ECE 4110/5110	Thursday, 10/19/23
Homework Set 4 (Due: Monday, 10/30/23, 11:59 pm)	
Dr. Kevin Tang	HW 4

P1. Branching Processes

In this problem, we will simulate a branching process. Let $Z_0, Z_1, ...$ be a branching process where the number of children of each node comes from a Poisson distribution with parameter λ.

(a) First assume $\lambda = 2$, simulate this branching process for 5 generations (until Z_5) and record the number of nodes (Z_5). Repeat this simulation 20 times and compute the average number of children at 5th generation. Does your result match with the theoretical result?

(b) Now, assume $\lambda = \frac{1}{2}$, simulate this branching process until the branching process is extinct. Repeat this simulation 20 times and record their extinction times. How many times was the process extinct after 1 generation? Does that match the theoretical results. What is the average time to extinction?

Solution.

(a) We have

$$E[Z_5] = E[Z_1]^5 = 2^5 = 32$$

(b) We have

$$P(\text{Extinct after 1 generation}) = P(Z_1 = 0) = e^{-0.5} \simeq 0.6$$

P2. Consider a branching process whose family sizes have the geometric mass function $f(k) = qp^k$, $k \geq 0$, where $p + q = 1$, and let Z_n be the size of the nth generation. Let $T = min\{n : Z_n = 0\}$ be the extinction time, and suppose that $Z_0 = 1$. Find P(T = n).

Solution.
Suppose that each family size has the mass function $f(k) = qp^k$, for $k \geq 0$, where $q = 1 - p$. Then $G(s) = q(1 - ps)^{-1}$, and each family size is one member less than a geometric variable. We can show by induction that

$$G_n(s) = \begin{cases} \frac{n-(n-1)s}{n+1-ns}, & p = q = \frac{1}{2} \\ \frac{q[p^n - q^n - ps(p^{n-1} - q^{n-1}]}{p^{n+1} - q^{n+1} - ps(p^n - q^n)}, & p \neq q \end{cases}$$

Thus, we have:

$$P(Z_n = 0) = G_n(0) = \begin{cases} \frac{n}{n+1}, & p = q = \frac{1}{2} \\ \frac{q[p^n - q^n]}{p^{n+1} - q^{n+1}} & p \neq q \end{cases}$$

Now let T be the time of extinction. We want to compute $P(T = n)$. We have:

$$P(T = n) = P(Z_n = 0 \& Z_{n-1} \neq 0) = P(Z_n = 0) - P(Z_{n-1} = 0)$$

This can be computed from the above results:

$$P(T = n) = \begin{cases} \frac{1}{n(n+1)}, & p = q = \frac{1}{2} \\ \frac{p^{n-1} q^n (p-q)^2}{(p^n - q^n)(p^{n+1} - q^{n+1})} & p \neq q \end{cases}$$

P3. Monte Carlo Methods
We are trying to approximate the number π. One way of doing so is using Monte Carlo Methods and Law of Large Numbers. We will choose two random numbers (x and y) from a uniform distribution U(-1,1). This way we have a random point in the square $[-1, 1] \times [-1, 1]$. Then, we will check whether that point is inside the unit circle.

(a) What is the probability of a random point chosen in the manner above being in the unit circle.

(b) Propose a way to use this fact to estimate π.

(c) Write a code to simulate this experiment (at least 1000 samples) and report the results.

Solution

(a) The probability of the point falling in to the circle is equal to:

$$\frac{\text{area of the circle}}{\text{area of the square}} = \frac{\pi}{4}$$

(b) Repeat the experiment above N times and count the number of times the point falls in the circle(n). By law of large numbers we know that:

$$\lim_{N\to\infty} 4\frac{n}{N} = \pi$$

Thus if repeat the experiment for a large N, we can approximate π.

P4. Metropolis Hastings Algorithm
The Metropolis Hastings algorithm is a beautifully simple algorithm for producing samples from distributions that may otherwise be difficult to sample from.

Suppose we want to sample from a distribution π, which we will call the "target" distribution. For simplicity we assume that π is a one-dimensional distribution on the real line, although it is easy to extend to more than one dimension (see below).

The MH algorithm works by simulating a Markov Chain, whose stationary distribution is π. This means that, in the long run, the samples from the Markov chain look like the samples from π. As we will see, the algorithm is incredibly simple and flexible. Its main limitation is that, for difficult problems, "in the long run" may mean after a very long time. However, for simple problems the algorithm can work well.

