Probability and Random Process

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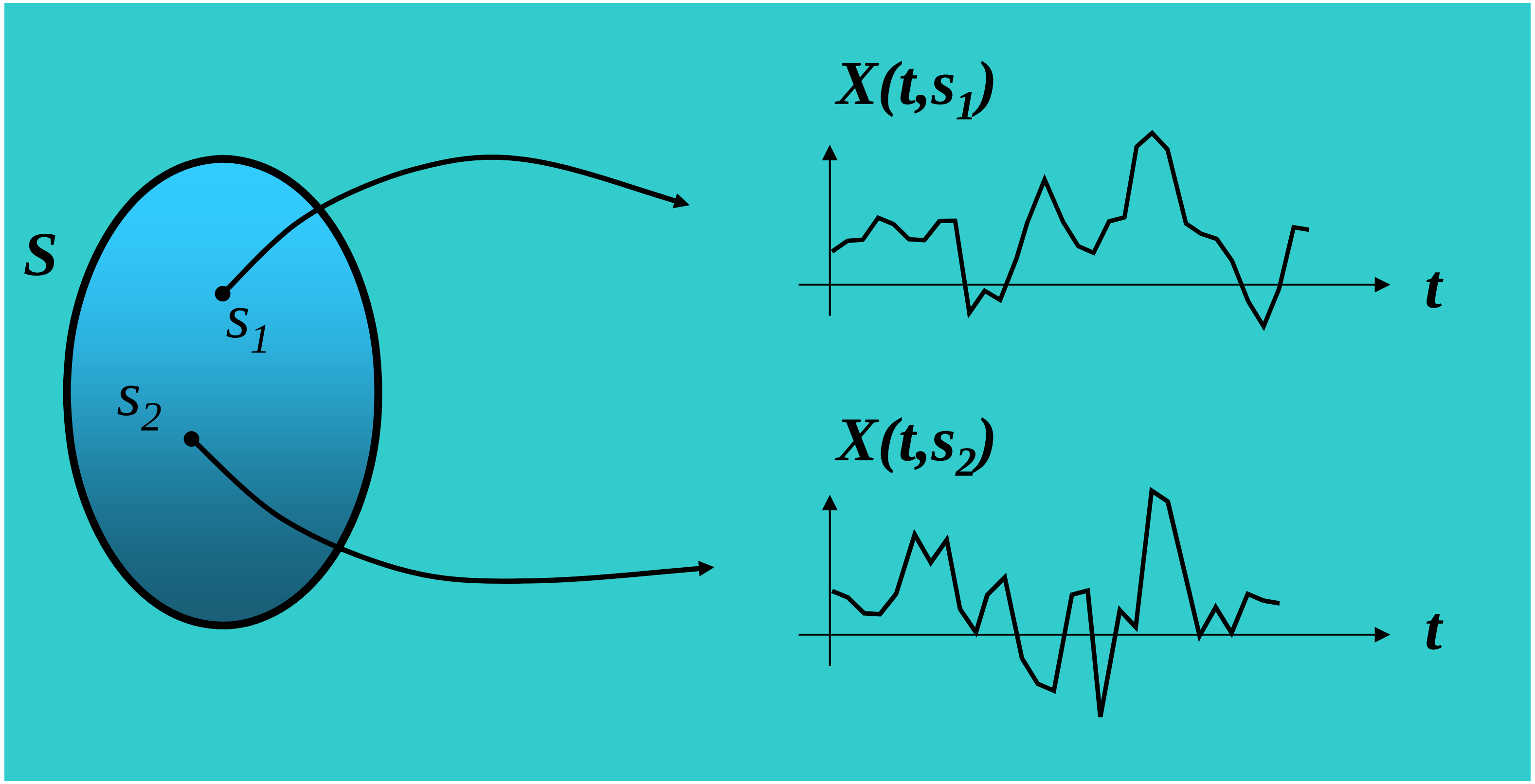
Outline

- 4. Random Process-I
  - Introduction to Random Processes
  - Brownian Motion//Wiener Process
  - Poisson Process
  - Complex RV and RP
  - Stationarity
  - PSD, QAM, White Noise
  - Response of Systems
  - LTI Systems and RPs

Introduction to Random Processes

Random Processes

A random process (RP) is a function that maps each outcome from a sample space **S** to a function of time.



