ECE 4110/ ECE 5110

Random Signals in Communication and Signal Processing

Homework 6

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P1 . Solution:

(i) $E\{(X_{n+1}-\alpha X_n)^2\} = var(X_{n+1} - \alpha X_n)$

$= var(X_{n+1}) - 2\alpha\, var(X_{n+1}, X_n) + \alpha^2 var(X_n)$

$= c(0)\ \cancel{-2\alpha\, var(t} - 2\alpha c(1) + \alpha^2 c(0) \quad = (1+\alpha^2)c(0) - 2\alpha c(1)$

$\cancel{c(0) = \frac{\alpha^0}{1-\alpha^2} = \frac{1}{1-\alpha^2} \quad c(1) = \frac{\alpha^1}{1-\alpha^2} = \frac{\alpha}{1-\alpha}} \implies \alpha = \frac{c(1)}{c(0)}$

Thus, $\hat{X}_{n+1} = \frac{c(1)}{c(0)} X_n$ $\qquad \sum_{i=0}^{s} a_i c(|i-j|) = c(k+j)$ for $0 \le j \le s$

(ii) $E\{(X_{n+1}-\beta X_n - \gamma X_{n-1})^2\} = var(X_{n+1} - \beta X_n - \gamma X_{n-1}) \quad (\beta = a_0, \gamma = a_1)$

$= (1+\beta^2+\gamma^2)c(0) - 2\beta c(1) - 2\gamma c(2) + 2\beta\gamma c(1)$

$= (1+\beta^2+\gamma^2)c(0) + 2\beta(\gamma-1)c(1) - 2\gamma c(2)$

$\tilde{X}_{n+1} = \sum_{i=0}^{1} a_i X_{n-1} \implies \begin{cases} \beta c(0) + \gamma c(1) = c(1) \\ \beta c(1) + \gamma c(0) = c(2) \end{cases} \implies \begin{cases} \beta = \frac{c(1)(c(0)-c(2))}{c(0)^2 - c(1)^2} \\ \gamma = \frac{c(0)c(2) - c(1)^2}{c(0)^2 - c(1)^2} \end{cases}$

$\tilde{X}_{n+1} = \frac{c(1)(c(0)-c(2))}{c(0)^2-c(1)^2} X_n + \frac{c(0)c(2)-c(1)^2}{c(0)^2-c(1)^2} X_{n-1}$

(iii) From (i) and (ii) we have

$E(X_{n+1}-\hat{X}_{n+1})^2 = (1+\frac{c(1)^2}{c(0)^2})c(0) - 2\frac{c(1)}{c(0)} \cdot c(1) = \frac{c(0)^2 - c(1)^2}{c(0)}$

$E(X_{n+1}-\tilde{X}_{n+1})^2 = \frac{c(0)c(1)^2[c(0)-c(2)]^2 + c(0)^3[c(2)-c(1)^2]^2 + [c(0)-c(1)]^2[c(0)+c(1)]^2 + 2c(1)^2(c(0)-c(2))[c(0)c(2)-c(0)^2] + 2c(2)[c(0)c(2)-c(1)^2][c(0)^2-c(1)^2]}{[c(0)^2-c(1)^2]^2}$

$\implies D = \frac{[c(1)^2 - c(0)c(2)]^2}{c(0)[c(0)^2-c(1)^2]}$

(a) In this case, $c(0) = \frac{1}{2}$, $c(1) = 0$, $c(2) = 0 \implies \hat{X}_{n+1} = \tilde{X}_{n+1} = 0$, $D = 0$

(b) In this case, $c(0) = 1$, $c(1) = \alpha$, $c(2) = \alpha^2 \implies D = 0$

P2. Solution:

Let $\{Z_n : n = \cdots, -1, 0, 1, \cdots\}$ be independent R.V.s with zero mean and unit variances, define $X_n = \frac{Z_n + aZ_{n-1}}{\sqrt{1+a^2}}$ satisfies the autocovariance.

