Probability and Random Process

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Outline

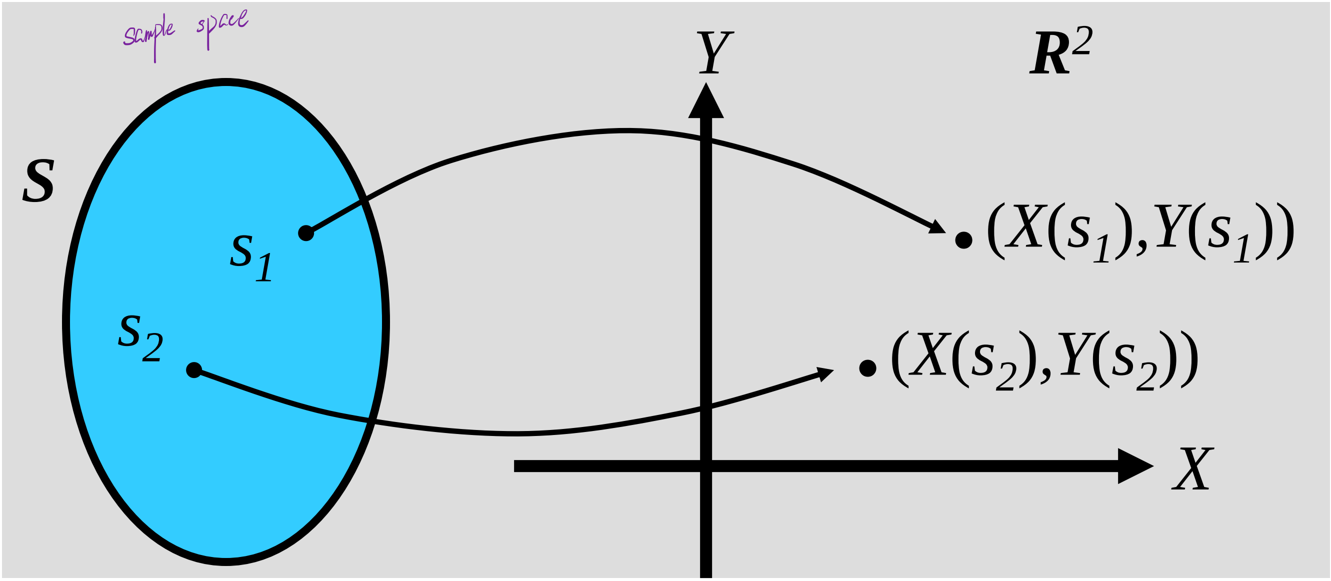
- 3. Multiple Random Variables
  - Two Random Variables
  - Marginal PDF
  - Functions of Two Random Variables
  - Conditional PDF
  - Joint Moments
  - Mean Square Error Estimation
  - Probability bound
  - Random Vectors
  - Sample Mean
  - Convergence of Random Sequences
  - Central Limit Theorem

Two Random Variables

Two Random Variables

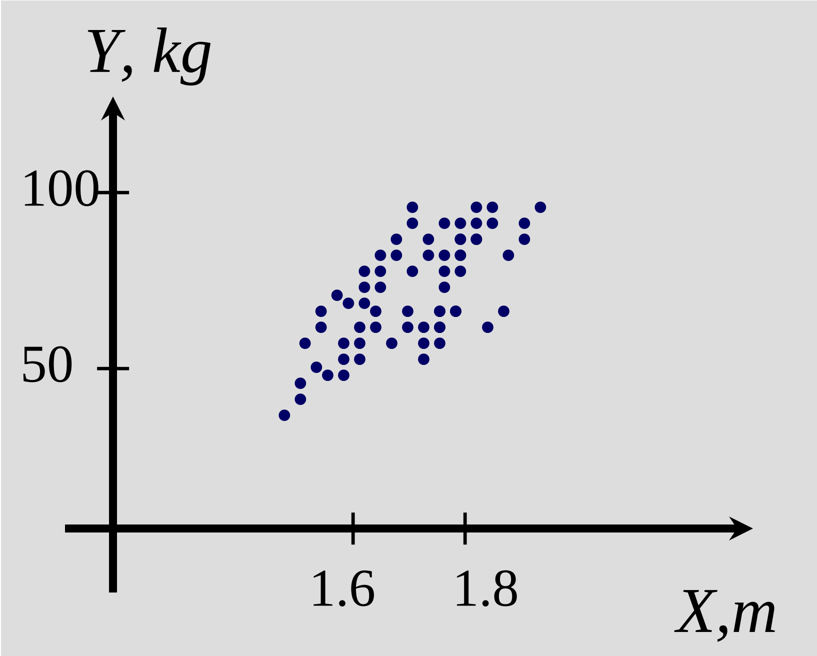
Also known as:

- Two-dimensional random variable
- Bivariate random variable



Example of Two RVs

Let S be all the students at JI. Let $(X(s), Y(s))$ be the $(height, weight)$ of students.



