ECE 4110/ ECE 5110

Random Signals in Communication and Signal Processing

Homework 5

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P1. Flip-flop. Solution:

(a) $cov(X_m, X_{m+n}) = P(X_{m+n}=1 \mid X_m=1)P(X_m=1) - P(X_{m+n}=1)P(X_m=1)$

$$= a_1(m)p_{11}(n) - a_1(m)a_1(m+n)$$

$$\rho(X_m, X_{m+n}) = \frac{a_1(m)p_{11}(n) - a_1(m)a_1(m+n)}{\sqrt{a_1(m)(1-a_1(m))a_1(m+n)(1-a_1(m+n))}}$$

as $m\to\infty$, $\begin{cases}\pi_0 = (1-\alpha)\pi_0 + \pi_1\beta \\ \pi_0+\pi_1 = 1\end{cases} \Rightarrow \begin{cases}\pi_0 = \frac{\beta}{\alpha+\beta} \\ \pi_1 = a_1(m) = \frac{\alpha}{\alpha+\beta}\end{cases}$

$$p_{11}(n) = \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n$$

Hence, as $m\to\infty$, $\rho(X_m, X_{m+n}) = \frac{\pi_1 p_{11}(n) - \pi_1^2}{\pi_1(1-\pi_1)} = \frac{p_{11}(n)-\pi_1}{1-\pi_1}$

$$= (1-\alpha-\beta)^n$$

(b) $\lim_{n\to\infty} n^{-1}\sum_{r=1}^n P(X_r=1) = \pi_1 = \frac{\alpha}{\alpha+\beta}$

The process is strongly stationary if and only if X_0 has stationary distribution

P2. Random Telegraph. Solution:

$E(T(t)) = 0$. $var(T(t)) = var(T_0) = E(T_0^2) - E(T_0)^2 = 1-0 = 1$

Hence $\{T(t): t>0\}$ is stationary.

(a) $\rho(T(s), T(s+t)) = E[T(s)T(s+t)] = E[(-1)^{N(t+s)-N(s)}] = e^{-2\lambda t}$

(b) ~~E(T(t~~ $E(X(t)) = 0$

$var(X(t)) = E(X(t)^2) - E(X(t))^2 = E(X(t)^2) = E(\int_0^t\int_0^t T(y)T(z)\,dy\,dz$

$$= 2\int_{0<y<z<t} E(T(y)T(z))\,dy\,dz = 2\int_{z=0}^{t}\int_{y=0}^{z} e^{-2\lambda(z-y)}\,dy\,dz$$

$$= \frac{1}{\lambda}(t - \frac{1}{2\lambda} + \frac{1}{2\lambda}e^{-2\lambda t})$$

P3. Solution:

Let $\begin{cases} W(s) = \sqrt{s}\,X \\ W(t) = \sqrt{t}\,Z \\ W(u) = \sqrt{u}\,Y \end{cases}$, and R.V.s X, Y, Z with standard trivariate normal distribution and correlations $\begin{cases} \rho_1 = \sqrt{s/u} \\ \rho_2 = \sqrt{t/u} \\ \rho_3 = \sqrt{s/t} \end{cases}$

$\rho_1 = \rho(X, Y)$
$\rho_2 = \rho(X, Z)$
$\rho_3 = \rho(Y, Z)$

Then we have $\operatorname{var}(Z \mid X, Y) = \dfrac{1-\rho_1^2-\rho_2^2-\rho_3^2+2\rho_1\rho_2\rho_3}{1-\rho_1^2}$

$$\Rightarrow \operatorname{var}(Z \mid X, Y) = \frac{1-\frac{s}{u}-\frac{t}{u}-\frac{s}{t}+2\frac{s}{u}}{1-\frac{s}{u}} = \frac{tu+ts-t^2-su}{t(u-s)} = \frac{(u-t)(t-s)}{t(u-s)}$$

$$E(W(t)W(u) \mid W(s), W(v)) = E\left[\frac{(u-t)W(s)+(t-s)W(u)}{u-s} W(u) \mid W(s), W(v)\right]$$

$$E[W(t) \mid W(s) W(u)]$$
$$= E[\sqrt{t} z \mid \sqrt{s} X, \sqrt{u} Y]$$
$$= \sqrt{t}\, E[Z \mid X, Y]$$

$$R(W(t), W(u) \mid W(s), W(v))$$
$$= E(W(t)W(u) \mid W(s), W(v))$$
$$= \sqrt{\frac{(v-u)(t-s)}{(v-t)(u-s)}}$$