By the projection theorem, $X_n - \hat{X}_n$ is orthogonal to $\{X_{n-r} : r \geq 1\}$

Hence $E\{(X_n - \hat{X}_n)X_{n-r}\} = 0$, $r \geq 1$ and set $\hat{X}_n = \sum_{s=1}^{\infty} b_s X_{n-s}$ to obtain that

$$\begin{cases} \alpha = b_1 + b_2\alpha \\ 0 = b_{s-1}\alpha + b_s + b_{s+1}\alpha, \ (\text{for } s \geq 2) \end{cases} \text{ where } \alpha = \frac{a}{1+a^2}$$

$\Rightarrow b_s = (-1)^{s+1} a^s$

Hence $\hat{X}_n = \sum_{s=1}^{\infty} (-1)^{s+1} a^s X_{n-s}$, then the mean square error of prediction is

$$E\{(X_n - \hat{X}_n)^2\} = E\left\{\left(\sum_{s=0}^{\infty} (-a)^s X_{n-s}\right)^2\right\} = \frac{1}{1+a^2} E(Z_n^2) = \frac{1}{1+a^2}$$

$E(\hat{X}_n) = 0$

~~So~~ $\operatorname{cov}(\hat{X}_n, \hat{X}_{n-m}) = \sum_{r,s=1}^{\infty} b_r b_s c(m+r-s)$, $m \geq 0$ depends on m alone.

Thus $\hat{X}$ is weakly stationary.

~~P3~~

~~(a) $p(t) = e^{-\frac{1}{2}t^2}$~~

~~(b) $p(t) = \frac{1}{2}\left(\frac{1}{1-it} + \frac{1}{1+it}\right) = \frac{1}{1+t^2}$~~

P3. Solution:

$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{2j\pi f\tau} df$$

(a) For $S_X(f) = N(0,1) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(f)^2}$

Then $R_X(\tau) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}f^2} \cdot e^{2j\pi f\tau} df \cancel{= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{2j\pi\tau f - \frac{1}{2}f^2} df}$

$= \cancel{e^{-2\pi^2\tau^2}} \; e^{-2\pi^2\tau^2}$ (from table)

(b) From table we know $\int_{-\infty}^{\infty} e^{-2\pi k|\tau|} e^{-2j\pi f(\tau)} d\tau = \frac{1}{\pi} \cdot \frac{k}{f^2+k^2}$

Let $k = \frac{1}{2\pi}$, then we have $\int_{-\infty}^{\infty} e^{-|\tau|} e^{-2j\pi f\tau} d\tau = \frac{4\pi}{1+4\pi^2 f^2}$

To calulate $R_X(\tau) = \int_{-\infty}^{\infty} S_X(f) e^{2j\pi f\tau} df$

$= \int_{-\infty}^{\infty} e^{-|f|} e^{2j\pi f\tau} df \xrightarrow{f=-f} \frac{4\pi}{1+4\pi^2(-f)^2}$

$= \frac{4\pi}{1+4\pi^2 f^2}$

~~(b) For $S_X(f) = e^{-|f|}$~~

$\cancel{\text{Then } R_X(\tau) = \int_{-\infty}^{\infty} e^{-|f|} e^{2j\pi f\tau} df = \int_{-\infty}^{0} e^{(2j\pi\tau+1)f} df + \int_{0}^{\infty} e^{(2j\pi\tau-1)f} df}$

$\cancel{= \frac{1}{2j\pi\tau+1} e^{(2j\pi\tau+1)f}\Big|_{-\infty}^{0} + \frac{1}{2j\pi\tau-1} e^{(2j\pi\tau-1)f}\Big|_{0}^{+\infty}}$

$\cancel{= \frac{1}{2j\pi\tau+1} - \frac{1}{2j\pi\tau-1} = -\frac{4j\pi\tau}{4\pi^2\tau^2+1}}$

P4. Solution:

(a) $\cancel{\hat{X}_t = aX_{t-1} + bX_{t-2}}$ $Y_n = X_n + \alpha \sum_{i=1}^{\infty} \beta^{i-1} X_{n-i} = X_n + \alpha X_{n-1} + \alpha\beta X_{n-2} + \alpha\beta^2 X_{n-3} \cdots$

