Probability and Random Process

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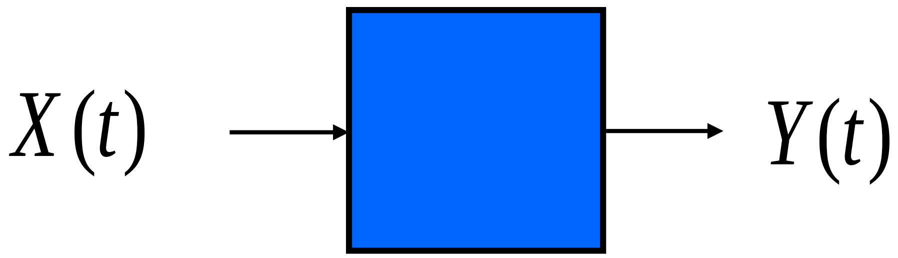
Outline

- 4. Random Process-I
  - Introduction to Random Processes
  - Brownian Motion
  - Poisson Process
  - Complex RV and RP
  - Stationarity
  - PSD, White Noise
  - Response of Systems
  - LTI Systems and RPs

Response of Systems

Response of Systems

- The objective is to determine the statistics of the output of a system, given the statistics of the input to the system



Nonlinear Memoryless

$$X(t) \longrightarrow \boxed{G(X)} \longrightarrow Y(t)$$

- Can use "function of a RV" methods to get PDF for $Y(t)$ and joint PDFs for $Y(t_1)$, $Y(t_2)$, etc.

Linear Time-invariant

- Linear time-invariant (LTI) systems are described by their impulse response $h(t)$ or their frequency response $H(\omega)$

- This topic is treated in the next module

Systems Described by Differential Equations

- Let $L_t(X(t))$ be the linear operator operating on the function with t-dependence.

- **Ex:** Let $Y(t) = L_t(X(t))$, where the operator is represented by the differential equation:

$$a(t)\dot{Y}(t) + b(t)Y(t) = X(t)$$

Initial conditions can be random vectors or variables

$$Y(0) = A \qquad \dot{Y}(0) = B$$

X(t) is a white noise: $R(\tau) = q\delta(\tau)$

Non-zero Initial Conditions

- In a linear system, the output signal can be decomposed into the sum of these:
  - Zero-input solution
    - system response to non-zero initial conditions; these initial conditions can be RVs
  - Zero-state solution
    - assumes that the system is at rest prior to application of the stimulus (the input RP), i.e. the initial conditions are zero

Non-zero Initial Conditions, Concluded

- One approach: characterize these two responses separately. Convenient if the initial conditions are non-random or independent of the input RP.

- Alternatively, under certain conditions (e.g. jointly Gaussian initial conditions and input RP), the output RP can be analyzed as a Markov Process.

Zero Initial Conditions

- A differential equation with zero initial conditions is equivalent to an impulse response
- If the differential equation has constant coefficients and initial conditions are zero, the system is LTI
- Zero initial conditions are assumed in this course

Mean and Autocorrelation

The first and second order moments depend on the linear operator as follows

$$m_Y(t) = L_t\left[m_X(t)\right]$$

$$R_{XY}(t_1,t_2) = L_{t_2}\left[R_{XX}(t_1,t_2)\right]$$

First, t_1 is considered a fixed parameter

$$R_{YY}(t_1,t_2) = L_{t_1}\left[R_{XY}(t_1,t_2)\right]$$

Second, t_2 is considered a fixed parameter

Example: Derivative of Wiener Process

Let X(t) be a Wiener process. Then,

$$R_X(t_1, t_2) = \sigma^2 \min(t_1, t_2), \quad \begin{matrix} t_1 > 0 \\ t_2 > 0 \end{matrix}$$

and

$$E[X(t)] = 0 \qquad X(0) = 0 \qquad \text{for } t \text{ £ } 0$$

and X(t) is Gaussian for t>0.

