Probability and Random Process

Aimin Tang

The University of Michigan- Shanghai Jiao Tong University Joint Institute
Shanghai Jiao Tong University

Sep. 14 2021

Ve501 2021-2022 Fall

Based on Lecture Notes by Prof. Yong Long

Outline

- 1. Introduction to Probability
  - Application example
  - Review of set and functions
  - Models of random experiments
  - Axioms and properties of probability
  - Conditional probability
  - Independence of events
  - Combinatorics and probability

Application Areas

- Application areas of probability and random processes
  - Signal processing
  - Communications
  - Control
  - Industrial engineering
  - Economics
  - Aerospace
  - Information science
  - Computer science
  - …

Example: Signal Processing

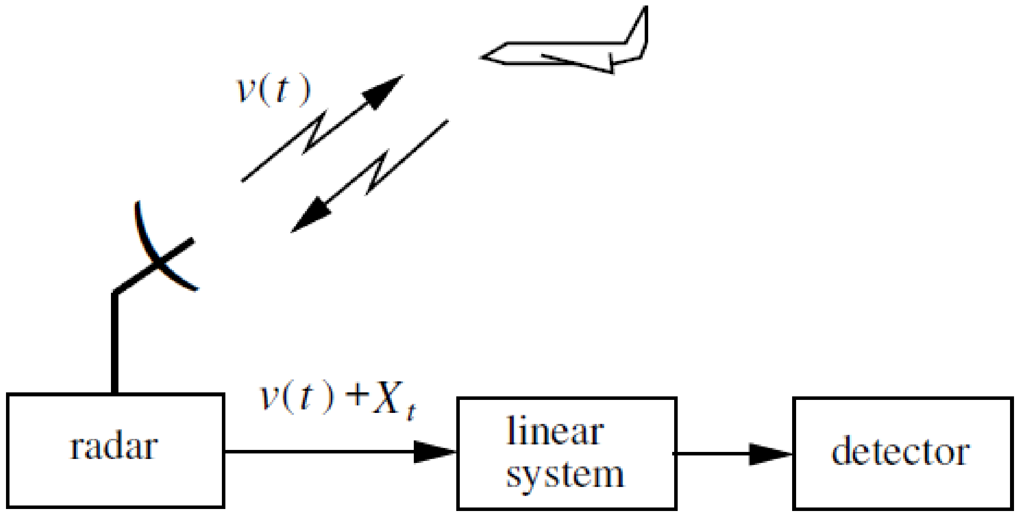


Figure **1.1.** Block diagram of radar detection system.

- To determine the presence of an aircraft, a known radar pulse $v(t)$ is sent out.
- The overall goal is to decide whether the received waveform is noise only or signal plus noise.
  - No object in range of radar, noise waveform only X_t.
  - An object in range, reflected radar pulse plus noise $v(t) + X_t$.

