ECE 4110/5110	Friday, 09/22/23
Homework Set 3(Due: Friday, 10/06/23, 11:59 pm)	
Dr. Kevin Tang	HW 3

1 Problems from Bertsekas and Tsitsiklis, 2nd edition - Chapter 7

1. **Problem 2.** A mouse moves along a tiled corridor with 2m tiles, where $m > 1$. From each tile $i \neq 1, 2m$. it moves to either tile i - l or i + 1 with equal probability. From tile 1 or tile 2m, it moves to tile 2 or 2m - 1, respectively, with probability 1. Each time the mouse moves to a tile $i \leq m$ or $i > m$, an electronic device outputs a signal L or R, respectively. Can the generated sequence of signals L and R be described as a Markov chain with states L and R?

2. **Problem 4.** A spider and a fly move along a straight line in unit increments. The spider always moves towards the fly by one unit. The fly moves towards the spider by one unit with probability 0.3, moves away from the spider by one unit with probability 0.3, and stays in place with probability 0.4. The initial distance between the spider and the fly is integer. When the spider and the fly land in the same position, the spider captures the fly.

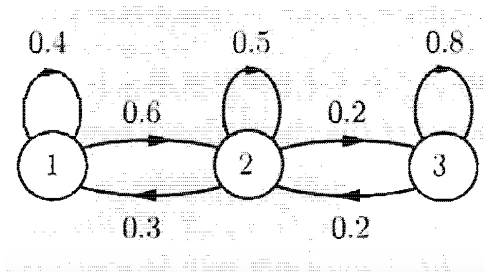
   (a) Construct a Markov chain that describes the relative location of the spider and fly.

   (b) Identify the transient and recurrent states.

3. **Problem 5.** Consider a Markov chain with states 1 , 2, . . . , 9. and the following transition probabilities: $p_{12} = p_{17} = 1/2$, $p_{i(i+1)} = 1$ for $i \neq 1, 6, 9$, and $p_{61} = p_{91} = 1$. Is the recurrent class of the chain periodic or not?

4. **Problem 11.** A professor gives tests that are hard, medium, or easy. If she gives a hard test, next test will be either medium or easy, with equal probability. However, if she gives a medium or easy test, there is a 0.5 probability that her next test will be the same difficulty, and a 0.25 probability for of the other two levels of difficulty. Construct an appropriate Markov chain and find the steady-state probabilities.

5. Problem 13. Consider the Markov Chain below. Let us refer to a transition that results in a state with a higher (respectively, lower) index as a birth (respectively death). Calculate the following quantities, assuming that when we start observing the chain, it is already in steady-state:



(a) For each state i, the probability that the current state is i.

(b) The probability that the first transition to observe is a birth.

(c) The probability that the first change of state to observe is a birth.

(d) The conditional probability that the process was in state 2 before the first transition that we observe, given that this transition was a birth.

(e) The conditional probability that the process was in state 2 before the first change of state that we observe, given that this change of state was a birth.

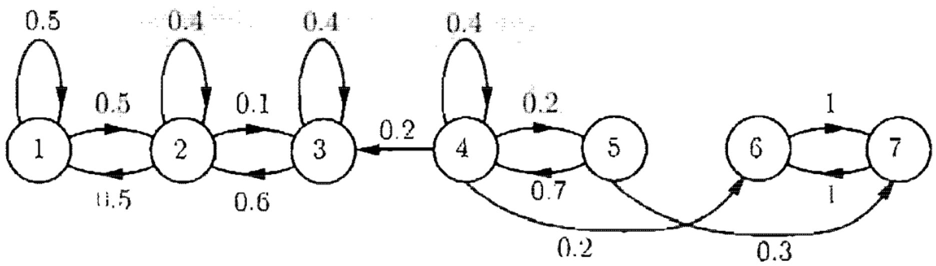
(f) The conditional probability that the first observed transition is a birth given that it resulted in a change of state.

(g) The conditional probability that the first observed transition leads to state 2, given that it resulted in a change of state.

6. Problem 15. Ehrenfest model of diffusion. We have a total of n balls, some of them black, some white. At each time step, we either do nothing, which happens with probability ϵ, where $0 < \epsilon < 1$, or we select a ball at random, so that each ball has probability $(1-\epsilon)/n > 0$ of being selected. In the latter case, we change the color of the selected ball (if white it becomes black, and vice versa), and the process is repeated indefinitely. What is the steady-state distribution of the number of white balls?

7. Problem 16. Bernoulli-Laplace model of diffusion. Each of two urns contains m balls. Out of the total of the 2m balls, m are white and m are black. A ball is simultaneously selected from each urn and moved to the other urn, and the process is indefinitely repeated. What is the steady-state distribution of the number of white balls in each urn?

8. Problem 30 Consider the Markov chain below



(a) Identify the transient and recurrent states. Also, determine the recurrent classes and indicate which ones, if any, are periodic.

(b) Do there exist steady state probabilities given that the process start in state 1? If so, what are they?

(c) Do there exist steady state probabilities given that the process start in state 6? If so, what are they?

(d) Assume that the process starts in state 1 but we begin observing it after it reaches steady-state.

  (i) Find the probability that the state increases by one during the first transition we observe.

  (ii) Find the conditional probability that the process was in state 2 when we started observing it, given that the state increased by one during the first transition that we observed.

  (iii) Find the probability that the state increased by one during the first change of state that we observed.

(e) Assume that the process starts in state 4.

  (i) For each recurrent class, determine the probability that we eventually reach that class.

  (ii) What is the expected number of transitions up to and including the transition at which we reach a recurrent state for the first time?

9. Problem 37 Empty taxis pass by a street corner at a Poisson rate of two per minute and pick up a passenger if one is waiting there. Passengers arrive at the street corner at a Poisson rate of one per minute and wait for a taxi only if there are less than four persons waiting; otherwise they leave and never return. Penelope arrives at the street corner at a given time. Find her expected waiting time, given that she joins the queue. Assume that the process is in steady-state.

2 Extra Problems:

10. I have 3 umbrellas, some at home, some in the office. I keep moving between home and office. I take an umbrella with me only if it rains. If it does not rain I leave the umbrella behind (at home or in the office). It may happen that all umbrellas are in one place, I am at the other, it starts raining and must leave, so I get wet. If the probability of rain is p, what is the probability that I get wet?

11. A fair coin is tossed repeatedly and independently.

(a) Find the expected number of tosses till the pattern "HT" appears.

(b) Find the expected number of tosses till the pattern "HH" appears.

(c) Find the expected number of tosses till the pattern "HTHT" appears.

(d) Find the expected number of tosses till the pattern "HHHH" appears.

(e) Write a code simulating the coin toss. Confirm the results in previous parts by repeating the experiment 100 times and averaging over them.

12. A certain experiment is believed to be described by a two-state Markov chain with the transition matrix P, where

$$P = \begin{bmatrix} 0.5 & 0.5 \\ p & 1-p \end{bmatrix}$$

and the parameter p is not known. When the experiment is performed many times, the chain ends in state one approximately 10 percent of the time and in state two approximately 90 percent of the time. Compute a sensible estimate for the unknown parameter p and explain how you found it.