Probability and Random Process

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Dec. 3 2020

Ve501 2020-2021 Fall

Based on Lecture Notes by Prof. Chong Han

Outline

- 4. Random Process-II
  - Introduction to Markov Processes
  - Classifications of States and MCs
  - Computing State Probabilities
  - Continuous-time MC
  - Ergodicity Theorems
  - Series Expansions

Ergodicity Theorems

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Ergodicity Theorems

- A process is ergodic in some statistic if that statistic can be estimated using a time average of an observation (i.e. using one outcome) of the process.

- If we weight the outcomes of a random process with their probabilities, we have the true mean of the RP

$$m_X(t) = \sum_i X(t, s_i) P(s_i)$$

- The mean is also called the ensemble average
  - The ensemble is another word for the collection of all outcomes

Time Average

Next suppose we use only one outcome. Then we might compute its time average:

$$\left\langle X(t) \right\rangle_T = \frac{1}{2T} \int_{-T}^{T} X(t, s_i)\, dt$$

When does a limit of $\left\langle X(t) \right\rangle_T$ equal $m_X(t)$?

Example

Let $X(t) = A + \sin(\omega t + \theta), \quad \theta \sim U\,[0,2\pi] \quad A \sim N\,(3,\sigma^2)$

A & θ independent

$$E\left[X(t)\right] = 3$$

But

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} X(t, s_i)\,dt = A(s_i) \neq 3$$

With probability 1

Limit in the Mean

- We can derive the condition for

$$\underset{T\to\infty}{l.i.m.}\langle X(t)\rangle_T = m_X$$

where $l.i.m.$ is the limit in the mean-square convergence

$$\lim_{T\to\infty} E\left\{\left(\langle X(t)\rangle_T - m_X\right)^2\right\} = 0$$

- The RP $X(t)$ satisfying this is "mean ergodic"

Mean-Ergodic Processes

- By expanding the expectation on the previous slide, a process X(t) is mean-ergodic if and only if its autocovariance $C_X(t_1, t_2)$ satisfies:

$$\lim_{T\to\infty} \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} C_X(t_1, t_2)\, dt_1\, dt_2 \xrightarrow[T\to\infty]{} 0$$

- If X(t) is WSS, this condition becomes

$$\frac{1}{2T} \int_{-2T}^{2T} C_X(\tau) \left(1 - \frac{|\tau|}{2T}\right) d\tau \xrightarrow[T\to\infty]{} 0$$

Integrate over the time difference

Example, Mean Ergodic

Let $X(t) = \eta + v(t)$ where $C_v(\tau) = q\delta(\tau)$

and η is non-random.

$$C_X(\tau) = C_v(\tau)$$

$$\frac{1}{2T}\int_{-2T}^{2T} q\delta(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau = \frac{q}{2T} \xrightarrow[T\to\infty]{} 0$$

Observations: Note that $1 - \frac{|\tau|}{2T} \le 1$ for $|\tau| \le 2T$,

a sufficient condition for mean-ergodicity is

$$\int_{-\infty}^{\infty} |C_X(\tau)|\, d\tau < \infty$$

Correlation-Ergodic

We want to estimate $R_X(\tau) = E\{X(t+\tau)X(t)\}$ by averaging $Z(t) = X(t+\tau)X(t)$ over a long period of time

Apply previous result to $Z(t)$:

$$\underset{T\to\infty}{l.i.m.}\frac{1}{2T}\int_{-T}^{T} X(t+\tau)X(t)dt = R_X(\tau)$$

iff
$$\frac{1}{4T^2}\int_{-T}^{T}\int_{-T}^{T} C_Z(t_1,t_2)dt_1dt_2 \xrightarrow[T\to\infty]{} 0$$

$$C_Z(t_1,t_2) = E\left[X(t_1+\tau)X(t_1)X(t_2+\tau)X(t_2)\right] - R_X^2(\tau)$$

Example, Correlation Ergodic

Suppose $X(t)$ is Gaussian, zero-mean with

$$R_X(\tau) = e^{-\alpha|\tau|} \quad \alpha > 0$$

We observe that $Z(t)$ is WSS. Let $\gamma = t_1 - t_2$, then

$$C_Z(\gamma) = R_X^2(\gamma) + R_X(\gamma+\tau)R_X(\gamma-\tau)$$
$$= e^{-\alpha 2|\gamma|} + e^{-\alpha[|\gamma+\tau|+|\gamma-\tau|]}$$