A scatter diagram is used to display measured data

Definition of Joint CDF -- I

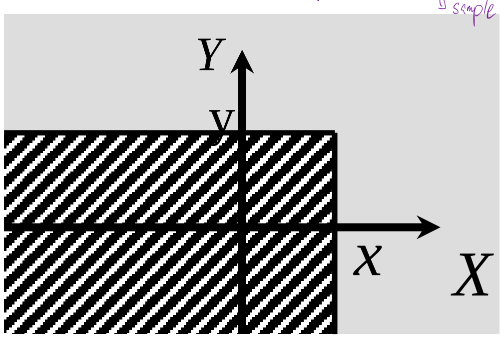
Definition of the joint cumulative distribution function (CDF):

$$F_{XY}(x, y) = P(X \le x \cap Y \le y)$$

Properties:

1. $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0$
2. $F_{XY}(x, +\infty) = F_X(x), \quad F_{XY}(+\infty, y) = F_Y(y)$

$P(X \le x \cap \Omega)$ marginal
Ω sample space



$F_{XY}(x, y) =$ Probability that (X,Y) is in this region

How about $F_{XY}(x_1, y_1) - F_{XY}(x_2, y_2)$?

Definition of Joint PDF -- II

Definition of the joint probability density function (PDF):

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

$$F_{XY}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(u,v)dvdu$$

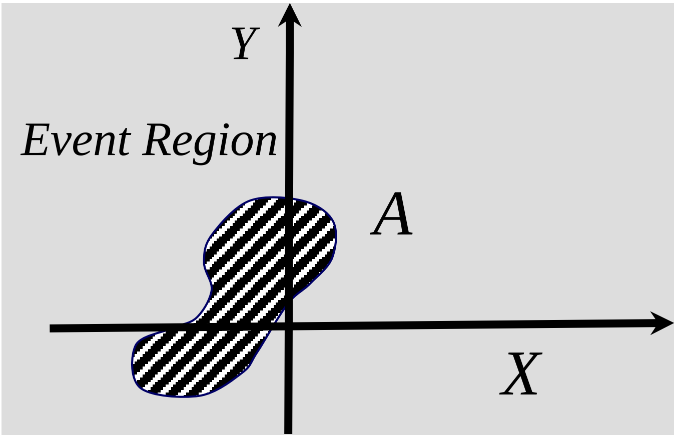
The joint PDF is a surface. The volume under $f_{XY}(x,y)$ is probability.

Properties of the Joint PDF

1. $f_{XY}(x,y) \geq 0$ for all $(\text{"}\forall\text{"})$ $x, y \in R^2$

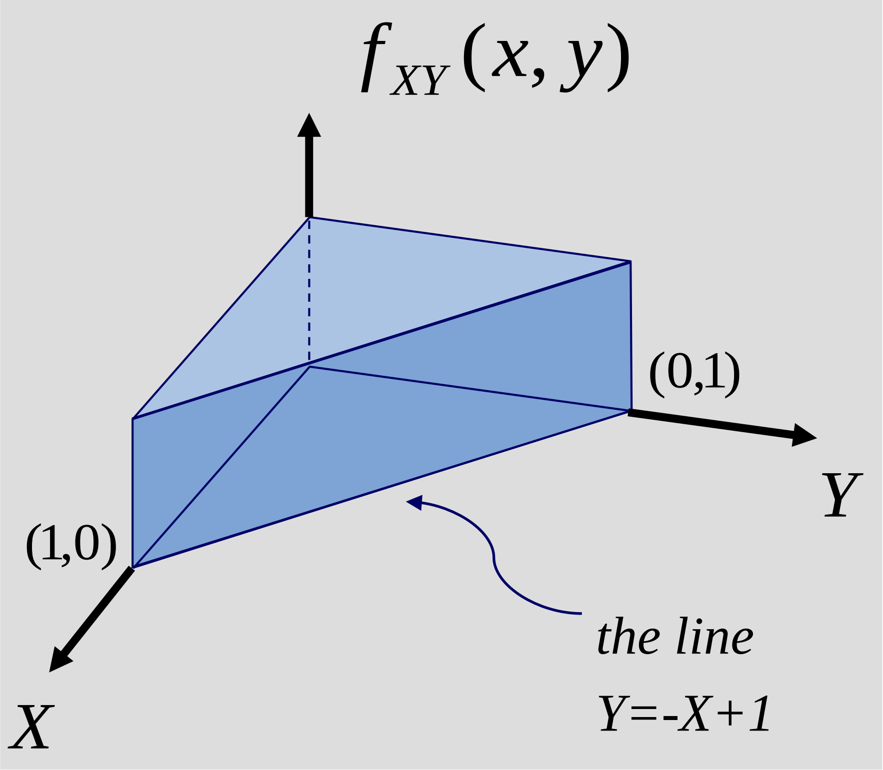
2. $$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dxdy = 1$$

3. $$P\big((X,Y) \in A\big) = \iint_A f_{XY}(x,y)\,dxdy$$



Example of a Joint PDF

Suppose the joint PDF is flat ("uniform") over the triangle with vertices (0,0), (0,1), (1,0).

