ECE 4110/5110	Monday, 11/13/23

Homework Set 6 (Due: Monday, 11/27/23, 11:59 pm)

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Note: Note that there are multiple definitions of the spectral density function. We will use the definition given below (this is a different definition from the Grimmet book):

$$S_X(f) = \mathcal{F}\{R_x(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau) e^{-2\pi f \tau} d\tau$$

Where $\mathcal{F}$ is the Fourier transform of function R and:

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{2\pi f \tau} df$$

Also a table of Fourier transforms is provided here:

f(x)	$\mathcal{F}\{f(x)\}$
$e^{-2\pi k\|x\|}$	$\frac{1}{\pi}\frac{k}{f^2+k^2}$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}} e^{-\pi^2 f^2/a}$

For the discrete time case:

$$S_X(f) = \mathcal{F}\{R_x(\tau)\} = \sum_{\tau=-\infty}^{\infty} R_X(\tau) e^{-2\pi f \tau}$$

Where $\mathcal{F}$ is the Fourier transform of function R and:

$$R_X(\tau) = \int_{-0.5}^{0.5} S_X(f) e^{2\pi f \tau} df$$

1 Problems from "Probability and Random Processes" Geoffrey Grimmet

P1. (Section 9.2 - Problem 1) Let X be a (weakly) stationary sequence with zero mean and autocovariance function c(m).

(i) Find the best linear predictor $\hat{X}_{n+1}$ of X_{n+1} given X_n.

(ii) Find the best linear predictor $\tilde{X}_{n+1}$ of X_{n+1} given X_n and X_{n-1}.

(iii) Find an expression for $D = E(X_{n+1} - \hat{X}_{n+1})^2 - E(X_{n+1} - \tilde{X}_{n+1})^2$, and evaluate this expression when:

  (a) $X_n = cos(nU)$ where U is uniform on $[-\pi, \pi]$,

  (b) X is an autoregressive scheme with $c(k) = \alpha^{|k|}$ where $|\alpha| < 1$.

Solution.

(i) We have that
$$E\{(X_{n+1} - aX_n)^2\} = (1 + a^2)c(0) - 2ac(1),$$
which is minimized by setting $a = c(1)/c(0)$. Hence $\hat{X}_{n+1} = c(1)X_n/c(0)$.

(ii) Similarly
$$E[(X_{n+1} - \beta X_n - \gamma X_{n-1})^2] = (1 + \beta^2 + \gamma^2)c(0) + 2\beta(\gamma - 1)c(1) - 2\gamma c(2)$$
an expression which is minimized by the choice
$$\beta = \frac{c(1)(c(0) - c(2))}{c(0)^2 - c(1)^2}, \gamma = \frac{c(0)c(2) - c(1)^2}{c(0)^2 - c(1)^2}$$
$\tilde{X}_{n+1}$ is given accordingly.

(iii) Substitute α, β, γ into the above equations, and subtract to obtain, after some manipulation,
$$D = \frac{(c(1)^2 - c(0)c(2))^2}{c(0)(c(0)^2 - c(1)^2)}$$

  (a) In this case c(0) =0.5 and c(1) =c(2) =0. Therefore $\hat{X}_{n+1} = \tilde{X}_{n+1} = 0$, and D=0.

  (b) Also, in this case D = 0

P2. (Section 9.2 - Problem 2) Suppose $|a| < 1$. Does there exist a (weakly) stationary sequence $\{X_n : -\infty < n < \infty\}$ with zero means and autocovariance function

$$c(k) = \begin{cases} 1, & k = 0 \\ \frac{a}{1+a^2}, & |k| = 1 \\ 0, & OW \end{cases}$$

Assuming that such a sequence exists, find the best linear predictor $\hat{X}_n$ of X_n given $X_{n-1}, X_{n-2}, \cdots$, and show the mean square of prediction is $(1 + a^2)^{-1}$.

