Probability and Random Process

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Outline

- 4. Random Process
  - Introduction to Random Processes ✓
  - Brownian Motion/Wiener Process ✓
  - Poisson Process ✓
  - Complex RV and RP ✗ not covered
  - Stationarity ✓
  - PSD ✓, QAM ✓, White Noise ✓ No problem about "show real value"
  - Response of Systems ✓ No problem about { "Non-linear" / "solving differential equation" }
  - LTI Systems and RPs ✓
    - Input WSS ⇒ Output WSS
    - how to find mean and correlation $*R_Y*$
    - PSD & R_Y/S_Y

Need $\mathcal{F}$ transform

* Example.

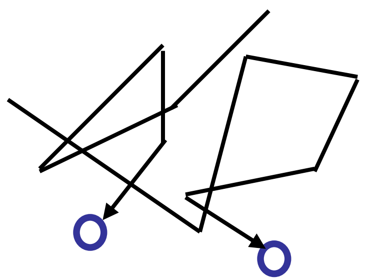
Brownian Motion/Wiener Process

Brownian Motion

In 1857, R. Brown observed that small particles immersed in a liquid exhibit ceaseless irregular motions.

Einstein described the phenomenon mathematically from the laws of physics.

The first concise mathematical formulation was given by Wiener in 1918; the process is also called the "Wiener Process."



Elastic collisions of gas particles

Defining Properties

properties defined

R rather than R^2

The one-dimensional Brownian Motion, X(t), has

a) Independent increments → continuous but not differential

b) Stationary increments

c) * important The increment X(s+t) - X(s) is normally distributed with mean zero and variance αt

d) X(0) = 0 and X(t) is continuous for $t \geq 0$

If $\alpha = 1$, X(t) is a Standard Brownian Motion.

Alternative Definition

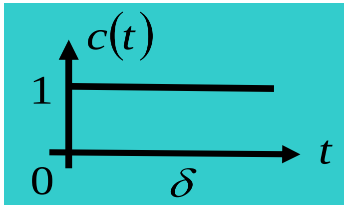
The Brownian Motion is the limit of the continuous-time random walk (CTRW)

Definition of CTRW:

Let $\{B_i\}$ be the same iid sequence used to define the discrete-time random walk.

Let $c(t)$ be the unit step function.

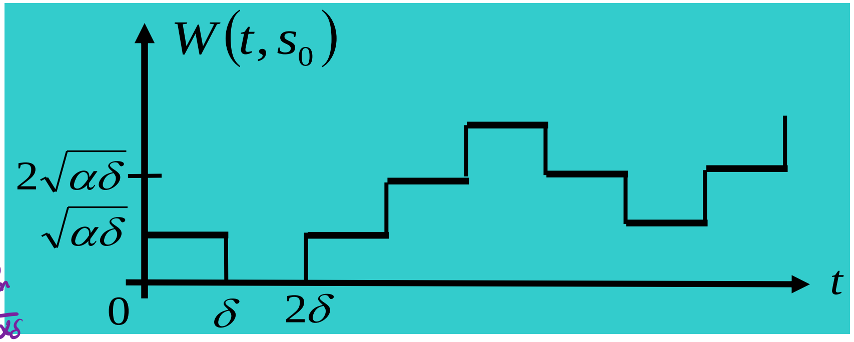
" $u(t)$



Then the CTRW is:

$$W(t)=\begin{cases} 0 & t<0 \\ \sum_{i=1}^{\infty}\sqrt{\alpha\delta}\,B_i c(t-i\delta) & t\geq 0 \end{cases}$$

$\sigma = \sqrt{\alpha}$, Bernoulli => a step for $t=\delta$: $\sqrt{\alpha\delta}$



Statistics of the CTRW

$$W(t)=\begin{cases}0 & t<0\\ \sum_{i=1}^{\infty}\sqrt{\alpha\delta}\,B_i\,c(t-i\delta) & t\geq 0\end{cases}$$

B_i: $+1$, -1

$$E[W(t)]= 0$$

$$Var[W(t)]=\begin{cases}\alpha\delta m \to \alpha t & m\delta \leq t \leq (m+1)\delta,\ m\geq 0\\ 0 & t<0\end{cases}$$

