ECE 4110/5110	Tuesday, 10/31/23

Homework Set 5 Solution

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P1. (Section 8.2 - Problem 1) Flip–flop. Let $\{X_n\}$ be a Markov chain on the state space $S = \{0, 1\}$ with transition matrix

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

where $\alpha + \beta > 0$. Find:

(a) the correlation $\rho(X_m, X_{m+n})$, and its limit as $m \to \infty$ with n remaining fixed. Note that rho is defined as the correlation and is equal to $\frac{cov(X_m, X_{m+n})}{\sqrt{var(X_m)var(X_{m+n})}}$.

(b) $\lim_{n\to\infty} n^{-1} \sum_{r=1}^{n} P(X_r = 1)$

Under what condition is the process strongly stationary?

Solution. Define $a_i(n) = P(X_n = i)$, and $p_{11}(n) = P(X_{n+m} = 1|X_n = 1)$. We have that

$$\begin{aligned} cov(X_m, X_{m+n}) &= P(X_{m+n} = 1|X_m = 1)P(X_m = 1) - P(X_{m+n} = 1)P(X_m = 1) \\ &= a_1(m)p_{11}(n) - a_1(m)a_1(m+n), \end{aligned}$$

and therefore,

$$\rho(X_m, X_{m+n}) = \frac{a_1(m)p_{11}(n) - a_1(m)a_1(m+n)}{\sqrt{a_1(m)(1-a_1(m))a_1(m+n)(1-a_1(m+n))}}$$

Now, $a_1(m) \to \alpha/(\alpha+\beta)$ as $m \to \infty$, and

$$p_{11}(n) = \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n$$

The above equation is extra and you didn't need to know it. You could leave it as p11. whence $\rho(X_m, X_{m+n}) \to (1-\alpha-\beta)^n$ as $m \to \infty$. Finally,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} P(X_r = 1) = \frac{\alpha}{\alpha+\beta}$$

The process is strictly stationary if and only if X_0 has the stationary distribution.

P2. (Section 8.2 - Problem 2) - Random telegraph. Let $\{N(t) : t \geq 0\}$ be a Poisson process of intensity λ, and let T_0 be an independent random variable such that $P(T_0 = \pm 1) = \frac{1}{2}$. Define $T(t) = T_0(-1)^{N(t)}$. Show that $\{T(t) : t \geq 0\}$ is stationary and find:

(a) $\rho(T(s), T(s+t))$

(b) the mean and variance of $X(t) = \int_0^t T(s)ds$

[The so-called Goldstein–Kac process X(t) denotes the position of a particle moving with unit speed, starting from the origin along the positive x-axis, whose direction is reversed at the instants of a Poisson process.]

Solution. We have that $E(T(t)) = 0$ and $var(T(t)) = var(T_0) = 1$. Hence:

(a) $\rho(T(s), T(s+t)) = E(T(s)T(s+t)) = E[(-1)^{N(t+s)-N(t)}] = e^{-2\lambda t}$

(b) Evidently, $E(X(t)) = 0$, and

$$\begin{aligned} E[X(t)^2] &= E(\int_0^t \int_0^t T(u)T(v)dudv) \\ &= 2\int_{0<u<v<t} E(T(u)T(v))dudv = 2\int_{v=0}^t \int_{u=0}^v e^{-2\lambda(v-u)}dudv \\ &= \frac{1}{\lambda}(t - \frac{1}{2\lambda} + \frac{1}{2\lambda}e^{-2\lambda t}) \end{aligned}$$

P3. (Section 8.5 - Problem 2) Let W be a Wiener process. Show that, for $s < t < u$, the conditional distribution of W(t) given W(s) and W(u) is normal

$$N\left(\frac{(u-t)W(s)+(t-s)W(u)}{u-s}, \frac{(u-t)(t-s)}{u-s}\right)$$

Deduce that the conditional correlation between W(t) and W(u), given W(s) and W(v), where $s < t < u < v$, is

$$\sqrt{\frac{(v-u)(t-s)}{(v-t)(u-s)}}$$

Solution. We will use the following lemma: Let X, Y, Z have the standard trivariate normal density, with $\rho_1 = \rho(X,Y), \rho_2 = \rho(X,Z), \rho_3 = \rho(Y,Z)$.

