- (Partial) **Moments** of a RP:
  - Mean:
    $m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx$
  - Auto-correlation / Auto-covariance:
    - $R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1)X(t_2)}(x,y)dxd$
    - $C_X(t_1,t_2) = E[(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))]$
      $= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x - m_X(t_1))(y - m_X(t_2))f_{X(t_1)X(t_2)}(x$
    - $C_X(t_1,t_2) = R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$
    - variance: $\sigma_X^2(t_1) = C_X(t_1,t_1)$
- Cross-correlation:
  $R_{XY}(t,s) = E[X_t Y_s], t, s \in \mathcal{T}$
- Cross-covariance:
  $C_{XY} = R_{XY}(t,s) - m_X(t)m_Y(s) = E[(X(t) - m_X(t))(Y(s) - m_Y(s))]$

*Discrete time random walk

- $X[n] = \Sigma_{k=1}^{n} B[k]$ is the discrete time random walk
  $B[k]$: $P(B[n] = 1) = P(B[n] = -1) = \frac{1}{2}$
  - $m_X(n) = 0$
  - $C_X(n_1,n_2) = min(n_1,n_2)$
    $\sigma_X^2(n) = n$
  - $p_{X_n}(l) = P(X(n) = l) = \binom{n}{\frac{l+n}{2}}\frac{1}{2^n}$,with $p = \frac{1}{2}$
  - $F_{X_n}(l) \to \Phi(\frac{l}{\sqrt{n}})$ by the Central Limit Theorem
- Discrete time random walk is with **independent increments**

Brownian Motion/Wiener Process (

Defining Properties

The one-dimensional Brownian motion, $X(t)$, has

- Independent increments
- Stationary increments
- The increment $X(s+t) - X(s)$ is **normally distributed** with
  - mean = 0
  - variance = αt
- $X(0) = 0$ and $X(t)$ is continuous for $t \geq 0$

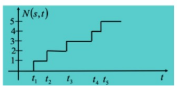
If diffusion constant $\alpha = 1$, $X(t)$ is a Standard Brownian Motion

Properties

- $R_X(t_1,t_2) = \alpha min(t_1,t_2)\ for \begin{cases} 0 < t_1 \\ 0 < t_2 \end{cases}$
- $Y(t) = \frac{dX(t)}{dt}$ is Gaussian White Noise with
  - $E[Y(t)] = 0$
  - $R_{XY}(t_1,t_2) = \frac{dR_X(t_1,t_2)}{dt_2} = \begin{cases} \alpha & t_2 < t_1 \\ 0 & o.w. \end{cases}$
  - $R_{YY}(t_1,t_2) = \frac{d}{dt}R_{XY}(t_1,t_2) = \alpha\delta(t_1 - t_2)$
  - Any integral of $Y(t)$ is Gaussian

Poisson Process (lec.17)

- $N(t)$ has independent increments
- $N(t)$ has stationary increments (homogeneous Poisso
- $P(N(h) \geq 1) = \lambda h + o(h)$
- $P(N(h) \geq 2) = o(h)$



Variations

- **Homogenous Poisson Process**: λ independent of t
- Volumn Poisson Process: $P(N(V) = m) = \frac{(\lambda V)^m}{m!}e^{-\lambda V}$
- Standard Poisson Process:
- $P(N(t+\Delta) - N(t) = m) = \frac{(\int_t^{t+\Delta}\lambda(u)du)^m}{m!}e^{-\int_t^{t+\Delta}\lambda(u)du}$
  - Not stationary increments