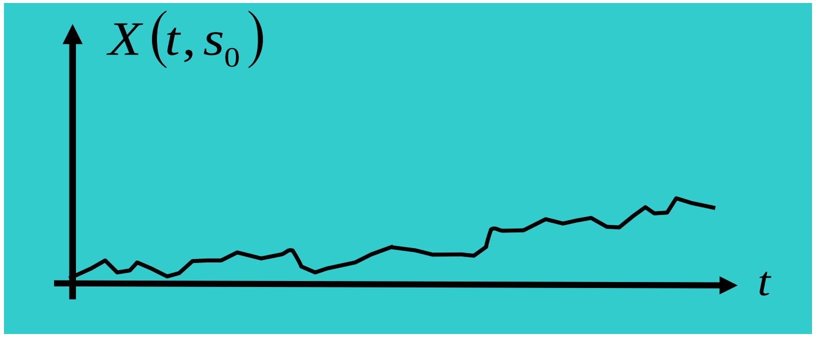
Now let $\delta \to 0$. Both distance-step and time-step shrink together in a certain way: $\delta m \to t$

$$X(t)=\lim_{\delta\to 0}W(t) \text{ and } \mathrm{Var}(X(t))=\sigma_X^2(t)=\alpha t$$

$\mu_{X(t)}=0$ still.

Limit of the CTRW: Wiener Process

$$X(t) = \lim_{\delta \to 0} W(t)$$



X(t) is the Wiener Process, or one-dimensional Brownian motion

Now any interval [0,t], t>0, has an infinite number of steps in it

non-differentiable

By the Central Limit Theorem, X(t) is Gaussian: $\mathcal{N}(0, \alpha t)$

α is called the diffusion constant

X(t) is continuous everywhere, but nowhere differentiable

Wiener Process Autocorrelation

Suppose $0 < t_1 < t_2$

$$R_X(t_1, t_2) = E\{X(t_1)X(t_2)\}$$
$$= E\{X(t_1)[X(t_1) + (X(t_2) - X(t_1))]\}$$
$$= E\{X^2(t_1)\} + E\{X(t_1)[(X(t_2) - X(t_1))]\}$$

$= \sigma_X^2(t_1)$

$= \alpha t_1$

independent

(independent increment)

$= E[X(t_1)]E[(X(t_2) - X(t_1))]$

$= 0 \times 0$

$= 0$

$R_X(t_1, t_2) = \alpha \min(t_1, t_2)$ for $\begin{cases} 0 < t_1 \\ 0 < t_2 \end{cases}$

$\rho = \dfrac{cov}{\sqrt{\alpha t_1}\sqrt{\alpha t_2}}$

Example: Derivative of Wiener Process

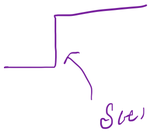
Extended Derivative

Let X(t) be a Wiener process such that

$$R_X(t_1,t_2)=\sigma^2\min(t_1,t_2),\quad \begin{matrix} t_1>0 \\ t_2>0 \end{matrix}$$

Observations: $E\left[X(t)\right]=0,\quad X(t)=0,\ \text{for } t\le 0$

and X(t) is Normal for t>0.



Let

Gaussian White Noise

$$Y(t)=L_t\left[X(t)\right]=\frac{dX(t)}{dt}$$

$$m_Y(t)=L_t\left[m_X(t)\right]=0$$

$$R_{XY}(t_1,t_2)=\underbrace{L_{t_2}\left[R_X(t_1,t_2)\right]}_{\frac{d\,R_X(t_1,t_2)}{dt_2}}$$

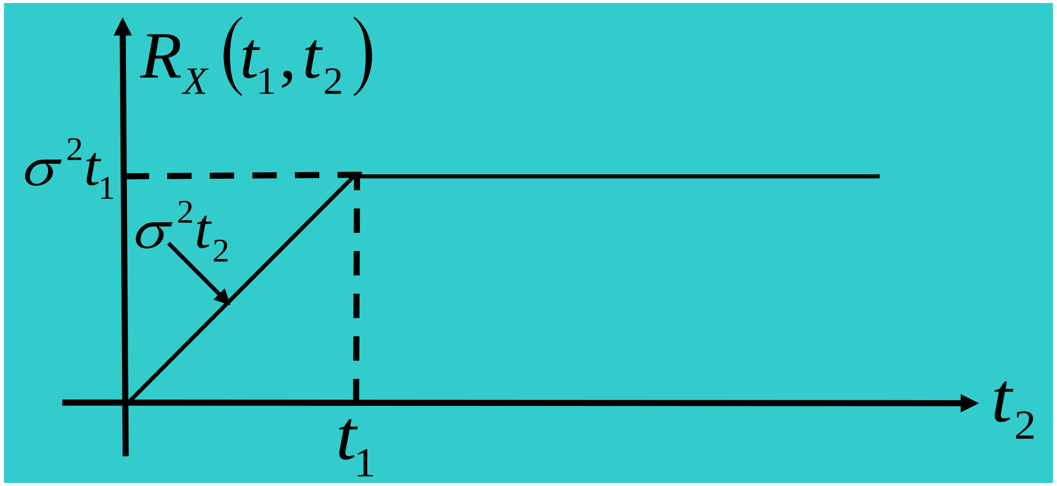
Weiner

New defined Gaussian

$X(t_1)\ Y(t_2)$

Derivative of Wiener Process

View $R_X(t_1,t_2)$ as a function of t_2 with t_1 fixed.