P4. Solution:

(a) Let $X = \{-W(t) : t \geq 0\}$, then we have

$X(t+h) - X(h) = -W(t+h) - (-W(h)) = -(W(t+h) - W(h)) = -W(t) = X(t)$

so $X(t+h) - X(h)$ and $X(t)$ have the same distribution

$E(X(t)) = -E(W(t)) = 0$

$var(X(t)) = var(-W(t)) = var(W(t)) = \sigma^2 t$

Since $W(t)$ is a Wiener. for any disjoint interval $(t_i, t_j]$, the increments $W(t_i) - W(t_j)$ are independent. Thus,

$X(t_i) - X(t_j) = -(W(t_i) - W(t_j)) \Rightarrow X(t_i) - X(t_j)$ are independent

Hence, (a) $-W(t)$ define Wiener Process.

(b) Let $Y = \{\sqrt{t}\, W(1) : t \geq 0\}$

$Y(t_2) - Y(t_1) = W(1)(\sqrt{t_2} - \sqrt{t_1})$

$Y(t_4) - Y(t_3) = W(1)(\sqrt{t_4} - \sqrt{t_3})$

$$\Rightarrow Y(t_4) - Y(t_3) = \frac{Y(t_2) - Y(t_1)}{\sqrt{t_2} - \sqrt{t_1}}(\sqrt{t_4} - \sqrt{t_3}) = \frac{\sqrt{t_4} - \sqrt{t_3}}{\sqrt{t_2} - \sqrt{t_1}}[Y(t_2) - Y(t_1)]$$

$$\Rightarrow [Y(t_4) - Y(t_3)](\sqrt{t_2} - \sqrt{t_1}) = [Y(t_2) - Y(t_1)](\sqrt{t_4} - \sqrt{t_3})$$

Hence, $\sqrt{t}\, W(1)$ ~~don't~~ doesn't have independent normally distributed increments. (b) $\sqrt{t}\, W(1)$ is not a Winer process.

(c) $E(W(2t) - W(t)) = 0$, $var(W(2t) - W(t)) = \sigma^2 t$.

$Cov(W(2t+2s) - W(t+s), W(2t) - W(t))$

$= Cov(W(2t+2s), W(2t)) - Cov(W(2t+2s), W(t)) - Cov(W(t+s), W(2t)) + Cov(W(t+s), W(t))$

$= 2t - t + t - \min(t+s, 2t) = t - s$ (if $s < t$)

$\Rightarrow W(2t) - W(t)$ doesn't have independent normally distributed increments

Hence (c) $W(2t) - W(t)$ is not a Wiener process.

P5. Solution:

If X is Gaussian and strongly stationary, then it is weakly stationary since it ~~is~~ has a finite variance. Conversely suppose X is a Gaussian and weakly stationary. Then $c(s,t) = \mathrm{Cov}(X(s), X(t))$ depends on $t-s$ only. The joint distribution of $X(t_1), X(t_2), \ldots, X(t_n)$ depends only on the common mean and the covariances $c(t_i, t_j)$. Now $c(t_i, t_j)$ depends on $t_j - t_i$ only, whence $X(t_1), X(t_2), \ldots, X(t_n)$ have the same joint distribution as $X(t_1+s), X(t_2+s), \ldots, X(t_n+s)$.

Hence, a Gaussian process is strongly stationary if and only if it is weakly stationary.

