Linear System Theory

Problem Set 1
Linear Spaces, Linear Maps, and Representations

Issue date: September 19, 2019
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Exercise 1. (Linear spaces [40 points])

1. [18 **points**] Let S be a set, and $F = \{f : S \to \mathbb{R}_+\}$ be the space of functions from S to the (strictly) positive reals. Let the operations $\oplus : F \times F \to F$, $\odot : \mathbb{R} \times F \to F$ be defined as follows:

$$[f_1 \oplus f_2](x) = f_1(x)f_2(x) \qquad \forall f_1, f_2 \in F, \forall x \in S$$
$$[\alpha \odot f](x) = f(x)^\alpha \qquad \forall \alpha \in \mathbb{R}, \forall f \in F, \forall x \in S$$

   - Show that $(F, \mathbb{R}, \oplus, \odot)$ is a linear space.
   - Identify the zero-vector.

2. [12 **points**] Let $S = \{a, b\}$, and let

$$f_1(a) = 2,\ \ f_1(b) = 1$$
$$f_2(a) = 1,\ \ f_2(b) = 3$$
$$f_3(a) = 4,\ \ f_3(b) = 1$$

   Show that $\{f_1, f_2\}$ are linearly independent and that $\{f_1, f_3\}$ are linearly dependent.

3. [10 **points**] Let $\varphi : F \to F$ be defined as follows:

$$[\varphi(f)](x) = \sqrt{f(x)} \qquad \forall f \in F, \forall x \in S$$

   Show that φ is a linear map over the space F on $(F, \mathbb{R}, \oplus, \odot)$.

Solution 1.1

- **vector addition**
  - **associative**:
    $\forall f_1, f_2, f_3 \in F, f_1 \oplus (f_2 \oplus f_3) = f_1 \oplus (f_2 f_3) = f_1(f_2 f_3) = f_1 f_2 f_3 = (f_1 f_2) f_3 = (f_1 \oplus f_2) \oplus f_3$
  - **commutative**:
    $\forall f_1, f_2 \in F, f_1 \oplus f_2 = f_1 f_2 = f_2 f_1 = f_2 \oplus f_1$
  - **identity**:
    Define $\theta(x) = 1, \forall x \in S$
    $F = \{f : S \to \mathbb{R}_+\} \Rightarrow \theta(x) \in F$
    $\forall f \in F, f \oplus \theta = f \cdot 1 = f$
  - **inverse**:
    $\forall f \in F, f(x) \in \mathbb{R}_+, f(x) > 0, f^-(x) = \frac{1}{f(x)} > 0, f^- \in F, f \oplus f^-(x) = f(x) \cdot \frac{1}{f(x)} = 1$
- **scalar multiplication**
  - **associative:**
    $\forall a, b \in \mathbb{R}, \forall f \in F, a \odot (b \odot f) = a \odot f^b = (f^b)^a = f^{ab} = (a \cdot b) \odot f$
  - **identity:**
    $\forall f \in F, 1 \odot f = f^1 = f$
- **distributive scalar multiplication**
  - $\forall a, b \in \mathbb{R}, \forall f \in F, (a + b) \odot f = f^{a+b} = f^a f^b = (f^a)(f^b) = (a \odot f) \oplus (b \odot f)$
  - $\forall a \in \mathbb{R}, \forall f_1, f_2 \in F, a \odot (f_1 \oplus f_2) = a \odot (f_1 f_2) = (f_1 f_2)^a = f_1^a f_2^a = (a \odot f_1) \oplus (a \odot f_2)$

Additionally, the addition and multiplication are closed in space F. Therefore, $(F, \mathbb{R}, \oplus, \odot)$ is a linear space and the zero-vector is 1.

