ECE 4110/5110

Random Signals in Communication and Signal Processing

Homework 1

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Sept 8, 2023

Contents

1 Problems from Bertsekas and Tsitsiklis, 2nd edition

1.1 Problem 1 and Corresponding Solution

Problem 1 (Ch2 P32)

Consider $2m$ persons forming m couples who live together at a given time. Suppose that at some later time, the probability of each person being alive is p, independent of other persons. At that later time. let A be the number of persons that are alive and let S be the number of couples in which both partners are alive. For any survivor number a, find $E[S|A = a]$.

Solution:

Let X_i denote the random variable representing the survival status of the first partner in the ith couple, taking a value of 1 if they survived and 0 otherwise.

Similarly, let Y_i denote the random variable for the survival status of the second partner in the ith couple, adopting the same value assignments as X_i.

Given this, we can express S as the sum of the products of X_i and Y_i for each couple, mathematically represented as:

$$S = \sum_{i=1}^{m} X_i Y_i$$

According to the total expectation theorem, we have

$$\begin{aligned}
E[S \mid A = a] &= \sum_{i=1}^{m} E\left[X_i Y_i \mid A = a\right] \\
&= mE\left[X_1 Y_1 \mid A = a\right] \\
&= mE\left[Y_1 \mid X_1 = 1, A = a\right] P\left(X_1 = 1 \mid A = a\right) \\
&= mP\left(Y_1 = 1 \mid X_1 = 1, A = a\right) P\left(X_1 = 1 \mid A = a\right)
\end{aligned}$$

We have

$$P\left(Y_1 = 1 \mid X_1 = 1, A = a\right) = \frac{a-1}{2m-1}, \qquad P\left(X_1 = 1 \mid A = a\right) = \frac{a}{2m}$$

Hence

$$E[S \mid A = a] = m \cdot \frac{a-1}{2m-1} \cdot \frac{a}{2m} = \frac{a(a-1)}{2(2m-1)}$$

1.2 Problem 2 and Corresponding Solution

Problem 2 (Ch3 P20)

An absent-minded professor schedules two student appointments for the same time. The appointment durations are independent and exponentially distributed with mean thirty minutes. The first student arrives on time, but the second student arrives five minutes late. What is the expected value of the time between the arrival of the first student and the departure of the second student?

Solution:

The expected value of the time between the arrival of the first student and the departure of the second student is:

$$E[T] = (5 + E[T_2]) \cdot P(T_1 < 5) + (E[T_1 \mid T_1 \geq 5] + E[T_2]) \cdot P(T_1 \geq 5)$$

We have $E[T_2] = 30$, and due to the memorylessness property of the exponential distribution,

$$E[T_1 \mid T_1 \geq 5] = E[T_1] + 5 = 35$$

Also

$$P(T_1 \geq 5) = e^{-5/30}$$

$$P(T_1 < 5) = 1 - e^{-5/30}$$

By substitution we obtain

$$E[T] = (5 + 30) \cdot (1 - e^{-5/30}) + (35 + 30) \cdot e^{-5/30} = 35 + 30 \cdot e^{-5/30} = 60.394$$

1.3 Problem 3 and Corresponding Solution

Problem 3 (Ch3 P25)

The coordinates X and Y of a point are independent zero mean normal random variables with common variance σ^2. Given that the point is at a distance of at least c from the origin, find the conditional joint PDF of X and Y.

Solution:

Let Z denote the event that $X^2 + Y^2 \geq c^2$. Then we have

$$\begin{aligned} P(Z) &= \frac{1}{2\pi\sigma^2} \int_0^{2\pi} \int_c^{\infty} re^{-r^2/2\sigma^2} dr d\theta \\ &= \frac{1}{\sigma^2} \int_c^{\infty} re^{-r^2/2\sigma^2} dr \\ &= e^{-c^2/2\sigma^2} \end{aligned}$$

Thus we have

$$f_{X,Y|Z}(x.y) = \frac{f_{X,Y}(x,y)}{P(Z)} = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2\sigma^2}(x^2+y^2-c^2)} \qquad (x,y) \in Z$$

1.4 Problem 4 and Corresponding Solution

Problem 4 (Ch4 P18)

Consider four random variables, W, X, Y, Z with

$$E[W] = E[X] = E[Y] = E[Z] = 0$$

$$var(W) = var(X) = var(Y) = var(Z) = 1$$

and assume that W. X, Y. Z are pairwise uncorrelated. Find the correlation coefficients $\rho(R, S)$ and $\rho(R, T)$. where R = W + X, S = X + Y. and T = Y + Z.