$$E(Z|X,Y) = \{(\rho_3 - \rho_1\rho_2)X + (\rho_2 - \rho_1\rho_3)Y\}/(1-\rho_1^2)$$

$$var(Z|X,Y) = \{1 - \rho_1^2 - \rho_2^2 - \rho_3^2 + 2\rho_1\rho_2\rho_3\}/(1-\rho_1^2).$$

For this problem we have: Writing $W(s) = \sqrt{s}X$, $W(t) = \sqrt{t}Z$, and $W(u) = \sqrt{u}Y$, we obtain random variables X, Y, Z with the standard trivariate normal distribution, with correlations $\rho_1 = \sqrt{\frac{s}{u}}$, $\rho_2 = \sqrt{\frac{t}{u}}$, $\rho_3 = \sqrt{\frac{s}{t}}$. We have:

$$E[Z|X,Y] = \frac{u}{u-s}[(\sqrt{\frac{s}{t}} - \frac{\sqrt{st}}{u})X + (\sqrt{\frac{t}{u}} - \frac{s}{\sqrt{ut}})Y]$$

Thus replacing X,Y,Z with the values of W, we get:

$$\begin{aligned} E[W(t)|W(s),W(u)] &= \sqrt{t}\frac{u}{u-s}[(\sqrt{\frac{s}{t}} - \frac{\sqrt{st}}{u})\frac{W(s)}{\sqrt{s}} + (\sqrt{\frac{t}{u}} - \frac{s}{\sqrt{ut}})\frac{W(u)}{\sqrt{u}}] \\ &= \frac{u}{u-s}[(1-\frac{t}{u})W(s) + (\frac{t}{u} - \frac{s}{u})W(u)] \\ &= \frac{(u-t)W(s)+(t-s)W(u)}{u-s} \end{aligned}$$

Similarly for variance:

$$var(Z|X,Y) = \frac{(u-t)(t-s)}{t(u-s)}$$

and

$$var(W(t)|W(s),W(u)) = t\frac{(u-t)(t-s)}{t(u-s)} = \frac{(u-t)(t-s)}{(u-s)}$$

To find the correlation, we have (using the law of iterated expectation):

$$E[W(t)W(u)|W(s),W(v)] = E[W(u)E[W(t)|W(s),W(v),W(u)]|W(s),W(v)]$$

Now as $v > u$, then $E[W(t)|W(s),W(v),W(u)] = E[W(t)|W(s),W(u)]$. Using the results above:

$$\begin{aligned} E[W(t)W(u)|W(s),W(v)] &= E\{[\frac{(u-t)W(s)+(t-s)W(u)}{u-s}]W(u)|W(s),W(v)\} \\ &= \frac{u-t}{u-s}W(s)E[W(u)|W(s),W(v)] + \frac{t-s}{u-s}E[W^2(u)|W(s),W(v)] \end{aligned}$$

Now, with a little bit of calculation, and using the relations above we will have the result required.

P4. (Section 8.5 - Problem 5) Let W be a Wiener process. Which of the following define Wiener processes?

(a) $-W(t)$

(b) $\sqrt{t}W(1)$

(c) $W(2t) - W(t)$

Solution. (a) is obviously still Wiener. It holds all the properties of the wiener process.
(b) is not Wiener cause it does not have independent increments. For example:

$$S_4 - S_1 = S_{16} - S_9$$

Thus, we don't have independent increments.
(c) is not Wiener cause it does not have independent increments. For example:

$$\begin{aligned} cov(S_3 - S_2, S_6 - S_4) &= cov(W(6) + W(4) - W(3) - W(2), W(12) + W(8) - W(6) - W(4)) \\ &= 6 + 4 - 3 - 2 + 6 + 4 - 3 - 2 - 6 - 4 + 3 + 2 - 4 - 4 + 3 + 2 \\ &= 10 - 5 - 3 = 2 \neq 0 \end{aligned}$$

Thus, we don't have independent increments.

P5. (Section 9.6 - Problem 3) Show that a Gaussian process is strongly stationary if and only if it is weakly stationary.

Solution. If X is Gaussian and strongly stationary, then it is weakly stationary since it has a finite variance.
Conversely suppose X is Gaussian and weakly stationary. Then $c(s,t) = cov(X(s), X(t))$ depends on t-s only. The joint distribution of $X(t_1), X(t_2), \cdots, X(t_n)$ depends only on the common mean and the covariances $c(t_i, t_j)$. Now $c(t_i, t_j)$ depends on $t_j - t_i$ only, whence $X(t_1), X(t_2), \cdots, X(t_n)$ have the same joint distribution as $X(s+t_1), X(s+t_2), \cdots, X(s+t_n)$. Therefore X is strongly stationary.