Solution 1.2

- $\{f_1, f_2\}$
  Assume, for the sake of contradiction, $\{f_1, f_2\}$ are linearly dependent such that
  $\exists c_1, c_2 \in \mathbb{R}$ which are not both 0 such that $(c_1 \odot f_1(x)) \oplus (c_2 \odot f_2(x)) = 1$
  $x = a$: $(c_1 \odot 2) \oplus (c_2 \odot 1) = 1 \Rightarrow 2^{c_1} \cdot 1^{c_2} = 2^{c_1} = 1 \Rightarrow c_1 = 0$
  $x = b$: $(0 \odot 1) \oplus (c_2 \odot 3) = 1 \Rightarrow 2^0 \cdot 3^{c_2} = 3^{c_2} = 1 \Rightarrow c_2 = 0$
  $c_1 = c_2 = 0$ contradicts our assumption $\Rightarrow \{f_1, f_2\}$ are linearly independent
- $\{f_1, f_3\}$
  $\{f_1, f_3\}$ are linearly dependent, if we can find $c_1, c_2 \in \mathbb{R}$ that are not both 0 such that
  $(c_1 \odot f_1(x)) \oplus (c_2 \odot f_3(x)) = 1$
  $x = a$: $(c_1 \odot 2) \oplus (c_2 \odot 4) = 1 \Rightarrow 2^{c_1} \cdot 4^{c_2} = 2^{c_1 + 2c_2} = 1$
  $x = b$: $(c_1 \odot 1) \oplus (c_2 \odot 1) = 1 \Rightarrow 1^{c_1} \cdot 1^{c_2} = 1^{c_1 + c_2} = 1$
  $1^{c_1+c_2} = 1$ implies c_1, c_2 can be any real number. We can choose $c_1 = 2, c_2 = -1$ to satisfy the two conditions above. Therefore, $\{f_1, f_3\}$ are linearly dependent.

Solution 1.3

$\forall a_1, a_2 \in \mathbb{R}, \forall f_1, f_2 \in F,$

$$\begin{aligned}
\varphi\left((a_1 \odot f_1) \oplus (a_2 \odot f_2)\right) &= \varphi({f_1}^{a_1} {f_2}^{a_2}) = \sqrt{{f_1}^{a_1} {f_2}^{a_2}} = \sqrt{{f_1}^{a_1}} \sqrt{{f_2}^{a_2}} \\
&= \sqrt{f_1}^{a_1} \sqrt{f_2}^{a_2} = \left(a_1 \odot \sqrt{f_1}\right)\left(a_2 \odot \sqrt{f_2}\right) \\
&= (a_1 \odot \varphi(f_1))\,(a_2 \odot \varphi f_2) = (a_1 \odot \varphi(f_1)) \oplus (a_2 \odot \varphi(f_2))
\end{aligned}$$

Therefore, φ is a linear map over the space F on $(F, \mathbb{R}, \oplus, \odot)$.

Exercise 2. (Range and null space [40 points])
Let $(F, +, \cdot)$ be a field and consider the linear maps $\mathcal{A} : (F^n, F) \to (F^m, F)$ and $\mathcal{B} : (F^m, F) \to (F^p, F)$. Show, without using the matrix representation of linear maps, that:

1. [10 **points**] $0 \leq dim(\textsc{Range}(\mathcal{A})) \leq \min\{m, n\}$.
2. [15 **points**] $dim(\textsc{Range}(\mathcal{A})) + dim(\textsc{Range}(\mathcal{B})) - m \leq dim(\textsc{Range}(\mathcal{B} \circ \mathcal{A}))$
3. [15 **points**] $dim(\textsc{Range}(\mathcal{B} \circ \mathcal{A})) \leq \min\{dim(\textsc{Range}(\mathcal{A})), dim(\textsc{Range}(\mathcal{B}))\}$.

Solution 2.1
The left inequality is trivial since the number of vectors of a basis should definitely be non-negative, i.e., $0 \leq dim(\textsc{Range}(\mathcal{A}))$. For the right inequality, we first recall a lemma.

Lemma 1 *If (V, F) has dimension n then any set of $n+1$ or more vectors is linearly dependent.*

Next, we prove $dim(\textsc{Range}(\mathcal{A})) \leq m$. Assume, for the sake of contradiction, $dim(\textsc{Range}(\mathcal{A})) = p > m$, and a basis of $\textsc{Range}(\mathcal{A})$ is $\{v_1, \dots, v_p\}$, which are linearly independent by definition.
The fact that linearly independent vectors $\{v_1, \dots, v_p\} \subseteq \textsc{Range}(\mathcal{A}) \subseteq (V, F)$ and the dimension of (V, F) is $m < p$ contradict the lemma. Therefore $\textsc{Range}(\mathcal{A}) \leq m$.