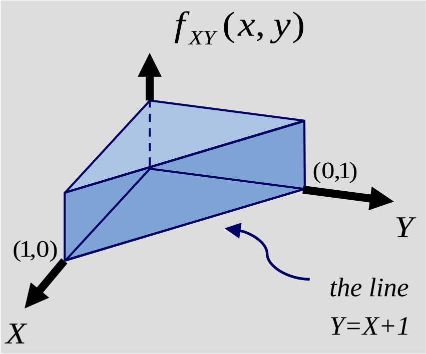


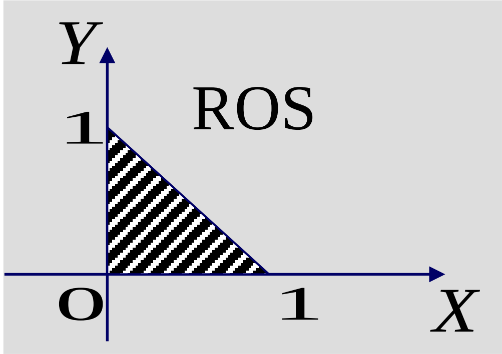
Definition of Region of Support

The region of support (ROS) is the set of all points (x,y) such that ("s.t.") $f_{XY}(x,y) > 0.$

strictly

In the previous example, the ROS is the shaded area:





The area of this ROS is ½. What is the PDF?

Example of Height Calculation

For a uniform joint PDF, the total volume is the height times the area of the ROS.

The total volume must be 1, so the height must be 2.

Symbolic description of the joint PDF:

$$f_{XY}(x,y)=\begin{cases}2 & x\geq 0, y\geq 0, y\leq -x+1\\ 0 & \text{otherwise}\end{cases}$$

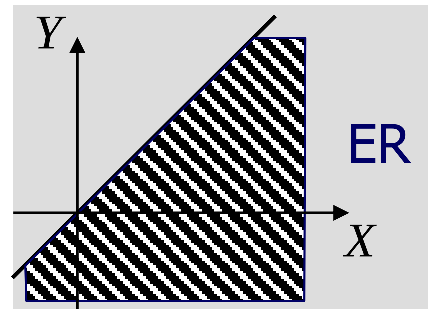
The "0 otherwise" is necessary for completeness.

Example Probability Calculation - I

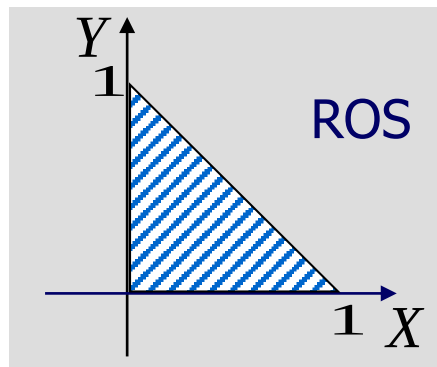
$\{s: X(s) > Y(s)\}$

- Calculate P(X>Y)
- General Procedures:

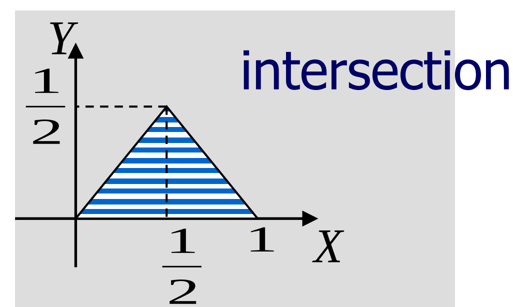
 1. Identify the event region (ER).



 2. Identify the ROS.



 3. Integrate the joint PDF over the intersection of the ER and ROS.



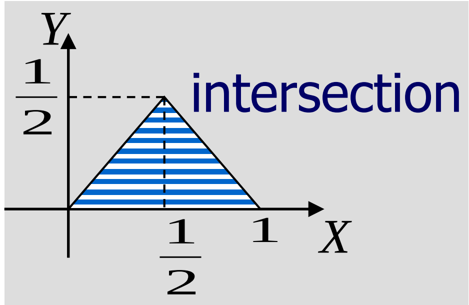
Example Probability Calculation - II

Because the joint PDF is uniform and the line $X=Y$ bisects the ROS, we observe that $\Pr(X > Y) = 1/2$.

In general, double integration must be performed to compute probability.

For this example:

$$P(X > Y) = \int_{y=0}^{1/2} \int_{x=y}^{1-y} 2\,dx\,dy = 1/2$$

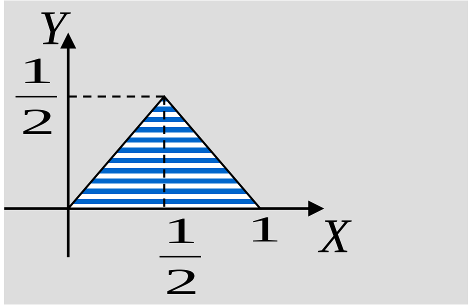


Integral Boundaries

How to determine the integral boundaries?

