# ECE 4110/5110

# Random Signals in Communication and Signal Processing

# Homework 1


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# Contents

# 1 Problems from Bertsekas and Tsitsiklis, 2nd edition

## 1.1 Problem 1 and Corresponding Solution

### Problem 1 (Ch2 P32)

Consider $2m$ persons forming m couples who live together at a given time. Suppose that at some later time, the probability of each person being alive is $p$, independent of other persons. At that later time. let $A$ be the number of persons that are alive and let $S$ be the number of couples in which both partners are alive. For any survivor number a, find $E[S|A = a]$.

### Solution:

Let $X_i$ be the random variable taking the value 1 or 0 depending on whether the first partner of the $i$th couple has survived or not. Let $Y_i$ be the corresponding random variable for the second partner of the $i$th couple. Then, we have $S = \sum_{i=1}^{m} X_i Y_i$, and by using the total expectation theorem,

$$\begin{aligned}
\mathbf{E}[S \mid A = a] &= \sum_{i=1}^{m} \mathbf{E}\left[X_i Y_i \mid A = a\right] \\
&= m\mathbf{E}\left[X_1 Y_1 \mid A = a\right] \\
&= m\mathbf{E}\left[Y_1 \mid X_1 = 1, A = a\right] \mathbf{P}\left(X_1 = 1 \mid A = a\right) \\
&= m\mathbf{P}\left(Y_1 = 1 \mid X_1 = 1, A = a\right) \mathbf{P}\left(X_1 = 1 \mid A = a\right)
\end{aligned}$$

We have

$$\mathbf{P}\left(Y_1 = 1 \mid X_1 = 1, A = a\right) = \frac{a-1}{2m-1}, \quad \mathbf{P}\left(X_1 = 1 \mid A = a\right) = \frac{a}{2m}$$

Thus

$$\mathbf{E}[S \mid A = a] = m\frac{a-1}{2m-1} \cdot \frac{a}{2m} = \frac{a(a-1)}{2(2m-1)}$$

Note that $E[S|A = a]$ does not depend on $p$.

## 1.2 Problem 2 and Corresponding Solution

## Problem 2 (Ch3 P20)

An absent-minded professor schedules two student appointments for the same time. The appointment durations are independent and exponentially distributed with mean thirty minutes. The first student arrives on time, but the second student arrives five minutes late. What is the expected value of the time between the arrival of the first student and the departure of the second student?

## Solution:

The expected value in question is

$$\begin{aligned}\mathbf{E}[\text{Time}] =& (5+\mathbf{E}[\text{stay of 2nd student}])\cdot\mathbf{P}(\text{1st stays no more than 5 minutes})\\ &+(\mathbf{E}[\text{stay of 1st}\mid\text{stay of 1st}\geq 5]+\mathbf{E}[\text{stay of 2nd}])\\ &\cdot\mathbf{P}(\text{1st stays more than 5 minutes}).\end{aligned}$$

We have $\mathbf{E}[\text{stay of 2nd student}] = 30$, and, using the memorylessness property of the exponential distribution,

$$\mathbf{E}[\text{stay of 1st}\mid\text{stay of 1st}\geq 5] = 5+\mathbf{E}[\text{stay of 1st}] = 35$$

Also

$$\mathbf{P}(\text{1st student stays no more than 5 minutes}) = 1-e^{-5/30}$$

$$\mathbf{P}(\text{1st student stays more than 5 minutes}) = e^{-5/30}$$

By substitution we obtain

$$\mathbf{E}[Time] = (5+30)\cdot(1-e^{-5/30})+(35+30)\cdot e^{-5/30} = 35+30\cdot e^{-5/30} = 60.394$$

### 1.3 Problem 3 and Corresponding Solution

## Problem 3 (Ch3 P25)

The coordinates X and Y of a point are independent zero mean normal random variables with common variance $\sigma^2$. Given that the point is at a distance of at least $c$ from the origin, find the conditional joint PDF of X and Y.

