
Linear System Theory
Problem Set 3
Linear Time-Varying Systems, Linear Time-Invariant Systems

Issue date: Oct. 21, 2019

Due date: Oct. 31, 2019

Exercise 1. (Linear Time-Varying Systems, [50 points in total])

Let $A_1(t), A_2(t), F(t) \in \mathbb{R}^{n \times n}$ be piecewise continuous matrix functions. Let Φ_i be the state transition matrix for $\dot{x}(t) = A_i(t)x(t)$, for $i = 1, 2$. Consider the matrix differential equation:

$$\dot{X}(t) = A_1(t)X(t) + X(t)A_2^T(t) + F(t), \quad X(t_0) = X_0,$$

where $X(t) \in \mathbb{R}^{n \times n}$ for any $t \geq t_0$.

1. [20 points] Show that this is an affine time-varying system. (Hint: An affine time-varying system is a system of the form $\dot{x}(t) = A(t)x(t) + b(t)$, where $x(t)$ and $b(t)$ are vectors.)
2. [30 points] Assume that the solution of the above system can be written as:

$$X(t) = \Phi_1(t, t_0)X_0\Phi_2^T(t, t_0) + \int_{t_0}^t \Phi_1(t, \tau)M(t, \tau)d\tau.$$

Express the matrix $M(t, \tau)$ as a function of $\Phi_1(t, \tau)$, $F(t)$, and $\Phi_2(t, \tau)$. (Hint: $\Phi_1(t, \tau)$, $F(t)$, and $\Phi_2(t, \tau)$ may not all appear in the expression of $M(t, \tau)$.)

Solution 1.1

Let $x_{ij}, a_{ij}^{(1)}, a_{ij}^{(2)}, f_{ij}$ denote the i^{th} row and j^{th} column element of matrix X, A_1, A_2, F respectively. According to the matrix differential equation, x_{ij} can be expressed as

$$\begin{aligned} \dot{x}_{ij} &= \sum_{k=1}^n a_{ik}^{(1)} x_{kj} + \sum_{k=1}^n x_{ik} a_{jk}^{(2)} + f_{ij} \\ &= \sum_{k=1}^n \left(a_{ik}^{(1)} x_{kj} + a_{jk}^{(2)} x_{ik} \right) + f_{ij} \end{aligned} \quad (1)$$

Let x_{ci} and f_{ci} denote the i^{th} column of matrix X and F respectively. Define vector $\tilde{x}, \tilde{f} \in \mathbb{R}^{n^2}$ by rearranging x_{ci} and f_{ci} :

$$\tilde{x} = \begin{bmatrix} x_{c1} \\ \vdots \\ x_{cn} \end{bmatrix}, \quad \tilde{f} = \begin{bmatrix} f_{c1} \\ \vdots \\ f_{cn} \end{bmatrix}.$$

Based on the equation (1), elements of $\dot{\tilde{x}}$ can be expressed as

$$\dot{\tilde{x}}_{i+n(j-1)} = \sum_{k=1}^n a_{ik}^{(1)} \tilde{x}_{k+n(j-1)} + \sum_{k=1}^n a_{jk}^{(2)} \tilde{x}_{i+n(k-1)} + f_{ij}, \quad (2)$$

which defines an affine time-varying system. To express it more explicitly, we define augmented matrix \tilde{A}_1 and \tilde{A}_2 to facilitate further notation:

$$\tilde{A}_1 = \text{diag}(A, \dots, A) = \begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2},$$

$$\tilde{A}_2 = \begin{bmatrix} \text{diag}(a_{11}^{(2)}) & \text{diag}(a_{12}^{(2)}) & \cdots & \text{diag}(a_{1n}^{(2)}) \\ \text{diag}(a_{21}^{(2)}) & \text{diag}(a_{22}^{(2)}) & \cdots & \text{diag}(a_{2n}^{(2)}) \\ \vdots & \vdots & \ddots & \vdots \\ \text{diag}(a_{n1}^{(2)}) & \text{diag}(a_{n2}^{(2)}) & \cdots & \text{diag}(a_{nn}^{(2)}) \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2},$$

where $\text{diag}(a_{ij}^{(2)})$ denotes an $n \times n$ diagonal matrix whose diagonal elements are all $a_{ij}^{(2)}$. Then the system can be fully described by an affine time-varying system in the standard form:

$$\dot{\tilde{x}}(t) = (\tilde{A}_1(t) + \tilde{A}_2(t)) \tilde{x}(t) + \tilde{f}(t)$$

