ECE 4110/5110	Friday, 09/22/23
Homework Set 3(Due: Friday, 10/06/23, 11:59 pm)	
Dr. Kevin Tang	HW 3

1 Problems from Bertsekas and Tsitsiklis, 2nd edition - Chapter 7

1. Problem 2. A mouse moves along a tiled corridor with 2m tiles, where $m > 1$. From each tile $i \neq 1, 2m$. it moves to either tile i - l or i + 1 with equal probability. From tile 1 or tile 2m, it moves to tile 2 or 2m - 1, respectively, with probability 1. Each time the mouse moves to a tile $i \leq m$ or $i > m$, an electronic device outputs a signal L or R, respectively. Can the generated sequence of signals L and R be described as a Markov chain with states L and R?

Solution.
It cannot be described as a Markov chain with states L and R, because
$P(X_{n+1} = L|X_n = R, X_{n-1} = L) = 1/2$, while
$P(X_{n+1} = L|X_n = R, X_{n-1} = R, X_{n-2} = L) = 0$.

2. Problem 4. A spider and a fly move along a straight line in unit increments. The spider always moves towards the fly by one unit. The fly moves towards the spider by one unit with probability 0.3, moves away from the spider by one unit with probability 0.3, and stays in place with probability 0.4. The initial distance between the spider and the fly is integer. When the spider and the fly land in the same position, the spider captures the fly.

(a) Construct a Markov chain that describes the relative location of the spider and fly.

(b) Identify the transient and recurrent states.

Solution.
(a) We introduce a Markov chain with state equal to the distance between spider and fly. Let n be the initial distance. Then, the states are $0, 1, ..., n$, and we have

$$p_{00} = 1, p_{0i} = 0, i \neq 0$$
$$p_{10} = 0.4, p_{11} = 0.6, p_{1i} = 0, i \neq 0, 1$$

and for all $i \neq 0, 1$,

$$p_{i(i-2)} = 0.3, p_{i(i-1)} = 0.4, p_{ii} = 0.3,$$

(b) All states are transient except for state 0 which forms a recurrent class.

3. Problem 5. Consider a Markov chain with states 1 , 2, . . . , 9. and the following transition probabilities: $p_{12} = p_{17} = 1/2$, $p_{i(i+l)} = 1$ for $i \neq 1, 6, 9$, and $p_{61} = p_{91} = 1$. Is the recurrent class of the chain periodic or not?

Solution.
It is periodic with period 2. The two corresponding subsets are
{2, 4, 6, 7, 9} and {1, 3, 5, 8}.

4. Problem 11. A professor gives tests that are hard, medium, or easy. If she gives a hard test, next test will be either medium or easy, with equal probability. However, if she gives a medium or easy test, there is a 0.5 probability that her next test will be the same difficulty, and a 0.25 probability for of the other two levels of difficulty. Construct an appropriate Markov chain and find the steady-state probabilities.

Solution. We use a Markov chain model with 3 states, H, M, and E, where the state reflects the difficulty of the most recent exam. We are given the transition probabilities

$$\begin{bmatrix} r_{HH} & r_{HM} & r_{HE} \\ r_{MH} & r_{MM} & r_{ME} \\ r_{EH} & r_{EM} & r_{EE} \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.5 & 0.25 \\ 0.25 & 0.25 & 0.5 \end{bmatrix}$$

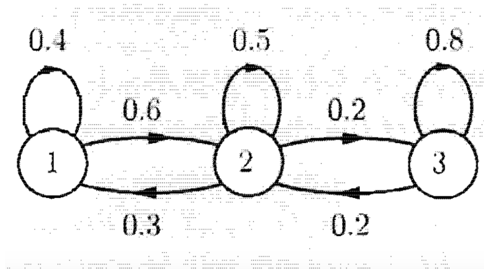
It is easy to see that our Markov chain has a single recurrent class, which is aperiodic. The balance equations take the form

$$\begin{aligned} \pi_1 &= 1/4(\pi_2 + \pi_3), \\ \pi_2 &= 1/2(\pi_1 + \pi_2) + 1/4\pi_3, \\ \pi_3 &= 1/2(\pi_1 + \pi_3) + 1/4\pi_2, \end{aligned}$$

and solving these with the constraint $\sum \pi_i = 1$ gives

$$\pi_1 = \frac{1}{5}, \qquad \pi_2 = \pi_3 = \frac{2}{5}$$

5. Problem 13. Consider the Markov Chain below. Let us refer to a transition that results in a state with a higher (respectively, lower) index as a birth (respectively death). Calculate the following quantities, assuming that when we start observing the chain, it is already in steady-state:



(a) For each state i, the probability that the current state is i.

