ECE 4110/5110

Homework Set 2 Solution

Dr. Kevin Tang HW 2

1 Problems from Bertsekas and Tsitsiklis, 2nd edition

1. Problem 2. Dave fails quizzes with probability 1/4, independent of other quizzes.

1. What is the probability that Dave fails exactly two of the next six quizzes?
2. What is the expected number of quizzes that Dave will pass before he has failed three times?
3. What is the probability that the second and third time Dave fails a quiz will occur when he takes his eighth and ninth quizzes, respectively?
4. What is the probability that Dave fails two quizzes in a row before he passes two quizzes in a row?

Solution

1. Failed quizzes are a Bernoulli process with parameter p = 1/4. The desired probability is given by the binomial formula:

$$\binom{6}{2}p^2(1-p)^4 = \frac{6!}{4!2!}(\frac{1}{4})^2(\frac{3}{4})^4$$

2. The expected number of quizzes up to the third failure is the expected value of a Pascal random variable of order three, with parameter 1/4, which is $3 \cdot 4 = 12$. Subtracting the number of failures, we have that the expected number of quizzes that Dave will pass is 12 - 3 = 9.

3. The event of interest is the intersection of the following three independent events:
   A: there is exactly one failure in the first seven quizzes.
   B: quiz eight is a failure.
   C: quiz nine is a failure.
   We have:

$$P(A) = \binom{7}{1}(\frac{1}{4})(\frac{3}{4})^6, \qquad P(B) = P(C) = \frac{1}{4}$$

   so the desired probability is

$$P(A \cap B \cap C) = 7(\frac{1}{4})^3(\frac{3}{4})^6$$

4. Let B be the event that Dave fails two quizzes in a row before he passes two quizzes in a row. Let us use F and S to indicate quizzes that he has failed or passed, respectively. We then have

$$\begin{aligned} P(B) &= P(FF \cup SFF \cup FSFF \cup SFSFF \cup FSFSFF \cup ...) \\ &= P(FF) + P(SFF) + P(FSFF) + P(SFSFF) + P(FSFSFF) + ... \\ &= (\frac{1}{4})^2 + \frac{3}{4}(\frac{1}{4})^2 + \frac{1}{4}\frac{3}{4}(\frac{1}{4})^2 + \frac{3}{4}\frac{1}{4}\frac{3}{4}(\frac{1}{4})^2 + \frac{1}{4}\frac{3}{4}\frac{1}{4}\frac{3}{4}(\frac{1}{4})^2 + ... \\ &= [(\frac{1}{4})^2 + \frac{1}{4}\frac{3}{4}(\frac{1}{4})^2 + \frac{1}{4}\frac{3}{4}\frac{1}{4}\frac{3}{4}(\frac{1}{4})^2 + ...] + [\frac{3}{4}(\frac{1}{4})^2 + \frac{3}{4}\frac{1}{4}\frac{3}{4}(\frac{1}{4})^2 + ...] \end{aligned}$$

   Therefore, P(B) is the sum of two infinite geometric series, and

$$P(B) = \frac{(\frac{1}{4})^2}{1 - \frac{1}{4}\frac{3}{4}} + \frac{\frac{3}{4}(\frac{1}{4})^2}{1 - \frac{1}{4}\frac{3}{4}} = \frac{7}{52}$$

2. Problem 3. A computer system carries out tasks submitted by two users. Time is divided into slots. A slot can be idle, with probability $P_I = 1/6$, and busy with probability $P_B = 5/6$. During a busy slot, there is probability $P_{1|B} = 2/5$ (respectively, $P_{2|B} = 3/5$) that a task from user 1 (respectively, 2) is executed. We assume that events related to different slots are independent.

(a) Find the probability that a task from user 1 is executed for the first time during the 4th slot.

(b) Given that exactly 5 out of the first 10 slots were idle, find the probability that the 6th idle slot is slot 12.

(c) Find the expected number of slots up to and including the 5th task from user 1.

(d) Find the expected number of busy slots up to and including the 5th task from user 1.

(e) Find the PMF, mean, and variance of the number of tasks from user 2 until the time of the 5th task from user 1.