- Determine y first, since dy is the outer integral
- Fix y, obtain the boundaries of x

$$P(X>Y)=\int_{y=0}^{1/2}\int_{x=y}^{1-y}2dxdy=1/2$$



Joint PMF for discrete case

- **Joint probability mass function (pmf)** of X and Y is defined by

$$p_{XY}(x,y) = \Pr(X = x, Y = y)$$

- Properties
  1. $p_{XY}(x,y) \geq 0$
  2. $\sum_X \sum_Y p_{XY}(x,y) = 1$
  3. $\Pr(X \in A, Y \in B) = \sum_{X\in A} \sum_{Y\in B} p_{XY}(x,y)$
  4. $\Pr((X,Y) \in G) = \sum_{(X,Y)\in G} p_{XY}(x,y)$

Example

- X and Y are integer valued.

$$P_r(1 \leq X \leq 3, 2 \leq Y \leq 4) = \sum_{x=1}^{3} \sum_{y=2}^{4} p_{XY}(x, y)$$

Marginal PDF

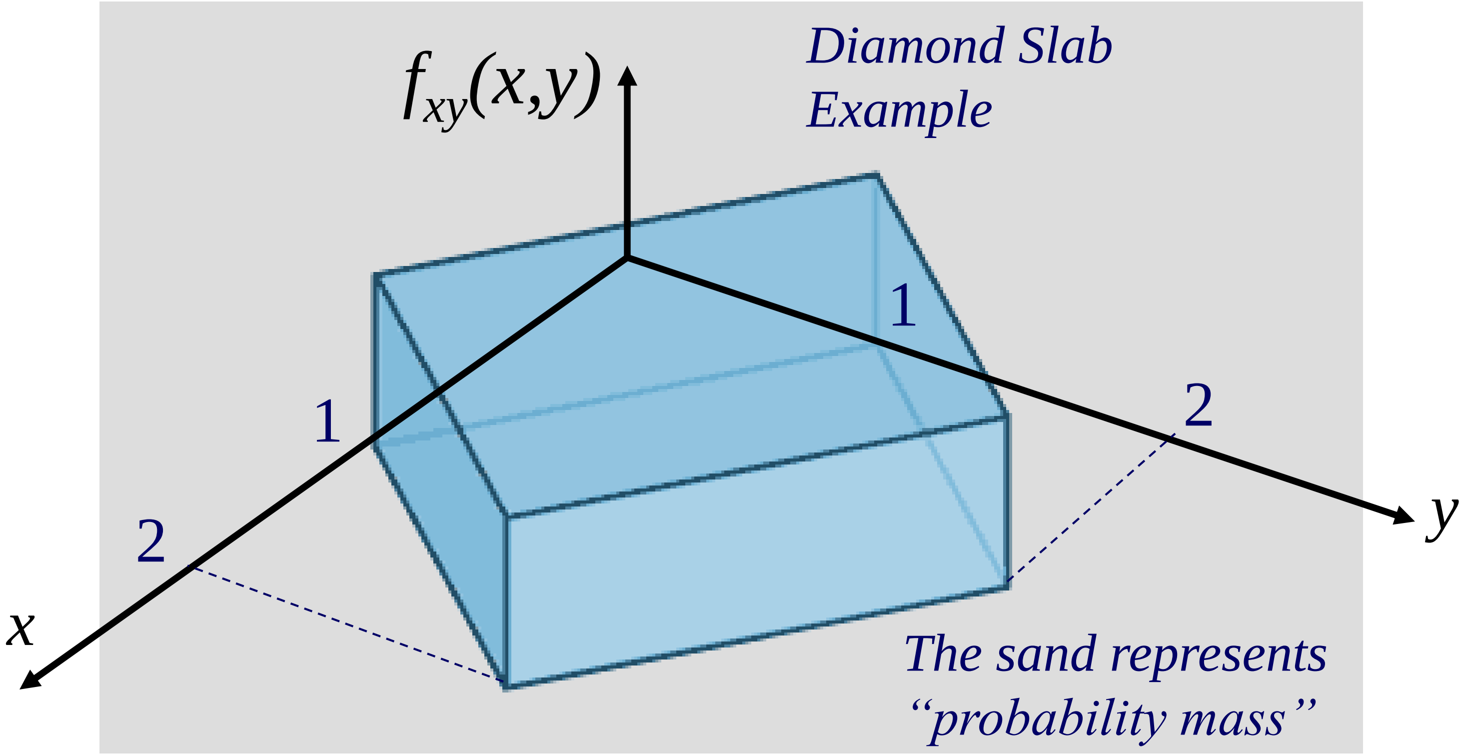
Marginal PDFs

A marginal PDF is obtained by "integrating out" the unwanted variable in a joint PDF:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY(x,y)}(x, y)dy$$

