SUSTech-25Fall-MAE5009-Note

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1 Stress Analysis

1.1 Stress State

- Normal Stress
- Shear Stress
- Stress Transformation

1.2 Equilibrium Equation of Stress

- Body Force: Gravitational Force, Magnetic Force, Inertial Force
- Surface Force: Friction Force, Pressure, Viscous Force(Fluid Flow)

$$Stress = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{1}$$

The equation means the force at the per unit surface area. The unit is Pascal($Pa = N/m^2$).
Other common units:

- $1atm \approx 10^5 Pa = 0.1MPa$
- $1bar \approx 0.98atm \approx 1atm = 0.1MPa$

Stress is a kind of tensor, different from **scalar** and **vector**.

- Scalar: only have magnitude, e.g. temperature, density;
- Vector: have both magnitude and direction, e.g. velocity, force;
- Tensor: magnitude and direction in multiple directions, e.g. stress, strain.

At a reference plane, the force F is vertically upward, the normal stress σ is perpendicular to the plane, while the shear stress τ is parallel to the plane.

1.3 2D Stress

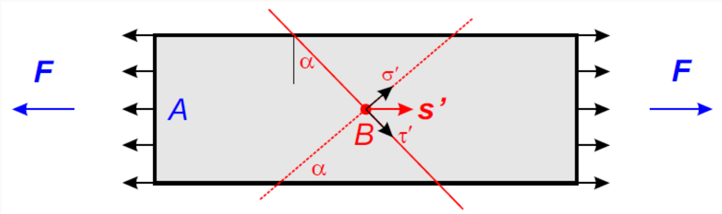

Figure 1: Stress components on a reference plane

For the circumstances shown in Figure 1, the normal stress and shear stress on the reference plane can be determined.

$$A' = \frac{A}{\sin\alpha}, S = \frac{F}{A'} = \frac{F}{A}\sin\alpha, \begin{cases} \tau = S\cdot\cos\alpha = \frac{F}{A}\sin\alpha\cos\alpha \\ \sigma = S\cdot\sin\alpha = \frac{F}{A}\sin^2\alpha \end{cases} \tag{2}$$

1.4 3D Stress

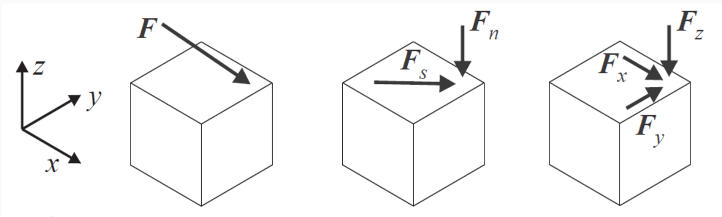

Figure 2: Decomposition of an external force$F$

As shown in Figure 2, the force F applied at an arbitrary angle to the x-y plane can be resolved into a normal component F_n and a shear component F_s. The shear component can be further decomposed into Cartesian components F_x and F_y.

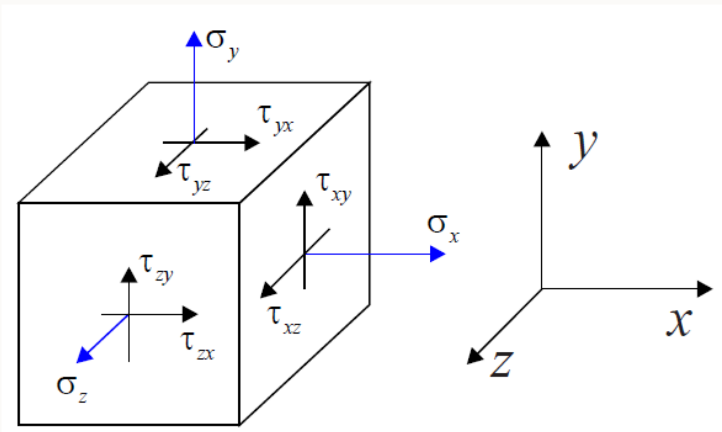

Figure 3: Three-Dimensional State of Stress

As shown in the Figure 3, in every face has three stress components, with 1 normal stress($\sigma_x, \sigma_y, \sigma_z$) and 2 shear stresses($\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy}$). Thus the components of stress can be expressed in a matrix form:

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{3}$$

The sign convention is that normal stresses causing tension are positive, while those causing compression are negative. If we consider rotational equilibrium of the infinitesimal square shown as Figure 4, we can calculate the moment with respect to lower left corner:

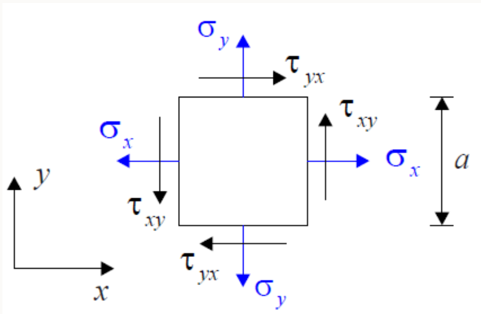

Figure 4: Rotational Equilibrium of an Infinitesimal Element

$$\sigma_x \cdot a(a/2) - \sigma_x \cdot a(a/2) + \sigma_y \cdot a(a/2) - \sigma_y \cdot a(a/2) + \tau_{xy} \cdot a \cdot a - \tau_{yx} \cdot a \cdot a = 0 \tag{4}$$

Thus we have:

$$\tau_{xy} = \tau_{yx} \tag{5}$$

Similarly, we can have:

$$\tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz} \tag{6}$$

Which means, the stress matrix is symmetric, and there are 3 normal stresses and 3 shear stresses, totally 6 independent stress components in 3D stress state.

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ Sym. & & \sigma_z \end{bmatrix} \tag{7}$$

1.5 2D Stress Transformation

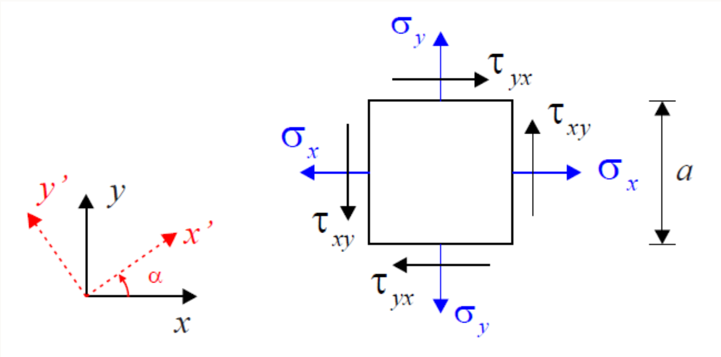

Figure 5: Stress Transformation on an Arbitrary Plane

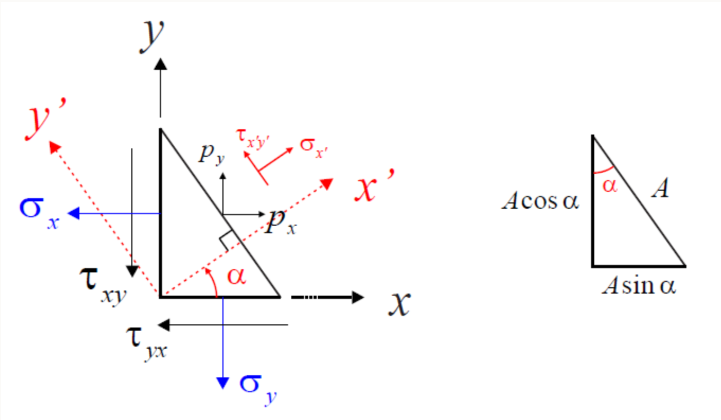

Figure 6: Stress Components on an Inclined Plane in 2D Stress State

After the transformation shown in Figure 5, the new stress components are shown in Figure 6. The stress transformation equations may be derived based on force equilibrium analysis:

$$\sum F_x = 0 \Rightarrow p_x = \sigma_x \cos\alpha + \tau_{yx} \sin\alpha \tag{8}$$

$$\sum F_y = 0 \Rightarrow p_y = \sigma_y \sin\alpha + \tau_{xy} \cos\alpha \tag{9}$$

Thus we have:

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha \\ \sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2} \cos 2\alpha - \tau_{xy} \sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha \end{cases} \tag{10}$$

In 2D circumstances, σ' can be calculated by $\sigma' = \mathbf{R}\sigma\mathbf{R}^T$, in which:

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{yx} \\ \tau_{xy} & \sigma_y \end{bmatrix}, \mathbf{R} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & -\cos\alpha \end{bmatrix}, \sigma' = \begin{bmatrix} \sigma'_x & \tau'_{yx} \\ \tau'_{xy} & \sigma'_y \end{bmatrix} \tag{11}$$

Also, the rotation angle α (principle directions)can be calculated by:

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} (\alpha \in [0, \pi]) \tag{12}$$

$$\sin 2\alpha = \pm \frac{2\tau_{xy}}{\sqrt{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}}, \quad \cos 2\alpha = \frac{\sigma_x - \sigma_y}{\sqrt{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}} \tag{13}$$

The principle stress can be calculated by:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{14}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{15}$$

$$\tau_{x'y'max} = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{16}$$

1.6 Mohr's circle of stress

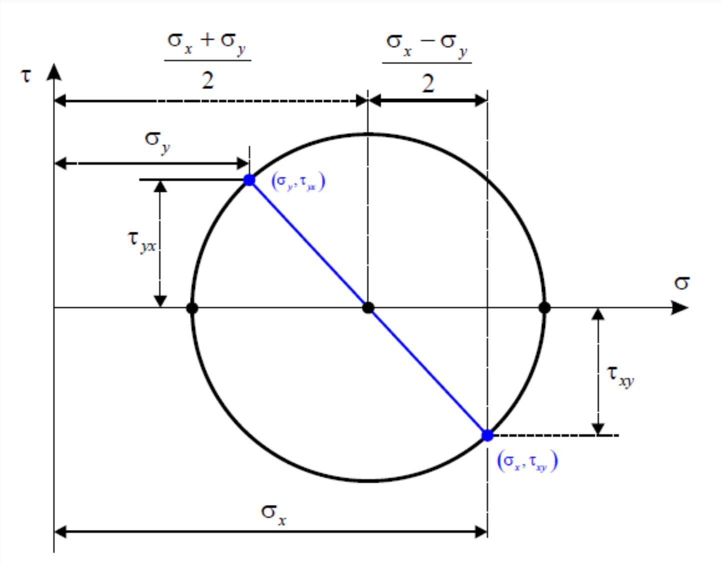

Figure 7: Mohr's Circle of Stress

As shown in the Figure 7, 2D stress transformation can also be conveniently represented graphically in a circle. And from Equation 10, we can calculate that the equation of the circle is

$$\left(\sigma - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \tag{17}$$

From the Equation 17 we can get the center is $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$ and the radius is $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$.

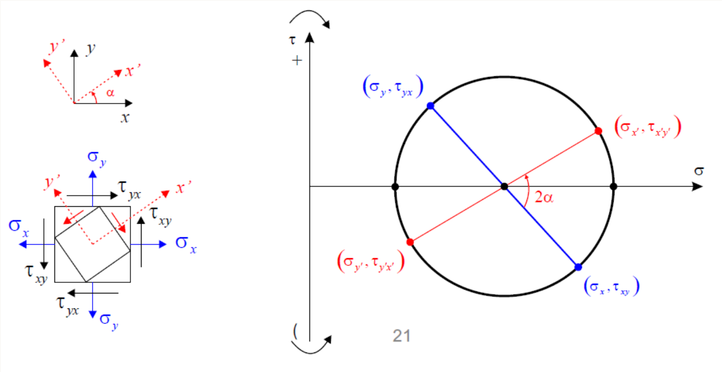

Figure 8: Rotation in the Mohr's Circle of Stress

After rotating the x-y axis as shown in the Figure 8, the stress would be:

$$\begin{cases} \sigma'_x = \dfrac{\sigma_1 + \sigma_2}{2} + \dfrac{\sigma_1 - \sigma_2}{2} \cos 2\alpha \\ \sigma'_y = \dfrac{\sigma_1 + \sigma_2}{2} - \dfrac{\sigma_1 - \sigma_2}{2} \cos 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_1 - \sigma_2}{2} \sin 2\alpha \end{cases} \tag{18}$$

1.7 3D stress transformation

Same to 2D stress transformation, the 3D stress transformation can also be expressed in matrix form:

$$\sigma' = \mathbf{R}\sigma\mathbf{R}^T \tag{19}$$

where

$$\mathbf{R} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos(x', x) & \cos(x', y) & \cos(x', z) \\ \cos(y', x) & \cos(y', y) & \cos(y', z) \\ \cos(z', x) & \cos(z', y) & \cos(z', z) \end{bmatrix} \tag{20}$$

which means the direction cosines between the old and new coordinate axes. And in each columns and rows of matrix $\mathbf{R}$, we have:

$$a_{1i}^2 + a_{2i}^2 + a_{3i}^2 = 1, \quad i = 1, 2, 3 \tag{21}$$

$$a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1, \quad i = 1, 2, 3 \tag{22}$$