# Probability and Random Process

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# Outline

- 4. Random Process
  - Introduction to Random Processes
  - Brownian Motion/Wiener Process
  - Poisson Process
  - Complex RV and RP
  - Stationarity
  - PSD, QAM, White Noise
  - Response of Systems
  - LTI Systems and RPs

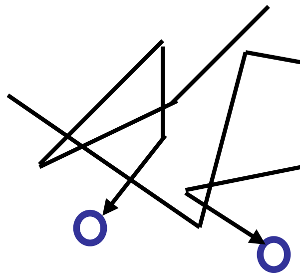
# Brownian Motion/Wiener Process

# Brownian Motion

In 1857, R. Brown observed that small particles immersed in a liquid exhibit ceaseless irregular motions.

Einstein described the phenomenon mathematically from the laws of physics.

The first concise mathematical formulation was given by Wiener in 1918; the process is also called the "Wiener Process."



Elastic collisions of gas particles

# Defining Properties

The one-dimensional Brownian Motion, $X(t)$, has

a) Independent increments
b) Stationary increments
c) The increment $X(s+t) - X(s)$ is normally distributed with mean zero and variance $\alpha t$
d) $X(0) = 0$ and $X(t)$ is continuous for $t \geq 0$

If $\alpha = 1$, $X(t)$ is a Standard Brownian Motion.

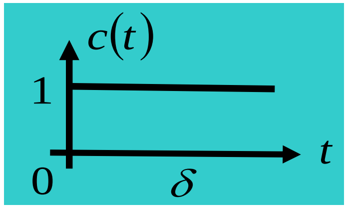
# Alternative Definition

The Brownian Motion is the limit of the continuous-time random walk (CTRW)

Definition of CTRW:

Let $\{B_i\}$ be the same iid sequence used to define the discrete-time random walk.

Let $c(t)$ be the unit step function.

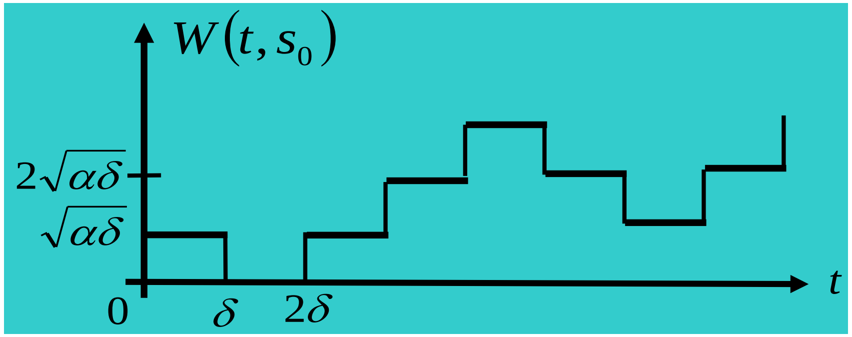



Then the CTRW is:

$$W(t) = \begin{cases} 0 & t < 0 \\ \sum_{i=1}^{\infty} \sqrt{\alpha\delta}\, B_i\, c(t - i\delta) & t \geq 0 \end{cases}$$

0

$0 < t \leq \delta$



# Statistics of the CTRW

$$W(t)=\begin{cases} 0 & t<0 \\ \sum_{i=1}^{\infty}\sqrt{\alpha\delta}B_i c(t-i\delta) & t\geq 0 \end{cases}$$

$$E[W(t)] = 0$$

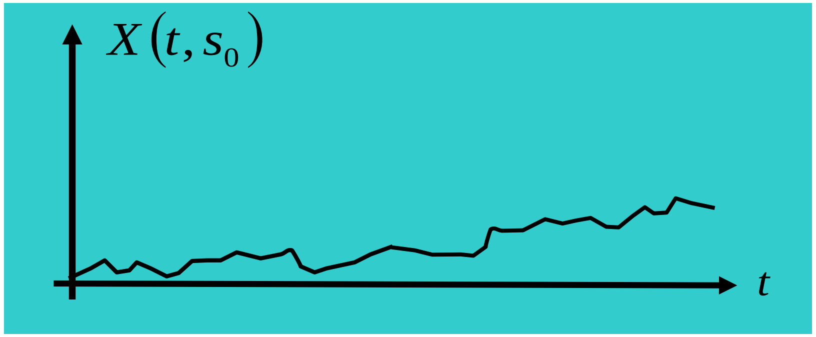
$$Var[W(t)] = \begin{cases} \alpha\delta m & m\delta \leq t < (m+1)\delta,\ m>0 \\ 0 & t<0 \end{cases}$$