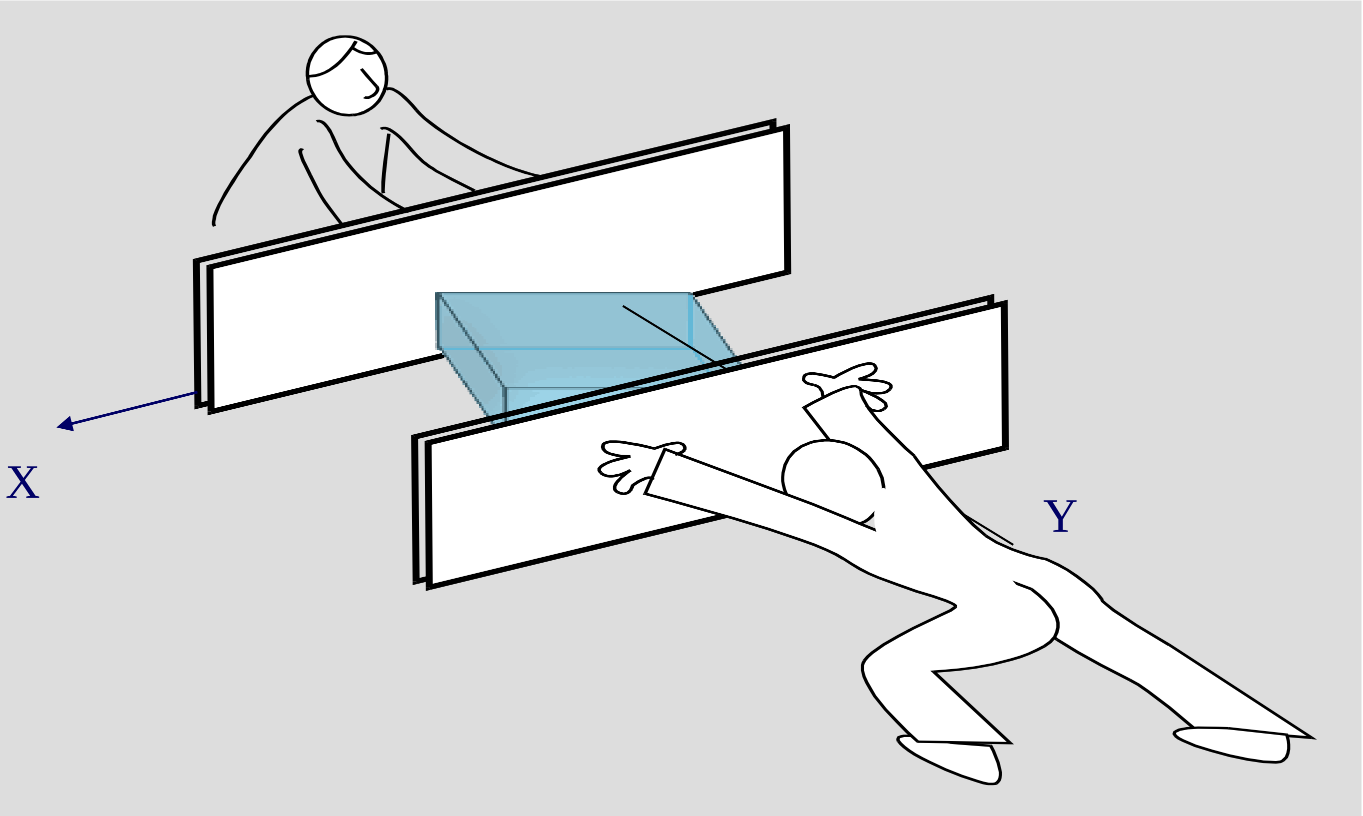
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY(x,y)}(x, y)dx$$

Sand Analogy

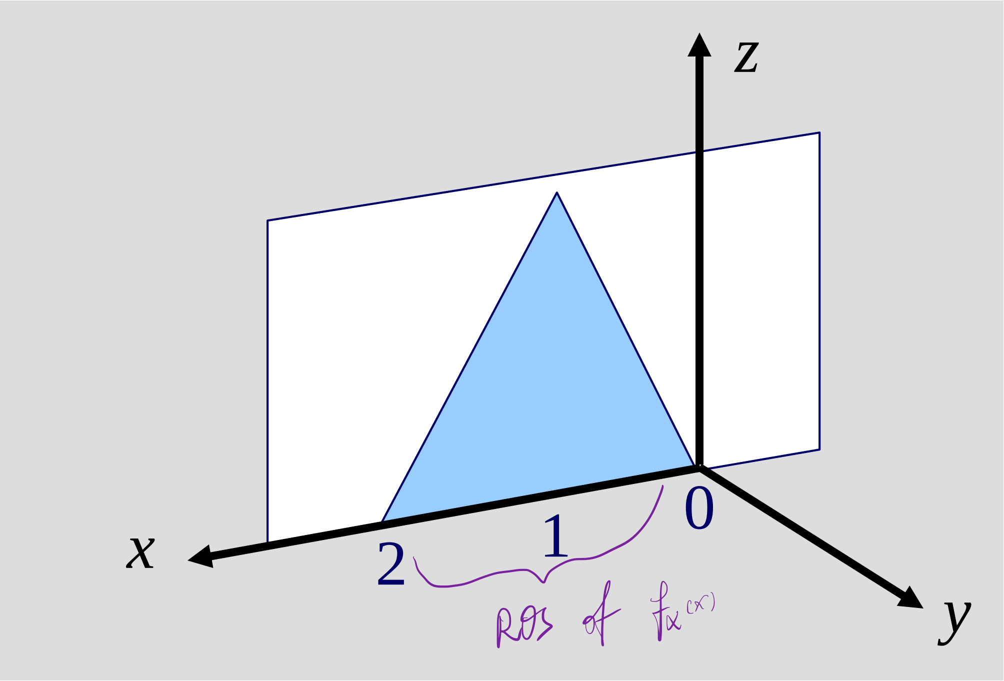


Sand Analogy II

To get the marginal for X, imagine using boards that are parallel to the X-*axis* to push the sand **onto** the X-*axis* :