## Solution:

Let $C$ denote the event that $X^2+Y^2 \geq c^2$. The probability $P(C)$ can be calculated using polar coordinates, as follows:

$$\begin{aligned} P(C) &= \frac{1}{2\pi\sigma^2}\int_0^{2\pi}\int_c^{\infty} re^{-r^2/2\sigma^2}drd\theta \\ &= \frac{1}{\sigma^2}\int_c^{\infty} re^{-r^2/2\sigma^2}dr \\ &= e^{-c^2/2\sigma^2} \end{aligned}$$

Thus, for $(x,y) \in C$,

$$f_{X,Y|C}(x.y) = \frac{f_{X,Y}(x,y)}{P(C)} = \frac{1}{2\pi\sigma^2}e^{-\frac{1}{2\sigma^2}(x^2+y^2-c^2)}$$

## 1.4 Problem 4 and Corresponding Solution

## Problem 4 (Ch4 P18)

Consider four random variables, W, X, Y, Z with

$$E[W] = E[X] = E[Y] = E[Z] = 0$$

$$var(W) = var(X) = var(Y) = var(Z) = 1$$

and assume that W. X, Y. Z are pairwise uncorrelated. Find the correlation coefficients $\rho(R, S)$ and $\rho(R, T)$. where R = W + X, S = X + Y. and T = Y + Z.

## Solution:

We have

$$cov(R, S) = E[RS] - E[R]E[S] = E[WX + WY + X^2 + XY] = E[X^2] = 1$$

and

$$var(R) = var(S) = 2$$

so

$$\rho(R, S) = \frac{cov(R, S)}{\sqrt{var(R)var(S)}} = \frac{1}{2}$$

We also have

$$cov(R, T) = E[RT] - E[R]E[T] = E[WY + WZ + XY + XZ] = 0$$

so that

$$\rho(R, T) = 0$$

## 1.5 Problem 5 and Corresponding Solution

## Problem 5 (Ch4 P19)

Suppose that a random variable X satisfies

$$E[X] = 0, \quad E[X^2] = 1, \quad E[X^3] = 0, \quad E[X^4] = 3$$

and let $Y = a + bX + cX^2$. Find the correlation coefficient p(X, Y).

## Solution:

To compute the correlation coefficient

$$\rho(X, Y) = \frac{cov(X, Y)}{\sigma_X \sigma_Y}$$

we first compute the covariance:

$$\begin{aligned} cov(X, Y) &= E[XY] - E[X]E[Y] \\ &= E[aX + bX^2 + cX^3] - E[X]E[Y] \\ &= aE[X] + bE[X^2] + cE[X^3] \\ &= b \end{aligned}$$

we also have

$$\begin{aligned} var(Y) &= var(a + bX + cX^2) \\ &= E[(a + bX + cX^2)^2] - (E[a + bX + cX^2])^2 \\ &= (a^2 + 2ac + b^2 + 3c^2) - (a^2 + c^2 + 2ac) \\ &= b^2 + 2c^2 \end{aligned}$$

and therefore, using the fact var(X) = 1,

$$\rho(X, Y) = \frac{b}{\sqrt{b^2 + 2c^2}}$$

### 1.6 Problem 6 and Corresponding Solution

## Problem 6 (Ch4 P24)

A retired professor comes to the office at a time that is uniformly distributed between 9 a.m. and 1 p.m., performs a single task and leaves when the task is completed. The duration of the task is exponentially distributed with parameter, $\lambda(y) = 1/(5-y)$, where y is the length of the time interval between 9 a.m. and the time of his arrival:

(a) What is the expected amount of time that the professor devotes to the task?

(b) What is the expected time at which the task is completed?

(c) The professor has a Ph.D. student who on a given day comes to see him at a time that is uniformly distributed between 9 a.m. and 5 p.m. If the student does not find the professor, he leaves and does not return. If he finds the professor, he spends an amount of time that is uniformly distributed between 0 and 1 hour. The professor will spend the same total amount of time on his task regardless of whether he is interrupted by the student. What is the expected amount of time that the professor will spend with the student and what is the expected time at which he will leave his office?