(b) The probability that the first transition to observe is a birth.

(c) The probability that the first change of state to observe is a birth.

(d) The conditional probability that the process was in state 2 before the first transition that we observe, given that this transition was a birth.

(e) The conditional probability that the process was in state 2 before the first change of state that we observe, given that this change of state was a birth.

(f) The conditional probability that the first observed transition is a birth given that it resulted in a change of state.

(g) The conditional probability that the first observed transition leads to state 2, given that it resulted in a change of state.

Solution.

(a) The local balance equations take the form

$$0.6\pi_1 = 0.3\pi_2, 0.2\pi_2 = 0.2\pi_3.$$

They can be solved, together with the normalization equation, to yield

$$\pi_1 = \frac{1}{5}, \qquad \pi_2 = \pi_3 = \frac{2}{5}$$

(b) The probability that the first transition is a birth is

$$0.6\pi_1 + 0.2\pi_2 = 0.12 + 0.08 = 0.2$$

(c) If the state is 1, which happens with probability 1/5, the first change of state is certain to be a birth. If the state is 2, which happens with probability 2/5, the probability that the first change of state is a birth is equal to $0.2/(0.3 + 0.2) = 2/5$. Finally, if the state is 3, the probability that the first change of state is a birth is equal to 0. Thus, the probability that the first change of state that we observe is a birth is equal to

$$1 \cdot \frac{1}{5} + \frac{2}{5} \cdot \frac{2}{5} + 0 = \frac{9}{25}$$

(d) We have

$$\begin{aligned} P(\text{state was 2} \mid \text{first transition is a birth}) &= \frac{P(\text{state was 2 and first transition is a birth})}{P(\text{first transition is a birth})} \\ &= \frac{\pi_2 \cdot 0.2}{0.2} = \frac{2}{5} \end{aligned}$$

(e) As shown in part (c), the probability that the first change of state is a birth is 9/25. Furthermore, the probability that the state is 2 and the first change of state is a birth is $2\pi_2/5 = 4/25$. Therefore, the desired probability is

$$\frac{4/25}{9/25} = \frac{4}{9}$$

(f) In a birth-death process, there must be as many births as there are deaths, plus or minus 1. Thus, the steady-state probability of births must be equal to the steady-state probability of deaths. Hence, in steady-state, half of the state changes are expected to be births. Therefore, the conditional probability that the first observed transition is a birth, given that it resulted in a change of state, is equal to 1/2. This answer can also be obtained algebraically:

$$P(birth|\text{change of state}) = \frac{P(birth)}{P\text{change of state}} = \frac{1/5}{0.2 \cdot 0.6 + 0.4 \cdot 0.5 + 0.4 \cdot 0.2} = 0.5$$

(g) We have

$$P(\text{leads to state 2}|change) = \frac{P(\text{change that leads to state 2})}{P(change)} = \frac{0.6\pi_1 + 0.2\pi_3}{0.4} = 0.5$$

This is intuitive because for every change of state that leads into state 2, there must be a subsequent change of state that leads away from state 2.

6. Problem 15. Ehrenfest model of diffusion. We have a total of n balls, some of them black, some white. At each time step, we either do nothing, which happens with probability ϵ, where $0 < \epsilon < 1$, or we select a ball at random, so that each ball has probability $(1-\epsilon)/n > 0$ of being selected. In the latter case, we change the color of the selected ball (if white it becomes black, and vice versa), and the process is repeated indefinitely. What is the steady-state distribution of the number of white balls?
Solution.
Let $i = 0, 1..., n$ be the states, with state i indicating that there are exactly i white balls. The nonzero transition probabilities are

$$p_{00} = \epsilon, p_{01} = 1-\epsilon, p_{nn} = \epsilon, p_{n,n-1} = 1-\epsilon,$$
$$p_{i,i-1} = (1-\epsilon)\frac{i}{n}, p_{ii} = \epsilon, p_{i,i+1} = (1-\epsilon)\frac{n-i}{n}, \qquad i = 1, ..., n-1$$