Sand Analogy III



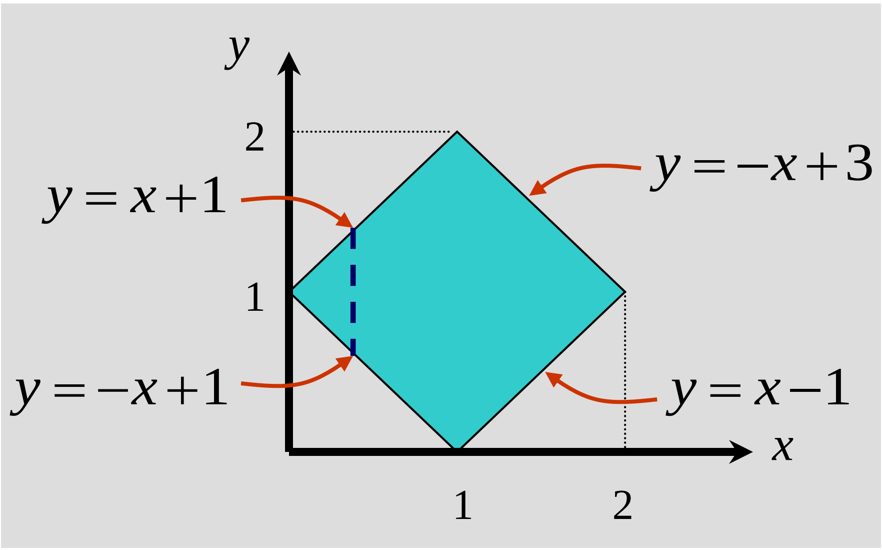
Use this idea to:

- Predict the ROS of $f_X(x)$
- Predict the profile of $f_X(x)$

Integral Limits Can Be Tricky

The tough part is to set up the integral correctly.

It is very helpful to sketch ROS of the joint PDF to see the limits of integration.



$$for \quad 0<x\le 1:$$

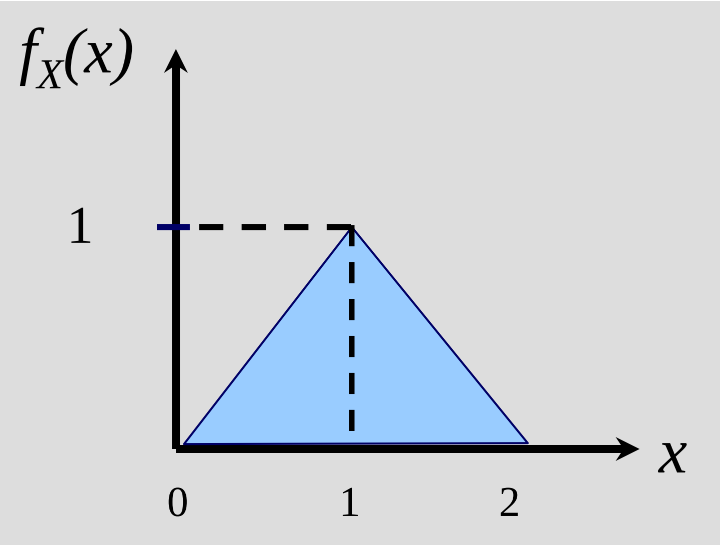
$$f_X(x)=\int_{-x+1}^{x+1}\frac{1}{2}\,dy=x$$

$$for \quad 1<x\le 2:$$

$$f_X(x)=\int_{x-1}^{-x+3}\frac{1}{2}\,dy=2-x$$

Marginal for Diamond Slab

$$f_X(x) = \begin{cases} x & 0 < x \le 1 \\ 2 - x & 1 < x \le 2 \\ 0 & \text{ow} \end{cases}$$



Two check-points:

- Does $f_X(x)$ integrate to 1?
- $f_X(x)$ should have no y dependence.

