Probability and Random Process

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Sep. 28 2021

Ve501 2021-2022 Fall

Based on Lecture Notes by Prof. Yong Long

Outline

- 1. Introduction to Probability
  - Application example
  - Review of set and functions
  - Models of random experiments
  - Axioms and properties of probability
  - Conditional probability
  - Independence of events
  - Combinatorics and probability

Combinatorics and probability

Motivating Example

- A communication system consists of n identical antennas that are to be lined up in a linear order
- The resulting system will be **functional**, i.e., it will be able to receive all incoming signals, as long as **no two consecutive antennas** are defective
- If it turns out that m out of the n antennas, $m < n$, are defective. What is the probability that the resulting system is functional?
- Consider the case where $n = 4, m = 2$.

Motivating Example: Solution

- There are 6 possible configurations (Here we have assumed that all configurations are equally likely)
  - 0 1 1 0
  - 0 1 0 1
  - 1 0 1 0
  - 0 0 1 1
  - 1 0 0 1
  - 1 1 0 0
  - where “1” denotes that the antenna is working and “0” denotes that the antenna is defective.
- The system will be functional in the first three arrangements and not functional in the remaining three.
- Thus it is reasonable to take 3/6 = 1/2 as the probability that the system is functional.

Combinatorial Analysis

- We can solve this problem for general n and $m < n$, we would count the number of configurations that result in a functional system and then divide by the total number of possible configurations
- From the above example we see that it would be useful to have an effective method for counting the number of ways in which thing can occur
- In fact, many problems in probability theory can be solved simply by counting the number of different ways that a certain event can occur
- The mathematical theory of counting is called **combinatorial analysis**

Sampling

- $P(A) = \frac{|A|}{|\Omega|}$ or $P(A)$ depends on cardinality |A| in some other way
- Basic experiment
  - $\Omega_0 = \{a_1, a_2, \ldots, a_m\}$
  - $\mathcal{A}_0 = 2^{\Omega_0}$
  - $P_0(\{a_k\}) = \frac{1}{m}, k = 1,2, \ldots$
- Repeating the basic experiment *n* times is called **"sampling"**
- **Combinatorics** is the study of systematic counting methods, which we will be using to find the cardinalities of various sets that arise in probability.

Four Kinds of Counting Problems

- We will study four kinds of counting/combinatorics problems
  - ordered sampling with replacement
  - ordered sampling without replacement
  - unordered sampling without replacement
  - unordered sampling with replacement

The Basic Principle of Counting

- If k experiments that are to be performed are such that the first one may result in any of n_1 possible outcomes.
- If for each of the n_1 possible outcomes there are n_2 possible outcomes of the second experiment.
- If for each of the possible outcomes of the first two experiments, there are n_3 possible outcomes of the third experiment and so on …
- Then there is a total of $n_1 \cdot n_2 \cdot n_3 \dots n_k$ possible outcomes of the k experiments.
- We will see how this principle can be used to deal with each of the four kinds of counting problems mentioned above.

Ordered Sampling with Replacement (1)

- Example
  - Suppose that to send an Internet packet from the east coast of the United States to the west coast, the packet must go through
    - a major east-coast city (Boston, New York, Washington, D.C., or Atlanta),
    - a major mid-west city (Chicago, St. Louis, or New Orleans),
    - and a major west-coast city (San Francisco, Los Angeles, or San Diego).
- How many possible routes are there?

Ordered Sampling with Replacement (2)

- Solution
  - We can think of the sets
    - A = {Boston, New York, Washington D.C., Atlanta}
    - B = {Chicago, St. Louis, New Orleans}
    - C = {San Francisco, Los Angeles, San Diego}
  - and ask the question
  - "how many triples are there of the form (a, b, c) where $a \in A$, $b \in B, c \in C$?"
  - There are 4 choices of a, for each choice of a there are 3 choices of b, and for each choice of (a, b) there are three choices of c.
  - Hence the number of triples (a, b, c) is $4 \cdot 3 \cdot 3 = 36 \Rightarrow$ there are 36 possible routes.

Ordered Sampling with Replacement (3)

- Key feature is ordered sampling.
- The order of appearance of a sample does matter.
- The samples consisting of the same elements appearing in different order are distinct.
  - 3311246 is different from 3123146.