Index Set

In general, t belongs to an index set I.

$I = \mathbf{R}$: $X(t,s)$ is a continuous-time random process.

$I = \mathbf{Z}$ (the integers): $X(n,s)$ is a discrete-time RP, also known as a random sequence.

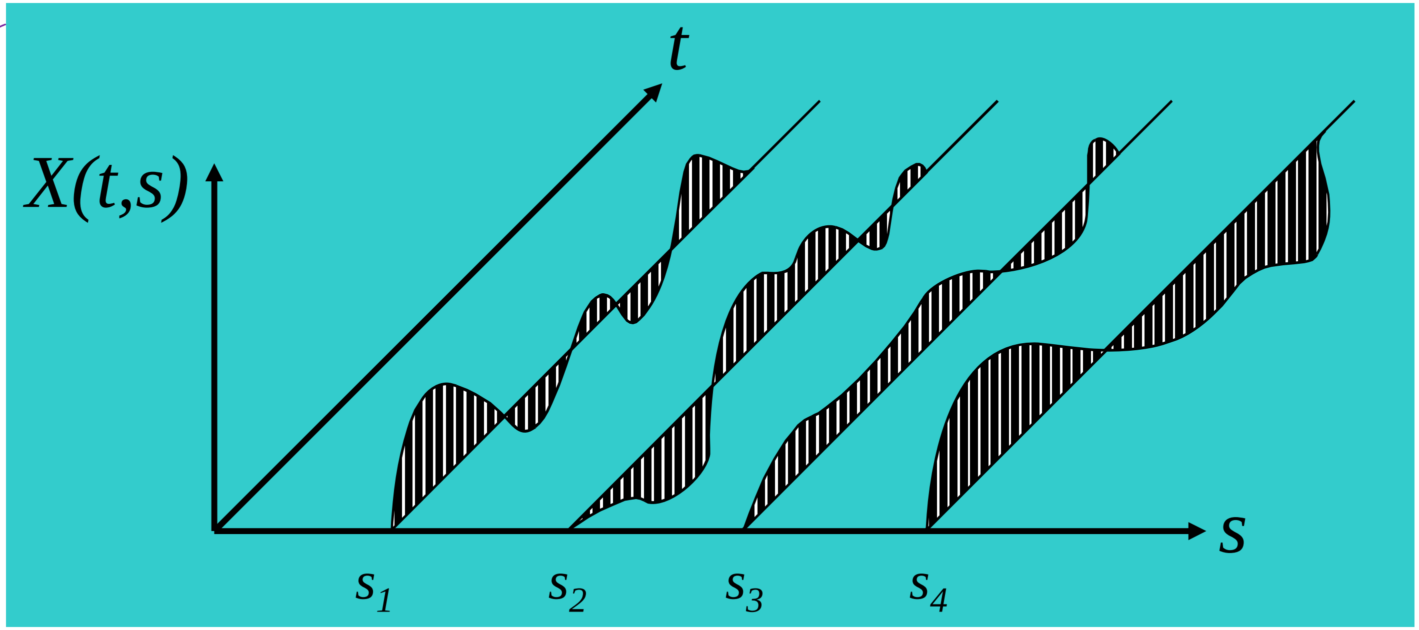
$I = \{1,2,\ldots,N\}$: $X(n,s)$ is a random vector.