Marginal PMF for discrete case

- Relationship between joint pmf and marginal (individual) pmf's.

$$p_X(x) = \sum_y p_{XY}(x,y), \quad p_Y(y) = \sum_x p_{XY}(x,y)$$

Short Summary

- Marginal PDF
  - Obtained by integrating out the unwanted variable in a joint PDF

- Remarks
  - Be careful about the limits of integral
  - Two check-points

Independence

Definition of Equality

- Two random variables X and Y are said to be **equal (almost surely)**, written as $X = Y$, if the event $\{X = Y\}$ occurs with probability one, i.e.

$$\Pr(X = Y) = P(\{s: X(s) = Y(s)\}) = 1$$

Definition of identical

- The random variables X and Y are **identically distributed** if

$$\Pr(X \in B) = \Pr(Y \in B), \forall B \in \mathcal{B}(\boldsymbol{R})$$

$\Pr(\{w: X(w) \in B\})$

同分布

equal => identical ✓

identical $\not\Rightarrow$ equal

Example

Consider the experiment of tossing a fair coin three times.
Define the random variables

identical but not equal

$X \triangleq$ number of heads observed $\quad Y \triangleq$ number of tails observed

Y is equal to 3-X

The distributions of X and Y are:

ω	HHH	HHT	HTH	THH	TTH	THT	HTT	TTT
$X(\omega)$	3	2	2	2	1	1	1	0
$Y(\omega)$	0	1	1	1	2	2	2	3

Definition of independence

- If the events $\{X \in B\}$ and $\{Y \in C\}$ are independent for all sets $B \in \mathcal{B}(\boldsymbol{R})$ and $C \in \mathcal{B}(\boldsymbol{R})$, we say that X and Y are **independent random variables**. In light of this definition, we see that X and Y are independent random variables if and only if

$$\Pr(X \in B, Y \in C) = \Pr(X \in B) \Pr(Y \in C)$$

for all Borel sets B and C.

Definition of independence

- Given any finite number of random variables, say $X_1, \ldots, X_n$, we say they are **independent** if

$$\Pr\left(\bigcap_{j=1}^{n}\{X_j \in B_j\}\right) = \prod_{j=1}^{n} \Pr(X_j \in B_j)$$

- for all choices of the Borel sets $B_1, \ldots, B_n$. If $X_1, \ldots, X_n$ are independent, then so is any subset of them, e.g., X_1, X_3 and X_4.

To prove: Set $B_2 = \Omega$

i.i.d. random variables

- If $X_1, X_2, \ldots$ are independent and every X_i and X_j are identically distributed for $i \neq j$,$i, j = 1, 2, \ldots$, we say that $X_1, X_2, \ldots$ are **independently and identically distributed (i.i.d.)**.

Independent RVs

Two RVs, X and Y are independent iff their joint PDF factors into a product of the marginal PDFs

Mathematically:

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

Or

$$p_{XY}(x, y) = p_X(x) p_Y(y)$$

Example for Independence

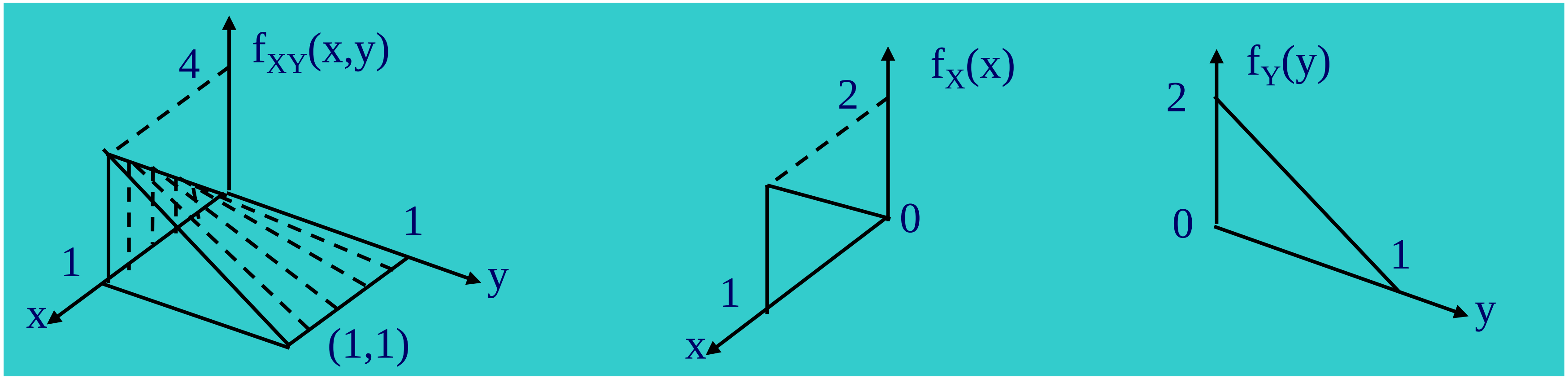
$$f_{XY}(x,y)=\begin{cases}4x(1-y) & 0\le x\le 1 \\ & 0\le y\le 1 \\ 0 & otherwise\end{cases}$$

marginal　　marginal

is the product of

$$f_X(x)=\begin{cases}2x & 0\le x\le 1\\ 0 & o.w.\end{cases}$$ and $$f_Y(y)=\begin{cases}2(1-y) & 0\le y\le 1\\ 0 & o.w.\end{cases}$$

therefore, X and Y are independent.