Example: Signal Processing

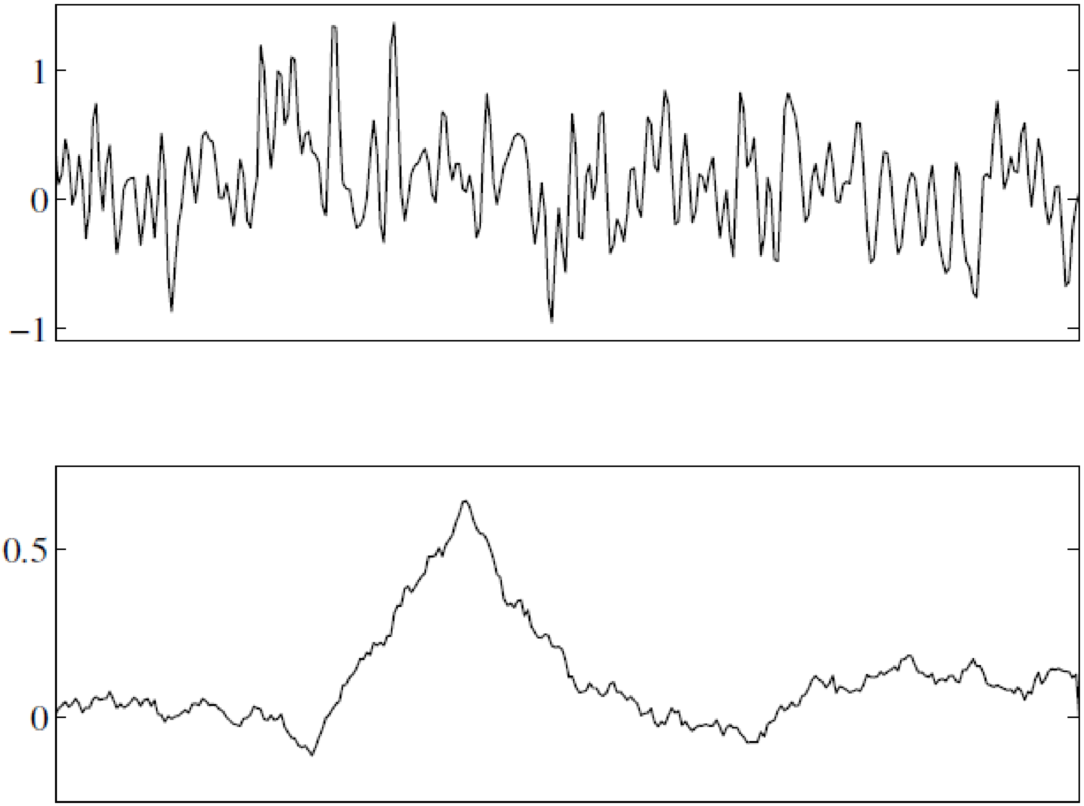


Figure 1.2. Matched filter input (top) in which the signal is hidden by noise. Matched filter output (bottom) in which the signal presence is obvious.

Describe Uncertainty

- How to describe/capture uncertainty in the behavior of engineering systems?
- What type of calculus does one develop to quantify uncertainty and show how uncertainty propagates through time?
- One way is through probability theory, random variables, and random processes.

Review of set and functions

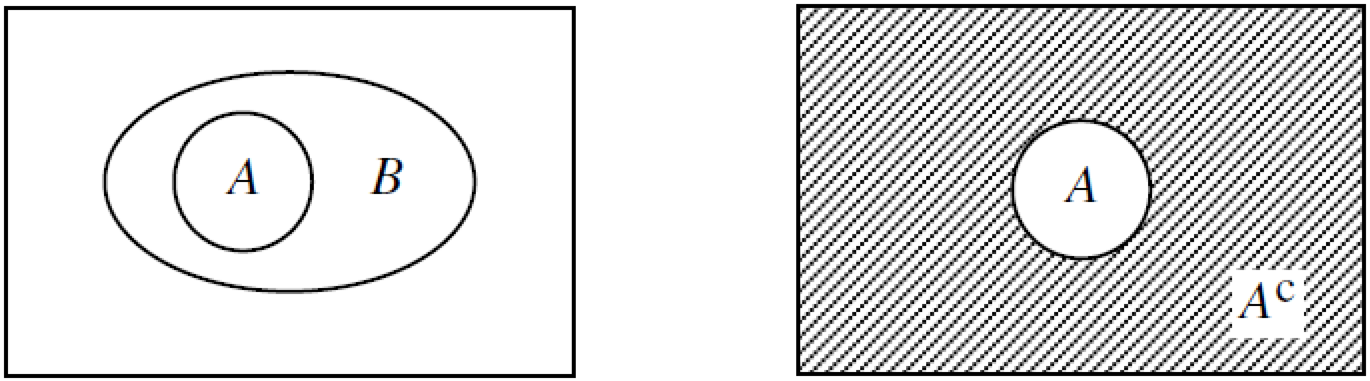
Set Definition and Representations

- A set is a collection of objects called elements or members of the set.
- Methods of specifying a set
  1. list them in curly brackets separately by commas $\{a, b, t, \ldots\}$
  2. predicate: $\{\text{real number } X : 0 \leq X \leq 1\}$ (colon means such that)
  3. intervals of the real line

