ECE 4110/5110 Wednesday, 08/30/23

Lecture 4: Some Inequalities and Limit Theorems

Dr. Kevin Tang Handout 5

Related Reading

Bertsekas and Tsitsiklis Chapter 5

Markov Inequality

For a nonnegative random variable X and any $a > 0$, we have

$$P(X \geq a) \leq \frac{E(X)}{a} \tag{1}$$

Chebyshev Inequality

For a random variable X with expected value μ and variance σ^2, then for any $c > 0$ we have

$$P(|X - \mu| \geq c) \leq \frac{\sigma^2}{c^2} \tag{2}$$

The Chernoff Bound

$$P(X \geq a) \leq E(e^{\lambda(X-a)}) \tag{3}$$

where $\lambda \geq 0$. Therefore, we can minimize the right hand side over $\lambda \geq 0$ to find the tightest upper bound.

The Weak Law of Large Numbers

$$\lim_{n\to\infty} P(|M_n - \mu| \geq \epsilon) = 0 \tag{4}$$

In other words, the same mean converges to the true mean in probability

Proof follows from Chebyshev's Inequality fairly easily.

The Strong Law of Large Numbers

$$P(\lim_{n\to\infty} M_n = \mu) = 1 \tag{5}$$

In other words, the sequence of sample means converges to the true mean almost surely (with probability 1).

Proper Scaling

$$Z_n = \frac{\sum_{i=1}^{n} X_i - n\mu}{\sqrt{n}\sigma} \tag{6}$$

The Central Limit Theorem

$$\lim_{n\to\infty} P(Z_n \le z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx \tag{7}$$

Approximation Based on the CLT

$$P(S_n \le c) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{c-n\mu}{\sqrt{n}\sigma}} e^{-\frac{x^2}{2}}\, dx \tag{8}$$