Derivative of Wiener Process, Cont'd

- Let

$$Y(t) = L_t[X(t)] = \frac{dX(t)}{dt}$$

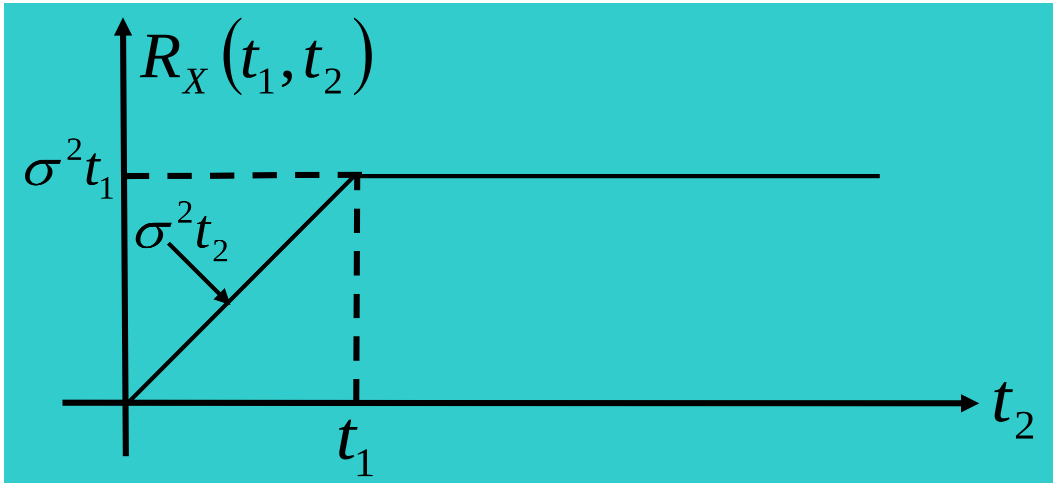
then

$$m_Y(t) = L_t[m_X(t)] = 0$$

$$R_{XY}(t_1, t_2) = L_{t_2}[R_X(t_1, t_2)] = \frac{dR_X(t_1, t_2)}{dt_2}$$

Derivative of Wiener Process, Cont'd

View $R_X(t_1,t_2)$ as a function of t_2 with t_1 fixed.

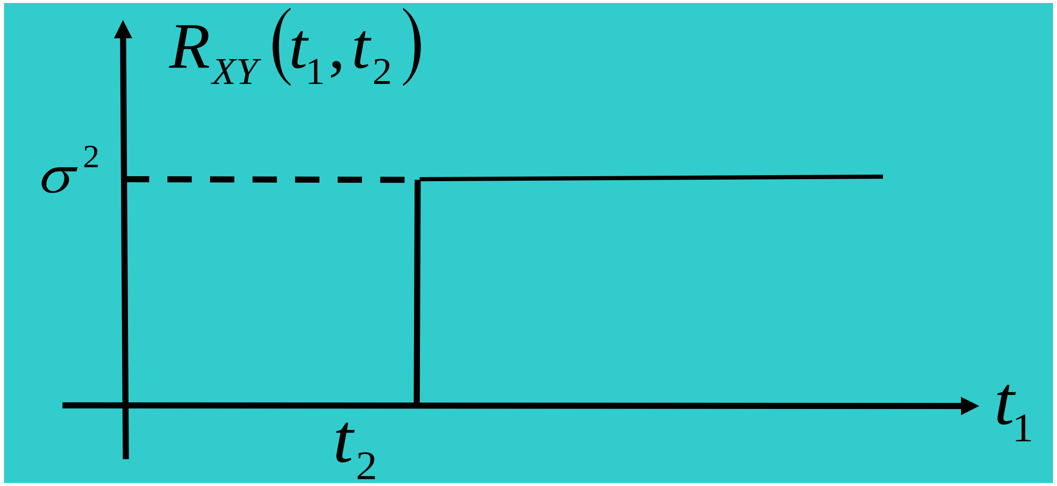


$$R_{XY}(t_1,t_2)=\frac{dR_X(t_1,t_2)}{dt_2}=\begin{cases}\sigma^2 & t_2<t_1\\ 0 & ow\end{cases}$$