Now let $\delta \to 0$. Both distance-step and time-step shrink together in a certain way: $\delta m \to t$

$$X(t)=\lim_{\delta\to 0}W(t) \text{ and } \mathrm{Var}(X(t))=\sigma_X^2(t)=\alpha t$$

# Limit of the CTRW: Wiener Process

$$X(t) = \lim_{\delta \to 0} W(t)$$




- $X(t)$ is the Wiener Process, or one-dimensional Brownian motion
- Now any interval $[0,t]$, $t>0$, has an infinite number of steps in it
- By the Central Limit Theorem, $X(t)$ is Gaussian: $\mathcal{N}(0, \alpha t)$
- $\alpha$ is called the diffusion constant
- $X(t)$ is continuous everywhere, but nowhere differentiable

# Wiener Process Autocorrelation

Suppose $0 < t_1 < t_2$

$$\begin{aligned} R_X(t_1, t_2) &= E\{X(t_1)X(t_2)\} \\ &= E\{X(t_1)[X(t_1) + (X(t_2) - X(t_1))]\} \\ &= E\{X^2(t_1)\} + E\{X(t_1)[(X(t_2) - X(t_1))]\} \\ &= \sigma_X^2(t_1) = \alpha t_1 \end{aligned}$$

Independent increments

In general,

$$R_X(t_1, t_2) = \alpha \min(t_1, t_2) \quad \text{for} \begin{cases} 0 < t_1 \\ 0 < t_2 \end{cases}$$

# Example: Derivative of Wiener Process

Let $X(t)$ be a Wiener process such that

$$R_X(t_1,t_2)=\sigma^2\min(t_1,t_2), \quad \begin{matrix} t_1>0 \\ t_2>0 \end{matrix}$$

Observations: $E\left[X(t)\right]=0, \quad X(t)=0, \text{ for } t\le 0$

and $X(t)$ is Normal for $t>0$.

Let

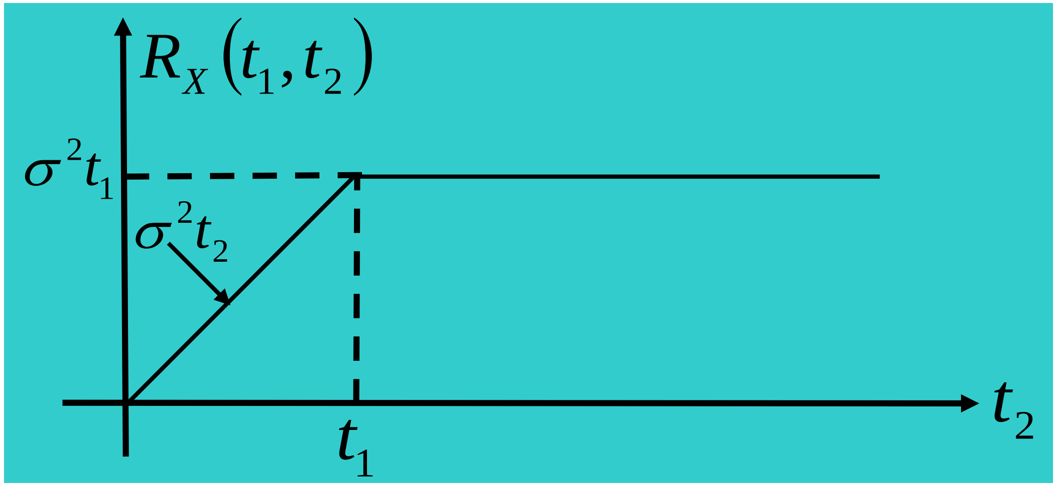
$$Y(t)=L_t\left[X(t)\right]=\frac{dX(t)}{dt}$$

$$m_Y(t)=L_t\left[m_X(t)\right]=0$$

$$R_{XY}(t_1,t_2)=L_{t_2}\left[R_X(t_1,t_2)\right]$$

# Derivative of Wiener Process

View $R_X(t_1,t_2)$ as a function of $t_2$ with $t_1$ fixed.