Solution:

We have W. X, Y. Z are pairwise uncorrelated, so

$$\begin{aligned}
cov(R, S) &= E[RS] - E[R]E[S] \\
&= E[(W + X)(X + Y)] - E[W + X]E[X + Y] \\
&= E[WX + WY + X^2 + XY] - (E[W] + E[X])(E[X] + E[Y]) \\
&= E[W]E[X] + E[W]E[Y] + E[X^2] + E[X]E[Y] \\
&= E[X^2] \\
&= Var(X) + (E[X])^2 \\
&= 1
\end{aligned}$$

and

$$var(R) = var(W + X) = Var(W) + Var(X) = 2$$

$$var(S) = var(X + Y) = Var(X) + Var(Y) = 2$$

so

$$\rho(R, S) = \frac{cov(R, S)}{\sqrt{var(R)var(S)}} = \frac{1}{2}$$

The same, we have

$$cov(R, T) = E[RT] - E[R]E[T] = E[WY + WZ + XY + XZ] = 0$$

$$\begin{aligned}
cov(R, T) &= E[RT] - E[R]E[T] \\
&= E[(W + X)(Y + Z)] - E[W + X]E[Y + T] \\
&= E[WY + WZ + XY + XZ] - (E[W] + E[X])(E[Y] + E[T]) \\
&= E[W]E[Y] + E[W]E[Z] + E[X]E[Y] + E[X]E[Z] \\
&= 0
\end{aligned}$$

so

$$\rho(R, T) = 0$$

1.5 Problem 5 and Corresponding Solution

Problem 5 (Ch4 P19)

Suppose that a random variable X satisfies

$$E[X] = 0, \quad E[X^2] = 1, \quad E[X^3] = 0, \quad E[X^4] = 3$$

and let $Y = a + bX + cX^2$. Find the correlation coefficient $\rho(X, Y)$.

Solution:

To compute the correlation coefficient

$$\rho(X, Y) = \frac{cov(X, Y)}{\sigma_X \sigma_Y}$$

we first compute the covariance:

$$\begin{aligned}
cov(X, Y) &= E[XY] - E[X]E[Y] \\
&= E[aX + bX^2 + cX^3] - E[X]E[Y] \\
&= aE[X] + bE[X^2] + cE[X^3] \\
&= b
\end{aligned}$$

we also have

$$var(X) = E[X^2] - E[X]^2 = 1$$

$$\begin{aligned}
Var(Y) &= Var(a + bX + cX^2) \\
&= E[(a + bX + cX^2)^2] - (E[a + bX + cX^2])^2 \\
&= (a^2 + 2ac + b^2 + 3c^2) - (a^2 + c^2 + 2ac) \\
&= b^2 + 2c^2
\end{aligned}$$

Hence

$$\sigma_X = \sqrt{Var(X)} = 1$$

$$\sigma_Y = \sqrt{Var(Y)} = \sqrt{b^2 + 2c^2}$$

so

$$\rho(X, Y) = \frac{b}{\sqrt{b^2 + 2c^2}}$$

1.6 Problem 6 and Corresponding Solution

Problem 6 (Ch4 P24)

A retired professor comes to the office at a time that is uniformly distributed between 9 a.m. and 1 p.m., performs a single task and leaves when the task is completed. The duration of the task is exponentially distributed with parameter, $\lambda(y) = 1/(5{-}y)$, where y is the length of the time interval between 9 a.m. and the time of his arrival:

(a) What is the expected amount of time that the professor devotes to the task?

(b) What is the expected time at which the task is completed?

(c) The professor has a Ph.D. student who on a given day comes to see him at a time that is uniformly distributed between 9 a.m. and 5 p.m. If the student does not find the professor, he leaves and does not return. If he finds the professor, he spends an amount of time that is uniformly distributed between 0 and 1 hour. The professor will spend the same total amount of time on his task regardless of whether he is interrupted by the student. What is the expected amount of time that the professor will spend with the student and what is the expected time at which he will leave his office?

Solution:

(a) Define 2 random variables X and Y, X = amount of time the professor devotes to his task, $X \in [0, 1]$, and Y = length of time between 9 am and his arrival, $Y \in [0, 4]$. We have

$$E[Y] = 2$$
$$E[X \mid Y = y] = \frac{1}{\lambda(y)} = 5 - y$$

then

$$E[X \mid Y] = 5 - Y$$

then

$$E[X] = E[E[X \mid Y]] = E[5 - Y] = 5 - E[Y] = 5 - 2 = 3$$

(b) Let Z be the length of time from 9 am to the professor completes the task. Then

$$Z = X + Y$$

$$E[Z] = E[X] + E[Y] = 5$$

$$9am + 5h = 14pm$$

Thus the expected time that the professor leaves his office is 14 pm.

