# Probability and Random Process

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# Outline

- 3. Multiple Random Variables
  - Two Random Variables
  - Marginal PDF
  - Functions of Two Random Variables
  - Conditional PDF
  - Joint Moments
  - Mean Square Error Estimation
  - Probability bound
  - Random Vectors
  - Sample Mean
  - Convergence of Random Sequences
  - Central Limit Theorem

# Functions of Two Random Variables

# Two Functions of Two RVs

Suppose $X$ and $Y$ are jointly distributed RVs with joint PDF $f_{XY}(x,y)$ and

$$Z = G(X,Y)$$
$$W = H(X,Y)$$

**Examples:**

Rectangular-to-Polar conversion

Rotation of Coordinates

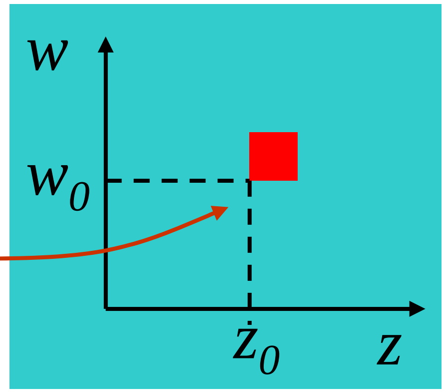
Then one might wish to find $f_{ZW}(z,w)$

# Graphical Example

Consider the approximation:

$$P\left(\{z_0 < Z < z_0 + dz\} \cap \{w_0 < W < w_0 + dw\}\right)$$
$$\approx f_{zw}(z_0, w_0)\,dzdw$$

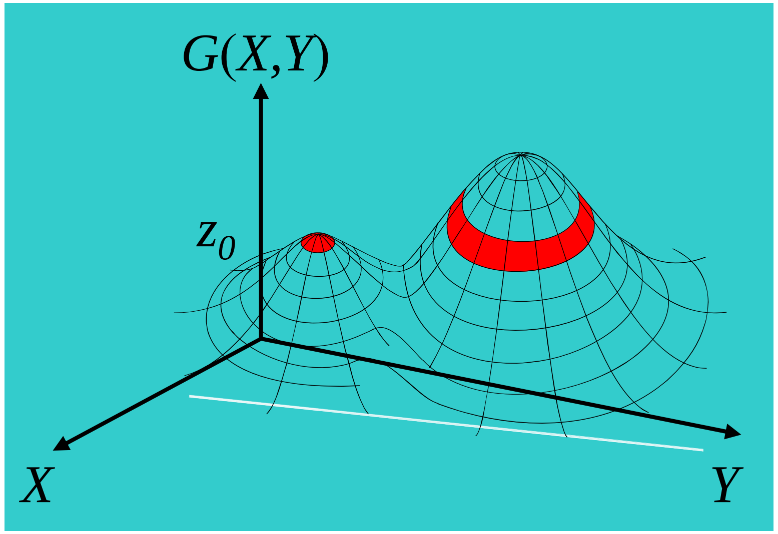
= The probability that $(Z,W)$ is in this small square

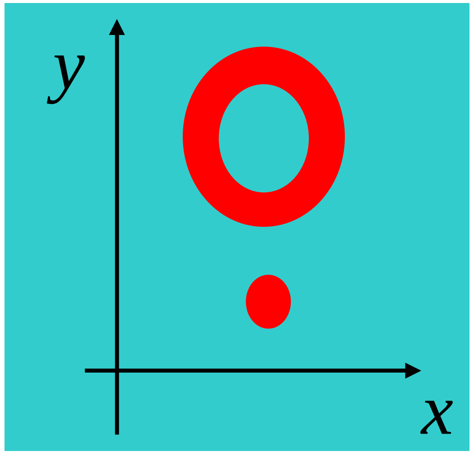


# Graphical Example, Cont'd

First consider the event $\{z_0 < G(X,Y) < z_0 + dz\}$

Example:

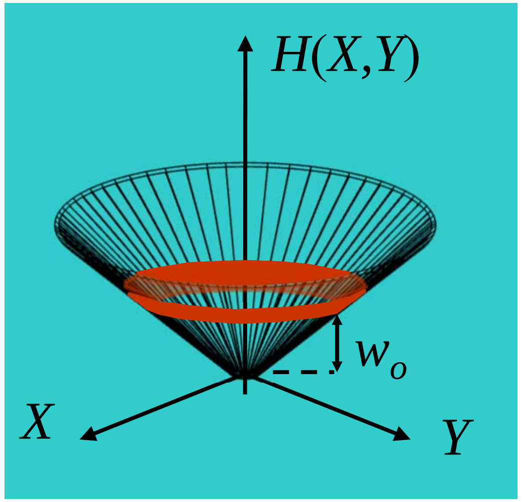


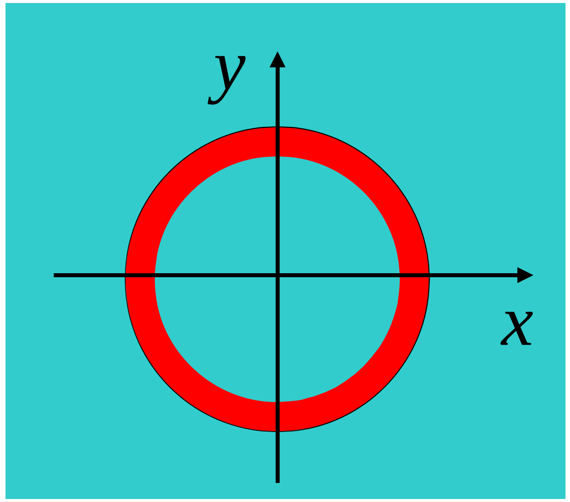

Event Region

# Graphical Example, Cont'd

Similarly, $\{w_0 < H(X,Y) < w_0 + dw\}$ also has an event region.




Event Region



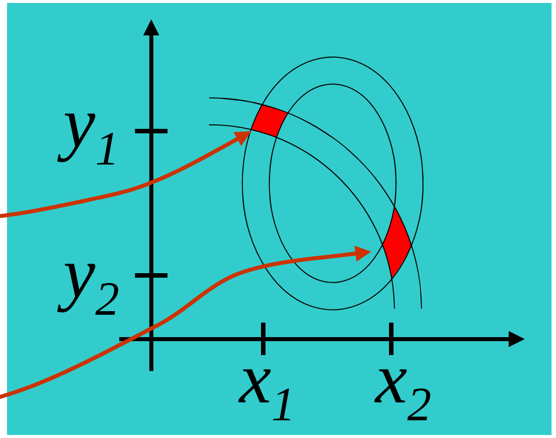
# Graphical Example, Concluded

In this example,

$$P\left(\{z_0 < Z < z_0 + dz\} \cap \{w_0 < W < w_0 + dw\}\right)$$

corresponds to the intersection of these two events regions.

Differential Area $dA_1$




Differential Area $dA_2$

$(X_1, Y_1)$ and $(X_2, Y_2)$ are two solutions to the equations:

$$z_0 = G(X, Y)$$

$$w_0 = H(X, Y)$$

# The Jacobian

Then we have

$$f_{zw}(z_0, w_0)dzdw \approx f_{XY}(x_1, y_1)dA_1 + f_{XY}(x_2, y_2)dA_2$$

It happens that $$\frac{\partial A_j}{\partial z \partial w} \approx \frac{1}{\left|J(x_i, y_i)\right|},$$

where $$J(x,y) = \det\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} =$$ Jacobian of $G$ and $H$

Recall the derivative in one RV case

# General Formula

- In general, if there are $n$ solutions to $z = G(X,Y)$ and

  $w = H(X,Y), (x_i, y_i), \; i = 1,2, \dots . n$

- Then

$$f_{ZW}(z,w) = \sum_{i=1}^{n} \frac{f_{XY}(x_i, y_i)}{\left| J(x_i, y_i) \right|}$$

- This is similar to the formula for the function of a RV

# Linear Example

Invertible Linear Transformation
$$\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

Only one solution $A^{-1} = \frac{1}{|A|} A^{*}$

$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \begin{bmatrix} Z \\ W \end{bmatrix} = \frac{1}{ad - cb} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} Z \\ W \end{bmatrix}$$

So, $X = \dfrac{dZ - bW}{ad - cb}$ $\qquad Y = \dfrac{-cZ + aW}{ad - cb}$

# Linear Example, Concluded

Jacobian

$$\det\begin{bmatrix} \dfrac{\partial z}{\partial x} & \dfrac{\partial z}{\partial y} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} \end{bmatrix} = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

$$f_{ZW}(z,w) = \frac{f_{XY}\left(\dfrac{dz - bw}{ad - cb}, \dfrac{-cz + aw}{ad - cb}\right)}{|ad - cb|}$$