Finally, we prove $dim(\textsc{Range}(\mathcal{A})) \leq n$. $\forall v \in \textsc{Range}(\mathcal{A}), \exists u \in U, \mathcal{A}(u) = v$. Let $\{e_1, \dots, e_n\}$ be a basis of U. $\forall u \in U, \exists a_1, \dots, a_n, u = a_1 e_1 + \cdots + a_n e_n$, where a_i are not all 0.
Therefore, $\forall v \in \textsc{Range}(\mathcal{A}), v = \mathcal{A}(u) = \mathcal{A}(a_1 e_1 + \cdots + a_n e_n) = a_1 \mathcal{A}(e_1) + \cdots + a_n \mathcal{A}(e_n)$. If $\{\mathcal{A}(e_1), \dots, \mathcal{A}(e_n)\}$ are linearly independent, then $dim(\textsc{Range}(\mathcal{A})) = n$, otherwise $dim(\textsc{Range}(\mathcal{A})) < n$.

$$\begin{cases} dim(\textsc{Range}(\mathcal{A})) \leq n \\ dim(\textsc{Range}(\mathcal{A})) \leq m \end{cases} \Rightarrow dim(\textsc{Range}(\mathcal{A})) \leq \min\{m, n\}, \text{ which completes the proof.}$$

Solution 2.2
Define $\tilde{\mathcal{B}}$ as a restriction of $\mathcal{B}$, $\tilde{\mathcal{B}} : \textsc{Range}(\mathcal{A}) \to F^p$
$\textsc{Range}(A) \subseteq F^m \Rightarrow \textsc{Null}(\tilde{\mathcal{B}}) \subseteq \textsc{Null}(\mathcal{B})$
Further, $\textsc{Range}(\tilde{\mathcal{B}}) = \textsc{Range}(\mathcal{B} \circ \mathcal{A})$ by construction

$$\begin{aligned} dim(\textsc{Range}(\mathcal{A})) &= dim(\textsc{Null}(\tilde{\mathcal{B}})) + dim(\textsc{Range}(\tilde{\mathcal{B}})) \\ &\leq dim(\textsc{Null}(\mathcal{B})) + dim(\textsc{Range}(\tilde{\mathcal{B}})) \\ &= dim(\textsc{Null}(\mathcal{B})) + dim(\textsc{Range}(\mathcal{B} \circ \mathcal{A})) \\ &= m - dim(\textsc{Range}(\mathcal{B})) + dim(\textsc{Range}(\mathcal{B} \circ \mathcal{A})) \end{aligned}$$

Therefore, $dim(\textsc{Range}(\mathcal{A})) + dim(\textsc{Range}(\mathcal{B})) - m \leq dim(\textsc{Range}(\mathcal{B} \circ \mathcal{A}))$.

Solution 2.3
$\textsc{Range}(\mathcal{B} \circ \mathcal{A}) \subseteq \textsc{Range}(\mathcal{B}) \Rightarrow dim(\textsc{Range}(\mathcal{B} \circ \mathcal{A})) \leq dim(\textsc{Range}(\mathcal{B}))$
Let $\{v_1, \dots, v_r\}$ be a set of basis of $\textsc{Range}(\mathcal{A})$.
$\textsc{Range}(\mathcal{B} \circ \mathcal{A}) = \{l \in F^p | \exists w \in \textsc{Range}(\mathcal{A}), \mathcal{B}(w) = l\}$.
$\textsc{Range}(\mathcal{A}) = span(\{v_1, \dots, v_r\}) \Rightarrow \forall w \in \textsc{Range}(\mathcal{A}), \exists c_1, \dots, c_r \in F, w = c_1 v_1 + \cdots + c_r v_r$
$\mathcal{B}(w) = c_1 \mathcal{B}(v_1) + \cdots + c_r \mathcal{B}(v_r)$.
$\forall l \in F^p, \exists w \in \textsc{Range}(\mathcal{A}), \exists c_1, \dots, c_r, l = \mathcal{B}(w) = \mathcal{B}(c_1 v_1 + \cdots + c_r v_r) = c_1 \mathcal{B}(v_1) + \cdots + c_r \mathcal{B}(v_r)$
If $\{\mathcal{B}(v_1), \dots, \mathcal{B}(v_r)\}$ are linearly independent, $dim(\textsc{Range}(\mathcal{B} \circ \mathcal{A})) = r = dim(\textsc{Range}(\mathcal{A}))$, otherwise $dim(\textsc{Range}(\mathcal{B} \circ \mathcal{A})) < r = dim(\textsc{Range}(\mathcal{A}))$.

