Linear System Theory

Problem Set 2
Normed Spaces, ODEs, and Linear Time-Varying Systems

Issue date: October 7, 2019
Due date: October 21, 2019

Exercise 1. (Norms, [45 points in total])

1. [15 **points**] Let $C([t_0, t_1], \mathbb{R}^n)$ be the space of all continuous functions from $[t_0, t_1]$ to $\mathbb{R}^n$. Prove that for $f \in C([t_0, t_1], \mathbb{R}^n)$, $\|f\|_\infty := \max_{t\in[t_0,t_1]} \|f(t)\|_p$ satisfies the axioms of the norm, where $\|x\|_p$ is the p-norm of $x \in \mathbb{R}^n$.

2. [10 **points**] Given a matrix $A \in \mathbb{R}^{m\times n}$, verify that the induced matrix norms $\|A\|_2, \|A\|_\infty$ are equivalent, by showing that they satisfy the following inequalities:

$$\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty.$$

Hint: The induced p-norm of a matrix A is given by:

$$\|A\|_p = \sup_{\|x\|_p\neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

3. [20 **points**] Consider a set of functions f_n in $C([0, 1], \mathbb{R})$ defined as:

$$f_n : [0, 1] \to \mathbb{R} \quad \text{s.t.} \quad f_n(x) = \begin{cases} n - n^2x & 0 \leq x \leq \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases}$$

for $n \in \mathbb{N}$. Compute the 1-norm, 2-norm and ∞-norm of the functions for each n, as defined below:

$$\|f\|_1 := \int_0^1 |f(x)|\, dx\,, \quad \|f\|_2 := \sqrt{\int_0^1 |f(x)|^2\, dx}\,, \quad \|f\|_\infty := \max_{t\in[0,1]} |f(t)|.$$

Based on your computations, what can you say about equivalence of these norms?

Solution 1.1

- $\forall f, g \in C([t_0, t_1], \mathbb{R}^n), \|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

$$\begin{aligned}\forall f, g \in C([t_0, t_1], \mathbb{R}^n), \|f + g\|_\infty &= \max_{t\in[t_0,t_1]} \|f + g\|_p \\ &\leq \max_{t\in[t_0,t_1]} \{\|f\|_p + \|g\|_p\} \\ &\leq \max_{t\in[t_0,t_1]} \|f\|_p + \max_{t\in[t_0,t_1]} \|g\|_p \\ &= \|f\|_\infty + \|g\|_\infty\end{aligned}$$

- $\forall f \in C([t_0, t_1], \mathbb{R}^n), \forall a \in \mathbb{R}, \|af\| = a\|f\|$

$$\begin{aligned}\forall f \in C([t_0, t_1], \mathbb{R}^n), \forall a \in \mathbb{R}, \|af\| &= \max_{t\in[t_0,t_1]} \|af\|_p \\ &= \max_{t\in[t_0,t_1]} a\|f\|_p \\ &= a \max_{t\in[t_0,t_1]} \|f\|_p \\ &= a\|f\|_\infty\end{aligned}$$

- $\|f\| = 0 \Leftrightarrow f = 0$
  $\Rightarrow$: $\|f\| = 0 \Rightarrow \max_{t\in[t_0,t_1]} \|f\|_p = 0$, $\|f\|_p \geq 0 \Rightarrow \|f\|_p = 0, \forall t \in [t_0, t_1] \Rightarrow f = 0$
  $\Leftarrow$: $f = 0 \Rightarrow \|f\|_p = 0, \forall t \in [t_0, t_1] \Rightarrow \max_{t\in[t_0,t_1]} \|f\|_p = 0 \Rightarrow \|f\| = 0$

Solution 1.2
Define x_{max} and $(Ax)_{max}$ to simplify further notation:

$$\begin{aligned} x_{max} &= \max_{i\in\{1,\dots,n\}} x_i \\ (Ax)_{max} &= \max_{j\in\{1,\dots,m\}} (Ax)_j \end{aligned}$$

