ECE 4110/5110	Monday, 10/23/23

Lecture 15: The Wiener Process (Brownian Motion)

Dr. Kevin Tang — Handout 16

Related Reading

Grimmett and Stirzaker Section 8.5

An Intuitive Picture

A "continuous" analogue of a Simple Random Walk.

Suppose in Δt time units, we take a step of size Δx either to the left or to the right with equal probability.

Assume $X(0) = 0$, we have

$$X(t) = \Delta x \sum_{i=1}^{t/\Delta t} X_i \tag{1}$$

where $X_i = 1$ or -1 with equal probability.

Let $\Delta x = \sigma\sqrt{\Delta t} \to 0$, we have $E(X(t)) = 0$ and $Var(X(t)) = \sigma^2 t$. Furthermore, by the Central Limit Theorem, we know $X(t)$ is a Gaussian random variable.

Definition

$\{X(t), t \geq 0\}$ is a Wiener process if $X(0) = 0$; $\{X(t), t \geq 0\}$ has stationary and independent increments; and for any $t > 0$, $X(t)$ follows $N(0, \sigma^2 t)$.

Finite-dimensional Distribution (assuming $\sigma = 1$)

$X(t_1)$, $X(t_2)$,..., $X(t_n)$ are jointly Gaussian with the mean vector being zero and the (i, j)th entry of the covariance matrix equals to $\min(t_i, t_j)$.