Linear System Theory

Problem Set 4
Stability of LTI and LTV, Inner product spaces

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Exercise 1, (Lyapunov stability and salmon extinction, [50 points in total])
Consider a system $\dot{x}(t) = f(x(t))$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous. Suppose we have

$$\frac{d}{dt}\big(x(t)^T P x(t)\big) \leq -x(t)^T Q x(t)\,, \tag{1}$$

where P and Q are symmetric positive definite matrices (here the time-derivative is taken along the state trajectories of the system).

1. [20 points] Prove that under condition (1) the system is exponentially stable, in the sense that its solutions satisfy $\|x(t)\| \leq k\mathrm{e}^{-\mu t}\|x(0)\|$ for some $k, \mu > 0$. Then, show that if $f(x) = Ax$, the condition (1) is equivalent to

$$A^T P + PA \leq -Q\,. \tag{2}$$

   Hints: You can use the fact that, for any positive definite matrix N, there exists $c, \rho > 0$ such that $cI - N$ and $N - \rho I$ are positive definite. Further, you can use Gronwall's Lemma, which states that for differentiable functions $g, h : \mathbb{R} \to \mathbb{R}$ it holds that

$$\frac{dg(t)}{dt} \leq h(t)g(t) \quad \forall t \in \mathbb{R} \implies g(t) \leq g(0)e^{\int_0^t h(s)ds}, \quad \forall t \in \mathbb{R}\,.$$

2. [10 points] As in the midterm: consider the sequence $\{x(k)\}_{k=0}^{\infty} \subseteq \mathbb{R}^n$ defined inductively by $x(k+1) = f(x(k))$ for some $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, starting with some $x(0) \in \mathbb{R}^n$. The so-called *Lyapunov's second method* for the discrete-time system $x(k+1) = f(x(k))$ states that if there exists some function $V(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$ such that $V(0) = 0$, $V(x) > 0$ for all $x \neq 0$ and $\Delta V(x) := V(f(x)) - V(x) < 0$ for all $x \neq 0$, then $\lim_{t\to\infty} x(t) = 0$ for all $x(0) \in \mathbb{R}^n$. Based on this insight, derive the counterpart of the Lyapunov equation (2) for the sequence generated by $x(k+1) = Ax(k)$.

3. Salmons give birth when they reach a certain age $n \in \mathbb{N}$. The spawning process involves returning to their home-stream, making eggs and digging a nest. Since this requires a significant amount of energy, they die soon after, still at age n. Let $x_i(k) \in \mathbb{R}$ represent a measure quantifying the number of salmons of age i alive at year k, where $k = 1, \ldots, n$. Let us denote the percentage of salmons of age j that survive to the next year as $s_j \in (0, 1)$, where $j = 0, \ldots n-1$. Moreover, let $F \in \mathbb{R}_{>0}$ denote the fertility rate of salmons of age n, that is $Fx_n(t)$ new salmons are born every year (but $(100 \times s_0)\%$ of them immediately die). Specifically, we can write the system as follows:

$$\begin{cases} x_1(k+1) = s_0 F x_n(k)\,, \\ x_2(k+1) = s_1 x_1(k)\,, \\ \vdots \\ x_n(k+1) = s_{n-1} x_{n-1}(k)\,. \end{cases}$$

   a. [10 points] Let $x(k) = \left[x_1^{\mathsf{T}}(k), \ldots, x_n^{\mathsf{T}}(k)\right]^{\mathsf{T}}$. Using your result from part 2, determine conditions on the parameters $s_0, s_1, \ldots, s_n, F$ such that the salmons go extinct, that is, $x(k) \to 0$ for $k \to \infty$.
   Hint: Letting $V(x(k)) = x(k)^{\mathsf{T}} P x(k)$, find conditions such that there exists $P \succ 0$ that guarantees

$$V(x(k+1)) - V(x(k)) = -x(k)^{\mathsf{T}} x(k)\,, \quad \forall k \in \mathbb{N}\,.$$

b. [5 points] Using your preferred method, determine a set of values $s_0, \dots, s_{n-1}, F$ such that salmons will not go extinct, but will not grow unbounded either. Explain your reasoning. Recall that $0 < s_j < 1$ for every $j = 0, \dots, n-1$ and $F > 0$.

c. [5 **bonus** points] Assuming that $n = 3$ and starting from an initial amount of salmons $x(0) \in \mathbb{R}^3$, plot the salmon trajectory over the years for choices of parameters such that i) salmons go extinct, ii) salmons do not go extinct and do not grow unbounded either, and iii) salmons grow unbounded. Explain your reasoning.

Solution 1.1

Exercise 2, (LTV stability [30 points in total])

1. [15 points] Consider a time-varying matrix $A(t)$ in the form

$$A(t) = \begin{bmatrix} \lambda & \beta(t) \\ 0 & \lambda \end{bmatrix}, \tag{3}$$

where $\lambda \in \mathbb{R}$ and $\beta \colon \mathbb{R} \to \mathbb{R}$ is given by $\beta(t) = (1-\lambda)\, e^{(1-\lambda)\, t}$. Show that even if the eigenvalues of $A(t)$ in (3) have strictly negative real part, the dynamical system $\dot{x}(t) = A(t)\, x(t)$ can be unstable.