finite number of rvs

Random Processes - Second View

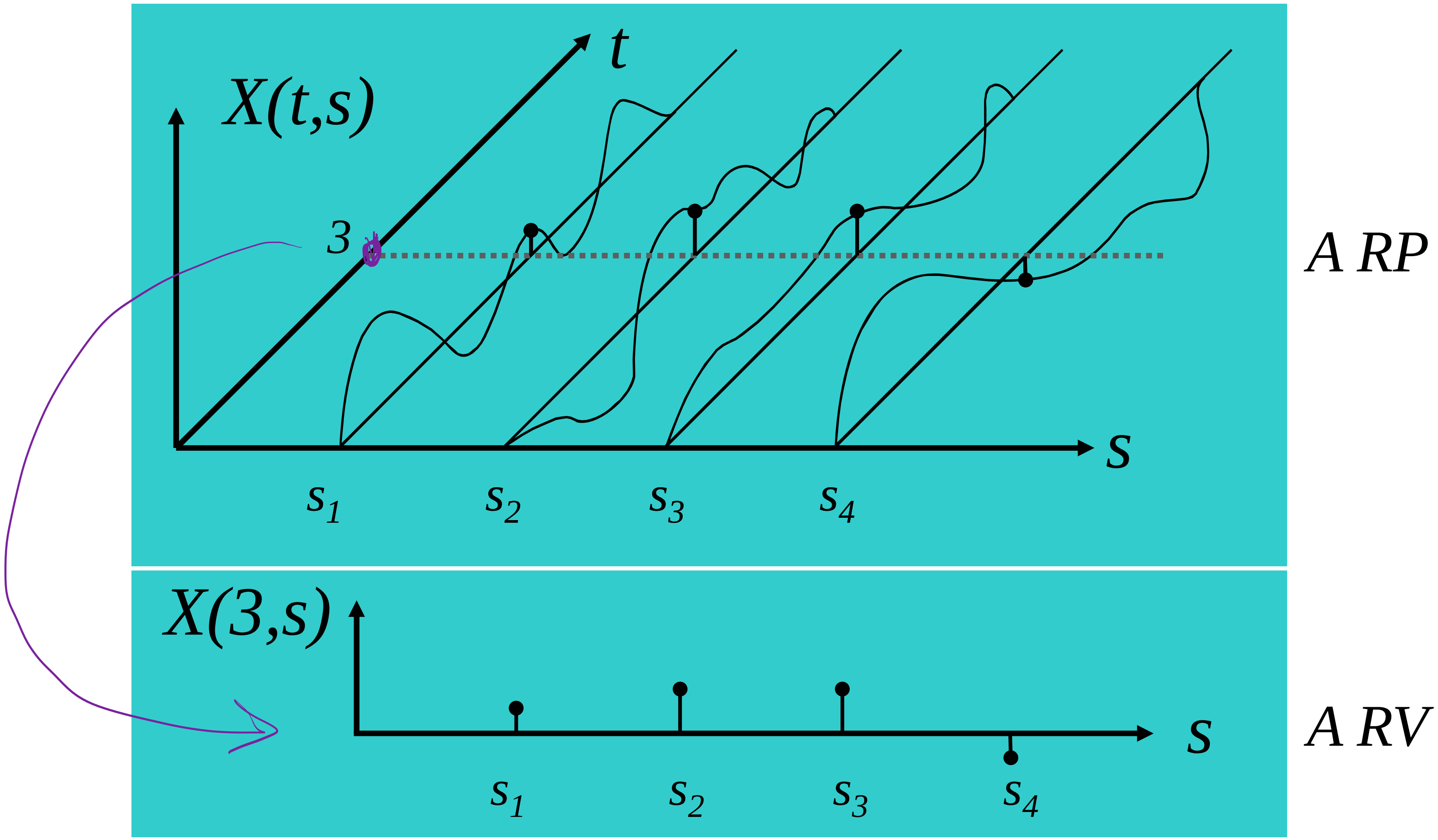
A RP is a function that maps from the cartesian product $S \times I$ to the real numbers.