     $$[a, b) = \{\text{real number } x : a \leq x < b\}$$

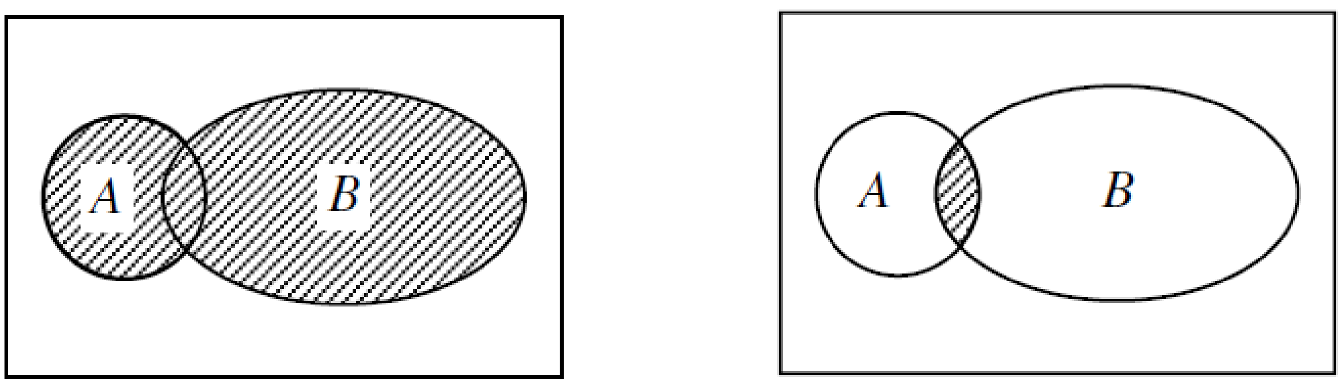
     $$(a, b), (a, b], [a, b], (a < b)$$
  4. in terms of other sets: $A = B \cup C$
  5. Venn diagrams (Picture)

Venn Diagram



(a) (b)

Figure **1.7.** (a) Venn diagram of $A \subset B$. (b) The complement of the disk A, denoted by A^c, is the shaded part of the diagram.



(a) (b)

Figure **1.8.** (a) The shaded region is $A \cup B$. (b) The shaded region is $A \cap B$.

Set Notations

- Let Ω be a set of points. Let A and B be two collections of points in Ω.
  - element/member/point of a set

    $\omega \in \Omega$, ω is an element of Ω, Ω contains ω
  - subset
    - $A \subset B$. A is contained in B (A is a subset of B). Every element of A is also in B.
    - $A \supset B$ (superset). B is a subset of A (A contains B)
  - equality

    $A = B$. A equals B. A and B have the same elements.

    $A \subset B$ and $A \supset B$ is a way to prove $A = B$

    $A \subset A$
  - proper subset
    - If $A \subset B$ but $A \neq B$, we say that A is a proper subset of B

Set Operations: Complement

- If $A \subset \Omega$ and $\omega \in \Omega$ does not belong to A, we write $\omega \notin A$. The set of all such ω is called the **complement** of A, i.e. $A^c = \{\omega \in \Omega: \omega \notin A\}$
- The empty set or null set contains no points in Ω. It is denoted $\emptyset$
  - for any $A \subset \Omega$, $\emptyset \subset A$
  - $\Omega^c = \emptyset$

Set Operations: Union

- The **union** of two subsets A and B is

$$A \cup B \triangleq \{\omega \in \Omega\colon\ \omega \in A \text{ or } \omega \in B\}$$

It is a set contains all elements of A and all elements of B.

