ECE 4110/5110	Friday, 09/08/23

Homework Set 2 (Due: Friday, 09/22/23, 11:59 pm)

Dr. Kevin Tang	HW 2

1 Problems from Bertsekas and Tsitsiklis, 2nd edition - Chapter 6

1. Problem 2. Dave fails quizzes with probability 1/4, independent of other quizzes.

1. What is the probability that Dave fails exactly two of the next six quizzes?
2. What is the expected number of quizzes that Dave will pass before he has failed three times?
3. What is the probability that the second and third time Dave fails a quiz will occur when he takes his eighth and ninth quizzes, respectively?
4. What is the probability that Dave fails two quizzes in a row before he passes two quizzes in a row?

2. Problem 3. A computer system carries out tasks submitted by two users. Time is divided into slots. A slot can be idle, with probability $P_I = 1/6$, and busy with probability $P_B = 5/6$. During a busy slot, there is probability $P_{1|B} = 2/5$ (respectively, $P_{2|B} = 3/5$) that a task from user 1 (respectively, 2) is executed. We assume that events related to different slots are independent.

(a) Find the probability that a task from user 1 is executed for the first time during the 4th slot.

(b) Given that exactly 5 out of the first 10 slots were idle, find the probability that the 6th idle slot is slot 12.

(c) Find the expected number of slots up to and including the 5th task from user 1.

(d) Find the expected number of busy slots up to and including the 5th task from user 1.

(e) Find the PMF, mean, and variance of the number of tasks from user 2 until the time of the 5th task from user 1.

3. Problem 14. Beginning at time t = 0. we start using bulbs. one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a type-A bulb and a type-B bulb. The lifetime, X, of any particular bulb of a particular type is a random variable, independent of everything else, with the following PDF:

$$\text{for type-A Bulbs:} f_x(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & ow \end{cases}$$

$$\text{for type-B Bulbs:} f_x(x) = \begin{cases} 3e^{-3x}, & x \geq 0 \\ 0, & ow \end{cases}$$

(a) Find the expected time until the first failure.

(b) Find the probability that there are no bulb failures before time t.

(c) Given that there are no failures until time t, determine the conditional probability that the first bulb used is a type-A bulb.

(d) Find the variance of the time until the first bulb failure.

(e) Find the probability that the 12th bulb failure is also the 4th type-A bulb failure.

(f) Up to and including the 12th bulb failure, what is the probability that a total of exactly 4 type-A bulbs have failed?

(g) Determine either the PDF or the transform associated with the time until the 12th bulb failure.

(h) Determine the probability that the total period of illumination provided by the first two type-B bulbs is longer than that provided by the first type-A bulb.

(i) Suppose the process terminates as soon as a total of exactly 12 bulb failures have occurred. Determine the expected value and variance of the total period of illumination provided by type-B bulbs while the process is in operation.

(j) Given that there are no failures until time t, find the expected value of the time until the first failure.

4. Problem 15. A service station handles jobs of two types, A and B. (Multiple jobs can be processed simultaneously.) Arrivals of the two job types are independent Poisson processes with parameters $\lambda_A = 3$ and $\lambda_B = 4$ per minute, respectively. Type A jobs stay in the service station for exactly one minute. Each type B job stays in the service station for a random but integer amount of time which is geometrically distributed, with mean equal to 2, and independent of everything else. The service station started operating at some time in the remote past.

(a) What is the mean, variance, and PMF of the total number of jobs that arrive within a given three-minute interval?

(b) We are told that during a 10-minute interval, exactly 10 new jobs arrived. What is the probability that exactly 3 of them are of type A?

(c) At time 0, nojob is present in the service station. What is the PMF ofthe number of type B jobs that arrive in the future, but before the first type A arrival?

(d) At time t = 0, there were exactly two type A jobs in the service station. What is the PDF of the time of the last (before time 0) type A arrival?

(e) At time 1, there was exactly one type B job in the service station. Find the distribution of the time until this type B job departs.

5. Problem 16. Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal in your neighborhood, and police cars drive by according to a Poisson process with rate A. You decide to make a U-turn once you see that the road has been clear of police cars for T time units. Let N be the number of police cars you see before you make the U-turn.

(a) Find E[N].

(b) Find the conditional expectation of the time elapsed between police cars n - 1 and n, given that $N \geq n$.

(c) Find the expected time that you wait until you make the U-turn. Hint: Condition on N.

6. Problem 17. A wombat in the San Diego zoo spends the day walking from a burrow to a food tray, eating, walking back to the burrow, resting, and repeating the cycle. The amount of time to walk from the burrow to the tray (and also from the tray to the burrow) is 20 secs. The amounts of time spent at the tray and resting are exponentially distributed with mean 30 secs. The wombat, with probability 1/3, will momentarily stand still (for a negligibly small time) during a walk to or from the tray, with all times being equally likely (and independent of what happened in the past). A photographer arrives at a random time and will take a picture at the first time the wombat will stand still. What is the expected value of the length of time the photographer has to wait to snap the wombat's picture?

2 Problems from Grimmett and Stirzaker, 3.9/3.10

7. Chapter 3.9 Problem 7 - Returns and visits by random walk. Consider a simple symmetric random walk on the set $\{0, 1, 2, ..., a\}$ with absorbing barriers at 0 and a, and starting at k where $0 < k < a$. Let r_k be the probability the walk ever returns to k, and let v_k, be the mean number of visits to point x before absorption. Find r_k, and hence show that,

$$v_k = \begin{cases} 2x(a-k)/a & 0 < x < k \\ 2k(a-x)/a & k < x < a \end{cases}$$

8. Chapter 3.10 Problem 3 For a symmetric simple random walk starting at 0, show that the probability of the first visit to S_{2n} takes place at time $2k$ equals the product $P(S_{2k} = 0)P(S_{2n-2k} = 0)$, for $0 \leq k \leq n$.

9. Chapter 3.11 Problem 29 Let S be a symmetric random walk with $S_0 = 0$, and let N_n be the number of points that have been visited by S exactly once up to time n. Show that $E(N_n) = 2$.

3 Extra Problems:

10. Gambler's Ruin Revisited In this problem we revisit the gambler's ruin problem from the first recitation. Two gamblers, A and B, bet on the outcomes of successive flips of a coin. On each flip, if the coin comes up heads, A collects 1 unit from B, whereas if it comes up tails, A pays 1 unit to B. They continue to do this until one of them runs out of money. If it is assumed that the successive flips of the coin are independent and fair, (Note that this game is a random walk.)

(a) What is the probability that A ends up with all the money if he starts with i units and B starts with N-i units?

(b) What is the expected length of the game? (On average how long it takes for one of them to go broke?)

11. Customers arrive at a bank according to a Poisson process with rate λ.

(a) Suppose exactly one customer arrived during the first hour. What is the probability that he/she arrived during the first 20 minutes?

(b) Suppose that exactly two customers arrived during the first hour. What is the probability that exactly one had arrived by 20 minutes?

(c) Suppose that exactly two customers arrived during the first hour. What is the probability that at least one arrived in the first 20 minutes?