not necessarily number



Getting a RV from a RP

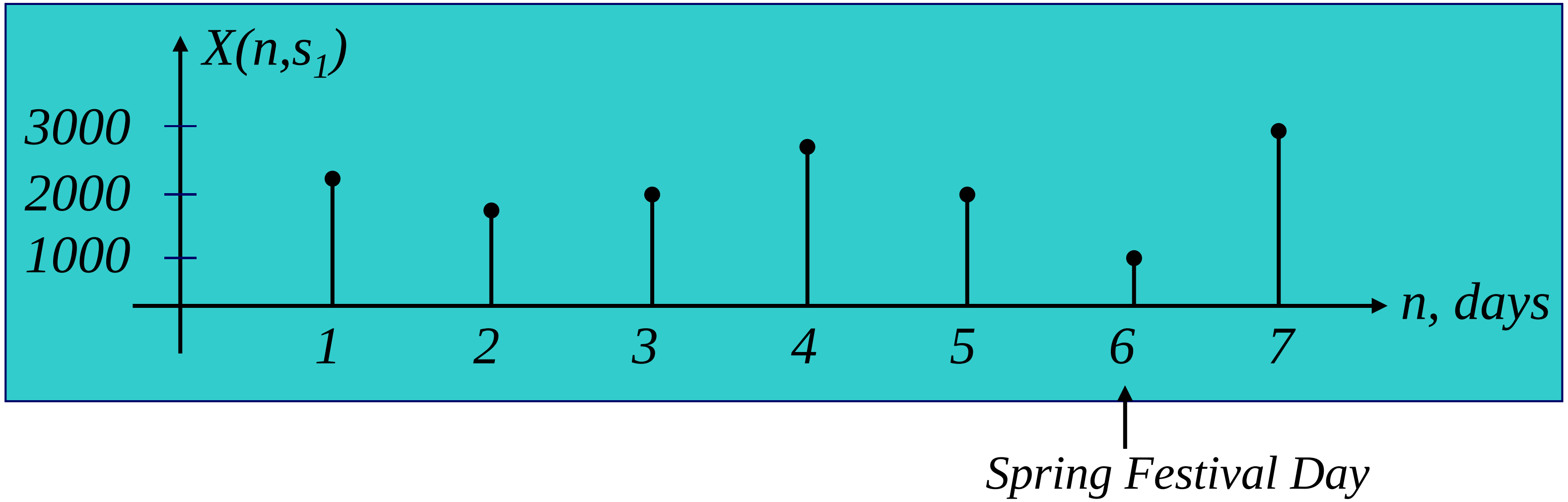
Specifying a particular point in time in a RP yields a RV.



Web Page Hits Example

Define $X(n,s)$ to be the number of visits to the JI home page on day n.

Example sample function



One possible question: What is the expected value of the number of visits on Spring Festival day?

State Space

The set of all possible values (or states) that a RP may take is its state space

A discrete state space RP (chain) occupies only a finite or countable number of states

Ex: Number of objects in a queue

A continuous state space RP can take any real value

Ex: Temperature

Discrete vs. Continuous Valued r.p.'s

	Discrete-valued	Continuous-valued
Discrete -time	seq of stock prices	seq of temp's in time or distance
Continuous -time	number of customers in line at time	waveform from microphone

It is essential that you keep separate track of time and value axes and properties.

Lessons

A random process maps from $S \times I$ to the real numbers

Random processes are classified as being

- discrete or continuous state
- discrete or continuous time

A Random process at a fixed time is a random variable

Characterization of a RP

Characterization of a RV: pdf, cdf — mean, variance

two RV: ← 同左; correlation, cover...; joint pdf, cdf

Most general: joint CDF for any subset of the index set

$t \in I$

Ex: For a continuous time RP X(t), given the subset {6, 10, 110, 200}, produce the function:

$$F_{X(6)X(10)X(110)X(200)}(x_1, x_2, x_3, x_4)$$

The joint PDF may also be used

As with RVs, the s-dependence is often dropped from the notation $X(t,s) = X(t)$, or simply use X_t

often dropped

a RV.

Distribution

1. **1st-order distribution**
   This is the distribution of X_t for all $t \in \mathcal{T}$, *e.g.*

$$F_{X_t}(x), \quad \forall t \in \mathcal{T}, \forall x$$

   It tells us nothing about dependence among variables.

2. **2nd-order distribution**
   The joint distribution of X_t, X_s for all $t, s \in \mathcal{T}$.

3. **nth-order distribution**
   The joint distribution of $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ for all $t_1, \ldots, t_n \in \mathcal{T}$.
   This can become overwhelming as n increases.