Solution

(a) During each slot, the probability of a task from user 1 is given by $p_1 = p_{1|B}p_B = (5/6)(2/5) = 1/3$. Tasks from user 1 form a Bernoulli process and

$$P(\text{first user 1 task occurs in slot 4}) = p_1(1-p_1)^3 = \frac{1}{3}(\frac{2}{3})^3.$$

(b) This is the probability that slot 11 was busy and slot 12 was idle, given that 5 out of the 10 first slots were idle. Because of the fresh-start property, the conditioning information is immaterial, and the desired probability is

$$p_B.p_I = \frac{5}{6}.\frac{1}{6}.$$

(c) Each slot contains a task from user 1 with probability $p_1 = 1/3$, independent of other slots. The time of the 5th task from user 1 is a Pascal random variable of order 5, with parameter $p_1 = 1/3$. Its mean is given by

$$\frac{5}{p_1} = 15$$

(d) Each busy slot contains a task from user 1 with probability $p_{1|B} = 2/5$, independent of other slots. The random variable of interest is a Pascal random variable of order 5, with parameter $p_{1|B} = 2/5$. Its mean is

$$\frac{5}{p_{1|B}} = \frac{5}{2/5} = \frac{25}{2}$$

(e) The number T of tasks from user 2 until the 5th task from user 1 is the same as the number B of busy slots until the 5th task from user 1, minus 5. The number of busy slots ("trials") until the 5th task from user 1 ("success") is a Pascal random variable of order 5, with parameter $p_{1|B} = 2/5$. Thus,

$$p_B(t) = \binom{t-1}{4}(\frac{2}{5})^5(1-\frac{2}{5})^{t-5}, \qquad t = 5, 6, ...$$

Since T =B-5, we have $p_T(t) = p_B(t+5)$, and we obtain

$$p_T(t) = \binom{t+4}{4}(\frac{2}{5})^5(1-\frac{2}{5})^{t}, \qquad t = 0, 1, ...$$

Using the formulas for the mean and the variance of the Pascal random variable B, we obtain

$$E[T] = E[B] - 5 = 12.5 - 5 = 7.5,$$

and

$$var(T) = var(B) = \frac{5(1-2/5)}{(2/5)^2}$$

3. Problem 14. Beginning at time t = 0. we start using bulbs. one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a type-A bulb and a type-B bulb. The lifetime, X, of any particular bulb of a particular type is a random variable, independent of everything else, with the following PDF:

$$\text{for type-A Bulbs:} f_x(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & ow \end{cases}$$

$$\text{for type-B Bulbs:} f_x(x) = \begin{cases} 3e^{-3x}, & x \geq 0 \\ 0, & ow \end{cases}$$

(a) Find the expected time until the first failure.

(b) Find the probability that there are no bulb failures before time t.

(c) Given that there are no failures until time t, determine the conditional probability that the first bulb used is a type-A bulb.

(d) Find the variance of the time until the first bulb failure.

(e) Find the probability that the 12th bulb failure is also the 4th type-A bulb failure.

(f) Up to and including the 12th bulb failure, what is the probability that a total of exactly 4 type-A bulbs have failed?

(g) Determine either the PDF or the transform associated with the time until the 12th bulb failure.

(h) Determine the probability that the total period of illumination provided by the first two type-B bulbs is longer than that provided by the first type-A bulb.

(i) Suppose the process terminates as soon as a total of exactly 12 bulb failures have occurred. Determine the expected value and variance of the total period of illumination provided by type-B bulbs while the process is in operation.

(j) Given that there are no failures until time t, find the expected value of the time until the first failure.

Solution

(a) Let X be the time until the first bulb failure. Let A (respectively, B) be the event that the first bulb is of type A (respectively, B). Since the two bulb types are equally likely, the total expectation theorem yields

$$E[X] = E[X|A]P(A) + E[X|B]P(B) = 1.\frac{1}{2} + \frac{1}{3}.\frac{1}{2} = \frac{2}{3}$$

(b) Let D be the event of no bulb failures before time t. Using the total probability theorem, and the exponential distributions for bulbs of the two types, we obtain

$$P(D) = P(D|A)P(A) + P(D|B)P(B) = \frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}$$

(c) We have

$$P(A|D) = \frac{P(A \cap D)}{P(D)} = \frac{\frac{1}{2}e^{-t}}{\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t}} = \frac{1}{1 + e^{-2t}}$$

(d) We first find $E[X^2]$. We use the fact that the second moment of an exponential random variable T with parameter λ is equal to $E[T^2] = E[T]^2 + var(T) = 1/\lambda^2 + 1/\lambda^2 = 2/\lambda^2$. Conditioning on the two possible types of the first bulb, we obtain

$$E[X^2] = E[X^2|A]P(A) + E[X^2|B]P(B) = 2.\frac{1}{2} + \frac{2}{9}.\frac{1}{2} = \frac{10}{9}$$