Check textbook: Papoulis "Probability, Random Variables and Stochastic Processes", Fourth edition, Page. 532

Example, Cont'd

Use $|\gamma| - |\tau| \le |\gamma + \tau|$

and $|\gamma| - |\tau| \le |\gamma - \tau|$ $\Rightarrow 2|\gamma| - 2|\tau| \le |\gamma + \tau| + |\gamma - \tau|$

$$C_Z(\gamma) \le e^{-\alpha 2|\gamma|} + e^{-\alpha[2|\gamma| - 2|\tau|]}$$
$$= e^{-\alpha 2|\gamma|}\left(1 + e^{\alpha 2|\tau|}\right)$$

Use the sufficient condition

$$\int_{-\infty}^{\infty} |C_Z(\gamma)|\, d\gamma \le \left(1 + e^{\alpha 2|\tau|}\right)\int_0^{\infty} 2e^{-\alpha 2|\gamma|}\, d\gamma$$

$$= \left(1 + e^{\alpha 2|\tau|}\right)\frac{e^{-\alpha 2\gamma}}{-\alpha}\Bigg|_0^{\infty} = \left(1 + e^{\alpha 2|\tau|}\right)\frac{1}{\alpha} < \infty$$

Summary

- Determined condition for estimating an ensemble average using a time average
- Determined condition for estimating autocorrelation by applying the formula for the mean but replacing the autocovariance

Series Expansions

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The Benefit of Series Expansions

In general, a continuous-time RP is an uncountable number of RVs

If the RP is limited in either time or frequency, or is mean-square periodic, then the RP can be specified by a countable number of RVs:

$$X(t) = \sum_{n=-\infty}^{\infty} a_n \phi_n(t) \quad \text{expansions}$$

where the basis functions $\phi_n(t)$ are not random and the coefficients a_n's are a random sequence

Often, the coefficients have properties that aid analysis

Three Expansions

- Sampling Theorem for band-limited RPs
- Fourier Series for mean square periodic RPs
- KL (Karhunen-Loéve) expansion for time-limited RPs

Sampling Theorem

- $X(t)$ is bandlimited (BL) if it has finite power and if

$$S_X(\omega) = 0 \quad \text{for} \quad |\omega| > B$$

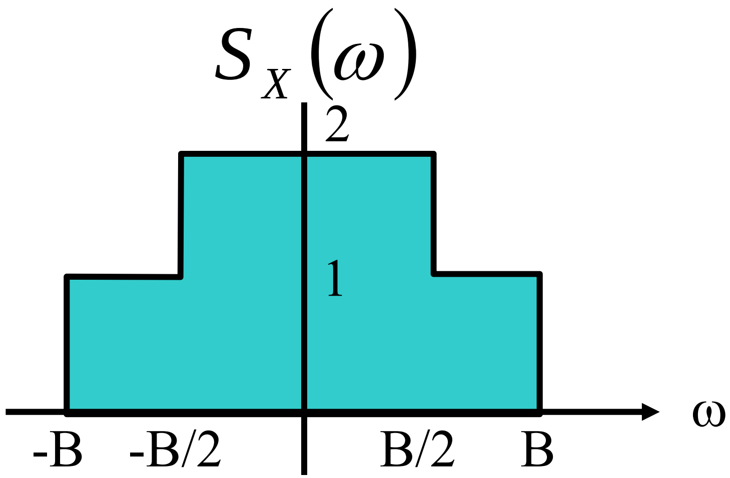
- Given this, the following expansion holds:

$$\boxed{X(t) = \sum_{n=-\infty}^{\infty} X(nT) \frac{\sin[B(t-nT)]}{B(t-nT)} \quad \text{(m.s.)} \quad T = \frac{\pi}{B}}$$

- The samples are correlated in general

Sampling Theorem Example

- Let $X(t)$ have the PSD shown:



$$R_X(\tau) = \frac{B}{\pi}\mathrm{sinc}\left(\frac{B}{\pi}\tau\right) + \frac{B}{2\pi}\mathrm{sinc}\left(\frac{B}{2\pi}\tau\right)$$