Properties

- $P(N(t+\Delta) - N(t) = m) = \frac{\Lambda^m}{m!}e^{-\Lambda}$
  - $\Lambda = \lambda\Delta$,
  - λ is the average rate of occurence
  - Δ is the length of the interval of observation
- $p_{N(t_1)N(t_2)}(i,j) = P[N(t_1) = i \cap N(t_2) = j]$
  $= P[N(t_1) = i] \cap P[N(t_2) - N(t_1) = j - i]$
  $= P[N(t_1) = i]P[N(t_2 - t_1) = j - i]$
- $E[N_t] = \lambda t$ and $var[N_t] = \lambda t$
- $R_{N_t N_s}(t,s) = E[N_t N_s] = (\lambda t)(\lambda s) + \lambda t$ for $t < s$
- $cov(N_t, N_s) = \lambda t$
- $T = t_2 - t_1$ where t_1 and t_2 are two consecutive event time:
  - $F_T(t) = 1 - e^{-\lambda t}$
  - $f_T(t) = \frac{d}{dt}F_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases}$

Stationarity (lec.17)

Strict-Sense Stationary (SSS)

- $E[X^k]$ is time-invariant for any k

Properties

- All n-tumples $(X_{t_1}, X_{t_2}, X_{t_3}, \dots, X_{t_n})$ and
  $(X_{t_1+\tau}, X_{t_2+\tau}, X_{t_3+\tau}, \dots, X_{t_n+\tau})$
  are identical
- $E[g(X_{t_1}, X_{t_2}, X_{t_3}, \dots, X_{t_n})] = E[g(X_{t_1+\tau}, X_{t_2+\tau}, X_{t_3+\tau}, \dots, X_{t_n+\tau})]$
- Moments are identical for all t
  - $m_X(t) = E[X_t]$ is the same for all t
  - $E[X_t^2]$ is the same for all t
  - $R_X(t+\tau, t)$ is the same for all t
  - $R_X(\tau) := E[X_{t+\tau}X_t]$
  - $R_X(t_1 - t_2) := E[X_{t_1}X_{t_2}]$

Wide-Sense Stationary (WSS)

Definition

- $E[X]$ and $E[X^2]$ are time-invariant
  - $E[X^2]$ is usually replaced by
    - $Cov[X_t, X_s]$ or
    - $R_X(\tau)$

Joint WSS

- X_t and Y_t are jointly WSS if
  - X_t is WSS; Y_t is WSS
  - $R_{XY}(t+\tau, t) = R_{XY}(\tau)$

Wide-Sense Cyclostationarity

Definition

(m is any integer and T is the period)

- $m_X(t + mT) = m_X(t)$
- $C_X(t_1 + mT, t_2 + mT) = C_X(t_1,t_2)$
  Or
  $R_X(t_1 + mT, t_2 + mT) = R_X(t_1,t_2)$

PSD, QAM, White Noise (lec.18)

Power Spectral Density (PSD)

- $S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau)e^{-j\omega\tau}d\tau = \int_{-\infty}^{\infty} R_X^e(\tau)cos(\omega\tau)d\tau - j\int_{-\infty}^{\infty} R_X^o(\tau)sin(\omega\tau)d\tau$
- $R_X(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)e^{j\omega\tau}d\omega$ = $\mathcal{F}^{-1}[S_X(\omega)]$
- If $X(t)$ is real, $R_X(\tau)$ is real and even => $S_X(\omega)$ is real and even

Average power

If $X(t)$ is WSS voltage waveform with $S_X(\omega)$,

- Average power(Watts): $P_{avg}[\omega_a, \omega_b] = \frac{1}{2\pi}\int_{-\omega_b}^{-\omega_a} S_X(\omega)d\omega + \frac{1}{2\pi}\int_{\omega_a}^{\omega_b} S_X(\omega)d\omega$
- Total average power: $E[|X(t)|^2] = E[X(t)X^*(t)] = R_X(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)d\omega$

Quadrature Amplitude Modulation (QAM)

$A(t)$ and $B(t)$ are real, jointly WSS RPs with zero mean;

ω_0 is the carrier frequency in radians/sec

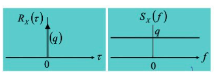
- QAM-modulated signal: $X(t) = A(t)cos(\omega_0 t) - B(t)sin(\omega_0 t)$
- $X(t)$ is WSS <=
  - $R_A(\tau) = R_B(\tau)$
  - $R_{AB}(\tau) = -R_{BA}(\tau)$
  where $R_{AB}(\tau) = E[A(t+\tau)B^*(t)]$
- $S_X(\omega) = \frac{1}{2}[S_A(\omega - \omega_0) - jS_A(\omega - \omega_0)] + \frac{1}{2}[S_A(\omega + \omega_0) + jS_{AB}(\omega + \omega_0)]$