  – Here "or" is inclusive; i.e., if $\omega \in A \cup B$, we permit ω to belong either to A or to B or to both.

Set Operations: Infinite Union

- Suppose $A_i \subset \Omega, i = 1,2, \ldots$. Then the **infinite union** is

$$\bigcup_{i=1}^{\infty} A_i \triangleq \{\omega \in \Omega, \omega \in A_i \text{ for some } 1 \leq i < \infty\}$$

$\omega \in \bigcup_{i=1}^{\infty} A_i$ iff for at least one integer i satisfying $1 \leq i < \infty$, $\omega \in A_i$.

  - This definition admits the possibility that $\omega \in A_i$ for more than one value of i.

Set Operations: Intersection

- The **intersection** of two subsets A and B is
$$A \cap B \triangleq \{\omega \in \Omega\colon\ \omega \in A \text{ and } \omega \in B\}$$
$\omega \in A \cap B$ iff ω belongs to both A and B.

- Suppose $A_i \subset \Omega, i = 1,2, \ldots$. Then the **infinite intersection** is
$$\bigcap_{i=1}^{\infty} A_i \triangleq \{\omega \in \Omega, \omega \in A_i \text{ for all } 1 \le i < \infty\}$$
$\omega \in \bigcap_{i=1}^{\infty} A_i$ iff for every integer i satisfying $1 \le i < \infty$, $\omega \in A_i$.

Example

- $\bigcap_{n=1}^{\infty}(0,\frac{1}{n}) =?$
- $\bigcup_{n=1}^{\infty}\left(\frac{1}{n},2\right] =?$

- $\bigcap_{n=1}^{\infty}(0,\frac{1}{n}) = \emptyset$
- $\bigcup_{n=1}^{\infty}\left(\frac{1}{n},2\right] = (0,2]$

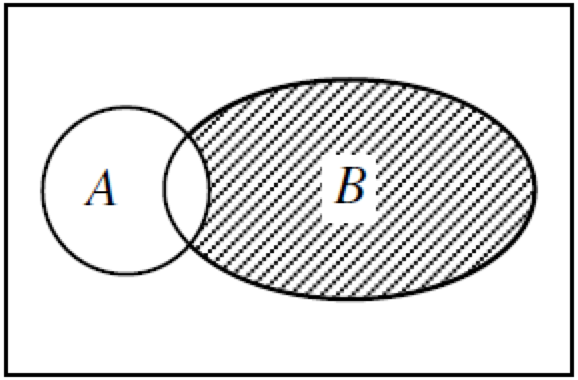
Set Operations: Difference

- The **difference** of two subsets A and B is

$$B \backslash \mathrm{A} \triangleq B - A \triangleq B \cap A^c = \{\omega \in \Omega: \omega \in B \text{ and } \omega \notin A\}$$

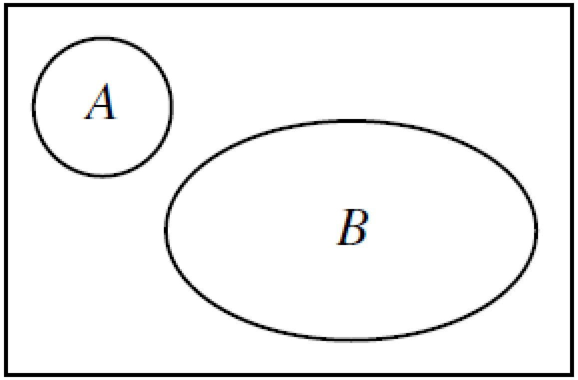
$B \cap A^c$ is a set ω ∈ B that do not A.

- $B \backslash \mathrm{A}$ is found by starting with all the points in B and then removing those that belong to A.



Set Operations: Disjoint

- Two subsets A and B are **disjoint** or **mutually exclusive** if $A \cap B = \emptyset$, i.e., there is not point in Ω that belongs to both A and B.