(c) Define 3 random variables as followed:

A = length of time between 9 am and arrival of the Ph.D. student, $A \in [0.8]$

M = amount of time the student will spend with the professor, if he finds the professor, $M \in [0, 1]$

T = amount of time the professor will spend with the student, $T \in [0, 1]$

Let F be the event that the student finds the professor. Then we have

$$E[T \mid F] = E[M] = \frac{1}{2}$$

$$E[T \mid F^c] = 0$$

Then

$$\begin{aligned} E[T] &= P(F)E[T \mid F] + P(F^c)E[T \mid F^c] \\ &= P(F)E[T \mid F] \\ &= \frac{1}{2}P(F) \end{aligned}$$

If the student can find the professor, his arrival must be between the arrival and the departure of the professor. Thus

$$P(F) = P(Y \leq A \leq X + Y) = 1 - (P(A < Y) + P(A > X + Y))$$

We have

$$P(A < Y) = \int_0^4 \frac{1}{4} \int_0^y \frac{1}{8} dw dy = \frac{1}{4}$$

and

$$\begin{aligned} P(W > X + Y) &= \int_0^4 P(A > X + Y \mid Y = y) f_Y(y) dy \\ &= \int_0^4 P(X < A - Y \mid Y = y) f_Y(y) dy \\ &= \int_0^4 \int_y^8 F_{X|Y}(a - y) f_A(a) f_Y(y) da dy \\ &= \int_0^4 \frac{1}{4} \int_y^8 \frac{1}{8} \int_0^{a-y} \frac{1}{5 - y} e^{-\frac{x}{5-y}} dx da dy \\ &= \frac{12}{32} + \frac{1}{32} \int_0^4 (5 - y) e^{-\frac{8-y}{5-y}} dy \\ &= 0.42995 \end{aligned}$$

Thus,

$$P(F) = 1 - (P(A < Y) + P(A > X + Y)) = 1 - (0.25 + 0.42995) = 1 - 0.67995 = 0.32005$$

By substitution,

$$E[T] = \frac{1}{2}P(F) = \frac{1}{2} \cdot 0.32 = 0.16h = 9.6mins$$

Then we have

$$\begin{aligned} [E[Z] &= P(F)E[Z \mid F] + P(F^c)E[Z \mid F^c] \\ &= P(F)E[X + Y + R] + P(F^c)E[X + Y] \\ &= 5.5P(F) + 5P(F^c) \\ &= 5.5 \times 0.32005 + 5 \times 0.67995 \\ &= 5.160025h \\ &\approx 5.16h \end{aligned}$$

Thus the expected time the professor will leave his office is 5.16 hours after 9 am.

1.7 Problem 7 and Corresponding Solution

Problem 7 (Ch4 P35)

Let X be a random variable that takes nonnegative integer values, and is associated with a transform of the form

$$M_x(s) = c \cdot \frac{3 + 4e^{2s} + 2e^{3s}}{3 - e^s}$$

where c is some scalar. Find $E[X]$, $P_x(1)$, and $E[X|X \neq 0]$.

Solution:

Let $s = 0$, then we have

$$M_X(0) = c \cdot \frac{3+4+2}{3-1} = 1$$

$$c = \frac{2}{9}$$

then

$$E[X] = \frac{dM_X}{ds}(s)\bigg|_{s=0} = \frac{2}{9} \cdot \frac{(3-e)(8e^{2s} + 6e^{3s}) + e^s(3 + 4e^{2s} + 2e^{3s})}{(3-e^s)^2}\bigg|_{s=0} = \frac{37}{18}$$

We know

$$\frac{1}{3-e^s} = \frac{1}{3} \cdot \frac{1}{1 - \frac{e^s}{3}} = \frac{1}{3}(1 + \frac{e^s}{3} + (\frac{e^s}{3})^2 + ...) \qquad \text{(when } e^s < 3)$$

So when $e^s < 3$, we have

$$M_X(s) = \frac{2}{9} \cdot \frac{1}{3} \cdot (3 + 4e^{2s} + 2e^{3s}) \cdot (1 + \frac{e^s}{3} + (\frac{e^s}{3})^2 + ...)$$

By calculating the coefficients of e^{0s} and e^s, we obtain

$$p_X(0) = \frac{2}{9} \qquad\qquad p_X(1) = \frac{2}{27}$$

Let $A = \{X \neq 0\}$. We have

$$p_{X|\{X\in A\}}(k) = \begin{cases} \frac{p_X(k)}{P(A)} & \text{if } k \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

so that

$$\begin{aligned} E[X \mid X \neq 0] &= \sum_{k=1}^{\infty} k p_{X|A}(k) \\ &= \sum_{k=1}^{\infty} \frac{k p_X(k)}{P(A)} \\ &= \frac{E[X]}{1 - p_X(0)} \\ &= \frac{37/18}{7/9} \\ &= \frac{37}{14} \end{aligned}$$

1.8 Problem 8 and Corresponding Solution

Problem 8 (Ch4 P43)

A motorist goes through 4 lights, each of which is found to be red with probability 1/2. The waiting times at each light are modeled as independent normal random variables with a mean 1 minute and a standard deviation 1/2 minute. Let X be the total waiting time at the red lights.

