Homework 4 Solutions

1.

(a) Since X and Y are Gaussian and independent, they are jointly Gaussian. Since Z is a linear combination of jointly Gaussian random variables, it is Gaussian. We need to find its mean and variance.

$E[Z] = E[3X + 4Y] = 3E[X] + 4E[Y] = 0$

$E[Z^2] = E[9X^2 + 16Y^2 + 24XY] = 9E[X^2] + 16E[Y^2] + 24E[XY] = 9 + 16 + 0 = 25$

$\sigma_Z^2 = E[Z^2] - E[Z]^2 = 25$

Thus $Z \sim N(0, 25)$.

$P_r(Z \geq 5) = 1 - P_r(Z \leq 5) = 1 - P_r(\frac{Z}{5} \leq 1) = 1 - \Phi(1) = 1 - 0.8413 = \boxed{0.1587}$.

(b) $Cov\{X, Z\} = E[(X - m_X)(Z - m_Z)] = E[XZ] = E[X(3X + 4Y)] = 3E[X^2] + 4E[XY] = 3$

$\rho_{XZ} = \frac{Cov\{X,Z\}}{\sigma_X \sigma_Z} = \frac{3}{1\times 5} = \boxed{\frac{3}{5}}$

(c) $W = X - Z$ is a Gaussian random variable with mean $E[W] = E[X] - E[Z] = 0$.

The Gaussian density with mean zero is symmetric in the sense that $f_W(-w) = f_W(w)$. It follows that $E[(X - Z)^3] = E[W^3] = \int_{-\infty}^{\infty} w^3 f_W(w)dw = \boxed{0}$. Since the integral is zero because the function $w^3 f_W(w)$ is odd symmetric, i.e.$(-w)^3 f_W(-w) = -w^3 f_W(w)$.

2. (a)

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

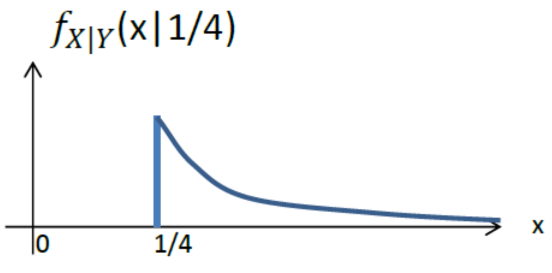
Therefore,

$$f_{X|Y}(x|1/4) = \frac{f_{XY}(x, 1/4)}{f_Y(1/4)}$$

Notice that the denominator is just a normalizing factor to make $f_{X|Y}(x|1/4)$ have unit area. The numerator is just an exponential PDF translated to the right by 1/4, so it happens that $f_{XY}(x, 1/4)$ already has unit area.

$$f_{X|Y}(x|1/4) = f_{XY}(x, 1/4) = \begin{cases} 3e^{-3(x-1/4)}u(x - 1/4), & 0 \leq x < +\infty \\ 0, & otherwise \end{cases}$$

The sketch of the conditional PDF is as follow,



(b)

Ans: For every $0 \leq y \leq 1$, the resulted conditional density conditioned on y are equivalent exponentials translated to the right by y. Since all the conditional densities integrates to 1 for all $y \in [0, 1]$. Therefore, the PDF of Y is uniform over $[0, 1]$.

(c)

Ans: Recall $Z = XY^4$. Hence, $E\{Z|Y\} = E\{XY^4|Y\}$. Conditioning on Y makes Y effectively a constant with respect to expectation, so the Y^4 factors out: $E\{Z|Y\} = E\{X|Y\}Y^4$. As for $E\{X|Y\}$, we use that $f_{X|Y}(x|y)$ is $3e^{-3x}u(x)$ shifted to the right by y. Since the mean of the RV with PDF $3e^{-3x}u(x)$ is $\frac{1}{3}$, it follows that $E\{X|Y\} = \frac{1}{3}+Y$. Therefore, $E\{Z|Y\} = (\frac{1}{3}+Y)Y^4 = \frac{Y^4}{3}+Y^5$.

(d) Ans: $$E\{Z\} = E\{E\{Z|Y\}\} = E\{\frac{Y^4}{3} + Y^5\} = \int_0^1 \frac{1}{3}y^4 + y^5 \text{ d}y = \frac{7}{30}$$

3. (a) The MMSE estimator for X based on Y is $\hat{\text{X}}= g(y) = E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx$. We need to find $f_{X|Y}(x|y)$ first.