- Subsets $A_i \subset \Omega, i = 1,2, \ldots$ are **pairwise disjoint** if

$A_i \cap A_j = \emptyset$, for all $i \neq j$.

Set Identities

Let A, B and C be subsets of Ω.

- communicative law

$$A \cup B = B \cup A, \qquad A \cap B = B \cap A$$

- associative law

$$A \cap (B \cap C) = (A \cap B) \cap C$$
$$A \cup (B \cup C) = (A \cup B) \cup C$$

- distributive law

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Set Identities

- generalized distributive law

$$B \cap \left(\bigcup_{i=1}^{\infty} A_i \right) = \bigcup_{i=1}^{\infty} (B \cap A_i), B \cup \left(\bigcap_{i=1}^{\infty} A_i \right) = \bigcap_{i=1}^{\infty} (B \cup A_i)$$

- De Morgan's law

$$(A \cup B)^c = (A^c \cap B^c), \qquad (A \cap B)^c = (A^c \cup B^c)$$

- generalized De Morgan's law

$$\left(\bigcap_{i=1}^{\infty} A_i \right)^c = \bigcup_{i=1}^{\infty} A_i^c, \qquad \left(\bigcup_{i=1}^{\infty} A_i \right)^c = \bigcap_{i=1}^{\infty} A_i^c$$

Countable Sets

- **Set size/cardinality** is the number of elements in a set A, denoted by $|A|$.
  - finite: countably finite
  - infinite: countably/uncountably infinite
- A set A is said to be **countable** iff it is either **finite**, or its elements can be **enumerated** or listed in a sequence: a_1, a_2, ..., i.e., A can be written in the form

$$A = \bigcup_{k=1}^{\infty} \{a_k\}$$

  - In other words, there is a **one-to-one** correspondence between elements of the set and **positive integers**

Uncountable Sets

- A set is uncountably infinite if its cardinality is infinite but not countably.

- Example:
  - Real number **R**
  - The interval of real number [0, 1)

Example

- Which of the following sets are countable? Enumerate the countable sets.
  1. $\{1,3,5,7,9,\ldots\}$
  2. $\{\ldots,-2,-1,0,1,2,\ldots\}$
  3. {positive rational numbers X: $X=m/n$, m, n are integers, $n \neq 0$}
  4. $B \cup C$, where $B = \{b_1, b_2, b_3, \ldots\}$, $C = \{c_1, c_2, c_3, \ldots\}$

Solution(1)

$$B = \{1, 3, 5, 7, 9, \ldots\}$$

$$a_1 \triangleq 1, \quad a_2 \triangleq 3, \quad a_3 \triangleq 5$$

$$a_4 \triangleq 7, \quad a_5 \triangleq 9, \quad a_6 \triangleq 11$$

$$\ldots$$

This shows

$$B = \bigcup_{k=1}^{\infty} \{a_k\}$$

and so B is a countable set.

Solution(2)

$$B = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$$

$$a_1 \triangleq 0, \quad a_2 \triangleq 1, \quad a_3 = -1,$$

$$a_4 \triangleq 2, \quad a_5 \triangleq -2, \quad a_6 = 3,$$

$$\cdots$$

This shows

$$B = \bigcup_{k=1}^{\infty} \{a_k\}$$

and so B is a countable set.