Derivative of Wiener Process, Cont'd

$$R_{YY}(t_1,t_2)=\frac{d}{dt_1}R_{XY}(t_1,t_2)$$

Now view $R_{XY}(t_1,t_2)$ as a function of t_1 with t_2 fixed.



$$R_{YY}(t_1,t_2)=\sigma^2\delta(t_1-t_2)$$

Summary

- For nonlinear, memoryless systems, use "functions of RVs" approach
- For general linear systems, mean and autocorrelation can be derived using the same linear operator on mean, auto- and cross-correlation functions
- The derivative of the Wiener process is Gaussian white noise (GWN)

LTI Systems and RPs

Linear Time-Invariant Systems and RPs

- LTI systems are described by their
  - Impulse response, $h(t)$, or their
  - Frequency response, $H(f) = \Im\{h(t)\}$

- Our goal: Suppose $X(t)$ is an input RP with mean $m_X(t)$ and correlation $R_X(t_1, t_2)$. What are the mean and autocorrelation of the output $Y(t)$?

$$X(t) \longrightarrow \boxed{h(t)} \longrightarrow Y(t)$$

The Mean of the Output

$$Y(t) = \int_{-\infty}^{+\infty} h(s) X(t-s) ds$$

$$E\{Y(t)\} = \int_{-\infty}^{+\infty} h(s) E\{X(t-s)\} ds$$

$$m_Y(t) = \int_{-\infty}^{+\infty} h(s) m_X(t-s) ds = h(t) * m_X(t)$$

The Autocorrelation of the Output

$$R_Y(t_1,t_2) = E\{Y(t_1)Y^*(t_2)\} = E\left\{\int_{-\infty}^{+\infty} h(s)X(t_1-s)ds\, Y^*(t_2)\right\}$$

$$= \int_{-\infty}^{+\infty} h(s)E\{X(t_1-s)Y^*(t_2)\}ds$$

$$= \int_{-\infty}^{+\infty} h(s)R_{XY}(t_1-s,t_2)ds = h(t_1) * R_{XY}(t_1,t_2)$$

Cross-correlation of Input and Output

$$R_{XY}(t_1,t_2)=E\left\{X(t_1)\int_{-\infty}^{+\infty}h^*(\alpha)X^*(t_2-\alpha)d\alpha\right\}$$

$$=\int_{-\infty}^{+\infty}h^*(\alpha)R_X(t_1,t_2-\alpha)d\alpha$$

$$=h^*(t_2)*R_X(t_1,t_2)$$

$$R_Y(t_1,t_2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}h(s)h^*(\alpha)R_X(t_1-s,t_2-\alpha)dsd\alpha$$

$$=h(t_1)*h^*(t_2)*R_X(t_1,t_2)$$

WSS Input - Mean

Mean:

$$m_X(t) = m_X$$

$$m_Y(t) = \int_{-\infty}^{+\infty} h(s) m_X(t-s) ds = m_X \int_{-\infty}^{+\infty} h(s) ds$$

Recall the DC response of the system:

$$H(0) = \left[\int_{-\infty}^{+\infty} h(t) e^{-jwt} dt \right]\Bigg|_{w=0}$$

Then,

$$m_Y = m_X H(0)$$

WSS Case - Autocorrelation

$$R_Y(t_1,t_2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}R_X(t_1-s,t_2-\alpha)h(s)h^*(\alpha)ds\,d\alpha$$

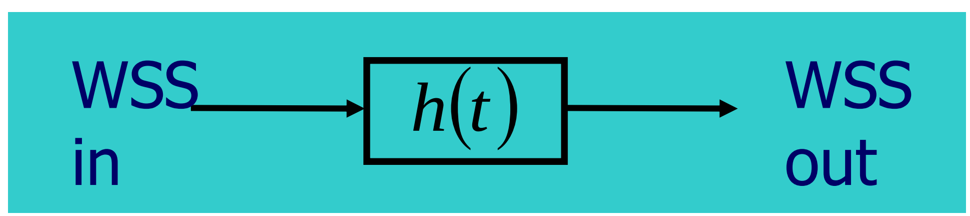
$$=\underbrace{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}R_X(t_1-s-t_2+\alpha)h(s)h^*(\alpha)ds\,d\alpha}_{\text{Whole thing is a function of } t_1-t_2}$$

So far …

$$m_Y(t)=m_Y \quad \text{(constant)}$$

$$R_Y(t_1,t_2)=R_Y(t_1-t_2)$$

Y(t) is WSS.



WSS Autocorrelation Simplification

$$R_Y(\tau) = E\{Y(t+\tau)Y^*(t)\} = E\left\{\int_{-\infty}^{+\infty} h(s)X(t+\tau-s)ds\,Y^*(t)\right\}$$

$$= \int_{-\infty}^{+\infty} h(s)R_{XY}(\tau-s)ds$$

$$= h(\tau) * R_{XY}(\tau)$$

WSS Cross-correlation Simplification

$$R_{XY}(\tau) = E\left\{X(t+\tau)Y^*(t)\right\}$$

$$= E\left\{X(t+\tau)\int_{-\infty}^{+\infty} h^*(\alpha)X^*(t-\alpha)d\alpha\right\}$$

$$= \int_{-\infty}^{+\infty} h^*(\alpha)R_X(\tau+\alpha)d\alpha$$

$$= h^*(-\tau) * R_X(\tau)$$

WSS Output Autocorrelation, Concluded

Putting the results

$$R_Y(\tau) = h(\tau) * R_{XY}(\tau)$$

$$R_{XY}(\tau) = h^*(-\tau) * R_X(\tau)$$

together, yields

$$R_Y(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_X(\tau - s + \alpha) h(s) h^*(\alpha) \, ds \, d\alpha$$

$$= h(t) * h^*(-t) * R_X(t)$$

PSD of the Output

- We know Y(t) is WSS. The PSD is

$$
\begin{aligned}
S_Y(\omega) &= \Im\{R_Y(\tau)\} \\
&= \Im\{h(\tau) * h^*(-\tau) * R_X(\tau)\} \\
&= \Im\{h(\tau)\}\Im\{h^*(-\tau)\}\Im\{R_X(\tau)\} \\
&= H(\omega)\Im\{h^*(-\tau)\}S_X(\omega)
\end{aligned}
$$

Final Results

- Change of variables

$$\Im\left\{h^*(-\tau)\right\} = \int_{-\infty}^{+\infty} h^*(-\tau) e^{-j\omega\tau} d\tau$$

傅立叶变换

Let $s = -\tau$

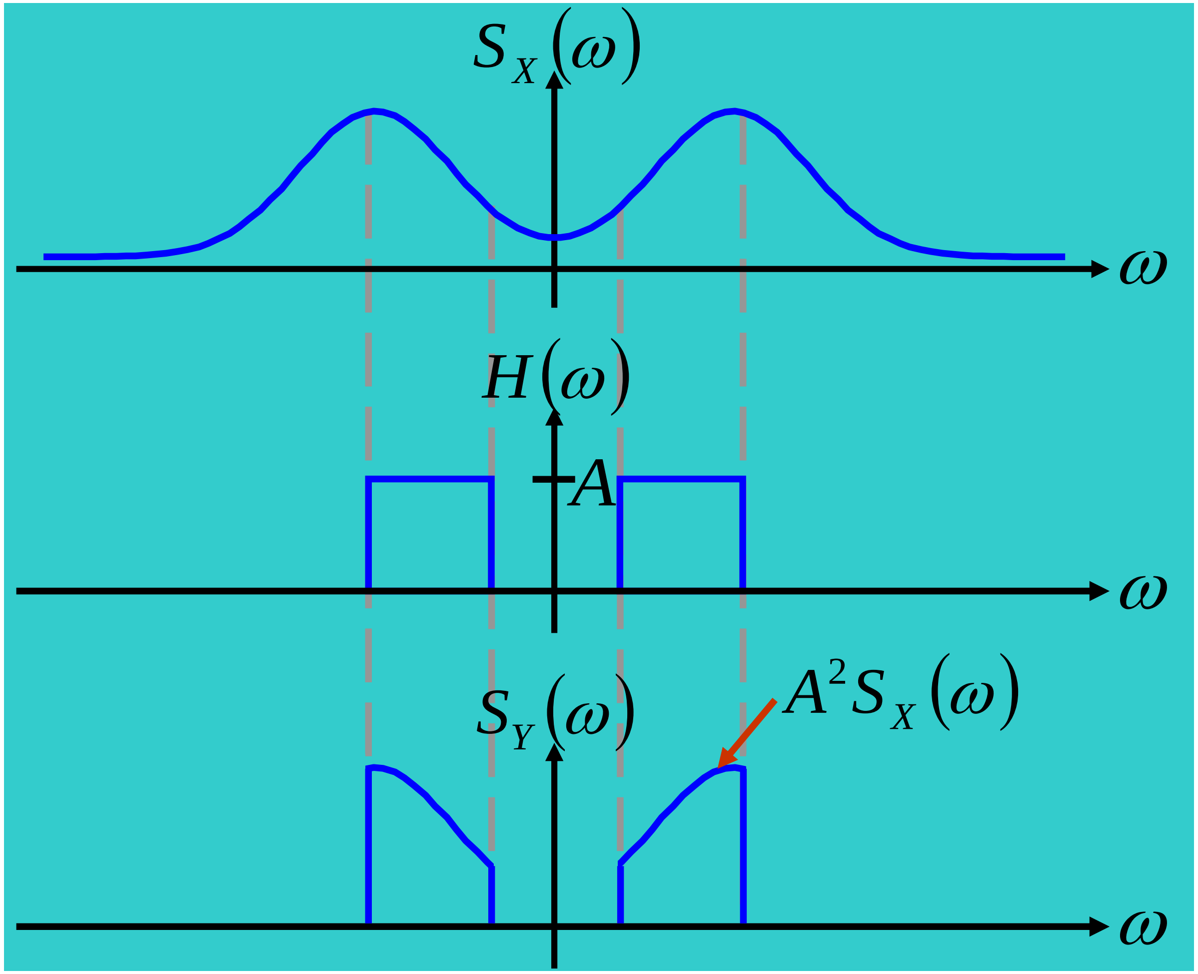
$$\Im\left\{h^*(-\tau)\right\} = \left[\int_{-\infty}^{+\infty} h(s) e^{-j\omega s} ds\right]^* = H^*(\omega)$$

- Final answers

$$S_Y(w) = H(w) H^*(w) S_X(w)$$

$$S_Y(w) = \left|H(w)\right|^2 S_X(w)$$

Graphical Example



Example: White Noise

Ex: Input WSS white noise $S_X(\omega) = \dfrac{N_0}{2}$

$$S_Y(\omega) = \frac{N_0}{2}\left|H(\omega)\right|^2$$

Average power of the output:

$$R_Y(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_Y(\omega)d\omega = \frac{N_0}{2}\underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|H(\omega)\right|^2 d\omega\right]}_{\text{Energy of the system impulse response}}$$

Summary

- WSS in, WSS out, for LTI systems
- $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$

Example

- White noise with power spectral density $N_0/2$ is passed through a linear, time-invariant system with impulse response $h(t) = 1/(1+t^2)$. If Y_t denotes the filter output, find $E[Y_{t+\frac{1}{2}} Y_t]$.

Solution

We need to find $E[Y_{t+\frac{1}{2}}Y_t]$ which is $R_Y(\frac{1}{2})$. We find this by finding the power spectral density and then taking the inverse Fourier transfrom. The power spectral density is

$$S_Y(f) = S_X(f)|H(f)|^2 = \tfrac{N_0}{2}\pi^2 e^{-4\pi|f|},$$

where $H(f) = \pi e^{-2\pi|f|}$. Then

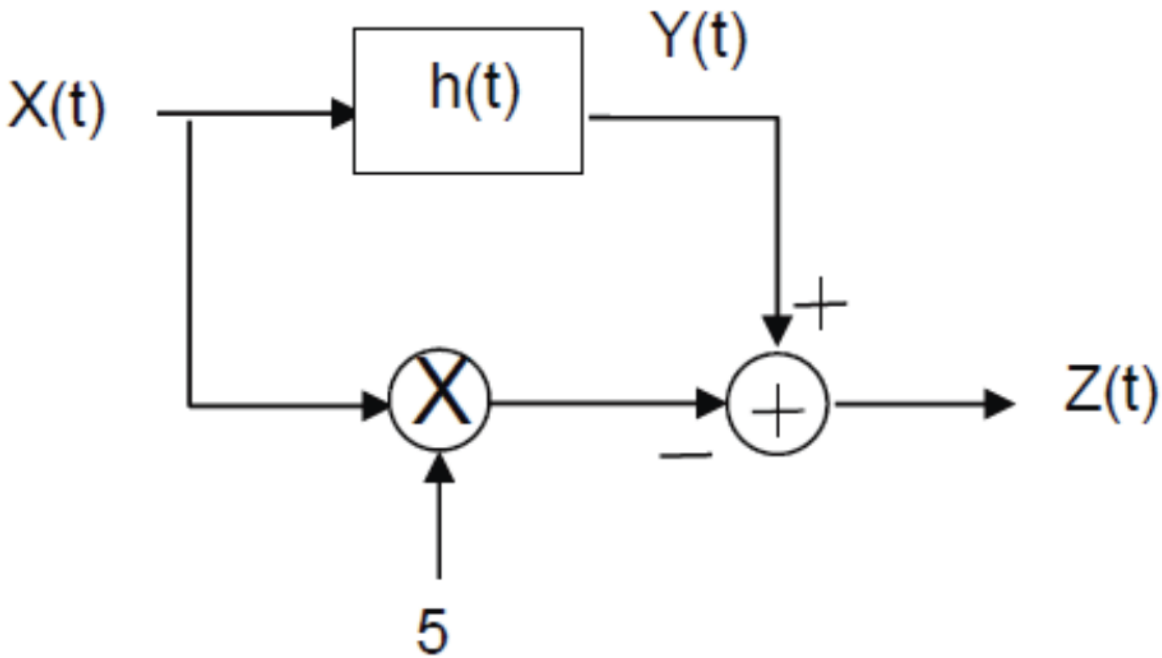
$$R_Y(\tau) = \mathcal{F}^{-1}(S_Y(f)) = \mathcal{F}^{-1}(\tfrac{N_0}{2}\pi^2 e^{-4\pi|f|}) = \tfrac{N_0\pi}{2}\tfrac{2}{4+t^2} = \tfrac{N_0\pi}{4+t^2}$$

and

$$E[Y_{t+\frac{1}{2}}Y_t] = R_Y(\tfrac{1}{2}) = \tfrac{N_0\pi}{4+\frac{1}{4}} = \boxed{\tfrac{4N_0\pi}{17}}.$$

Example

- Let Y(t)=h(t)*X(t) and Z(t)=-5X(t)+Y(t) as shown below.
- (a) Find $S_Z(\omega)$ in terms of $S_X(\omega)$. *Hint: represent everything between X(t) and Z(t) as one LTI system.*
- (b) Give an expression for $E\{Z^2(t)\}$.



Solution

- (a)

The easy way is to view everything between $X(t)$ and $Z(t)$ as one system with impulse response $\tilde{h}(t) = h(t) - 5\delta(t)$. Then you can write

$$S_Z(\omega) = S_X(\omega)|\tilde{H}(\omega)|^2 = S_X(\omega)|H(\omega) - 5|^2$$

- (b)

$$E\{Z^2(t)\} = R_Z(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_Z(\omega)\ \mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_X(\omega)|H(\omega) - 5|^2\ \mathrm{d}\omega$$

Thank You!