The chain has a single recurrent class, which is aperiodic. In addition, it is a birth-death process. The local balance equations take the form

$$\pi_i \frac{(1-\epsilon)(n-i)}{n} = \pi_{i+1}\frac{(1-\epsilon)(i+1)}{n}$$

which leads to

$$\pi_i = \frac{n(n-1)\cdots(n-i+1)}{1\cdot 2\cdots i}\pi_0 = \binom{n}{i}\pi_0$$

We recognize that this has the form of a binomial distribution, so that for the probabilities to add to 1, we must have $\pi_0 = 1/(2^n)$. Therefore, the steady-state probabilities are given by

$$\pi_j = \frac{\binom{n}{i}}{2^n}$$

7. Problem 16. Bernoulli-Laplace model of diffusion. Each of two urns contains m balls. Out of the total of the 2m balls, m are white and m are black. A ball is simultaneously selected from each urn and moved to the other urn, and the process is indefinitely repeated. What is the steady-state distribution of the number of white balls in each urn?

Solution.
Let j = 0,1...,m be the states, with state j corresponding to the first urn containing j white balls. The nonzero transition probabilities are

$$p_{j,j-1} = (\frac{j}{m})^2, \qquad p_{j,j+1} = (\frac{m-j}{m})^2, \qquad p_{jj} = \frac{2j(m-j)}{m^2}$$

The chain has a single recurrent class that is aperiodic. This chain is a birth-death process and the steady-state probabilities can be found by solving the local balance equations:

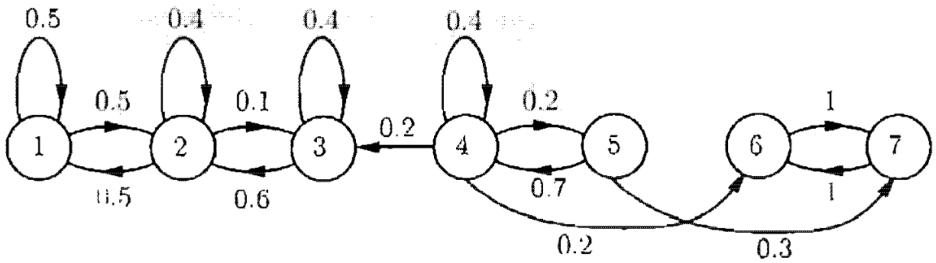
$$\pi_j(\frac{m-j}{m})^2 = \pi_{j+1}(\frac{j+1}{m})^2$$

The solution is of the form

$$\pi_j = \pi_0(\frac{m(m-1)\cdots(m-j+1)}{1\cdot 2\cdots j})^2 = \pi_0\binom{m}{j}^2$$

We recognize this as $\sum_{j=0}^{m}\binom{m}{j}^2 = \binom{2m}{m}$, which implies that

8. Problem 30 Consider the Markov chain below



(a) Identify the transient and recurrent states. Also, determine the recurrent classes and indicate which ones, if any, are periodic.

(b) Do there exist steady state probabilities given that the process start in state 1? If so, what are they?

(c) Do there exist steady state probabilities given that the process start in state 6? If so, what are they?

(d) Assume that the process starts in state 1 but we begin observing it after it reaches steady-state.

  (i) Find the probability that the state increases by one during the first transition we observe.

  (ii) Find the conditional probability that the process was in state 2 when we started observing it, given that the state increased by one during the first transition that we observed.

  (iii) Find the probability that the state increased by one during the first change of state that we observed.

(e) Assume that the process starts in state 4.

  (i) For each recurrent class, determine the probability that we eventually reach that class.

  (ii) What is the expected number of transitions up to and including the transition at which we reach a recurrent state for the first time?

Solution.

(a) States 4 and 5 are transient, and all other states are recurrent. There are two recurrent classes. The class $\{1, 2, 3\}$ is aperiodic, and the class $\{6, 7\}$ is periodic.

(b) If the process starts at state 1, it stays within the aperiodic recurrent class {1, 2, 3}, and the n-step transition probabilities converge to steady-state probabilities π_i. We have $\pi_i = 0$ for $i \notin \{1, 2, 3\}$. The local balance equations take the form

$$\pi_1 = \pi_2, \pi_2 = 6\pi_3$$

Using the normalization equation we get $\pi_1 = \pi_2 = \frac{6}{13}$ and $\pi_3 = \frac{1}{13}$

(c) Because the class {6,7} is periodic, there are no steady-state probabilities. In particular, the sequence $r_{66}(n)$ alternates between 0 and 1, and does not converge.