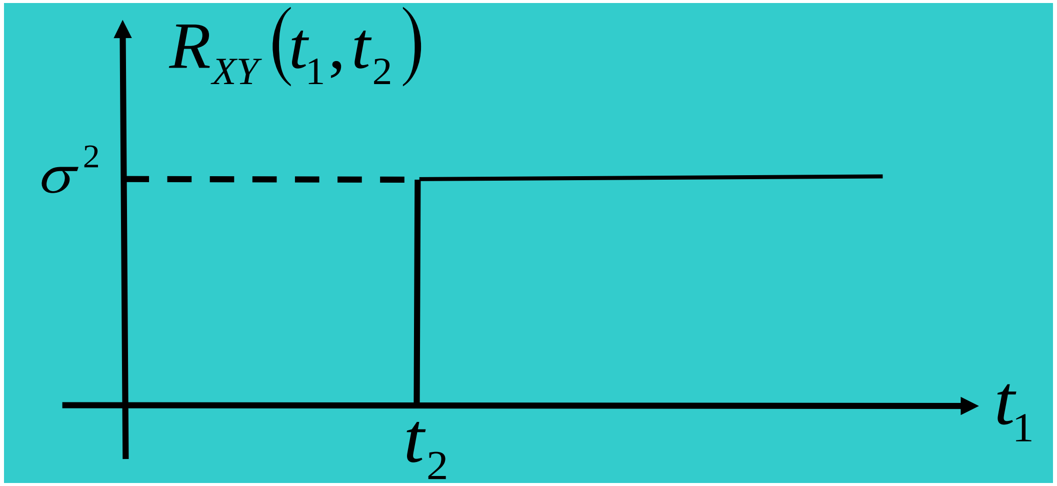


$$R_{XY}(t_1,t_2) = \frac{dR_X(t_1,t_2)}{dt_2} = \begin{cases} \sigma^2 & t_2 < t_1 \\ 0 & ow \end{cases}$$

Derivative of Wiener Process

$$R_{YY}(t_1,t_2)=\frac{d}{dt_1}R_{XY}(t_1,t_2)$$

Now view $R_{XY}(t_1,t_2)$ as a function of t_1 with t_2 fixed.



$$R_{YY}(t_1,t_2)=\sigma^2\delta(t_1-t_2)$$

Short Summary

- The Wiener Process is one of the most fundamental RPs

- The derivative of a Wiener process

$$Y(t) = \frac{dX(t)}{dt}$$ is white noise with

$$E[Y(t)] = 0$$

$$R_{YY}(t_1, t_2) = \sigma^2 \delta(t_1 - t_2), \quad 0 < t_1, \quad 0 < t_2$$

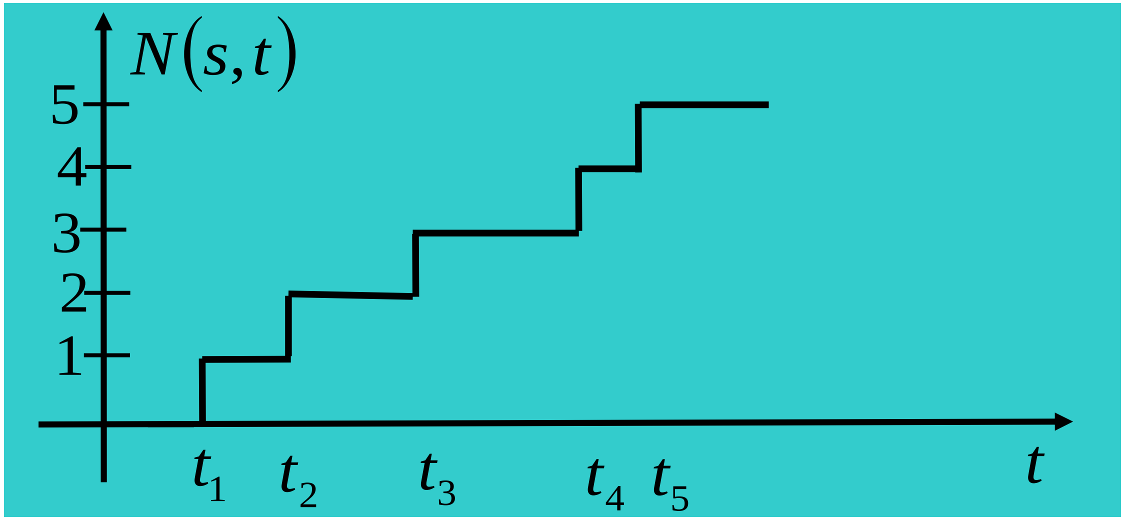
- It can be shown that any integral of Y(t) is Gaussian, therefore we call Y(t) Gaussian White Noise (GWN)