$Y_{n-1} = X_{n-1} + \alpha X_{n-2} + \alpha\beta X_{n-3} + \alpha\beta^2 X_{n-4} \cdots\cdots$

Hence. $Y_n = X_n + \beta Y_{n-1} + (\alpha-\beta) X_{n-1} = X_n + (\alpha-\beta) X_{n-1} + \beta Y_{n-1}$

$E(Y_n Y_n) = E[(X_n + (\alpha-\beta) X_{n-1} + \beta Y_{n-1})(X_n + (\alpha-\beta) X_{n-1} + \beta Y_{n-1})]$

$= E(X_n^2) + (\alpha-\beta)^2 E(X_{n-1}^2) + \beta^2 E(Y_{n-1}^2) + 2(\alpha-\beta) E(X_n X_{n-1})$

$+ 2(\alpha-\beta)\beta E(X_{n-1} Y_{n-1}) + 2\beta E(X_n Y_{n-1})$

$= 1 + (\alpha-\beta)^2 + \beta^2 R_Y(0) + 2\beta(\alpha-\beta)$

Then we have $(1-\beta^2) R_Y(0) = 1 + (\alpha-\beta)^2 + 2\beta(\alpha-\beta) = 1 + \alpha^2 - \beta^2$

$\Rightarrow C_Y(0) = R_Y(0) = \dfrac{1+\alpha^2-\beta^2}{1-\beta^2}$

~~For $|k|>1$~~. $E(Y_{n+k} Y_n) = E[(X_{n+k} + (\alpha-\beta) X_{n+k-1} + \beta Y_{n+k-1}) Y_n]$

$= E(X_{n+k} Y_n) + (\alpha-\beta) E(X_{n+k-1} Y_n) + \beta E(Y_{n+k-1} Y_n)$

For $|k|>1$, we have $C_Y(k) = \beta C_Y(k-1)$

For $k=1$, we have $C_Y(1) = (\alpha-\beta) + \beta C_Y(0)$

Hence, $C_Y(k) = \begin{cases} \dfrac{1+\alpha^2-\beta^2}{1-\beta^2}, & \text{for } k=0 \\ \beta^{|k|-1} \left\{ \dfrac{\alpha(1+\alpha\beta-\beta^2)}{1-\beta^2} \right\}, & \text{for } k \neq 0 \end{cases}$

Set $\hat{Y}_{n+1} = \sum_{i=0}^{\infty} a_i Y_{n-i}$, then we have $C_Y(k+1) = \sum_{i=0}^{\infty} a_i C_Y(k-i)$, $k \geq 0$

$\Rightarrow a_i = \alpha(\beta-\alpha)^i$, for $i \geq 0$.

P5. Solution:

(a) $\hat{X}_t = aX_{t_1} + bX_{t_2}$

$$\begin{cases} E[(X_t - aX_{t_1} - bX_{t_2})X_{t_1}] = 0 \\ E[(X_t - aX_{t_1} - bX_{t_2})X_{t_2}] = 0 \end{cases} \Rightarrow \begin{cases} c(t-t_1) - a c(0) - b c(t_1-t_2) = 0 \\ c(t-t_2) - a c(t_1-t_2) - b c(0) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} a = \dfrac{c(0)c(t-t_1) - c(t-t_2)c(t_2-t_1)}{c(0)^2 - c(t_2-t_1)^2} \\[2ex] b = \dfrac{c(0)c(t-t_2) - c(t-t_1)c(t_2-t_1)}{c(0)^2 - c(t_2-t_1)^2} \end{cases}$$