White Noise

Definition

- $X(t)$ is a white-noise process
  <=> $C(t_1,t_2) = q(t_1)\delta(t_1 - t_2)$
  <=> Any two samples of a white noise are uncorrelated
- Usually, white noise is **assumed to have zero mean**
  => $R(\tau) = q\delta(\tau)$

Properties

- PSD: flat power spectral density
- 

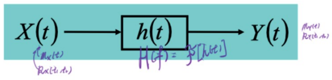
- Total average power: $E[|X(t)|^2] = \int_{-\infty}^{+\infty} S_X(f)df = +\infty$

Response of Systems (lec.19)

Zero Initial Conditions

- $m_Y(t) = L_t[m_X(t)]$
- $R_{XY}(t_1,t_2) = L_{t_2}[R_{XX}(t_1,t_2)]$
- $R_{YY}(t_1,t_2) = L_{t_1}[R_{XY}(t_1,t_2)]$

LTI Systems and RPs (lec.19)



- $Y(t) = \int_{-\infty}^{\infty} h(s)X(t-s)ds$
- $m_Y(t) = \int_{-\infty}^{\infty} h(s)m_X(t-s)ds = h(t) * m_X(t)$
- $R_{XY}(t_1,t_2) = h^*(t_2) * R_X(t_1,t_2)$
- $R_Y(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(s)h^*(\alpha)R_X(t_1 - s, t_2 - \alpha)dsd\alpha = h(t_1) * h^*(t_2) * R_{XY}(t_1,t_2)$
- WSS Input => WSS Output
  - $R_Y(\tau) = \int_{-\infty}^{\infty} h(s)R_{XY}(\tau - s)ds = h(\tau) * R_{XY}(\tau) = h(\tau) * h^*(-\tau) * R_X(\tau)$
  - $R_{XY}(\tau) = \int_{-\infty}^{\infty} h^*(\alpha)R_X(\tau + \alpha)d\alpha = h^*(-\tau) * R_X(\tau)$
  - $S_Y(\omega) = \mathcal{F}[R_Y(\tau)] = H(\omega)\mathcal{F}[h^*(-\tau)]S_X(\omega) = |H(\omega)|^2 S_X(\omega)$

Introduction to Markov Processes (lec.20)

General Properties

- $P(X_n = i_n | X_0 = i_0, X_1 = i_1, \dots, X_{n-1} = i_{n-1}) = P(X_n = i_n | X_{n-1} = i_{n-1})$
- $E[X_n | X_0, X_1, \dots, X_{n-1}] = E[X_n | X_{n-1}]$
- $P(X_n = i_n | X_{n+1} = i_{n+1}, \dots, X_{n+k} = i_{n+k}) = P(X_n = i_n | X_{n+1} = i_{n+1})$
- Let $k < m < n$, then
  $P(X_k = i_k \cap X_n = i_n | X_m = i_m) = P(X_k = i_k | X_m = i_m)P(X_n = i_n | X_m = i_m)$
- Chapman-Kolmogorov Equation:
  Let $k < m < n$, then
  - $P(X_n = i_n | X_k = i_k) = \Sigma_{all\ i_m} P(X_n = i_n | X_m = i_m)P(X_m = i_m | X_k = i_k)$
  - If homogenous: $P_{ij}^{(n)} = \Sigma_k P_{ik}^{(m)} P_{kj}^{(n-m)}$