Solution. Let $(Z_n : n = ..., -1, 0, 1, ...)$ be independent random variables with zero means and unit variances, and define the moving-average process

$$Xn = \frac{Z_n + aZ_{n-1}}{\sqrt{1 + a^2}}$$

It is easily checked that X has the required autocovariance function. By the orthogonality theorem, $X_n - \hat{X}$, is orthogonal to the collection $(X_{n-r} : r \geq 1)$, so that $E[(X_n - \hat{X}_n)X_{n-r}] = 0, r \geq 1$. Set $\hat{X}_n = \sum_{s=1}^{\infty} b_s X_{n-s}$ to obtain that

$$\alpha = b_1 + b_2\alpha, \qquad 0 = b_{s-1}\alpha + b_s + b_{s+1}\alpha, \forall s \geq 2,$$

where $\alpha = a/(1 + a^2)$. The unique bounded solution to the above difference equation is $b_s = (-1)^{s+1}a^s$, and therefore

$$\hat{X}_n = \sum_{s=1}^{\infty}(-1)^{s+1}a^s X_{n-s}$$

The mean squared error of prediction is

$$E[(X_n - \hat{X}_n)^2] = E[(\sum_{s=0}^{\infty}(-a)^s X_{n-s})^2] = \frac{1}{1 + a^2}E[Z_n^2] = \frac{1}{1 + a^2}$$

P3. (Section 9.3 - Problem 3) Find the autocorrelation function of the stationary process $\{X(t) : -\infty < t < \infty\}$ whose spectral density function is:

(a) N(0, 1)

(b) $f(x) = e^{-|x|}$

Solution.

(a) Looking at the table at the table and plugging in a=$2\pi^2$. We get the result needed. $R_X(t) = e^{-2\pi^2 t^2}$.

(b) Similarly, looking at the table, and plugging in $k = \frac{1}{2\pi}$ we get that $R_X(t) = \frac{2}{4\pi^2 t^2+1}$. Note that you can use the table from both sides. Cause $\mathcal{F}(\mathcal{F}(f(x)) = f(x)$

P4. (Section 9.7 - Problem 1) Let $\cdots, X_{-1}, X_0, X_1, \cdots$ be uncorrelated random variables with zero means and unit variances, and define

$$Y_n = X_n + \alpha \sum_{i=1}^{\infty} \beta^{i-1} X_{n-i}, \qquad -\infty < n < \infty$$

where α and β are constants satisfying $|\beta| < 1$, $|\beta - \alpha| < 1$. Find the best linear predictor of Y_{n+1} given the entire past $Y_n, Y_{n-1}, \cdots$

Solution. It is easily seen that $Y_n = X_n + (\alpha - \beta) X_{n-1} + \beta Y_{n-1}$, whence the autocovariance function c of Y is given by

$$c(k) = \begin{cases} \frac{1+\alpha^2-\beta^2}{1-\beta^2} & k = 0 \\ \beta^{|k|-1} \{ \frac{\alpha(1+\alpha\beta-\beta^2)}{1-\beta^2} \} & k \neq 0 \end{cases}$$

Set $\hat{Y}_{n+1} = \sum_{i=0}^{\infty} a_i Y_{n-i}$ and find the a_i for which it is the case that $E\{(Y_{n+1} - \hat{Y}_{n+1}) Y_{n-k}\} = 0$ for $k > 0$. These equations yield

$$c(k+1) = \sum_{i=0}^{\infty} a_i c(k-i), \qquad k \geq 0$$

which have solution $a_i = \alpha(\beta - \alpha)^i$ for $i \geq 0$.

2 Extra Problems.

P5. Let X(t) be a WSS, continuous-time process.

(a) Use the orthogonality principle to find the best estimator for X_t of the form

$$\hat{X}_t = aX_{t_1} + bX_{t_2},$$

where t_1 and t_2 are given time instants.

(b) Find the mean square error of the optimum estimator.

(c) Check your work by evaluating the answer in part b for $t = t_1$ and $t = t_2$. Is the answer what you would expect?

Solution.