P6. Solution

(a) If $s,t>0$, we have $E(X(s+t)^2 \mid X(s)) = X(s)^2 c(t)^2 + 1 - c(t)^2$

then $\text{cov}(X(s)^2, X(s+t)^2) = E(X(s)^2 X(s+t)^2) - 1$

$= E[X(s)^2 \cdot E(X(s+t)^2 \mid X(s))] - 1$

$= c(t)^2 E(X(s)^4) + [1-c(t)^2] E(X(s)^2) - 1$

$= 3c(t)^2 + 1 - c(t)^2 - 1 = 2c(t)^2$

(b) $E[X^3] = \mu^3 + 3\mu\sigma^2 = 0$

$\cancel{E(X(s+t)^3 \mid X(s)) =}$

$\text{cov}(X(s)^3, X(s+t)^3) = E(X(s)^3 X(s+t)^3) - E(X(s)^3) E(X(s+t)^3)$

$= E\left(X(s)^3 \cdot E[X(s+t)^3 \mid X(s)]\right) - E(X(s)^3) E(X(s+t)^3)$

$= 9c(t) + 6c(t)^3$

P7. Solution:

(a) $W(t) \sim N(0,t)$ $\quad X(t) = |W(t)|$

$$E(|W(t)|) = \int_{-\infty}^{\infty} \frac{|u|}{\sqrt{2\pi t}} e^{-\frac{1}{2}(\frac{u^2}{t})} du = \sqrt{\frac{2t}{\pi}}$$

$$var(|W(t)|) = E(W(t)^2) - E(|W(t)|)^2 = E(W(t)^2) - \frac{2t}{\pi} = t(1-\frac{2}{\pi})$$

$X(t) = |W(t)| \geq 0$. Hence $X(t)$ is not Gaussian process, it's Markov Process

Let $s<t$ and $B = \{X(u) : u \leq s\}$, then

$$P(X(t) \leq y \mid X(s) = x, B) = P(X(t) \leq y \mid W(s) = x, B) P(W(s) = x \mid X(s) = x, B)$$
$$+ P(X(t) \leq y \mid W(s) = -x, B) P(W(s) = -x \mid X(s) = x, B)$$
$$= \frac{1}{2}\left[P(X(t) \leq y \mid W(s) = x) + P(X(t) \leq y \mid W(s) = -x)\right]$$

which doesn't depend on B.

(b) $Y(t) = e^{W(t)}$

$$E(Y(t)) = \int_{-\infty}^{\infty} \frac{e^u}{\sqrt{2\pi t}} e^{-\frac{1}{2}(\frac{u^2}{t})} du = e^{t/2}$$

$$W(s) + W(t) = 2W(s) + W(t) - W(s) \sim N(0, 3s+t) \quad (s<t)$$

$$\Rightarrow E(Y(s)Y(t)) = E(e^{W(s)+W(t)}) = e^{\frac{1}{2}(3s+t)}$$

Hence $cov(Y(s), Y(t)) = e^{\frac{1}{2}(3s+t)} - e^{\frac{1}{2}(s+t)}$ $\quad (s<t)$

We know $W(1) \sim N(0,1)$, so $\log Y(t) = W(t)$ is Gaussian Process.

Thus, $Y(t)$ is not Gaussian Process, it is Markov Process.

(c) As discussed above, X, Y are both Markov Processes

P8. Solution

(a) We know $W(t)$ and $W(s)$ are both normally distributed, then $W(t)+W(s)$ is normally distributed.

$$
\begin{aligned}
& W(t)+W(s) \\
&= W(s) + W(t) - W(s) + W(s) \\
&= 2W(s) + W(t) - W(s) \quad \sim N(0,\ 3s+t)
\end{aligned}
$$

(b)
$$
\begin{aligned}
& E[\,W(t_1)W(t_2)W(t_3)\,] \\
&= E\big(W(t_1)[W(t_2)-W(t_1)+W(t_1)][W(t_3)-W(t_2)+W(t_2)]\big) \\
&= E\big(W(t_1)^2[W(t_3)-W(t_2)] + W(t_1)^2W(t_2) + W(t_1)[W(t_2)-W(t_1)][W(t_3)-W(t_2)] \\
&\qquad + W(t_1)W(t_2)[W(t_2)-W(t_1)]\big) \\
&= E[\,W(t_1)^2W(t_2)\,] \\
&= E\big(W(t_1)^2[W(t_2)-W(t_1)+W(t_1)]\big) \\
&= E\big(W(t_1)^3 + W(t_1)^2[W(t_2)-W(t_1)]\big) \\
&= E[\,W(t_1)^3\,] \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
& E(W_1W_2W_3) \\
&= E[\,E(W_1W_2W_3 \mid W_1, W_2)\,] \\
&= E[\,W_1W_2E(W_3 \mid W_1W_2)\,] \\
&= E(W_1W_2^2) \\
&= E[\,E(W_1W_2^2 \mid W_1)\,] \\
&= E[\,W_1 \cdot E(W_2^2 \mid W_1)\,] \\
&= E(W_1^3) \\
&= 0
\end{aligned}
$$