Ordered Sampling with Replacement (4)

- Example
  - We have a deck of 52 cards. We draw 5 cards with replacement, i.e., we draw a card make a note of it, put the card back in the deck, and reshuffle the deck before choosing the next card.
  - Each draw has 52 possibilities
  - How many sequences of 5 cards can we draw this way?
    - The answer is 52^5.

- In general if we sample k times from a set of n elements, and sampling with replacement, the number of possible ordered samples is n^k

Ordered Sampling without Replacement (1)

- Example
  - Considering a deck of 52 cards and draw 5 without replacement, how many possible ordered samples can we have?
    - $52 \cdot 51 \cdot 50 \cdot 49 \cdot 48$
  - Contrast this with 52^5 we obtained when replacement is allowed.

Ordered Sampling without Replacement (2)

- Example
  - Let A be a finite set of n elements. How may k-tuples $(a_1, \ldots, a_k)$ of distinct entries $a_i \in A$ can be formed?

$$n \cdot (n-1) \cdot (n-2) \ldots (n-k+1) = \frac{n!}{(n-k)!}$$

Ordered Sampling without Replacement (3)

- Combining counting with probability
- Example
  - Compute the probability that in a group of k people two or more people have the same birthday.

Combining Counting with Probability (1)

- *probability model*

$$\Omega = \{(b_1, b_2, \ldots, b_i, \ldots, b_k) : b_i \in D\}, \quad D = \{1, 2, 3, 4, \ldots, 365\}$$

$$\mathscr{A} = 2^{\Omega}$$

where 1 $\Rightarrow$ Jan. 1 . . . 365 $\Rightarrow$ Dec. 31

- *Let* $B = \{(b_1, b_2, \ldots, b_k) : \text{two or more } b_i \text{ are the same}\}$
- *then* $\mathrm{P}(B) = \frac{|B|}{|D|^k}$
- *also* $\mathrm{P}(B) = 1 - \mathrm{P}(B^c)$. *It is easier to compute* $\mathrm{P}(B^c)$.

$$B^c = \{(b_1, b_2, \ldots, b_k) : b_i \neq b_j, \text{for any } i \neq j\}$$

Combining Counting with Probability (2)

$$|B^c| = 365 \cdot 364 \ldots (365 - (k-1)) = \frac{365!}{(365-k)!}$$

(ordered sampling without replacement) then

$$P(B^c) = \frac{365!}{(365)^k (365-k)!}$$

and

$$P(B) = 1 - P(B^c) = \boxed{1 - \frac{365!}{365^k (365-k)!}}$$

If $k = 40$,

$$P(B) = 1 - \frac{326 \cdots 365}{(365)^40} \approx \boxed{0.89}$$

Ordered Sampling without Replacement (4)

- Important remark: In ordered sampling the order in which items appear in a sample does matter.
- Example
  - If we draw 4 cards out of a deck of 52 and the sample is

    K hearts, K spades, K clubs, K diamonds
  - this sample is different from

    K spades, K hearts, K clubs, K diamonds
  - and so on.....

Unordered Sampling without Replacement (1)

- Question
  - What happens if in the previous problems of drawing 4 cards out of a deck of 52, order does not matter?

- Before we had that

  K hearts, K spades, K clubs, K diamonds

  is distinct from

  K spades, K hearts, K clubs, K diamonds

  and so on...
- Now all samples of 4 Kings are indistinguishable.

Unordered Sampling without Replacement (2)

- Question
  - How can we then determine how many "unordered samples" of 4 cards out of a deck of 52 cards we have? (without replacement)

  - Before we computed that we can have 52*51*50*49 =52! /48! ordered samples
  - Consider a sample of 4 cards. The same 4 cards form 4*3*2*1 "distinct ordered" samples, but all of these "ordered samples" corresponds to the same "unordered sample". (the example of 4 Kings)
  - Thus we expect to have a smaller number of "unordered samples".

Unordered Sampling without Replacement (3)

- To compute this number we use the argument above. A set of k objects can be ordered in $k!$ different ways (without replacement).
- This is true because when we consider a set of k objects and its orderings, we have k choices for the first element of the order, $(k - 1)$ choices for the second element of the order, $(k - 2)$ choices for the third elements, ... so we have totally $k!$ distinct orderings and all of them are one unordered sample.
- So if we have 52 cards and we want to compute the number of unordered samples of 4 cards without replacement, we find that this number is

$$\frac{52!}{48!4!} = \binom{52}{4}, \text{“52 choose 4”}$$

Unordered Sampling without Replacement (4)

- By the same argument we find that if we have n objects and we want to compute the number of unordered samples of k objects without replacement, this number is

$$\frac{n!}{(n-k)!k!} = \binom{n}{k}, \text{ “}n\text{ choose }k\text{”}$$

and this is called the binomial coefficient because it arises in the binomial theorem (which we will discuss later).