Moments of a RP

Moments provide partial characterization:

Mean (first moment):

$$m_X(t) = E\{X(t)\} = \int_{-\infty}^{+\infty} x f_{X(t)}(x)dx$$

Auto-correlation:

Correlation

$$R_X(t_1, t_2) = \underbrace{E\{X(t_1)X(t_2)\}}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} xy f_{X(t_1)X(t_2)}(x, y)dxdy$$

Auto-correlation properties

Property

1. symmetric function of t and s.

$$R_X(\underset{\text{time } t}{t}, \underset{\text{time } s}{s}) = \mathrm{E}[X_t X_s] = \mathrm{E}[X_s X_t] = R_X(s, t)$$

2. $R_X(t, t) = \mathrm{E}\left[X_t^2\right] \geq 0$

$$|R_X(t, s)| \leq \sqrt{\mathrm{E}\left[X_t^2\right] \mathrm{E}\left[X_s^2\right]}$$

Cauchy-Schwarz inequality

$$|R_X(t, s)| = |\mathrm{E}[X_t X_s]| \leq \sqrt{\mathrm{E}\left[X_t^2\right] \mathrm{E}\left[X_s^2\right]}$$

Auto-covariance

Covariance

$$C_X(t_1,t_2) = E\{(X(t_1)-m_X(t_1))(X(t_2)-m_X(t_2))\}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(x-m_X(t_1))(y-m_X(t_2))f_{X(t_1)X(t_2)}(x,y)\,dx\,dy$$

Autocovariance can also be expressed:

$$C_X(t_1,t_2) = R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$$

Variance (second moment):

$$\sigma_X^2(t_1) = C_X(t_1,t_1)$$

Cross-correlation & covariance

Let $\{X_t, t \in \mathcal{T}\}$ and $\{Y_t, t \in \mathcal{T}\}$ be random processes.
Their cross-correlation is defined as

$$R_{XY}(t,s) = E[X_t Y_s], t, s \in \mathcal{T}$$

Let $\{X_t, t \in \mathcal{T}\}$ and $\{Y_t, t \in \mathcal{T}\}$ be random processes.
Their cross-covariance is defined as mean at t.

$$C_{XY}(t,s) = Cov\{X_t Y_s\} = E\big[\big(X_t - m_X(t)\big)\big(Y_s - m_Y(s)\big)\big]$$
$$= R_{XY}(t,s) - m_X(t) m_Y(s)$$

Random Line Example - I

Let A and B be independent random variables. Then, let

$$X(t) = A + Bt$$

Find $m_X(t)$, $C_X(t_1,t_2)$, and $f_{X(t)}(x)$.

↳ mean ↳ covariance ↳ first-order pdf.

$$m_X(t) = E[X(t)] = E[A+Bt] = \mu_A + \mu_B \cdot t$$

$$\begin{aligned} C_X(t_1,t_2) &= E[(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))] \\ &= E[(A+Bt_1 - \mu_A - \mu_B t_1)(A+Bt_2 - \mu_A - \mu_B t_2)] \\ &= E[(A-\mu_A + t_1(B-\mu_B))(A-\mu_A + t_2(B-\mu_B))] \\ &= E[(A-\mu_A)^2 + (A-\mu_A)(B-\mu_B)(t_1+t_2) + t_1 t_2 (B-\mu_B)^2] \\ &= Var[A] + (t_1+t_2)\underbrace{Cov(A,B)}_{0} + t_1 t_2 Var[B] \\ &= Var[A] + t_1 t_2 Var[B] \end{aligned}$$

$$f_{X(t)}(x) = f_{A+Bt}(x) = f_A(x) * \frac{1}{|t|} f_B\left(\frac{x}{t}\right)$$

Random Line Example - I

Let A and B be independent random variables. Then, let

$$X(t) = A + Bt$$

Find $m_X(t)$, $C_X(t_1,t_2)$, and $f_{X(t)}(x)$.

$$m_X(t) = E\{X(t)\} = m_A + m_B t$$

$$C_X(t_1,t_2) = E\{[A + Bt_1 - m_A - m_B t_1][A + Bt_2 - m_A - m_B t_2]\}$$

$$= E\{[(A - m_A) + (B - m_B)t_1][(A - m_A) + (B - m_B)t_2]\}$$

$$= \sigma_A^2 + \sigma_B^2 t_1 t_2$$

Why did the cross terms drop out?

Random Line Example - II

X(t) is a function of two RVs.