(d) (i) The probability that the state increases by one during the first transition is equal to

$$0.5\pi_1 + 0.1\pi_2 = \frac{18}{65}$$

(ii) The probability that the process is in state 2 and that the state increases is $0.1\pi_2 = \frac{0.6}{13}$. Thus, the desired conditional probability is equal to

$$\frac{0.6/13}{18/65} = \frac{1}{6}$$

(iii) If the state is 1 (probability 6/13), it is certain to increase at the first change of state. if the state is 2 (probability 6/13), it has probability 1/6 of increasing at the first change of state. Finally, if the state is 3, it cannot increase at the first change of state. Therefore, the probability that the state increases at the first change of state is equal to

$$\frac{6}{13} + \frac{1}{6} \cdot \frac{6}{13} = \frac{7}{13}$$

(e) (i) Let a_4 and a_5 be the probability that the class {1,2,3} is eventually reached, starting from state 4 and 5, respectively. We have

$$a_4 = 0.2 + 0.4a_4 + 0.2a_5$$
$$a_5 = 0.7a_4$$

which yields

$$a_4 = 0.2 + 0.4a_4 + 0.14a_4$$

and $a_4 = 10/23$. Also, the probability that the class {6, 7} is reached, starting from state 4, is 1-(10/23) = 13/23.

(ii) Let μ_4 and μ_5 be the expected times until a recurrent state is reached, starting from state 4 and 5, respectively. We have

$$\mu_4 = 1 + 0.4\mu_4 + 0.2\mu_5$$
$$\mu_5 = 1 + 0.7\mu_4$$

Substituting the second equation into the first, and solving for μ_4, we obtain

$$\mu_4 = \frac{60}{23}$$

9. Problem 37 Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. Penelope arrives at the street corner at a given time. Find her expected waiting time, given that she joins the queue. Assume that the process is in steady-state.

Solution.
We consider a continuous-time Markov chain with state n = 0,1,...,4, where
n = number of people waiting. For n = 0,1,2,3, the transitions from n to n+1 have rate 1, and the transitions from n+1 to n have rate 2.
The balance equations are

$$\pi_n = \frac{\pi_{n-1}}{2}$$

so that $\pi_n = \pi_0/2^n$, n=1,...,4. Using the normalization equation, we obtain

$$\pi_0 = \frac{1}{1 + 2^{-1} + 2^{-2} + 2^{-3} + 2^{-4}} = \frac{16}{31}$$

A passenger who joins the queue (in steady-state) will find n other passengers with probability $\pi_n/(\pi_0 + \pi_1 + \pi_2 + \pi_3)$, for n = 0, 1, 2, 3. The expected number of passengers found by Penelope is

$$E[N] = \frac{\pi_1 + 2\pi_2 + 3\pi_3}{\pi_0 + \pi_1 + \pi_2 + \pi_3} = \frac{8/31 + 2 \cdot 4/31 + 3 \cdot 2/31}{16/31 + 8/31 + 4/31 + 2/31} = \frac{22}{30}$$

Thus, including penelope herself there will be $\frac{26}{15}$ people in queue. Since the expected waiting time for a new taxi is 1/2 minute, the expected waiting time (by the law of iterated expectations) is

$$E[T] = E[N] \cdot \frac{26}{15} = \frac{13}{15}$$

2 Extra Problems:

10. I have 3 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet. If the probability of rain is p, what is the probability that I get wet?

Solution.

To solve the problem, consider a Markov chain taking values in the set S = {i : i = 0,1,2,3}, where i represents the number of umbrellas in the place where I am currently at (home or office). If i = 1 and it rains then I take the umbrella, move to the other place, where there are already 2 umbrellas, and, including the one I bring, I have next 3 umbrellas. Thus, $p_{1,3} = p$, because p is the probability of rain. If i = 1 but does not rain then I do not take the umbrella, I go to the other place and find 2 umbrellas. Thus, $p_{1,2} = 1 - p$. Continuing in the same manner, I form a Markov chain with the following transition matrix:

$$P = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1-p & p \\ 0 & 1-p & p & 0 \\ 1-p & p & 0 & 0 \end{bmatrix}$$