Unordered Sampling without Replacement (5)

- Combining counting with probability
- Example
  - A box contains n balls of which n_r are red and $n_b = n - n_r$ are black. If k of these balls are drawn one after another without replacement, with each selection being equally likely to be any of the balls that remain at that time. What is the probability that I of the k balls are red?

Solution

- *There are* $|\Omega| = \binom{n}{k}$ *possible unordered samples of* k *balls out of* n *balls*
- *Event of interest*

$$A = \{l \text{ of } k \text{ balls are red}\}, \quad |A| = \binom{n_r}{l}\binom{n-n_r}{k-l}$$

- *The probability is*

$$\mathrm{P}(A) = \frac{|A|}{|\Omega|} = \frac{\binom{n_r}{l}\binom{n-n_r}{k-l}}{\binom{n}{k}}$$

Unordered Sampling without Replacement (6)

- Example
  - A five-card poker hand is said to be full-house if it consists of 3 cards of the same denomination and 2 cards of the same denomination (e.g., 3 9's and 2 8's). What is the probability that one is having a full house?

Solution

- *The number of possible hands is* $\binom{52}{5}$. *and are all equally likely.*
- *There are 13 ways to pick denomination of 3s. After the denomination is chosen, there are 12 ways to pick the denomination of 2's, so* $13 \cdot 12$.
- *There are* $\binom{4}{3} \cdot \binom{4}{2}$ *combinations of say 3 9's and 2 8's. (4: hearts, spades, clubs, diamonds)*
- *So* $13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}$ *hands that are a full house*
- *Hence*

$$\mathrm{P_r}(\textit{full house}) = \frac{13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}}{\binom{52}{5}}$$

Question

- A set of n distinct items is to be divided into r distinct groups $n_1, n_2, \ldots, n_r$ such that

$$\sum_{k=1}^{r} n_k = n$$

- *How many different divisions are possible?*

Solution

- There are $\binom{n}{n_1}$ possible choices for the first group.
- For each choice of the first group there are $\binom{n-n_1}{n_2}$ possible choices for the second group
- For each choice of the first + second group there are $\binom{n-n_1-n_2}{n_3}$ choices for the third group, and so on.

Solution

$$\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n_2}{n_3}\cdots\binom{n-n_1-n_2-\ldots-n_{r-1}}{n_r}$$

$$= \frac{n!}{(n-n_1)!n_1!}\frac{(n-n_1)!}{(n-n_1-n_2)!n_2!}\frac{(n-n_1-n_2)!}{(n-n_1-n_2-n_3)!n_3!}$$

$$\cdots\frac{(n-n_1-n_2-\ldots-n_{r-1})!}{\underbrace{(n-n_1-n_2-\ldots-n_r)!}_{0!=1}\,n_r!}$$

$$\frac{n!}{n_1!n_2!\ldots n_r!} \triangleq \binom{n}{n_1 n_2 \ldots n_r}$$

This is the ***multinomial coefficient****.*

Example

- A police department in a small city has 10 policemen. If the department policy is to have 5 men patrolling the city streets, 2 men working full time at the police station and 3 men on reserve at the station. How many different divisions of the 10 policemen into the 3 groups are possible?

- Solution: There are $\frac{10!}{5!3!2!}$ possible divisions

Unordered Sampling with Replacement (1)

- Example
  - An automated snack machine dispenses apples, bananas, and oranges.
  - For a fixed price, the customer gets totally five items from the above.
  - It is assumed that there is an infinite supply of the above items.
  - How many customer choices are there?