If $0 \le y \le 1$, $f_Y(y) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = \int_0^1 2xydx = x^2y|_0^1 = y$.

If $-1 \le y \le 0$, $f_Y(y) = \int_{-\infty}^{\infty} f_{X|Y}(x|y)dx = \int_{-1}^0 2xydx = x^2y|_{-1}^0 = -y$.

Otherwise, $f_Y(y) = 0$.

Then

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \begin{cases} 2x, & 0 \le x \le 1, 0 \le y \le 1 \\ -2x, & -1 \le x \le 0, -1 \le y \le 0 \\ \text{undefined}, & y < -1 \text{ or } y > 1 \text{ or } y = 0 \\ 0, & \text{otherwise} \end{cases}$$

The MMSE estimate for X based on Y is

$$g(y) = E[X|Y = y] = \int_{-\infty}^{\infty} x f_{X|Y}(x|y)dx = \begin{cases} \frac{2}{3}, & 0 \le y \le 1 \\ -\frac{2}{3}, & -1 \le y \le 0 \\ \text{undefined}, & \text{else} \end{cases}$$

The resulting MSE is

$$\text{MSE}= E[(X - g(y))^2] = E[X^2] - 2E[Xg(y)] + E[(g(y))^2].$$

Consider each term:

By symmetry

$$f_X(x) = f_Y(y) = \begin{cases} x, & 0 \le x \le 1 \\ -x, & -1 \le x \le 0 \\ 0, & \text{otherwise} \end{cases}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)dx = \int_{-1}^1 x^2|x|dx = 2\int_0^1 x^3dx = \frac{1}{2}.$$

Since $g(y) = \pm\frac{2}{3}$, $E[(g(y))^2] = \frac{4}{9}$.

$$\begin{aligned} E[Xg(y)] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xg(y)f_{XY}(x,y)dydx \\ &= \int_0^1\int_0^1 x\tfrac{2}{3}2xydydx + \int_{-1}^0\int_{-1}^0 x(-\tfrac{2}{3})2xydydx \\ &= \tfrac{2}{3}\int_0^1 x^2y^2|_0^1dx - \tfrac{2}{3}\int_{-1}^0 x^2y^2|_{-1}^0dx \\ &= \tfrac{2}{3}\int_0^1 x^2dx + \tfrac{2}{3}\int_{-1}^0 x^2dx \\ &= \tfrac{4}{9} \end{aligned}$$

Substituting into the MSE give

MSE=$\frac{1}{2} - 2\frac{4}{9} + \frac{4}{9} = \boxed{\frac{1}{18}}$.

(b) The MMSE linear estimator is

$$g(y) = \frac{Cov\{X,Y\}}{\sigma_Y^2}(y - E[Y]) + E[X].$$

From the density of X and Y found in part (a)

$$\begin{aligned}
E[X] = E[Y] &= \textstyle\int_{-1}^{1} x|x|dx = \int_{-1}^{0} -x^2 dx + \int_0^1 x^2 dx = 0 \\
\sigma_Y^2 &= E[Y^2] = E[X^2] = \tfrac{1}{2} \\
\text{Cov}\{X,Y\} &= E[XY] = \textstyle\int_{-\infty}^{\infty} xy f_{XY}(x,y)dxdy \\
&= \textstyle\int_0^1 \int_0^1 xy \cdot 2xydxdy + \int_{-1}^{0}\int_{-1}^{0} xy \cdot 2xydxdy \\
&= \textstyle\int_{-1}^{0} \frac{2}{3} y^2 x^3|_{-1}^{0} dy + \int_0^1 \frac{2}{3} y^2 x^3|_0^1 dy \\
&= \textstyle\int_{-1}^{0} \frac{2}{3} y^2 dy + \int_0^1 \frac{2}{3} y^2 dy \\
&= \tfrac{4}{9}
\end{aligned}$$

Substituting gives

$$\boxed{g(y) = \tfrac{8}{9} y}$$

The resulting MSE is

$$\begin{aligned}
\text{MSE}= E[(X - \tfrac{8}{9}Y)^2] &= E[X^2] - \tfrac{16}{9}E[XY] + \tfrac{64}{81}E[Y^2] \\
&= \tfrac{1}{2} - \tfrac{16}{9}\tfrac{4}{9} + \tfrac{64}{81}\tfrac{1}{2} \\
&= \boxed{\tfrac{17}{162} = 0.105}
\end{aligned}$$

4.