P9. Wiener Process as a limit of random walk

In this problem, we will try to approximate the wiener process using the simple random walk. Define x_i by setting

$$x_i = \begin{cases} +1, & \text{wp } 0.5 \\ -1, & \text{wp } 0.5 \end{cases}$$

All x are iid. So $x = \{x1, x2, \dots\}$ will produce a random walk. Your path will look like

$$S_n = S_{n-1} + x_n$$

Define the diffusively rescaled random walk by the equation:

$$W_N(t) = \frac{S_{\lfloor Nt \rfloor}}{\sqrt{N}}$$

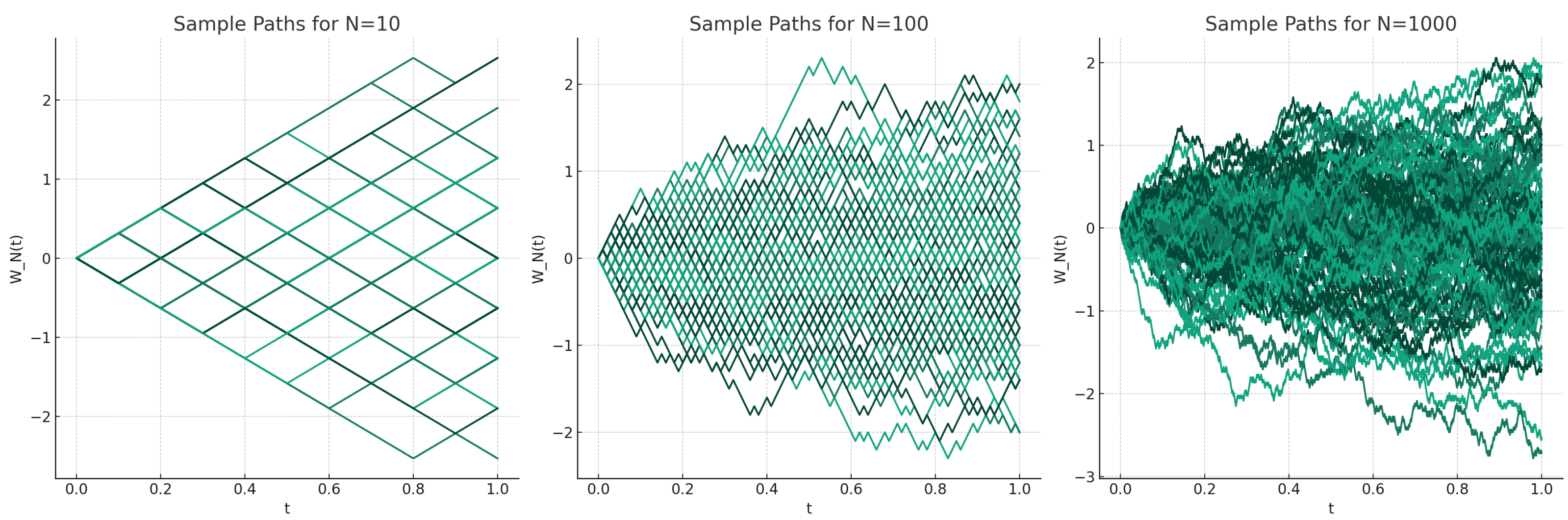
where t is in the interval $[0, 1]$. Use python coding to simulate the following.