Solution

- *Consider any sample and group together the apples, the bananas, and the oranges. Separate these groups by inserting, in the following order, a separator symbol (▲) as shown below.*

$$\underbrace{0,0}_{apple} \blacktriangle \underbrace{0,0}_{banana} \blacktriangle \underbrace{0}_{orange}$$

- *Since there are three distinct items we used two separators, we can create a group of*
  - *6 positions to choose for the first separator*
    ␣0▲0␣0␣0␣0␣
  - *7 positions to choose for the second separator*
    ␣0␣▲␣0␣0␣0␣0␣

Solution

- So now we have an array of $5 + (3 - 1) = 7$ elements and the question is in how many ways can we arrange 2 ”separators” or 5 “objects” in an array of totally 7 objects (including the separators)
- $\frac{6 \cdot 7}{2} = \binom{7}{2}$ Dividing by 2 because the order of separators not being important
- Note that this problem allows for the possible choices that we select 5 apples or 5 bananas or 5 oranges

$$\underbrace{0,0,0,0,0}_{\text{apple}} \blacktriangle \underbrace{\quad\quad}_{\text{banana}} \blacktriangle \underbrace{\quad\quad}_{\text{orange}}$$

Unordered Sampling with Replacement (2)

- Example: General problem
  - We have a set of n objects $A = \{1, 2, \ldots, n\}$.

    Previous example $n = 3$ (apples, bananas and oranges)
  - We select out of this set k objects with replacement. ($k = 5$)
  - How many of different (unordered) sets of k objects can we obtain in this way?

Solution

- *We agree as before we consider one possible sample and form groups such that each group contains the same object.*
- *There are n possible group at most and we need $n-1$ "separators" to separate these groups*
- *By inserting the "separators" at the appropriate places we now have an array of $k+n-1$ elements.*
- *We always obtain an array of $k+n-1$ elements using the above approach*
- *The question is then "in how many ways can we arrange $(n-1)$ "separator" in an array of length $k+n-1$".*
- *The answer is*

$$\boxed{\binom{k+n-1}{k} = \binom{k+n-1}{n-1}}$$

Unordered Sampling with Replacement

- Example
  - Ten passengers get on an airport shuttle at the airport. The shuttle has a route that includes 5 hotels, and each passenger gets off the shuttle at his/her hotel. The driver records how many passengers leave the shuttle at each hotel. How many different possibilities exist?

$$\binom{5+10-1}{10} = \binom{5+10-1}{5-1} = \binom{14}{4} = \frac{14!}{4!10!}$$

Summary

- Assuming that we have a set with n elements, and we want to draw k samples from the set, then the total number of ways we can do this is given by the following.
  - ordered sampling with replacement
  $$n^k$$
  - ordered sampling without replacement
  $$\frac{n!}{(n-k)!}$$
  - unordered sampling without replacement
  $$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$
  - unordered sampling with replacement
  $$\binom{k+n-1}{k} = \binom{k+n-1}{n-1}$$

GAMBLER'S RUIN PROBLEM

- Two players A and B play a game consecutively till one of them loses all his capital. Suppose A starts with a capital of $$a$ and B with a capital of $$b$ and the loser pays \$1 to the winner in each game. Let p represent the probability of winning each game for A and $q = 1 - p$ for player B. Find the probability of ruin for each player if no limit is set for the number of games.

- Example:
  - $p = q = 0.5,\ \mathrm{a} = 9, b = 1$
  - $p = 0.45, q = 0.55,\ \mathrm{a} = 9, b = 1$

Solution

TABLE 3-1
Gambler's ruin

p	q	Capital, a	Gain, b	Probability of Ruin, P_a	Probability of Success, $1 - P_a$	Average duration, N_a
0.50	0.50	9	1	0.100	0.900	9
0.50	0.50	90	10	0.100	0.900	900
0.50	0.50	90	5	0.053	0.947	450
0.50	0.50	500	100	0 167	0.833	50,000
0.45	0.55	9	1	0.210	0.790	11
0.45	0.55	50	10	0.866	0.134	419
0.45	0.55	90	5	0.633	0.367	552
0.45	0.55	90	10	0.866	0.134	765
0.45	0.55	100	5	0.633	0.367	615
0.45	0.55	100	10	0.866	0.134	852

Solution

$p \neq q$:

$$P_a = \frac{1-\left(\frac{p}{q}\right)^b}{1-\left(\frac{p}{q}\right)^{a+b}}$$

$$Q_b = \frac{1-\left(\frac{q}{p}\right)^a}{1-\left(\frac{q}{p}\right)^{a+b}}$$

$p = q$:

$$P_a = \frac{b}{a+b}$$

$$Q_b = \frac{a}{a+b}$$

$$N_a = \begin{cases} \dfrac{b}{2p-1} - \dfrac{a+b}{2p-1} \dfrac{1-\left(\frac{p}{q}\right)^b}{1-\left(\frac{p}{q}\right)^{a+b}} & p \neq q \\ ab & p = q = \dfrac{1}{2} \end{cases}$$

Thank You!