Finally, using the fact $E[X] = 2/3$ from part (a),

$$var(X) = E[X^2] - E[X]^2 = \frac{10}{9} - \frac{2^2}{3^2} = \frac{2}{3}.$$

(e) This is the probability that out of the first 11 bulbs, exactly 3 were of type A and that the 12th bulb was of type A. It is equal to

$$\binom{11}{3}(\frac{1}{2})^{12}$$

(f) This is the probability that out of the first 12 bulbs, exactly 4 were of type A, and is equal to

$$\binom{12}{4}(\frac{1}{2})^{12}$$

(g) The PDF of the time between failures is $(e^{-x}+3e^{-3x})/2$, for $x \geq 0$, and the associated transform is

$$[\frac{1}{2}(\frac{1}{1-s}+\frac{3}{3-s})]$$

Since the times between successive failures are independent, the transform associated with the time until the 12th failure is given by

$$[\frac{1}{2}(\frac{1}{1-s}+\frac{3}{3-s})]^{12}$$

(h) Let Y be the total period of illumination provided by the first two type-B bulbs. This has an Erlang distribution of order 2, and its PDF is

$$f_Y(y) = 9ye^{-3y}, y \geq 0.$$

Let T be the period of illumination provided by the first type-A bulb. Its PDF is

$$f_T(t) = e^{-t}, t \geq 0.$$

We are interested in the event $T < Y$. We have

$$P(T < Y|Y = y) = 1 - e^{-y}, y \geq 0.$$

Thus,

$$P(T < Y) = \int_0^\infty f_Y(y)P(T < Y|Y = y)dy = \int_0^\infty 9ye^{-3y}(1-e^{-y})dy = \frac{7}{16},$$

as can be verified by carrying out the integration.

(i) Let V be the total period of illumination provided by type-B bulbs while the process is in operation. Let N be the number of light bulbs, out of the first 12, that are of type B. Let X_i be the period of illumination from the ith type-B bulb. We then have $V = X_1 + + X_N$. Note that N is a binomial random variable, with parameters n=12 and p = 1/2, so that

$$E[N] = 6, var(N) = 3$$

Furthermore, $E[X_i] = 1/3$ and $var(X_i) = 1/9$. Using the formulas for the mean and variance of the sum of a random number of random variables, we obtain

$$E[V] = E[N]E[X_i] = 2,$$

and

$$var(V) = var(X_i)E[N] + E[X_i]^2 var(N) = \frac{1}{9}.6 + \frac{1}{9}.3 = 1.$$

(j) Using the notation in parts (a)-(c), and the result of part (c), we have

$$\begin{aligned} E[T|D] &= t + E[T-t|D\cap A]P(A|D) + E[T-t|D\cap B]P(B|D) \\ &= t + 1.\frac{1}{1+e^{-2t}} + \frac{1}{3}(1-\frac{1}{1+e^{-2t}}) = t + \frac{1}{3} + \frac{2}{3(1+e^{-2t})} \end{aligned}$$

4. Problem 15. A service station handles jobs of two types, A and B. (Multiple jobs can be processed simultaneously.) Arrivals of the two job types are independent Poisson processes with parameters $\lambda_A = 3$ and $\lambda_B = 4$ per minute, respectively. Type A jobs stay in the service station for exactly one minute. Each type B job stays in the service station for a random but integer amount of time which is geometrically distributed, with mean equal to 2, and independent of everything else. The service station started operating at some time in the remote past.

(a) What is the mean, variance, and PMF of the total number of jobs that arrive within a given three-minute interval?

(b) We are told that during a 10-minute interval, exactly 10 new jobs arrived. What is the probability that exactly 3 of them are of type A?

(c) At time 0, no job is present in the service station. What is the PMF of the number of type B jobs that arrive in the future, but before the first type A arrival?

(d) At time t = 0, there were exactly two type A jobs in the service station. What is the PDF of the time of the last (before time 0) type A arrival?

(e) At time 1, there was exactly one type B job in the service station. Find the distribution of the time until this type B job departs.