(b) $MSE = E[(X_t - \hat{X}_t)^2] = E[(X_t - aX_{t_1} - bX_{t_2})^2]$

$$= E(X_t^2) + a^2E(X_{t_1}^2) + b^2E(X_{t_2}^2) - 2bE(X_tX_{t_2}) - 2aE(X_tX_{t_1}) + 2abE(X_{t_1}X_{t_2})$$

$$= (1+a^2+b^2)c(0) - 2a\,c(t-t_1) - 2b\,c(t-t_2) + 2ab\,c(t_1-t_2)$$

Then we can just substitue a and b form part (a) to get the resulc of MSE

(c) when $t = t_1$, then $MSE = (1+a^2+b^2)c(0) - 2a\,c(0) + (2ab-2b)c(t_1-t_2)$

$$\begin{cases} a = \dfrac{c(0)^2 - c(t_1-t_2)^2}{c(0)^2 - c(t_2-t_1)^2} = 1 \\[2ex] b = \dfrac{c(0)c(t_1-t_2) - c(0)c(t_1-t_2)}{c(0)^2 - c(t_2-t_1)^2} = 0 \end{cases}$$

$$= (1+a^2+b^2-2a)c(0) + 2b(a-1)c(t_1-t_2) = 0.$$

when $t = t_2$,
$$\begin{cases} a = \dfrac{c(0)c(t_2-t_1) - c(0)c(t_2-t_1)}{c(0)^2 - c(t_2-t_1)^2} = 0 \\[2ex] b = \dfrac{c(0)c(0) - c(t_2-t_1)c(t_2-t_1)}{c(0)^2 - c(t_2-t_1)^2} = 1 \end{cases}$$

then $MSE = (1+a^2+b^2-2b)c(0) + 2a(b-1)c(t_1-t_2) = 0$.

The Answer is what we expect.

P6. Solution:

(a) The power density function $S_X(f) = \mathcal{F}\{R_X(k)\} = \sum_{k=-\infty}^{\infty} R_X(k) e^{-2j\pi f k}$

$$\Rightarrow S_X(f) = \sum_{k=-\infty}^{\infty} 4\alpha^{|k|} e^{-2j\pi f k}$$

For $|\alpha|<1$, the sum involves an infinite geometric series and the series converges. then we have

$$\cancel{S_X(f) = \frac{4(1-\alpha^2)e^{2\pi i f}}{\alpha^2 e^{2\pi i f} - \alpha e^{4\pi i f} - \alpha + e^{2\pi i f}}}$$

$$S_X(f) = 4\left(1 + \sum_{k=1}^{\infty} \alpha^k e^{-2j\pi f k} + \sum_{k=-1}^{-\infty} \alpha^{-k} e^{-2j\pi f k}\right)$$

$$= 4\left[1 + \sum_{k=1}^{\infty} (\alpha e^{-2j\pi f})^k + \sum_{k=1}^{\infty} (\alpha e^{2j\pi f})^k\right]$$

$$= 4\left(1 + \frac{\alpha e^{-2j\pi f}}{1-\alpha e^{-2j\pi f}} + \frac{\alpha e^{2j\pi f}}{1-\alpha e^{2j\pi f}}\right)$$