# Nonlinear Example

S-plane to Z-plane mapping:

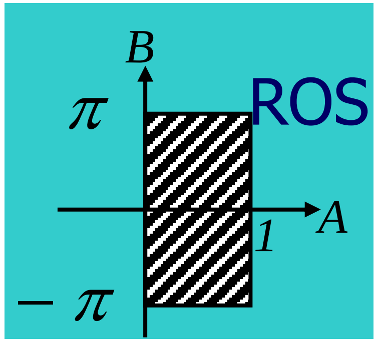
$$S = A + jB$$

$$Z = C + jD = e^{ST} = e^{(A+jB)T}$$

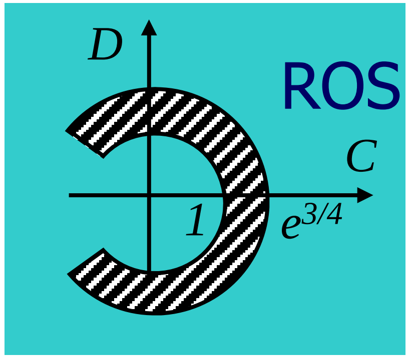
$$= e^{AT} e^{jBT} = e^{AT} \cos BT + je^{AT} \sin BT$$

$$\therefore C = e^{AT} \cos BT = G(A, B)$$

$$D = e^{AT} \sin BT = H(A, B)$$

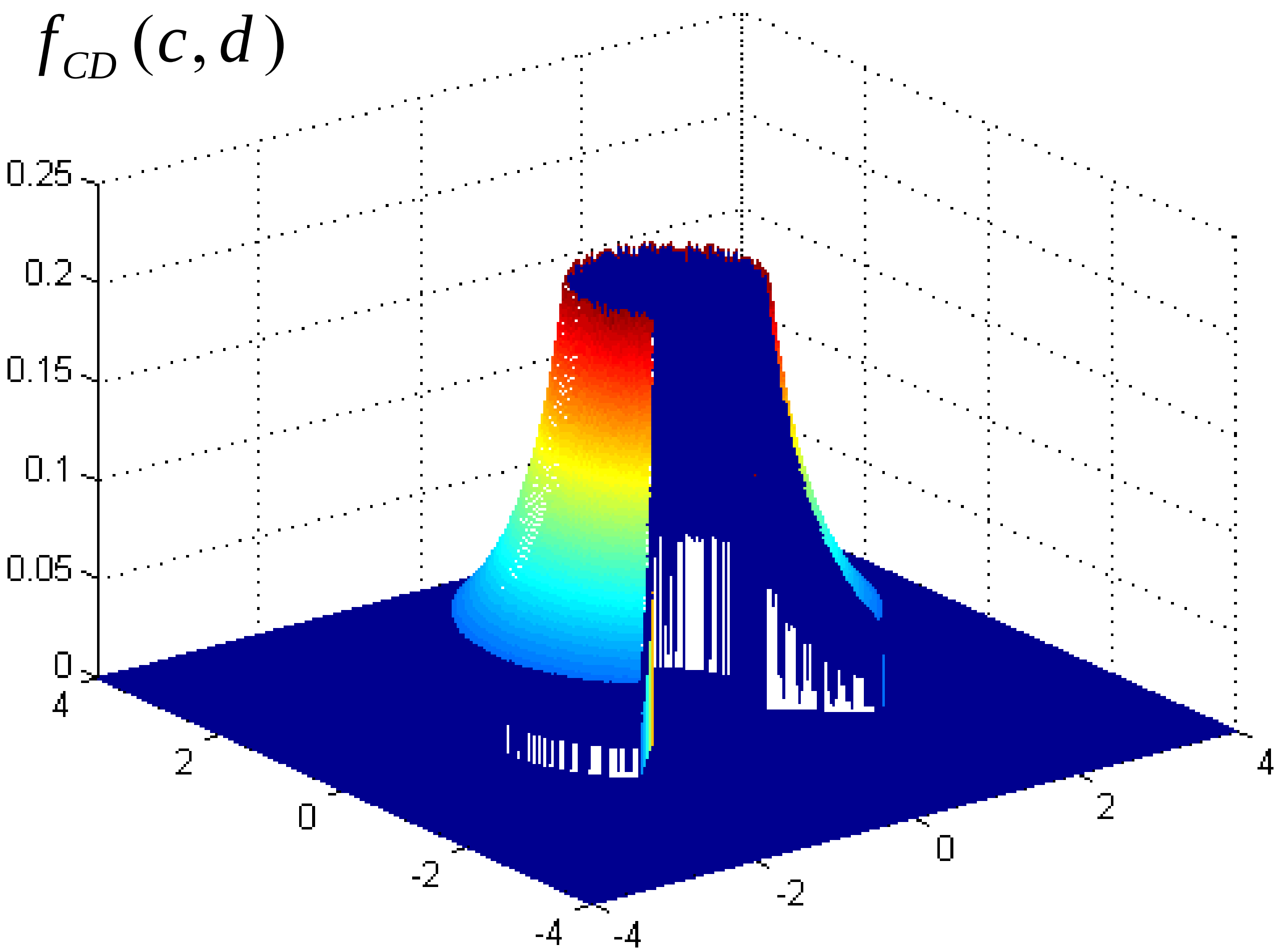



Let $f_{AB}(a,b) = \begin{cases} \dfrac{1}{2\pi} & \begin{matrix} 0 < a < 1 \\ -\pi < b < \pi \end{matrix} \end{cases}$




and $T=3/4$

# Nonlinear Example, Concluded



# One Function of Two RVs

Two equally valuable approaches:

CDF Approach

Auxiliary RV approach

CDF approach: Given $f_{XY}(x, y)$ and $Z = G(X, Y)$.
Find $F_Z(z)$, then differentiate to get $f_Z(z)$.

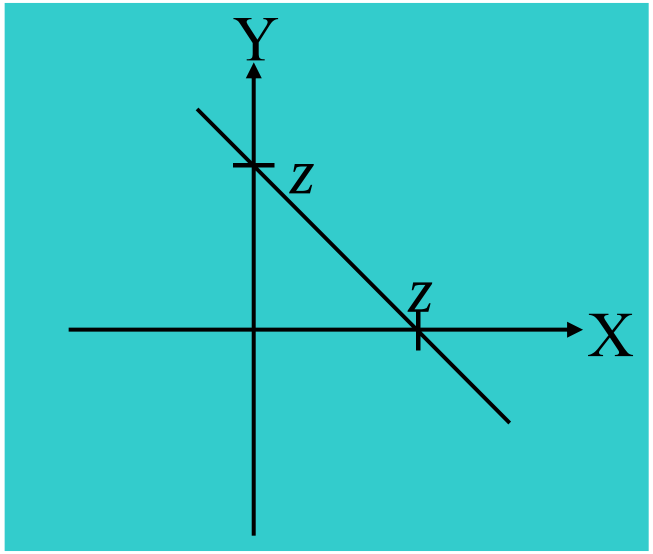
Auxiliary RV approach: Define an auxiliary or "dummy" RV $W$ as either $W = X$ or $W = Y$. Use "two-functions-of-two-RVs" approach to get $f_{ZW}(z, w)$, then get marginal $f_Z(z)$.

# Example: Adding RVs

$$Z = X + Y$$

