SUSTech - 25Fall - MAE5009 - Note

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1 Stress Analysis

1.1 Stress State

- Normal Stress
- Shear Stress
- Stress Transformation

1.2 Equilibrium Equation of Stress

- Body Force: Gravitational Force, Magnetic Force, Inertial Force
- Surface Force: Friction Force, Pressure, Viscous Force(Fluid Flow)

$$\sigma = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A} \tag{1-1}$$

The equation means the force at the per unit surface area. The unit is Pascal($Pa = N/m^2$).

Other common units:

- $1atm \approx 10^5 Pa = 0.1MPa$
- $1bar \approx 0.98atm \approx 1atm = 0.1MPa$

Stress is a kind of tensor, different from **scalar** and **vector**.

- Scalar: only have magnitude, e.g. temperature, density;
- Vector: have both magnitude and direction, e.g. velocity, force;
- Tensor: magnitude and direction in multiple directions, e.g. stress, strain.

At a reference plane, the force F is vertically upward, the normal stress σ is perpendicular to the plane, while the shear stress τ is parallel to the plane.

1.3 2D Stress

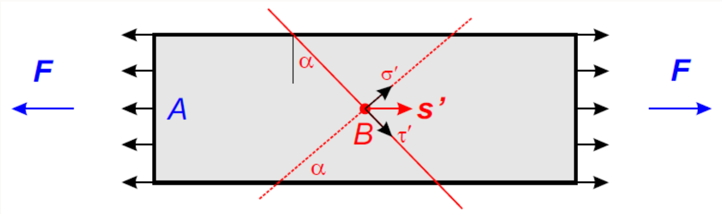

Figure 1: Stress components on a reference plane

For the circumstances shown in Fig. 1, the normal stress and shear stress on the reference plane can be determined.

$$A' = \frac{A}{\sin\alpha}, \quad S = \frac{F}{A'} = \frac{F}{A}\sin\alpha, \quad \begin{cases} \tau = S \cdot \cos\alpha = \dfrac{F}{A}\sin\alpha\cos\alpha \\ \sigma = S \cdot \sin\alpha = \dfrac{F}{A}\sin^2\alpha \end{cases} \tag{1-2}$$

1.4 3D Stress

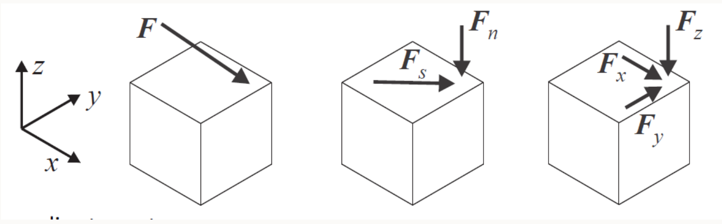

Figure 2: Decomposition of an external force$F$

As shown in Fig. 2, the force F applied at an arbitrary angle to the x-y plane can be resolved into a normal component F_n and a shear component F_s. The shear component can be further decomposed into Cartesian components F_x and F_y.

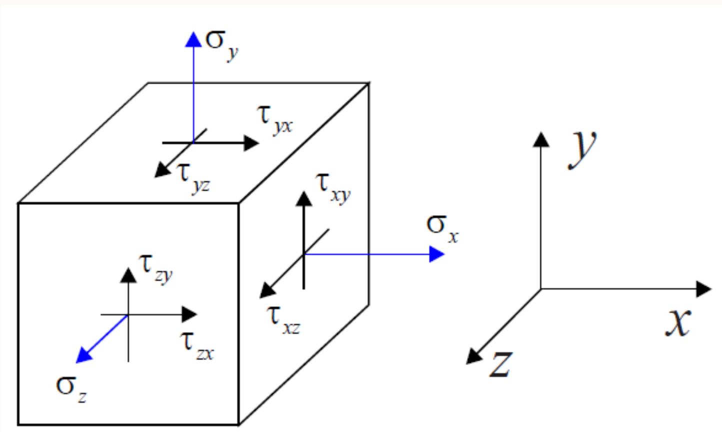

Figure 3: Three-Dimensional State of Stress

As shown in the Fig. 3, in every face has three stress components, with 1 normal stress($\sigma_x, \sigma_y, \sigma_z$) and 2 shear stresses($\tau_{xy}, \tau_{xz}, \tau_{yx}, \tau_{yz}, \tau_{zx}, \tau_{zy}$). Thus the components of stress can be expressed in a matrix form:

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} \tag{1-3}$$

The sign convention is that normal stresses causing tension are positive, while those causing compression are negative. If we consider rotational equilibrium of the infinitesimal square shown as Fig. 4, we can calculate the moment with respect to lower left corner:

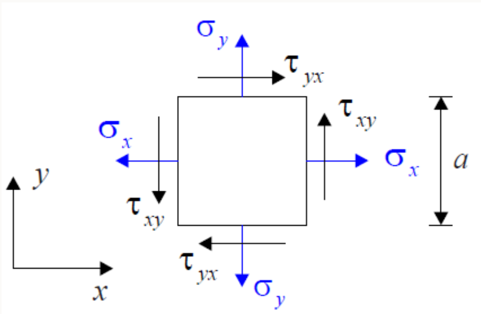

Figure 4: Rotational Equilibrium of an Infinitesimal Element

$$\sigma_x \cdot a(a/2) - \sigma_x \cdot a(a/2) + \sigma_y \cdot a(a/2) - \sigma_y \cdot a(a/2) + \tau_{xy} \cdot a \cdot a - \tau_{yx} \cdot a \cdot a = 0 \tag{1-4}$$

Thus we have:

$$\tau_{xy} = \tau_{yx} \tag{1-5}$$

Similarly, we can have:

$$\tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz} \tag{1-6}$$

Which means, the stress matrix is symmetric, and there are 3 normal stresses and 3 shear stresses, totally 6 independent stress components in 3D stress state.

$$[\sigma] = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ & \sigma_y & \tau_{yz} \\ Sym. & & \sigma_z \end{bmatrix} \tag{1-7}$$

1.5 2D Stress Transformation

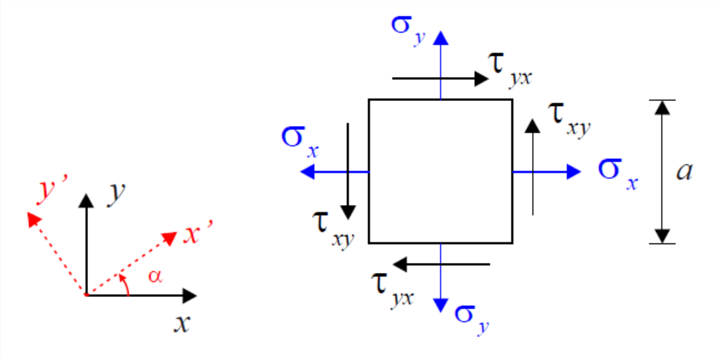

Figure 5: Stress Transformation on an Arbitrary Plane

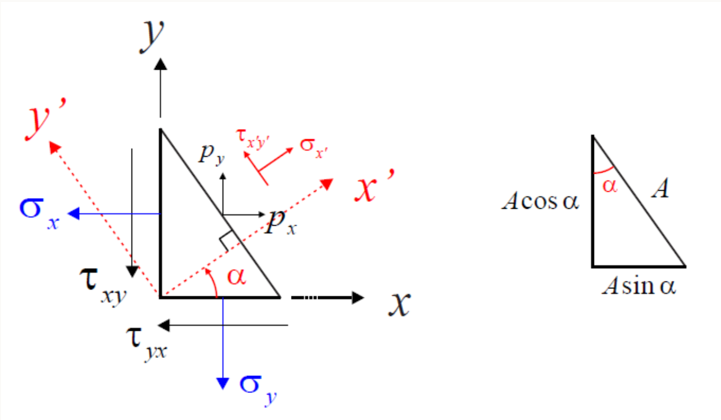

Figure 6: Stress Components on an Inclined Plane in 2D Stress State

After the transformation shown in Fig. 5, the new stress components are shown in Fig. 6. The stress transformation equations may be derived based on force equilibrium analysis:

$$\sum F_x = 0 \Rightarrow p_x = \sigma_x \cos\alpha + \tau_{yx} \sin\alpha \tag{1-8}$$

$$\sum F_y = 0 \Rightarrow p_y = \sigma_y \sin\alpha + \tau_{xy} \cos\alpha \tag{1-9}$$

Thus we have:

$$\begin{cases} \sigma_{x'} = \dfrac{\sigma_x + \sigma_y}{2} + \dfrac{\sigma_x - \sigma_y}{2} \cos 2\alpha + \tau_{xy} \sin 2\alpha \\ \sigma_{y'} = \dfrac{\sigma_x + \sigma_y}{2} - \dfrac{\sigma_x - \sigma_y}{2} \cos 2\alpha - \tau_{xy} \sin 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_x - \sigma_y}{2} \sin 2\alpha + \tau_{xy} \cos 2\alpha \end{cases} \tag{1-10}$$

In 2D circumstances, σ' can be calculated by $\sigma' = \mathbf{R}\sigma\mathbf{R}^T$, in which:

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{yx} \\ \tau_{xy} & \sigma_y \end{bmatrix}, \mathbf{R} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & -\cos\alpha \end{bmatrix}, \sigma' = \begin{bmatrix} \sigma'_x & \tau'_{yx} \\ \tau'_{xy} & \sigma'_y \end{bmatrix} \tag{1-11}$$

Also, the rotation angle α (principle directions)can be calculated by:

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} (\alpha \in [0, \pi]) \tag{1-12}$$

$$\sin 2\alpha = \pm \frac{2\tau_{xy}}{\sqrt{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}}, \quad \cos 2\alpha = \frac{\sigma_x - \sigma_y}{\sqrt{4\tau_{xy}^2 + (\sigma_x - \sigma_y)^2}} \tag{1-13}$$

The principle stress can be calculated by:

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{1-14}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{1-15}$$

$$\tau_{x'y'max} = \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \tag{1-16}$$

1.6 Mohr's circle of stress

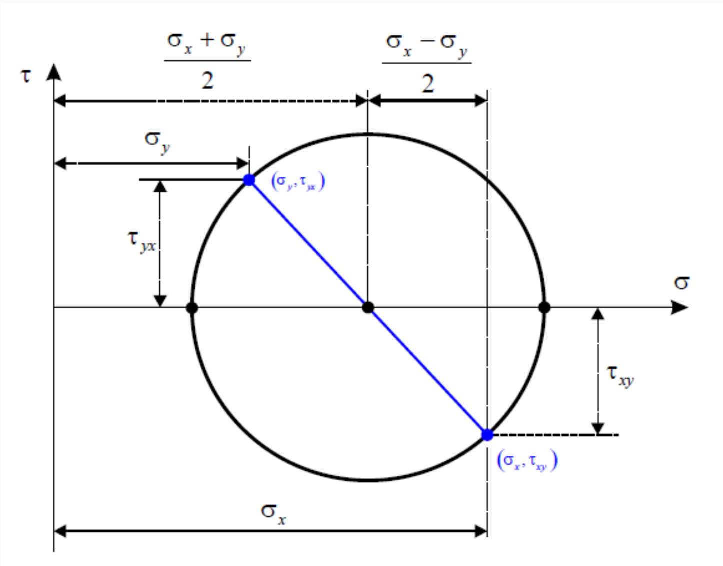

Figure 7: Mohr's Circle of Stress

As shown in the Fig. 7, 2D stress transformation can also be conveniently represented graphically in a circle. And from Eq. (1-10), we can calculate that the equation of the circle is

$$\left(\sigma - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2 \tag{1-17}$$

From the Eq. (1-17) we can get the center is $\left(\frac{\sigma_x + \sigma_y}{2}, 0\right)$ and the radius is $R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$.

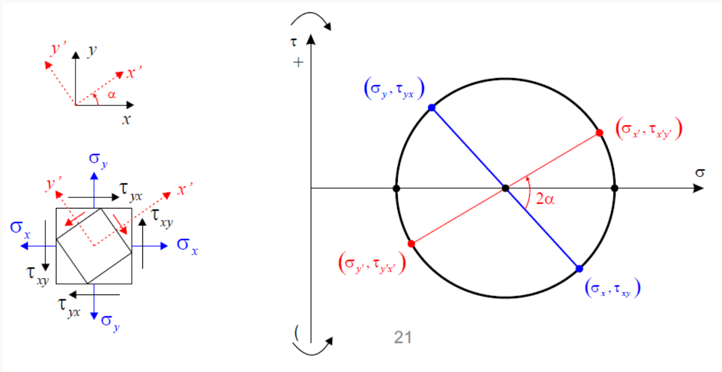

Figure 8: Rotation in the Mohr's Circle of Stress

After rotating the x-y axis as shown in the Fig. 8, the stress would be:

$$\begin{cases} \sigma_x' = \dfrac{\sigma_1 + \sigma_2}{2} + \dfrac{\sigma_1 - \sigma_2}{2}\cos 2\alpha \\ \sigma_y' = \dfrac{\sigma_1 + \sigma_2}{2} - \dfrac{\sigma_1 - \sigma_2}{2}\cos 2\alpha \\ \tau_{x'y'} = -\dfrac{\sigma_1 - \sigma_2}{2}\sin 2\alpha \end{cases} \tag{1-18}$$

1.7 3D stress transformation

Same to 2D stress transformation, the 3D stress transformation can also be expressed in matrix form:

$$\sigma' = \mathbf{R}\sigma\mathbf{R}^T \tag{1-19}$$

where

$$\mathbf{R} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} \cos(x', x) & \cos(x', y) & \cos(x', z) \\ \cos(y', x) & \cos(y', y) & \cos(y', z) \\ \cos(z', x) & \cos(z', y) & \cos(z', z) \end{bmatrix} \tag{1-20}$$

which means the direction cosines between the old and new coordinate axes. And in each columns and rows of matrix $\mathbf{R}$, we have:

$$a_{1i}^2 + a_{2i}^2 + a_{3i}^2 = 1, \quad i = 1,2,3 \tag{1-21}$$

$$a_{i1}^2 + a_{i2}^2 + a_{i3}^2 = 1, \quad i = 1,2,3 \tag{1-22}$$

1.8 3D principal stress

For symmetric matrix $\mathbf{A}$, its eigenvalue and eigenvector satisfy:

$$\mathbf{A}\mathbf{v} = \lambda\mathbf{v} \tag{1-23}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = 0 \tag{1-24}$$

As the equation has non-zero $\mathbf{v}$ if and only if:

$$|\mathbf{A} - \lambda\mathbf{I}| = 0 \tag{1-25}$$

Then we have the **Characteristic Equation** for the principle stress(in Eqs. (1-23) to (1-25), the matrix $\mathbf{A}$ can be replaced by stress matrix $[\sigma]$):

$$\det(\sigma - \lambda\mathbf{I}) = 0 \tag{1-26}$$

Expand the determinant, we have:

$$\lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{1-27}$$

The invariants I_1, I_2, I_3 are **Stress Invariants**, which means they are independent of the coordinate system:

$$\begin{cases} I_1 = \sigma_x + \sigma_y + \sigma_z = \sigma_1 + \sigma_2 + \sigma_3 \\ I_2 = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 \\ I_3 = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2 = \sigma_1\sigma_2\sigma_3 \end{cases} \tag{1-28}$$

1.9 Differential equation of equilibrium

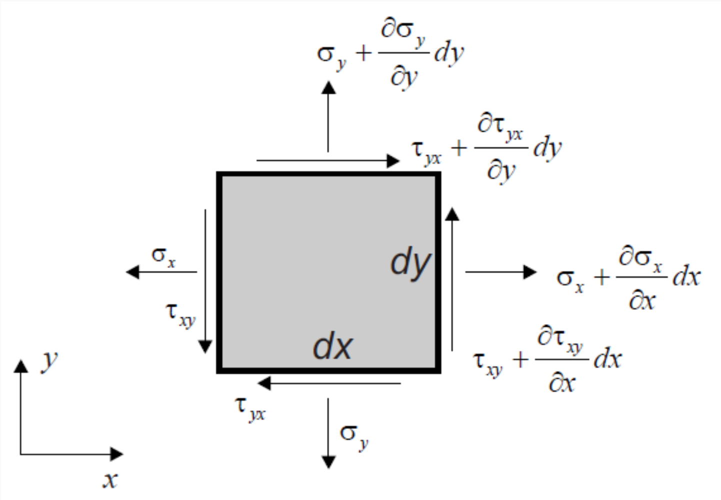

Figure 9: Differential Element under Stress and Body Force

For the force balance $\sum F = 0$ in the circumstances shown in Fig. 9, we have:

$$\begin{cases} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + f_x = 0 \\ \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + f_y = 0 \\ \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + f_z = 0 \end{cases} \tag{1-29}$$

Which can be expressed like:

$$\nabla \cdot \sigma + f = 0 \tag{1-30}$$

In the Eqs. (1-29) and (1-30), f is the body force intensities(per unit volume), e.g. gravitational force, magnetic force, inertial force.

And when the body has acceleration, the Eq. (1-30) can be expressed as:

$$\nabla \cdot \sigma + f = \rho \frac{\partial^2 u}{\partial t^2} \tag{1-31}$$

And for the torque balance $\sum M = 0$ in the circumstances shown in Fig. 9, we have:

$$\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz} \tag{1-32}$$

From the Eq. (1-29), there are 6 independent unkowns in total, while only 3 equations. Thus additional equations are needed to complete the solutions of the stress distribution in a body. For example, Strain-displacement, Generalized Hooke's Law, etc.

2 Strain Analysis

2.1 Assumptions

- Infinitesimal deformation (1% - 5%)
- Continuous materials
  - Continuous displacement
  - No gap/discontinuities after displacement
- Displacement functions must be single-valued

2.2 Strain & Displacement

Same to stress, strain also can be devided into:

- **Normal Strain**: relative changes of the **length** of the objects
- **Shear Strain**: changes of the **angle** of the two sides of the objects

And the definition of the strain is:

$$\varepsilon = \frac{\Delta L}{L} \tag{2-1}$$

which is dimensionless.

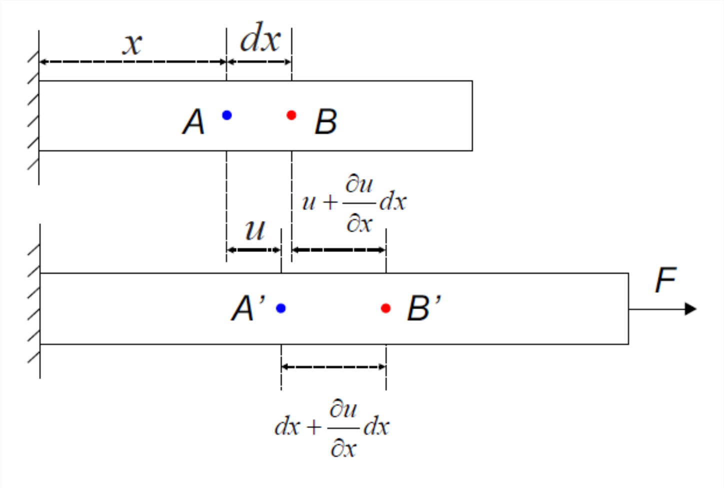

Figure 10: Deformation of a body under load F

As shown in Fig. 10, under the load F, the strain ε can be calculated by:

$$\varepsilon = \frac{A'B' - AB}{AB} = \frac{dx + \frac{\partial u}{\partial x}dx - dx}{dx} = \frac{\frac{\partial u}{\partial x} \cdot dx}{dx} = \frac{\partial u}{\partial x} \tag{2-2}$$

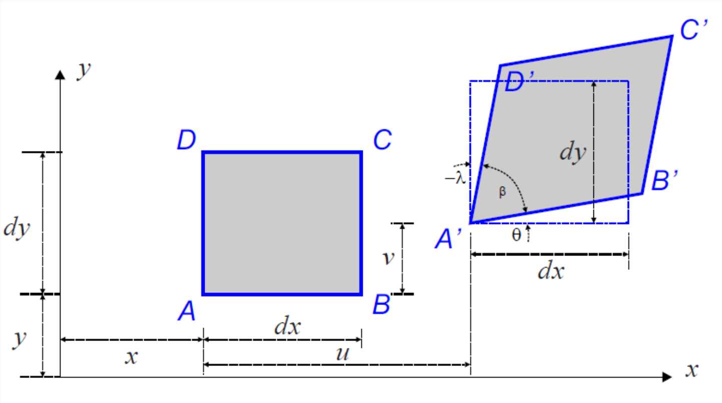

Figure 11: 2D deformation of a body under load

As shown in Fig. 11, the normal strain and the shear strain can be calculated by Eqs. (2-3) to (2-5), and the deformation of the body in both horizontal and vertical can be calculated by Tab. 1.

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{A'B' - dx}{dx} \tag{2-3}$$

$$\varepsilon_y = \frac{A'D' - AD}{AD} = \frac{A'D' - dy}{dy} \tag{2-4}$$

$$\gamma = \frac{\pi}{2} - \beta \tag{2-5}$$

Table 1: Deformation of a body in 2D

	Horizontal	Vertical
A	$u(x,y)$	$v(x,y)$
B	$u(x,y) + \frac{\partial u}{\partial x}dx$	$v(x,y) + \frac{\partial v}{\partial x}dx$
D	$u(x,y) + \frac{\partial u}{\partial y}dy$	$v(x,y) + \frac{\partial v}{\partial y}dy$

From Tab. 1, we can calculate the normal strain and shear strain in 2D:

$$(A'B')^2 = (dx + \frac{\partial u}{\partial x}dx)^2 + (\frac{\partial v}{\partial x}dx)^2 = (dx(1+\varepsilon_x)^2) \Rightarrow (1+\varepsilon_x)^2 = (1 + \frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 \tag{2-6}$$

As $\frac{\partial v}{\partial x} \approx 1\% - 5\%$, we have: $\varepsilon_x = \frac{\partial u}{\partial x}$, and similarly, $\varepsilon_y = \frac{\partial v}{\partial y}$. And for 3D circumstances, we have $\varepsilon_z = \frac{\partial w}{\partial z}$.

For the defoemation of angle, we have:

$$\theta \to 0, \theta = \tan\theta = \frac{\frac{\partial v}{\partial x}dx}{dx + \frac{\partial u}{\partial x}dx} = \frac{\frac{\partial v}{\partial x}}{1 + \frac{\partial u}{\partial x}} = \frac{\partial v}{\partial x} \tag{2-7}$$

$$\lambda \to 0, -\lambda = -\tan\lambda = \frac{\frac{\partial u}{\partial y}dy}{dy + \frac{\partial v}{\partial y}dy} = \frac{\frac{\partial u}{\partial y}}{1 + \frac{\partial v}{\partial y}} = \frac{\partial u}{\partial y} \tag{2-8}$$

Thus, the shear strain can be expressed as:

$$\gamma_{xy} = \frac{\pi}{2} - \beta = \theta + (-\lambda) = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \tag{2-9}$$

Generalization to three-dimensional case, we have:

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} \\ \varepsilon_y = \dfrac{\partial v}{\partial y} \\ \varepsilon_z = \dfrac{\partial w}{\partial z} \end{cases}, \quad \begin{cases} \gamma_{xy} = \dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y} \\ \gamma_{yz} = \dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z} \\ \gamma_{zx} = \dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x} \end{cases}, \quad \begin{cases} \gamma_{xy} = \gamma_{yx} \\ \gamma_{yz} = \gamma_{zy} \\ \gamma_{zx} = \gamma_{xz} \end{cases} \tag{2-10}$$

2.3 Strain compatibility equation

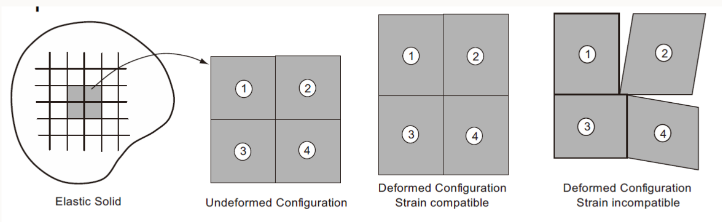

Figure 12: Strain Compatibility

Six equations for the strain components are functions of only three displacement components. If six strain components are known, we have six equations for only three unknowns. There must be additional equations relate the six strain components, which are called **Strain Compatibility Equations**.

$$\begin{cases} \dfrac{\partial^2 \epsilon_x}{\partial y^2} + \dfrac{\partial^2 \epsilon_y}{\partial x^2} = \dfrac{\partial^2 \gamma_{xy}}{\partial x \partial y} \\ \dfrac{\partial^2 \epsilon_y}{\partial z^2} + \dfrac{\partial^2 \epsilon_z}{\partial y^2} = \dfrac{\partial^2 \gamma_{yz}}{\partial y \partial z} \\ \dfrac{\partial^2 \epsilon_z}{\partial x^2} + \dfrac{\partial^2 \epsilon_x}{\partial z^2} = \dfrac{\partial^2 \gamma_{zx}}{\partial z \partial x} \end{cases}, \quad \begin{cases} 2\dfrac{\partial^2 \epsilon_x}{\partial y \partial z} = \dfrac{\partial}{\partial x}\left(-\dfrac{\partial \gamma_{yz}}{\partial x} + \dfrac{\partial \gamma_{zx}}{\partial y} + \dfrac{\partial \gamma_{xy}}{\partial z}\right) \\ 2\dfrac{\partial^2 \epsilon_y}{\partial z \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \gamma_{yz}}{\partial x} - \dfrac{\partial \gamma_{zx}}{\partial y} + \dfrac{\partial \gamma_{xy}}{\partial z}\right) \\ 2\dfrac{\partial^2 \epsilon_z}{\partial x \partial y} = \dfrac{\partial}{\partial z}\left(\dfrac{\partial \gamma_{yz}}{\partial x} + \dfrac{\partial \gamma_{zx}}{\partial y} - \dfrac{\partial \gamma_{xy}}{\partial z}\right) \end{cases} \tag{2-11}$$

The compatibility equation Eq. (2-11) can also be called as **Saint-Venant's compatibility equation**. In this equation, only three compatibility equations are independent. If the displacement components are single-valued, continuous functions, the strain components will automatically satisfy the compatibility equations.

2.4 2D Strain Transformation

The transformation of strain is almost same as stress transformation.

