ECE 4110/5110	Tuesday, 10/31/23

Homework Set 5 (Due: Friday, 11/10/23, 11:59 pm)

Dr. Kevin Tang — HW 5

1 Problems from "Probability and Random Processes" Geoffrey Grimmet

P1. (Section 8.2 - Problem 1) Flip–flop. Let $\{X_n\}$ be a Markov chain on the state space $S = \{0, 1\}$ with transition matrix

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

where $\alpha + \beta > 0$. Find:

(a) the correlation $\rho(X_m, X_{m+n})$, and its limit as $m \to \infty$ with n remaining fixed. Note that rho is defined as the correlation and is equal to $\frac{cov(X_m, X_{m+n})}{\sqrt{var(X_m) var(X_{m+n})}}$.

(b) $\lim_{n\to\infty} n^{-1} \sum_{r=1}^{n} P(X_r = 1)$

Under what condition is the process strongly stationary?

P2. (Section 8.2 - Problem 2) - Random telegraph. Let $\{N(t) : t \geq 0\}$ be a Poisson process of intensity λ, and let T_0 be an independent random variable such that $P(T_0 = \pm 1) = \frac{1}{2}$. Define $T(t) = T_0(-1)^{N(t)}$. Show that $\{T(t) : t \geq 0\}$ is stationary and find:

(a) $\rho(T(s), T(s+t))$

(b) the mean and variance of $X(t) = \int_0^t T(s)ds$

[The so-called Goldstein–Kac process X(t) denotes the position of a particle moving with unit speed, starting from the origin along the positive x-axis, whose direction is reversed at the instants of a Poisson process.]

P3. (Section 8.5 - Problem 2) Let W be a Wiener process. Show that, for $s < t < u$, the conditional distribution of W(t) given W(s) and W(u) is normal

$$N\left(\frac{(u-t)W(s)+(t-s)W(u)}{u-s}, \frac{(u-t)(t-s)}{u-s}\right)$$

Deduce that the conditional correlation between W(t) and W(u), given W(s) and W(v), where $s < t < u < v$, is

$$\sqrt{\frac{(v-u)(t-s)}{(v-t)(u-s)}}$$

P4. (Section 8.5 - Problem 5) Let W be a Wiener process. Which of the following define Wiener processes?

(a) $-W(t)$

(b) $\sqrt{t}W(1)$

(c) $W(2t) - W(t)$

P5. (Section 9.6 - Problem 3) Show that a Gaussian process is strongly stationary if and only if it is weakly stationary.

P6. (Section 9.6 - Problem 4) Let X be a stationary Gaussian process with zero mean, unit variance, and autocovariance function c(t). Find the autocovariance functions of the processes $X^2 = \{X(t)^2 : -\infty < t < \infty\}$ $X^3 = \{X(t)^3 : -\infty < t < \infty\}$.

P7. (Section 9.7 - Problem 20) Let W be a standard Wiener process. Find the means of the following three processes, and the autocovariance functions in case (b) :
Note that in the book there are more questions. You don't need to do them.

(a) $X(t) = |W(t)|$

(b) $Y(t) = e^{W(t)}$

(c) Which of X, and Y are Gaussian processes? Which of these are Markov processes?

2 Extra Problems.

P8. Let W be the standard Wiener Process. Answer the following questions about it:

(a) What is the distribution of W(s) + W(t), $s \leq t$?

(b) Compute $E[W(t_1)W(t_2)W(t_3)]$ for $t_1 < t_2 < t_3$.

P9. Wiener Process as a limit of random walk In this problem, we will try to approximate the wiener process using the simple random walk. Define x_i by setting

$$x_i = \begin{cases} +1, & \text{wp } 0.5 \\ -1, & \text{wp } 0.5 \end{cases}$$

All x are iid. So $x = \{x_1, x_2, ...\}$ will produce a random walk. Your path will look like

$$S_n = S_{n-1} + x_n$$

Define the diffusively rescaled random walk by the equation:

$$W_N(t) = \frac{S_{\lfloor Nt \rfloor}}{\sqrt{N}}$$

where t is in the interval [0,1]. Use coding to simulate the following.

(a) Generate 100 sample paths for N=10,100,1000 respectively.

(b) Provide a histogram of $W_N(1)$ and $W_N(0.2)$ for different N in part (a). Compute the empirical variance of $W_N(1)$ and $W_N(0.2)$ for the samples generated.

(c) What is the theoretical variance of $W_N(0.2)$ and $W_N(1)$ for different N?

(d) What is the variance of W(0.2) and W(1) for the standard Wiener process.

(e) Compare the results of part (b), (c), and (d).

P10. Consider the random process $\{X(t), t \in R\}$ defined as $X(t) = cos(t + U)$, where $U \sim Uniform(0, 2\pi)$. Show that X(t) is a weakly stationary process.