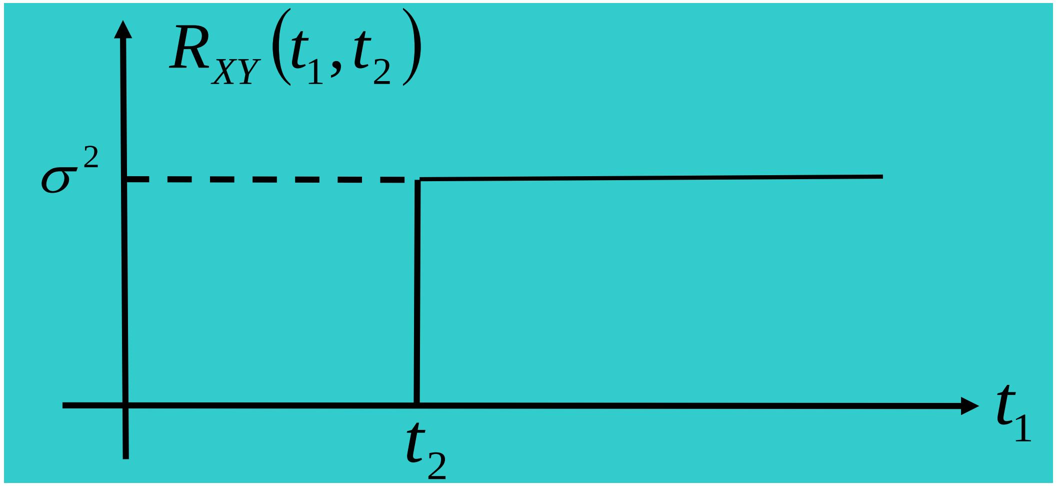



$$R_{XY}(t_1,t_2) = \frac{dR_X(t_1,t_2)}{dt_2} = \begin{cases} \sigma^2 & t_2 < t_1 \\ 0 & ow \end{cases}$$

# Derivative of Wiener Process

$$R_{YY}(t_1,t_2)=\frac{d}{dt_1}R_{XY}(t_1,t_2)$$

Now view $R_{XY}(t_1,t_2)$ as a function of $t_1$ with $t_2$ fixed.




$$R_{YY}(t_1,t_2)=\sigma^2\delta(t_1-t_2)$$

# Short Summary

- The Wiener Process is one of the most fundamental RPs
- The derivative of a Wiener process $Y(t) = \dfrac{dX(t)}{dt}$ is white noise with

$$E[Y(t)] = 0$$

$$R_{YY}(t_1, t_2) = \sigma^2 \delta(t_1 - t_2), \quad 0 < t_1, \quad 0 < t_2$$

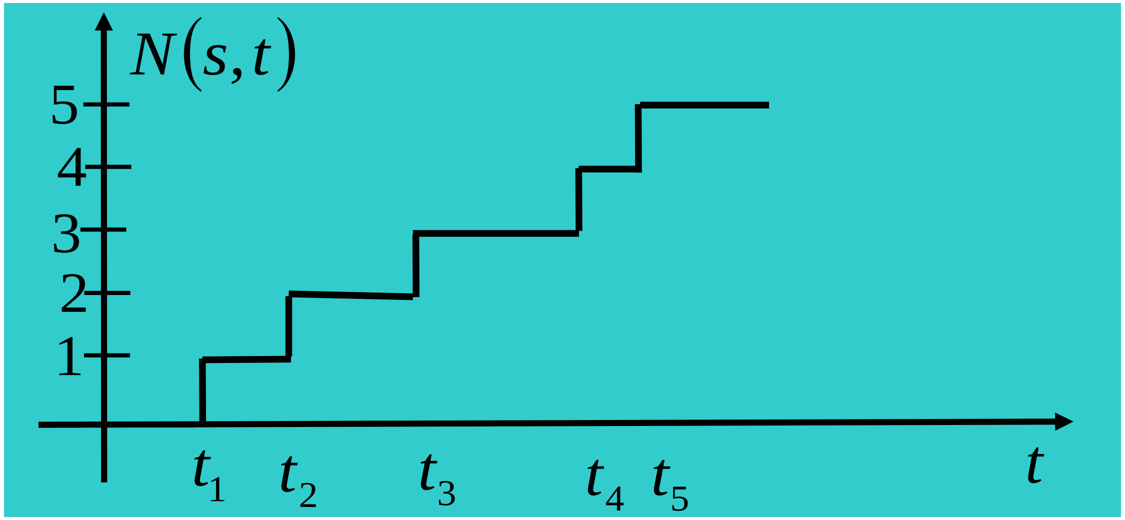
- It can be shown that any integral of Y(t) is Gaussian, therefore we call Y(t) Gaussian White Noise (GWN)

# Poisson Process

# The Poisson Counting Process

The Poisson counting process, N(t), counts the number of times a specified event occurs during the time from 0 to t. Thus, each sample function is a non-decreasing step function.