Writing the balance equations we get:

$$\begin{aligned} \pi_0 &= (1-p)\pi_3 \\ \pi_1 &= (1-p)\pi_2 + p\pi_3 \\ \pi_2 &= (1-p)\pi_1 + p\pi_2 \\ \pi_3 &= \pi_0 + p\pi_1 \end{aligned}$$

We get :

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{1-p}\pi_0$$

Thus:

$$\pi = (\frac{1-p}{4-p}, \frac{1}{4-p}, \frac{1}{4-p}, \frac{1}{4-p})$$

I get wet every time I happen to be in state 0 and it rains. The chance I am in state 0 is π_0. The chance it rains is p. Hence

$$P(WET) = \pi_0 p = \frac{p(1-p)}{4-p}$$

11. A fair coin is tossed repeatedly and independently.

(a) Find the expected number of tosses till the pattern "HT" appears.

(b) Find the expected number of tosses till the pattern "HH" appears.

(c) Find the expected number of tosses till the pattern "HTHT" appears.

(d) Find the expected number of tosses till the pattern "HHHH" appears.

(e) Write a code simulating the coin toss. Confirm the results in previous parts by repeating the experiment 100 times and averaging over them.

Solution.

(a) The markov chain has 3 states:

- $\emptyset$
- H
- HT

The Markov chain is as follows:

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}$$

We find the time to absorption:

$$\mu_1 = 1 + 0.5\mu_1 + 0.5\mu_2, \mu_2 = 1 + 0.5\mu_2$$

Thus $\mu_2 = 2$, and $\mu_1 = 4$

(b) Similar to above the matrix is:

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0 & 1 \end{bmatrix} \tag{1}$$

We find the time to absorption:

$$\mu_1 = 1 + 0.5\mu_1 + 0.5\mu_2, \mu_2 = 1 + 0.5\mu_1$$

Thus $\mu_2 = 4$, and $\mu_1 = 6$

(c) Now, we have 5 states: The time to absorption in this case is 20.

- $\emptyset$
- H
- HT
- HTH
- HTHT

The Markov chain is as follows:

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We find the time to absorption:

$$\begin{aligned} \mu_1 &= 1 + 0.5\mu_1 + 0.5\mu_2, \\ \mu_2 &= 1 + 0.5\mu_2 + 0.5\mu_3 \\ \mu_3 &= 1 + 0.5\mu_1 + 0.5\mu_4 \\ \mu_4 &= 1 + 0.5\mu_2 \end{aligned}$$

Thus $\mu_4 = 10$, and $\mu_2 = 18$ and $\mu_1 = 20$. The time to absorption in this case is 20.

(d) The Markov chain is as follows:

$$P = \begin{bmatrix} 0.5 & 0.5 & 0 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 & 0 \\ 0.5 & 0 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We find the time to absorption:

$$\begin{aligned} \mu_1 &= 1 + 0.5\mu_1 + 0.5\mu_2, \\ \mu_2 &= 1 + 0.5\mu_1 + 0.5\mu_3 \\ \mu_3 &= 1 + 0.5\mu_1 + 0.5\mu_4 \\ \mu_4 &= 1 + 0.5\mu_1 \end{aligned}$$

Thus $\mu_4 = 16$, and $\mu_1 = 30$. The time to absorption in this case is 30.

12. A certain experiment is believed to be described by a two-state Markov chain with the transition matrix P, where

$$P = \begin{bmatrix} 0.5 & 0.5 \\ p & 1-p \end{bmatrix}$$

and the parameter p is not known. When the experiment is performed many times, the chain ends in state one approximately 10 percent of the time and in state two approximately 90 percent of the time. Compute a sensible estimate for the unknown parameter p and explain how you found it.

Solution.
The statement above is equivalent to saying $\pi_1 = 0.1$ and $\pi_2 = 0.9$. Thus, we first find the stationary distribution of the above Markov chain:

$$\pi_1 = 0.5\pi_1 + p\pi_2, \pi_2 = 0.5\pi_1 + (1-p)\pi_2$$

Thus, we have $\pi_1 = 2p\pi_2$ and using the normalization equation, we have:

$$\pi_1 = \frac{2p}{1+2p}, \pi_2 = \frac{1}{1+2p}$$

Thus we should have $1 + 2p = \frac{10}{9}$ and $p = \frac{1}{18}$.