Let $D = Bt$, then $f_D(d) = \dfrac{f_B(\frac{d}{t})}{|t|}$

$f_z(z) = \dfrac{f_X(x)}{\frac{dz}{dx}}$

$f_{ZW}(z,w) = \sum \dfrac{f_{XY}(x,y)}{J(x,y)}$

Since $X(t) = A + D$ and A and D are independent,

求pdf要 convolution

$$f_{X(t)}(x) = f_A(x) * f_D(x) = f_A(x) * \frac{f_B(\frac{x}{t})}{|t|}$$

convolution

Discrete-time Random Walk

Let B[n] be a sequence of iid RVs, such that:

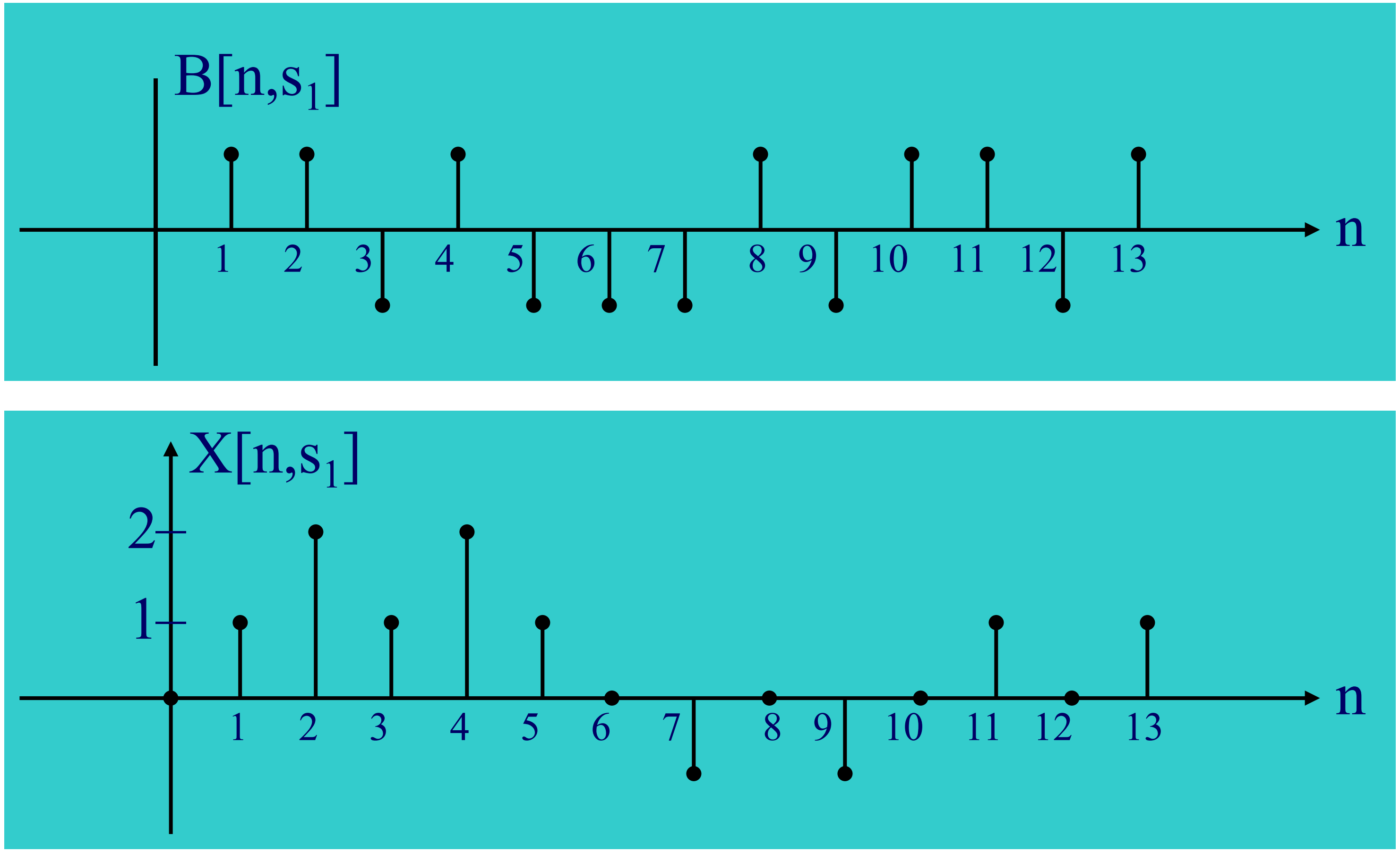
$$P(B[n]=1) = P(B[n]=-1) = 1/2$$

Let $X[n] = \sum_{k=1}^{n} B[k]$, and set X[0]=0

X[n] is an example of a sum process

X[n] is the discrete time random walk

Discrete-time Random Walk - II



DT Random Walk Moments - I

$$m_X[n] = \quad = E\{X[n]\}$$

$$= \sum E\{B[k]\}$$

$$C_X[n_1, n_2] = \quad \sum^{n_1} \sum^{n_2} C_B[k, j]$$

DT Random Walk Moments - IV

To get the PMF of X[n], observe that it can be expressed as a function of a Binomial RV

Let D[n] be a Bernoulli sequence with p=1/2. Then B[k]=2D[k]-1, and

transform

$$X[n] = \sum_{k=1}^{n} B[k] = 2\underbrace{\left(\sum_{k=1}^{n} D[k]\right)}_{\text{Binomial} = Z[n]} - n$$

DT Random Walk Moments - V

$$p_{X[n]}[l] = P\left(X[n] = l\right) = P\left(2Z[n] - n = l\right)$$

$$= P\left[Z[n] = \frac{l+n}{2}\right] = \binom{n}{\frac{l+n}{2}}\frac{1}{2^n} \qquad ,\text{with } p = \frac{1}{2}$$