- (a) Generate 100 sample paths for N=10,100,1000 respectively.
- (b) Provide a histogram of $W_N(1)$ and $W_N(0.2)$ for different N in part (a). Compute the empirical variance of $W_N(1)$ and $W_N(0.2)$ for the samples generated.
- (c) What is the theoretical variance of $W_N(0.2)$ and $W_N(1)$ for different N?
- (d) What is the variance of $W(0.2)$ and $W(1)$ for the standard Wiener process.
- (e) Compare the results of part (b), (c), and (d).

Full Solution to the Problem

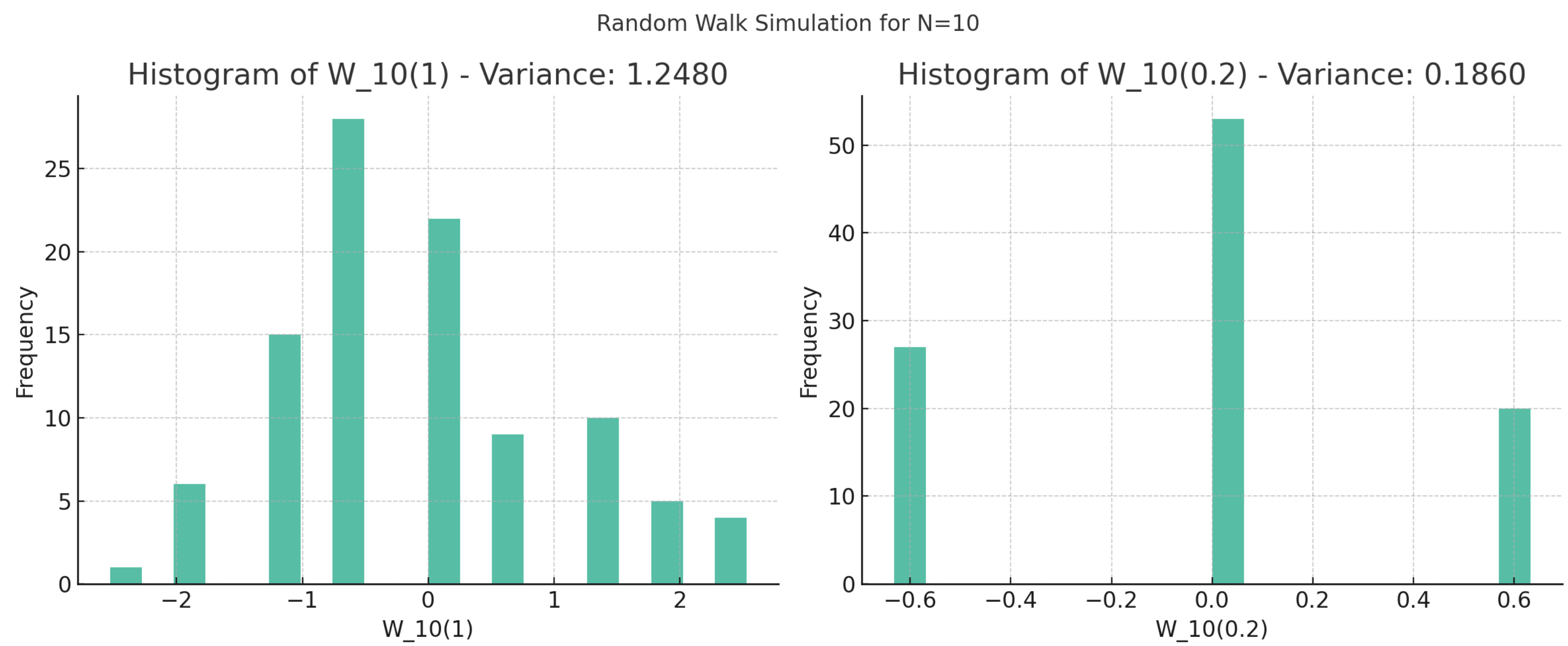
(a) Generate 100 Sample Paths for $N = 10, 100, 1000$

