Linear System Theory

Problem Set 2
Normed Spaces, ODEs, and Linear Time-Varying Systems

Issue date: October 7, 2019
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Exercise 1. (Norms, [45 points in total])

1. [15 **points**] Let $C([t_0,t_1],\mathbb{R}^n)$ be the space of all continuous functions from $[t_0,t_1]$ to $\mathbb{R}^n$. Prove that for $f \in C([t_0,t_1],\mathbb{R}^n)$, $\|f\|_\infty := \max_{t\in[t_0,t_1]} \|f(t)\|_p$ satisfies the axioms of the norm, where $\|x\|_p$ is the p-norm of $x \in \mathbb{R}^n$.

2. [10 **points**] Given a matrix $A \in \mathbb{R}^{m\times n}$, verify that the induced matrix norms $\|A\|_2, \|A\|_\infty$ are equivalent, by showing that they satisfy the following inequalities:
$$\frac{1}{\sqrt{n}}\|A\|_\infty \le \|A\|_2 \le \sqrt{m}\|A\|_\infty.$$
Hint: The induced p-norm of a matrix A is given by:
$$\|A\|_p = \sup_{\|x\|_p\neq 0} \frac{\|Ax\|_p}{\|x\|_p}$$

3. [20 **points**] Consider a set of functions f_n in $C([0,1],\mathbb{R})$ defined as:
$$f_n : [0,1] \to \mathbb{R} \quad \text{s.t.} \quad f_n(x) = \begin{cases} n - n^2 x & 0 \le x \le \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases}$$
for $n \in \mathbb{N}$. Compute the 1-norm, 2-norm and ∞-norm of the functions for each n, as defined below:
$$\|f\|_1 := \int_0^1 |f(x)|\,dx\,, \quad \|f\|_2 := \sqrt{\int_0^1 |f(x)|^2\,dx}\,, \quad \|f\|_\infty := \max_{t\in[0,1]} |f(t)|.$$
Based on your computations, what can you say about equivalence of these norms?

Solution 1.1

- $\forall f, g \in C([t_0, t_1], \mathbb{R}^n), \|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

$$\begin{aligned}\forall f, g \in C([t_0, t_1], \mathbb{R}^n), \|f+g\|_\infty &= \max_{t\in[t_0,t_1]} \|f+g\|_p \\ &\leq \max_{t\in[t_0,t_1]} \{\|f\|_p + \|g\|_p\} \\ &\leq \max_{t\in[t_0,t_1]} \|f\|_p + \max_{t\in[t_0,t_1]} \|g\|_p \\ &= \|f\|_\infty + \|g\|_\infty\end{aligned}$$

- $\forall f \in C([t_0, t_1], \mathbb{R}^n), \forall a \in \mathbb{R}, \|af\| = a\|f\|$

$$\begin{aligned}\forall f \in C([t_0, t_1], \mathbb{R}^n), \forall a \in \mathbb{R}, \|af\| &= \max_{t\in[t_0,t_1]} \|af\|_p \\ &= \max_{t\in[t_0,t_1]} a\|f\|_p \\ &= a \max_{t\in[t_0,t_1]} \|f\|_p \\ &= a\|f\|_\infty\end{aligned}$$

- $\|f\| = 0 \Leftrightarrow f = 0$
  $\Rightarrow$: $\|f\| = 0 \Rightarrow \max_{t\in[t_0,t_1]} \|f\|_p = 0,\ \|f\|_p \geq 0 \Rightarrow \|f\|_p = 0, \forall t \in [t_0, t_1] \Rightarrow f = 0$
  $\Leftarrow$: $f = 0 \Rightarrow \|f\|_p = 0, \forall t \in [t_0, t_1] \Rightarrow \max_{t\in[t_0,t_1]} \|f\|_p = 0 \Rightarrow \|f\| = 0$

Solution 1.2

Define x_{max} and $(Ax)_{max}$ to simplify further notation:

$$x_{max} = \max_{i\in\{1,\ldots,n\}} x_i$$
$$(Ax)_{max} = \max_{j\in\{1,\ldots,m\}} (Ax)_j$$