## Solution:

(a) Consider the following two random variables:

$X$ = amount of time the professor devotes to his task — exponentially distributed with parameter $\lambda(y) = 1/(5-y)$

$Y$ = length of time between 9 am and his arrival (uniformly distributed between 0 and 4)

Note that E[Y] = 2. We have

$$E[X \mid Y = y] = \frac{1}{\lambda(y)} = 5 - y$$

which implies that

$$E[X \mid Y] = 5 - Y$$

and

$$E[X] = E[E[X \mid Y]] = E[5 - Y] = 5 - E[Y] = 5 - 2 = 3$$

(b) Let $Z$ be the length of time with 9 am until the professor completes the task. Then

$$Z = X + Y$$

We already know from part(a) that $E[X] = 3$ and $E[Y] = 2$, so that

$$E[Z] = E[X] + E[Y] = 5$$

Thus the expected time that the professor leaves his office is 5 hours after 9 am.

(c) We define the following random variables:

$W$ = length of time between 9 am and arrival of the Ph.D. student (uniformly distributed between 9 am and 5 pm).

$R$ = amount of time the student will spend with the professor, if he finds the professor (uniformly distributed between 0 and 1 hour).

$T$ = amount of time the professor will spend with the student.

Let also $F$ be the event that the student finds the professor.

To find E[T], we write

$$E[T] = P(F)E[T \mid F] + P(F^c)E[T \mid F^c]$$

Using the problem data,

$$E[T \mid F] = E[R] = \frac{1}{2}$$

(this is the expected value of a uniformly distribution ranging from 0 to 1),

$$E[T \mid F^c] = 0$$

(since the student leaves if he does not find the professor). We have

$$E[T] = P(F)E[T \mid F] = \frac{1}{2}P(F)$$

so we need to find $P(F)$.

In order for the student to find the professor, his arrival should be between the arrival and the departure of the professor. Thus

$$P(F) = P(Y \leq W \leq X + Y)$$

We have that $W$ can be between 0 (9 am) and 8 (5 pm), but $X + Y$ can be any value greater than 0. In particular, it may happen that the sum is greater than the upper bound for $W$. We write

$$P(F) = P(Y \leq W \leq X + Y) = 1 - (P(W < Y) + P(W > X + Y))$$

We have

$$P(W < Y) = \int_0^4 \frac{1}{4} \int_0^y \frac{1}{8} dw dy = \frac{1}{4}$$

and

$$\begin{aligned}
P(W > X + Y) &= \int_0^4 P(W > X + Y \mid Y = y) f_Y(y) dy \\
&= \int_0^4 P(X < W - Y \mid Y = y) f_Y(y) dy \\
&= \int_0^4 \int_y^8 F_{X|Y}(w - y) f_W(w) f_Y(y) dw dy \\
&= \int_0^4 \frac{1}{4} \int_y^8 \frac{1}{8} \int_0^{w-y} \frac{1}{5-y} e^{-\frac{x}{5-y}} dx dw dy \\
&= \frac{12}{32} + \frac{1}{32} \int_0^4 (5 - y) e^{-\frac{8-y}{5-y}} dy
\end{aligned}$$

Integrating numerically, we have

$$\int_0^4 (5 - y) e^{-\frac{8-y}{5-y}} dy = 1.7584$$

Thus,

$$P(Y \leq W \leq X + Y) = 1 - (P(W < Y) + P(W > X + Y)) = 1 - 0.68 = 0.32$$

The expected amount of time the professor will spend with the student is then

$$E[T] = \frac{1}{2}P(F) = \frac{1}{2} \cdot 0.32 = 0.16h = 9.6mins$$

Next, we want to find the expected time the professor will leave his office. Let $Z$ be the length of time measured from 9 am until he leaves his office. If the professor doesn't spend any time with the student, then $Z$ will be equal to $X+Y$. On the other hand, if the professor is interrupted by the student, then the length of time will be equal to $X + Y + R$. This is because the professor will spend the same amount of total time on the task regardless of whether he is interrupted by the student. Therefore,

$$E[Z] = P(F)E[Z \mid F] + P(F^c)E[Z \mid F^c] = P(F)E[X + Y + R] + P(F^c)E[X + Y]$$

Using the results of the earlier calculations,

$$E[X + Y] = 5$$

$$E[X + Y + R] = E[X + Y] + E[R] = 5 + \frac{1}{2} = 5.5$$

Therefore,

$$E[Z] = 0.68 \times 5 + 0.32 \times 5.5 = 5.16$$

Thus the expected time the professor will leave his office is 5.16 hours after 9 am.