Solution

(a) The total arrival process corresponds to the merging of two independent Poisson processes, and is therefore Poisson with rate $\lambda = \lambda_A + \lambda_B = 7$. Thus, the number N of jobs that arrive in a given three-minute interval is a Poisson random variable, with $E[N] = 3\lambda = 21$, var(N) = 21, and PMF

$$p_N(n) = \frac{(21)^n e^{-21}}{n!}$$

(b) Each of these 10 jobs has probability $\lambda_A/(\lambda_A + \lambda_B) = 3/7$ of being of type A, independently of the others. Thus, the binomial PMF applies and the desired probability is equal to

$$\binom{10}{3}(\frac{3}{7})^3(\frac{4}{7})^7$$

(c) Each future arrival is of type A with probability $\lambda_A/(\lambda_A+\lambda_B) = 3/7$, independently of other arrivals. Thus, the number K of arrivals until the first type A arrival is geometric with parameter 3/7. The number of type B arrivals before the first type A arrival is equal to K -1, and its PMF is similar to a geometric, except that it is shifted by one unit to the left. In particular,

$$p_K(k) = \frac{3}{7}(\frac{4}{7})^k, k = 0, 1, 2, ...$$

(d) The fact that at time 0 there were two type A jobs in the system simply states that there were exactly two type A arrivals between time -1 and time 0. Let X and Y be the arrival times of these two jobs. From recitation, each time instant is equally likely to contain an arrival and since the arrival times are independent, it follows that X and Y are independent uniform random variables. We are interested in the PDF of $Z = max\{X, Y\}$. We first find the CDF of Z. We have, for $z \in [-1, 0]$,

$$P(Z \leq z) = P(X \leq z \& Y \leq z) = (1 + z)^2$$

By differentiating, we obtain

$$f_Z(z) = 2(1 + z), -1 \leq z \leq 0.$$

(e) Let T be the arrival time of this type B job. We can express T in the form $T = -K + X$, where K is a nonnegative integer and X lies in [0,1]. We claim that X is independent from K and that X is uniformly distributed. Indeed, conditioned on the event K = k, we know that there was a single arrival in the interval [-k, -k + 1]. Conditioned on the latter information, the arrival time is uniformly distributed in the interval [-k, -k + 1], which implies that X is uniformly distributed in [0,1]. Since this conditional distribution of X is the same for every k, it follows that X is independent of -K.
Let D be the departure time of the job of interest. Since the job stays in the system for an integer amount of time, we have that D is of the form D = L + X, where L is a nonnegative integer. Since the job stays in the system for a geometrically distributed amount of time, and the geometric distribution has the memorylessness property, it follows that L is also memoryless. In particular, L is similar to a geometric random variable, except that its PMF starts at zero. Furthermore, L is independent of X, since X is determined by the arrival process, whereas the amount of time a job stays in the system is independent of the arrival process. Thus, D is the sum of two independent random variables, one uniform and one geometric. Therefore, D has “geometric staircase” PDF, given by

$$f_D(d) = (\frac{1}{2})^{\lfloor d \rfloor}, \qquad d \geq 0$$

5. Problem 16. Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal in your neighborhood, and police cars drive by according to a Poisson process with rate λ. You decide to make a U-turn once you see that the road has been clear of police cars for T time units. Let N be the number of police cars you see before you make the U-turn.

(a) Find E[N].

(b) Find the conditional expectation of the time elapsed between police cars n - 1 and n, given that $N \geq n$.

(c) Find the expected time that you wait until you make the U-turn. Hint: Condition on N.

Solution

(a) The random variable N is equal to the number of successive interarrival intervals that are smaller than T. Interarrival intervals are independent and each one is smaller than τ with probability $1 - e^{-T\lambda}$. Therefore,

$$P(N = 0) = e^{-T\lambda}, \quad P(N = 1) = e^{-T\lambda}(1 - e^{-T\lambda}), \quad P(N = k) = e^{-T\lambda}(1 - e^{-T\lambda})^k,$$

so that N has a distribution similar to a geometric one, with parameter $p = e^{-T\lambda}$, except that it shifted one place to the left, so that it starts out at 0. Hence,

$$E[N] = \frac{1}{p} - 1 = e^{T\lambda} - 1$$

(b) Let T_n be the nth interarrival time. The event $\{N \geq n\}$ indicates that the time between cars n- 1 and n is less than or equal to T, and therefore $E[T_n|N \geq n] = E[T_n|T_n \leq T]$. Note that the conditional PDF of T_n is the same as the unconditional one, except that it is now restricted to the interval [0,T], and that it has to be suitably renormalized so that it integrates to 1. Therefore, the desired conditional expectation is

$$E[T_n|T_n \leq T] = \frac{\int_0^T s\lambda e^{-\lambda s} ds}{\int_0^T \lambda e^{-\lambda s} ds}$$