(a) Ans:

$$E\{Z\} = E\{X_1\} + E\{X_2\} = 10$$

As for the variance,

$$Var\{Z\} = E\{Z^2\} - (E\{Z\})^2$$

$$E\{Z^2\} = E\{(X_1 + X_2)^2\} = E\{X_1^2\} + 2E\{X_1X_2\} + E\{X_2^2\} = 180.4$$

Hence,

$$Var\{Z\} = 180.4 - 100 = 80.4$$

(b) Ans:

$$\hat{X}_{2,LH} = a_{LH}X_1, \ a_{LH} = \frac{E\{X_1X_2\}}{E\{X_1^2\}} = \frac{-6.3}{13} = -0.4846$$

Therefore, $\hat{X}_{2,LH} = -0.4846X_1$.

(c)

Ans: The optimal linear nonhomogeneous estimator becomes the linear homogeneous estimator when the means are zero. Therefore, the MSE for the linear homogeneous estimator is the MSE for the linear nonhomogeneous estimator with the means set to zero. Doing that yields

$$MSE_{LH} = E\{X_2^2\}\left(1 - \left[\frac{(E\{X_1X_2\})^2}{E\{X_1^2\}E\{X_2^2\}}\right]\right) = 180\left(1 - \frac{(-6.3)^2}{13 \cdot 180}\right) = 176.94$$

(d)

Ans: The estimator is in the space spanned by X_1. Therefore, the angle is zero.

(e) Give the optimal linear nonhomogeneous estimator of X_2, given X_1.
Ans:

$$\hat{X}_{2,LNH} = a_{LNH}X_1 + b_{LNH}$$

Let σ_1 be the standard deviation for X_1 and let ρ be the normalized correlation coefficient for X_2 and X_1. Then, a_{LNH} can be expressed

$$a_{LNH} = \frac{\sigma_2}{\sigma_1}\rho = \frac{\text{Cov}(X_1, X_2)}{\sigma_1^2} = \frac{E\{X_1X_2\} - E\{X_1\}E\{X_2\}}{E\{X_1^2\} - (E\{X_1\})^2} = \frac{-6.3 - (-2 \cdot 12)}{13 - (-2)^2} = 1.96$$

$$b_{LNH} = E\{X_2\} - a_{LNH}E\{X_1\} = 12 - 1.96(-2) = 15.92$$

Hence,

$$\hat{X}_{2,LNH} = 1.96X_1 + 15.92$$

(f) Give the MSE performance of the estimator of part (e).
Ans:

$$MSE_{LNH} = \sigma_2^2(1 - \rho^2)$$

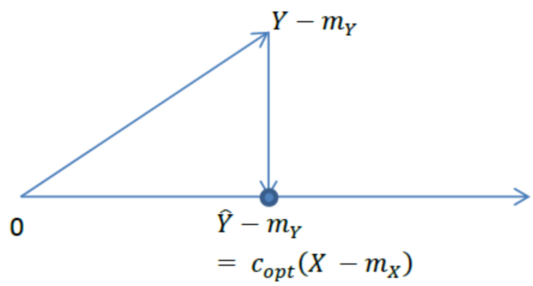
From (e), we have $\text{Cov}(X_1, X_2) = 17.7$, $\sigma_1^2 = 9$ and $\sigma_2^2 = E\{X_2^2\} - (E\{X_2\})^2 = 180 - 12^2 = 36$. Therefore,

$$\rho = \frac{\text{Cov}(X_1, X_2)}{\sigma_1\sigma_2} = \frac{17.7}{3.6} = 0.98$$

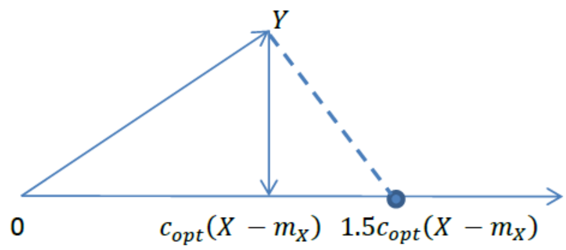
$$MSE_{LNH} = 36(1 - 0.96) = 1.42$$

5.