## 1.7 Problem 7 and Corresponding Solution

## Problem 7 (Ch4 P35)

Let X be a random variable that takes nonnegative integer values, and is associated with a transform of the form

$$M_x(s) = c \cdot \frac{3 + 4e^{2s} + 2e^{3s}}{3 - e^s}$$

where c is some scalar. Find $E[X]$, $P_x(1)$ , and $E[X|X \neq 0]$ .

## Solution:

We first find c by using the equation

$$1 = M_X(0) = c \cdot \frac{3 + 4 + 2}{3 - 1}$$

so that $c = \frac{2}{9}$. We then obtain

$$E[X] = \frac{dM_X}{ds}(s)\bigg|_{s=0} = \frac{2}{9} \cdot \frac{(3 - e)(8e^{2s} + 6e^{3s}) + e^s(3 + 4e^{2s} + 2e^{3s})}{(3 - e^s)^2}\bigg|_{s=0} = \frac{37}{18}$$

We now use the identity

$$\frac{1}{3 - e^s} = \frac{1}{3} \cdot \frac{1}{1 - \frac{e^s}{3}} = \frac{1}{3}(1 + \frac{e^s}{3} + \frac{e^{2s}}{9} + ...)$$

which is valid as long as $s$ is small enough so that $e^s < 3$. It follows that

$$M_X(s) = \frac{2}{9} \cdot \frac{1}{3} \cdot (3 + 4e^{2s} + 2e^{3s}) \cdot (1 + \frac{e^s}{3} + \frac{e^{2s}}{9} + ...)$$

By identifying the coefficients of $e^{0s}$ and $e^s$, we obtain

$$p_X(0) = \frac{2}{9} \qquad p_X(1) = \frac{2}{27}$$

Let $A = \{X \neq 0\}$. We have

$$p_{X|\{X \in A\}}(k) = \begin{cases} \frac{p_X(k)}{P(A)} & \text{if } k \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

so that

$$\begin{aligned} E[X \mid X \neq 0] &= \sum_{k=1}^{\infty} k p_{X|A}(k) \\ &= \sum_{k=1}^{\infty} \frac{k p_X(k)}{P(A)} \\ &= \frac{E[X]}{1 - p_X(0)} \\ &= \frac{37/18}{7/9} \\ &= \frac{37}{14} \end{aligned}$$

## 1.8 Problem 8 and Corresponding Solution

## Problem 8 (Ch4 P43)

A motorist goes through 4 lights, each of which is found to be red with probability 1/2. The waiting times at each light are modeled as independent normal random variables with a mean 1 minute and a standard deviation 1/2 minute. Let X be the total waiting time at the red lights.

(a) Use the total probability theorem to find the PDF and the transform associated with X, and the probability that X exceeds 4 minutes. Is X normal?

(b) Find the transform associated with X by viewing X as a sum of a random number of random variables.

## Solution:

(a) Using the total probability theorem, we have

$$P(X > 4) = \sum_{k=0}^{4} P(k\ lights\ are\ red)P(X > 4 \mid k\ lights\ are\ red)$$

We have

$$P(k\ lights\ are\ red) = \begin{pmatrix} 4 \\ k \end{pmatrix} (\frac{1}{2})^4$$

The conditional PDF of $X$ given that $k$ lights are red, is normal with mean $k$ minutes and standard deviation $(1/2)\sqrt{k}$. Thus, $X$ is a mixture of normal random variables and the transform associated with its (unconditional) PDF is the corresponding mixture of the transforms associated with the (conditional) normal PDFs. However, X is not normal, because a mixture of normal PDFs need not be normal. The probability $P(X > 4 \mid k\ lights\ are\ red)$ can be computed from the normal tables for each $k$, and $P(X > 4)$ is obtained by substituting the results in the total probability formula above.