(a) Use the total probability theorem to find the PDF and the transform associated with X, and the probability that X exceeds 4 minutes. Is X normal?

(b) Find the transform associated with X by viewing X as a sum of a random number of random variables.

Solution:

(a) Using the total probability theorem, we have

$$\begin{aligned} P(X > 4) &= \sum_{k=0}^{4} P(\text{k lights are red}) P(X > 4 \mid \text{k lights are red}) \\ &= \sum_{k=0}^{4} \left(\begin{array}{c} 4 \\ k \end{array} \right) (\frac{1}{2})^k (\frac{1}{2})^{4-k} P(X > 4 \mid \text{k lights are red}) \\ &= \sum_{k=0}^{4} \left(\begin{array}{c} 4 \\ k \end{array} \right) (\frac{1}{2})^4 P(X > 4 \mid \text{k lights are red}) \end{aligned}$$

$P(X > 4 \mid \text{k lights are red})$ is normal with k minutes and standard deviation $\frac{\sqrt{k}}{2}$

$$P(X > 4) = \sum_{k=0}^{4} \frac{6}{(4-k)!k!} \left(1 - \Phi(\frac{2(4-k)}{\sqrt{k}}) \right) \approx 0.25$$

X is a mixture of normal random variables, but **X is not normal.**

(b) Let K denote the number of red lights encountered.

The transform of X is then obtained by substituting the normal transform corresponding to a mean (μ) of 1 and a standard deviation (σ) of $\frac{1}{2}$ into the binomial transform

$$M_K(s) = \left(\frac{1}{2} + \frac{1}{2} e^s \right)^4$$

:

$$M_X(s) = \left(\frac{1}{2} + \frac{1}{2} e^{\frac{1}{8}s^2 + s} \right)^4$$

1.9 Problem 9 and Corresponding Solution

Problem 9 (Ch5 P5)

Let X_1, X_2, ... be independent random variables that are uniformly distributed over [-1,1]. Show that the sequence Y_1, Y_2, ... Converges in probability to some limit, and identify the limit, for each of the following cases:

(a) $Y_n = X_n/n$

(b) $Y_n = (X_n)^n$

(c) $Y_n = X_1 X_2 ... X_n$

(d) $Y_n = max(X_1, ..., X_n)$

Solution:

(a) For any $\epsilon > 0$, we have

$$P(|Y_n| \geq \epsilon) = 0$$

for all n with $\frac{1}{n} < \epsilon$, so

$$P(|Y_n| \geq \epsilon) \to 0$$

(b) For all $\epsilon \in (0,1)$, we have

$$P(|Y_n| \geq \epsilon) = P(|X_n|^n \geq \epsilon) = P(X_n \geq \epsilon^1/n) + P(X_n \leq -\epsilon^1/n) = 1 - \epsilon^{\frac{1}{n}} \to 0$$

(c) Since $X_1, X_2...$ are independent random variables, we have

$$E[Y_n] = E[X_1]...E[X_n] = 0$$

Then

$$Var(Y_n) = E[Y_n^2] = E[X_1^2]...E[X_n^2] = Var(X_1)^n = (\frac{1}{3})^n \to 0$$

We know $E(Y_n) = 0$,according to Chebychev's inequality,

$$P(|Y_n - E(Y_n)| \geq c) \leq \frac{Var(K)}{c^2}$$

$$P(Y_n \geq c) \to 0$$

(d) We have for all $\epsilon \in (0,1)$, using the independence of $X_1, X_2, ...,$

$$\begin{aligned} P[|Y_n - 1| \geq \epsilon] &= P(max\{X_1, ...X_n\} \geq 1 + \epsilon) + P(max\{X_1, ...X_n\} \leq 1 - \epsilon) \\ &= P(X_1 \leq 1 - \epsilon, ..., X_n \leq 1 - \epsilon) \\ &= (P(X_1 \leq 1 - \epsilon))^n \\ &= \left(1 - \frac{\epsilon}{2}\right)^n \to 0 \end{aligned}$$