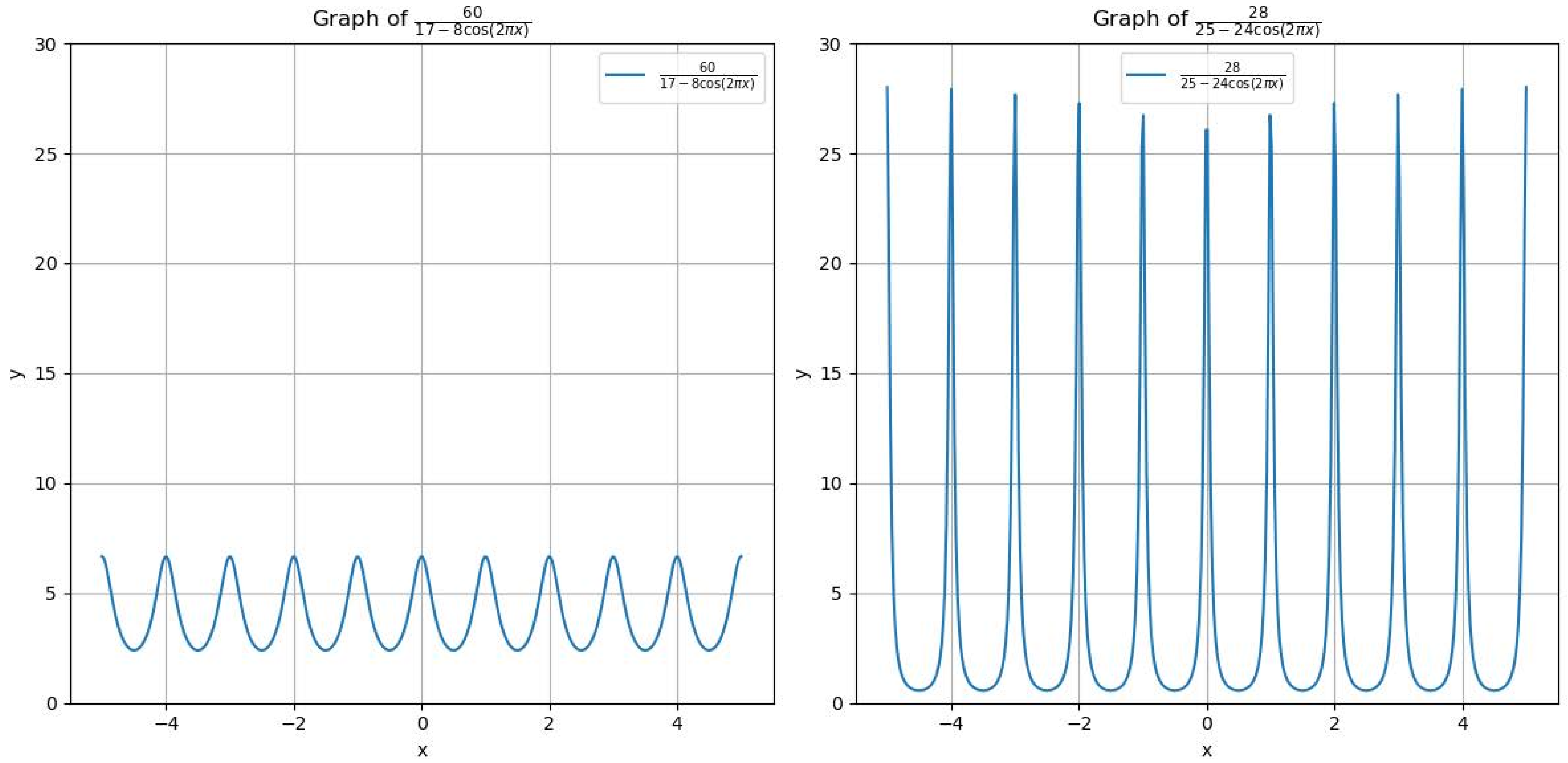
~~(b) if α = 0.25, Sx(f) =~~ $= \dfrac{4(1-\alpha^2)}{1-2\alpha\cos 2\pi f + \alpha^2}$

(b) if $\alpha = 0.25$, $S_X(f) = \cancel{\dfrac{60}{17/4 - 2\cos 2\pi f}} = \dfrac{60}{17-8\cos 2\pi f}$

if $\alpha = 0.75$, $S_X(f) = \dfrac{28}{25-24\cos 2\pi f}$

The figure is shown below.

for $|\alpha|<1$, as α increase, ~~it effects~~ the strength of the signal increase. The figure becomes more "steep".



Graph of $\frac{60}{17-8\cos(2\pi x)}$
$\frac{60}{17-8\cos(2\pi x)}$
Graph of $\frac{28}{25-24\cos(2\pi x)}$
$\frac{28}{25-24\cos(2\pi x)}$
y
x
0
5
10
15
20
25
30
−4
−2
0
2
4

P7. Solution:

(a) We can know from the problem that $R_X(k) = E(X_n X_{n-k}) = \begin{cases} 0, & \text{for } k \neq 0 \\ 1, & \text{for } \cancel{k\neq 0}\ k=0 \end{cases}$

Then we can get the spectral density function

$$S_X(f) = \int_{-\infty}^{\infty} R_X(k) e^{-2j\pi fk} dk = \int_{-\infty}^{\infty} 0 \cdot e^{-2j\pi fk} dk + 1 \cdot e^{-2j\pi f \cdot 0} d(0) = 1$$