$$F_z(z) = P(Z \leq z)$$
$$= P(X + Y \leq z)$$

$$F_z(z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{z-y} f_{XY}(x, y)\,dxdy$$



## Recall Leibniz's Rule

If $\Phi(t) = \int_{a(t)}^{b(t)} f(x)dx,$

then $\Phi'(t) = f(b(t))b'(t) - f(a(t))a'(t)$

so $\dfrac{d}{dz} F_Z(z) = \int_{-\infty}^{+\infty} \left( \dfrac{d}{dz} \int_{-\infty}^{z-y} f_{XY}(x, y)dx \right) dy$

$f(z-y, y) - f(-\infty, y) \times 0$

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{XY}(z-y, y)dy$$

# Adding Independent RVs

Suppose, $Z=X+Y$ and $X$ and $Y$ are independent. Then

$$f_Z(z) = \int_{-\infty}^{+\infty} f_{XY}(z-y, y)dy$$

$$= \int_{-\infty}^{+\infty} f_X(z-y) f_y(y)dy$$

$$= f_X(z) * f_Y(z)$$

*Convolution*

REMEMBER

To add independent RVs, convolve their PDFs.

# Example for discrete case

- Let $X, Y$ be **non-negative integer valued** (discrete) r.v.'s that are **independent** and have pmfs $p_X(x)$ and $p_Y(y)$ respectively. Let $Z = X + Y$. Determine the pmf $p_Z(z)$.