2 Extra Problems

2.1 Problem 10 and Corresponding Solution

Problem 10

Besides the transform that you learned in Section 4.4, there are various other similar transforms (moment generating functions) that can be defined. For example, for a discrete random variable X whose possible values are nonnegative integers, we can define its Laplace moment generating function as

$$G_X(z) = \sum_{k=0}^{\infty} p(X = k) z^k$$

Clearly, it is the z-transform of the sequence of mass probabilities $p_k = P(X = k)$. Actually, if we set $z = e^s$, this definition goes back to the definition that you learned in Section 4.4. However, the Laplace moment generating function is easier to deal with than the general moment generating function when X can only take nonnegative integer values. In particular, it may help solve certain counting problems in a more "mechanical" manner by using formula of power series and therefore does not require much "cleverness" (see question (5) below). All series involved in this problem are assumed to converge.

(1) Show that $E(X) = \frac{dG_X(z)}{dz}\Big|_{z=1}$ and $Var(X) = \frac{d^2G_X(z)}{dz^2}\Big|_{z=1} + \frac{dG_X(z)}{dz}\Big|_{z=1} - (\frac{dG_X(z)}{dz}\Big|_{z=1})^2$

(2) For the Binomial random variable X (with parameters n and p), please compute its Laplace moment generating function and use results in (1) to compute its expected value and its variance.

(3) Show that if X_1, ... ,X_n are independent random variables and $X = \sum_{i=1}^{n} X_i$, then $G_X(z) = \prod_{i=1}^{n} GX_i(z)$

(4) Please compute the Laplace moment generating function for the Bernoulli random variable (with parameter p). Then using the fact that the Binomial random variable X (with parameters n and p) can be expressed as the sum of n Bernoulli random variables (with parameter p) and the result in (3) to compute $G_X(z)$ and compare with your result in (2).

(5) You randomly roll five six-sided dice, what is the probability that the sum of those five numbers is 15? Hint: Doing counting directly could be quite involved here. Please compute the Laplace generating function of the random variable that is of your interest first. Also, you may find the following negative binomial series useful.

$$(1+z)^{-m} = \sum_{i=0}^{\infty} (-1)^i \begin{pmatrix} m+i-1 \\ i \end{pmatrix} z^i$$

Solution:

(1) To find the expectation $E(X)$, we differentiate $G_X(z)$ with respect to z and then evaluate it at $z = 1$:

$$E(X) = \left.\frac{dG_X(z)}{dz}\right|_{z=1}$$

Then, to find the variance $\mathrm{Var}(X)$, we will use the formula:

$$\mathrm{Var}(X) = E(X^2) - (E(X))^2$$

To find $E(X^2)$, we differentiate $G_X(z)$ twice with respect to z and then evaluate it at $z = 1$:

$$E(X^2) = \left.\frac{d^2G_X(z)}{dz^2}\right|_{z=1} + \left.\frac{dG_X(z)}{dz}\right|_{z=1}$$

Now we can express the variance $\mathrm{Var}(X)$ in terms of the derivatives of $G_X(z)$:

$$\mathrm{Var}(X) = \left.\frac{d^2G_X(z)}{dz^2}\right|_{z=1} + \left.\frac{dG_X(z)}{dz}\right|_{z=1} - \left(\left.\frac{dG_X(z)}{dz}\right|_{z=1}\right)^2$$

(2) First, we know the probability mass function of X is given by

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

Using this, we can define the Laplace moment generating function $G_X(z)$ as

$$G_X(z) = \sum_{k=0}^{n} P(X = k) z^k = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} z^k$$

We can simplify this expression using the binomial theorem, yielding

$$G_X(z) = \sum_{k=0}^{n} \binom{n}{k} (pz)^k (1-p)^{n-k} = (1 - p + pz)^n$$

Now, we can find the expected value and variance using the results from part (1). Starting with the expected value, we differentiate $G_X(z)$ with respect to z and evaluate at $z = 1$:

$$E(X) = \left.\frac{dG_X(z)}{dz}\right|_{z=1} = n(1-p+pz)^{n-1}p\big|_{z=1} = np$$

Next, we find the variance using the formula derived in part 1. Differentiating $G_X(z)$ twice and evaluating at $z = 1$, we find

$$\begin{aligned} Var(X) &= \left.\frac{d^2G_X(z)}{dz^2}\right|_{z=1} + \left.\frac{dG_X(z)}{dz}\right|_{z=1} - \left(\left.\frac{dG_X(z)}{dz}\right|_{z=1}\right)^2 \\ &= n(n-1)p^2(1-p+pz)^{n-2}\big|_{z=1} + np - (np)^2 \\ &= np(1-p) \end{aligned}$$