Then, start with $\|A\|_2^2$:

$$\begin{aligned} \|A\|_2^2 &= \left(\sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}\right)^2 = \sup_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_i^2}{\sum_{i=1}^n x_i^2} \geq \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_i^2}{\sum_{i=1}^n x_{max}^2} \\ &= \sup_{x\neq 0} \frac{1}{n} \frac{\sum_{i=1}^m (Ax)_i^2}{x_{max}^2} \geq \sup_{x\neq 0} \frac{1}{n} \frac{(Ax)_{max}^2}{x_{max}^2} = \sup_{x\neq 0} \frac{1}{n} \left(\frac{(Ax)_{max}}{x_{max}}\right)^2 \\ &= \frac{1}{n} \left(\sup_{x\neq 0} \frac{(Ax)_{max}}{x_{max}}\right)^2 = \frac{1}{n} \left(\sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}\right)^2 = \frac{1}{n}\|Ax\|_\infty \\ &\Rightarrow \|A\|_2 \geq \frac{1}{\sqrt{n}}\|Ax\|_\infty \end{aligned}$$

$$\begin{aligned} \|A\|_2^2 &= \left(\sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}\right)^2 = \sup_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_i^2}{\sum_{i=1}^n x_i^2} \leq \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_{max}^2}{\sum_{i=1}^n x_i^2} \\ &= \sup_{x\neq 0} \frac{m\,(Ax)_{max}^2}{\sum_{i=1}^n x_i^2} = m \sup_{x\neq 0} \frac{(Ax)_{max}^2}{\sum_{i=1}^n x_i^2} \leq m \sup_{x\neq 0} \frac{(Ax)_{max}^2}{x_{max}^2} = m \sup_{x\neq 0} \left(\frac{(Ax)_{max}}{x_{max}}\right)^2 \\ &= m \left(\sup_{x\neq 0} \frac{(Ax)_{max}}{x_{max}}\right)^2 = m \left(\sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}\right)^2 = m\|Ax\|_\infty \\ &\Rightarrow \|A\|_2 \leq \sqrt{m}\|Ax\|_\infty \end{aligned}$$

Solution 1.3
Although $n \in \mathbb{N}$ can be any non-negative integer, we additionally require $n > 0$ to make sure the piece-wise function $f_0(x)$ is well defined, otherwise $\frac{1}{0}$ does not make sense.

$$\forall x \in \left[0, \frac{1}{n}\right], \forall n \in \mathbb{N}, n - n^2 x \geq 0 \Rightarrow f_n(x) \geq 0, \forall x \in [0,1] \Rightarrow |f_n(x)| = f_n(x), \forall x \in [0,1]$$

- $\|f\|_1$

$$\|f\|_1 = \int_0^1 |f_n(x)|\,\mathrm{d}x = \int_0^1 f_n(x)\,\mathrm{d}x = \int_0^{\frac{1}{n}} \left(n - n^2 x\right)\,\mathrm{d}x = nx - \frac{1}{2}n^2x^2\Big|_0^{\frac{1}{n}} = \frac{1}{2}$$

- $\|f\|_2$

$$\begin{aligned} \|f\|_2^2 &= \int_0^1 |f_n(x)|^2\,\mathrm{d}x = \int_0^{\frac{1}{n}} \left(n - n^2 x\right)^2\,\mathrm{d}x \\ &= \int_0^{\frac{1}{n}} \left(n^4x^2 - 2n^3x + n^2\right)\,\mathrm{d}x = \frac{1}{3}n^4x^3 - n^3x^2 + n^2x\Big|_0^{\frac{1}{n}} = \frac{1}{3}n \\ \|f\|_2 &= \sqrt{\frac{1}{3}n} \end{aligned}$$

