Probability and Random Process

Aimin Tang

The University of Michigan- Shanghai Jiao Tong University Joint Institute
Shanghai Jiao Tong University

Oct. 20 2020

Ve501 2010-2021 Fall

Based on Lecture Notes by Prof. Chong Han

Outline

- 3. Multiple Random Variables
  - Two Random Variables
  - Marginal PDF
  - Functions of Two Random Variables
  - Conditional PDF
  - Joint Moments
  - Mean Square Error Estimation
  - Probability bound
  - Random Vectors
  - Sample Mean
  - Convergence of Random Sequences
  - Central Limit Theorem

Functions of Two Random Variables

Two Functions of Two RVs

Suppose X and Y are jointly distributed RVs with joint PDF $f_{XY}(x, y)$ and

$$Z = G(X, Y)$$

$$W = H(X, Y)$$

Examples:

Rectangular-to-Polar conversion

Rotation of Coordinates

Then one might wish to find $f_{ZW}(z, w)$

$$F_{ZW}(z, w) = P(Z \le z \wedge W \le w)$$

$$= P\{s: Z(s) \le z \wedge W(s) \le w\}$$

Graphical Example

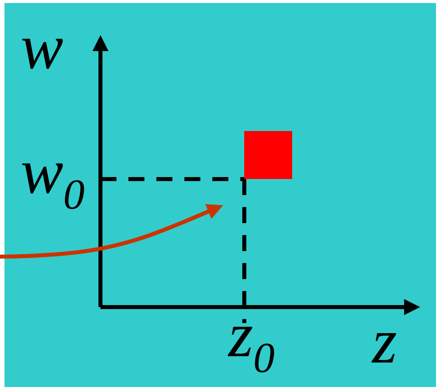
Consider the approximation:

CDF

$$P\left(\{z_0 < Z < z_0 + dz\} \cap \{w_0 < W < w_0 + dw\}\right)$$

$$\approx f_{zw}(z_0, w_0)\,dz\,dw$$

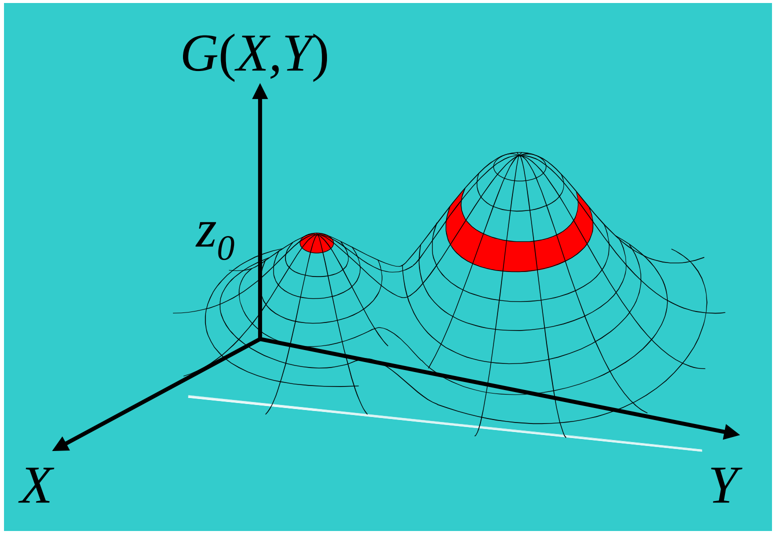
= The probability that (Z,W) is in this small square

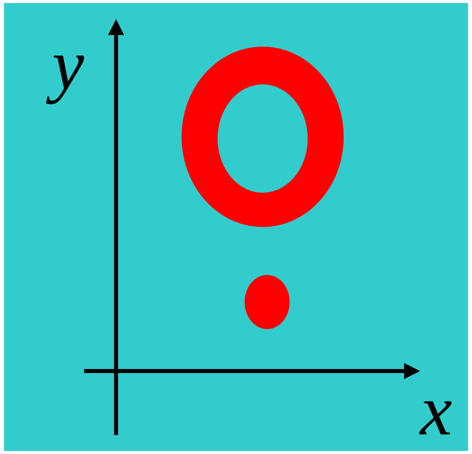


Graphical Example, Cont'd

First consider the event $\{z_0 < G(X,Y) < z_0 + dz\}$

Example:



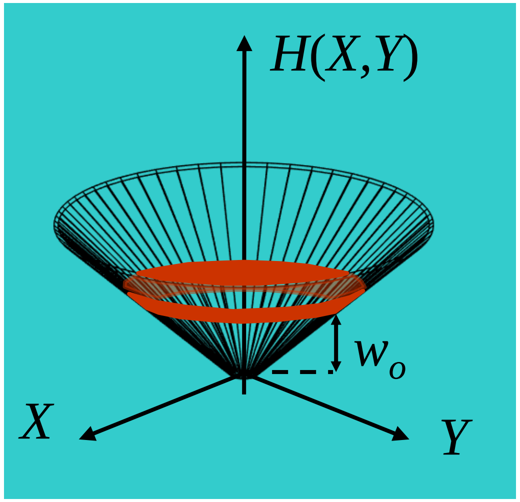


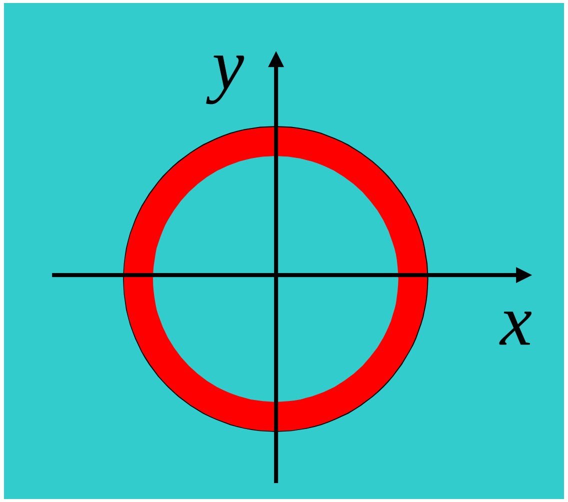
Event Region

Graphical Example, Cont'd

Similarly, $\{w_0 < H(X,Y) < w_0 + dw\}$ also has an event region.

Event Region





Graphical Example, Concluded

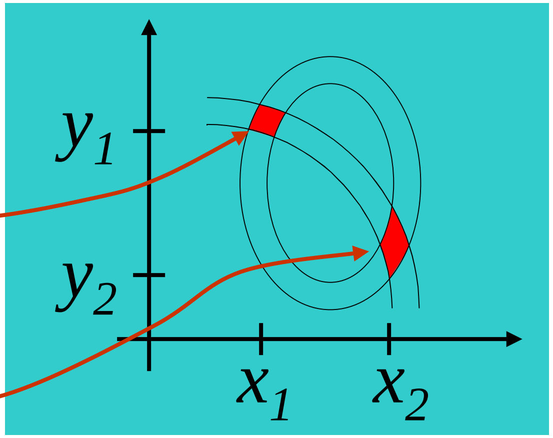
In this example,

$$P\left(\{z_0 < Z < z_0 + dz\} \cap \{w_0 < W < w_0 + dw\}\right)$$

corresponds to the intersection of these two events regions.

Differential Area dA_1

Differential Area dA_2



(X_1, Y_1) and (X_2, Y_2) are two solutions to the equations:

$$z_0 = G(X, Y)$$

$$w_0 = H(X, Y)$$

The Jacobian

Then we have

$$f_{zw}(z_0, w_0)dzdw \approx f_{XY}(x_1, y_1)\overset{\approx dzdw}{dA_1} + f_{XY}(x_2, y_2)dA_2$$

It happens that $$\frac{\partial A_j}{\partial z \partial w} \approx \frac{1}{|J(x_i, y_i)|},$$

$\to G$

$\to H$

where $$J(x, y) = \det\begin{bmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{bmatrix} =$$ Jacobian of G and H