Ans: Recall that MSE is defined as $E\{[\hat{Y} - Y]^2\}$. Consider the illustration of the orthogonality principle:



The horizontal axis represents the collection of all points of the form $c(X - m_X)$, and the dot is the one point from the collection that is closest to $(Y - m_Y)$. Length in this space is square root of the mean square value. For instance, the distance from the origin to the point $(Y - m_Y)$ is $\sqrt{E\{(Y - m_Y)^2\}}$, which is σ_Y. Also, the distance between the point $(Y - m_Y)$ and $\hat{Y} - m_Y$ is the error of the optimal MSE estimate, and is the square root of the MSE of the optimal estimate.
Now consider the point in the collection at $1.5c_{opt}(X - m_X)$. This point must be 1.5 times as far from the origin as $c_{opt}(X - m_X)$. Therefore, it must be at the location of the dot in the figure below:



The error of this estimate is indicated by the dashed line.

Here is the derivation of this MSE:

$$E\{\epsilon^2\} = E\{[(Y-m_Y)-1.5c_{opt}(X-m_X)][(Y-m_Y)-1.5c_{opt}(X-m_X)]\} = \sigma_Y^2 - 3c_{opt}\,\mathrm{Cov}(Y,X) + 2.25c_{opt}^2\sigma_X^2$$

Next, substitute in $c_{opt} = \frac{\sigma_Y}{\sigma_X}\rho_{XY}$:

$$E\{\epsilon^2\} = \sigma_Y^2 - 3\frac{\sigma_Y}{\sigma_X}\rho_{XY}\,\mathrm{Cov}(Y,X) + 2.25\left(\frac{\sigma_Y}{\sigma_X}\rho_{XY}\right)^2\sigma_X^2$$

Recall that $\mathrm{Cov}(Y,X) = \rho_{XY}\sigma_Y\sigma_X$, substitute it and simplify:

$$E\{\epsilon^2\} = \sigma_Y^2 - 3\frac{\sigma_Y}{\sigma_X}\rho_{XY}[\rho_{XY}\sigma_Y\sigma_X] + 2.25\left(\frac{\sigma_Y}{\sigma_X}\rho_{XY}\right)^2\sigma_X^2 = \sigma_Y^2(1 - 0.75\rho_{XY}^2)$$

6. (a)

Ans: Let the sample mean of n samples be denoted by $\bar{X}_n$. The mean of $\bar{X}_n$ is the true mean μ and the variance is $\sigma^2_{\bar{X}_n}$. We want the minimum n that satisfies

$$P\left(|\bar{X}_n - \mu| \leq \frac{\sigma}{4}\right) \geq 0.98$$

The direction of the inequality does not match Chebyshev's inequality. Let's negate both sides and add 1 to both sides.

$$P\left(|\bar{X}_n - \mu| \geq \frac{\sigma}{4}\right) \leq 0.02$$

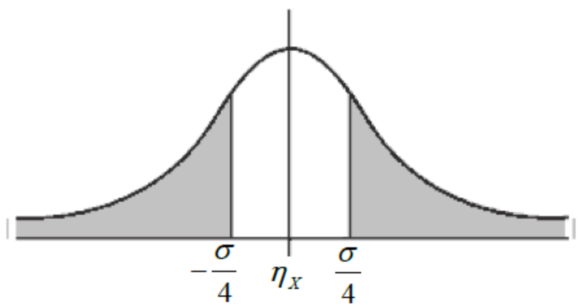
Then, by Chebyshev's Inequality, we have,

$$\frac{\sigma^2_{\bar{X}_n}}{(\frac{\sigma}{4})^2} = \frac{\frac{\sigma^2}{n}}{(\frac{\sigma}{4})^2} = \frac{16}{n} = 0.02$$

Therefore, we select n as $n \geq 800$.

(b)

Ans: To apply the CLT, we assume that $\bar{X}_n$ is Gaussian with mean μ (this is the true mean becasue $\bar{X}_n$ is unbiased) and variance $\frac{\sigma^2}{n}$. The probability that $\bar{X}_n$ deviates from its mean by less than $\frac{\sigma}{4}$ is the white part of the area shown below.



Using the definition of erf in the book, the area of the white part can be expressed as

$$2\,\mathrm{erf}\left(\frac{\frac{\sigma}{4}}{\frac{\sigma}{\sqrt{n}}}\right) = 2\,\mathrm{erf}\left(\frac{\sqrt{n}}{4}\right) > 0.98$$

According to the table we have when $\frac{\sqrt{n}}{4} > 2.327$ (approximately), or $n > 86$ we achieve the desired inequality. We obeserve that this bound is an order of magnitude lower than the one based on the Chebyshev Inequality.