$$\begin{cases} dim(\textsc{Range}(\mathcal{B} \circ \mathcal{A})) \leq dim(\textsc{Range}(\mathcal{A})) \\ dim(\textsc{Range}(\mathcal{B} \circ \mathcal{A})) \leq dim(\textsc{Range}(\mathcal{B})) \end{cases} \Rightarrow dim(\textsc{Range}(\mathcal{B} \circ \mathcal{A})) \leq \min\{dim(\textsc{Range}(\mathcal{A})), dim(\textsc{Range}(\mathcal{B}))\}$$

Exercise 3. (Linear maps and matrix representations [20 points])
Consider a linear map $\mathcal{A} : (U, F) \to (U, F)$ where U has finite dimension n.

1. [10 **points**] Assume there exists a basis ν_i, $i = 1, \dots, n$ for U such that $\mathcal{A}(\nu_n) = \lambda \nu_n$ and $\mathcal{A}(\nu_i) = \lambda \nu_i + \nu_{i+1}$, $i = 1, \dots, n-1$. Derive the representation of $\mathcal{A}$ with respect to this basis.
2. [10 **points**] Assume there exists a vector $b \in U$ such that the set $\{b, \mathcal{A}(b), \mathcal{A} \circ \mathcal{A}(b), \dots, \mathcal{A}^{n-1}(b)\}$ is linearly independent. Derive the representation of $\mathcal{A}$ with respect to this basis.

Solution 3.1

$$\begin{aligned}
y_1 = \mathcal{A}(\nu_1) &= \lambda \nu_1 + \nu_2 \Rightarrow a_{11} = \lambda, a_{21} = 1 \\
y_2 = \mathcal{A}(\nu_2) &= \lambda \nu_2 + \nu_3 \Rightarrow a_{22} = \lambda, a_{32} = 1 \\
&\vdots \\
y_{n-1} = \mathcal{A}(\nu_{n-1}) &= \lambda \nu_{n-1} + \nu_n \Rightarrow a_{n-1,n-1} = \lambda, a_{n,n-1} = 1 \\
y_n = \mathcal{A}(\nu_n) &= \lambda \nu_n \Rightarrow a_{nn} = \lambda \\
\Rightarrow A &= \begin{bmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & 1 & \ddots & & \\ & & \ddots & \lambda & \\ & & & 1 & \lambda \end{bmatrix}
\end{aligned}$$

Solution 3.2

$\mathcal{A} : (U, F) \to (U, F) \Rightarrow \mathcal{A}^n(b) \in U$.
Since $\{b, \mathcal{A}(b), \mathcal{A} \circ \mathcal{A}(b), \dots, \mathcal{A}^{n-1}(b)\}$ are linearly independent and $dim(U) = n$, $\exists c_1, \dots, c_n \in F$, which are not all zero, such that $\mathcal{A}^n(b) = c_1 b + \cdots + c_n \mathcal{A}^{n-1}(b)$

$$\begin{aligned}
y_1 = \mathcal{A}(b) &= \mathcal{A}(b) \Rightarrow a_{21} = 1 \\
y_2 = \mathcal{A}(\mathcal{A}(b)) &= \mathcal{A}^2(b) \Rightarrow a_{32} = 1 \\
&\vdots \\
y_{n-1} = \mathcal{A}(\mathcal{A}^{n-2}(b)) &= \mathcal{A}^{n-1}(b) \Rightarrow a_{n,n-1} = 1 \\
y_n = \mathcal{A}(\mathcal{A}^{n-1}(b)) &= \mathcal{A}^n(b) \Rightarrow a_{jn} = c_j, j = 1, \dots, n \\
\Rightarrow A &= \begin{bmatrix} & & & & c_1 \\ 1 & & & & \vdots \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & c_n \end{bmatrix}
\end{aligned}$$