This integral can be evaluated by parts. We will provide, however, an alternative approach that avoids integration. We use the total expectation formula

$$E[T_n] = E[T_n|T_n \leq T]P(T_n \leq T) + E[T_n|T_n > T]P(T_n > T).$$

We have $E[T_n] = 1/\lambda$, $P(T_n \leq T) = 1 - e^{-\lambda T}$, $P(T_n > T) = e^{-T\lambda}$, and $E[T_n|T_n > T] = T + (1/\lambda)$. (The last equality follows from the memorylessness of the exponential PDF.) Using these equalities, we obtain

$$\frac{1}{\lambda} = E[T_n|T_n \leq T](1 - e^{-T\lambda}) + (T + \frac{1}{\lambda})e^{-T\lambda}$$

which yields

$$E[T_n|T_n \leq T] = \frac{\frac{1}{\lambda} - (T + \frac{1}{\lambda})e^{-T\lambda}}{1 - e^{-T\lambda}}$$

(c) Let T be the time until the U-turn. Note that $T = T_1 + \ + T_N + T$. Let v denote the value of $E[T_n|T_n \leq T]$. We find E[T] using the total expectation theorem:

$$\begin{aligned}
E[T] &= T + \sum_{n=0}^{\infty} P(N = n)E[T_1 + \ + T_N|N = n] \\
&= T + \sum_{n=0}^{\infty} P(N = n) \sum_{i=1}^{n} E[T_i|T_1 \leq T, ..., T_n \leq T, T_{n+1} > T] \\
&= T + \sum_{n=0}^{\infty} P(N = n) \sum_{i=1}^{n} E[T_i|T_i \leq T] \\
&= T + \sum_{n=0}^{\infty} P(N = n)nv \\
&= T + vE[N]
\end{aligned}$$

where E[N] was found in part (a) and v was found in part (b). The second equality used the fact that the event $\{N = n\}$ is the same as the event $\{T_1 \leq T, ..., T_n \leq T, T_{n+1} > T\}$. The third equality used the independence of the interarrival times T_i.

6. Problem 17. A wombat in the San Diego zoo spends the day walking from a burrow to a food tray, eating, walking back to the burrow, resting, and repeating the cycle. The amount of time to walk from the burrow to the tray (and also from the tray to the burrow) is 20 secs. The amounts of time spent at the tray and resting are exponentially distributed with mean 30 secs. The wombat, with probability 1/3, will momentarily stand still (for a negligibly small time) during a walk to or from the tray, with all times being equally likely (and independent of what happened in the past). A photographer arrives at a random time and will take a picture at the first time the wombat will stand still. What is the expected value of the length of time the photographer has to wait to snap the wombat's picture?

Solution

We will calculate the expected length of the photographer's waiting time T conditioned on each of the two events: A, which is that the photographer arrives while the wombat is resting or eating, and A^c, which is that the photographer arrives while the wombat is walking. We will then use the total expectation theorem as follows:

$$E[T] = P(A)E[T|A] + P(A^c)E[T|A^c].$$

The conditional expectation E[T—A] can be broken down in three components:

(i) The expected remaining time up to when the wombat starts its next walk; by the memorylessness property, this time is exponentially distributed and its expected value is 30 secs.

(ii) A number of walking and resting/eating intervals(each of expected length 50 secs) during which the wombat does not stop; if N is the number of these intervals, then N + 1 is geometrically distributed with parameter 1/3. Thus the expected length of these intervals is (3 - 1) · 50 = 100 secs.

(iii) The expected waiting time during the walking interval in which the wombat stands still. This time is uniformly distributed between 0 and 20, so its expected value is 10 secs. Collecting the above terms, we see that $E[T|A] = 30 + 100 + 10 = 140$.

The conditional expectation $E[T|A^c]$ can be calculated using the total expectation theorem, by conditioning on three events: B1, which is that the wombat does not stop during the photographer's arrival interval (probability 2/3); B2, which is that the wombat stops during the photographer's arrival interval after the photographer arrives (probability 1/6); B3, which is that the wombat stops during the photographer's arrival interval before the photographer arrives (probability 1/6). We have

$E[T|A^c, B1] = E[$photographer's wait up to the end of the interval$] + E[T|A] = 10+140 = 150$.

