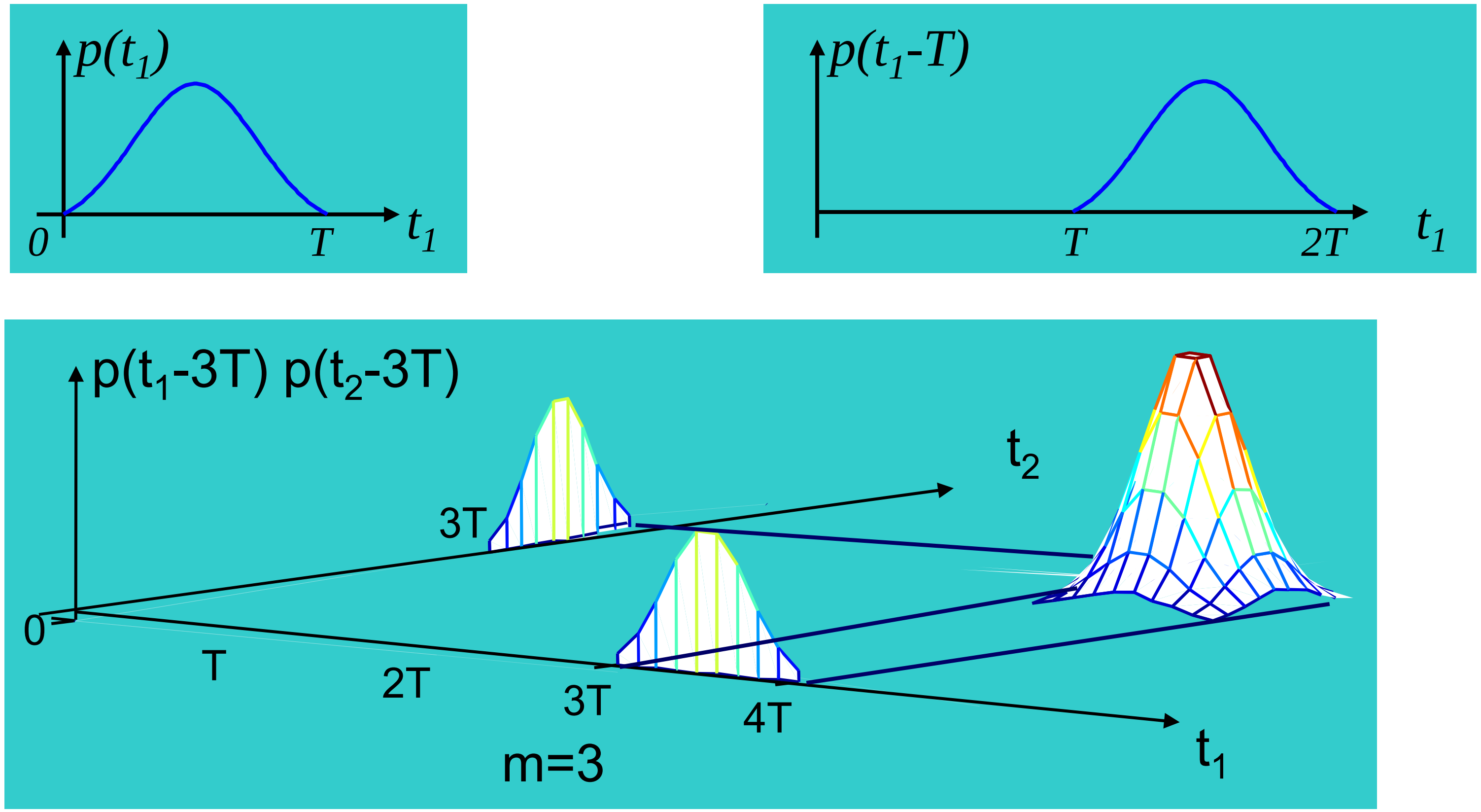
$$= \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} E\{A_m A_k\} p(t_1 - mT) p(t_2 - kT)$$