Poisson Process

The Poisson Counting Process

The Poisson counting process, N(t), counts the number of times a specified event occurs during the time from 0 to t. Thus, each sample function is a non-decreasing step function.

Ex:



$\{t_1, t_2, \ldots\}$ are called the Poisson Points

Postulates

properties

Karlin and Taylor, A First Course in Stochastic Processes

The following postulates define the process:

1. N(t) has independent increments.
2. N(t) has stationary increments.
3. $P(N(h)\geq 1)=\lambda h+o(h)$ → if h is small o(h)=0
4. $P(N(h)\geq 2)=o(h)$

Interpretation: the probability of two or more changes in a sufficiently small interval is essentially 0

$$g(t)=o(t),\ t\to 0$$

is the usual way of writing $\lim_{t\to 0}\frac{g(t)}{t}=0$

Postulate 4

4. $P\left(N\left(h\right)\geq 2\right)=o\left(h\right)$

Postulate 4 implies that the events do not happen simultaneously

This is reasonable for:

1. Breakdowns of a machine
2. Customer arrivals
3. Photoelectron generation in an optical detector

Homogenous Poisson Process

Based on these postulates, one can show that

PMP:

$$P\big(N(t+\Delta)-N(t)=m\big)=\frac{\Lambda^m}{m!}e^{-\Lambda}$$

That is, the increments are Poisson-distributed with parameter $\Lambda=\lambda\Delta$, where

λ is the average rate of occurrence

Δ the length of the interval of observation

λ being not a function of t makes $\mathrm{N(t)}$ a homogenous Poisson Process

if λ changes, i.e. non-homo.

句表讨论略

Volume Poisson Process

A homogenous Poisson Process may also describe random points in space, for example, locations of stars in the galaxy, or point defects in a solid material.

For example, let V be a volume in $\mathcal{R}^3$, and $N(V)$ = the number of points in V

$$P(N(V)=m)=\frac{(\lambda V)^m}{m!}e^{-\lambda V}, \quad \lambda > 0$$

Standard Poisson Process

A Standard Poisson Process allows the rate, $\lambda(t)$, to vary with time

$$P\left(N(t+\Delta)-N(t)=m\right)=\frac{\left(\int_{t}^{t+\Delta}\lambda(u)du\right)^{m}}{m!}e^{-\int_{t}^{t+\Delta}\lambda(u)du}$$

This process has independent but not stationary increments

λ changes as t changes

A homogenous Poisson Process is a special case of the Standard Poisson Process

Joint PMF

Consider the joint PMF of the homogenous Poisson Process. Let $t_1 < t_2, i \leq j$

$t_1 \rightsquigarrow t_2$ follows poisson.

$$p_{N(t_1)N(t_2)}(i, j) = P\{N(t_1) = i \cap N(t_2) = j\}$$

Write $N(t_2)$ as a sum of independent increments

$$N(t_2) = [N(t_2) - N(t_1)] + N(t_1)$$

Then, we can write

$$\{N(t_1) = i \cap N(t_2) = j\}$$
$$= \{N(t_1) = i\} \cap \{[N(t_2) - N(t_1)] = j - i\}$$

Joint PMF, cont'd

$$\{N(t_1)=i \cap N(t_2)=j\}$$
$$=\{N(t_1)=i\}\cap\{[N(t_2)-N(t_1)]=j-i\}$$

Now exploit the independent increments

$$p_{N(t_1)N(t_2)}(i,j)=P\{N(t_1)=i\}\,P\{N(t_2)-N(t_1)=j-i\}$$

And finally exploit the stationary increments

$$p_{N(t_1)N(t_2)}(i,j)=P\{N(t_1)=i\}\,P\{N(t_2-t_1)=j-i\}$$

Joint PMF, Final Expression

poisson

$$\therefore p_{N(t_1)N(t_2)}(i,j) = \left[\frac{(\lambda t_1)^i}{i!} e^{-\lambda t_1} \right] \frac{(\lambda[t_2 - t_1])^{(j-i)}}{(j-i)!} e^{-\lambda[t_2-t_1]}$$

