ECE 4110/5110	Monday, 11/13/23

Lecture 21: Wiener Filter

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Problem Setup

Let $X(t) = S(t) + N(t)$ pass through a LTI system with impulse response function $h(t)$. Let $Y(t)$ be the output of the system. We want to use $Y(t)$ as $\hat{S}(t+\alpha)$, i.e., the estimation of $S(t+\alpha)$.

We seek estimation optimality in the MMSE sense. In other words, we want to

$$\min_{h(t) \in LTI} E(((S(t+\alpha) - \hat{S}(t+\alpha))^2)) \tag{1}$$

If $\alpha > 0$, the problem is optimal prediction; If $\alpha = 0$, we call the problem optimal filtering; And if $\alpha < 0$, we are considering optimal smoothing.

Wiener-Hopf Equation

We assume $E(S(t)) = E(N(t)) = 0$, and $E(S(t)N(t)) = 0$.

By the orthogonality principle, we should minimize

$$E((S(t+\alpha) - Y(t))Y(\theta)) = 0 \tag{2}$$

for any $-\infty < \theta < t$.

This yields the following Wiener-Hopf equation.

$$C_S(u+\alpha) = \int_{-\infty}^{\infty} h(\tau) C_X(u-\tau) d\tau \tag{3}$$

Its frequency domain version is

$$S_S(\omega) e^{i\alpha\omega} = H(\omega) S_X(\omega) \tag{4}$$

Hence we have

$$H(\omega) = \frac{S_S(\omega) e^{i\omega\alpha}}{S_X(\omega)} = \frac{S_S(\omega) e^{i\omega\alpha}}{S_S(\omega) + S_N(\omega)} \tag{5}$$