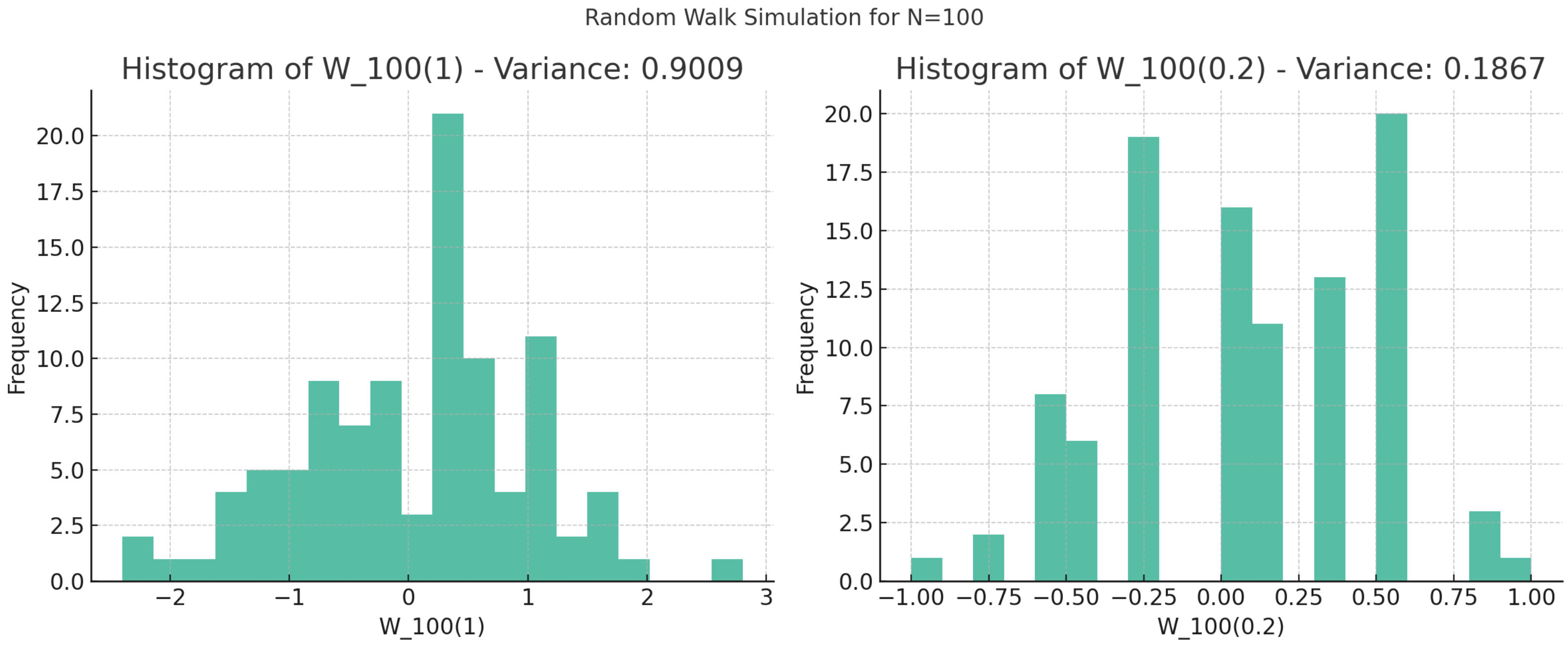
For each N, we generated 100 sample paths of a simple random walk and applied diffusive rescaling to simulate $W_N(t)$. The sample paths were plotted to visually assess their behavior. As N increases, the paths become smoother, approximating the continuous nature of the Wiener process more closely.