(b) Let K be the number of traffic lights that are found to be red. We can view $X$ as the sum of $K$ independent normal random variables. Thus the transform associated with $X$ can be found by replacing in the binomial transform $M_K(s) = (1/2 + (1/2)e^s)^4$ the occurrence of $e^s$ by the normal transform corresponding to $\mu = 1$ and $\sigma = 1/2$. Thus

$$M_X(s) = \left(\frac{1}{2} + \frac{1}{2}(e^{\frac{(1/2)^2 s^2}{2} + s})\right)^4$$

Note that by using the formula for the transform, we cannot easily obtain the probability $P(X > 4)$.

### 1.9 Problem 9 and Corresponding Solution

## Problem 9 (Ch5 P5)

Let $X_1$, $X_2$, ... be independent random variables that are uniformly distributed over [-1,1]. Show that the sequence $Y_1$, $Y_2$, ... Converges in probability to some limit, and identify the limit, for each of the following cases:

(a) $Y_n = X_n/n$

(b) $Y_n = (X_n)^n$

(c) $Y_n = X_1X_2...X_n$

(d) $Y_n = max(X_1, ..., X_n)$

## Solution:

In cases (a), (b), and (c), we show that $Y_n$ converges to 0 in probability. In case (d), we show that $Y_n$ converges to 1 in probability.

(a) For any $\epsilon > 0$, we have

$$P(|Y_n| \geq \epsilon) = 0$$

for all $n$ with $\frac{1}{n} < \epsilon$, so

$$P(|Y_n| \geq \epsilon) \rightarrow 0$$

(b) For all $\epsilon \in (0, 1)$, we have

$$P(|Y_n| \geq \epsilon) = P(|X_n|^n \geq \epsilon) = P(X_n \geq \epsilon^1/n) + P(X_n \leq -\epsilon^1/n) = 1 - \epsilon^{1-n}$$

and the two terms in the right-hand side converge to 0, since $\epsilon^{1-n} \rightarrow 1$

(c) Since $X_1, X_2...$ are independent random variables, we have

$$E[Y_n] = E[X_1]...E[X_n] = 0$$

Also

$$var(Y_n) = E[Y_n^2] = E[X_1^2]...E[X_n^2] = var(X_1)^n = (\frac{4}{12})^n$$

so $var(Y_n) \rightarrow 0$. Since all $Y_n$ have 0 as a common mean, from Chebyshev's inequality it follows that $Y_n$ converges to 0 in probability.

(d) We have for all $\epsilon \in (0, 1)$, using the independence of $X_1, X_2, ...$,

$$\begin{aligned}
P[|Y_n - 1| \geq \epsilon] &= P(max\{X_1, ...X_n\} \geq 1 + \epsilon) + P(max\{X_1, ...X_n\} \leq 1 - \epsilon) \\
&= P(X_1 \leq 1 - \epsilon, ..., X_n \leq 1 - \epsilon) \\
&= (P(X_1 \leq 1 - \epsilon))^n \\
&= \left(1 - \frac{\epsilon}{2}\right)^n
\end{aligned}$$

Hence $P(|Y_n| - 1 \geq \epsilon) \rightarrow 0$

# 2 Extra Problems

## 2.1 Problem 10 and Corresponding Solution

### Problem 10

Besides the transform that you learned in Section 4.4, there are various other similar transforms (moment generating functions) that can be defined. For example, for a discrete random variable $X$ whose possible values are nonnegative integers, we can define its Laplace moment generating function as

$$G_X(z) = \sum_{k=0}^{\infty} p(X = k) z^k$$

Clearly, it is the $z$-transform of the sequence of mass probabilities $p_k = P(X = k)$. Actually, if we set $z = e^s$, this definition goes back to the definition that you learned in Section 4.4. However, the Laplace moment generating function is easier to deal with than the general moment generating function when $X$ can only take nonnegative integer values. In particular, it may help solve certain counting problems in a more "mechanical" manner by using formula of power series and therefore does not require much "cleverness" (see question (5) below). All series involved in this problem are assumed to converge.