2. [15 points] Consider the linear time-varying system $\dot{x}(t) = A(t)\, x(t)$, where $A(t) = A^T(t) \in \mathbb{R}^{n \times n}$ and $\lambda(t) \leq -\epsilon < 0$ for all $t \geq t_0 \in \mathbb{R}$ and all $\lambda(t) \in \mathrm{Spec}[A(t)]$. Show that this system is asymptotically stable.

   Hint: Consider the function $V(x) = x^T x$ and use Gronwall's Lemma (see Hint in Exercise 1 part 1).

Exercise 3, (Inner Product Spaces, [25 points in total])
Let $\mathcal{H}$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal collection of vectors in $\mathcal{H}$, i.e., $\|v_i\| = 1$ for all $i \in \{1, 2, ..., n\}$, and $\langle v_i, v_j \rangle = 0$ for all $j \neq i$.

1. [15 points] Prove that, for all $f \in \mathcal{H}$,

$$\sum_{i=1}^{n} |\langle f, v_i \rangle|^2 \leq \|f\|^2 .$$

   Hint. For all $f \in \mathcal{H}$, consider $g = \sum_{i=1}^{n} \langle f, v_i \rangle v_i \in \mathcal{H}$.

2. [10 points] Assume now that $\mathcal{H}$ is finite dimensional and let $\{b_1, b_2, ..., b_n\}$ be an orthonormal basis for $\mathcal{H}$. Prove that, for all $f \in \mathcal{H}$,

$$\sum_{i=1}^{n} |\langle f, b_i \rangle|^2 = \|f\|^2 .$$

Linear System Theory
Hints for Problem Set 4

The following results would be useful.

Fact 1: Consider the system $\dot{x}(t) = f(x(t))$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz. For any differentiable function $V : \mathbb{R}^n \to \mathbb{R}$, by using the chain rule, we get that $\frac{dV(x(t))}{dt} = \dot{x}(t)^\top \frac{\partial V(x(t))}{\partial x} = f(x(t))^\top \frac{\partial V(x(t))}{\partial x}$.

Fact 2: For any symmetric square matrix $A \in \mathbb{R}^{n\times n}$, it holds that $\lambda_{\min}(A)\|x\|_2^2 \leq x^\top Ax \leq \lambda_{max}(A)\|x\|_2^2$, where $\lambda_{\min}(A)$ and $\lambda_{max}(A)$ denote the minimum and maximum eigenvalues of A, respectively. **Try to prove this claim on your own**. To this end, remember that a square matrix A is symmetric and positive semidefinite (resp. positive definite) if and only if all its eigenvalues are nonnegative (resp. strictly positive).

Fact 3: A symmetric positive definite $n \times n$ matrix $P = P^\top > 0$ defines an inner product in $\mathbb{R}^n$ by $\langle x, y\rangle_P := x^\top Py$, for all $x, y \in \mathbb{R}^n$. **Try to prove this claim on your own**[1], by verifying the conditions in Definition 7.1. Therefore, the map $\|\cdot\|_P : \mathbb{R}^n \to \mathbb{R}_+$ defined by $\|x\|_P = \sqrt{x^\top Px}$ is a norm in $\mathbb{R}^n$ by Theorem 7.1. In particular, since $\mathbb{R}^n$ is finite-dimensional, for any $P = P^\top > 0$ and $Q = Q^\top > 0$, the norms $\|\cdot\|_P$, $\|\cdot\|_Q$ and $\|\cdot\|_2$ are equivalent. Thus for example, there exist positive constants[2] $m, M > 0$ such that $m\|x\|_P \leq \|x\|_Q \leq M\|x\|_P$, for all $x \in \mathbb{R}^n$.

Exercise 1 (Lyapunov stability and salmon extinction)

(1) Use Fact 3 and Gronwalll's Lemma. For the second part, use Fact 1 for $V(x) = x^\top Px$ and $f(x) = Ax$.
(2) Use $V(x) = x^\top Px$ and $f(x) = Ax$.

Exercise 2 (LTV stability)

(1) Solve analytically the ODE and show that even if $\mathrm{Re}(\lambda) < 0$ the system is unstable.
(2) Consider $V(x) = x^\top x$. Then, use Fact 1, Fact 2 and Gronwall's Lemma.

Exercise 3 (Inner product spaces)

(1) For a fixed $f \in \mathcal{H}$, consider $g = \sum_{i=1}^n \langle v_i, f\rangle v_i$. Argue that this is an element in $\mathcal{H}$. Then show that $\langle g, g - f\rangle = 0$. Finally, use the Pythagoras Theorem 7.2 twice.

[1] We consider the vector field to be $\mathbb{R}$. That is, condition 4 in Definition 7.1 is $\langle x, y\rangle = \langle y, x\rangle$. Moreover $\langle ax, y\rangle = a\langle x, y\rangle$, for all $a \in \mathbb{R}$

[2] You can explicity determine the constants in the corresponding inequalities, by using the fact that for any positive definite matrix N, there exist $c, \rho > 0$, such that $cI - N$ and $N - \rho I$ are positive definite.

(2) First show that if $\mathcal{B} = \{b_1, \ldots, b_n\}$ is an orthonormal basis for $\mathcal{H}$, then every $f \in \mathcal{H}$ is written as $f = \sum_{i=1}^{n} \langle b_i, f \rangle b_i$.