**Ex:**




$\{t_1, t_2, \ldots\}$ are called the Poisson Points

# Postulates

Karlin and Taylor, A First Course in Stochastic Processes

The following postulates define the process:

1. N(t) has independent increments.
2. N(t) has stationary increments.
3. $P(N(h) \geq 1) = \lambda h + o(h)$ $h \to 0$
4. $P(N(h) \geq 2) = o(h)$ $h \to 0$

h : a significantly small interval.

*Interpretation*: the probability of two or more changes in a sufficiently small interval is essentially 0

$$g(t) = o(t),\ t \to 0$$

is the usual way of writing $\lim_{t \to 0} \frac{g(t)}{t} = 0$

# Postulate 4

4. $P(N(h) \geq 2) = o(h)$

Postulate 4 implies that the events do not happen simultaneously

This is reasonable for:

1. Breakdowns of a machine
2. Customer arrivals
3. Photoelectron generation in an optical detector

# Homogenous Poisson Process

Based on these postulates, one can show that

$$P\left(N\left(t+\Delta\right)-N\left(t\right)=m\right)=\frac{\Lambda^{m}}{m!}e^{-\Lambda}$$

That is, the increments are Poisson-distributed with parameter $\Lambda = \lambda\Delta$, where

$\lambda$ is the average rate of occurrence

$\Delta$ the length of the interval of observation

$\lambda$ being not a function of $t$ makes $N(t)$ a homogenous Poisson Process

# Volume Poisson Process

A homogenous Poisson Process may also describe random points in space, for example, locations of stars in the galaxy, or point defects in a solid material.

For example, let V be a volume in $\mathcal{R}^3$, and N(V) = the number of points in V

$$P(N(V)=m)=\frac{(\lambda V)^m}{m!}e^{-\lambda V}, \quad \lambda>0$$

# Standard Poisson Process

A Standard Poisson Process allows the rate, $\lambda(t)$, to vary with time

$$P\big(N(t+\Delta)-N(t)=m\big)=\frac{\left(\int_t^{t+\Delta}\lambda(u)du\right)^m}{m!}e^{-\int_t^{t+\Delta}\lambda(u)du}$$

This process has independent but not stationary increments

A homogenous Poisson Process is a special case of the Standard Poisson Process

# Joint PMF

Consider the joint PMF of the homogenous Poisson Process. Let $t_1 < t_2, i < j$

$$p_{N(t_1)N(t_2)}(i, j) = P\{N(t_1) = i \cap N(t_2) = j\}$$

Write $N(t_2)$ as a sum of independent increments

$$N(t_2) = [N(t_2) - N(t_1)] + N(t_1)$$

Then, we can write

$$\{N(t_1) = i \cap N(t_2) = j\}$$
$$= \{N(t_1) = i\} \cap \{[N(t_2) - N(t_1)] = j - i\}$$

# Joint PMF, cont'd

$$\{N(t_1)=i \cap N(t_2)=j\}$$
$$=\{N(t_1)=i\} \cap \{[N(t_2)-N(t_1)]=j-i\}$$

Now exploit the independent increments

$$p_{N(t_1)N(t_2)}(i,j)=P\{N(t_1)=i\}\,P\{N(t_2)-N(t_1)=j-i\}$$

And finally exploit the stationary increments

$$p_{N(t_1)N(t_2)}(i,j)=P\{N(t_1)=i\}\,P\{N(t_2-t_1)=j-i\}$$

# Joint PMF, Final Expression

$$\therefore p_{N(t_1)N(t_2)}(i,j)=\left[\frac{(\lambda t_1)^i}{i!}e^{-\lambda t_1}\right]\frac{\left(\lambda[t_2-t_1]\right)^{(j-i)}}{(j-i)!}e^{-\lambda[t_2-t_1]}$$

# Mean, Correlation, and Covariance

$E[N_t] = \lambda t$ and $var(N_t) = \lambda t$.