- $\|f\|_\infty$

$$\|f\|_\infty = \max_{t\in[0,1]} |f_n(t)| = \max_{t\in[0,1/n]} n - n^2 t = n$$

- Equivalence of $\|f_n\|_1$ and $\|f_n\|_\infty$

$$\exists m_u = 4n \geq m_l = n \geq 0, \forall f_n \in C([0,1],\mathbb{R}), m_l\|f_n\|_1 = \frac{1}{2}n \leq n = \|f_n\|_\infty \leq 2n = m_u\|f_n\|_1$$

- Equivalence of $\|f_n\|_2$ and $\|f_n\|_\infty$

$$\exists m_u = \sqrt{27n} \geq m_l = \sqrt{\frac{1}{3}n} \geq 0, \forall f_n \in C([0,1],\mathbb{R}), m_l\|f_n\|_2 = \frac{1}{3}n \leq n = \|f_n\|_\infty \leq 3n = m_u\|f_n\|_2$$

Therefore, $\|f_n\|_1$, $\|f_n\|_2$ and $\|f_n\|_\infty$ are equivalent.

Exercise 2. (Banach fixed point theorem [25 points in total])

1. [**20 points**] Let $(X, \|\cdot\|)$ be a Banach space, and $f : X \to X$. Assume that there exists $\alpha \in [0, 1)$ such that, for all $x, y \in X$,

$$\|f(x) - f(y)\| \leq \alpha \|x - y\| .$$

   Show that there exists a unique point $\bar{x}$ such that $f(\bar{x}) = \bar{x}$.

   Hint: Given an arbitrary initial point x, consider the sequence of iterates $f^{[n]}(x) = f(f^{[n-1]}(x))$, where the first iterate is given by $f^{[0]}(x) = x$. You can start by showing that this sequence is Cauchy.

2. [**5 points**] Now assume f is a linear map. Given the condition in the first part of Exercise 2, what can you conclude about the induced norm of f?

Solution 2.1

$$\begin{aligned}
f^{[0]}(x) &= x \\
f^{[1]}(x) &= f\left(f^{[0]}(x)\right) = f(x) \\
f^{[2]}(x) &= f\left(f^{[1]}(x)\right) = f(f(x)) \\
&\vdots \\
f^{[n]}(x) &= f\left(f^{[n-1]}(x)\right) = f(\cdots f(f(x)))
\end{aligned}$$

$\forall p \geq q \geq 0, p, q \in N :$

$$\begin{aligned}
\|f^{[p]}(x) - f^{[q]}(x)\| &= \left\|f\left(f^{[p-1]}(x)\right) - f\left(f^{[q-1]}(x)\right)\right\| \\
&\leq \alpha \left\|f^{[p-1]}(x) - f^{[q-1]}(x)\right\| \\
&\vdots \\
&\leq \alpha^q \left\|f^{[p-q]}(x) - f^{[0]}(x)\right\| \\
&= \alpha^q \left\|f^{[p-q]}(x) - x\right\|
\end{aligned}$$

$\forall k \in N :$

$$\begin{aligned}
\left\|f^{[k]}(x) - x\right\| &= \left\|f^{[k]}(x) - f(x) + f(x) - x\right\| \\
&\leq \left\|f^{[k]}(x) - f(x)\right\| + \|f(x) - x\| \\
&\leq \alpha \left\|f^{[k-1]}(x) - x\right\| + \|f(x) - x\| \\
&\leq \alpha^2 \left\|f^{[k-2]}(x) - x\right\| + (1 + \alpha) \|f(x) - x\| \\
&\vdots \\
&\leq \alpha^{k-1} \left\|f^{[1]}(x) - x\right\| + \left(1 + \alpha + \cdots + \alpha^{k-2}\right) \|f(x) - x\| \\
&\leq \left(1 + \alpha + \cdots + \alpha^{k-2} + \alpha^{k-1}\right) \|f(x) - x\| \\
&= \frac{1 - \alpha^k}{1 - \alpha} \|f(x) - x\|
\end{aligned}$$