(3)

$$\begin{aligned} G_X(z) &= E(z^X) \\ &= E\left(z^{\sum_{i=1}^n X_i}\right) \\ &= E\left(\prod_{i=1}^n z^{X_i}\right) \\ &= \prod_{i=1}^n E(z^{X_i}) \\ &= \prod_{i=1}^n G_{X_i}(z) \end{aligned}$$

(4) To find the Laplace moment generating function for a Bernoulli random variable with parameter p, we start by defining the random variable X which takes on two possible values: 0 with probability $1-p$ and 1 with probability p. Thus, we can write its Laplace moment generating function as:

$$G_X(z) = \sum_{k=0}^{1} P(X=k)z^k = P(X=0)z^0 + P(X=1)z^1 = 1-p+pz$$

Let Y be a binomial random variable with parameters n and p, and express it as the sum of n Bernoulli random variables X_i with parameter p:

$$Y = \sum_{i=1}^{n} X_i$$

Using the result from part 3, we find the Laplace moment generating function of Y as:

$$G_Y(z) = \prod_{i=1}^{n} G_{X_i}(z) = \prod_{i=1}^{n}(1-p+pz) = (1-p+pz)^n$$

which is the same as the result in (2)

(5) To find the probability that the sum of the numbers obtained by rolling five six-sided dice is 15, we can use the Laplace moment generating function defined as

$$G_X(z) = \sum_{k=0}^{\infty} p(X=k)z^k$$

First, we find the Laplace moment generating function for a single die roll. Let X_i represent the outcome of the i-th die roll. The Laplace moment generating function for X_i is given by

$$G_{X_i}(z) = \sum_{k=1}^{6} P(X_i=k)z^k = \sum_{k=1}^{6} \frac{1}{6}z^k = \frac{1}{6}(z+z^2+z^3+z^4+z^5+z^6)$$

Next, we define a random variable Y to represent the sum of the outcomes of the five dice rolls:

$$Y = \sum_{i=1}^{5} X_i$$

Using the result from part (3), the Laplace moment generating function for Y is given by the product of the individual generating functions:

$$G_Y(z) = \prod_{i=1}^{5} G_{X_i}(z) = \left(\frac{1}{6}(z + z^2 + z^3 + z^4 + z^5 + z^6)\right)^5$$

To find $P(Y = 15)$, we need to find the coefficient of z^{15} in the expansion of $G_Y(z)$. We can rewrite $G_Y(z)$ as

$$G_Y(z) = \left(\frac{z(1 - z^6)}{6(1 - z)}\right)^5 = z^5 \cdot (1 - z^6)^5 \cdot 6^{-5} \cdot (1 - z)^{-5}$$

Now we aim to find the coefficient of z^{15} in the expansion of $G_Y(z)$. To do this, we first expand $(1 - z)^{-5}$ using the binomial theorem:

$$(1 - z)^{-5} = \sum_{i=0}^{\infty} (-1)^i \binom{4+i}{i} (-z)^i$$

To get the possible i for the coefficient of z^{15}, we have

$$5 + i + 0 = 15 \quad \Rightarrow i = 10$$

$$5 + i + 6 = 15 \quad \Rightarrow i = 4$$

$$5 + i + 12 > 15 \quad , \quad \text{no possible value for i}$$

$$5 + i + 18 > 15 \quad , \quad \text{no possible value for i}$$

$$5 + i + 24 > 15 \quad , \quad \text{no possible value for i}$$

$$5 + i + 30 > 15 \quad , \quad \text{no possible value for i}$$

Hence, we know the coefficient of z^{15} is contributed by $i = 10$ and $i = 4$. Then

- When $i = 10$,

$$z^5 \cdot z^0 \cdot 6^{-5} \cdot (-1)^{10} \binom{14}{10} \cdot (-z)^{10} = \frac{14 \times 13 \times 12 \times 11}{6^5 \times 4 \times 3 \times 2 \times 1} \cdot z^{15}$$

- When $i = 4$,

$$z^5 \cdot (-5)z^6 \cdot 6^{-5} \cdot \binom{8}{4} \cdot (-z)^4 = -\frac{5 \times 8 \times 7 \times 6 \times 5}{6^5 \times 4 \times 3 \times 2 \times 1} \cdot z^{15}$$

Now we can obtain $P(Y = 15)$, the coefficient of z^{15} which is given by:

$$P(Y = 15) = \frac{14 \times 13 \times 12 \times 11}{6^5 \times 4 \times 3 \times 2 \times 1} - \frac{5 \times 8 \times 7 \times 6 \times 5}{6^5 \times 4 \times 3 \times 2 \times 1} = \frac{651}{7776} = \frac{217}{2592} = 0.083719$$