$$= \sum_{m=-\infty}^{\infty} E\{A_m^2\} p(t_1 - mT) p(t_2 - mT)$$

$$= 5\sum_{m=-\infty}^{\infty} p(t_1 - mT) p(t_2 - mT)$$

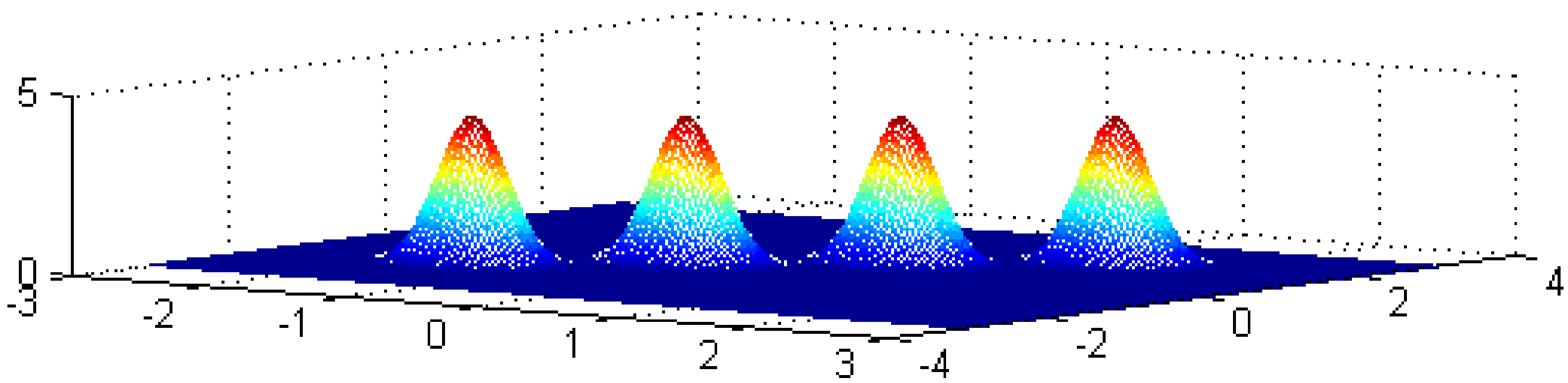
When $|t_1 - t_2| < T$ for some $m$,

# Consider $p(t_1 - 3T)p(t_2 - 3T)$



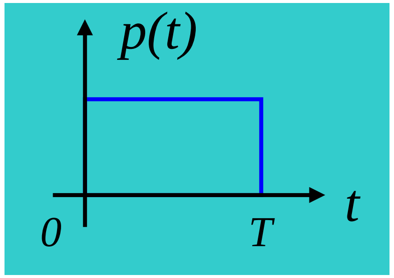
# Example – Graphical Illustration, Cont'd

$$C_X(t_1,t_2) = 5\sum_{k=-\infty}^{\infty} p(t_1 - mT)\,p(t_2 - mT)$$

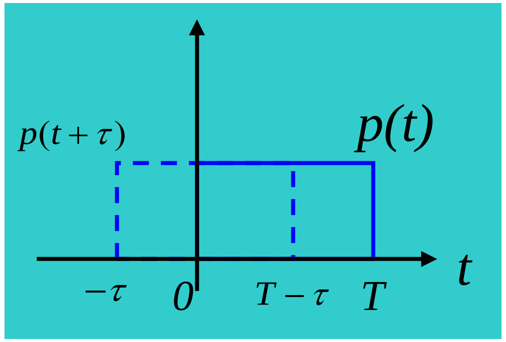


# Pulse Amplitude Modulation Example, Cont'd

If $p(t)$ is as shown on the right, then only the $m=0$ term contributes to the integral.




For $\tau > 0$




$$C_Y(\tau) = \frac{5}{T}\int_0^T p(t)p(t+\tau)dt = \frac{5}{T}(T-\tau)$$

Since $C_Y(\tau)$ is even,

$$C_Y(\tau) = \begin{cases} \frac{5}{T}(T-|\tau|) & |\tau| \le T \\ 0 & o.w. \end{cases}$$

# Generating a WSS process from a WS Cyclostationary Process

Suppose $X(t)$ is WS cyclostationary with period T

We can create a WSS process from $X(t)$ by randomizing the time origin over an interval as long as the period:

$$Y(t) = X(t+\theta) \quad \theta \sim U[0,T]$$

# Cy-WSS to WSS: Mean

$$E\{Y(t)\} = E\{E\{X(t+\theta)\,|\,\theta\}\}$$

$$= \int_0^T m_X(t+\theta)\frac{1}{T}d\theta$$

$$= m_Y$$ Because $m_x(t)$ is periodic

Integrating over $m_x(t)$ over one period yields a constant

# Cy-WSS to WSS : Autocorrelation

$$
\begin{aligned}
R_Y(t_1, t_2) &= E\{Y(t_1)Y^*(t_2)\} = E\{X(t_1+\theta)X^*(t_2+\theta)\} \\
&= E\{E[X(t_1+\theta)X^*(t_2+\theta) \mid \theta]\} \\
&= E\{R_X(t_1+\theta, t_2+\theta)\} \\
&= \frac{1}{T}\int_0^T R_X(t_1+\theta, t_2+\theta)\,d\theta
\end{aligned}
$$

Let $t = t_1 + \theta, \quad dt = d\theta$

Can you further derive it?

# Cy-WSS to WSS : Autocorrelation, Cont'd

Let $t = t_1 + \theta, \quad dt = d\theta$

$$R_Y(t_1, t_2) = \frac{1}{T} \int_{t_1}^{t_1+T} R_X(t, t_2 + t - t_1)\, dt$$

Let $\tau = t_1 - t_2$, and observe that $R_X(t, t+\tau)$ is periodic in $t$ with period $T$, so

$$R_Y(t_1, t_2) = \frac{1}{T} \int_{0}^{T} R_X(t, t+\tau)\, dt \longleftarrow \text{A function only of } \tau$$

$$= R_Y(\tau)$$

Therefore, $Y$ is WSS

# Summary

Stationary

SSS: all statistics are time-invariant

WSS: first and second order statistics are time-invariant

Cyclostationary: all statistics of cyclostationary random processes (CyRP) are invariant to periodic shifts of the time origin

A WSS CyRP has a periodic mean and a periodic autocovariance

Any WSS CyRP can be converted into a WSS RP by time-shifting via a RV uniformly distributed over an interval as long as the period

# Thank You!