P6. (Section 9.6 - Problem 4) Let X be a stationary Gaussian process with zero mean, unit variance, and autocovariance function c(t). Find the autocovariance functions of the processes $X^2 = \{X(t)^2 : -\infty < t < \infty\}$ $X^3 = \{X(t)^3 : -\infty < t < \infty\}$.

Solution. If $s, t > 0$, we have that

$$E(X(s+t)^2|X(s)) = X(s)^2 c(t)^2 + 1 - c(t)^2$$

We have:

$$\begin{aligned}
cov(X(s)^2, X(s+t)^2) &= E(X(s)^2 X(s+t)^2) - E(X^2(s))E(X^2(s+t)) \\
&= E(X(s)^2 X(s+t)^2) - 1 \\
&= E[E(X(s)^2 X(s+t)^2|X(s))] - 1 \\
&= c(t)^2 E[X^4(s)] + E[X^2(s)](1 - c(t)^2) - 1 \\
&= 3c(t)^2 + (1 - c(t)^2) - 1 = 2c(t)^2
\end{aligned}$$

We have:

$$E(X(s+t)|X(s)) = c(t)X(s), var(X(s+t)|X(s)) = 1 - c(t)^2$$

Thus:

$$E(X^3(s+t)|X(s)) = c(t)^3 X(s)^3 + 3(1 - c(t)^2)c(t)X(s)$$

We have:

$$\begin{aligned}
cov(X(s)^3, X(s+t)^3) &= E(X(s)^3 X(s+t)^3) - E(X^3(s))E(X^3(s+t)) \\
&= E(X(s)^3 X(s+t)^3) \\
&= E[E(X(s)^3 X(s+t)^3|X(s))] - 1 \\
&= c(t)^3 E[X^6(s)] + E[X^4(s)]3(1 - c(t)^2)c(t) \\
&= 15c(t)^3 + 9(1 - c(t)^2)c(t) \\
&= 3c(t)(2c(t)^2 + 3)
\end{aligned}$$

P7. (Section 9.7 - Problem 20) Let W be a standard Wiener process. Find the means of the following three processes, and the autocovariance functions in case (b) :
Note that in the book there are more questions. You don't need to do them.

(a) $X(t) = |W(t)|$

(b) $Y(t) = e^{W(t)}$

(c) Which of X, and Y are Gaussian processes? Which of these are Markov processes?

Solution.

(a) W(t) is N(0,t), so that

$$E[X(t)] = \int_{-\infty}^{\infty} \frac{|u|}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}du} = \sqrt{\frac{2t}{\pi}}$$

The process X is never negative, and therefore it is not Gaussian. It is Markov since, if $s < t$ and B is an event defined in terms of $(X(u) : u < s)$, then the conditional distribution function of X(t) satisfies

$$\begin{aligned} P(X(t) \leq y|X(s) = x, B) &= P(X(t) \leq y|W(s) = x, B)P(W(s) = x|X(s) = x, B) \\ &+ P(X(t) \leq y|W(s) = -x, B)P(W(s) = -x|X(s) = x, B) \\ &= \frac{1}{2}\{P(X(t) \leq y|W(s) = x) + P(X(t) \leq y|W(s) = -x)\} \end{aligned}$$

which does not depend on B.

(b) Certainly,

$$E[Y(t)] = \int_{-infty}^{\infty} \frac{e^u}{\sqrt{2\pi t}} e^{-\frac{u^2}{2t}du} = e^{\frac{t}{2}}$$

Secondly, $W(s) + W(t) = 2W(s) + (W(t) - W(s)$ is $N(0, 3s + t)$ if $s < t$, implying that

$$E(Y(s)Y(t)] = E(e^{W(s)+W(t)}) = e^{\frac{3s+t}{2}},$$

and therefore

$$cov(Y(s), Y(t)] = E(Y(s)Y(t)] - E(Y(s)]E[Y(t)] = e^{\frac{3s+t}{2}} - e^{\frac{s+t}{2}},$$

W(1) is N(0, 1), and therefore Y(1) has the log-normal distribution. Therefore Y is not Gaussian. It is Markov since W is Markov, and Y(t) is a one-one function of W(t).

(c) Answered above.