Also, it can be shown that if two points are randomly chosen in an interval of length l, the expected distance between the two points is l/3, and using this fact, we have

$$E[T|A^c, B2] = E[\text{photographer's wait up to the time when the wombat stops}] = \frac{20}{3}$$

Similarly, it can be shown that if two points are randomly chosen in an interval of length l, the expected distance between each point and the nearest endpoint of the interval is l/3. Using this fact, we have

$$E[T|A^c, B3] = E[\text{photographer's wait up to the end of the interval}] + E[T|A] = \frac{20}{3} + 140.$$

Applying the total expectation theorem, we see that

$$E[T|A^c] = \frac{2}{3}.150 + \frac{1}{6}.\frac{20}{3} + \frac{1}{6}(\frac{20}{3} + 140) = 125.55.$$

To apply the total expectation theorem and obtain E[T], we need the probability P(A) that the photographer arrives during a resting/eating interval. Since the expected length of such an interval is 30 seconds and the length of the complementary walking interval is 20 seconds, we see that P(A) = 30/50 = 0.6. Substituting in the equation we obtain

$$E[T] = P(A)E[T|A] + (1 - P(A))E[T|A^c] = 0.6 \times 140 + 0.4 \times 125.55 = 134.22.$$

2 Problems from Grimmett and Stirzaker, 3.9/3.10

7. Chapter 3.9 Problem 7 - Returns and visits by random walk. Consider a simple symmetric random walk on the set $\{0, 1, 2, ..., a\}$ with absorbing barriers at 0 and a, and starting at k where $0 < k < a$. Let r_k be the probability the walk ever returns to k, and let v_k, be the mean number of visits to point x before absorption. Find r_k, and hence show that,

$$v_k = \begin{cases} 2x(a-k)/a & 0 < x < k \\ 2k(a-x)/a & k < x < a \end{cases}$$

Solution.

The first step of the random walk is either k+1 or k-1. We can condition the probability of returning to k on the first step.

$$r_k = P(return|x_1 = k-1)P(x_1 = k-1) + P(return|x_1 = k+1)P(x_1 = k+1)$$

First, assume $x_1 = k - 1$. The probability of returning to k is equal to the probability that the walk will go to k before going to origin. This is a gambler's ruin problem. From recitation 1, we know that this probability is equal to $1 - \frac{1}{k}$. Thus,

$$P(return|x_1 = k-1) = 1 - \frac{1}{k}$$

Now, assume $x_1 = k + 1$. The probability of returning to k is equal to the probability that the walk will go to k before going to a. This is a gambler's ruin problem. From recitation 1, we know that this probability is equal to $1 - \frac{1}{a-k}$. Thus,

$$P(return|x_1 = k-1) = 1 - \frac{1}{a-k}$$

Hence:

$$r_k = \frac{1}{2}(1 - \frac{1}{k}) + \frac{1}{2}(1 - \frac{1}{a-k}) = 1 - \frac{1}{2}(\frac{1}{k} + \frac{1}{a-k})$$

Now, we compute v_k. Consider the case where $0 < x < k$. First, we compute the probability that we will visit x at least once. The probability of visiting x is equal to the probability that the walk will go to x before going to a. This is a gambler's ruin problem. From recitation 1, we know that this probability is equal to $\frac{a-k}{a-x}$.

Now after reaching x, we will revisit x with probability r_x. Thus, the expected number of times we visit x, comes from a geometric distribution with probability of success equal to 1-r_x. Hence, the expected number of visits to x given we visit it at least once is equal to $\frac{1}{1-r_x}$. Thus

$$v_k = \frac{a-k}{a-x}\frac{1}{1-r_x} = \frac{a-k}{a-x}\frac{2x(a-x)}{a} = \frac{2x(a-k)}{a}$$

The case where $k < x < a$ can be computed similarly.

8. Chapter 3.10 Problem 3 For a symmetric simple random walk starting at 0, show that the probability of the first visit to S_{2n} takes place at time $2k$ equals the product $P(S_{2k} = 0)P(S_{2n-2k} = 0)$, for $0 \leq k \leq n$.

Solution
By considering the random walk reversed, we see that the probability of a first visit to S_{2n} at time 2k is the same as the probability of a last visit to S_0 at time 2n - 2k. The result is then immediate from the arc sine law (3.10.19) for the last visit to the origin.