# Solution

$$p_Z(z) = P_r(Z = z) = P_r(X + Y = z)$$

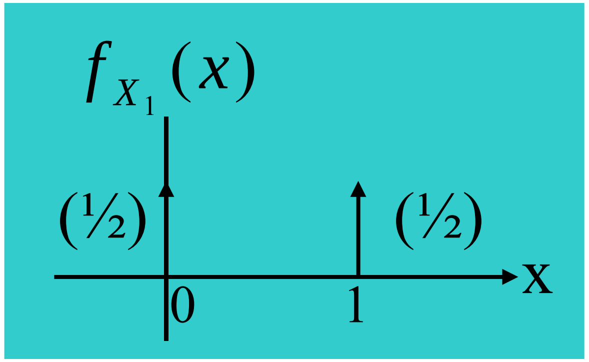
$$= \sum_{k=0}^{z} P_r(X = k, Y = z - k), \quad (\textit{non-negative integer valued r.v.'s})$$

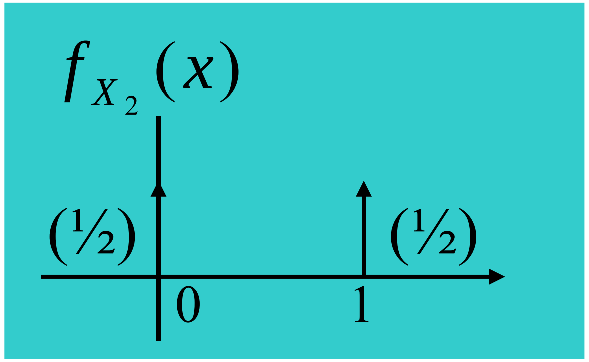
$$= \sum_{k=0}^{z} P_r(X = k) P_r(Y = z - k) \quad (\textit{independence})$$

$$= \sum_{k=0}^{z} p_X(k) p_Y(z - k)$$

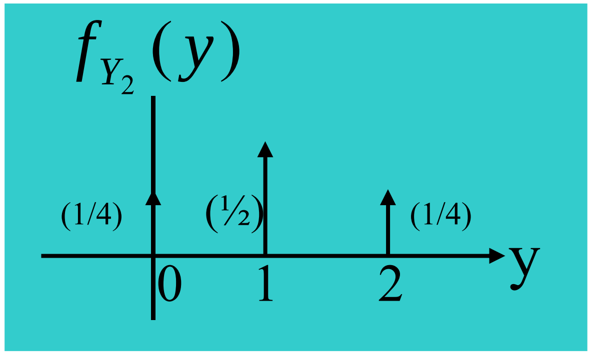
# Adding Independent Bernoulli RVs

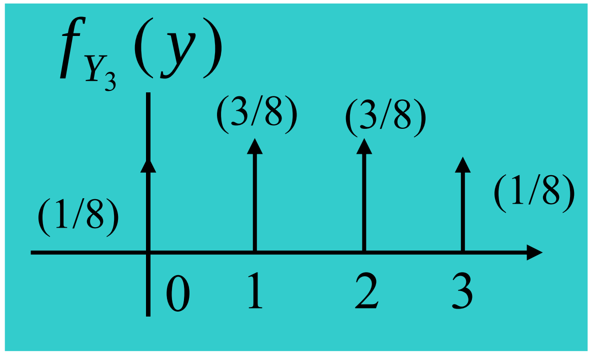
Let $X_1$, $X_2$ and $X_3$ be iid Bernoulli RVs with $p=1/2$