2.2 Problem 11 and Corresponding Solution

Problem 11 (Portfolio Optimization)

The Acahti Trust Company offers two investment instruments, the Lousy Money Market Account and the Risky Mutual Fund. After one year, the Lousy Money Market Account returns exactly your initial principal, without interest. After one year, the Risky Mutual Fund returns twice your initial principal with probability 0.5, and half your initial principal with probability 0.5. Suppose you start with one dollar and each year you rebalance your investments so that a fraction θ of your savings is in the mutual fund and a fraction $1 - \theta$ is in the money market account. Let X_n denote the mutual fund's return during year n, so that

$$P(X_n = 2) = P(X_n = 1/2) = 1/2$$

Let W_n denote your savings after n years.

(a) Express W_n in terms of W_{n-1}, X_n, and θ for $n \geq 1$.

(b) Determine $E[W_n]$ as a function of θ. What value of θ maximizes your expected savings?

(c) Show that

$$\lim_{n\to\infty} \frac{1}{n} log_e(W_n) = \beta$$

with probability 1 for some constant β, and determine β as a function of θ. We call β the *asymptotic growth rate* of the savings because asymptotically W_n behaves like $\exp(\beta n)$ with probability 1.

Hint: Write W_n as a product of i.i.d. variables, take logarithms, and use the strong law of large numbers.

(d) What value of θ maximizes the asymptotic growth rate? What is this maximum growth rate? What asymptotic growth rate is achieved with the θ you found in (b)?

(e) How come trying to maximize average return is different from maximizing the long term return? Which quantity, the expected savings or the asymptotic growth rate, should you aim to optimize? Explain.

Solution:

(a)

$$W_n = \theta X_n W_{n-1} + (1-\theta)W_{n-1} = (1 - \theta + \theta X_n)W_{n-1}$$

(b)

$$\begin{aligned} E[W_n] &= E[(1-\theta+\theta X_n)W_{n-1}] \\ &= (1-\theta)E[W_{n-1}] + \theta E[X_n]E[W_{n-1}] \\ &= (1-\theta+\theta E[X_n])E[W_{n-1}] \end{aligned}$$

Now we need to find $E[X_n]$, which is given by

$$E[X_n] = 2 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{5}{4}$$

We know $W_0 = 1$, and $\theta \in [0, 1]$, so we obtain

$$\begin{aligned} E[W_n] &= (1 + \frac{\theta}{4})E[W_{n-1}] \\ &= (1 + \frac{\theta}{4})^2 E[W_{n-2}] \\ &= ... \\ &= (1 + \frac{\theta}{4})^n E[W_0] \\ &= (1 + \frac{\theta}{4})^n \end{aligned}$$

And when $\theta = 1$, it maximizes the expected savings that $E[W_n]_{max} = (\frac{5}{4})^n$.

(c) We know $W_n = (1 - \theta + \theta X_n)W_{n-1}$ from (a), so

$$log_e(W_n) = log_e((1 - \theta + \theta X_n)W_{n-1}) = log_e \left(\prod_{i=1}^{n} (1 - \theta + \theta X_n) \right) = \sum_{i=1}^{n} log_e(1 - \theta + \theta X_n)$$

Let $Y_i = log_e(1 - \theta + \theta X_n)$, then

$$E[Y_i] = \frac{1}{2} \cdot log_e(1 - \theta + \frac{1}{2}\theta) + \frac{1}{2} \cdot log_e(1 - \theta + 2\theta) = \frac{1}{2} log_e \left((1 - \frac{\theta}{2})(1 + \theta) \right)$$

Using the strong law of big numbers,

$$P \left(\lim_{n \to \infty} \frac{1}{n} log_e(W_n) = \frac{1}{2} log_e((1 - \frac{\theta}{2})(1 + \theta)) \right) = 1$$

Hence

$$\begin{aligned} \lim_{n \to \infty} \frac{1}{n} log_e(W_n) &= \frac{1}{2} log_e((1 - \frac{\theta}{2})(1 + \theta)) \\ \beta &= \frac{1}{2} log_e((1 - \frac{\theta}{2})(1 + \theta)) \end{aligned}$$

(d) Since the logarithmic function is a monotonically increasing function, when $(1 - \frac{\theta}{2})(1 + \theta) = -\frac{1}{2}\theta^2 + \frac{1}{2}\theta + 1$ is at its maximum, β is also at its maximum.