Two Kinds of Products

Observe that the factorization of $f_{XY}(x,y)$ occurs in two places:

Product of expressions

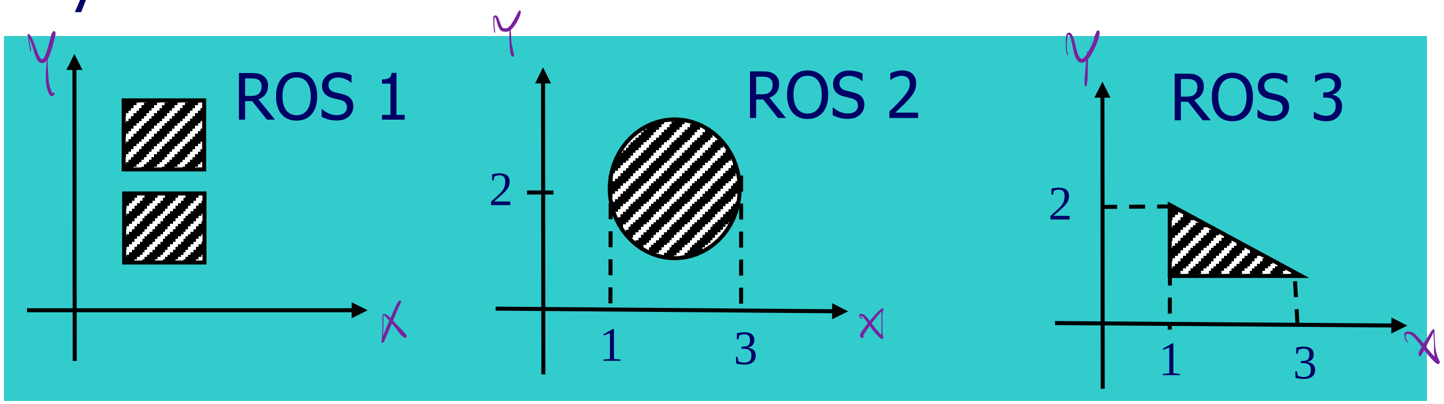
Cartesian Product of sets

$$f_{XY}(x,y)=\begin{cases}[2x][2(1-y)] & \{0\le x\le 1\}\times\{0\le y\le 1\}\\ 0 & o.w.\end{cases}$$

If either the expression or the ROS for $f_{XY}(x,y)$ is NOT factorable, then the RVs are dependent

Easy Decision

A joint PDF ROS that is not a Cartesian product is easy to identify:



In ROS 2, X=1 tells us Y=2. Dependent!

In ROS 3, X=3 tells us Y=1. Dependent!

★REMEMBER

If a joint PDF ROS has a sloped or curved boundary, the RVs are dependent.

But Be Careful

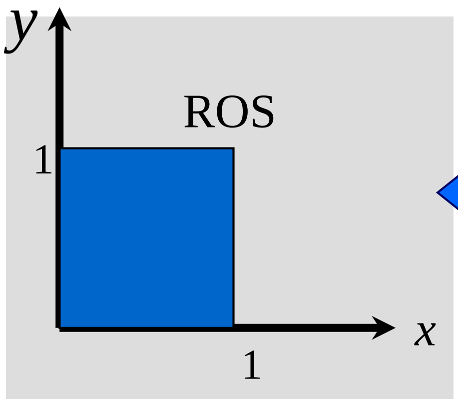
A Cartesian product ROS does not imply independence.

Example:

$$f_{XY}(x,y) = \begin{cases} Ke^{xy} & \begin{matrix} 0 \le x \le 1 \\ 0 \le y \le 1 \end{matrix} \\ 0 & o.w. \end{cases}$$

where K is a proper constant of normalization.

Marginal PDF



$$f_{XY}(x,y) = \begin{cases} Ke^{xy} & 0 \le x \le 1 \\ & 0 \le y \le 1 \\ 0 & o.w. \end{cases}$$

$$\frac{1}{x} e^{xy} \Big|_0^1$$

$$= \frac{1}{x} e^{x} - \frac{1}{x}$$

$$f_X(x) = \int_0^1 Ke^{xy}\, dy = \frac{K}{x}(e^x - 1) \quad 0 < x \le 1$$

$$f_Y(y) = \frac{K}{y}(e^y - 1) \quad 0 < y \le 1$$

$$f_X(x) f_Y(y) = \frac{K^2}{xy}(e^{x+y} - e^x - e^y + 1) \qquad (0,1] \times (0,1]$$

① $F_X(x) = F_{XY}(x, +\infty) = 1 - e^{-x} \quad x > 0.$

$F_Y(y) = F_{XY}(+\infty, y) = \frac{y}{y+1} \quad y > 0.$

② $f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \, \partial y} = \frac{\partial}{\partial y}\left(\frac{1}{y+1} \cdot e^{-x(y+1)} \cdot -(y+1) + e^{-x}\right)$

$= -(-x) \cdot e^{-x(y+1)}$

$= x e^{-x(y+1)} \quad x, y > 0.$

$f_X(x) = e^{-x}. \quad x > 0.$

$f_Y(y) = \left(1 - \frac{1}{y}\right)'$

$= y^{-2} \quad y > 0$

Thank You!