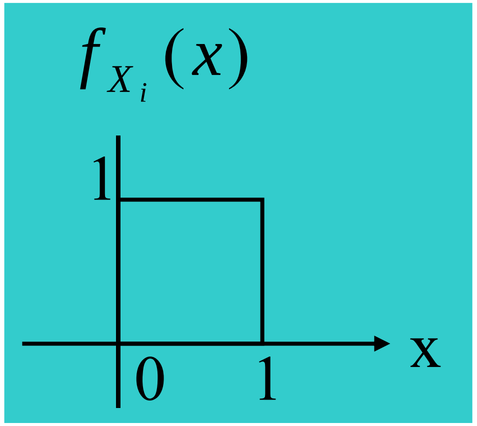

Let $Y_2=X_1+X_2$ and $Y_3=Y_2+X_3$



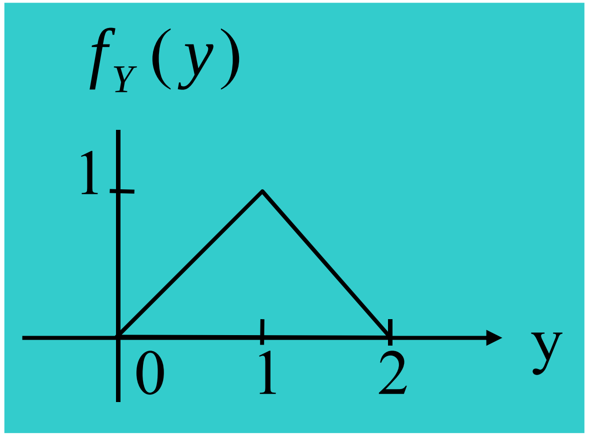



# Adding Uniform RVs

Let $X_1$ and $X_2$ be i.i.d. with




Then $Y=X_1+X_2$ has the PDF ?



# Example

- A random, continuous-valued signal $X$ is transmitted over a channel subject to multiplicative, continuous-valued noise $Y$. The received signal is $Z = XY$. Find the cdf and density of $Z$ if $X$ and $Y$ has a joint density $f_{XY}(x, y)$.

# Solution

$$F_Z(z) = \mathsf{P}(Z \le z) = \mathsf{P}(XY \le z) = \mathsf{P}((X,Y) \in A_z),$$

where $A_z := \{(x,y) : xy \le z\}$ is partitioned into two disjoint regions, $A_z = A_z^+ \cup A_z^-$, as sketched in Figure 7.2. Next, since

$$F_Z(z) = \mathsf{P}((X,Y) \in A_z^-) + \mathsf{P}((X,Y) \in A_z^+),$$

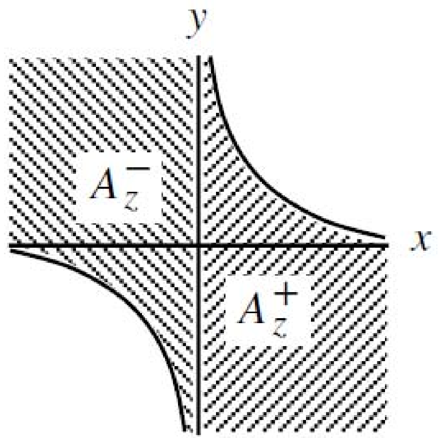



***Figure* 7.2.** The curve is $y = z/x$. The shaded region to the left of the vertical axis is $A_z^- = \{(x,y) : y \ge z/x, x < 0\}$, and the shaded region to the right of the vertical axis is $A_z^+ = \{(x,y) : y \le z/x, x > 0\}$. The sketch is for the case $z > 0$. How would the sketch need to change if $z = 0$ or if $z < 0$?

# Solution

$$\mathsf{P}((X,Y)\in A_z^+) = \int_0^\infty \left[\int_{-\infty}^{z/x} f_{XY}(x,y)\,dy\right]dx$$

and

$$\mathsf{P}((X,Y)\in A_z^-) = \int_{-\infty}^0 \left[\int_{z/x}^{\infty} f_{XY}(x,y)\,dy\right]dx.$$

It follows that[b]

$$f_Z(z) = \int_0^\infty f_{XY}(x,\tfrac{z}{x})\tfrac{1}{x}\,dx - \int_{-\infty}^0 f_{XY}(x,\tfrac{z}{x})\tfrac{1}{x}\,dx.$$

In the first integral on the right, the range of integration implies $x$ is positive, and so we can replace $1/x$ with $1/|x|$. In the second integral on the right, the range of integration implies $x$ is negative, and so we can replace $1/(-x)$ with $1/|x|$. Hence,

$$f_Z(z) = \int_0^\infty f_{XY}(x,\tfrac{z}{x})\tfrac{1}{|x|}\,dx + \int_{-\infty}^0 f_{XY}(x,\tfrac{z}{x})\tfrac{1}{|x|}\,dx.$$

Now that the integrands are the same, the two integrals can be combined to get

$$f_Z(z) = \int_{-\infty}^\infty f_{XY}(x,\tfrac{z}{x})\tfrac{1}{|x|}\,dx.$$

# Example

- Let $Y = \max(X_1, X_2)$, where $X_1$, and $X_2$ are independent discrete r.v.'s with the given joint pmf

$$p_{X_1X_2}(x_1, x_2) = p_{X_1}(x_1)p_{X_2}(x_2)$$

Let $D_Y$ be the range space of Y.

$$D_Y = \{y_1, y_2, y_3, \dots\}, y_1 \le y_2 \le \cdots$$

Compute the pmf of $Y$, i.e., $p_Y(y_i)$.