P8. Let W be the standard Wiener Process. Answer the following questions about it:

(a) What is the distribution of W(s) + W(t), $s \leq t$?

(b) Compute $E[W(t_1)W(t_2)W(t_3)]$ for $t_1 < t_2 < t_3$.

Solution.

(a) $W(s) + W(t) = 2W(s) + W(t) - W(s)$. Now 2W(s) is normal with mean 0 and variance 4s and $W(t) - W(s)$ is normal with mean 0 and variance $t - s$. As X(s) and $W(t) - W(s)$ are independent, it follows that $W(s) + W(t)$ is normal with mean 0 and variance $4s + t - s = 3s + t$.

(b)

$$\begin{aligned}
&E[W(t_1)W(t_2)W(t_3)] \\
&= E[E[W(t_1)W(t_2)W(t_3)|W(t_1), W(t_2)]] \\
&= E[W(t_1)W(t_2)E[W(t_3)|W(t_1), W(t_2)]] \\
&= E[W(t_1)W(t_2)W(t_2)] \\
&= E[E[W(t_1)W^2(t_2)|W(t_1)]] \\
&= E[W(t_1)E[W^2(t_2)|W(t_1)]] \\
&= E[W(t_1)\{(t_2 - t_1) + W^2(t_1)\}] \\
&= E[W^3(t_1)] + (t_2 - t_1)E[W(t_1)] = 0
\end{aligned}$$

P9. Wiener Process as a limit of random walk In this problem, we will try to approximate the wiener process using the simple random walk. Define X_i by setting

$$X_i = \begin{cases} +1, & \text{wp } 0.5 \\ -1, & \text{wp } 0.5 \end{cases}$$

All X_i are iid. So $X = \{X_1, X_2, ...\}$ will produce a random walk. Your path will look like

$$S_n = S_{n-1} + x_n$$

Define the diffusively rescaled random walk by the equation:

$$W_N(t) = \frac{S_{\lfloor Nt \rfloor}}{\sqrt{N}}$$

where t is in the interval [0,1]. Use coding to simulate the following.

(a) Generate 100 sample paths for N=10,100,1000 respectively.

(b) Provide a histogram of $W_N(1)$ and $W_N(0.2)$ for different N in part (a). Compute the empirical variance of $W_N(1)$ and $W_N(0.2)$ for the samples generated.

(c) What is the theoretical variance of $W_N(0.2)$ and $W_N(1)$ for different N?

(d) What is the variance of W(0.2) and W(1) for the standard wiener process?

(e) Compare the results of part (b), (c), and (d).

Solution (c) N=10:

$$var(W_N(0.2)) = \frac{1}{10}\sum_{i=1}^{2} var(X_i) = 0.2$$

$$var(W_N(1)) = \frac{1}{10}\sum_{i=1}^{10} var(X_i) = 1$$

Same for other N.
(d)

$$var(W(0.2)) = 0.2$$

$$var(W(1)) = 1$$

Other parts are coding.

P10. Consider the random process $\{X(t), t \in R\}$ defined as $X(t) = cos(t + U)$, where $U \sim Uniform(0, 2\pi)$. Show that X(t) is a weakly stationary process.

Solution We need to check two conditions:

1. $\mu_X(t) = \mu_X$, for all $t \in R$, and
2. $R_X(t_1, t_2) = R_X(t_1 - t_2)$, for all $t_1, t_2 \in R$.

We have

$$\mu_X(t) = E[X(t)] = E[cos(t + U)] = \int_0^{2\pi} \frac{1}{2\pi} cos(t + u) du = 0, \forall t \in R.$$

We can also find $R_X(t_1, t_2)$ as follows

$$\begin{aligned}
R_X(t_1, t_2) &= E[X(t_1)X(t_2)] \\
&= E[cos(t_1 + U)cos(t_2 + U)] \\
&= E[\frac{1}{2}cos(t_1 + t_2 + 2U) + \frac{1}{2}cos(t_1 - t_2)] \\
&= E[\frac{1}{2}cos(t_1 + t_2 + 2U)] + E[\frac{1}{2}cos(t_1 - t_2)] \\
&= \int_0^{2\pi} cos(t_1 + t_2 + 2u)\frac{1}{2\pi}du + \frac{1}{2}cos(t_1 - t_2) \\
&= \frac{1}{2}cos(t_1 - t_2), \forall t_1, t_2 \in R.
\end{aligned}$$

As we see, both conditions are satisfied, thus X(t) is a WSS process.