Recall the derivative in one RV case

General Formula

- In general, if there are n solutions to $z = G(X,Y)$ and $w = H(X,Y), (x_i, y_i),\ i = 1,2,\dots.n$
- Then

$$f_{ZW}(z,w) = \sum_{i=1}^{n} \frac{f_{XY}(x_i, y_i)}{|J(x_i, y_i)|}$$

- This is similar to the formula for the function of a RV

Linear Example

$$\begin{cases} Z = aX + bY \\ W = cX + dY \end{cases}$$

Invertible Linear Transformation

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

Only one solution

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} Z \\ W \end{bmatrix} = \frac{1}{\underbrace{ad - cb}_{\text{det.}}} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} Z \\ W \end{bmatrix}$$

So, $X = \dfrac{dZ - bW}{ad - cb}$ $\qquad$ $Y = \dfrac{-cZ + aW}{ad - cb}$

Linear Example, Concluded

Jacobian

对 adx - bc y

用 x= ... , y= ... 替换掉

$$\det\begin{bmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \\[2ex] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{bmatrix} = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$f_{ZW}(z,w) = \frac{f_{XY}\left(\dfrac{dz - bw}{ad - cb}, \dfrac{-cz + aw}{ad - cb}\right)}{|ad - cb|}$$

Nonlinear Example

S-plane to Z-plane mapping:

$$S = A + jB$$

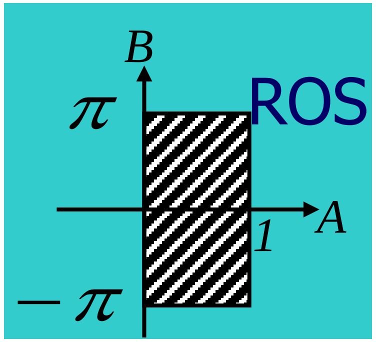
$$Z = C + jD = e^{ST} = e^{(A+jB)T}$$

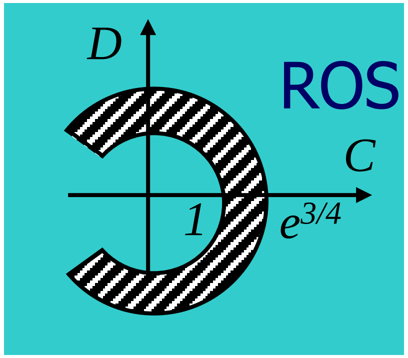
$$= e^{AT} e^{jBT} = e^{AT} \cos BT + j e^{AT} \sin BT$$