Then, start with $\|A\|_2^2$:

$$\begin{aligned}\|A\|_2^2 &= \left(\sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}\right)^2 = \sup_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_i^2}{\sum_{i=1}^n x_i^2} \geq \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_i^2}{\sum_{i=1}^n x_{max}^2} \\ &= \sup_{x\neq 0} \frac{1}{n} \frac{\sum_{i=1}^m (Ax)_i^2}{x_{max}^2} \geq \sup_{x\neq 0} \frac{1}{n} \frac{(Ax)_{max}^2}{x_{max}^2} = \sup_{x\neq 0} \frac{1}{n} \left(\frac{(Ax)_{max}}{x_{max}}\right)^2 \\ &= \frac{1}{n} \left(\sup_{x\neq 0} \frac{(Ax)_{max}}{x_{max}}\right)^2 = \frac{1}{n} \left(\sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}\right)^2 = \frac{1}{n}\|A\|_\infty \\ &\Rightarrow \|A\|_2 \geq \frac{1}{\sqrt{n}}\|A\|_\infty\end{aligned}$$

$$\begin{aligned}\|A\|_2^2 &= \left(\sup_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2}\right)^2 = \sup_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} = \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_i^2}{\sum_{i=1}^n x_i^2} \leq \sup_{x\neq 0} \frac{\sum_{i=1}^m (Ax)_{max}^2}{\sum_{i=1}^n x_i^2} \\ &= \sup_{x\neq 0} \frac{m\,(Ax)_{max}^2}{\sum_{i=1}^n x_i^2} = m \sup_{x\neq 0} \frac{(Ax)_{max}^2}{\sum_{i=1}^n x_i^2} \leq m \sup_{x\neq 0} \frac{(Ax)_{max}^2}{x_{max}^2} = m \sup_{x\neq 0} \left(\frac{(Ax)_{max}}{x_{max}}\right)^2 \\ &= m \left(\sup_{x\neq 0} \frac{(Ax)_{max}}{x_{max}}\right)^2 = m \left(\sup_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}\right)^2 = m\|A\|_\infty \\ &\Rightarrow \|A\|_2 \leq \sqrt{m}\|A\|_\infty\end{aligned}$$

Solution 1.3

$$\forall x \in \left[0, \frac{1}{n}\right], \forall n \in \mathbb{N}, n - n^2 x \geq 0 \Rightarrow f_n(x) \geq 0, \forall x \in [0,1] \Rightarrow |f_n(x)| = f_n(x), \forall x \in [0,1]$$

Compute norms:

- $\|f\|_1$

$$\|f\|_1 = \int_0^1 |f_n(x)|\,\mathrm{d}x = \int_0^1 f_n(x)\,\mathrm{d}x = \int_0^{\frac{1}{n}} \left(n - n^2 x\right)\,\mathrm{d}x = nx - \frac{1}{2}n^2x^2\Big|_0^{\frac{1}{n}} = \frac{1}{2}$$

- $\|f\|_2$

$$\begin{aligned}\|f\|_2^2 &= \int_0^1 |f_n(x)|^2\,\mathrm{d}x = \int_0^{\frac{1}{n}} \left(n - n^2x\right)^2\,\mathrm{d}x \\ &= \int_0^{\frac{1}{n}} \left(n^4x^2 - 2n^3x + n^2\right)\,\mathrm{d}x = \frac{1}{3}n^4x^3 - n^3x^2 + n^2x\Big|_0^{\frac{1}{n}} = \frac{1}{3}n \\ \|f\|_2 &= \sqrt{\frac{1}{3}n}\end{aligned}$$

- $\|f\|_\infty$

$$\|f\|_\infty = \max_{t\in[0,1]} |f_n(t)| = \max_{t\in[0,1/n]} n - n^2 t = n$$

Equivalence test:

- Equivalence of $\|f_n\|_1$ and $\|f_n\|_\infty$

$$\forall m_l > 0, \exists f_n \in C([0,1],\mathbb{R}) \text{ and } n > \frac{1}{2m_l}, m_l\,\|f_n\|_\infty = m_l \cdot n > m_l \cdot \frac{1}{2m_l} = \frac{1}{2} = \|f_n\|_1$$

  Therefore, $\|f_n\|_1$ and $\|f_n\|_\infty$ are not equivalent.