Mean, Correlation, and Covariance

自己算下
补全
mid-steps

$$E[N_t] = \lambda t \text{ and } var(N_t) = \lambda t.$$

$$E[N_t N_s] = (\lambda t)(\lambda s) + \lambda t \text{ for } t < s$$

$$\begin{aligned} cov(N_t, N_s) &= E[(N_t - \lambda t)(N_s - \lambda s)] \\ &= E[N_t N_s] - (\lambda t)(\lambda s) \\ &= \lambda t. \end{aligned}$$

Example

Suppose fish bite with a Poisson distribution, with an average rate of one per 20 min. What is the probability that at least one fish will bite in the next 5 min given that no fish has bitten in the last 20 min?

$$P\left(N(t+5)-N(t)\geq 1 \mid N(t)-N(t-20)=0\right)$$

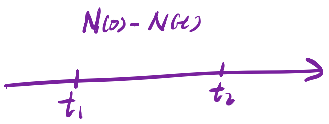
$$=P\left(N(t+5)-N(t)\geq 1\right)=1-\frac{(5/20)^0}{0!}e^{-5/20}=0.22$$

Because of independent increments, there is "No premium for waiting"

Poisson Inter-arrival Times

Let $T = t_2 - t_1$, where t_1 and t_2 are two consecutive event times. Characterize T.

CDF or PDF



CDF: $P(T \leq t)$

$= 1 - P(N(t_1 + t) - N(t_1) = 0)$

$= 1 - P(N(t) = 0)$

$= 1 - e^{-\lambda t} \quad t \geq 0 \quad t \geq 0$

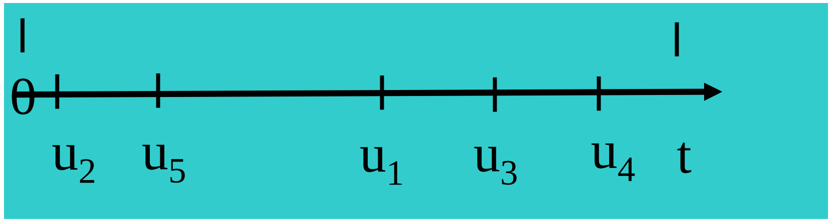
$$f_T(t) = \frac{d}{dt} F_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

Random Points Interpretation

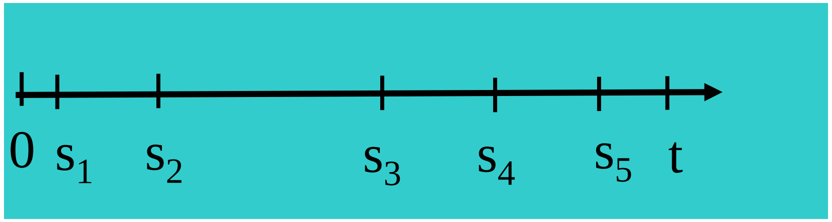
Given $N(t) = k$, the k event times are equivalent to k independent RVs that are uniformly distributed over $[0, t]$, given that they are ordered.

In other words, let $u_1, u_2, \ldots, u_k$ be k independent trials of a RV uniformly distributed over $[0, t]$.

Ex: $k = 5$



Then, define $s_1, s_2, \ldots, s_k$ to be the ordered version of the u's.



Example

Suppose two customers arrive at a shop during a two-minute period. Find the probability that one arrived in the first minute and the other arrived in the second minute.

Poisson approach:

$$P(N(1)=1 \mid N(2)=2) = \frac{P(N(1)=1 \cap N(2)=2)}{P(N(2)=2)} = \frac{P(N(1)=1 \cap N(2)-N(1)=1)}{P(N(2)=2)}$$

Poisson Process Summary

Definition of Poisson Process

- Postulates
- Homogeneous
- Standard

Inter-arrival Times are Exponential

Random Points

Comparison (1)

Similarities of Wiener and Poisson processes:

- Stationary and independent increments
- Variance are scalar of time
- Initial is 0

Comparison (2)

Differences between Wiener and Poisson processes:

- Continuous and discrete
- Gaussian and Poisson
- Mean: 0 and λt

Thank You!