# Solution

- $p_Y(y_i) = \Pr(Y = y_i) = \Pr(Y \le y_i) - \Pr(Y \le y_{i-1})$
- $\Pr(Y \le y_i)$

$= \Pr(\max(X_1, X_2) \le y_i)$

$= \Pr(X_1 \le y_i, X_2 \le y_i)$

$= \mathrm{P}(\{s: X_1(s) \le y_i, X_2(s) \le y_i\})$

$= P(\{s: X_1(s) \le y_i\} \cap \{s: X_2(s) \le y_i\})$

$= P(\{s: X_1(s) \le y_i\}) P(\{s: X_2(s) \le y_i\})$ ---independent

$= \Pr(X_1 \le y_i) \Pr(X_2 \le y_i)$

$= [\sum_{x_i \le y_i} p_{X_1}(x_i)][\sum_{x_i \le y_i} p_{X_2}(x_i)]$

# Auxiliary RV Example

Let $U = +\sqrt{XY}$, where $X$ and $Y$ are iid

$$f_X(x) = \begin{cases} \frac{1}{x^2} & x \geq 1 \\ 0 & else \end{cases} \qquad f_y(y) = \begin{cases} \frac{1}{y^2} & y \geq 1 \\ 0 & else \end{cases}$$

Let $V=X$ be the auxiliary RV.

1. The solution is:

$$X = V$$

$$Y = \frac{U^2}{V}$$

# Auxiliary RV Example, Cont'd

$$U = +\sqrt{XY}$$
$$V = X$$

2. Find Jacobian

$$J(x,y) = \det \begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

$$= \det \begin{bmatrix} \text{don't care} & \dfrac{1}{2}\sqrt{\dfrac{x}{y}} \\ 1 & 0 \end{bmatrix} = -\frac{1}{2}\sqrt{\frac{x}{y}}$$

3. Plug solution into Jacobian

$$X = V, \quad Y = \frac{U^2}{V}$$

$$-\frac{1}{2}\sqrt{\frac{x}{y}} = -\frac{1}{2}\sqrt{\frac{v}{\dfrac{u^2}{v}}} = -\frac{1}{2}\frac{v}{u}$$

# Auxiliary RV Example, Cont'd

Plug arguments into ROS for $f_{XY}(x,y)$

4. PDF formula

$$f_{UV}(u,v) = \frac{f_{XY}\left(v, \frac{u^2}{v}\right)}{\left|-\frac{1}{2}\cdot\frac{v}{u}\right|} = \begin{cases} \dfrac{\frac{1}{v^2}\cdot\frac{v^2}{u^4}}{\left|-\frac{1}{2}\frac{v}{u}\right|} & v \geq 1, \quad \frac{u^2}{v} \geq 1 \\ 0 & o.w. \end{cases}$$

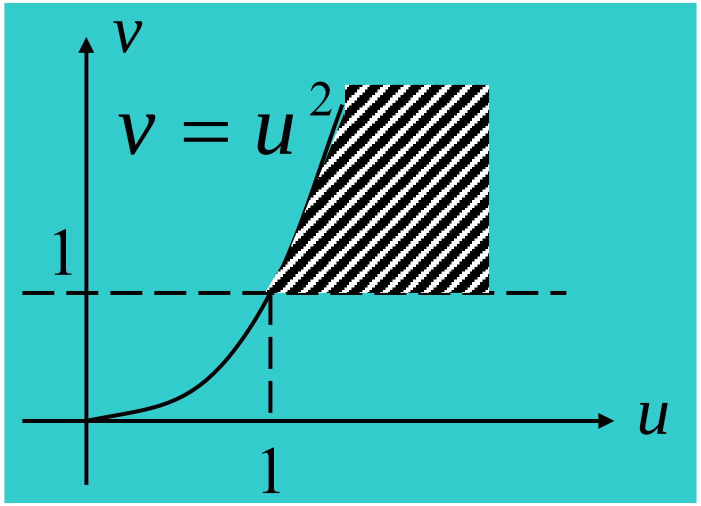
$$= \begin{cases} \dfrac{2}{u^3 v} & v \geq 1, \quad u^2 \geq v \\ 0 & o.w. \end{cases}$$