Arc sine law Suppose that p = 1/2 and $S_0 = 0$. The probability that the last visit to 0 up to time 2n occurred at time 2k is $P(S_{2k} = 0)P(S_{2n-2k} = 0)$.

9. Chapter 3.11 Problem 29 Let S be a symmetric random walk with $S_0 = 0$, and let N_n be the number of points that have been visited by S exactly once up to time n. Show that $E(N_n) = 2$.

Solution

Let $u_k = P(S_k = 0)$, $f_k = P(S_k = 0, S_i \neq 0, 1 < i < k)$, and use conditional probability to obtain

$$u_{2n} = \sum_{k=1}^{n} u_{2n-2k} f_{2k}$$

Now $N_1 = 2$, and therefore it suffices to prove that $E(N_n) = E(N_{n-1})$ for $n \geq 2$ Let N'_{n-1} be the number of points visited by the walk $S_1, S_2, ..., S_n$ exactly once (we have removed S_0). Then

$$N_n = \begin{cases} N'_{n-1} + 1 & \text{if } S_k \neq S_0 \text{ for } 1 \leq k \leq n \\ N'_{n-1} - 1 & \text{if } S_k = S_0 \text{ for exactly one k in } \{1,2,...,n\} \\ N'_{n-1} & \text{otherwise} \end{cases}$$

Hence, writing $\alpha_n = P(Sk \neq 0, 1 < k \leq n)$,

$$\begin{aligned} E(N_n) &= E(N'_{n-1}) + \alpha_n - P(S_k = S_0 \text{ exactly once} \\ &= E(N_{n-1}) + \alpha_n - \{f_2\alpha_{n-2} + f_4\alpha_{n-4} + ... + f_{2\lfloor n/2 \rfloor}\} \end{aligned}$$

where $\lfloor x \rfloor$ is the integer part of x. Now $\alpha_{2m} = \alpha_{2m+1} = u_{2m}$. If n = 2k is even, then

$$E(N_{2k}) - E(N_{2k-1}) = u_{2k} - (f_2 u_{2k-2} + ... + f_{2k}) = 0$$

If n=2k +1 is odd, then

$$E(N_{2k+1}) - E(N_{2k}) = u_{2k} - (f_2 u_{2k-2} + ... + f_2 k) = 0$$

In either case the claim is proved.

3 Extra Problems:

10. Gambler's Ruin Revisited In this problem we revisit the gambler's ruin problem from the first recitation. Two gamblers, A and B, bet on the outcomes of successive flips of a coin. On each flip, if the coin comes up heads, A collects 1 unit from B, whereas if it comes up tails, A pays 1 unit to B. They continue to do this until one of them runs out of money. If it is assumed that the successive flips of the coin are independent and fair, (Note that this game is a random walk.)

(a) What is the probability that A ends up with all the money if he starts with i units and B starts with N-i units?

(b) What is the expected length of the game? (On average how long it takes for one of them to go broke?)

Solution

(a) Let E denote the event that A ends up with all the money when he starts with i and B starts with N - i, and to make clear the dependence on the initial fortune of A, let $P_i = P(E)$. We shall obtain an expression for P(E) by conditioning on the outcome of the first flip as follows: Let H denote the event that the first flip lands on heads; then

$$\begin{aligned} P_i = P(E) &= P(E|H)P(H) + P(E|H^c)P(H^c) \\ &= pP(E|H) + (1-p)P(E|H^c) \end{aligned}$$

Now, given that the first flip lands on heads, the situation after the first bet is that A has i + 1 units and B has N - (i + 1). Since the successive flips are assumed to be independent with a common probability p of heads, it follows that, from that point on, A's probability of winning all the money is exactly the same as if the game were just starting with A having an initial fortune of i + 1 and B having an initial fortune of N - (i + 1). Therefore

$$P(E|H) = P_{i+1}$$

and similarly,

$$P(E|H^c) = P_{i-1}$$

Hence, letting q = 1 - p, we obtain

$$P_i = 0.5P_{i+1} + 0.5P_{i-1} \qquad i = 1, 2, ..., N-1$$

By making use of the obvious boundary conditions $P_0 = 0$ and $P_N = 1$, we shall now solve Equation above.

$$0.5P_i + 0.5P_i = 0.5P_{i+1} + 0.5P_{i-1}$$

or

$$P_{i+1} - P_i = P_i - P_{i-1}, \qquad i = 1, 2, ..., N-1$$

Hence, since $P_0 = 0$, we obtain, from Equation above,

$$P_2 - P_1 = P_1 - P_0 = P_1 \qquad .$$

.