$$\therefore C = e^{AT} \cos BT = G(A, B)$$

$$D = e^{AT} \sin BT = H(A, B)$$

non-linear

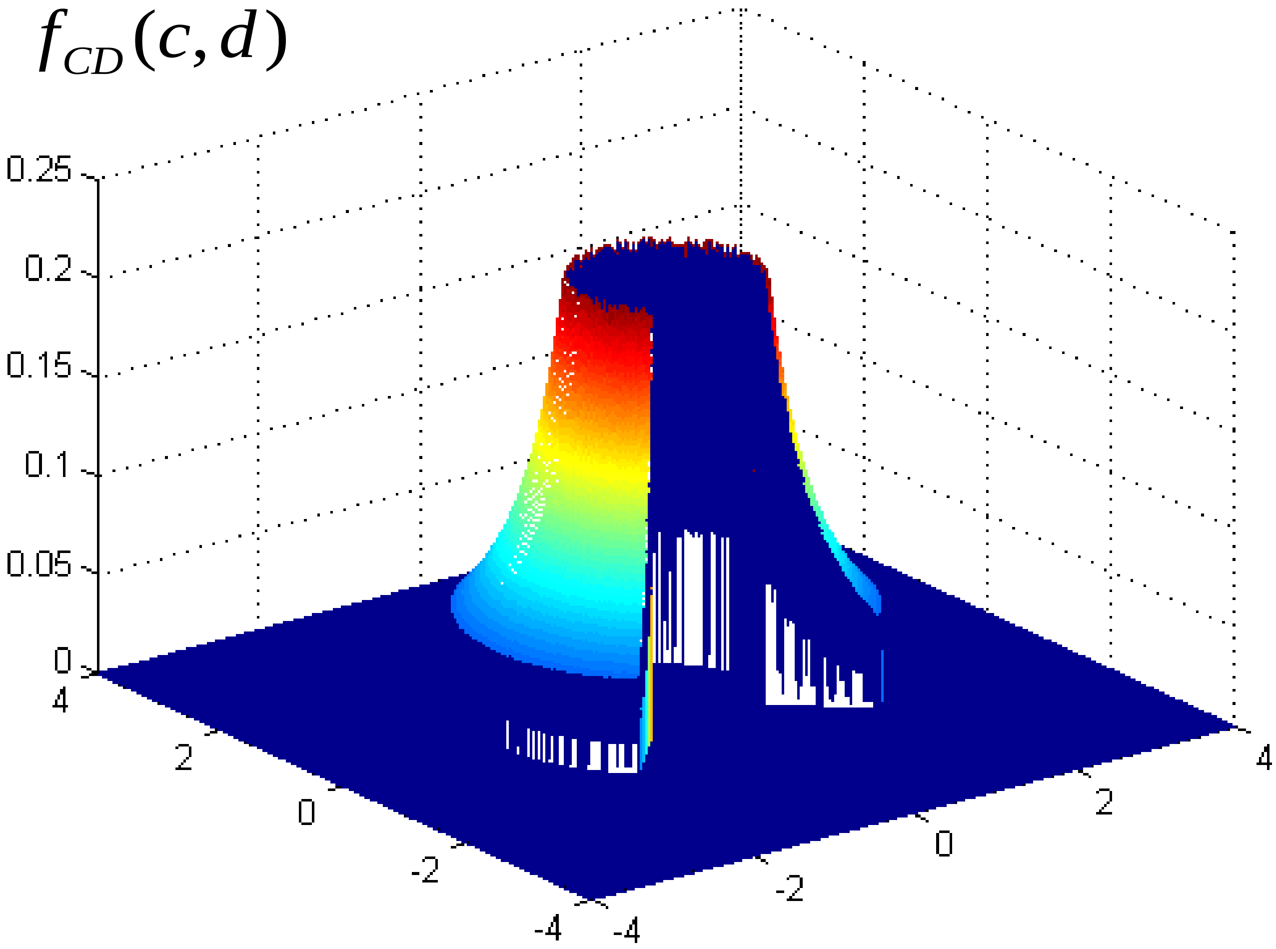




Let $f_{AB}(a,b) = \begin{cases} \dfrac{1}{2\pi} & 0 < a < 1 \\ & -\pi < b < \pi \end{cases}$

and $T=3/4$

Nonlinear Example, Concluded



One Function of Two RVs

Two equally valuable approaches:

CDF Approach

Auxiliary RV approach

CDF approach: Given $f_{XY}(x, y)$ and $Z = G(X, Y)$.
Find $F_Z(z)$, then differentiate to get $f_Z(z)$.

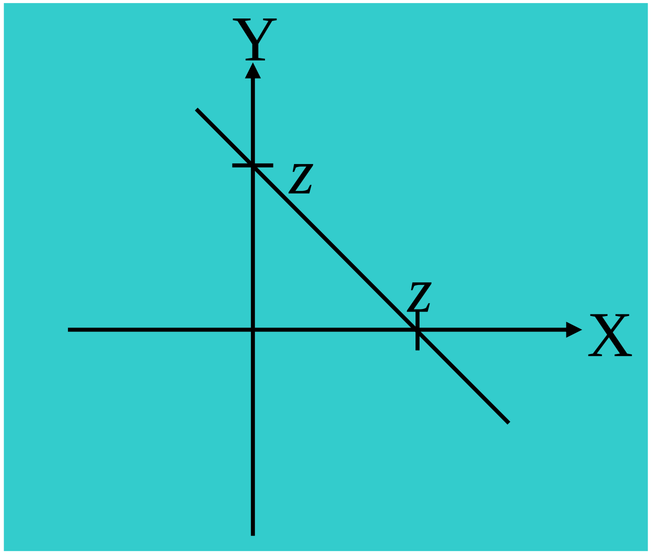
Auxiliary RV approach: Define an auxiliary or "dummy" RV W as either $W = X$ or $W = Y$. Use "two-functions-of-two-RVs" approach to get $f_{ZW}(z, w)$, then get marginal $f_Z(z)$.