Terms

- Transition Probabilities / Transition Densities:
  - $P(X_n = i_n | X_m = i_m)$
  - $f_{X(t_n)|X(t_m)}(x|y)$
- Homogeneity:
  - $P(X_n = i | X_k = j) = P(X_{n+m} = i | X_{k+m} = j)$
- Transition Probability Matrix
  - $\mathbf{\Pi} = \begin{bmatrix} P_{00} & P_{01} & \dots & P_{0N} \\ P_{10} & & & P_{1N} \\ \dots & & & \dots \\ P_{N0} & & \dots & P_{NN} \end{bmatrix}$
  - Each row sums to 1.
- State Probability Vector
  - $\pi_j^{(n)} = P(X_n = j)$
  - $\boldsymbol{\pi}^{(n)} = [\pi_1^{(n)} \quad \pi_2^{(n)} \quad \pi_3^{(n)} \quad \dots \quad \pi_N^{(n)}]$
  - Total Probability Theorem:
    $\boldsymbol{\pi}^{(n)} = \boldsymbol{\pi}^{(n-k)}\mathbf{\Pi}^k$
- P(First return to state j occurs n steps after leaving it)
  - $f_j^{(n)}$
  - $f_j^1 = [\Pi]_{jj}$
- P(ever returning to state j)
  - $f_j = \Sigma_{n=1}^{\infty} f_j^{(n)}$
- Recurrence Formula
  - $[\Pi^n]_{jj} = \Sigma_{k=1}^{n} f_j^{(k)}[\Pi^{n-k}]_{jj}$
- Mean Recurrence Time: the average time to return to state
  - $M_j = \Sigma_{n=1}^{\infty} n f_j^{(n)}$

Classification of States

Transient	Recurrent			
	Recurrent Null		Recurrent Non-null	
	Periodic	Aperiodic	Periodic	Aperiodic (Ergodic)

- Transient: $f_j < 1$
- Recurrent: $f_j = 1$
  - Recurrent Null: $M_j = +\infty$
  - Recurrent Non-null: $M_j < +\infty$
  - Periodic: the only possible numbers of steps for returning to state
  - Aperiodic: $\gamma = 1$
  - Ergodicity: Recurrent Non-null + Aperiodic => Steady States exist
- Absorbing

Poisson

- Suppose fish bite with a Poisson distribution, with an average rate of one per 20 min. What is the probability that at least one fish will bite in the next 5 min given that no fish has bitten in the last 20 min?

$$P(N(t+5)-N(t)\ge 1 \mid N(t)-N(t-20)=0)$$
$$=P(N(t+5)-N(t)\ge 1)=1-\frac{(5/20)^0}{0!}e^{-5/20}=0.22$$

- Suppose two customers arrive at a shop during a two-minute period. Find the probability that one arrived in the first minute and the other arrived in the second minute.

Poisson approach:

$$P(N(1)=1 \mid N(2)=2)=\frac{P(N(1)=1\cap N(2)=2)}{P(N(2)=2)}$$
$$=\frac{P(N(1)=1\cap N(2)-N(1)=1)}{P(N(2)=2)}$$

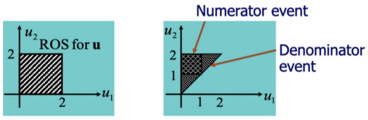
From previous slide,

$$P(N(1)=1 \mid N(2)=2)=\frac{P(N(1)=1\cap N(2)-N(1)=1)}{P(N(2)=2)}$$

Using increment properties,

$P(N(2)-N(1)=1)$
$=P(N(1)-N(0)=1)$, $N(0)=0$

$$P(N(1)=1 \mid N(2)=2)=\frac{[P(N(1)=1)]^2}{P(N(2)=2)}=\frac{\left[\frac{(\lambda\cdot 1)^1}{1!}e^{-\lambda\cdot 1}\right]^2}{\left[\frac{(\lambda\cdot 2)^2}{2!}e^{-\lambda\cdot 2}\right]}=\frac{1}{2}$$



$$P(N(1)=1 \mid N(2)=2)=P(u_1\le 1, u_2\ge 1 \mid u_1\le u_2)$$
$$=\frac{P(u_1\le 1, u_2\ge 1\cap u_1\le u_2)}{P(u_1\le u_2)}=\frac{1}{2}$$