(a) Using the orthogonality principle we should have:

$$E[(X_t - \hat{X}_t)X_{t_1}] = 0$$
$$E[(X_t - \hat{X}_t)X_{t_2}] = 0$$

Which gives us the equations:

$$E[X_t X_{t_1} - aX_{t_1}X_{t_1} - bX_{t_2}X_{t_1}] = 0$$
$$E[X_t X_{t_2} - aX_{t_1}X_{t_2} - bX_{t_2}X_{t_2}] = 0$$

Thus:

$$c(t - t_1) - ac(0) - bc(t_2 - t_1) = 0$$
$$c(t - t_2) - ac(t_1 - t_2) - bc(0) = 0$$

Which gives us:

$$a = \frac{c(0)c(t - t_1) - c(t - t_2)c(t_2 - t_1)}{c(0)^2 - c(t_2 - t_1)^2}$$

and

$$b = \frac{c(0)c(t - t_2) - c(t - t_1)c(t_2 - t_1)}{c(0)^2 - c(t_2 - t_1)^2}$$

(b) Now, we find the mean square error.

$$
\begin{aligned}
&E[(X_t - \hat{(X)}_t)^2] \\
&= E[X_t^2] + a^2E[X_{t_1}^2] + b^2E[X_{t_2}^2] - 2aE[X_tX_{t_1}] - 2bE[X_tX_{t_2}] + 2abE[X_{t_1}X_{t_2}] \\
&= (1 + a^2 + b^2)c(0) - 2ac(t - t_1) - 2bc(t - t_2) + 2abc(t_2 - t_1)
\end{aligned}
$$

Where a and b are as defined in part (a).

(c) If $t = t_1$, plugging in the result from part (a), we will have $a = 1, b = 0$. The error will be (using the result from part (b):

$$(1 + 1)c(0) - 2c(0) = 0$$

Similarly, if $t = t_2$, plugging in the result from part (a), we will have $a = 0, b = 1$. The error will be:

$$(1 + 1)c(0) - 2c(0) = 0$$

P6. Let $R_X(k) = 4\alpha^{|k|}$, $|\alpha| < 1$.

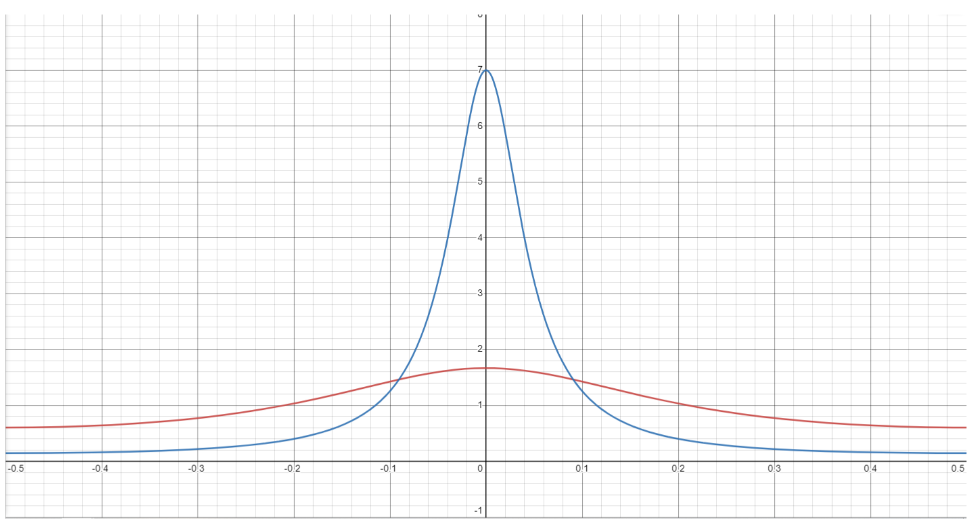
(a) Find the power spectral density function.

(b) Plot the function found above for $\alpha = 0.25$ and $\alpha = 0.75$, and comment on the effect of the value of α.

Solution. Plugging in the definition for Fourier transform we should have:

$$\begin{aligned}
&S_X(f) \\
&= \sum_{k=-\infty}^{\infty} R_X(k)e^{-j2\pi fk} \\
&= \sum_{k=-\infty}^{\infty} 4\alpha^{|k|}e^{-j2\pi fk} \\
&= \sum_{k=0}^{\infty} 4\alpha^k e^{-j2\pi fk} + \sum_{k=0}^{\infty} 4\alpha^k e^{j2\pi fk} - 4 \\
&= 4(\frac{1}{1-\alpha e^{-j2\pi f}} + \frac{1}{1-\alpha e^{j2\pi f}} - 1) \\
&= 4(\frac{1-\alpha^2}{1+\alpha^2 - 2\alpha cos(2\pi f)})
\end{aligned}$$

Plotting it for $\alpha = 0.25, 0.75$. The blue line is 0.75 and red is 0.25.