Example: Adding RVs

$$Z = X + Y$$

$$F_z(z) = P(Z \le z)$$
$$= P(X + Y \le z)$$

$$F_z(z) = \int_{-\infty}^{+\infty}\int_{-\infty}^{z-y} f_{XY}(x, y)\,dx\,dy$$



Leibniz's Rule

Recall Leibniz's Rule

If $\Phi(t) = \int_{a(t)}^{b(t)} f(x)dx,$

then $\Phi'(t) = f(b(t))b'(t) - f(a(t))a'(t)$

so $\frac{d}{dz}F_Z(z) = \int_{-\infty}^{+\infty}\left(\frac{d}{dz}\int_{-\infty}^{z-y} f_{XY}(x,y)dx\right)dy$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{XY}(z-y,y)dy$$

Adding Independent RVs

Suppose, $Z=X+Y$ and X and Y are independent. Then

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{XY}(z-y, y)dy$$

$$= \int_{-\infty}^{+\infty} f_X(z-y) f_y(y)dy$$

$$= f_X(z) * f_Y(z)$$

Convolution

★ REMEMBER

To add independent RVs, convolve their PDFs.

Example for discrete case

- Let X, Y be **non-negative integer valued** (discrete) r.v.'s that are **independent** and have pmfs $p_X(x)$ and $p_Y(y)$ respectively. Let $Z = X + Y$. Determine the pmf $p_Z(z)$.

$$p_Z(z) = \sum_{k=0}^{z} P_r(X=k, Y=z-k)$$

$$= \sum_{k=0}^{z} P_r(X=k) P_r(Y=z-k)$$

$$= \sum_{k=0}^{z} p_X(k) p_Y(z-k)$$

$p_X(0) p_Y(0-0)$

$= \frac{1}{4}$

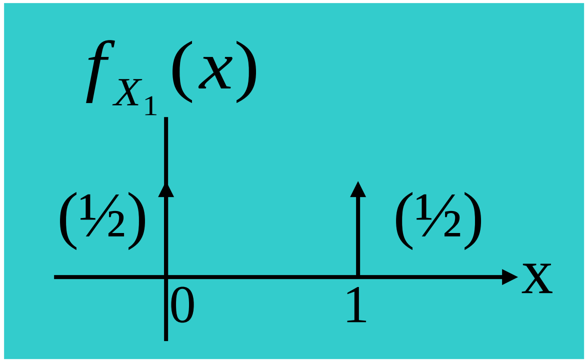
$p_X(0) p_Y(1-0) + p_X(1) p_Y(1-1) = \frac{1}{2}.$

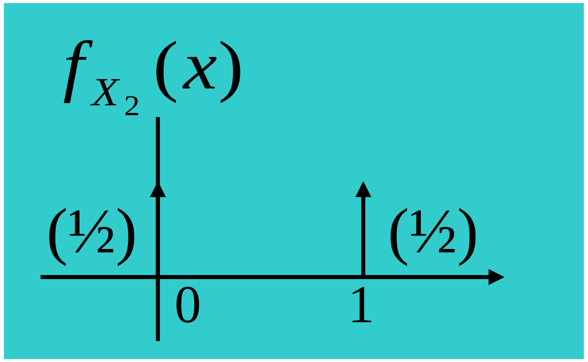
$p_X(0) p_Y(2-0) + p_X(1) p_Y(2-1) + p_X(2) p_Y(2-2)$ $=0$ $=0$