Must count all permutations

$$P(N(1)=1 \mid N(2)=2)$$
$$=P(\{u_1\le 1\cap u_2\ge 1\}\cup\{u_2\le 1\cap u_1\ge 1\})$$
$$=P\{u_1\le 1\cap u_2\ge 1\}+P\{u_2\le 1\cap u_1\ge 1\}$$
$$=1/4+1/4=1/2$$

- Micrometeors strike the space shuttle according to a Poisson process. The expected time between strikes is 30 minutes. Find the probability that during at least one hour out of five consecutive hours, three or more micrometeors strike the shuttle.

$$\mathrm{P}\left(\bigcup_{i=1}^{5}\{N_i-N_{i-1}\ge 3\}\right)=1-\mathrm{P}\left(\bigcap_{i=1}^{5}\{N_i-N_{i-1}<3\}\right)$$
$$=1-\prod_{i=1}^{5}\mathrm{P}(N_i-N_{i-1}\le 2),$$

where the last step follows by the independent increments property of the Poisson process. Since $N_i-N_{i-1}\sim \text{Poisson}(\lambda[i-(i-1)])$, or simply $\text{Poisson}(\lambda)$,

$$\mathrm{P}(N_i-N_{i-1}\le 2)=e^{-\lambda}(1+\lambda+\lambda^2/2)=5e^{-2},$$

and we have

$$\mathrm{P}\left(\bigcup_{i=1}^{5}\{N_i-N_{i-1}\ge 3\}\right)=1-(5e^{-2})^5\approx 0.86.$$

LTI system

Example

- White noise with power spectral density $N_0/2$ is passed through a linear, time-invariant system with impulse response $h(t)=1/(1+t^2)$. If Y_t denotes the filter output, find $E[Y_{t+\frac{1}{2}}Y_t]$.

We need to find $E[Y_{t+\frac{1}{2}}Y_t]$ which is $R_Y(\frac{1}{2})$. We find this by finding the power spectral de taking the inverse Fourier transfrom. The power spectral density is

$$S_Y(f)=S_X(f)|H(f)|^2=\frac{N_0}{2}\pi^2e^{-4\pi|f|},$$

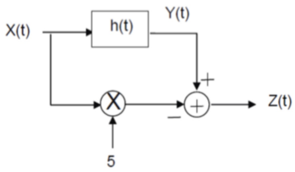
where $H(f)=\pi e^{-2\pi|f|}$. Then

$$R_Y(\tau)=\mathcal{F}^{-1}(S_Y(f))=\mathcal{F}^{-1}(\tfrac{N_0}{2}\pi^2e^{-4\pi|f|})=\frac{N_0\pi}{2}\frac{2}{4+t^2}=\frac{N_0\pi}{4+t^2}$$

and

Example

- Let Y(t)=h(t)*X(t) and Z(t)=-5X(t)+Y(t) as shown below.
- (a) Find $S_Z(\omega)$ in terms of $S_X(\omega)$. *Hint: represent everything between X(t) and Z(t) as one LTI system.*
- (b) Give an expression for $E\{Z^2(t)\}$.



- (a)

The easy way is to view everything between $X(t)$ and $Z(t)$ as one system with impulse response $\tilde{h}(t)=h(t)-5\delta(t)$. Then you can write

$$S_Z(\omega)=S_X(\omega)|\tilde{H}(\omega)|^2=S_X(\omega)|H(\omega)-5|^2$$

- (b)

$$E\{Z^2(t)\}=R_Z(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_Z(\omega)\,d\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}S_X(\omega)|H(\omega)-5|^2\,d\omega$$

6. Suppose $X(t)$ is a stationary RP. Prove that $X(t)$ and its derivative are orthogonal and uncorrelated.

Hint: Consider the chain rule and represent $X(t)\frac{dX(t)}{dt}$ as a derivative.

6. $\int X(t)\frac{dX(t)}{dt}dt = \frac{1}{2}X^2(t)$

$X(t)\cdot\frac{dX(t)}{dt} = \frac{d}{dt}\frac{1}{2}X^2(t)$

$E[X(t)\frac{dX(t)}{dt}] = E[\frac{d}{dt}\frac{1}{2}X^2(t)]$

$= \frac{d}{dt}\frac{1}{2}E[X^2(t)]$

$= \frac{d}{dt}\cdot\frac{1}{2}R_X(0)$ ⊥ t.