The transition kernel: To implement the MH algorithm, the user (you!) must provide a "transition kernel", Q. A transition kernel is simply a way of moving, randomly, to a new position in space (y say), given a current position (x say). That is, Q is a distribution on y given x, and we will write it $Q(y|x)$.

The MH algorithm: for sampling from a target distribution π, using transition kernel Q, consists of the following steps:

- Initialize, $X_1 = x_1$ say.
- For t=1,2,...
  - sample y from $Q(y|x_t)$. Think of y as a "proposed" value for x_{t+1}.
  - Compute
  $$A = min(1, \frac{\pi(y)Q(x_t|y)}{\pi(x_t)Q(y|x_t)}).$$
  - A is often called the "acceptance probability". with probability A "accept" the proposed value, and set $x_{t+1} = y$. Otherwise set $x_{t+1} = x_t$.

Example (You will code this part): Assume, we want to sample from the following distribution:

$$\pi(i) = \begin{cases} \frac{1}{2^{6-i}}, & \text{for i=1,2,...,5} \\ \frac{1}{2^5}, & \text{for i=0} \end{cases}$$

We do that using the following kernel:

$$Q(y|x) = \frac{1}{6}, \qquad \forall y, x \in \{0, 1, 2, 3, 4, 5\}$$

Simulate the above Markov Chain Monte Carlo for n=100,1000,10000 steps. Find the frequency of visiting each state and plot the histogram. Compare the results for different n. Does it get closer to π?
Solution. As n grows larger the distribution will get closer to π.

P5. MDP

A stuntman performs stunts for a living. At each day, she is either "Healthy", has "Minor injuries", or has "Major injuries".

- If she is healthy , She can choose whether to do a low risk stunt or a high risk stunt.
  - Low Risk Stunt: This stunt Pays 100 Dollars (reward). She will stay Healthy with probability 0.7, Sustain minor injuries with probability 0.3. (She will never sustain a major injury.)
  - High Risk Stunt: This stunt pays 400$. But she will sustain major injuries.
- If she has minor injuries she will be healthy the next day with probability 1.
- If she has major injuries she will still have major injuries the day after with probability 0.5. She will heal a little bit and have minor injuries with probability 0.5.

The days she is in hospital/injured, she will make no money. Now, answer the following questions:

(a) Assume, the stunt woman is retiring tomorrow and she is healthy today. What should she do?

(b) Now assume she has n=2 days till retirement and she is healthy today. What should she do?

(c) Answer The above question for n=3.

(d) Now, write a code to find the optimal policy for n=7.

(e) One day, While wandering the wilderness, she comes across a potion. She drinks it and becomes immortal! Now she faces a predicament. She is debating (for the maximum average reward) is it better to perform high risk stunts every time or perform low risk stunts all the time (all the times when she healthy!). Help her decide!

Solution.

(a) Define $r('H', i), r('Min', i), r('Maj', i)$ as the maximum reward obtainable if there are i days to go and stunt woman is at state above. For 1 day we have:

$$r('H', 1) = max(400, 100) = 400$$

$$r('Min', 1) = 0$$

$$r('Maj', 1) = 0$$

Thus, the most she can earn is 400 dollars by performing a high risk stunt.

(b) For 2 days we have.

$$r('H',2) = max(400{+}r('Maj',1), 100{+}0.7r('H',1){+}0.3r('Min',1) = max(400,380) = 400$$

$$r('Min',2) = r('H',1) = 400$$

$$r('Maj',2) = 0.5r('Maj',1) + 0.5r('Min',1) = 0$$

Thus, the most she can earn is 400 dollars by performing a high risk stunt at day 1.

(c) For 3 days we have.

$$r('H',3) = max(400{+}r('Maj',2), 100{+}0.7r('H',2){+}0.3r('Min',2) = max(400,500) = 500$$

$$r('Min',3) = r('H',2) = 400$$

$$r('Maj',3) = 0.5r('Maj',2) + 0.5r('Min',2) = 200$$

Thus, the most she can earn is 500 dollars by performing a low risk stunt at day 1.

(d) Code and you will see The best strategy is do high risk stunt at day 1.

(e) If she does the high risk stunt the Markov chain looks like:

$$\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

The stationary distribution is (0.25,0.25,0.5). Thus the average reward is equal to $400 \times 0.25 = 100$.

If she does low risk stunts the Markov chain looks like this:

$$\begin{bmatrix} 0.7 & 0.3 & 0 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}$$

The stationary distribution is (0.77,0.23,0). Thus the average reward is equal to $100 \times 0.77 = 77$.
Thus, doing high risk stunts is a better strategy.