Solution(3)

$$B = \left\{ \text{positive rational numbers } X : X = \frac{m}{n}, m, n \text{ are inegers}, n \neq 0 \right\}$$

Rewrite the set as

$$B = \bigcup_{m,n=1}^{\infty} \{b_{mn}\}, \quad b_{mn} \triangleq \frac{m}{n}$$

The point here is that a sequence that is doubly indexed by positive integers forms a countable set. To see this, consider the array

Solution(3)

$$\begin{array}{ccccc} b_{11} & b_{12} & b_{13} & b_{14} & \cdots \\ b_{21} & b_{22} & b_{23} & b_{24} & \cdots \\ b_{31} & b_{32} & b_{33} & b_{34} & \cdots \\ b_{41} & b_{42} & b_{43} & b_{44} & \cdots \\ \vdots & & & & \end{array}$$

$$\begin{array}{cccccc} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & \cdots \\ b_{21} & \cancel{b_{22}} & b_{23} & \cancel{b_{24}} & b_{25} & \cdots \\ b_{31} & b_{32} & \cancel{b_{33}} & b_{34} & b_{35} & \cdots \\ b_{41} & \cancel{b_{42}} & b_{43} & \cancel{b_{44}} & b_{45} & \cdots \\ b_{51} & b_{52} & b_{53} & b_{54} & \cancel{b_{55}} & \cdots \\ \vdots & & & & & \end{array}$$

Solution(3)

Now list the array elements along antidiagonals from lower left to upper right, skip equivalent rational numbers (fractions) and define

$$
\begin{array}{lllll}
a_1 = b_{11} & & & & \\
a_2 = b_{21} & a_3 = b_{12} & & & \\
a_4 = b_{31} & \cancel{b_{22}} & a_5 = b_{31} & & \\
a_6 = b_{41} & a_7 = b_{32} & a_8 = b_{23} & a_9 = b_{14} & \\
a_{10} = b_{51} & \cancel{b_{42}} & \cancel{b_{33}} & \cancel{b_{24}} & a_{11} = b_{15} \\
\vdots & & & &
\end{array}
$$

This shows

$$B = \bigcup_{k=1}^{\infty} \{a_k\}$$

and so B is a countable set.

Solution(4)

$B \cup C$, *where* $B = \{b_1, b_2, b_3, b_4, \ldots\}$, $\quad C = \{c_1, c_2, c_3, c_4, \ldots\}$

Both B and C are countable. We must show that $A = B \cup C$ *is countable.*

1. *B and C are disjoint.*
2. *B and C are not disjoint.*

Solution(4)

- If B and C are **disjoint**, we can write
  $B \cup C = \{b_1, c_1, b_2, c_2, b_3, c_3, \ldots\} = \{a_1, a_2, a_3, a_4, \ldots\}$
  - Where $a_{2k} = c_k, a_{2k-1} = b_k$

- We have now established that the elements of A = B ∪ C can be indexed by the positive integers, which implies that B ∪ C is countable

Solution(4)

B and C are not disjoint. Then

$$B \cup C = B \cup C', \quad \text{where } C' \triangleq C - B.$$

B and C' are disjoint and C' is countable (since it is a subset of C).

1. When C' is countably infinite, we can apply the previous case to deduce that $B \cup C'$ is countable.
2. When C' is finite, i.e., $|C'| = N < \infty$, we can count the elements in C' first, and then count the elements in B.

$$\begin{aligned} a_k &= c'_k, \quad k = 1, \ldots, N \\ a_{N+l} &= b_l, \quad l = 1, 2, \ldots \end{aligned}$$

Since $B \cup C = B \cup C'$, we conclude that $B \cup C$ is countable.

Solution(4)

- Since it is a pretty obvious statement, it is not necessary to prove the statement that a subset of countable set is countable
- But for completeness, here's a proof

- Let $C = \{c_1, c_2, \ldots\}$. Then index the members of C' by
  1. Finding the smallest k such that $c_k \in C'$ and call that c_1'
  2. Find the next smallest k s.t. $c_k \in C'$ and call that c_2'
  3. And so on
- This leads to an indexing of C', which demonstrates that C' is countable.