$= 0$

Properties of Integrated White Noise

$Y(t)=\int_0^t X(s)ds=X(t)*u(t)$

- $E[|Y(t)|^2]=\int_0^t q(v)dv$
- $Y(t)$ has uncorrelated increments:
  - $X(t)$ samples independent => increments are indep
  - $X(t)$ follows Gaussian distribution => increments a

Gaussian White Noise

- Each sample is independent + normal distribution with
  => Gaussian white noise
- Any integral of Gaussian White noise (GWN) is Gaussiar
- $X(t)$ is WSS GWN =>
  - $Y(t)=\int_0^t X(s)ds$ has independent increments
  - $Y(t)$ is a Wiener Process
- If $Y(t)=\int_0^t X(s)ds$ is a Wiener process =>
  - $X(t)$ is Gaussian white noise with
    - $E[X(t)]=0$
    - $R_{XX}(t_1,t_2)=\sigma^2\delta(t_1-t_2)$

Formula

- $\Sigma_{k=1}^n a^k=\frac{1-a^k}{1-a}\quad(a\ne 1)$
- $\Sigma_{k=0}^{\infty}\frac{\Lambda^k}{k!}=e^{\Lambda}$
- $1-e^{-x}\approx x\ when\ x\to 0$
- $e^{jA}=cosA+jsinA$
- $cosA=\frac{e^{jA}+e^{-jA}}{2}$
- $sinA=\frac{e^{jA}-e^{-jA}}{2j}$
- $cosAcosB=\frac{1}{2}[cos(A+B)+cos(A-B)]$
- $cos(A+B)=cosAcosB-sinAsinB$
- $cos(A-B)=cosAcosB+sinAsinB$
- $sin(A+B)=sinAcosB+cosAsinB$
- $sin(A-B)=sinAcosB-cosAsinB$

If $\Phi(t)=\int_{a(t)}^{b(t)}f(x)dx,$

then $\Phi'(t)=f(b(t))b'(t)-f(a(t))a'(t)$

$\int ln(x)dx=xln(x)-x$

- $f_Z(z)=f_X(z)*f_Y(z)=\int_{-\infty}^{+\infty}f_X(z-u)f_Y(u)du$
- If $X(t)=sin(\omega_0t+\theta)\quad\theta\sim U[-\pi,\pi]$
  - $R_X(\tau)=\frac{1}{2}cos(\omega_0\tau)$

	$x(t)$	$X(\omega)$
1.	1	$2\pi\delta(\omega)$
2.	$u(t)$	$\pi\delta(\omega)+\frac{1}{j\omega}$
3.	$\delta(t)$	1
4.	$\delta(t-t_0)$	$\exp[-j\omega t_0]$
5.	$\text{rect}(t/\tau)$	$\tau\,\text{sinc}\,\frac{\omega\tau}{2\pi}=\frac{2\sin\omega\tau/2}{\omega}$
6.	$\frac{\omega_B}{\pi}\text{sinc}\,\frac{\omega_Bt}{\pi}=\frac{\sin\omega_Bt}{\pi t}$	$\text{rect}(\omega/2\omega_B)$
7.	$\text{sgn}\,t$	$\frac{2}{j\omega}$
8.	$\exp[j\omega_0t]$	$2\pi\delta(\omega-\omega_0)$
9.	$\sum_{n=-\infty}^{\infty}a_n\exp[jn\omega_0t]$	$2\pi\sum_{n=-\infty}^{\infty}a_n\delta(\omega-n\omega_0)$
10.	$\cos\omega_0t$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
11.	$\sin\omega_0t$	$\frac{\pi}{j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
12.	$(\cos\omega_0t)u(t)$	$\frac{\pi}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]+\frac{j\omega}{\omega_0^2-\omega^2}$
13.	$(\sin\omega_0t)u(t)$	$\frac{\pi}{2j}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]+\frac{\omega_0}{\omega_0^2-\omega^2}$
14.	$\cos\omega_0t\,\text{rect}(t/\tau)$	$\tau\,\text{sinc}\,\frac{(\omega-\omega_0)\tau}{2\pi}$
15.	$\exp[-at]u(t),\quad \text{Re}\{a\}>0$	$\frac{1}{a+j\omega}$
16.	$t\exp[-at]u(t),\quad \text{Re}\{a\}>0$	$\left(\frac{1}{a+j\omega}\right)^2$
17.	$\frac{t^{n-1}}{(n-1)!}\exp[-at]u(t),\quad \text{Re}\{a\}>0$	$\frac{1}{(a+j\omega)^n}$
18.	$\exp[-a\lvert t\rvert],\quad a>0$	$\frac{2a}{a^2+\omega^2}$
19.	$\lvert t\rvert\exp[-a\lvert t\rvert],\quad \text{Re}\{a\}>0$	$\frac{4aj\omega}{a^2+\omega^2}$

