Linear System Theory
Problem Set 2
Normed Spaces, ODEs, and Linear Time-Varying Systems
Issue date: October 7, 2019
Due date: October 21, 2019

Exercise 1. (Norms, [45 points in total])

1. [15 **points**] Let $C([t_0, t_1], \mathbb{R}^n)$ be the space of all continuous functions from $[t_0, t_1]$ to $\mathbb{R}^n$. Prove that for $f \in C([t_0, t_1], \mathbb{R}^n)$, $\|f\|_\infty := \max_{t \in [t_0, t_1]} \|f(t)\|_p$ satisfies the axioms of the norm, where $\|x\|_p$ is the p-norm of $x \in \mathbb{R}^n$.

2. [10 **points**] Given a matrix $A \in \mathbb{R}^{m \times n}$, verify that the induced matrix norms $\|A\|_2, \|A\|_\infty$ are equivalent, by showing that they satisfy the following inequalities:

$$\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty.$$

Hint: The induced p-norm of a matrix A is given by:

$$\|A\|_p = \sup_{\|x\|_p \neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

3. [20 **points**] Consider a set of functions f_n in $C([0,1], \mathbb{R})$ defined as:

$$f_n : [0,1] \to \mathbb{R} \quad \text{s.t.} \quad f_n(x) = \begin{cases} n - n^2 x & 0 \leq x \leq \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases}$$

for $n \in \mathbb{N}$. Compute the 1-norm, 2-norm and ∞-norm of the functions for each n, as defined below:

$$\|f\|_1 := \int_0^1 |f(x)|\, dx\,, \quad \|f\|_2 := \sqrt{\int_0^1 |f(x)|^2\, dx}\,, \quad \|f\|_\infty := \max_{t \in [0,1]} |f(t)|.$$

Based on your computations, what can you say about equivalence of these norms?

Solution 1.1

- $\forall f, g \in C([t_0, t_1], \mathbb{R}^n), \|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

$$\begin{aligned} \forall f, g \in C([t_0, t_1], \mathbb{R}^n), \|f+g\|_\infty &= \max_{t \in [t_0, t_1]} \|f + g\|_p \\ &\leq \max_{t \in [t_0, t_1]} \{\|f\|_p + \|g\|_p\} \\ &\leq \max_{t \in [t_0, t_1]} \|f\|_p + \max_{t \in [t_0, t_1]} \|g\|_p \\ &= \|f\|_\infty + \|g\|_\infty \end{aligned}$$

- $\forall f \in C([t_0, t_1], \mathbb{R}^n), \forall a \in \mathbb{R}, \|af\| = a\|f\|$

$$\begin{aligned} \forall f \in C([t_0, t_1], \mathbb{R}^n), \forall a \in \mathbb{R}, \|af\| &= \max_{t \in [t_0, t_1]} \|af\|_p \\ &= \max_{t \in [t_0, t_1]} a\|f\|_p \\ &= a \max_{t \in [t_0, t_1]} \|f\|_p \\ &= a\|f\|_\infty \end{aligned}$$

- $\|f\| = 0 \Leftrightarrow f = 0$
  $\Rightarrow$: $\|f\| = 0 \Rightarrow \max_{t\in[t_0,t_1]} \|f\|_p = 0,\ \|f\|_p \geq 0 \Rightarrow \|f\|_p = 0, \forall t \in [t_0, t_1] \Rightarrow f = 0$
  $\Leftarrow$: $f = 0 \Rightarrow \|f\|_p = 0, \forall t \in [t_0, t_1] \Rightarrow \max_{t\in[t_0,t_1]} \|f\|_p = 0 \Rightarrow \|f\| = 0$

Solution 1.2
Define x_{max} and $(Ax)_{max}$ to simplify further notation:

$$\begin{aligned} x_{max} &= \max_{i\in\{1,\dots,n\}} x_i \\ (Ax)_{max} &= \max_{j\in\{1,\dots,m\}} (Ax)_j \end{aligned}$$

Then, start with $\|A\|_2^2$:

$$\begin{aligned} \|A\|_2^2 &= \left(\sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}\right)^2 = \sup_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_i^2}{\sum_{i=1}^n x_i^2} \geq \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_i^2}{\sum_{i=1}^n x_{max}^2} \\ &= \sup_{x\neq 0} \frac{1}{n} \frac{\sum_{i=1}^m (Ax)_i^2}{x_{max}^2} \geq \sup_{x\neq 0} \frac{1}{n} \frac{(Ax)_{max}^2}{x_{max}^2} = \sup_{x\neq 0} \frac{1}{n} \left(\frac{(Ax)_{max}}{x_{max}}\right)^2 \\ &= \frac{1}{n} \left(\sup_{x\neq 0} \frac{(Ax)_{max}}{x_{max}}\right)^2 = \frac{1}{n} \left(\sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}\right)^2 = \frac{1}{n} \|Ax\|_\infty \\ &\Rightarrow \|A\|_2 \geq \frac{1}{\sqrt{n}} \|Ax\|_\infty \end{aligned}$$

$$\begin{aligned} \|A\|_2^2 &= \left(\sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}\right)^2 = \sup_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_i^2}{\sum_{i=1}^n x_i^2} \leq \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_{max}^2}{\sum_{i=1}^n x_i^2} \\ &= \sup_{x\neq 0} \frac{m\,(Ax)_{max}^2}{\sum_{i=1}^n x_i^2} = m \sup_{x\neq 0} \frac{(Ax)_{max}^2}{\sum_{i=1}^n x_i^2} \leq m \sup_{x\neq 0} \frac{(Ax)_{max}^2}{x_{max}^2} = m \sup_{x\neq 0} \left(\frac{(Ax)_{max}}{x_{max}}\right)^2 \\ &= m \left(\sup_{x\neq 0} \frac{(Ax)_{max}}{x_{max}}\right)^2 = m \left(\sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}\right)^2 = m \|Ax\|_\infty \\ &\Rightarrow \|A\|_2 \leq \sqrt{m} \|Ax\|_\infty \end{aligned}$$