P7.

(a) Let the process X_n be a sequence of uncorrelated random variables with zero mean and variance 1. Find the spectral density function.

(b) Let the process Y_n be defined by

$$Y_n = X_n + \alpha X_{n-1}$$

where is the white noise process of part a. Find the spectral density function.

Solution

(a) We need to find the autocovariance function first. For k=0, we have:

$$R_X(0) = E[X_n^2] = 1$$

and for $k \neq 0$:

$$R_X(k) = E[X_n X_{n+k}] = 0$$

We used independence to get the result. Now plugging in the definition we have:

$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi fk} = 1$$

(b) We need to find the autocovariance function first. For k=0, we have:

$$R_X(0) = E[Y_n^2] = E[X_n^2] + \alpha^2 E[X_{n-1}^2] + 2\alpha E[X_n X_{n-1}] = 1 + \alpha^2$$

and for $|k| = 1$:

$$R_X(k) = E[X_n X_{n-1}] = E[X_n X_{n-1}] + \alpha^2 E[X_{n-1} X_{n-2}] + \alpha E[X_{n-1}^2] + \alpha E[X_{n-2} X_n] = \alpha$$

and for $|k| > 1$:

$$R_X(k) = E[X_n X_{n+k}] = 0$$

We used independence to get the result. Now plugging in the definition we have:

$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi fk} = 1 + \alpha^2 + \alpha(e^{-j2\pi f} + e^{j2\pi f}) = 1 + \alpha^2 + 2\alpha cos(2\pi f)$$

P8. (Discrete Wiener Filter) In class, we discussed the Wiener Filter for the continuous time. Using the similar approach, we can find a filter for the discrete time. More specifically, given $X(n) = Z(n)+N(n)$ where $n \in Z$ and $Z(n)$ and $N(n)$ are independent, zero- mean random processes, the optimum filter for estimating Z(k) from the entire values of X is:

$$H(f) = \frac{S_Z(f)}{S_Z(f) + S_N(f)}$$

This is the same as continuous time, but the signals are discrete.
Using the relation above find the optimal filter for when N(k) is zero-mean white noise density 1 and Z(n) is a first-order autoregressive process with $\sigma_x^2 = 1$ and $\alpha = 0.5$ which is described below. (you only need to find the frequency response of the filter $H(f)$)
A first-order autoregressive process Y_n with zero mean is defined by $Y_n = \alpha Y_{n-1} + X_n$, where X_n is a zero-mean white noise input random process with average power σ_x^2.

Solution. First, we find the autocorrelation function of Z(n). We have: For k=0, we have:

$$\begin{aligned}
&R_Z(0) = E[Z_n^2] \\
&= E[(\alpha Z_{n-1} + X_n)(\alpha Z_{n-1} + X_n)] \\
&= \alpha^2 E[Z_{n-1}^2] + 2\alpha E[Z_{n-1}X_n] + E[X_n^2] \\
&= \alpha^2 R_Z(0) + 1 \\
&= \frac{1}{1-\alpha^2}
\end{aligned}$$

For $k > 0$, we have:

$$\begin{aligned}
&R_Z(k) = E[Z_n Z_{n+k}] \\
&= E[(\alpha Z_{n+k-1} + X_{n+k})Z_n)] \\
&= \alpha E[Z_{n+k-1}Z_n] + E[X_{n+k}Z_n] \\
&= \alpha R_Z(k-1)
\end{aligned}$$

Thus, by symmetry of autocorrelation function we have:

$$R_Z(k) = \frac{\alpha^{|k|}}{1-\alpha^2}$$

This is similar to P6, we have:

$$S_Z(f) = \frac{1}{1+\alpha^2 - 2\alpha cos(2\pi f)} = \frac{1}{1.25 - cos(2\pi f)}$$

Thus, the optimal filter is:

$$H(f) = \frac{\frac{1}{1.25-cos(2\pi f)}}{1 + \frac{1}{1.25-cos(2\pi f)}} = \frac{1}{2.25 - cos(2\pi(f))}$$