ECE 4110/5110	Monday, 11/13/23

Homework Set 6 (Due: Monday, 11/27/23, 11:59 pm)

Dr. Kevin Tang	HW 6

Note: Note that there are multiple definitions of the spectral density function. We will use the definition given below (this is a different definition from the Grimmet book):

$$S_X(f) = \mathcal{F}\{R_x(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau)e^{-2\pi f\tau}d\tau$$

Where $\mathcal{F}$ is the Fourier transform of function R and:

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)e^{2\pi f\tau}df$$

Also a table of Fourier transforms is provided here:

f(x)	$\mathcal{F}\{f(x)\}$
$e^{-2\pi k\lvert x\rvert}$	$\frac{1}{\pi}\frac{k}{f^2+k^2}$
e^{-ax^2}	$\sqrt{\frac{\pi}{a}}e^{-\pi^2 f^2/a}$

For the discrete time case:

$$S_X(f) = \mathcal{F}\{R_x(\tau)\} = \sum_{\tau=-\infty}^{\infty} R_X(\tau)e^{-2\pi f\tau}$$

Where $\mathcal{F}$ is the Fourier transform of function R and:

$$R_X(\tau) = \int_{-0.5}^{0.5} S_X(f)e^{2\pi f\tau}df$$

1 Problems from "Probability and Random Processes" Geoffrey Grimmet

P1. (Section 9.2 - Problem 1) Let X be a (weakly) stationary sequence with zero mean and autocovariance function c(m).

(i) Find the best linear predictor $\hat{X}_{n+1}$ of X_{n+1} given X_n.

(ii) Find the best linear predictor $\tilde{X}_{n+1}$ of X_{n+1} given X_n and X_{n-1}.

(iii) Find an expression for $D = E(X_{n+1} - \hat{X}_{n+1})^2 - E(X_{n+1} - \tilde{X}_{n+1})^2$, and evaluate this expression when:

   (a) $X_n = cos(nU)$ where U is uniform on $[-\pi, \pi]$,

   (b) X is an autoregressive scheme with $c(k) = \alpha^{|k|}$ where $|\alpha| < 1$.

P2. (Section 9.2 - Problem 2) Suppose $|a| < 1$. Does there exist a (weakly) stationary sequence $\{X_n : -\infty < n < \infty\}$ with zero means and autocovariance function

$$c(k) = \begin{cases} 1, & k = 0 \\ \frac{a}{1+a^2}, & |k| = 1 \\ 0, & OW \end{cases}$$

Assuming that such a sequence exists, find the best linear predictor $\hat{X}_n$ of X_n given $X_{n-1}, X_{n-2}, \cdots$, and show the mean square of prediction is $(1 + a)^{-1}$.

P3. (Section 9.3 - Problem 3) Find the autocorrelation function of the stationary process $\{X(t) : -\infty < t < \infty\}$ whose spectral density function is:

(a) N(0, 1)

(b) $f(x) = e^{-|x|}$

P4. (Section 9.7 - Problem 1) Let $\cdots, X_{-1}, X_0, X_1, \cdots$ be uncorrelated random variables with zero means and unit variances, and define

$$Y_n = X_n + \alpha \sum_{i=1}^{\infty} \beta^{i-1} X_{n-i}, \qquad -\infty < n < \infty$$

where α and β are constants satisfying $|\beta| < 1$, $|\beta - \alpha| < 1$. Find the best linear predictor of Y_{n+1} given the entire past $Y_n, Y_{n-1}, \cdots$

2 Extra Problems.

P5. Let X(t) be a WSS, continuous-time process.

(a) Use the orthogonality principle to find the best estimator for X_t of the form

$$\hat{X}_t = aX_{t_1} + bX_{t_2},$$

where t_1 and t_2 are given time instants.

(b) Find the mean square error of the optimum estimator.

(c) Check your work by evaluating the answer in part b for $t = t_1$ and $t = t_2$. Is the answer what you would expect?

P6. Let $R_X(k) = 4\alpha^{|k|}$, $|\alpha| < 1$.

(a) Find the power spectral density function.

(b) Plot the function found above for $\alpha = 0.25$ and $\alpha = 0.75$, and comment on the effect of the value of α.

P7.

(a) Let the process X_n be a sequence of uncorrelated random variables with zero mean and variance 1. Find the spectral density function.

(b) Let the process Y_n be defined by

$$Y_n = X_n + \alpha X_{n-1}$$

where is the white noise process of part a. Find the spectral density function.

P8. (Discrete Wiener Filter) In class, we discussed the Wiener Filter for the continuous time. Using the similar approach, we can find a filter for the discrete time. More specifically, given $X(n) = Z(n)+N(n)$ where $n \in Z$ and $Z(n)$ and $N(n)$ are independent, zero- mean random processes, the optimum filter for estimating Z(k) from the entire values of X is:

$$H(f) = \frac{S_Z(f)}{S_Z(f) + S_N(f)}$$

This is the same as continuous time, but the signals are discrete.
Using the relation above find the optimal filter for when N(k) is zero-mean white noise density 1 and Z(n) is a first-order autoregressive process with $\sigma_x^2 = 1$ and $\alpha = 0.5$ which is described below. (you only need to find the frequency response of the filter $H(f)$)
A first-order autoregressive process Y_n with zero mean is defined by $Y_n = \alpha Y_{n-1} + X_n$, where X_n is a zero-mean white noise input random process with average power σ_x^2.