$E[N_t N_s] = (\lambda t)(\lambda s) + \lambda t$ for $t < s$

$$\begin{aligned} cov(N_t, N_s) &= E[(N_t - \lambda t)(N_s - \lambda s)] \\ &= E[N_t N_s] - (\lambda t)(\lambda s) \\ &= \lambda t. \end{aligned}$$

# Example

$\frac{1}{4}$

Suppose fish bite with a Poisson distribution, with an average rate of one per 20 min. What is the probability that at least one fish will bite in the next 5 min given that no fish has bitten in the last 20 min?

$$P\left(N(t+5)-N(t)\geq 1 \mid N(t)-N(t-20)=0\right)$$

$$=P\left(N(t+5)-N(t)\geq 1\right)=1-\frac{(5/20)^0}{0!}e^{-5/20}=0.22$$

Because of independent increments, there is "No premium for waiting"

# Poisson Inter-arrival Times

Let $T = t_2 - t_1$, where $t_1$ and $t_2$ are two consecutive event times. Characterize $T$.

$$F_T(t) = P(T \le t) = 1 - P\left(N(t_1 + t) - N(t_1) = 0\right)$$

$$= 1 - P\left(N(t) = 0\right)$$

Because of stationary increments

$$= 1 - e^{-\lambda t} \quad t \ge 0$$

$$f_T(t) = \frac{d}{dt} F_T(t) = \begin{cases} \lambda e^{-\lambda t} & t \ge 0 \\ 0 & t < 0 \end{cases}$$

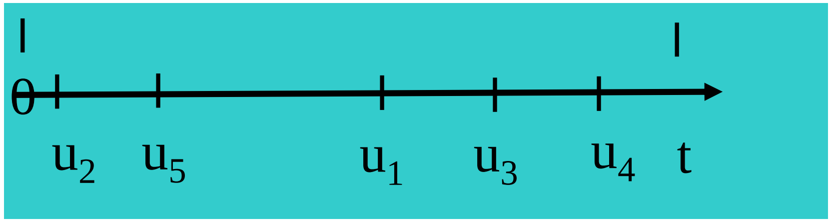
$T$ is exponential with parameter $\lambda$

# Random Points Interpretation

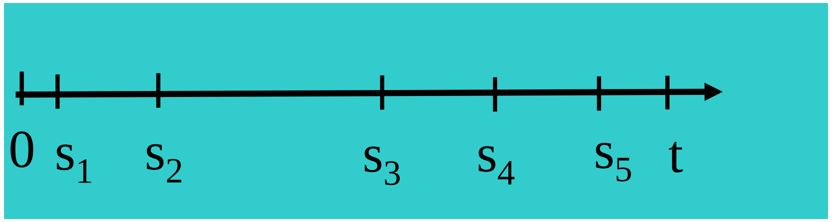
Given $N(t) = k$, the $k$ event times are equivalent to $k$ independent RVs that are uniformly distributed over $[0, t]$, given that they are ordered.

In other words, let $u_1, u_2, \ldots, u_k$ be $k$ independent trials of a RV uniformly distributed over $[0, t]$.

**Ex:** $k = 5$




Then, define $s_1, s_2, \ldots, s_k$ to be the ordered version of the u's.



# Example

Suppose two customers arrive at a shop during a two-minute period. Find the probability that one arrived in the first minute and the other arrived in the second minute.

Poisson approach:

$$P\left(N(1)=1 \mid N(2)=2\right)=\frac{P\left(N(1)=1 \cap N(2)=2\right)}{P\left(N(2)=2\right)}$$

$$=\frac{P\left(N(1)=1 \cap N(2)-N(1)=1\right)}{P\left(N(2)=2\right)}$$

# Poisson Approach – Cont.

From previous slide,

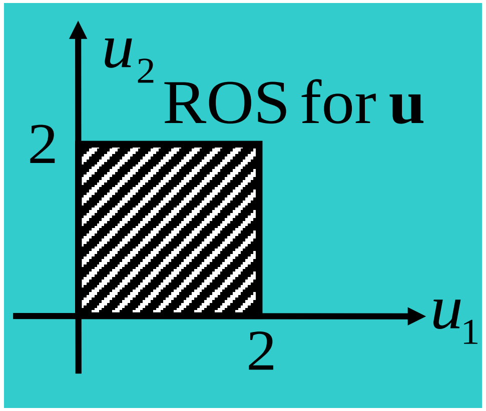
$$P\left(N(1)=1 \mid N(2)=2\right)=\frac{P\left(N(1)=1 \cap N(2)-N(1)=1\right)}{P\left(N(2)=2\right)}$$