binomial

Recall that $E(X[n]) = 0$ and $\sigma_X^2[n] = n$

By the Central Limit Theorem,

$$\boxed{F_{X[n]}(l) \to \Phi\left(\frac{l}{\sqrt{n}}\right)}$$

$t = \sqrt{n}$

Lessons

Some ways to characterize a RP:

- mean
- autocorrelation or autocovariance
- joint PMF (discrete space) or joint PDF

Discrete-time random walk

- zero mean
- variance=n
- CDF approaches Gaussian

Increments

An increment of a RP is

$$\overset{\text{time } b}{X(b)} - \overset{\text{time } a}{X(a)}$$

where a < b

Two types of increments that are useful in analysis are

- Independent increments
- Stationary increments

Independent Increments

A RP X(t) has independent increments if, for any $a < b \le c < d$, no matter the length of the time gap

$$\left[X(b) - X(a)\right] \text{ is independent from } \left[X(d) - X(c)\right]$$

All sum processes that sum over sequences of independent RVs are ind. inc.
independent increment

Example: Let $U_1, U_2, U_3, \ldots$ be a sequence of independent RVs, such that $U_i \sim U[0,1]$. Let

$$X(n) = \sum_{i=1}^{n} U_i$$

increment

X(n) is an ind. inc. discrete-time RP

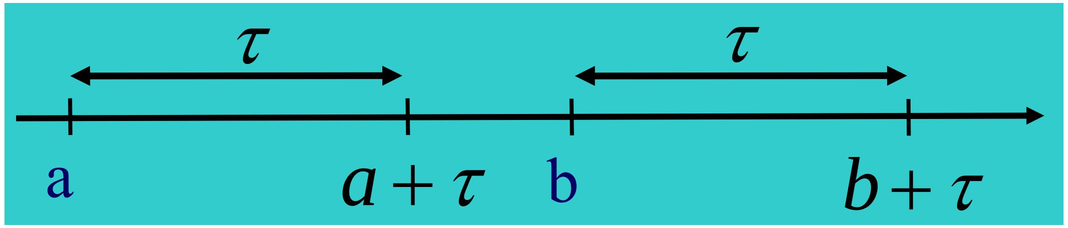
Discrete-time Random Walk is another example

Stationary Increments

A RP X(t) has stationary increments if, for any a, b and τ:

× equal
✓ identical

$[X(a+\tau) - X(a)]$ has the same PDF as $[X(b+\tau) - X(b)]$



Intervals are same length and may be overlapping

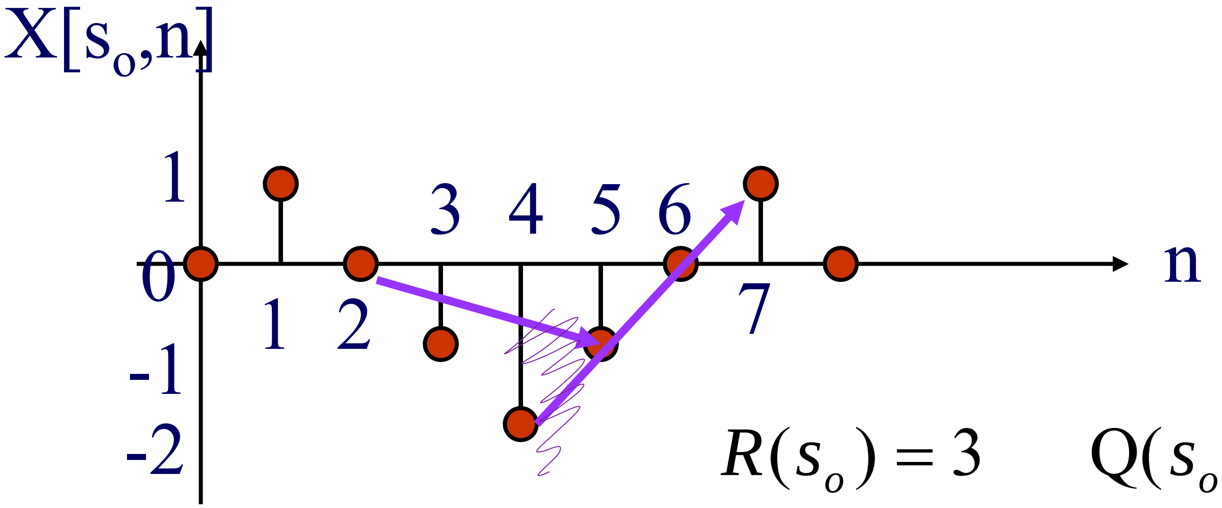
Caution: The values of the increments are not equal (in general); only their statistics are.

Stationary Increment Example

Discrete-time Random Walk:

Consider two intervals of length 3.

$$R = [X(7) - X(4)] \text{ and } Q = [X(5) - X(2)]$$



$$R(s_o) = 3 \qquad Q(s_o) = -1$$

Overlap

The overlap means that the increments are dependent.

For example, R = -3, Q = 3 is not possible.

To get joint PMF, apply Total Probability, conditioning on the overlap value.

$$p_{RQ}(r,q) = P(R = r \cap Q = q)$$

$$= \sum_{i=-1,1} P(R = r \cap Q = q \mid B_5 = i) P(B_5 = i)$$

These are conditionally independent

$$= \sum_{i=-1,1} P(R = r \mid B_5 = i) P(Q = q \mid B_5 = i) P(B_5 = i)$$

PDFs of the Increments

$$R = [X(7) - X(4)] \quad \text{and} \quad Q = [X(5) - X(2)]$$

$r=1$, $r=3$, $r=-1$, $r=-3$.

$r = 2k-3$.

$\frac{\binom{3}{\frac{r+3}{2}}}{2^3}$

$$R = \sum_{i=5}^{7} B(i) \quad f_R(r) = \sum_{k=0}^{3} \frac{\binom{3}{k}}{2^3} \delta(r - [2k-3])$$

$k = \frac{r+3}{2}$

filter among $k = 0, 1, 2, 3$

but not independent (overlap)

Same PDFs

$$Q = \sum_{i=3}^{5} B(i) \quad f_Q(q) = \sum_{k=0}^{3} \frac{\binom{3}{k}}{2^3} \delta(q - [2k-3])$$

What does overlap imply about R and Q?

Stationary but dependent increments

Lessons

Useful features are

- Independent increments
- Stationary increments

Do not confuse with stationary RPs

stationary increments ≠ stationary RP

Quiz.

$$m_{X_t} = E[X_t]$$
$$= E[\cos(2\pi f t + \theta)]$$
$$= E[\underline{\cos(2\pi f t)\sin\theta - \sin(2\pi f t)\cos\theta}]$$ → 好像这选择?
$$= \cos(2\pi f t)\, E[\sin\theta] - \sin(2\pi f t)\, E[\cos\theta]$$
$$= \cos(2\pi f t)\cdot 0 - \sin(2\pi f t)\cdot 0$$
$$= 0$$

$R_{X_t} =$

Solution in textbook

Thank You!