Linear System Theory
Hints for Problem Set 4

The following results would be useful.

Fact 1: Consider the system $\dot{x}(t) = f(x(t))$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz. For any differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, by using the chain rule, we get that $\frac{dV(x(t))}{dt} = \dot{x}(t)^\top \frac{\partial V(x(t))}{\partial x} = f(x(t))^\top \frac{\partial V(x(t))}{\partial x}$.

Fact 2: For any symmetric square matrix $A \in \mathbb{R}^{n\times n}$, it holds that $\lambda_{\min}(A)\|x\|_2^2 \leq x^\top Ax \leq \lambda_{max}(A)\|x\|_2^2$, where $\lambda_{\min}(A)$ and $\lambda_{max}(A)$ denote the minimum and maximum eigenvalues of A, respectively. **Try to prove this claim on your own**. To this end, remember that a square matrix A is symmetric and positive semidefinite (resp. positive definite) if and only if all its eigenvalues are nonnegative (resp. strictly positive).

Fact 3: A symmetric positive definite $n \times n$ matrix $P = P^\top > 0$ defines an inner product in $\mathbb{R}^n$ by $\langle x, y\rangle_P := x^\top Py$, for all $x, y \in \mathbb{R}^n$. **Try to prove this claim on your own**[1], by verifying the conditions in Definition 7.1. Therefore, the map $\|\cdot\|_P : \mathbb{R}^n \to \mathbb{R}_+$ defined by $\|x\|_P = \sqrt{x^\top Px}$ is a norm in $\mathbb{R}^n$ by Theorem 7.1. In particular, since $\mathbb{R}^n$ is finite-dimensional, for any $P = P^\top > 0$ and $Q = Q^\top > 0$, the norms $\|\cdot\|_P$, $\|\cdot\|_Q$ and $\|\cdot\|_2$ are equivalent. Thus for example, there exist positive constants[2] $m, M > 0$ such that $m\|x\|_P \leq \|x\|_Q \leq M\|x\|_P$, for all $x \in \mathbb{R}^n$.

Exercise 1 (Lyapunov stability and salmon extinction)

(1) Use Fact 3 and Gronwalll's Lemma. For the second part, use Fact 1 for $V(x) = x^\top Px$ and $f(x) = Ax$.
(2) Use $V(x) = x^\top Px$ and $f(x) = Ax$.

Exercise 2 (LTV stability)

(1) Solve analytically the ODE and show that even if $\mathrm{Re}(\lambda) < 0$ the system is unstable.
(2) Consider $V(x) = x^\top x$. Then, use Fact 1, Fact 2 and Gronwall's Lemma.

Exercise 3 (Inner product spaces)

(1) For a fixed $f \in \mathcal{H}$, consider $g = \sum_{i=1}^{n}\langle v_i, f\rangle v_i$. Argue that this is an element in $\mathcal{H}$. Then show that $\langle g, g - f\rangle = 0$. Finally, use the Pythagoras Theorem 7.2 twice.

[1] We consider the vector field to be $\mathbb{R}$. That is, condition 4 in Definition 7.1 is $\langle x, y\rangle = \langle y, x\rangle$. Moreover $\langle ax, y\rangle = a\langle x, y\rangle$, for all $a \in \mathbb{R}$

[2] You can explicity determine the constants in the corresponding inequalities, by using the fact that for any positive definite matrix N, there exist $c, \rho > 0$, such that $cI - N$ and $N - \rho I$ are positive definite.

(2) First show that if $\mathcal{B} = \{b_1, \ldots, b_n\}$ is an orthonormal basis for $\mathcal{H}$, then every $f \in \mathcal{H}$ is written as $f = \sum_{i=1}^{n} \langle b_i, f \rangle b_i$.