Set Operations

- **empty set $\emptyset$**
  - $\emptyset \subset A, A \cup \emptyset = A, A \cap \emptyset = \emptyset$
  - $A \cap B = \emptyset$ iff A, B are disjoint.
- **singleton**
  - $\{X\} =$ singleton set containing only X

- **power set 2^A** of a set A is a set of all subsets of A.
- Example: $2^{\{1,2,3\}} = ?$
  - $\{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\},\emptyset\}$
- cardinality: $|2^A| = 2^{|A|}$

Partition and Cartesian Product

- **Partition** of a set A is a set of sets (called cells or atoms of the partition) $\{B_1, B_2, \ldots\}$ s.t. (such that) B_i 's are disjoint and their union is A: $\bigcup_{i=1}^{\infty} B_i = A$
- **Cartesian product**: $A \times B = \{(X, Y) : X \in A, Y \in B\}$
- Example:
  $[0, 1] \times [2, 3] = \{(X, Y) : 0 \leq X \leq 1, 2 \leq Y \leq 3\}$

Function

- A function consists of a set X of inputs called the domain and a rule or mapping f that associates to each $x \in X$ a value $f(x)$ that belongs to a set Y called the co-domain.

We write

$$f : X \to Y$$

and say that f maps X into Y.

Range and Co-domain

- The set of all possible values of $f(x)$ is called the **range**. It is the set $\{f(x) : x \in X\}$.

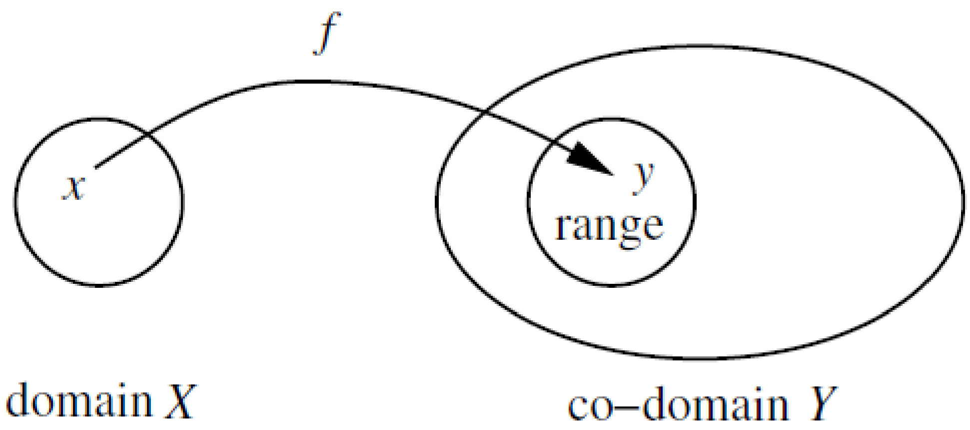
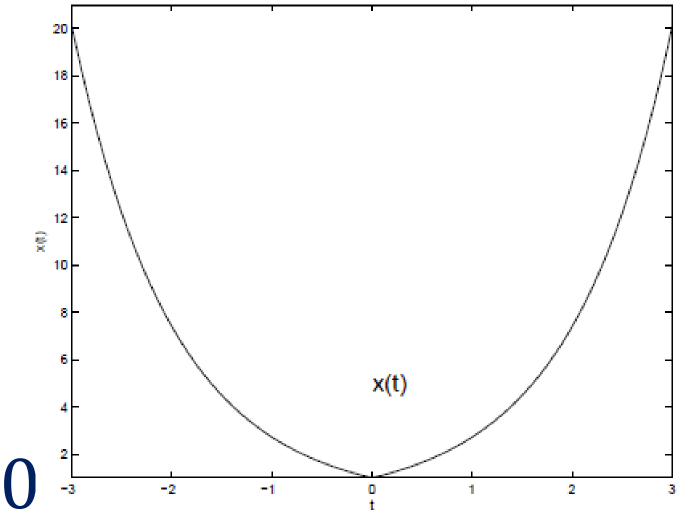


Figure **1.11.** The mapping f associates each x in the domain X to a point y in the co-domain Y. The range is the subset of Y consisting of those y that are associated by f to at least one $x \in X$. In general, the range is a proper subset of the co-domain.