- Equivalence of $\|f_n\|_2$ and $\|f_n\|_\infty$

$$\forall m_l > 0, \exists f_n \in C([0,1],\mathbb{R}) \text{ and } n > \frac{1}{3m_l^2}, m_l\,\|f_n\|_\infty = m_l \cdot n > \sqrt{\frac{1}{3n}} \cdot n = \sqrt{\frac{1}{3}n} = \|f_n\|_2$$

  Therefore, $\|f_n\|_1$ and $\|f_n\|_\infty$ are not equivalent.

- Equivalence of $\|f_n\|_1$ and $\|f_n\|_2$

$$\forall m_l > 0, \exists f_n \in C([0,1],\mathbb{R}) \text{ and } n > \frac{3}{4m_l^2}, m_l\,\|f_n\|_2 = m_l \cdot \sqrt{\frac{1}{3}n} > m_l \cdot \sqrt{\frac{1}{3} \cdot \frac{3}{4m_l^2}} = \frac{1}{2} = \|f_n\|_1$$

  Therefore, $\|f_n\|_1$ and $\|f_n\|_2$ are not equivalent.

To conclude, none of them is equivalent to another.

Exercise 2. (Banach fixed point theorem [25 points in total])

1. [20 **points**] Let $(X, \|\cdot\|)$ be a Banach space, and $f : X \to X$. Assume that there exists $\alpha \in [0, 1)$ such that, for all $x, y \in X$,
$$\|f(x) - f(y)\| \leq \alpha \|x - y\| .$$
Show that there exists a unique point $\bar{x}$ such that $f(\bar{x}) = \bar{x}$.

   Hint: Given an arbitrary initial point x, consider the sequence of iterates $f^{[n]}(x) = f(f^{[n-1]}(x))$, where the first iterate is given by $f^{[0]}(x) = x$. You can start by showing that this sequence is Cauchy.

2. [5 **points**] Now assume f is a linear map. Given the condition in the first part of Exercise 2, what can you conclude about the induced norm of f?

Solution 2.1

$$\begin{aligned}
f^{[0]}(x) &= x \\
f^{[1]}(x) &= f\left(f^{[0]}(x)\right) = f(x) \\
f^{[2]}(x) &= f\left(f^{[1]}(x)\right) = f(f(x)) \\
&\vdots \\
f^{[n]}(x) &= f\left(f^{[n-1]}(x)\right) = f(\cdots f(f(x)))
\end{aligned}$$

$\forall p \geq q \geq 0, p, q \in N :$

$$\begin{aligned}
\|f^{[p]}(x) - f^{[q]}(x)\| &= \left\|f\left(f^{[p-1]}(x)\right) - f\left(f^{[q-1]}(x)\right)\right\| \\
&\leq \alpha \left\|f^{[p-1]}(x) - f^{[q-1]}(x)\right\| \\
&\vdots \\
&\leq \alpha^q \left\|f^{[p-q]}(x) - f^{[0]}(x)\right\| \\
&= \alpha^q \left\|f^{[p-q]}(x) - x\right\|
\end{aligned}$$

$\forall k \in N :$

$$\begin{aligned}
\left\|f^{[k]}(x) - x\right\| &= \left\|f^{[k]}(x) - f(x) + f(x) - x\right\| \\
&\leq \left\|f^{[k]}(x) - f(x)\right\| + \|f(x) - x\| \\
&\leq \alpha \left\|f^{[k-1]}(x) - x\right\| + \|f(x) - x\| \\
&\leq \alpha^2 \left\|f^{[k-2]}(x) - x\right\| + (1 + \alpha) \|f(x) - x\| \\
&\vdots \\
&\leq \alpha^{k-1} \left\|f^{[1]}(x) - x\right\| + \left(1 + \alpha + \cdots + \alpha^{k-2}\right) \|f(x) - x\| \\
&\leq \left(1 + \alpha + \cdots + \alpha^{k-2} + \alpha^{k-1}\right) \|f(x) - x\| \\
&= \frac{1 - \alpha^k}{1 - \alpha} \|f(x) - x\|
\end{aligned}$$