$= \frac{1}{4}.$

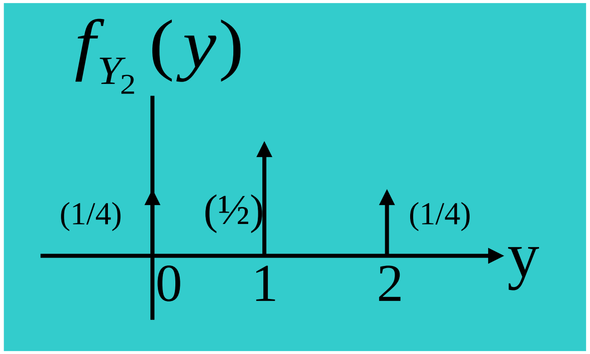
Adding Independent Bernoulli RVs

Let X_1 , X_2 and X_3 be iid Bernoulli RVs with $p=1/2$

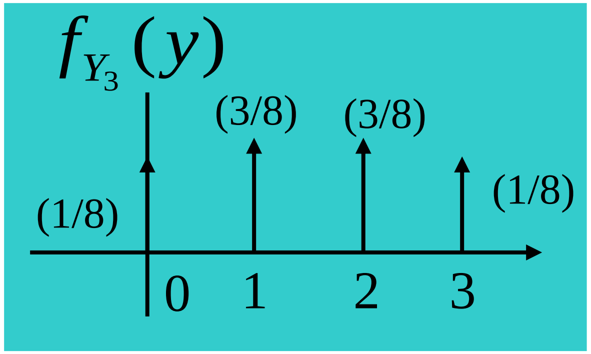




Let $Y_2=X_1 + X_2$ and $Y_3=Y_2 + X_3$



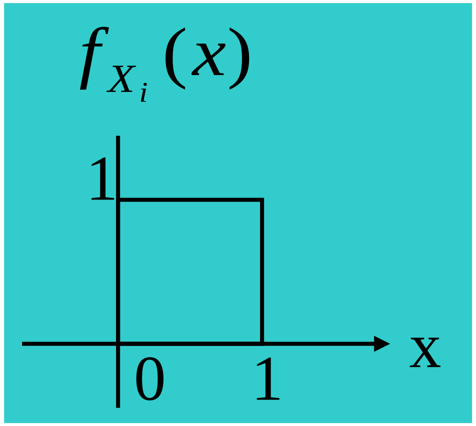
$P_{Y_3}(0) = \frac{1}{4}\times\frac{1}{2}$
$= \frac{1}{8}$
$P_{Y_3}(1) = \frac{1}{4}\times\frac{1}{2}$
$+ \frac{1}{2}\times\frac{1}{2}$
$= \frac{3}{8}$
$P_{Y_3}(2) = \frac{1}{4}\times 0$
$+ \frac{1}{2}\times\frac{1}{2}$
$+ \frac{1}{4}\times\frac{1}{2}$
$= \frac{3}{8}$



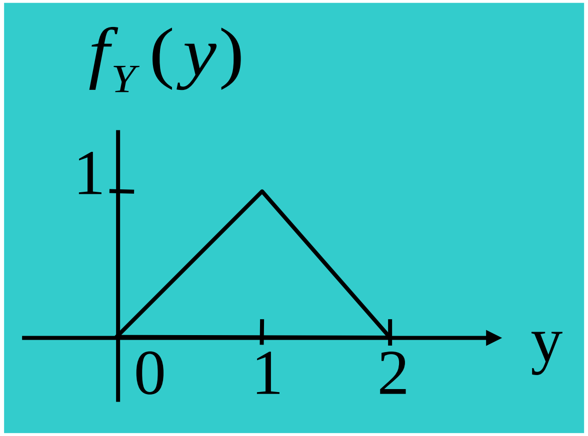
$P_{Y_3}(3) = \frac{1}{8}$
similarly.

Adding Uniform RVs

Let X_1 and X_2 be i.i.d. with



Then $Y=X_1+X_2$ has the PDF ?



Example

- A random, continuous-valued signal X is transmitted over a channel subject to multiplicative, continuous-valued noise Y. The received signal is $Z = XY$. Find the cdf and density of Z if X and Y has a joint density $f_{XY}(x, y)$.

Example

- Let $Y = \max(X_1, X_2)$, where X_1, and X_2 are independent discrete r.v.'s with the given joint pmf

$$p_{X_1X_2}(x_1, x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$$

Let D_Y be the range space of Y.

$$D_Y = \{y_1, y_2, y_3, \dots\}, y_1 \le y_2 \le \cdots$$

Compute the pmf of Y, i.e., $p_Y(y_i)$.