Some Selected Fourier Transform Pairs *(continued)*

	$x(t)$	$X(\omega)$
20.	$\frac{1}{a^2+t^2},\ \text{Re}\{a\}>0$	$\frac{\pi}{a}\exp[-a\lvert\omega\rvert]$
21.	$\frac{t}{a^2+t^2},\ \text{Re}\{a\}>0$	$\frac{-j\pi\omega\exp[-a\lvert\omega\rvert]}{2a}$
22.	$\exp[-at^2],\quad a>0$	$\sqrt{\frac{\pi}{a}}\exp\left[\frac{-\omega^2}{4a}\right]$
23.	$\Delta(t/\tau)=\begin{cases}\frac{1}{\tau}(\tau-\lvert t\rvert), & \lvert t\rvert\le\tau\\ 0, & \lvert t\rvert\ge\tau\end{cases}$	$\tau\,\text{sinc}^2\frac{\omega\tau}{2\pi},\quad \tau>0$
24.	$\sum_{n=-\infty}^{\infty}\delta(t-nT)$	$\frac{2\pi}{T}\sum_{n=-\infty}^{\infty}\delta\left(\omega-\frac{2n\pi}{T}\right)$

Some Selected Properties of the Fourier Transform

1.	Linearity	$\sum_{n=1}^{N}\alpha_nx_n(t)$	$\sum_{n=1}^{N}\alpha_nX_n(\omega)$
2.	Complex conjugation	$x^*(t)$	$X^*(-\omega)$
3.	Time shift	$x(t-t_0)$	$X(\omega)\exp[-j\omega t_0]$
4.	Frequency shift	$x(t)\exp[j\omega_0t]$	$X(\omega-\omega_0)$
5.	Time scaling	$x(at)$	$1/\lvert a\rvert\,X(\omega/a)$
6.	Differentiation	$d^nx(t)/dt^n$	$(j\omega)^nX(\omega)$
7.	Integration	$\int_{-\infty}^{t}x(\tau)d\tau$	$\frac{X(\omega)}{j(\omega)}+\pi X(0)\delta(\omega)$
8.	Parseval's relation	$\int_{-\infty}^{\infty}\lvert x(t)\rvert^2dt$	$\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert X(\omega)\rvert^2d\omega$
9.	Convolution	$x(t)*h(t)$	$X(\omega)H(\omega)$
10.	Duality	$X(t)$	$2\pi\,x(-\omega)$
11.	Multiplication by t (Differentiation in frequency domain)	$(-jt)^nx(t)$	$\frac{d^nX(\omega)}{d\omega^n}$
12.	Modulation (Multiplication in time domain)	$x(t)m(t)$	$\frac{1}{2\pi}X(\omega)*M(\omega)$

$$F(w)=\mathcal{F}[f(t)]=\int_{-\infty}^{\infty}f(t)e^{-jwt}dt$$
$$f(t)=\mathcal{F}^{-1}[F(w)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}F(w)e^{jwt}dw$$