Solution 1.3
Although $n \in \mathbb{N}$ can be any non-negative integer, we additionally require $n > 0$ to make sure the piece-wise function $f_0(x)$ is well defined, otherwise $\frac{1}{0}$ does not make sense.

$$\forall x \in \left[0, \frac{1}{n}\right], \forall n \in \mathbb{N}, n - n^2 x \geq 0 \Rightarrow f_n(x) \geq 0, \forall x \in [0,1] \Rightarrow |f_n(x)| = f_n(x), \forall x \in [0,1]$$

- $\|f\|_1$

$$\|f\|_1 = \int_0^1 |f_n(x)|\,\mathrm{d}x = \int_0^1 f_n(x)\,\mathrm{d}x = \int_0^{\frac{1}{n}} \left(n - n^2 x\right)\,\mathrm{d}x = nx - \frac{1}{2} n^2 x^2 \bigg|_0^{\frac{1}{n}} = \frac{1}{2}$$

- $\|f\|_2$

$$\begin{aligned} \|f\|_2^2 &= \int_0^1 |f_n(x)|^2\,\mathrm{d}x = \int_0^{\frac{1}{n}} \left(n - n^2 x\right)^2\,\mathrm{d}x \\ &= \int_0^{\frac{1}{n}} \left(n^4 x^2 - 2n^3 x + n^2\right)\,\mathrm{d}x = \frac{1}{3} n^4 x^3 - n^3 x^2 + n^2 x \bigg|_0^{\frac{1}{n}} = \frac{1}{3} n \\ \|f\|_2 &= \sqrt{\frac{1}{3} n} \end{aligned}$$

- $\|f\|_\infty$

$$\|f\|_\infty = \max_{t\in[0,1]} |f_n(t)| = \max_{t\in[0,1/n]} n - n^2 t = n$$

- Equivalence of $\|f_n\|_1$ and $\|f_n\|_\infty$

$$\exists m_u = 4n \geq m_l = n \geq 0, \forall f_n \in C([0,1],\mathbb{R}), m_l\|f_n\|_1 = \frac{1}{2}n \leq n = \|f_n\|_\infty \leq 2n = m_u\|f_n\|_1$$

- Equivalence of $\|f_n\|_2$ and $\|f_n\|_\infty$

$$\exists m_u = \sqrt{27n} \geq m_l = \sqrt{\frac{1}{3}n} \geq 0, \forall f_n \in C([0,1],\mathbb{R}), m_l\|f_n\|_2 = \frac{1}{3}n \leq n = \|f_n\|_\infty \leq 3n = m_u\|f_n\|_2$$

Therefore, $\|f_n\|_1$, $\|f_n\|_2$ and $\|f_n\|_\infty$ are equivalent.

Exercise 2. (Banach fixed point theorem [25 points in total])

1. [20 **points**] Let $(X, \|\cdot\|)$ be a Banach space, and $f : X \to X$. Assume that there exists $\alpha \in [0,\, 1)$ such that, for all $x, y \in X$,

$$\|f(x) - f(y)\| \leq \alpha \, \|x - y\| \, .$$

   Show that there exists a unique point $\bar{x}$ such that $f(\bar{x}) = \bar{x}$.

   Hint: Given an arbitrary initial point x, consider the sequence of iterates $f^{[n]}(x) = f(f^{[n-1]}(x))$, where the first iterate is given by $f^{[0]}(x) = x$. You can start by showing that this sequence is Cauchy.

2. [5 **points**] Now assume f is a linear map. Given the condition in the first part of Exercise 2, what can you conclude about the induced norm of f?

Exercise 3. (Ordinary differential equations [30 points in total])

1. [12 **points**] Consider the following ordinary differential equation (ODE)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -x_1(t) + e^t \, \cos(x_1(t) - x_2(t)) \\ -x_2(t) + \sin(x_1(t) - x_2(t)) \end{bmatrix},$$

   where $x_i(t) \in \mathbb{R}$, $\forall i$. Prove or disprove the following statements:

   (a) This system is globally Lipschitz,

   (b) This system admits a unique solution.
   Hint: You may assume that functions with bounded derivatives are Lipschitz.

2. [18 **points**] Consider the following ordinary differential equation (ODE)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3\sin(t)\, x_1(t) + x_1(t)\, x_2(t) \\ -2\, x_2(t) \end{bmatrix},$$

   where $x_i(t) \in \mathbb{R}$, $\forall i$. Prove or disprove the following statements:

   (a) This system is globally Lipschitz,

   (b) This system admits a unique solution.