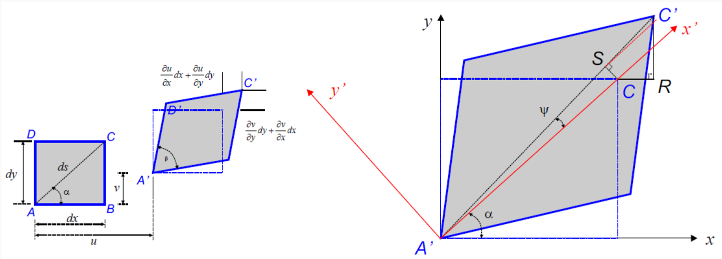

Figure 13: 2D Strain Transformation

Similar to Eq. (1-10), the strain transformation equations are:

$$\begin{cases} \varepsilon_{x'} = \dfrac{\varepsilon_x+\varepsilon_y}{2} + \dfrac{\varepsilon_x-\varepsilon_y}{2}\cos 2\alpha + \varepsilon_{xy}\sin 2\alpha \\ \varepsilon_{y'} = \dfrac{\varepsilon_x+\varepsilon_y}{2} - \dfrac{\varepsilon_x-\varepsilon_y}{2}\cos 2\alpha - \varepsilon_{xy}\sin 2\alpha \\ \varepsilon_{x'y'} = -\dfrac{\varepsilon_x-\varepsilon_y}{2}\sin 2\alpha + \varepsilon_{xy}\cos 2\alpha \end{cases} \tag{2-12}$$

Also, similar to Eq. (1-11), the strain transformation can be expressed by matrix form $\varepsilon' = \mathbf{R}\varepsilon\mathbf{R}^T$.

$$\varepsilon = \begin{bmatrix} \varepsilon_x & \varepsilon_{yx} \\ \varepsilon_{xy} & \varepsilon_y \end{bmatrix}, \mathbf{R} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & -\cos\alpha \end{bmatrix}, \varepsilon' = \begin{bmatrix} \varepsilon'_x & \varepsilon'_{yx} \\ \varepsilon'_{xy} & \varepsilon'_y \end{bmatrix} \tag{2-13}$$

Direction of principal strain can be calculated by:

$$\tan 2\alpha = \frac{2\varepsilon_{xy}}{\varepsilon_x-\varepsilon_y}(\alpha \in [0,\pi]) \tag{2-14}$$

And the magnitudes of principal strains are:

$$\varepsilon_{1,2} = \frac{\varepsilon_x+\varepsilon_y}{2} \pm \sqrt{\left(\frac{\varepsilon_x-\varepsilon_y}{2}\right)^2 + \varepsilon_{xy}^2} \tag{2-15}$$

For the strain tensor in 3D:

$$[\varepsilon] = \begin{bmatrix} \varepsilon_x & \varepsilon_{xy} & \varepsilon_{xz} \\ & \varepsilon_y & \varepsilon_{yz} \\ Sym. & & \varepsilon_z \end{bmatrix} \tag{2-16}$$

The calculation of strain invariants is similar to stress invariants in Sec. 1.8.

$$\det(\varepsilon - \lambda \mathbf{I}) = 0 \tag{2-17}$$

Thus we can get:

$$\begin{cases} I_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \\ I_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \varepsilon_{xy}^2 - \varepsilon_{yz}^2 - \varepsilon_{zx}^2 = \varepsilon_1\varepsilon_2 + \varepsilon_2\varepsilon_3 + \varepsilon_3\varepsilon_1 \\ I_3 = \varepsilon_x\varepsilon_y\varepsilon_z + 2\varepsilon_{xy}\varepsilon_{yz}\varepsilon_{zx} - \varepsilon_x\varepsilon_{yz}^2 - \varepsilon_y\varepsilon_{zx}^2 - \varepsilon_z\varepsilon_{xy}^2 = \varepsilon_1\varepsilon_2\varepsilon_3 \end{cases} \tag{2-18}$$