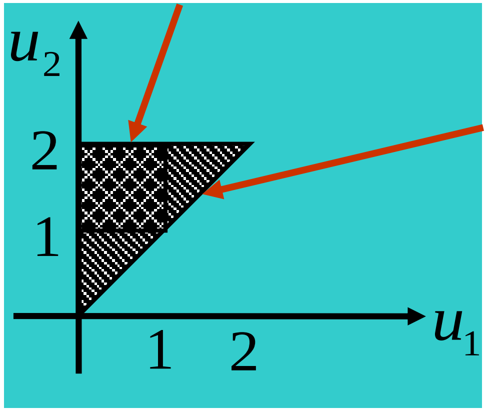
Using increment properties,

$P(N(2)-N(1)=1)$
$= P(N(1)-N(0)=1)$, with $N(0)=0$

$$P\left(N(1)=1 \mid N(2)=2\right)=\frac{\left[P\left(N(1)=1\right)\right]^2}{P\left(N(2)=2\right)}=\frac{\left[\frac{(\lambda\cdot 1)^1}{1!}e^{-\lambda\cdot 1}\right]^2}{\left[\frac{(\lambda\cdot 2)^2}{2!}e^{-\lambda\cdot 2}\right]}=\frac{1}{2}$$

# Ordered Uniform RV Approach




$$P\left(N(1)=1 \mid N(2)=2\right)=P\left(u_1 \le 1, u_2 \ge 1 \mid u_1 \le u_2\right)$$

$$=\frac{P\left(u_1 \le 1, u_2 \ge 1 \cap u_1 \le u_2\right)}{P\left(u_1 \le u_2\right)}=\frac{1}{2}$$

# Alternative Uniform RV Approach

Must count all permutations

$$
\begin{aligned}
&P\left(N(1)=1 \mid N(2)=2\right) \\
&\quad = P\left(\left\{u_1 \le 1 \cap u_2 \ge 1\right\} \cup \left\{u_2 \le 1 \cap u_1 \ge 1\right\}\right) \\
&\quad = P\left\{u_1 \le 1 \cap u_2 \ge 1\right\} + P\left\{u_2 \le 1 \cap u_1 \ge 1\right\} \\
&\quad = 1/4 + 1/4 = 1/2
\end{aligned}
$$

# Poisson Process Summary

Definition of Poisson Process

- Postulates
- Homogeneous
- Standard

Inter-arrival Times are Exponential

Random Points

# Comparison (1)

Similarities of Wiener and Poisson processes:

- Stationary and independent increments
- Variance are scalar of time
- Initial is 0

# Comparison (2)

Differences between Wiener and Poisson processes:

- Continuous and discrete
- Gaussian and Poisson
- Mean: 0 and $\lambda t$

# Thank You!

# quiz

- Micrometeors strike the space shuttle according to a Poisson process. The expected time between strikes is 30 minutes. Find the probability that during at least one hour out of five consecutive hours, three or more micrometeors strike the shuttle.

# Solution

***Solution.*** The problem statement is telling us that the expected interarrival time is 30 minutes. Since the interarrival times are $\exp(\lambda)$ random variables, their mean is $1/\lambda$. Thus, $1/\lambda = 30$ minutes, or 0.5 hours, and so $\lambda = 2$ strikes per hour. The number of strikes during the $i$th hour is $N_i - N_{i-1}$. The probability that during at least 1 hour out of five consecutive hours, three or more micrometeors strike the shuttle is

$$\begin{aligned}
\mathsf{P}\left(\bigcup_{i=1}^{5}\{N_i - N_{i-1} \geq 3\}\right) &= 1 - \mathsf{P}\left(\bigcap_{i=1}^{5}\{N_i - N_{i-1} < 3\}\right) \\
&= 1 - \prod_{i=1}^{5}\mathsf{P}(N_i - N_{i-1} \leq 2),
\end{aligned}$$

where the last step follows by the independent increments property of the Poisson process. Since $N_i - N_{i-1} \sim \text{Poisson}(\lambda[i-(i-1)])$, or simply Poisson$(\lambda)$,

$$\mathsf{P}(N_i - N_{i-1} \leq 2) = e^{-\lambda}(1 + \lambda + \lambda^2/2) = 5e^{-2},$$

and we have

$$\mathsf{P}\left(\bigcup_{i=1}^{5}\{N_i - N_{i-1} \geq 3\}\right) = 1 - (5e^{-2})^5 \approx 0.86.$$