Describing A Function

- Graphically :



- Braces or piecewise notation:

$$x(t) = \begin{cases} e^{-t}, t \geq 0 \\ e^{t}, t < 0 \end{cases}$$

- Formula: $x(t) = e^{-|t|}$
- In terms of other functions: $x(t) = s(t) + s(-t)$ where

$$s(t) = \begin{cases} e^{-t}, t > 0 \\ 1/2, t = 0 \\ 0, t < 0 \end{cases}$$

Function Classes

- $f(X)$ is **one-to-one** if $f(X_1) \neq f(X_2)$ where $X_1, X_2 \in A$ and $X_1 \neq X_2$

- $f(X)$ is **onto** if its range equal to its co-domain $f(X) = Y$

- $f(X)$ is **invertible** if it is one-to-one and onto, *i.e.*, for every $y \in Y$ there is a unique $x \in X$ with $f(x) = y$

Image and Inverse Image

- If $f: X \to Y$ and if $A \subset X$, then the **image** of A is
$$f(A) = \{f(x) : x \in A\}$$

- If $f: X \to Y$ and if $B \subset Y$, then the **inverse image** of B is
$$f^{-1}(B) = \{x \in X : f(x) \in B\}$$

- This concept applies to any function whether or not it is invertible

Monotonicity

- non-decreasing: $f(X_2) \geq f(X_1)$ whenever $X_2 > X_1$
- strictly increasing: $f(X_2) > f(X_1)$whenever $X_2 > X_1$
- non-increasing: $f(X_2) \leq f(X_1)$ whenever $X_2 > X_1$
- strictly decreasing: $f(X_2) < f(X_1)$ whenever $X_2 > X_1$

Continuity

- $f: X \to Y$ (X, Y are intervals of the real line)

- f is **continuous** if

$$x_n \to x \;\Rightarrow\; f(x_n) \to f(x)$$

- Equivalently, for all $\epsilon > 0$, there is a $\delta > 0$ $s.t.$

$|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$

Example

- For each of the following cases, determine if f is a **valid function** with domain A and co-domain B. For those that are valid functions, determine if they are **one-to-one, onto, continuous, monotonic** (if so, state the type of monotonicity), and find the **inverse image** of the set $(-0.1, 0.2)$

a) $A = [0,1], B = [-1,1], f(x)=\{y \in B: y^2 = x\}, \forall x \in A$

b) $A = [-1,1], B = [-\pi,\pi], f(x)=\{y \in B: \sin(y)= x\}, \forall x \in A$

c) $A = [0,1], B = [-1,1], f(x) = \begin{cases} 1, x \in [\frac{1}{4}, 1] \\ 0, \text{otherwise} \end{cases}, \forall x \in A$

Solution

(a) $\boxed{\textit{not a function}}$ – *except for* $x = 0$, *two values of* y *are assigned to each value of* x.

(b) $\boxed{\textit{not a function}}$ – *for example,* $x = \sqrt{2}/2$ *corresponds to two values of* y: $\pi/4$ *and* $3\pi/4$.

Solution

(c) $\boxed{\textit{a function}}$ – *because each x in A is associated with a single y that is in B.*

- $\boxed{\textit{not one-to-one}}$ *because for example,* $f(0.5) = f(0.6)$
- $\boxed{\textit{not onto}}$, *because for example, no value of x maps to* -1.
- $\boxed{\textit{not continuous}}$ – *for example* $f(1/4 - 1/n) \to 0 \neq f(1/4)$ *as* $n \to \infty$, *even though* $(1/4 - 1/n) \to 1/4$.
- $\boxed{\textit{monotonic nondecreasing}}$ – $f(x_2) \geq f(x_1)$ *whenever* $x_2 \neq x_1$.
- *inverse image* $f^{-1}((-0.1, 0.2)) = \boxed{[0, 1/4)}$.

Thank You!