Auxiliary RV Example

Let $U = +\sqrt{XY}$, where X and Y are iid

$$f_X(x) = \begin{cases} \dfrac{1}{x^2} & x \geq 1 \\ 0 & else \end{cases} \qquad f_y(y) = \begin{cases} \dfrac{1}{y^2} & y \geq 1 \\ 0 & else \end{cases}$$

Let $V=X$ be the auxiliary RV.

1. The solution is:

$$X = V$$

$$Y = \frac{U^2}{V}$$

Auxiliary RV Example, Cont'd

$$U = +\sqrt{XY}$$
$$V = X$$

2. Find Jacobian

$$J(x,y) = \det\begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$$

$$= \det\begin{bmatrix} \text{don't care} & \frac{1}{2}\sqrt{\frac{x}{y}} \\ 1 & 0 \end{bmatrix} = -\frac{1}{2}\sqrt{\frac{x}{y}}$$

3. Plug solution into Jacobian

$$X = V, \quad Y = \frac{U^2}{V}$$

$$-\frac{1}{2}\sqrt{\frac{x}{y}} = -\frac{1}{2}\sqrt{\frac{v}{\frac{u^2}{v}}} = -\frac{1}{2}\frac{v}{u}$$

Auxiliary RV Example, Cont'd

Plug arguments into ROS for $f_{XY}(x,y)$

4. PDF formula

$$f_{UV}(u,v)=\frac{f_{XY}\left(v,\frac{u^2}{v}\right)}{\left|-\frac{1}{2}\cdot\frac{v}{u}\right|}=\begin{cases}\dfrac{\frac{1}{v^2}\cdot\frac{v^2}{u^4}}{\left|-\frac{1}{2}\frac{v}{u}\right|} & v\geq 1,\quad \dfrac{u^2}{v}\geq 1\\ 0 & o.w.\end{cases}$$

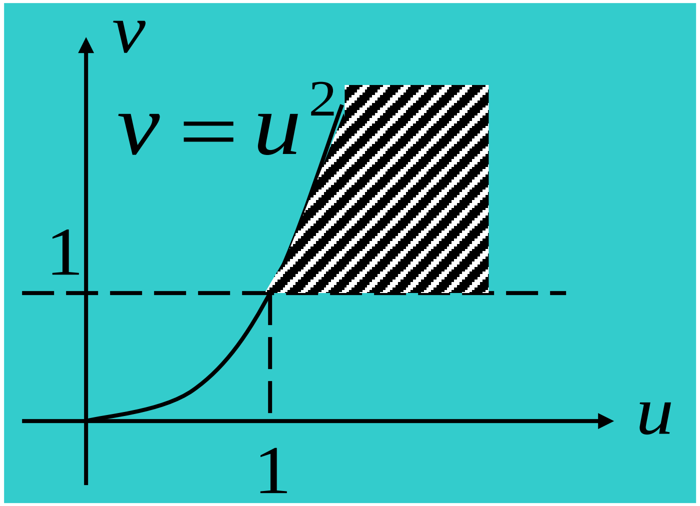
$$=\begin{cases}\dfrac{2}{u^3v} & v\geq 1,\quad u^2\geq v\\ 0 & o.w.\end{cases}$$

$u>0$ *is understood from the initial definition.*

Last Step

5. Find marginal $f_U(u)$.

Consider the ROS of $f_{UV}(u,v)$



$$f_U(u)=\int_{-\infty}^{+\infty} f_{UV}(u,v)dv=\int_{1}^{u^2}\frac{2}{u^3 v}dv$$

$$=\begin{cases}\dfrac{2\ln\left(u^2\right)}{u^3} & u\geq 1\\ 0 & o.w.\end{cases}$$

Expectation

- Given the joint pdf $f_{XY}(x,y)$, the **law of the unconscious statistician (LOTUS)** can easily be used to show that

$$E\big(g(X,Y)\big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\,dx\,dy$$