$\forall \epsilon > 0, \exists N = \left\lceil \log_\alpha \left(\frac{\epsilon(1-\alpha)}{\|f(x)-x\|} + \alpha^m \right) \right\rceil \in \mathbb{N}$ ($\lceil t \rceil$ means the smallest integer greater than t), $\forall m \geq N$,

$$\begin{aligned} \left\| f^{[m]}(x) - f^{[N]}(x) \right\| &\leq \alpha^N \left\| f^{[m-N]}(x) - x \right\| \\ &\leq \alpha^N \cdot \frac{1-\alpha^{m-N}}{1-\alpha} \cdot \|f(x) - x\| \\ &= \frac{\alpha^N - \alpha^m}{1-\alpha} \|f(x) - x\| \\ N &> \log_\alpha \left(\frac{\epsilon(1-\alpha)}{\|f(x) - x\|} + \alpha^m \right) \\ \Rightarrow \left\| f^{[m]}(x) - f^{[N]}(x) \right\| &< \epsilon \end{aligned}$$

Therefore, $\{f^{[i]}\}_{i=0}^{\infty}$ is a Cauchy sequence. Note that given an arbitrary initial point x, we can compute the value of $\|f(x) - x\|$ so we treat $\|f(x) - x\|$ as a known constant in the proof above. Because $(X, \|\cdot\|)$ is a Banach space, $\{f^{[i]}\}_{i=0}^{\infty}$ converges to a point f^*.

$$f^* = \lim_{n\to\infty} f^{[n]}(x) = \lim_{n\to\infty} f\left(f^{[n-1]}(x)\right) = f\left(\lim_{n\to\infty} f^{[n-1]}(x)\right) = f(f^*)$$

Therefore, there exist $\bar{x} = f^*$ such that $f(\bar{x}) = \bar{x}$. Next, we prove uniqueness: For the sake of contradiction, assume there exist $\bar{x}_1, \bar{x}_2 \in X, \bar{x}_1 \neq \bar{x}_2$ such that $f(\bar{x}_1) = \bar{x}_1$ and $f(\bar{x}_2) = \bar{x}_2$.

$$\|f(\bar{x}_2) - f(\bar{x}_1)\| = \|\bar{x}_2 - \bar{x}_1\| \leq \alpha \|\bar{x}_2 - \bar{x}_1\|$$

Since $\alpha \in [0, 1)$, $\alpha \neq 1$. The above inequality never holds, which leads to contradiction, so $\bar{x}$ is unique.

Solution 2.2

Let F be the representation of linear map $f(\cdot)$. $\forall x, y \in X$:

$$\|f(x) - f(y)\| = \|Fx - Fy\| = \|F(x - y)\| \leq \alpha \|x - y\| \Rightarrow \frac{\|F(x-y)\|}{\|x-y\|} \leq \alpha$$

$\forall u \in X, \exists x, y \in X$ such that $u = x - y$.

$$\|F\| = \sup_{u\neq 0} \frac{\|Fu\|}{\|u\|} \leq \sup_{u\neq 0} \alpha = \alpha$$

Therefor, the induced norm of f is bounded by α.

Exercise 3. (Ordinary differential equations [30 **points in total**]**)**

1. [12 **points**] Consider the following ordinary differential equation (ODE)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -x_1(t) + e^t \cos(x_1(t) - x_2(t)) \\ -x_2(t) + \sin(x_1(t) - x_2(t)) \end{bmatrix},$$

where $x_i(t) \in \mathbb{R}$, $\forall i$. Prove or disprove the following statements:

   (a) This system is globally Lipschitz,

   (b) This system admits a unique solution.
   Hint: You may assume that functions with bounded derivatives are Lipschitz.

2. [18 **points**] Consider the following ordinary differential equation (ODE)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3\sin(t)\, x_1(t) + x_1(t)\, x_2(t) \\ -2\, x_2(t) \end{bmatrix},$$

where $x_i(t) \in \mathbb{R}$, $\forall i$. Prove or disprove the following statements:

   (a) This system is globally Lipschitz,

   (b) This system admits a unique solution.