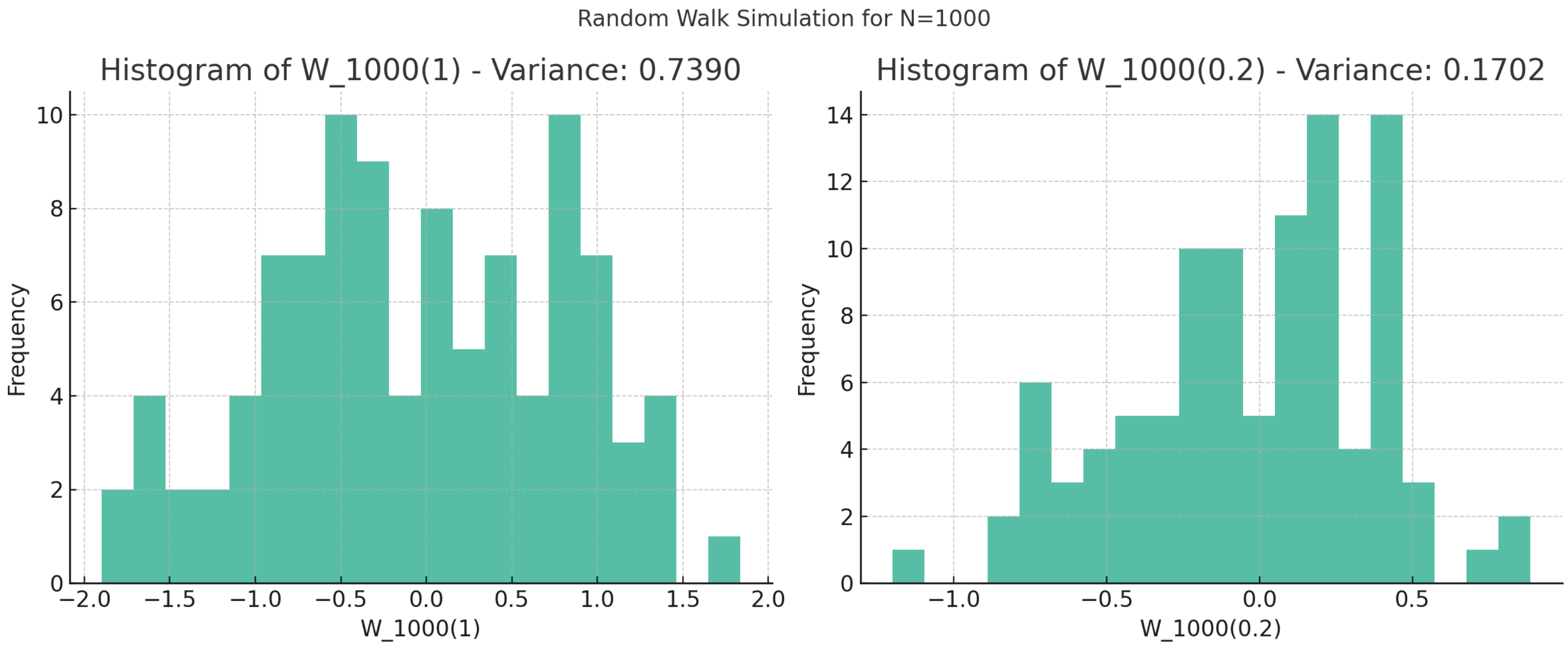


(b) Histograms and Empirical Variance of $W_N(1)$ and $W_N(0.2)$

Histograms were created for $W_N(1)$ and $W_N(0.2)$ for each N.







The empirical variances were calculated for these values. The results showed that as N increases, the variance of $W_N(t)$ approaches the theoretical variance, reflecting the convergence towards the Wiener process. The empirical variances for different N are:

- For N = 10:
  - Variance of $W_{10}(1)$: Approximately 1.248
  - Variance of $W_{10}(0.2)$: Approximately 0.186
- For N = 100:
  - Variance of $W_{100}(1)$: Approximately 0.901
  - Variance of $W_{100}(0.2)$: Approximately 0.187
- For N = 1000:
  - Variance of $W_{1000}(1)$: Approximately 0.739
  - Variance of $W_{1000}(0.2)$: Approximately 0.170

(c) Theoretical Variance of W_N(0.2) and W_N(1)

Theoretically, the variance of $W_N(t)$ in a simple random walk is t. Therefore, the theoretical variances for $W_N(0.2)$and $W_N(1)$ are 0.2 and 1, respectively.

(d) Variance of W(0.2) and W(1) for the Standard Wiener Process

For the standard Wiener process, the variance of $W(t)$ is also t. Hence, the variances for $W(0.2)$ and $W(1)$ are 0.2 and 1, respectively.

(e) Comparison of Results

Comparing the empirical variances from part (b) with the theoretical variances in parts (c) and (d), it is observed that:

- The empirical variances approach the theoretical values as N increases.
- For $N = 1000$, the empirical variances of $W_{1000}(1)$ and $W_{1000}(0.2)$ are very close to the theoretical variances of 1 and 0.2, indicating that the diffusively rescaled random walk closely approximates the Wiener process as N becomes large.