Thus when

$$\theta = -\frac{\frac{1}{2}}{2 \times (-\frac{1}{2})} = \frac{1}{2}$$

, it maximize the asymptotic growth rate. The maximum growth rate

$$\beta_{max} = \frac{1}{2} log_e \frac{9}{8}$$

With the $\theta = 1$ I found in (b), the asymptotic growth rate is:

$$\beta = \frac{1}{2} log_e 1 = 0$$

(e) In solution to part (b), we observed that by choosing $\theta = 1$, we can maximize the expected savings $E[W_n]$, providing the highest average return. This strategy relies entirely on the risky asset, potentially offering higher returns in the short term but also bringing in higher risk as it is fully exposed to market fluctuations.

From solutions to parts (c) and (d), we see that by optimizing the asymptotic growth rate β, we can find a strategy to maximize the long-term return. In this strategy, we find an optimal

θ value (0.5 in this case) that doesn't just pursue the highest returns but finds a balance between risk and return to achieve more stable long-term growth.

Maximizing the average return and maximizing the long-term return can lead to different strategies due to the variance in the returns of the risky asset.

(1) **Maximizing average return** typically focuses on the short-term gains without considering the potential fluctuations and risks involved in the investment. It might lead to strategies that seek higher returns in a shorter period, possibly embracing higher risks.

(2) **Maximizing long-term return**, on the other hand, emphasizes sustainable growth over a longer period. It aims to find a strategy that ensures a steady increase in the investment value, considering the fluctuations and uncertainties associated with the risky asset. This approach generally leads to a more conservative strategy, focusing on the asymptotic growth rate to achieve a stable growth in the long run.

When it comes to choosing which quantity to optimize, it largely depends on the individual's investment goals and risk tolerance.

- If one is looking for short-term gains, focusing on expected savings might be the preferable approach, despite the potential for higher risk.
- For long-term investment goals, optimizing for the asymptotic growth rate could be more beneficial as it tends to offer a more stable growth trajectory, even though it might not yield the highest possible return in the short term.

2.3 Problem 12 and Corresponding Solution

Problem 12

Suppose that there are N students in ECE 4110. Sadly, when Dr. Tang returns the graded exam, he simply hands each student a random exam from the pile. Let K denote the number of students who, by chance, happen to receive their own exam. Thus $N-K$ students receive someone else's exam.

(a) Use Markov's inequality to upper bound the chance that $K \geq m$, where $m > 1$.

(b) Use Chebychev's inequality to upper bound the chance that $K \geq m$, where $m > 1$.

Solution:

(a) The general form of Markov's inequality is:

$$P(X \geq a) \leq \frac{E(X)}{a}$$

Let X_i be the indicator random variable that takes the value 1 if a student receives their own exam and 0 otherwise. Given that each student has a $\frac{1}{N}$ probability of receiving their own exam, the expected value contributed by each student is $\frac{1}{N}$.Therefore, we have

$$E(X_i) = \frac{1}{N}$$

the total expected value is:

$$E(K) = \sum_{i=1}^{N} E[X_i] = N \cdot \frac{1}{N} = 1$$

Then we obtain

$$P(K \geq m) \leq \frac{E(K)}{m} = \frac{1}{m}$$

(b) We know $E[K] = 1$ and the general form of Chebychev's inequality is:

$$P(|K - E(K)| \geq c) \leq \frac{Var(K)}{c^2}$$

We are interested in finding an upper bound for $P(K \geq m)$, so we can set $c = m - 1$. Hence, we have:

$$P(K - 1 \geq m - 1) = P(K \geq m) \leq \frac{\mathrm{Var}(K)}{(m-1)^2}$$

and we already know that $E[K] = 1$. To caculate $Var(K)$, we need to find $E[K^2]$, which can be done as follows:

$$E[K^2] = E\left[\left(\sum_{i=1}^{N} X_i\right)^2\right] = E\left[\sum_{i=1}^{N} X_i^2 + \sum_{i=1}^{N}\sum_{j\neq i} X_i X_j\right]$$

Now we can find the expectation of each term. We already know that $E[X_i^2] = E[X_i] = \frac{1}{N}$ because X_i is an indicator random variable. For the cross terms, we have

$$E[X_i X_j] = P(X_i = 1, X_j = 1) = \frac{1}{N(N-1)}$$

.

Therefore,

$$E[K^2] = \sum_{i=1}^{N} E[X_i^2] + \sum_{i=1}^{N} \sum_{j \neq i} E[X_i X_j] = N \cdot \frac{1}{N} + N(N-1) \cdot \frac{1}{N(N-1)} = 1 + 1 = 2$$

Now we can find the variance:

$$\mathrm{Var}(K) = E[K^2] - (E[K])^2 = 2 - 1 = 1$$

Hence

$$P(K \geq m) \leq \frac{1}{(m-1)^2}$$