ECE 4110/5110

Random Signals in Communication and Signal Processing

Homework 2

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Sept 22, 2023

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1 Problems from Bertsekas and Tsitsiklis, 2nd edition - Chapter 6

1.1 Problem 1 and Corresponding Solution

Problem 2

Dave fails quizzes with probability 1/4, independent of other quizzes.

1. What is the probability that Dave fails exactly two of the next six quizzes?
2. What is the expected number of quizzes that Dave will pass before he has failed three times?
3. What is the probability that the second and third time Dave fails a quiz will occur when he takes his eighth and ninth quizzes, respectively?
4. What is the probability that Dave fails two quizzes in a row before he passes two quizzes in a row?

Solution:

1. This is a Bernoulli process with parameter $p = \frac{1}{4}$. Hence

   $$P(\text{Dave fails exactly two of the next six quizzes}) = \binom{6}{2} P^2(1-P)^4 = 15 \cdot (\frac{1}{4})^2 \cdot (\frac{3}{4})^4 = 0.297$$

2. The 3th arrival time of failure is equal to the sum of the first 3 interarrival times

   $$Y_3 = T_1 + T_2 + T_3$$

   $$E[Y_3] = E[T_1] + E[T_2] + E[T_3] = \frac{3}{p} = \frac{3}{1/4} = 12$$

   $$E[Pass] = E[Y_3] - 3 = 12 - 3 = 9$$

   Thus the expected number of quizzes that Dave will pass before he has failed 3 times is 9.

3. The event that the second and third time Dave fails a quiz will occur when he takes his eighth and ninth quizzes respectively means there is 1 failure in the first 7 trials. Let A denote this event. And let

   B: there is 1 failure in the first 7 trials

   C: Dave fails a quiz on the 8th trial.

   D: Dave fails a quiz on the 9th trial.

   $$P(B) = \binom{7}{1} P(1-P)^6 = 7 \cdot (\frac{1}{4}) \cdot (\frac{3}{4})^6$$

   $$P(C) = P(D) = \frac{1}{4}$$

   $$P(A) = P(B \cap C \cap D) = 7 \cdot (\frac{1}{4})^3 \cdot (\frac{3}{4})^6 = \frac{5103}{262144} = 0.019466$$

4. Let B be the event that Dave fails two quizzes in a row before he passes two quizzes in a row. Let F denote he failed the quiz, and S denote he passed the quiz. Hence,

$$
\begin{aligned}
P(B) &= P(FF \cup SFF \cup FSFF \cup SFSFF \cup FSFSFF...) \\
&= \left(\frac{1}{4}\right)^2 + \frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + ... \\
&= \left[\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \cdots\right] + \\
&\left[\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\left(\frac{1}{4}\right)^2 + \cdots\right]
\end{aligned}
$$

Hence, P(B) is the sum of the two infinite geometric series, and

$$P(B) = \frac{(1/4)^2}{1 - \frac{1}{4}\cdot\frac{3}{4}} + \frac{3/4\cdot(1/4)^2}{1 - \frac{3}{4}\cdot\frac{1}{4}} = \frac{7}{52}$$

1.2 Problem 2 and Corresponding Solution

Problem 3

A computer system carries out tasks submitted by two users. Time is divided into slots. A slot can be idle, with probability$P_I = 1/6$, and busy with probability $P_B = 5/6$. During a busy slot, there is probability $P_{1|B} = 2/5$ (respectively, $P_{2|B} = 3/5$) that a task from user 1 (respectively, 2) is executed. We assume that events related to different slots are independent.

(a) Find the probability that a task from user 1 is executed for the first time during the 4th slot.

(b) Given that exactly 5 out of the first 10 slots were idle, find the probability that the 6th idle slot is slot 12.

(c) Find the expected number of slots up to and including the 5th task from user 1.

(d) Find the expected number of busy slots up to and including the 5th task from user 1.

(e) Find the PMF, mean, and variance of the number of tasks from user 2 until the time of the 5th task from user 1.

Solution:

(a) This is a Bernoulli process, in each slot, we have

$$P_1 = P_B P_{1|B} = \frac{5}{6} \cdot \frac{2}{5} = \frac{1}{3}$$

$$P(\text{a task from user 1 is executed for the first time during the 4th slot}) = P_1(1-P_1)^3 = \frac{1}{3}\cdot(\frac{2}{3})^3 = \frac{8}{81}$$

(b) We already knew that 5 out of the first 10 slots were idle. Then the 6th idle slot is slot 12 means slot 11 is a busy slot. Due to the memorylessness property, we have

$$P(\text{the 6th idle slot is slot 12} \mid \text{5 out of the first 10 slots were idle}) = P_B \cdot P_I = \frac{5}{6} \cdot \frac{1}{6} = \frac{5}{36}$$

(c) We have $P_1 = \frac{1}{3}$ from (a). The time of the 5th task from user 1 is a Pascal RV with parameter $k = 5$ and $P_1 = \frac{1}{3}$, so

$$E[\text{slots up to and including the 5th task from user 1}] = \frac{5}{P_1} = 15$$

(d) This is also a Pascal RV with parameter k=5 and $P_{1|B} = \frac{2}{5}$, so

$$E[\text{busy slots up to and including the 5th task from user 1}] = \frac{5}{P_{1|B}} = 12.5$$

(e) Let X denote the number of tasks from user 2 until the 5th task from user 1, and let Y denote the number of busy slots until the 5th task from user 1. Then we have $X = Y - 5$.

The number of busy slots until the 5th task from user 1 is a Pascal RV with parameter $k = 5$ and $P_{1|B} = \frac{2}{5}$. Hence

$$p_Y(t) = \binom{t-1}{4}(\frac{2}{5})^5(\frac{3}{5})^{t-5} \qquad t = 5, 6, ...$$

Then

$$p_X(t) = p_Y(t+5) = \binom{t+4}{4}(\frac{2}{5})^5(\frac{3}{5})^t \qquad t = 0, 1, , ...$$

We already knew $E[Y] = 12.5$ from (d), hence

$$E[X] = E[Y-5] = E[Y] - 5 = 12.5 - 5 = 7.5$$

$$var(X) = var(Y-5) = var(Y) = k \cdot \frac{1 - P_{1|B}}{(P_{1|B})^2} = 5 \cdot \frac{3/5}{(2/5)^2} = 18.75$$

1.3 Problem 3 and Corresponding Solution

Problem 14

Beginning at time $t = 0$. we start using bulbs. one at a time, to illuminate a room. Bulbs are replaced immediately upon failure. Each new bulb is selected independently by an equally likely choice between a type-A bulb and a type-B bulb. The lifetime, X, of any particular bulb of a particular type is a random variable, independent of everything else, with the following PDF:

for type-A Bulbs: $$f_x(x) = \begin{cases} e^{-x} & x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

for type-A Bulbs: $$f_x(x) = \begin{cases} 3e^{-3x} & x \geq 0, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the expected time until the first failure.

(b) Find the probability that there are no bulb failures before time t.

(c) Given that there are no failures until time t, determine the conditional probability that the first bulb used is a type-A bulb.

(d) Find the variance of the time until the first bulb failure.

(e) Find the probability that the 12th bulb failure is also the 4th type-A bulb failure.

(f) Up to and including the 12th bulb failure, what is the probability that a total of exactly 4 type-A bulbs have failed?

(g) Determine either the PDF or the transform associated with the time until the 12th bulb failure.

(h) Determine the probability that the total period of illumination provided by the first two type-B bulbs is longer than that provided by the first type-A bulb.

(i) Suppose the process terminates as soon as a total of exactly 12 bulb failures have occurred. Determine the expected value and variance of the total period of illumination provided by type-B bulbs while the process is in operation.

(j) Given that there are no failures until time t, find the expected value of the time until the first failure.

Solution:

(a) Let T denote the time untile the first failure. Let A denote the event that the first failure comes from type-A, and B denote the event that the first failure comes from type-B. Since the property of exponential distribution, we have $\lambda_A = 1, \lambda_B = 3$, then

$$E[T \mid A] = \frac{1}{\lambda_A} = 1 \qquad E[T \mid B] = \frac{1}{\lambda_B} = \frac{1}{3}$$

Since the choice of two types of bulb are equally likely, so

$$P(A) = P(B) = \frac{1}{2}$$

$$E[T] = E[T \mid A]P(A) + E[T \mid B]P(B) = 1 \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2} = \frac{2}{3}$$

(b) Let S be the event that there are no bulb failures before time t.

$$P(S) = P(S \mid A)P(A) + P(S \mid B)P(B) = \frac{1}{2}(e^{-t} + e^{-3t})$$

(c)

$$P(A \mid D) = \frac{P(A \cap D)}{P(D)} = \frac{\frac{1}{2}e^{-t}}{\frac{1}{2}(e^{-t} + e^{-3t})} = \frac{1}{1 + e^{-3t}}$$

(d) We have $E[T] = \frac{2}{3}$ from (a).

$$\begin{aligned} E[T^2] &= E[T^2 \mid A]P(A) + E[T^2 \mid B]P(B) \\ &= \frac{2}{\lambda_A^2}P(A) + \frac{2}{\lambda_B^2}P(B) \\ &= 2 \times \frac{1}{2} + \frac{2}{9} \times \frac{1}{2} \\ &= \frac{10}{9} \end{aligned}$$

$$var(T) = E[T^2] - E[T]^2 = \frac{10}{9} - \frac{6}{9} = \frac{2}{3}$$

(e) the event that the 12th bulb failure is also the 4th type-A bulb failure means that there are 3 type-A bulb failures and 8 type-B bulb failures in the first 11th bulb failures. Then we have

$$P(\text{the 12th bulb failure is also the 4th type-A bulb failure}) = \binom{11}{3}(\frac{1}{2})^{11}(\frac{1}{2}) = \frac{165}{2^{12}}$$

(f) Up to and including the 12th bulb failure, a total of exactly 4 type-A bulbs have failed means that there are 4 type-A bulb failures in the first 12th bulb failure. Then we have

$$P(\text{4 type-A bulb failures in the first 12th bulb failure}) = \binom{12}{4}(\frac{1}{2})^{12} = \frac{165}{2^{12}} = \frac{495}{2^{12}}$$

(g) The PDF of the time between failures is $\frac{1}{2}(e^{-x} + 3e^{-3x})$, so the transform associated with the time until the 12 bulb failure is

$$M_X(s) = \frac{1}{2}(\frac{1}{1-s}) + \frac{3}{3-s} \qquad , \quad x \geq 0$$

(h) Let Y denote the total period of illumination provided by the first two type-B bulbs. According to the Erlang distribution of order 2, we have

$$f_Y(y) = 9ye^{-3y} \qquad , \quad y \geq 0$$

Let T_A denote the period of illumination provided by the first type-A bulb, then

$$P(T_A < Y \mid Y = y) = \int_0^y e^{-t}\, dy = -e^{-y} + 1$$

$$P(T_A < Y) = \int_0^\infty f_Y(y)P(T_A < Y \mid Y = y)\, dy = -e^{-y} + 1 \quad = \int_0^\infty 9ye^{-3y}(1 - e^{-y})\, dy = \frac{7}{16}$$

(i) Let Z denote the total period of illumination provided by type-B bulbs while the process is in operation. Let N_B denote the number of type-B. N_B is a binomial random variable, with parameter $n = 12$ and $p = \frac{1}{2}$. Thus,

$$E[N_B] = np = 12 \times \frac{1}{2} = 6$$

$$var[N_B] = np(1-p) = 12 \times \frac{1}{2} \times \frac{1}{2} = 3$$

Let T_{Bi} denote the interval period of illumination from i-th type-B bulb.

$$E[T_{Bi}] = \frac{1}{\lambda_B} = \frac{1}{3}$$

$$var[T_{Bi}] = \frac{1}{\lambda_B^2} = \frac{1}{9}$$

Then we have

$$E[Z] = E[N_B]E[T_{Bi}] = 6 \times \frac{1}{3} = 2$$

$$var[Z] = var(T_{Bi})E[N] + E[T_{Bi}]^2 var(N_B) = \frac{1}{9} \times 6 + \frac{1}{9} \times 3 = 1$$

(j) Given that there are no failures until time t, the expected value of the time until the first failure can be obtained by

$$\begin{aligned}
E[T \mid S] &= t + E[T - t \mid S \cap A] \cdot P(A \mid S) + E[T - t \mid S \cap B] \cdot P(B \mid S) \\
&= t + 1 \cdot \frac{1}{1 + e^{-2t}} + \frac{1}{3} \cdot (1 - \frac{1}{1 + e^{-2t}}) \\
&= t + \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{1 + e^{-2t}}
\end{aligned}$$

1.4 Problem 4 and Corresponding Solution

Problem 15

A service station handles jobs of two types, A and B. (Multiple jobs can be processed simultaneously.) Arrivals of the two job types are independent Poisson processes with parameters $\lambda_A = 3$ and $\lambda_B = 4$ per minute, respectively. Type A jobs stay in the service station for exactly one minute. Each type B job stays in the service station for a random but integer amount of time which is geometrically distributed, with mean equal to 2, and independent of everything else. The service station started operating at some time in the remote past.

(a) What is the mean, variance, and PMF of the total number of jobs that arrive within a given three-minute interval?

(b) We are told that during a 10-minute interval, exactly 10 new jobs arrived. What is the probability that exactly 3 of them are of type A?

(c) At time 0, no job is present in the service station. What is the PMF of the number of type B jobs that arrive in the future, but before the first type A arrival?

(d) At time t = 0, there were exactly two type A jobs in the service station. What is the PDF of the time of the last (before time 0) type A arrival?

(e) At time 1, there was exactly one type B job in the service station. Find the distribution of the time until this type B job departs.

Solution:

(a) Let N denote the total number of jobs that arrive within a given three-minute intervals. This is a Poisson RV with parameter $\lambda = 7$ by merging the two Poisson processes, we have

$$\lambda = \lambda_A + \lambda_B = 3 + 4 = 7$$

Thus,

$$E(N) = \lambda\tau = 7 \times 3 = 21$$

$$var(N) = \lambda\tau = 7 \times 3 = 21$$

$$p_N(n) = P(k, \tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^n}{n!} \qquad n = 0, 1, ...$$

(b) Arrival of jobs A of the merged process has probability $\lambda_A/(\lambda_A + \lambda_B) = \frac{3}{7}$.

Arrival of jobs B of the merged process has probability $\lambda_B/(\lambda_A + \lambda_B) = \frac{4}{7}$.

Thus, the probability of binomial distribution is

$$P(\text{exactly 3 of 10 new arrived jobs are of type A}) = \binom{10}{3}(\frac{3}{7})^3(\frac{4}{7})^7$$

(c) Let K denote the number of arrivals of type B until the first arrival of type A, then

$$p_K(k) = (\frac{4}{7})^k(\frac{3}{7})$$

(d) We know that Type A jobs stay in the service station for exactly one minute, so there were exactly 2 Type A jobs in the service station at time 0 if and only if there were exactly 2 Type A arrivals in the time interval [-1,0].

Let X and Y be the arrival times of these two jobs. Since each time instant is equally likely to contain an arrival and the arrival times are independent, it follows that X and Y are independent uniform random variables over the interval [-1,0].

Given this, the probability that X or Y is less than or equal to a specific value z within this interval is:

$$P(X \leq z) = P(Y \leq z) = z - (-1) = z + 1$$

Thus, the CDF of $Z = \max[X, Y]$ is:

$$\begin{aligned} P(Z \leq z) &= P(\max[X, Y] \leq z) \\ &= P(X \leq z \text{ and } Y \leq z) \\ &= P(X \leq z)P(Y \leq z) \\ &= (z+1)^2 \qquad\qquad \text{for } z \in [-1, 0]. \end{aligned}$$

Differentiating the CDF with respect to z, we obtain the PDF of Z:

$$f_Z(z) = 2(z+1)$$

for $z \in [-1, 0]$.

(e) Given that at time $t = 1$, there was exactly one type B job in the service station. The time until this type B job departs follows a geometric distribution with mean equal to 2. The parameter p of the geometric distribution is given by:

$$p = \frac{1}{\text{mean}} = \frac{1}{2}$$

Therefore, the PDF of the time T until the type B job departs is:

$$f_T(t) = \left(\frac{1}{2}\right)^{\lfloor t \rfloor} \qquad t \geq 0$$

where $\lfloor t \rfloor$ is the largest integer less than or equal to t.

1.5 Problem 5 and Corresponding Solution

Problem 16

Each morning, as you pull out of your driveway, you would like to make a U-turn rather than drive around the block. Unfortunately, U-turns are illegal in your neighborhood, and police cars drive by according to a Poisson process with rate A. You decide to make a U-turn once you see that the road has been clear of police cars for T time units. Let N be the number of police cars you see before you make the U-turn.

(a) Find $E[N]$.

(b) Find the conditional expectation of the time elapsed between police cars $n-1$ and n, given that $N \geq n$.

(c) Find the expected time that you wait until you make the U-turn. Hint: Condition on N.

Solution:

(a)

$$P(N=0) = e^{-\lambda\tau}$$

$$P(N=1) = e^{-\lambda\tau}(1-e^{-\lambda\tau})$$

$$P(N=k) = e^{-\lambda\tau}(1-e^{-\lambda\tau})^k$$

Then we have

$$E[N] = \frac{1}{p} - 1 = e^{\lambda\tau} - 1$$

(b) Let Tn be the nth interarrival time.

$$E[T_n \mid T_n \leq \tau] = \frac{\int_0^\tau s\lambda e^{-\lambda s}\, ds}{\int_0^\tau \lambda e^{-\lambda s}\, ds}$$

$$E\left[T_n\right] = \mathbf{E}\left[T_n \mid T_n \leq \tau\right] P\left(T_n \leq \tau\right) + E\left[T_n \mid T_n > \tau\right] P\left(T_n > \tau\right)$$

$$\frac{1}{\lambda} = E[T_n \mid T_n \leq \tau]\left(1-e^{-\lambda\tau}\right) + \left(\tau + \frac{1}{\lambda}\right) e^{-\lambda\tau}$$

$$E[T_n \mid T_n \leq \tau] = \frac{\frac{1}{\lambda} - \left(\tau + \frac{1}{\lambda}\right) e^{-\lambda\tau}}{1-e^{-\lambda\tau}}$$

(c) Let T be the time until the U-turn. Let v denote the value of $E[Tn \mid Tn \leq \tau]$.

$$\begin{aligned}
\mathbf{E}[T] &= \tau + \sum_{n=0}^{\infty} \mathbf{P}(N=n)\mathbf{E}\left[T_1 + \cdots + T_N \mid N=n\right] \\
&= \tau + \sum_{n=0}^{\infty} \mathbf{P}(N=n) \sum_{i=1}^{n} \mathbf{E}\left[T_i \mid T_1 \leq \tau, \ldots, T_n \leq \tau, T_{n+1} > \tau\right] \\
&= \tau + \sum_{n=0}^{\infty} \mathbf{P}(N=n) \sum_{i=1}^{n} \mathbf{E}\left[T_i \mid T_i \leq \tau\right] \\
&= \tau + \sum_{n=0}^{\infty} \mathbf{P}(N=n) n v \\
&= \tau + v\mathbf{E}[N]
\end{aligned}$$

1.6 Problem 6 and Corresponding Solution

Problem 17

A wombat in the San Diego zoo spends the day walking from a burrow to a food tray, eating, walking back to the burrow, resting, and repeating the cycle. The amount of time to walk from the burrow to the tray (and also from the tray to the burrow) is 20 secs. The amounts of time spent at the tray and resting are exponentially distributed with mean 30 secs. The wombat, with probability 1/3, will momentarily stand still (for a negligibly small time) during a walk to or from the tray, with all times being equally likely (and independent of what happened in the past). A photographer arrives at a random time and will take a picture at the first time the wombat will stand still. What is the expected value of the length of time the photographer has to wait to snap the wombat's picture?

Solution:

Let A denote the event that the photographer arrives while the wombat is resting or eating; And let B denote the event that the photographer arrives while the wombat is walking($B = A^c$). So

$$P(A) = \frac{30}{30+20} = \frac{3}{5}$$

$$P(B) = \frac{20}{30+20} = \frac{2}{5}$$

Now we assume that A happens. Then there are three steps,

1. Wait until the wombat finishes eating or resting According to the memorylessness property, the time remaining to wait is $30s$.
2. The wombat begins walking The wombat may not stop in the first walking period, then there is another eating or resting for the wombat. If so, the time until the next walk is $30+20 = 50s$. if there are N intervals, then $N+1$ is geometrically distributed with parameter $p = \frac{1}{3}$. Thus the time remaining to wait is $(\frac{1}{p} - 1) \cdot 50 = 100s$.
3. The wombat stop The wombat will stop at $(N+1)$-th interval. The time is uniformly distributed between 0 and 20. Thus the expected value is $10s$.

Then we have

$$E[T \mid A] = 30 + 100 + 10 = 140s$$

Now we assume that B happens. Then there are three possible events,

1. B_1: the wombat does not stop during the photographer's arrival interval (probability 2/3)

$$E[T \mid B, B_1] = E[\text{time wait to the end}] + E[T \mid A] = 10 + 140 = 150$$

2. B_2: the wombat stops during the photographer's arrival interval after the photographer arrives (probability 1/6)

$$E[T \mid B, B_2] = E[\text{time wait to the first wombat stop}] = \frac{20}{3}$$

3. B_3: the wombat stops during the photographer's arrival interval before the photographer arrives (probability 1/6)

$$E[T \mid B, B_3] = E[\text{time wait to the end}] + E[T \mid A] = \frac{20}{3} + 140$$

According to the total expectation theorem, we obtain

$$\begin{aligned}
E[T \mid B] &= E[T \mid B_1]P(B_1) + E[T \mid B_2]P(B_2) + E[T \mid B_3]P(B_3) \\
&= \frac{2}{3} \cdot 150 + \frac{1}{6} \cdot \frac{20}{3} + \frac{1}{6} \cdot (\frac{20}{3} + 140) = 125.55
\end{aligned}$$

Then we have

$$E[T] = E[T \mid A]P(A) + E[T \mid B]P(B) = \frac{3}{5} \cdot 140 + \frac{2}{5} \cdot 125.55 = 134.22$$

2 Problems from Grimmett and Stirzaker, 3.9/3.10

2.1 Problem 7 and Corresponding Solution

Chapter 3.9 Problem 7 - Returns and visits by random walk.

Consider a simple symmetric random walk on the set {0, 1, 2, ..., a} with absorbing barriers at 0 and a, and starting at k where $0 < k < a$. Let r_k be the probability the walk ever returns to k, and let v_k, be the mean number of visits to point x before absorption. Find r_k, and hence show that,

$$v_k = \begin{cases} 2x(a-k)/a & 0 < x < k \\ 2k(a-k)/a & k < x < a \end{cases}$$

Solution:

The number R_k of returns to k before absorption is geometrically distributed on {0, 1, 2,... } with mean

$$rk/(1{-}rk) = [2k(a{-}k)/a]{-}1$$

Let $x < k$. The mean number of visits to x is the probability of ever visiting x multiplied by $1 + R_x$, that is,

$$\frac{a-k}{a-x}\left(1 + \frac{2x(a-x)}{a} - 1\right) = \frac{2x(a-k)}{a}$$

The case $x > k$ follows by symmetry. Curiously, if the barrier at a is reflecting,

$$v_x = \begin{cases} 2x & \text{if } x \leq k \\ 2k & \text{if } x \geq k \end{cases}$$

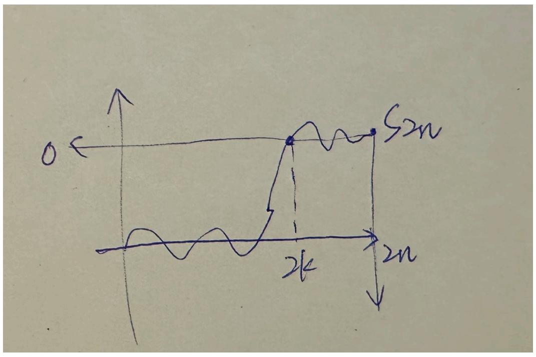
2.2 Problem 8 and Corresponding Solution

Chapter 3.10 Problem 3

For a symmetric simple random walk starting at 0, show that the probability of the first visit to S_{2n} takes place at time $2k$ equals the product $P(S_{2k} = 0)P(S_{2n-2k} = 0)$, for $0 \leq k \leq n$.

Solution:

By considering the random walk reversed, we see that the probability of a first visit to S_{2n} at time $2k$ is the same as the probability of a last visit to S_0 at time $2n-2k$. The result is then immediate from the arc sine law for the last visit to the origin.



By arc sine law, we have

$$P(S_{2k} = 0)P(S_{2n-2k} = 0), \text{for} 0 \leq k \leq n$$

2.3 Problem 9 and Corresponding Solution

Chapter 3.11 Problem 29

Let S be a symmetric random walk with $S_0 = 0$, and let N_n be the number of points that have been visited by S exactly once up to time n. Show that $E(N_n) = 2$.

Solution:

Let $u_k = P(S_k = 0)$, $f_k = P(S_k = 0, S_i \neq 0 \text{ for } 1 \leq i < k)$, and use conditional probability to obtain

$$u_{2n} = \sum_{k=1}^{n} u_{2n-2k} f_{2k}$$

Now $N_1 = 2$, and therefore it suffices to prove that $E(N_n) = E(N_{n-1})$ for $n \geq 2$. Let $N_{n-1}^{'}$ be the number of points visited by the walk $S_1, S_2, ..., S_n$ exactly once (we have removed S_0). Then

$$N_n = \begin{cases} N_{n-1}^{'} + 1 & \text{if } S_k \neq S_0 \text{ for } 1 \leq k \leq n \\ N_{n-1}^{'} - 1 & \text{if } S_k = S_0 \text{ for exactly one k in } \{1, 2, ..., n\} \\ N_{n-1}^{'} & \text{ow} \end{cases} \tag{1}$$

Hence, writing $\alpha_n = P(S_k \neq 0 \text{ for } 1 \leq k \leq n)$,

$$\begin{aligned} E[N_n] &= E[N_{n-1}^{'}] + \alpha_n - P(S_k = S_0 \text{ exactly once}) \\ &= E[N_{n-1}] + \alpha_n - \{f_2\alpha_{n-2} + f_4\alpha_{n-4} + ... + f_{2[n/2]}\} \end{aligned}$$

where $\lfloor x \rfloor$ is the integer part of x. Now $\alpha_{2m} = \alpha_{2m+1} = u_{2m}$. If n=2k is even, then

$$E[N_{2k}] - E[N_{2k-1}] = u_{2k} - \{f_2 u_{2k-2} + ... + f_{2k} = 0\} \qquad by(1)$$

If n=2k+1 is odd, then

$$E[N_{2k+1}] - E[N_{2k}] = u_{2k} - \{f_2 u_{2k-2} + ... + f_{2k} = 0\} \qquad by(1)$$

3 Extra Problems

3.1 Problem 10 and Corresponding Solution

Gambler's Ruin Revisited

In this problem we revisit the gambler's ruin problem from the first recitation. Two gamblers, A and B, bet on the outcomes of successive flips of a coin. On each flip, if the coin comes up heads, A collects 1 unit from B, whereas if it comes up tails, A pays 1 unit to B. They continue to do this until one of them runs out of money. If it is assumed that the successive flips of the coin are independent and fair, (Note that this game is a random walk.)

(a) What is the probability that A ends up with all the money if he starts with i units and B starts with $N - i$ units?

(b) What is the expected length of the game? (On average how long it takes for one of them to go broke?)

Solution:

(a) the probability that A ends up with all the money if he starts with i units and B starts with $N - i$ units is

$$P_A = \frac{i}{i + N - i} = \frac{i}{N}$$

(b) Let B_i denote the length of the games before one of them go broke.

For $1 \leq i \leq N - 1$, we have

$$B_i = \frac{1}{2}(1 + B_{i+1}) + \frac{1}{2}(1 + B_{i-1})$$

And for i=0 or i=N, we have

$$B_0 = B_N = 0$$

Hence

$$B_i = i(N - i)$$

3.2 Problem 11 and Corresponding Solution

Problem 11

Customers arrive at a bank according to a Poisson process with rate λ.

(a) Suppose exactly one customer arrived during the first hour. What is the probability that he/she arrived during the first 20 minutes?

(b) Suppose that exactly two customers arrived during the first hour. What is the probability that exactly one had arrived by 20 minutes?

(c) Suppose that exactly two customers arrived during the first hour. What is the probability that at least one arrived in the first 20 minutes?

Solution:

(a) Let A denote the event that one customer arrived during the first hour. According to the formula

$$P(k,\tau) = e^{-\lambda\tau}\frac{(\lambda\tau)^k}{k!} \qquad k = 0, 1, ...$$

Then we have

$$P(A) = P(1,1) = e^{-\lambda}\frac{\lambda}{1} = \lambda e^{-\lambda}$$

hence, the probability that the customer arrived during the first 20 minutes is

$$\begin{aligned} P(\text{the customer arrived during the first 20 minutes } | A) &= P(N_{\frac{1}{3}}(1) \mid N_1(1)) \\ &= \frac{P(1,\frac{1}{3})P(0,\frac{2}{3})}{P(1,1)} \\ &= \frac{\frac{1}{3}\lambda e^{-\frac{\lambda}{3}}e^{-\frac{2\lambda}{3}}}{\lambda e^{-\lambda}} \\ &= \frac{1}{3} \end{aligned}$$

(b) The probability that exactly one had arrived by 20 minutes is:

$$\begin{aligned} P(N_{\frac{1}{3}}(1) \mid N_1(2)) &= \frac{P(N_{\frac{1}{3}}(1) \cap N_{\frac{2}{3}}(1))}{P(2,1)} \\ &= \frac{P(1,\frac{1}{3}) \cdot P(1,\frac{2}{3})}{P(2,1)} \\ &= \frac{e^{-\frac{\lambda}{3}} \cdot \frac{\lambda}{3} \cdot e^{-\frac{2\lambda}{3}} \cdot \frac{2\lambda}{3}}{e^{-\lambda} \cdot \frac{\lambda^2}{2!}} \\ &= \frac{4}{9} \end{aligned}$$

(c)

$$\begin{aligned}
P(\text{Both arrived during the first 20 minutes}) &= P(N_{\frac{1}{3}}(2) \mid N_1(2)) \\
&= \frac{P(2, \frac{1}{3}) \cdot P(0, \frac{2}{3})}{P(2,1)} \\
&= \frac{1}{9}
\end{aligned}$$

hence,

$$\begin{aligned}
P(\text{at least one arrived in the first 20 minutes}) &= P(N_{\frac{1}{3}}(1) \mid N_1(2)) + P(N_{\frac{1}{3}}(2) \mid N_1(2)) \\
&= \frac{4}{9} + \frac{1}{9} \\
&= \frac{5}{9}
\end{aligned}$$