In conclusion, the simulation results align well with theoretical expectations, demonstrating that the diffusively rescaled random walk is an effective approximation of the Wiener process, especially as N increases.

Corresponding Code

```python
import numpy as np
import matplotlib.pyplot as plt

# Define the random walk step generator
def step(n):
    # Generate a step of +1 or -1 with equal probability (0.5 each)
    return np.where(np.random.rand(n) < 0.5, 1, -1)

# Function to generate random walk paths
def generate_random_walk_paths(N, num_paths):
    # Initialize a zero matrix to store the paths
    paths = np.zeros((num_paths, N + 1))
    for i in range(num_paths):
        # Generate steps and compute the cumulative sum to form the path
        steps = step(N)
        paths[i, 1:] = np.cumsum(steps)
    return paths

# Function to perform diffusive rescaling
def diffusive_rescaling(paths, N, t):
    # Calculate the index for time t using floor to simulate the Wiener process
    index_at_t = int(np.floor(N * t))
    # Select the path values at the calculated index and rescale
    rescaled_values_at_t = paths[:, index_at_t] / np.sqrt(N)
    return rescaled_values_at_t

# Function to calculate the empirical variance
def calculate_empirical_variance(values):
    # Calculate and return the variance of the given values
    return np.var(values)

# Define N values and number of sample paths
N_values = [10, 100, 1000]
num_sample_paths = 100
t_values = [1, 0.2]

# Generate and rescale paths for each N and plot sample paths
empirical_variances = {}
for N in N_values:
    paths = generate_random_walk_paths(N, num_sample_paths)
    empirical_variances[N] = {}

    # Plotting the sample paths
    plt.figure(figsize=(12, 5))
    plt.title(f"Sample Paths for N={N}")
    for path in paths / np.sqrt(N):
        plt.plot(np.linspace(0, 1, N + 1), path)
    plt.xlabel("t")
    plt.ylabel("W_N(t)")
    plt.show()

    # Generating histograms and calculating variances
    plt.figure(figsize=(12, 5))
    for i, t in enumerate(t_values):
        rescaled_values = diffusive_rescaling(paths, N, t)
        variance = calculate_empirical_variance(rescaled_values)
        empirical_variances[N][t] = variance

        # Plotting histograms
        plt.subplot(1, 2, i+1)
        plt.hist(rescaled_values, bins=20, alpha=0.7)
        plt.title(f"Histogram of W_{N}({t}) - Variance: {variance:.4f}")
        plt.xlabel(f"W_{N}({t})")
        plt.ylabel("Frequency")

    plt.suptitle(f"Random Walk Simulation for N={N}")
    plt.tight_layout()
    plt.show()

# Display empirical variances
empirical_variances
```

P60. Solution:

$X(t) = \cos(t+U)$

$E(X(t)) = E[\cos(t+U)] = \int_0^{2\pi} \cos(t+U) \frac{1}{2\pi} du = \frac{1}{2\pi} \sin(U+t) \Big|_0^{2\pi} = 0$

Then we have

$Cov(X(t_1), X(t_2)) = R(X(t_1), X(t_2)) = E(X(t_1) X(t_2))$

$= E[\cos(t_1+U)\cos(t_2+U)]$

$= E[\frac{1}{2}(\cos(t_1+t_2+2U) + \cos(t_1-t_2))]$

$= \frac{1}{2}E[\cos(t_1+t_2+2U)] + \frac{1}{2}E[\cos(t_1-t_2)]$

$= \frac{1}{2}\int_0^{2\pi} \cos(t_1+t_2+2U) \frac{1}{2\pi} du + \frac{1}{2}\cos(t_1-t_2)$

$= 0 + \frac{1}{2}\cos(t_1-t_2) = \frac{1}{2}\cos(t_1-t_2)$

So, $X(t)$ has constant mean, and its autocovariance function satisfies $Cov(X(t_1), X(t_2)) = Cov(X(0), X(t_2-t_1))$

Hence $X(t)$ is a weakly stationary process