(1) Show that $E(X) = \frac{dG_X(z)}{dz}\Big|_{z=1}$ and $Var(X) = \frac{d^2G_X(z)}{dz^2}\Big|_{z=1} + \frac{dG_X(z)}{dz}\Big|_{z=1} - (\frac{dG_X(z)}{dz}\Big|_{z=1})^2$

(2) For the Binomial random variable $X$ (with parameters $n$ and $p$), please compute its Laplace moment generating function and use results in (1) to compute its expected value and its variance.

(3) Show that if $X_1$, ... ,$X_n$ are independent random variables and $X = \sum_{i=1}^{n} X_i$, then $G_X(z) = \prod_{i=1}^{n} GX_i(z)$

(4) Please compute the Laplace moment generating function for the Bernoulli random variable (with parameter $p$). Then using the fact that the Binomial random variable $X$ (with parameters $n$ and $p$) can be expressed as the sum of n Bernoulli random variables (with parameter $p$) and the result in (3) to compute $G_X(z)$ and compare with your result in (2).

(5) You randomly roll five six-sided dice, what is the probability that the sum of those five numbers is 15? Hint: Doing counting directly could be quite involved here. Please compute the Laplace generating function of the random variable that is of your interest first. Also, you may find the following negative binomial series useful.

$$(1+z)^{-m} = \sum_{i=0}^{\infty} (-1)^i \begin{pmatrix} m+i-1 \\ i \end{pmatrix} z^i$$

**Solution:**

## 2.2 Problem 11 and Corresponding Solution

### Problem 11 (Portfolio Optimization)

The Acahti Trust Company offers two investment instruments, the Lousy Money Market Account and the Risky Mutual Fund. After one year, the Lousy Money Market Account returns exactly your initial principal, without interest. After one year, the Risky Mutual Fund returns twice your initial principal with probability 0.5, and half your initial principal with probability 0.5. Suppose you start with one dollar and each year you rebalance your investments so that a fraction $\theta$ of your savings is in the mutual fund and a fraction $1-\theta$ is in the money market account. Let $X_n$ denote the mutual fund's return during year $n$, so that

$$P(X_n = 2) = P(X_n = 1/2) = 1/2$$

Let $W_n$ denote your savings after $n$ years.

(a) Express $W_n$ in terms of $W_{n-1}$, $X_n$, and $\theta$ for $n \geq 1$.

(b) Determine $E[W_n]$ as a function of $\theta$. What value of $\theta$ maximizes your expected savings?

(c) Show that

$$\lim_{n\to\infty} \frac{1}{n} log_e(W_n) = \beta$$

with probability 1 for some constant $\beta$, and determine $\beta$ as a function of $\theta$. We call $\beta$ the *asymptotic growth rate* of the savings because asymptotically $W_n$ behaves like $\exp(\beta n)$ with probability 1.

Hint: Write Wn as a product of i.i.d. variables, take logarithms, and use the strong law of large numbers.

(d) What value of $\theta$ maximizes the asymptotic growth rate? What is this maximum growth rate? What asymptotic growth rate is achieved with the $\theta$ you found in (b)?

(e) How come trying to maximize average return is different from maximizing the long term return? Which quantity, the expected savings or the asymptotic growth rate, should you aim to optimize? Explain.

### Solution:

## 2.3 Problem 12 and Corresponding Solution

### Problem 12

Suppose that there are $N$ students in ECE 4110. Sadly, when Dr. Tang returns the graded exam, he simply hands each student a random exam from the pile. Let $K$ denote the number of students who, by chance, happen to receive their own exam. Thus $N-K$ students receive someone else's exam.

(a) Use Markov's inequality to upper bound the chance that $K \geq m$, where $m > 1$.

(b) Use Chebychev's inequality to upper bound the chance that $K \geq m$, where $m > 1$.

### Solution: