

# SUSTech - 25Fall - MAE5009 - Note

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## 1 Stress Analysis

### 1.1 Stress State

- Normal Stress
- Shear Stress
- Stress Transformation

### 1.2 Equilibrium Equation of Stress

- Body Force: Gravitational Force, Magnetic Force, Inertial Force
- Surface Force: Friction Force, Pressure, Viscous Force(Fluid Flow)

$$\sigma = \lim_{\Delta A \rightarrow 0} \frac{\Delta F}{\Delta A} \quad (1-1)$$

The equation means the force at the per unit surface area. The unit is Pascal( $Pa = N/m^2$ ).

Other common units:

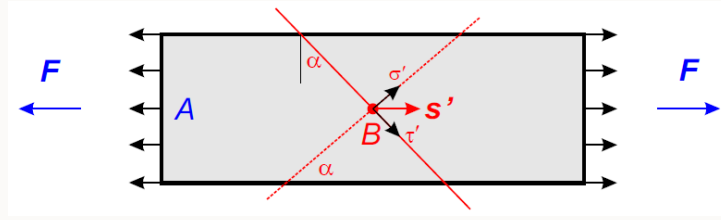
- $1atm \approx 10^5 Pa = 0.1MPa$
- $1bar \approx 0.98atm \approx 1atm = 0.1MPa$

Stress is a kind of tensor, different from **scalar** and **vector**.

- Scalar: only have magnitude, e.g. temperature, density;
- Vector: have both magnitude and direction, e.g. velocity, force;
- Tensor: magnitude and direction in multiple directions, e.g. stress, strain.

At a reference plane, the force  $F$  is vertically upward, the normal stress  $\sigma$  is perpendicular to the plane, while the shear stress  $\tau$  is parallel to the plane.

### 1.3 2D Stress

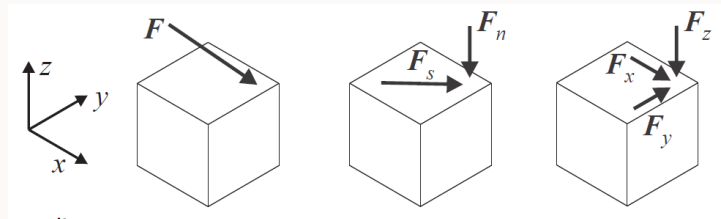


**Figure 1:** Stress components on a reference plane

For the circumstances shown in Fig. 1, the normal stress and shear stress on the reference plane can be determined.

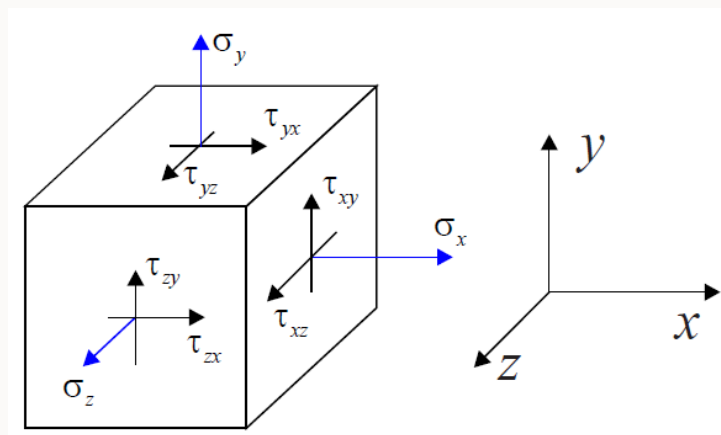
$$A' = \frac{A}{\sin \alpha}, \quad S = \frac{F}{A'} = \frac{F}{A} \sin \alpha, \quad \begin{cases} \tau = S \cdot \cos \alpha = \frac{F}{A} \sin \alpha \cos \alpha \\ \sigma = S \cdot \sin \alpha = \frac{F}{A} \sin^2 \alpha \end{cases} \quad (1-2)$$

### 1.4 3D Stress



**Figure 2:** Decomposition of an external force  $F$

As shown in Fig. 2, the force  $F$  applied at an arbitrary angle to the x-y plane can be resolved into a normal component  $F_n$  and a shear component  $F_s$ . The shear component can be further decomposed into Cartesian components  $F_x$  and  $F_y$ .

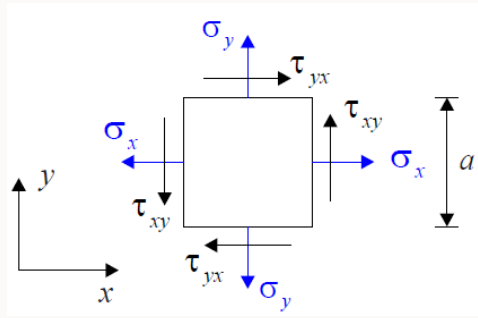


**Figure 3:** Three-Dimensional State of Stress

As shown in the Fig. 3, in every face has three stress components, with 1 normal stress( $\sigma_x, \sigma_y, \sigma_z$ ) and 2 shear stresses( $\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy}$ ). Thus the components of stress can be expressed in a matrix form:

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \quad (1-3)$$

The sign convention is that normal stresses causing tension are positive, while those causing compression are negative. If we consider rotational equilibrium of the infinitesimal square shown as Fig. 4, we can calculate the moment with respect to lower left corner:



**Figure 4:** Rotational Equilibrium of an Infinitesimal Element

$$\sigma_x \cdot a(a/2) - \sigma_x \cdot a(a/2) + \sigma_y \cdot a(a/2) - \sigma_y \cdot a(a/2) + \tau_{xy} \cdot a \cdot a - \tau_{yx} \cdot a \cdot a = 0 \quad (1-4)$$

Thus we have:

$$\tau_{xy} = \tau_{yx} \quad (1-5)$$

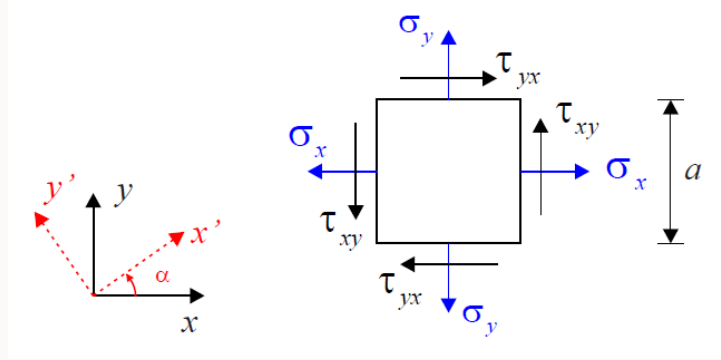
Similarly, we can have:

$$\tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz} \quad (1-6)$$

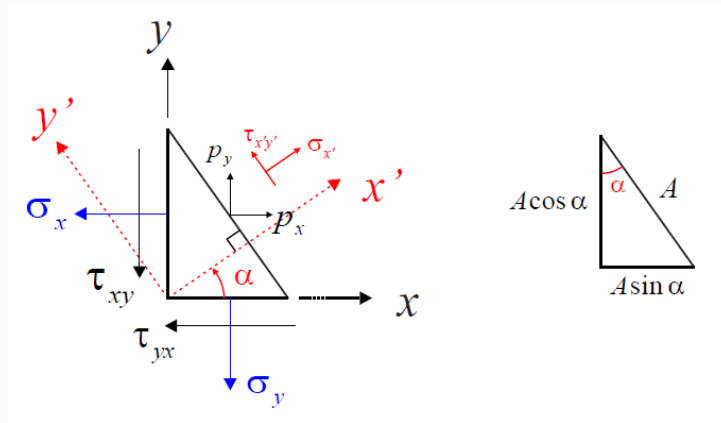
Which means, the stress matrix is symmetric, and there are 3 normal stresses and 3 shear stresses, totally 6 independent stress components in 3D stress state.

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ Sym. & & \sigma_z \end{bmatrix} \quad (1-7)$$

## 1.5 2D Stress Transformation



**Figure 5:** Stress Transformation on an Arbitrary Plane



**Figure 6:** Stress Components on an Inclined Plane in 2D Stress State

After the transformation shown in Fig. 5, the new stress components are shown in Fig. 6. The stress transformation equations may be derived based on force equilibrium analysis:

$$\sum F_x = 0 \Rightarrow p_x = \sigma_x \cos \alpha + \tau_{yx} \sin \alpha \quad (1-8)$$

$$\sum F_y = 0 \Rightarrow p_y = \sigma_y \sin \alpha + \tau_{xy} \cos \alpha \quad (1-9)$$

Thus we have:

$$\begin{cases} \sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha \\ \sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\alpha - \tau_{xy} \sin 2\alpha \\ \tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha \end{cases} \quad (1-10)$$

In 2D circumstances,  $\sigma'$  can be calculated by  $\sigma' = \mathbf{R}\sigma\mathbf{R}^T$ , in which:

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{yx} \\ \tau_{xy} & \sigma_y \end{bmatrix}, \mathbf{R} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & -\cos \alpha \end{bmatrix}, \sigma' = \begin{bmatrix} \sigma'_x & \tau'_{yx} \\ \tau'_{xy} & \sigma'_y \end{bmatrix} \quad (1-11)$$

Also, the rotation angle  $\alpha$  (principle directions) can be calculated by:

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} (\alpha \in [0, \pi]) \quad (1-12)$$

$$\sin 2\alpha = \pm \frac{2\tau_{xy}}{\sqrt{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}}, \quad \cos 2\alpha = \frac{\sigma_x - \sigma_y}{\sqrt{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}} \quad (1-13)$$

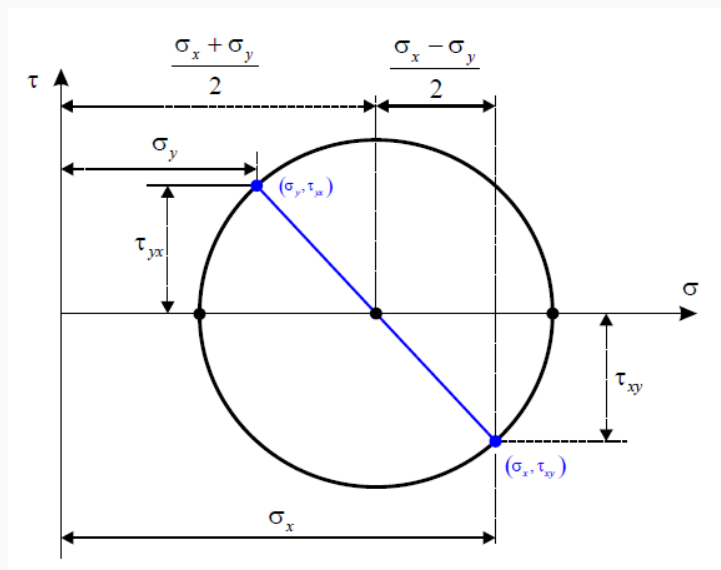
The principle stress can be calculated by:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (1-14)$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (1-15)$$

$$\tau_{x'y'max} = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \quad (1-16)$$

## 1.6 Mohr's circle of stress

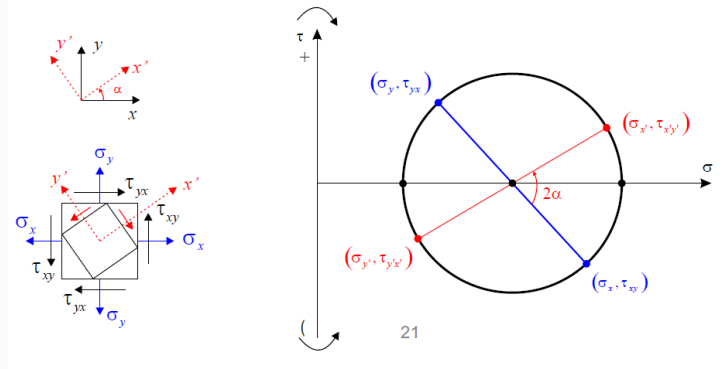


**Figure 7:** Mohr's Circle of Stress

As shown in the Fig. 7, 2D stress transformation can also be conveniently represented graphically in a circle. And from Eq. (1-10), we can calculate that the equation of the circle is

$$\left(\sigma - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \quad (1-17)$$

From the Eq. (1-17) we can get the center is  $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$  and the radius is  $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$ .



**Figure 8:** Rotation in the Mohr's Circle of Stress

After rotating the x-y axis as shown in the Fig. 8, the stress would be:

$$\begin{cases} \sigma'_x = \frac{\sigma_1 + \sigma_2}{2} + \frac{\sigma_1 - \sigma_2}{2} \cos 2\alpha \\ \sigma'_y = \frac{\sigma_1 + \sigma_2}{2} - \frac{\sigma_1 - \sigma_2}{2} \cos 2\alpha \\ \tau_{x'y'} = -\frac{\sigma_1 - \sigma_2}{2} \sin 2\alpha \end{cases} \quad (1-18)$$

## 1.7 3D stress transformation

Same to 2D stress transformation, the 3D stress transformation can also be expressed in matrix form:

$$\sigma' = \mathbf{R}\sigma\mathbf{R}^T \quad (1-19)$$

where

$$\mathbf{R} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos(x', x) & \cos(x', y) & \cos(x', z) \\ \cos(y', x) & \cos(y', y) & \cos(y', z) \\ \cos(z', x) & \cos(z', y) & \cos(z', z) \end{bmatrix} \quad (1-20)$$

which means the direction cosines between the old and new coordinate axes. And in each columns and rows of matrix  $\mathbf{R}$ , we have:

$$a_{1i}^2 + a_{2i}^2 + a_{3i}^2 = 1, \quad i = 1, 2, 3 \quad (1-21)$$

$$a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1, \quad i = 1, 2, 3 \quad (1-22)$$

## 1.8 3D principal stress

For symmetric matrix  $\mathbf{A}$ , its eigenvalue and eigenvector satisfy:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \quad (1-23)$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0 \quad (1-24)$$

As the equation has non-zero  $\mathbf{v}$  if and only if:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \quad (1-25)$$

Then we have the **Characteristic Equation** for the principle stress(in Eqs. (1-23) to (1-25), the matrix  $\mathbf{A}$  can be replaced by stress matrix  $[\sigma]$ ):

$$\det(\sigma - \lambda\mathbf{I}) = 0 \quad (1-26)$$

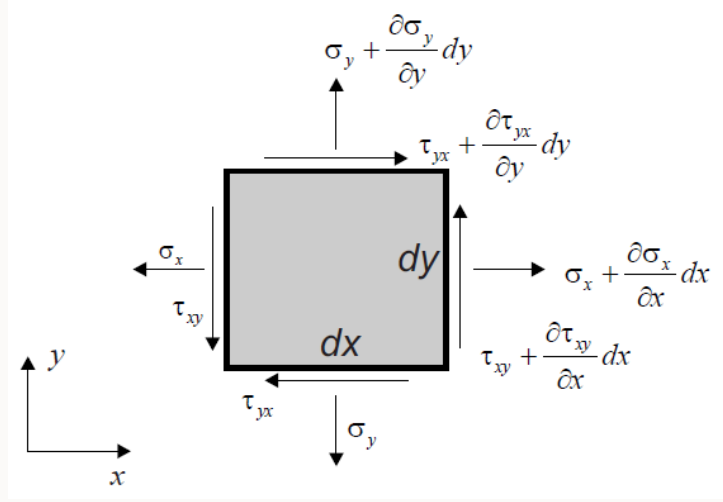
Expand the determinant, we have:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \quad (1-27)$$

The invariants  $I_1, I_2, I_3$  are **Stress Invariants**, which means they are independent of the coordinate system:

$$\begin{cases} I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 = \sigma_1\sigma_2\sigma_3 \end{cases} \quad (1-28)$$

## 1.9 Differential equation of equilibrium



**Figure 9:** Differential Element under Stress and Body Force

For the force balance  $\sum F = 0$  in the circumstances shown in Fig. 9, we have:

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \end{cases} \quad (1-29)$$

Which can be expressed like:

$$\nabla \cdot \sigma + f = 0 \quad (1-30)$$

In the Eqs. (1-29) and (1-30),  $f$  is the body force intensities(per unit volume), e.g. gravitational force, magnetic force, inertial force.

And when the body has acceleration, the Eq. (1-30) can be expressed as:

$$\nabla \cdot \sigma + f = \rho \frac{\partial^2 u}{\partial t^2} \quad (1-31)$$

And for the torque balance  $\sum M = 0$  in the circumstances shown in Fig. 9, we have:

$$\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz} \quad (1-32)$$

From the Eq. (1-29), there are 6 independent unknowns in total, while only 3 equations. Thus additional equations are needed to complete the solutions of the stress distribution in a body. For example,

Strain-displacement, Generalized Hooke's Law, etc.

## 2 Strain Analysis

### 2.1 Assumptions

- Infinitesimal deformation (1% - 5%)
- Continuous materials
  - Continuous displacement
  - No gap/discontinuities after displacement
- Displacement functions must be single-valued

### 2.2 Strain & Displacement

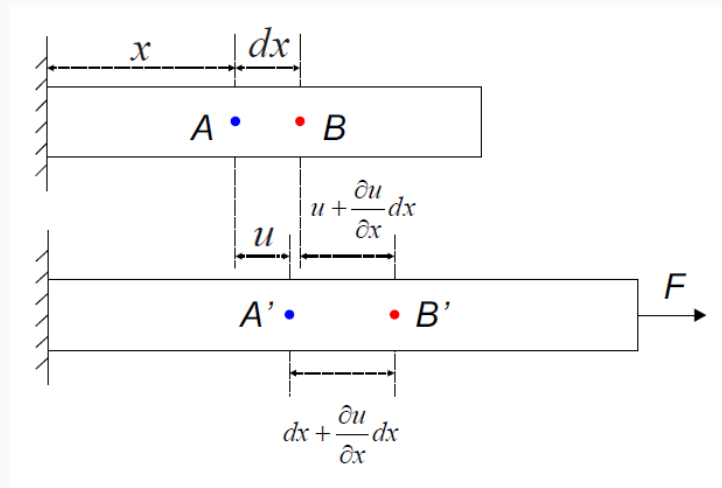
Same to stress, strain also can be divided into:

- **Normal Strain:** relative changes of the **length** of the objects
- **Shear Strain:** changes of the **angle** of the two sides of the objects

And the definition of the strain is:

$$\varepsilon = \frac{\Delta L}{L} \quad (2-1)$$

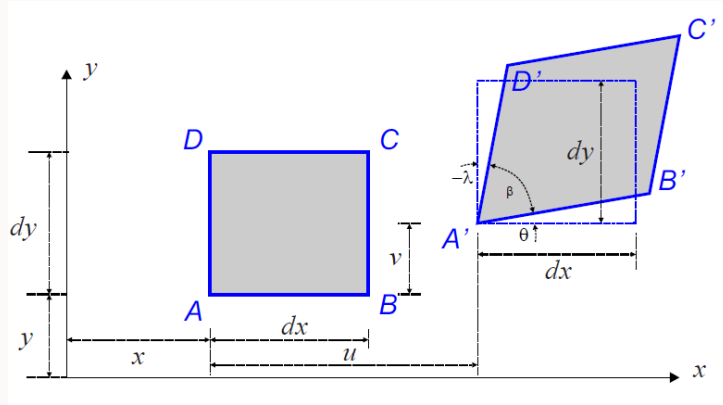
which is dimensionless.



**Figure 10:** Deformation of a body under load  $F$

As shown in Fig. 10, under the load  $F$ , the strain  $\varepsilon$  can be calculated by:

$$\varepsilon = \frac{A'B' - AB}{AB} = \frac{dx + \frac{\partial u}{\partial x} dx - dx}{dx} = \frac{\frac{\partial u}{\partial x} \cdot dx}{dx} = \frac{\partial u}{\partial x} \quad (2-2)$$



**Figure 11:** 2D deformation of a body under load

As shown in Fig. 11, the normal strain and the shear strain can be calculated by Eqs. (2-3) to (2-5), and the deformation of the body in both horizontal and vertical can be calculated by Tab. 1.

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{A'B' - dx}{dx} \quad (2-3)$$

$$\varepsilon_y = \frac{A'D' - AD}{AD} = \frac{A'D' - dy}{dy} \quad (2-4)$$

$$\gamma = \frac{\pi}{2} - \beta \quad (2-5)$$

**Table 1:** Deformation of a body in 2D

	Horizontal	Vertical
A	$u(x, y)$	$v(x, y)$
B	$u(x, y) + \frac{\partial u}{\partial x} dx$	$v(x, y) + \frac{\partial v}{\partial x} dx$
D	$u(x, y) + \frac{\partial u}{\partial y} dy$	$v(x, y) + \frac{\partial v}{\partial y} dy$

From Tab. 1, we can calculate the normal strain and shear strain in 2D:

$$(A'B')^2 = (dx + \frac{\partial u}{\partial x} dx)^2 + (\frac{\partial v}{\partial x} dx)^2 = (dx(1 + \varepsilon_x))^2 \Rightarrow (1 + \varepsilon_x)^2 = (1 + \frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 \quad (2-6)$$

As  $\frac{\partial v}{\partial x} \approx 1\% - 5\%$ , we have:  $\varepsilon_x = \frac{\partial u}{\partial x}$ , and similarly,  $\varepsilon_y = \frac{\partial v}{\partial y}$ . And for 3D circumstances, we have  $\varepsilon_z = \frac{\partial w}{\partial z}$ .

For the defoemation of angle, we have:

$$\theta \rightarrow 0, \theta = \tan \theta = \frac{\frac{\partial v}{\partial x} dx}{dx + \frac{\partial u}{\partial x} dx} = \frac{\frac{\partial v}{\partial x}}{1 + \frac{\partial u}{\partial x}} = \frac{\partial v}{\partial x} \quad (2-7)$$

$$\lambda \rightarrow 0, -\lambda = -\tan \lambda = \frac{\frac{\partial u}{\partial y} dy}{dy + \frac{\partial v}{\partial y} dy} = \frac{\frac{\partial u}{\partial y}}{1 + \frac{\partial v}{\partial y}} = \frac{\partial u}{\partial y} \quad (2-8)$$

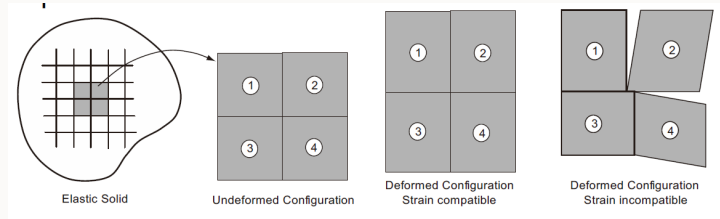
Thus, the shear strain can be expressed as:

$$\gamma_{xy} = \frac{\pi}{2} - \beta = \theta + (-\lambda) = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad (2-9)$$

Generalization to three-dimensional case, we have:

$$\begin{cases} \varepsilon_x = \frac{\partial u}{\partial x} \\ \varepsilon_y = \frac{\partial v}{\partial y} \\ \varepsilon_z = \frac{\partial w}{\partial z} \end{cases}, \quad \begin{cases} \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \\ \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \\ \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \end{cases}, \quad \begin{cases} \gamma_{xy} = \gamma_{yx} \\ \gamma_{yz} = \gamma_{zy} \\ \gamma_{zx} = \gamma_{xz} \end{cases} \quad (2-10)$$

## 2.3 Strain compatibility equation



**Figure 12: Strain Compatibility**

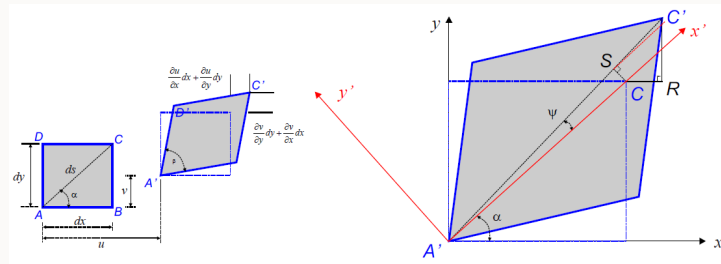
Six equations for the strain components are functions of only three displacement components. If six strain components are known, we have six equations for only three unknowns. There must be additional equations relate the six strain components, which are called **Strain Compatibility Equations**.

$$\begin{cases} \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\ \frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \end{cases}, \quad \begin{cases} 2 \frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x} \left( -\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\ 2 \frac{\partial^2 \epsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\ 2 \frac{\partial^2 \epsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right) \end{cases} \quad (2-11)$$

The compatibility equation Eq. (2-11) can also be called as **Saint-Venant's compatibility equation**. In this equation, only three compatibility equations are independent. If the displacement components are single-valued, continuous functions, the strain components will automatically satisfy the compatibility equations.

## 2.4 2D Strain Transformation

The transformation of strain is almost same as stress transformation.



**Figure 13: 2D Strain Transformation**

Similar to Eq. (1-10), the strain transformation equations are:

$$\begin{cases} \epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\epsilon_x - \epsilon_y}{2} \cos 2\alpha + \epsilon_{xy} \sin 2\alpha \\ \epsilon_{y'} = \frac{\epsilon_x + \epsilon_y}{2} - \frac{\epsilon_x - \epsilon_y}{2} \cos 2\alpha - \epsilon_{xy} \sin 2\alpha \\ \epsilon_{x'y'} = -\frac{\epsilon_x - \epsilon_y}{2} \sin 2\alpha + \epsilon_{xy} \cos 2\alpha \end{cases} \quad (2-12)$$

Also, similar to Eq. (1-11), the strain transformation can be expressed by matrix form  $\epsilon' = \mathbf{R}\epsilon\mathbf{R}^T$ .

$$\epsilon = \begin{bmatrix} \epsilon_x & \epsilon_{yx} \\ \epsilon_{xy} & \epsilon_y \end{bmatrix}, \mathbf{R} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & -\cos \alpha \end{bmatrix}, \epsilon' = \begin{bmatrix} \epsilon'_{xx} & \epsilon'_{yx} \\ \epsilon'_{xy} & \epsilon'_{yy} \end{bmatrix} \quad (2-13)$$

Direction of principal strain can be calculated by:

$$\tan 2\alpha = \frac{2\epsilon_{xy}}{\epsilon_x - \epsilon_y} \quad (\alpha \in [0, \pi]) \quad (2-14)$$

And the magnitudes of principal strains are:

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \sqrt{\left(\frac{\epsilon_x - \epsilon_y}{2}\right)^2 + \epsilon_{xy}^2} \quad (2-15)$$

For the strain tensor in 3D:

$$[\varepsilon] = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ Sym. & & \varepsilon_z \end{bmatrix} \quad (2-16)$$

The calculation of strain invariants is similar to stress invariants in Sec. 1.8.

$$\det(\varepsilon - \lambda \mathbf{I}) = 0 \quad (2-17)$$

Thus we can get:

$$\begin{cases} I_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ I_2 = \varepsilon_x \varepsilon_y + \varepsilon_y \varepsilon_z + \varepsilon_z \varepsilon_x - \varepsilon_{xy}^2 - \varepsilon_{yz}^2 - \varepsilon_{zx}^2 = \varepsilon_1 \varepsilon_2 + \varepsilon_2 \varepsilon_3 + \varepsilon_3 \varepsilon_1 \\ I_3 = \varepsilon_x \varepsilon_y \varepsilon_z + 2\varepsilon_{xy} \varepsilon_{yz} \varepsilon_{zx} - \varepsilon_x \varepsilon_{yz}^2 - \varepsilon_y \varepsilon_{zx}^2 - \varepsilon_z \varepsilon_{xy}^2 = \varepsilon_1 \varepsilon_2 \varepsilon_3 \end{cases} \quad (2-18)$$

### 3 Stress Strain Relations

So far we have unknowns for 6 stress, 6 strain and 3 displacement components:

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ Sym. & & \sigma_z \end{bmatrix}, \quad [\varepsilon] = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ Sym. & & \varepsilon_z \end{bmatrix}, \quad [u] = [u, v, w] \quad (3-1)$$

And we have 3 equilibrium equations(Eq. (1-29)), 6 strain-displacement equations(Eq. (2-10)) and 6 strain compatibility equations(Eq. (2-11))

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0, \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \end{cases}, \quad \begin{cases} \varepsilon_x = \frac{\partial u}{\partial x} & \varepsilon_y = \frac{\partial v}{\partial y} & \varepsilon_z = \frac{\partial w}{\partial z} \\ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} & \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} & \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \end{cases} \quad (3-2)$$

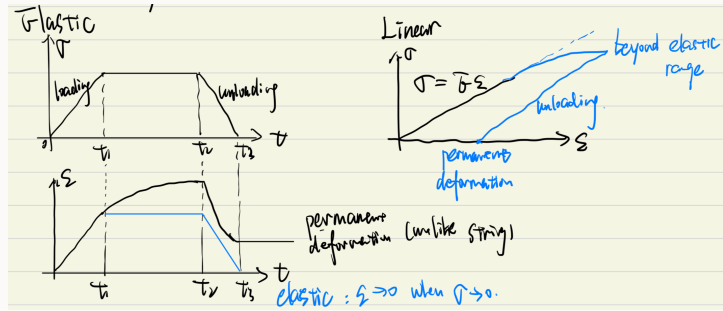
$$\begin{cases} \frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \frac{\partial^2 \varepsilon_y}{\partial z^2} + \frac{\partial^2 \varepsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\ \frac{\partial^2 \varepsilon_z}{\partial x^2} + \frac{\partial^2 \varepsilon_x}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x} \end{cases}, \quad \begin{cases} 2 \frac{\partial^2 \varepsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x} \left( -\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\ 2 \frac{\partial^2 \varepsilon_y}{\partial z \partial x} = \frac{\partial}{\partial y} \left( \frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right) \\ 2 \frac{\partial^2 \varepsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z} \left( \frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z} \right) \end{cases} \quad (3-3)$$

### 3.1 Assumptions

Linear elastic, isotropic, homogeneous, linear stress-strain relations.

- **Linear elastic:** Regain its original dimensions after unloading
- **Isotropic:** Properties are the same in any directions
- **Homogeneous:** Properties are independent of position

### 3.2 Generalized Hooke's Law



**Figure 14:** Differential Element under loading and unloading

For 1D elements, it follows Hooke's Law in 1D:

$$\sigma_x = E\epsilon_x \quad (3-4)$$

where,  $E$  is the **Modulus of Elasticity** or **Young's Modulus**.

And for 3D elements, it follows Generalized Hooke's Law in 3D:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{21} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{31} & c_{32} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{41} & c_{42} & c_{43} & c_{44} & c_{45} & c_{46} \\ c_{51} & c_{52} & c_{53} & c_{54} & c_{55} & c_{56} \\ c_{61} & c_{62} & c_{63} & c_{64} & c_{65} & c_{66} \end{bmatrix} \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} \quad (3-5)$$

For homogeneous material,  $c_{ij}$  are independent of position, and thus are constants, or elastic constants.

For anisotropic material,  $c_{ij} = c_{ji}$ , so there are in total 21 independent elastic constants.

For orthotropic material,  $c_{15} = c_{16} = c_{25} = c_{26} = 0$ ,  $c_{35} = c_{36} = c_{45} = c_{46} = 0$ ,  $c_{14} = c_{24} = c_{34} = c_{56} = 0$  so there are in total 9 independent elastic constants.

For transversely isotropic material,  $c_{11} = c_{33}$ ,  $c_{21} = c_{23}$ ,  $c_{44} = c_{55}$  so there are in total 5 independent elastic constants.

For isotropic material,  $c_{11} = c_{22} = c_{33}$ ,  $c_{12} = c_{13} = c_{23}$ ,  $c_{44} = c_{55} = c_{66}$  so there are in total 2 independent elastic constants, which is  $E$  and  $\nu$ .

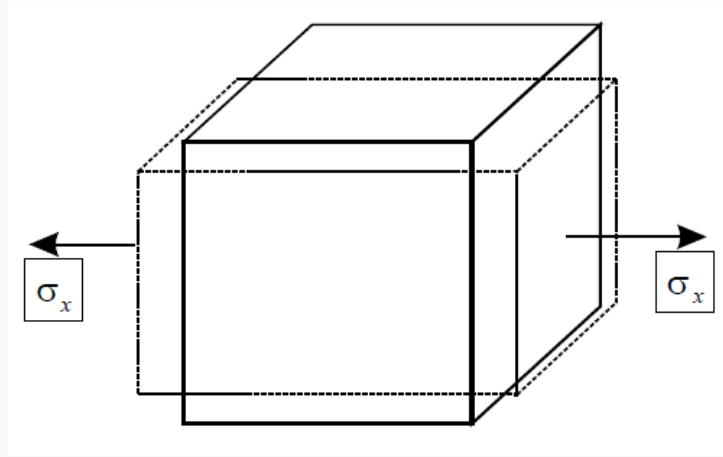
**Poisson's ratio**  $\nu$  is defined as:

$$\nu = -\frac{\varepsilon_{trans}}{\varepsilon_{axial}} \quad (3-6)$$

$\varepsilon_{trans}$  is the strain on the perpendicular direction of the compression.

$\varepsilon_{axial}$  is the strain on the axial direction of the compression.

The Poisson's ratio of a stable, isotropic, linear elastic material must be between  $-1.0$  and  $+0.5$ .



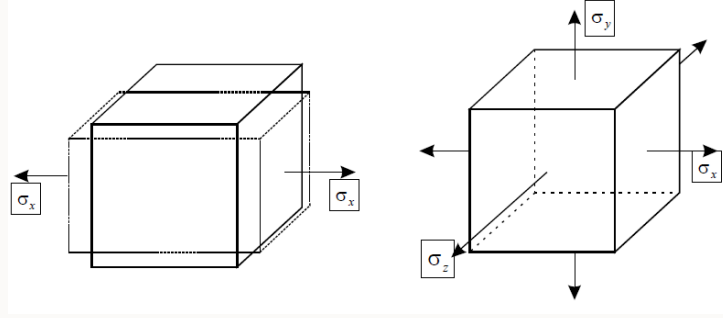
**Figure 15:** Axial load

As shown in Fig. 15, under axial load, the strain in three directions can be calculated by:

$$\begin{cases} \sigma_x = E \cdot \varepsilon_x \\ \nu = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x} \end{cases} \quad (3-7)$$

For isotropic materials, normal stress only cause normal strain, and shear stress only cause shear strain.

### 3.3 Normal stress VS Normal Strain



**Figure 16:** Normal Stress in 3 directions

Assuming that in the beginning, the dimensions of 3 axes are  $L_x = L_y = L_z = 1$  after applying three stress in x, y, z directions, the dimensions become  $L'_x, L'_y, L'_z$ . As shown in Fig. 16, we have:

$$\begin{cases} L'_x = (1 + \varepsilon_x)L_x = \left(1 - \nu \frac{\sigma_z}{E}\right) \left(1 - \nu \frac{\sigma_y}{E}\right) \left(1 + \frac{\sigma_x}{E}\right) \\ L'_y = (1 + \varepsilon_y)L_y = \left(1 - \nu \frac{\sigma_z}{E}\right) \left(1 + \frac{\sigma_y}{E}\right) \left(1 - \nu \frac{\sigma_x}{E}\right) \\ L'_z = (1 + \varepsilon_z)L_z = \left(1 + \frac{\sigma_z}{E}\right) \left(1 - \nu \frac{\sigma_y}{E}\right) \left(1 - \nu \frac{\sigma_x}{E}\right) \end{cases} \quad (3-8)$$

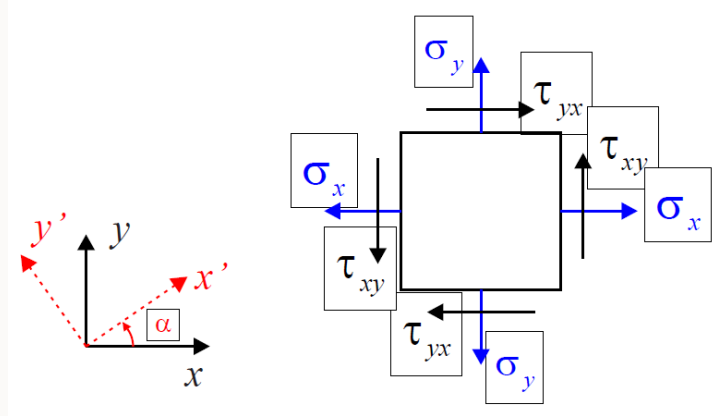
Thus, we can get the relation between normal stress and normal strain:

$$\begin{cases} \varepsilon_x = \frac{L_x - L_{x0}}{L_{x0}} = \frac{1}{E} (\sigma_x - \nu(\sigma_y + \sigma_z)) \\ \varepsilon_y = \frac{L_y - L_{y0}}{L_{y0}} = \frac{1}{E} (\sigma_y - \nu(\sigma_x + \sigma_z)) \\ \varepsilon_z = \frac{L_z - L_{z0}}{L_{z0}} = \frac{1}{E} (\sigma_z - \nu(\sigma_x + \sigma_y)) \end{cases} \quad (3-9)$$

Also, it can be written in the matrix form:

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix} = \begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix} \quad (3-10)$$

### 3.4 Shear stress VS Shear Strain



**Figure 17:** Shear Stress in a body

The relationship between shear stress and shear strain:

$$\begin{cases} \gamma_{xy} = \frac{1}{G} \tau_{xy} \\ \gamma_{yz} = \frac{1}{G} \tau_{yz} \\ \gamma_{zx} = \frac{1}{G} \tau_{zx} \end{cases} \quad (3-11)$$

where,  $G$  is the **Shear Modulus** or **Modulus of Rigidity**.

Also, it can be written in the matrix form:

$$\begin{bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \begin{bmatrix} \frac{1}{G} & 0 & 0 \\ 0 & \frac{1}{G} & 0 \\ 0 & 0 & \frac{1}{G} \end{bmatrix} \begin{bmatrix} \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} \quad (3-12)$$

Correspondingly, we have the relations between strain and stress:

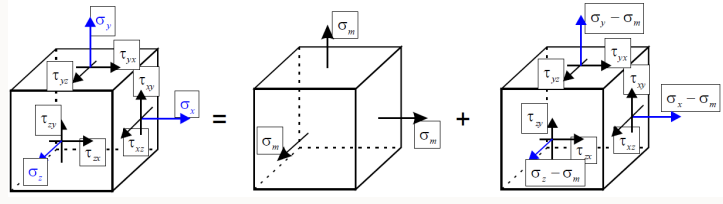
$$\begin{cases} \sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \\ \sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \\ \sigma_z = 2G\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \end{cases}, \quad \begin{cases} \tau_{xy} = G\gamma_{xy} \\ \tau_{yz} = G\gamma_{yz} \\ \tau_{zx} = G\gamma_{zx} \end{cases} \quad (3-13)$$

And for the relations between  $E$ ,  $G$ ,  $\nu$  and  $\lambda$ , we have:

$$G = \frac{E}{2(1+\nu)}, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} \quad (3-14)$$

where,  $\lambda$  is the **Lamé constants**.

### 3.5 Decomposition of stress



**Figure 18:** Decomposition of stress

As shown in Fig. 18, the stress tensor can be decomposed into hydrostatic stress and deviatoric stress in  $\sigma_{ij} = \sigma_m \cdot \delta_{ij} + S_{ij}$ ,  $i, j = x, y, z$ ,  $\delta_{ij}$  is Kronecker delta function, only when  $i = j$ ,  $\delta_{ij} = 1$ , otherwise,  $\delta_{ij} = 0$ .

$$\begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} \sigma_m & & \\ & \sigma_m & \\ & & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_x - \sigma_m & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y - \sigma_m & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z - \sigma_m \end{bmatrix} \quad (3-15)$$

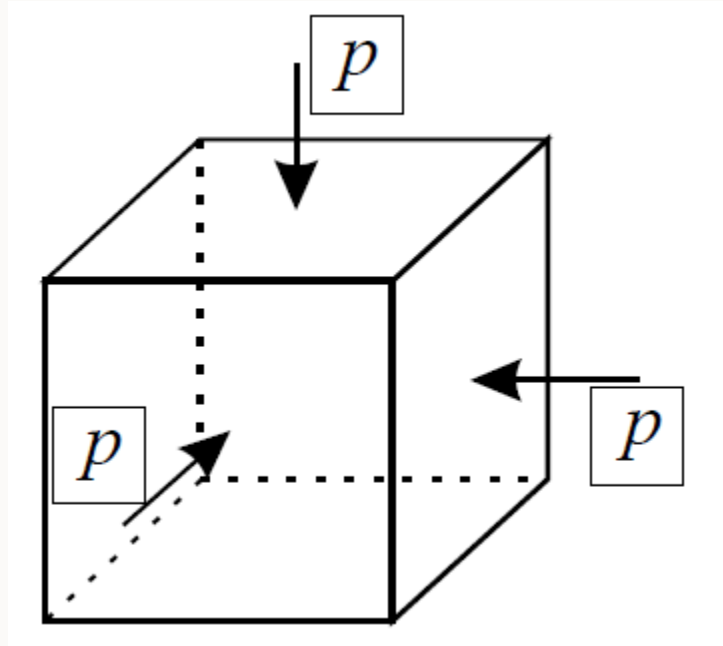
$$\text{Normally, } \sigma_m = \frac{\sigma_x + \sigma_y + \sigma_z}{3} = \frac{I_1}{3}.$$

Similarly, the strain tensor can also be decomposed into volumetric strain and deviatoric strain in  $\varepsilon_{ij} = \varepsilon_m \cdot \delta_{ij} + e_{ij}$ ,  $i, j = x, y, z$ .

$$\begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} \varepsilon_m & & \\ & \varepsilon_m & \\ & & \varepsilon_m \end{bmatrix} + \begin{bmatrix} \varepsilon_x - \varepsilon_m & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_y - \varepsilon_m & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_z - \varepsilon_m \end{bmatrix} \quad (3-16)$$

$$\text{Normally, } \varepsilon_m = \frac{\varepsilon_x + \varepsilon_y + \varepsilon_z}{3} = \frac{I'_1}{3}.$$

### 3.6 Bulk modulus of elasticity



**Figure 19:** Bulk modulus of elasticity

$$\sigma_x = \sigma_y = \sigma_z = -p \Rightarrow \varepsilon_x = \varepsilon_y = \varepsilon_z = -\frac{1-2\nu}{E}p \quad (3-17)$$

$$\tau_{xy} = \tau_{yz} = \tau_{zx} = 0 \Rightarrow \gamma_{xy} = \gamma_{yz} = \gamma_{zx} = 0 \quad (3-18)$$

After deformation,

$$\begin{cases} l_x = (1 + \varepsilon_x)dx \\ l_y = (1 + \varepsilon_y)dy \\ l_z = (1 + \varepsilon_z)dz \end{cases} \quad (3-19)$$

Thus, the new volume is:

$$V' = l_x l_y l_z = (1 + \varepsilon_x)(1 + \varepsilon_y)(1 + \varepsilon_z)dx dy dz \quad (3-20)$$

By ignoring the higher order small quantities, we have:

$$V' = (1 + \varepsilon_x + \varepsilon_y + \varepsilon_z)dx dy dz \quad (3-21)$$

Thus, the percentage change in volume is:

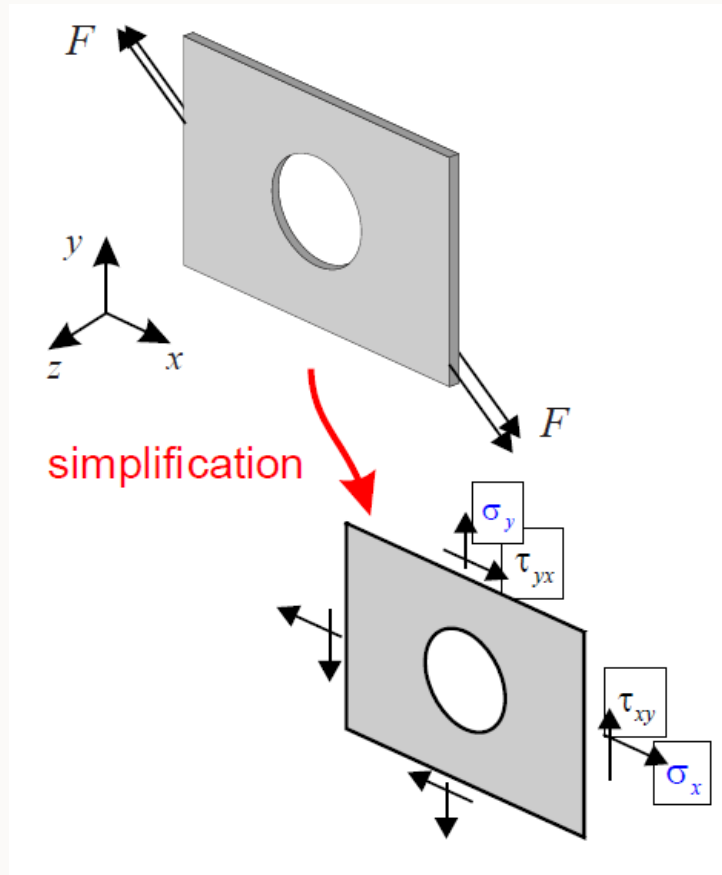
$$\begin{aligned}
\frac{\Delta V}{V} &= \frac{V' - V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z \\
&= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \quad \Rightarrow K = \frac{E}{3(1-2\nu)} \\
&= -\frac{3(1-2\nu)}{E} = -\frac{1}{K}p
\end{aligned} \tag{3-22}$$

where,  $K$  is the **Bulk Modulus of Elasticity**.

### 3.7 Plane stress/strain

For thin flat plates acted upon only by load forces that are parallel to them, the stress analysis can be considerably simplified to **plane stress**.

Stress components perpendicular to the plate are negligible compared to those parallel to it.

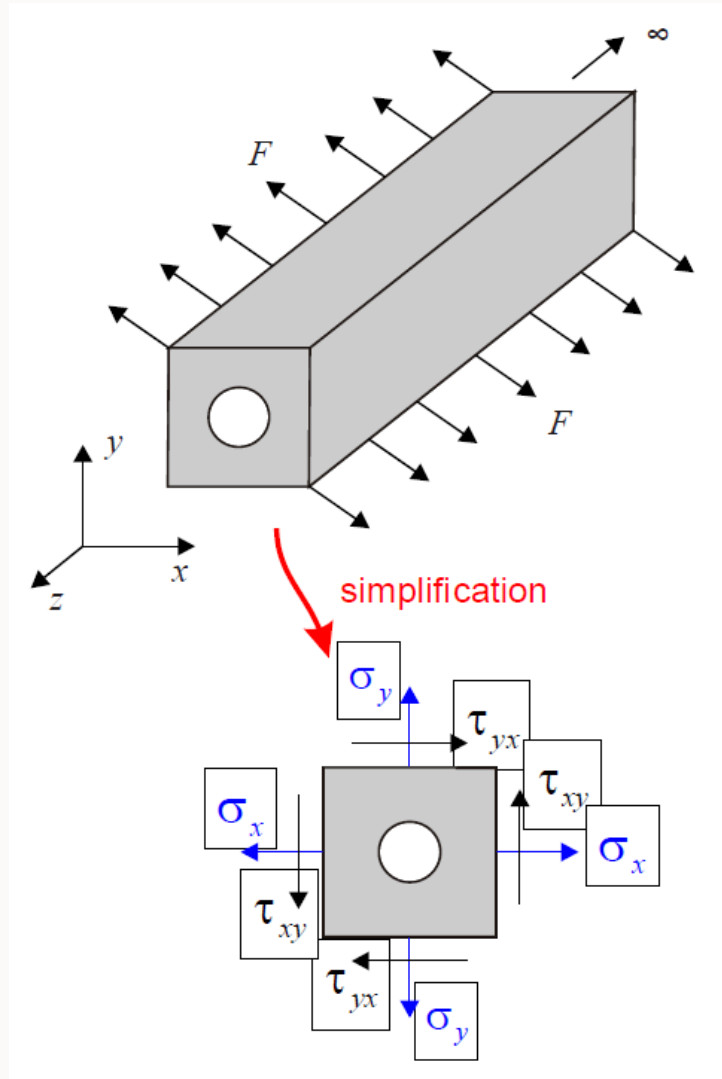


**Figure 20:** Plane stress

As  $x \gg z$ , we have  $\sigma_z = \tau_{xz} = \tau_{yz} = 0$ , thus, the strain components are:

$$\begin{cases} \varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \\ \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \\ \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \end{cases}, \quad \begin{cases} \gamma_{xy} = \frac{1}{G}\tau_{xy} \\ \gamma_{yz} = 0 \\ \gamma_{zx} = 0 \end{cases} \quad (3-23)$$

If one dimension is very large compared to the others, the principal strain in the direction of the longest dimension is constrained and can be assumed as zero, the stress analysis can be considerably simplified to **plane strain**.



**Figure 21:** Plane strain

As  $z \gg x, y$ , we have  $\varepsilon_z = \gamma_{yz} = \gamma_{zx} = 0$ , thus, the stress components are:

$$\begin{cases} \sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y) \\ \sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y) \\ \sigma_z = \lambda(\varepsilon_x + \varepsilon_y) \end{cases}, \quad \begin{cases} \tau_{xy} = G\gamma_{xy} \\ \tau_{yz} = 0 \\ \tau_{zx} = 0 \end{cases} \quad (3-24)$$

## 4 Formulation of Problems in Elasticity

### 4.1 Boundary conditions

There are two types of boundary conditions in elasticity problems: **Displacement Boundary** and **Stress Boundary**.

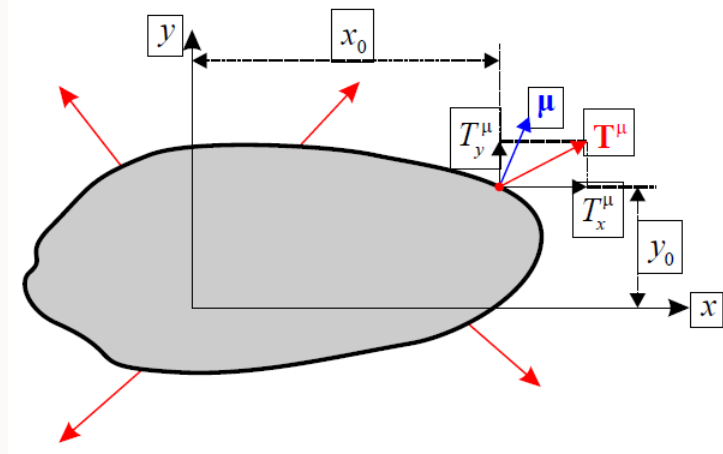


Figure 22: Boundary conditions

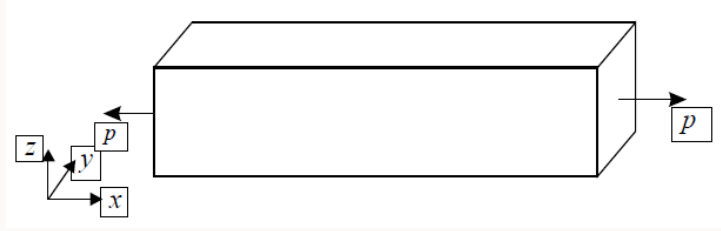
At boundary point  $(x_0, y_0, z_0)$ , we have unit outward normal  $\mu = (\mu_x, \mu_y, \mu_z)$  and surface force  $(T_x^\mu, T_y^\mu, T_z^\mu)$ .

The displacement boundary condition is:

$$\begin{cases} u(x_0, y_0, z_0) = u_b \\ v(x_0, y_0, z_0) = v_b \\ w(x_0, y_0, z_0) = w_b \end{cases} \quad (4-1)$$

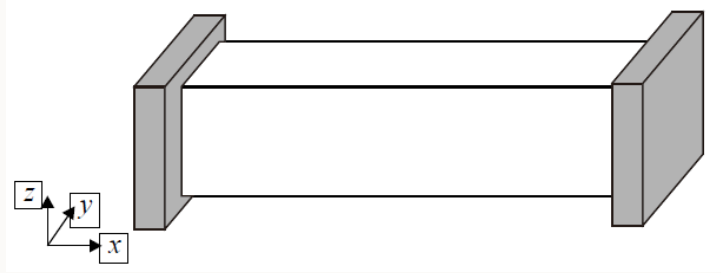
And the stress boundary condition is:

$$\begin{cases} T_x^\mu = \sigma_{xx}|_0 \mu_x + \tau_{yx}|_0 \mu_y + \tau_{zx}|_0 \mu_z \\ T_y^\mu = \tau_{xy}|_0 \mu_x + \sigma_{yy}|_0 \mu_y + \tau_{zy}|_0 \mu_z \\ T_z^\mu = \tau_{xz}|_0 \mu_x + \tau_{yz}|_0 \mu_y + \sigma_{zz}|_0 \mu_z \end{cases} \quad (4-2)$$



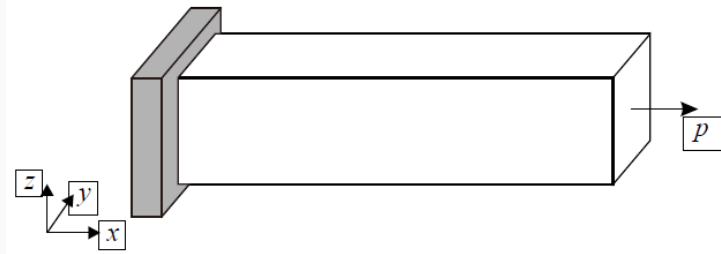
**Figure 23:** Stress boundary condition

For the condition shown in Fig. 23, it's first boundary-value problem, where:  $T_x^\mu = \pm p, T_y^\mu = T_z^\mu = 0$ .



**Figure 24:** Displacement boundary condition

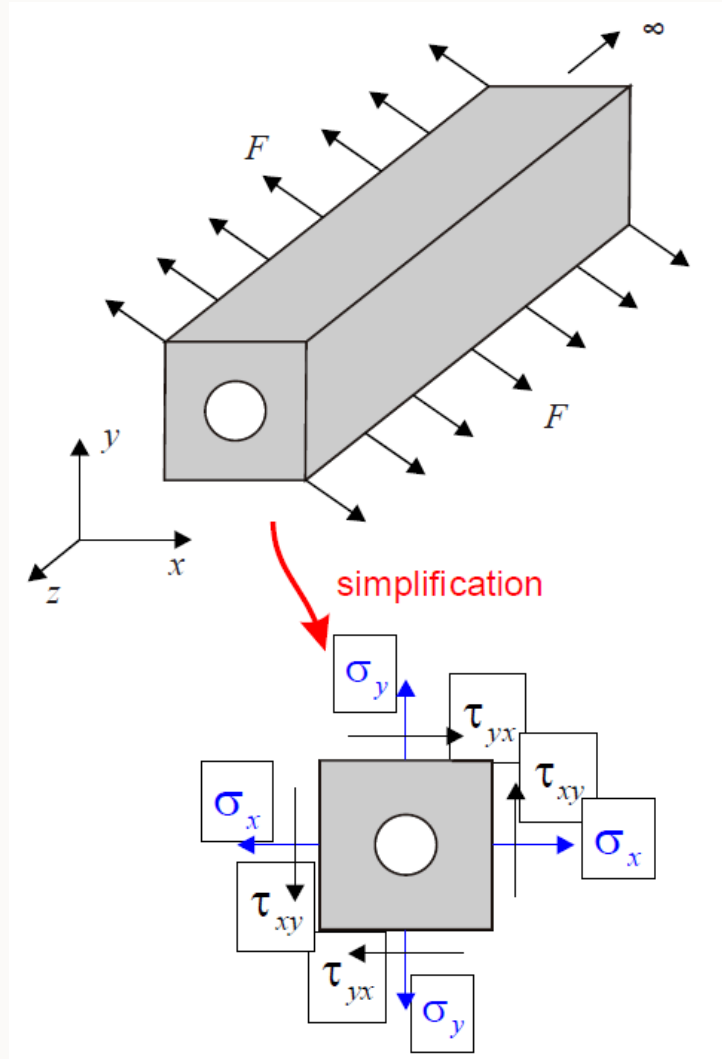
For the condition shown in Fig. 24, it's second boundary-value problem, where:  $u_b = v_b = w_b = 0$ .



**Figure 25:** Mixed boundary condition

For the condition shown in Fig. 25, it's mixed boundary-value problem.

## 4.2 Plane strain problem



**Figure 26:** Plane strain problem

As shown in the Eq. (3-24), and combining with the equilibrium equations, we can have the displacement equation:

$$\begin{cases} G\nabla^2 u + (\lambda + G)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_x = 0 \\ G\nabla^2 v + (\lambda + G)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + f_y = 0 \end{cases} \quad (4-3)$$

where,  $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ .

Also, the Eq. (4-3) can be written in:

$$G\nabla^2 \mathbf{u} + (\lambda + G)\nabla(\nabla \cdot \mathbf{u}) + \mathbf{f} = 0, \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \quad (4-4)$$

By combining with the stress-strain relations and the strain compatibility equations, we can have the stress equations:

$$\nabla^2(\sigma_x + \sigma_y) = -(1 + \nu) \left( \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right) \quad (4-5)$$

## 4.3 Miscellaneous

### 4.3.1 Principal of superposition

Two or more stresses fields may be superposed to yield the results for combined loads.

Only when displacements and strains are small and the strain-displacement, stress-strain equations are linear.

$$f_1 \rightarrow u_1, f_2 \rightarrow u_2 \Rightarrow f_1 + f_2 \rightarrow u_1 + u_2 \quad (4-6)$$

### 4.3.2 Uniqueness of elasticity solutions

For a given surface force and body force distribution, there is only one solution for the stress components consistent with equilibrium and compatibility

## 5 Two-Dimensional Problems

### 5.1 Plane Stress

For the plane stress problem shown in Fig. 26, we have displacement in  $u = u(x, y), v = v(x, y)$ , strain-displacement in:

$$\begin{cases} \varepsilon_x = \frac{\partial u}{\partial x} \\ \varepsilon_y = \frac{\partial v}{\partial y} \\ \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \end{cases}, \quad \begin{cases} \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \\ \gamma_{yz} = 0 \\ \gamma_{zx} = 0 \end{cases} \quad (5-1)$$

And stress-strain in:

$$\begin{cases} \sigma_x = 2G\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \\ \sigma_y = 2G\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) \\ \sigma_z = \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) = 0 \Rightarrow \varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \end{cases}, \quad \begin{cases} \tau_{xy} = G\gamma_{xy} \\ \tau_{yz} = 0 \\ \tau_{zx} = 0 \end{cases} \quad (5-2)$$

Thus, the equilibrium equations become:

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + f_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + f_y = 0 \end{cases} \quad (5-3)$$

With Eq. (5-1) and Eq. (5-2), we can get:

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} \quad (5-4)$$

And we can get compatibility equation in terms of stress in:

$$\nabla^2(\sigma_x + \sigma_y) = -(1 + \nu) \left( \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} \right) \quad (5-5)$$

For problems with no or constant body force intensities, the corresponding plane strain and plane stress problems are identical, which is **Harmonic function**:

$$\nabla^2(\sigma_x + \sigma_y) = 0 \quad (5-6)$$

If the body force intensity is conservative, there is a potential function  $V$  such that:

$$f_x = \frac{\partial V}{\partial x}, \quad f_y = \frac{\partial V}{\partial y} \quad (5-7)$$

Introduce a stress function  $\phi = \phi(x, y)$ , which is also known as Airy's stress function:

$$\begin{cases} \sigma_x + V = \frac{\partial^2 \phi}{\partial y^2} \\ \sigma_y + V = -\frac{\partial^2 \phi}{\partial x^2} \\ \tau_{xy} = \frac{\partial^2 \phi}{\partial x \partial y} \end{cases} \quad (5-8)$$

The introduction of  $\phi$  implies that the equilibrium equations are identically satisfied.

The compatibility equation in terms of stress becomes:

$$\nabla^4 \phi = \nabla^2(\nabla^2 \phi) = (1 - \nu) \nabla^2 V \quad (5-9)$$

where **Biharmonic operator** is:

$$\nabla^4 = \nabla^2(\nabla^2) = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \quad (5-10)$$

This is the governing field equation for plane stress problems in which the body forces are conservative. If the body force is constant, or if  $V$  is a harmonic function (e.g.  $\nabla^2 V = 0$ ):

$$\nabla^4 \phi = 0 \quad (5-11)$$