*$u>0$ is understood from the initial definition.*

# Last Step

5. Find marginal $f_U(u)$.

Consider the ROS of $f_{UV}(u,v)$




$$f_U(u) = \int_{-\infty}^{+\infty} f_{UV}(u,v)\,dv = \int_{1}^{u^2} \frac{2}{u^3 v}\,dv$$

$$= \begin{cases} \dfrac{2\ln\left(u^2\right)}{u^3} & u \geq 1 \\ 0 & o.w. \end{cases}$$

# Expectation

- Given the joint pdf $f_{XY}(x, y)$, the **law of the unconscious statistician (LOTUS)** can easily be used to show that

$$E\big(g(X,Y)\big) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)dxdy$$

- Discrete case:

$$E\big(g(X,Y)\big) = \sum_{x,y} g(x,y)\, p_{XY}(x,y)$$

# Example

$X_1$ and $X_2$ are discrete random variables. $Y = X_1 + X_2$.
$E[Y] = ?$

$$
\begin{aligned}
\mathrm{E}[Y] &= \mathrm{E}[X_1 + X_2] = \sum_{x_1, x_2} (x_1 + x_2) p_{X_1 X_2}(x_1, x_2) \\
&= \sum_{x_1} \sum_{x_2} x_1 p_{X_1 X_2}(x_1, x_2) + \sum_{x_1} \sum_{x_2} x_2 p_{X_1 X_2}(x_1, x_2) \\
&= \sum_{x_1} x_1 \sum_{x_2} p_{X_1 X_2}(x_1, x_2) + \sum_{x_2} x_2 \sum_{x_1} p_{X_1 X_2}(x_1, x_2) \\
&= \sum_{x_1} x_1 p_{X_1}(x_1) + \sum_{x_2} x_2 p_{X_2}(x_2) \\
&= \mathrm{E}[X_1] + \mathrm{E}[X_2]
\end{aligned}
$$

# Joint characteristic function

- For arbitrary random variables X and Y, their **joint characteristic function** is defined by

$$\varphi_{XY}(v_1, v_2) = E\left[e^{j(v_1 X + v_2 Y)}\right]$$

# Joint characteristic function

If $X$ and $Y$ have joint pdf $f_{XY}(x, y)$, then

$$\boxed{\varphi_{XY}(v_1, v_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)e^{j(v_1 x + v_2 y)}\,dx\,dy}$$

which is just the **2D Fourier transform** of $f_{XY}(x, y)$ evaluated at $(-v_1, -v_2)$.
Using the **inverse Fourier transform**,

$$\boxed{f_{XY}(x, y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi_{XY}(v_1, v_2)e^{-j(v_1 x + v_2 y)}\,dv_1\,dv_2}$$

# Independence

- X and Y are **independent** if and only if their joint characteristic function factors into the product of the marginal characteristic functions

$$\varphi_{XY}(v_1, v_2) = \varphi_X(v_1)\varphi_Y(v_2)$$

# Proof

*If X and Y are independent*

$$\varphi_{XY}(v_1, v_2) = \mathrm{E}\left[e^{j(v_1 X + v_2 Y)}\right]$$

$$= \mathrm{E}\left[e^{jv_1 X}\right]\mathrm{E}\left[e^{jv_2 Y}\right] \quad (\textit{independence})$$

$$= \varphi_X(v_1)\varphi_Y(v_2)$$

# Proof

$$f_{XY}(x,y) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_{XY}(v_1,v_2)e^{-j(v_1x+v_2y)}dv_1\,dv_2$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi_X(v_1)\varphi_Y(v_2)e^{-j(v_1x+v_2y)}dv_1\,dv_2$$

$$= \left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_X(v_1)e^{-jv_1x}dv_1\right]\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_Y(v_2)e^{-jv_2y}dv_2\right]$$

$$= f_X(x)f_Y(y)$$

# Short Summary

Two functions of two RVs

Using Jacobian

$$f_{ZW}(z,w)=\sum_{i=1}^{n}\frac{f_{XY}(x_i,y_i)}{\left|J(x_i,y_i)\right|}$$

One function of two RVs

To add independent RVs, convolve their PDFs

CDF Approach

Auxiliary RV approach

Expectation

# Thank You!