$\forall \epsilon > 0, \exists N = \left\lceil \log_\alpha \left(\frac{\epsilon(1-\alpha)}{\|f(x)-x\|} + \alpha^m \right) \right\rceil \in \mathbb{N}$ ($\lceil t \rceil$ means the smallest integer greater than t), $\forall m \geq N$,

$$\begin{aligned}
\left\| f^{[m]}(x) - f^{[N]}(x) \right\| &\leq \alpha^N \left\| f^{[m-N]}(x) - x \right\| \\
&\leq \alpha^N \cdot \frac{1-\alpha^{m-N}}{1-\alpha} \cdot \|f(x) - x\| \\
&= \frac{\alpha^N - \alpha^m}{1-\alpha} \|f(x) - x\| \\
N &> \log_\alpha \left(\frac{\epsilon(1-\alpha)}{\|f(x)-x\|} + \alpha^m \right) \\
\Rightarrow \left\| f^{[m]}(x) - f^{[N]}(x) \right\| &< \epsilon
\end{aligned}$$

Therefore, $\{f^{[i]}\}_{i=0}^\infty$ is a Cauchy sequence. Note that given an arbitrary initial point x, we can compute the value of $\|f(x) - x\|$ so we treat $\|f(x) - x\|$ as a known constant in the proof above. Because $(X, \|\cdot\|)$ is a Banach space, $\{f^{[i]}\}_{i=0}^\infty$ converges to a point f^*.

$$f^* = \lim_{n\to\infty} f^{[n]}(x) = \lim_{n\to\infty} f\left(f^{[n-1]}(x) \right) = f\left(\lim_{n\to\infty} f^{[n-1]}(x) \right) = f(f^*)$$

Therefore, there exist $\bar{x} = f^*$ such that $f(\bar{x}) = \bar{x}$. Next, we prove uniqueness: For the sake of contradiction, assume there exist $\bar{x}_1, \bar{x}_2 \in X, \bar{x}_1 \neq \bar{x}_2$ such that $f(\bar{x}_1) = \bar{x}_1$ and $f(\bar{x}_2) = \bar{x}_2$.

$$\|f(\bar{x}_2) - f(\bar{x}_1)\| = \|\bar{x}_2 - \bar{x}_1\| \leq \alpha \|\bar{x}_2 - \bar{x}_1\|$$

Since $\alpha \in [0, 1)$, $\alpha \neq 1$. The above inequality never holds, which leads to contradiction, so $\bar{x}$ is unique.

Solution 2.2
Let F be the representation of linear map $f(\cdot)$. $\forall x, y \in X$:

$$\|f(x) - f(y)\| = \|Fx - Fy\| = \|F(x-y)\| \leq \alpha \|x - y\| \Rightarrow \frac{\|F(x-y)\|}{\|x-y\|} \leq \alpha$$

$\forall u \in X, \exists x, y \in X$ such that $u = x - y$.

$$\|F\| = \sup_{u\neq 0} \frac{\|Fu\|}{\|u\|} \leq \sup_{u\neq 0} \alpha = \alpha$$

Therefore, the induced norm of f is bounded by α.

Exercise 3. (Ordinary differential equations [30 points in total])

1. [12 **points**] Consider the following ordinary differential equation (ODE)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -x_1(t) + e^t \cos(x_1(t) - x_2(t)) \\ -x_2(t) + \sin(x_1(t) - x_2(t)) \end{bmatrix},$$

where $x_i(t) \in \mathbb{R}$, $\forall i$. Prove or disprove the following statements:

   (a) This system is globally Lipschitz,

   (b) This system admits a unique solution.
   Hint: You may assume that functions with bounded derivatives are Lipschitz.