.

$$P_i - P_{i-1} = P_{i-1} - P_{i-2} = P_1$$

.

.

.

$$P_N - P_{N-1} = P_{N-1} - P_{N-2} = P_1$$

Adding the first i - 1 equations of yields

$$P_i = iP_1$$

Using the fact that $P_N = 1$, we obtain

$$P_i = \frac{i}{N},$$

(b) Let L_i be the expected length of game starting from i dollars. We shall find an expression for L_i by conditioning of the first flip as follows: Let H denote the event that the first flip lands on heads; then

$$L_i = 1 + 0.5(L_{i-1} + L_{i+1})$$

By making use of the obvious boundary conditions $P_0 = 0$ and $P_N = 0$, we shall now solve Equation above.

$$0.5L_i + 0.5L_i = 1 + 0.5L_{i+1} + 0.5L_{i-1}$$

or

$$L_{i+1} - L_i = L_i - L_{i-1} - 2, \qquad i = 1, 2, ..., N-1$$

Hence, since $P_0 = 0$, we obtain, from Equation above,

$$
\begin{aligned}
&L_2 - L_1 = L_1 - L_0 - 2 = L_1 - 2 \\
&L_3 - L_2 = L_2 - L_1 - 2 = L_1 - 4 \\
&. \\
&. \\
&. \\
&L_i - L_{i-1} = L_{i-1} - L_{i-2} = L_1 - 2(i-1) \\
&. \\
&. \\
&. \\
&L_N - L_{N-1} = L_{N-1} - L_{N-2} - 2 = L_1 - 2(N-1)
\end{aligned}
$$

Adding the first i - 1 equations of yields

$$L_i = iL_1 - i(i-1)$$

Using the fact that $P_N = 0$, we obtain

$$L_1 = N - 1$$

and

$$L_i = i(N-1) - i(i-1) = Ni - i^2 = i(N-i)$$

So the expected length of the game is i(N-i).

11. Customers arrive at a bank according to a Poisson process with rate λ.

(a) Suppose exactly one customer arrived during the first hour. What is the probability that he/she arrived during the first 20 minutes?

(b) Suppose that exactly two customers arrived during the first hour. What is the probability that exactly one had arrived by 20 minutes?

(c) Suppose that exactly two customers arrived during the first hour. What is the probability that at least one arrived in the first 20 minutes?

Solution

(a) Let T be the arrival time of the packet. We calculate $P(T \leq 20|N(0,60)=1)$. We have:

$$P(T \leq 20|N(0,60)=1) = \frac{P(T \leq 20, N(0,60)=1)}{P(N(0,60)=1)} = \frac{P(N(0,20]=1)P(N(20,60]=0)}{P(N(0,60)=1)}$$

We can compute each component now.

$$\frac{P(N(0,20]=1)P(N(20,60]=0)}{P((0,60)=1)} = \frac{(20\lambda)e^{-\lambda 20}e^{-\lambda(40)}}{60\lambda e^{-\lambda 60}} = \frac{20}{60} = \frac{1}{3}$$

Thus, we have:

$$P(T \leq 20|N(0,60)=1) = \frac{1}{3}$$

Which is uniform.

(b) Similar to part (a), we can compute the probability as:

$$\frac{P(N(0,20]=1)P(N(20,60]=1)}{P(N(0,60)=2)} = \frac{(20\lambda)e^{-\lambda 20}(40\lambda)e^{-\lambda(40)}}{\frac{(60\lambda)^2}{2}e^{-\lambda 60}} = \frac{800}{1800} = \frac{4}{9}$$

(c) we first compute the probability that both of the cars arrive in the next forty minutes. We compute the probability as:

$$\frac{P(N(0,20]=0)P(N(20,60]=2)}{P(N(0,60)=2)} = \frac{e^{-\lambda 20}(\frac{(40\lambda)^2}{2})e^{-\lambda(40)}}{\frac{(60\lambda)^2}{2}e^{-\lambda 60}} = \frac{800}{1800} = \frac{4}{9}$$

Thus, that at least one arrived in the first 20 minutes is equal to $1 - \frac{4}{9} = \frac{5}{9}$