- Discrete case:

$$E\big(g(X,Y)\big) = \sum_{x,y} g(x,y)\, p_{XY}(x,y)$$

Example

X_1 and X_2 are discrete random variables. $Y = X_1 + X_2$. $E[Y] = ?$

$$
\begin{aligned}
\mathrm{E}[Y] &= \mathrm{E}[X_1 + X_2] = \sum_{x_1, x_2} (x_1 + x_2) p_{X_1 X_2}(x_1, x_2) \\
&= \sum_{x_1} \sum_{x_2} x_1 p_{X_1 X_2}(x_1, x_2) + \sum_{x_1} \sum_{x_2} x_2 p_{X_1 X_2}(x_1, x_2) \\
&= \sum_{x_1} x_1 \sum_{x_2} p_{X_1 X_2}(x_1, x_2) + \sum_{x_2} x_2 \sum_{x_1} p_{X_1 X_2}(x_1, x_2) \\
&= \sum_{x_1} x_1 p_{X_1}(x_1) + \sum_{x_2} x_2 p_{X_2}(x_2) \\
&= \mathrm{E}[X_1] + \mathrm{E}[X_2]
\end{aligned}
$$

Joint characteristic function

- For arbitrary random variables X and Y, their **joint characteristic function** is defined by

$$\varphi_{XY}(v_1, v_2) = E[e^{j(v_1 X + v_2 Y)}]$$

Joint characteristic function

If X and Y have joint pdf $f_{XY}(x, y)$, then

$$\boxed{\varphi_{XY}(v_1, v_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) e^{j(v_1 x + v_2 y)}\, dx\, dy}$$

which is just the **2D Fourier transform** of $f_{XY}(x, y)$ evaluated at $(-v_1, -v_2)$.

Using the **inverse Fourier transform**,

$$\boxed{f_{XY}(x, y) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \varphi_{XY}(v_1, v_2) e^{-j(v_1 x + v_2 y)}\, dv_1\, dv_2}$$

Independence

- X and Y are **independent** if and only if their joint characteristic function factors into the product of the marginal characteristic functions

$$\varphi_{XY}(v_1, v_2) = \varphi_X(v_1)\varphi_Y(v_2)$$

Proof

If X and Y are independent

$$\varphi_{XY}(v_1, v_2) = \mathrm{E}\left[e^{j(v_1 X + v_2 Y)}\right]$$

$$= \mathrm{E}\left[e^{jv_1 X}\right]\mathrm{E}\left[e^{jv_2 Y}\right] \quad (\textit{independence})$$

$$= \varphi_X(v_1)\varphi_Y(v_2)$$

Proof

$$f_{XY}(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{XY}(v_1,v_2)e^{-j(v_1x+v_2y)}dv_1dv_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_X(v_1)\varphi_Y(v_2)e^{-j(v_1x+v_2y)}dv_1dv_2$$

$$= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_X(v_1)e^{-jv_1x}dv_1\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_Y(v_2)e^{-jv_2y}dv_2\right]$$

$$= f_X(x)f_Y(y)$$

Short Summary

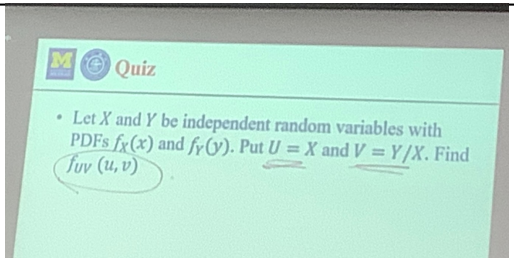
Two functions of two RVs

- Using Jacobian

$$f_{ZW}(z,w) = \sum_{i=1}^{n} \frac{f_{XY}(x_i, y_i)}{|J(x_i, y_i)|}$$

One function of two RVs

- To add independent RVs, convolve their PDFs
- CDF Approach
- Auxiliary RV approach
- Expectation



- $$f_{UV}(u,v) = \sum_{i=1}^{p} \frac{f_{XY}(x,y)}{|J(x_i, y_i)|}$$

$$= \sum_{i=1}^{n} \frac{f_X(x) f_Y(y)}{|J(x_i, y_i)|}$$

$$|J(x_i, y_i)| = \det \left[\frac{du}{dx} \quad \frac{du}{dt} \right.$$

$$= \frac{1}{x}$$

Thank You!