2. [18 **points**] Consider the following ordinary differential equation (ODE)

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -3\sin(t)\, x_1(t) + x_1(t)\, x_2(t) \\ -2\, x_2(t) \end{bmatrix},$$

where $x_i(t) \in \mathbb{R}$, $\forall i$. Prove or disprove the following statements:

   (a) This system is globally Lipschitz,

   (b) This system admits a unique solution.

Solution 3.1

$$\begin{aligned}
\dot{x} &= p(x,t) \\
\frac{\partial p}{\partial x} &= \begin{bmatrix} \frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -1 - e^t \sin(x_1 - x_2) & e^t \sin(x_1 - x_2) \\ \cos(x_1 - x_2) & -1 - \cos(x_1 - x_2) \end{bmatrix} \\
\left|\frac{\partial p_1}{\partial x_1}\right| &= \left|-1 - e^t \sin(x_1 - x_2)\right| \le 1 + \left|e^t \sin(x_1 - x_2)\right| \le 1 + e^t \\
\left|\frac{\partial p_1}{\partial x_2}\right| &= \left|e^t \sin(x_1 - x_2)\right| \le e^t \\
\left|\frac{\partial p_2}{\partial x_1}\right| &= |\cos(x_1 - x_2)| \le 1 \\
\left|\frac{\partial p_2}{\partial x_2}\right| &= |-1 - \cos(x_1 - x_2)| \le 1 + |-\cos(x_1 - x_2)| \le 2
\end{aligned}$$

The function p is differentiable in x_1, x_2 and the derivative is bounded by $1 + e^t$, which leads to the conclusion that the system is globally Lipschitz. It is also obvious that $p(x,t)$ is continuous in t. From (a) we know that $p(x,t)$ is globally Lipschitz with respect to x. Per **Theorem 3.6**, the system admits a unique solution.

Solution 3.2
Expand the second row of the system dynamics:

$$\begin{aligned}\dot{x}_2 &= -2x_2 \\ \frac{1}{x_2}\,\mathrm{d}x_2 &= -2\,\mathrm{d}t \\ \ln x_2 &= -2t + c_1 \\ x_2 &= Be^{-2t}\end{aligned}$$

Plug $x_2(t)$ into the system dynamics and solve $x_1(t)$:

$$\begin{aligned}\dot{x}_1 &= -3\sin(t) + x_2 \\ \frac{1}{x_1}\,\mathrm{d}x_1 &= -3\sin(t)\,\mathrm{d}t + Be^{-2t}\,\mathrm{d}t \\ \ln x_1 &= 3\cos(t) - \frac{1}{2}Be^{-2t} + c_2 \\ x_1 &= Ae^{3\cos(t)-\frac{1}{2}Be^{-2t}}\end{aligned}$$

Take the gradient of the system dynamics:

$$\begin{aligned}\dot{x} &= p(x,t) \\ \frac{\partial p}{\partial x} &= \begin{bmatrix}\frac{\partial p_1}{\partial x_1} & \frac{\partial p_1}{\partial x_2} \\ \frac{\partial p_2}{\partial x_1} & \frac{\partial p_2}{\partial x_2}\end{bmatrix} = \begin{bmatrix}-3\sin t + x_2 & x_1 \\ 0 & -2\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\left|\frac{\partial p_1}{\partial x_1}\right| &= |-3\sin(t) + x_2| \leq |-3\sin(t)| + |x_2| \leq 3 + Be^{-2t} := K_{11}(t) \\ \left|\frac{\partial p_1}{\partial x_2}\right| &= |x_1| \leq \left|Ae^{3\cos(t)-\frac{1}{2}Be^{-2t}}\right| := K_{12}(t) \\ \left|\frac{\partial p_2}{\partial x_1}\right| &= |0| \leq 0 := K_{21}(t) \\ \left|\frac{\partial p_2}{\partial x_2}\right| &= |-2| \leq -2 := K_{22}(t)\end{aligned}$$

Since the derivative is bounded, $p(x,t)$ is globally Lipschitz with respect to x. Obviously, $p(x,t)$ is continuous with respect to t. By existence and uniqueness of the linear ordinary differential equation, the system admits a unique solution.