(b) Given that $Y_n = X_n + aX_{n-1}$, we can get the autocorrelation function

$$R_Y(k) = E(Y_n Y_{n-k}) = E[(X_n + aX_{n-1})(X_{n-k} + aX_{n-k-1})]$$

$$= E(X_n X_{n-k} + a X_n X_{n-k-1} + a X_{n-1} X_{n-k} + a^2 X_{n-1} X_{n-k-1})$$

Since X_n are uncorrelated, $E(X_n X_{n-a}) = 0$ for $a \neq 0$ and $E(X_n^2) = 1$

Hence $R_Y(k) = \begin{cases} 1+a^2, & k=0 \\ a, & k=\pm 1 \\ 0, & ow \end{cases}$

Then the spectral density function $S_Y(f)$ is the Fourier Transform of $R_Y(k)$

$$S_Y(f) = \int_{-\infty}^{\infty} R_Y(k) e^{-2j\pi fk} dk$$

$$= \int_{-\infty}^{\infty} 0 \cdot e^{-2j\pi fk} dk + (1+a^2) \cdot e^{-2j\pi f \cdot 0} d(0) + a \cdot e^{-2j\pi f \cdot 1} d(1) + a \cdot e^{2j\pi f} d(-1)$$

$$= 1 + a^2 + a e^{-2j\pi f} + a e^{2j\pi f}$$

(Sorry for write " α " in " a ").

P8. Solution:

$$Y_n = \alpha Y_{n-1} + X_n$$

For $k=0$, $R_Y(k) = E[Y_n Y_n] = E[(\alpha Y_{n-1}+X_n)(\alpha Y_{n-1}+X_n)]$

$$= E[\alpha^2 Y_{n-1}^2 + 2Y_{n-1}X_n + X_n^2] \qquad (\sigma_x^2 = 1)$$

$$= \alpha^2 R_Y(0) + 1 \qquad (\alpha = \tfrac{1}{2})$$

$$\Longrightarrow R_Y(0) = \alpha^2 R_Y(0) + 1 \Longrightarrow R_Y(0) = \frac{1}{1-\alpha^2} = \frac{4}{3}$$

For $k>0$, $R_Y(k) = E[Y_{n+k}Y_n] = E[(\alpha Y_{n+k-1} + X_{n+k})Y_n]$

$$= \alpha E(Y_{n+k-1}Y_n) + E(X_{n+k}Y_n)$$

$$= \alpha R_Y(k-1)$$

$$\Longrightarrow R_Y(k) = \alpha R_Y(k-1) \Longrightarrow R_Y(k) = \frac{\alpha^{|k|}}{1-\alpha^2} = \frac{4}{3}\cdot\left(\frac{1}{2}\right)^{|k|}$$

$$S_Y(f) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi fk} = \sum_{k=-\infty}^{\infty} \frac{4}{3}\cdot\frac{1}{2^{|k|}}\cdot e^{-j2\pi fk} = \frac{4}{3}\sum_{k=-\infty}^{\infty}\frac{1}{2^{|k|}} e^{-j2\pi fk}$$

$$= \frac{4}{3}\left(1 + \sum_{k=1}^{\infty}\frac{1}{2^k}\cdot e^{-j2\pi fk} + \sum_{k=-1}^{-\infty}\frac{1}{2^{-k}}\cdot e^{-j2\pi fk}\right)$$

$$= \frac{4}{3}\left(1 + \sum_{k=1}^{\infty}\left(\frac{e^{-j2\pi f}}{2}\right)^k + \sum_{k=1}^{\infty}\left(\frac{e^{j2\pi f}}{2}\right)^k\right)$$

$$= \frac{4}{5-4\cos 2\pi f}$$

$$S_X(f) = \sigma_x^2 = 1$$

Hence. $$H(f) = \frac{S_Y(f)}{S_Y(f)+S_X(f)} = \frac{\frac{4}{5-4\cos 2\pi f}}{\frac{4}{5-4\cos 2\pi f}+1} = \frac{4}{9-4\cos 2\pi f}$$