Correlation Between Samples

- The correlation between samples is

$$E\{X(nT)X(mT)\} = R_X([n-m]T)$$

$$= \begin{cases} \dfrac{3B}{2\pi} & n = m \\ \dfrac{B}{2\pi}\operatorname{sinc}\left([n-m]\dfrac{1}{2}\right) & n \neq m \end{cases}$$

- Some pairs of samples are not orthogonal

Mean Square Equality

- If $X = Y$ (m.s.), then

$$E\left[(X - Y)^2\right] = 0$$

Fourier Series

- $X(t)$ is m.s. periodic with period T if

$$E\left[\left|X(t+mT)-X(t)\right|^2\right]=0$$

for all t and for all integers m

- Equivalently, $X(t)$ is m.s. periodic iff

$$R_X\left(t_1+mT,t_2+nT\right)=R_X\left(t_1,t_2\right)$$

for every integer m and n

Fourier Series, Cont'd

- Given that $X(t)$ is m.s. periodic with period T, then the following m.s. expansion holds

$$X(t) = \sum_{n=-\infty}^{\infty} c_n e^{\mathrm{j}n\omega_o t} \qquad c_n = \frac{1}{T}\int_0^T X(t) e^{-\mathrm{j}n\omega_o t} dt$$

for

$$\omega_o = 2\pi / T$$

Correlation of Fourier Series Coefficients

- The correlation of the c_n is

$$E(c_n c_m^*) = \gamma_n \delta(\mathrm{n-m})$$

- Orthogonal coefficients!

- What is γ_n?

Mean Square of Coefficient

$$E\left\{\left|c_n\right|^2\right\} = E\left\{\frac{1}{T^2}\int_0^T\int_0^T X(t)X(s)e^{-jn\omega_0 t}e^{jn\omega_0 s}dtds\right\}$$

$$= \frac{1}{T^2}\int_0^T\int_0^T R_X(t-s)e^{-jn\omega_0(t-s)}dtds$$

Both $R_X(t-s)$ and $e^{-jn\omega_0(t-s)}$ are periodic in t with period T, so the inner integral is invariant to s and equals $T\gamma_n$, where

$$\gamma_n = \frac{1}{T}\int_0^T R(\tau)e^{-jn\omega_0\tau}d\tau$$

$$\therefore E\left[\left|c_n\right|^2\right] = \gamma_n$$

Fourier Series Expansion of the Autocorrelation Function

- The previous slide implies that, for WSS Mean Square Periodic RPs,

$$R_X(\tau) = \sum_{n=-\infty}^{\infty} \gamma_n e^{\mathrm{j}n\omega_o \tau}$$

Karhunen-Loéve Expansion

- Applies to time-limited RPs with finite autocorrelation
- Definition

$$\boxed{X(t) = \sum_{n=-\infty}^{+\infty} c_n \phi_n(t) \qquad 0 < t < T}$$

where $\{\phi_1(t), \phi_2(t), \ldots\}$ is a complete ortho-normal (CON) set of deterministic functions

$$\boxed{c_n = \int_0^T X(t)\phi_n^*(t)dt \,, \quad \text{and} \quad E\left[c_n c_m^*\right] = \lambda_n \delta[n-m]}$$

Projection on the basis function

CON Set

- "Complete" means that there are enough $\phi(t)$ functions such that if g(t) is any deterministic, finite energy function over [0, T], then there exists a set of deterministic coefficients α_n such that

$$\int_0^T \left| g(t) - \sum_{n=1}^{\infty} \alpha_n \phi_n(t) \right|^2 dt = 0$$

- "Orthonormal" means

$$\int_0^T \phi_n(t) \phi_m^*(t) dt = \delta[n-m]$$

Basis Functions

- While there are many CON sets in general, for the KL expansion, the $\phi_n(t)$'s must satisfy

$$\int_0^T R_X(t_1,t_2)\phi(t_2)dt_2 = \lambda\phi(t_1)$$

- These $\phi_n(t)$'s are eigenfunctions of $R_X(t_1, t_2)$ with corresponding eigenvalues λ_n

- The $\phi_n(t)$'s are called basis functions

Summary

- There are several different kinds of series expansions that are possible, depending on how the RP is constrained (frequency, periodic, or time)

- The desirable feature is for the coefficients of the expansion to be statistically orthogonal

Thank You!