Solution 1.2

$$\begin{aligned} \frac{d}{dt} \left[\int_{t_0}^t \Phi_1(t, \tau) M(t, \tau) d\tau \right] &= \int_{t_0}^t \frac{d}{dt} (\Phi_1(t, \tau) M(t, \tau)) d\tau + \Phi_1(t, t) M(t, t) \cdot 1 \\ &= \int_{t_0}^t A_1(t) \Phi_1(t, \tau) M(t, \tau) d\tau + \int_{t_0}^t \Phi_1(t, \tau) \frac{d}{dt} M(t, \tau) d\tau + M(t, t) \\ &= A_1(t) \int_{t_0}^t \Phi_1(t, \tau) M(t, \tau) d\tau + \int_{t_0}^t \Phi_1(t, \tau) \frac{d}{dt} M(t, \tau) d\tau + M(t, t) \end{aligned}$$

$$\begin{aligned} \dot{X}(t) &= A_1(t) \Phi_1(t, t_0) X_0 \Phi_2^T(t, t_0) + \Phi_1(t, t_0) X_0 \Phi_2^T(t, t_0) A_2^T(t) + \frac{d}{dt} \left[\int_{t_0}^t \Phi_1(t, \tau) M(t, \tau) d\tau \right] \\ &= A_1(t) X(t) - A_1(t) \int_{t_0}^t \Phi_1(t, \tau) M(t, \tau) d\tau + X(t) A_2^T(t) - \left(\int_{t_0}^t \Phi_1(t, \tau) M(t, \tau) d\tau \right) A_2^T(t) \\ &\quad + A_1(t) \int_{t_0}^t \Phi_1(t, \tau) M(t, \tau) d\tau + \int_{t_0}^t \Phi_1(t, \tau) \frac{d}{dt} M(t, \tau) d\tau + M(t, t) \\ &= A_1(t) X(t) + X(t) A_2^T(t) - \left(\int_{t_0}^t \Phi_1(t, \tau) M(t, \tau) d\tau \right) A_2^T(t) + \int_{t_0}^t \Phi_1(t, \tau) \frac{d}{dt} M(t, \tau) d\tau + M(t, t) \end{aligned}$$

From the matrix differential equation, we also know that

$$\dot{X}(t) = A_1(t) X(t) + X(t) A_2^T(t) + F(t)$$

Compare these two equations, it follows immediately

$$F(t) = - \left(\int_{t_0}^t \Phi_1(t, \tau) M(t, \tau) d\tau \right) A_2^T(t) + \int_{t_0}^t \Phi_1(t, \tau) \frac{d}{dt} M(t, \tau) d\tau + M(t, t)$$

Guess a solution for $M(t, \tau)$:

$$M(t, \tau) = F(\tau) \Phi_2^T(t, \tau)$$

The right-hand side would thus be

$$\begin{aligned} R &= - \left(\int_{t_0}^t \Phi_1(t, \tau) M(t, \tau) d\tau \right) A_2^T(t) + \int_{t_0}^t \Phi_1(t, \tau) \frac{d}{dt} M(t, \tau) d\tau + M(t, t) \\ &= - \left(\int_{t_0}^t \Phi_1(t, \tau) M(t, \tau) d\tau \right) A_2^T(t) + \int_{t_0}^t \Phi_1(t, \tau) F(\tau) \Phi_2^T(t, \tau) A_2^T(t) d\tau + F(t) \Phi_2^T(t, t) \\ &= F(t) \\ &= L \end{aligned}$$

Therefore, $X(t) = \Phi_1(t, t_0)X_0\Phi_2^T(t, t_0) + \int_{t_0}^t \Phi_1(t, \tau)F(\tau)\Phi_2^T(t, \tau)d\tau$ is a solution to the differential equation. Next we prove the uniqueness of the solution.

Define $g(X) = \dot{X}$, $\forall X_1, X_2 \in \mathbb{R}^{n \times n}$, $\forall t \geq t_0$,

$$\begin{aligned}
\|g(X_1) - g(X_2)\| &= \|\dot{X}_1 - \dot{X}_2\| \\
&= \|(A_1X_1 + X_1A_2^T + F) - (A_1X_2 + X_2A_2^T + F)\| \\
&= \|A_1X_1 - A_1X_2 + X_1A_2^T - X_2A_2^T\| \\
&= \|A_1(X_1 - X_2) + (X_1 - X_2)A_2^T\| \\
&\leq \|A_1(X_1 - X_2)\| + \|(X_1 - X_2)A_2^T\| \\
&\leq \|A_1\| \|X_1 - X_2\| + \|X_1 - X_2\| \|A_2^T\| \\
&= (\|A_1\| + \|A_2^T\|) \|X_1 - X_2\| \\
&:= K(t) \|X_1(t) - X_2(t)\|
\end{aligned}$$

Therefore, $g(X)$ is globally Lipschitz with respect to X and piece-wise continuous with respect to t , which indicates the uniqueness of the solution to the matrix differential equation. Thus, the value of $\int_{t_0}^t \Phi_1(t, \tau)M(t, \tau)d\tau$ is unique and $\forall t$

$$\int_{t_0}^t \Phi_1(t, \tau)F(\tau)\Phi_2^T(t, \tau)d\tau = \int_{t_0}^t \Phi_1(t, \tau)M(t, \tau)d\tau.$$

However, we cannot make sure the uniqueness of $M(t, \tau)$. For example, any $\tilde{M}(t, \tau)$ defined as

$$\tilde{M}(t, \tau) = \begin{cases} F(\tau)\Phi_2^T(t, \tau) & t \notin \mathcal{D} \\ K & t \in \mathcal{D} \end{cases}$$

where K is any finite real number, and \mathcal{D} stands for the discontinuity set of $\tilde{M}(t, \tau)$. Then, $\tilde{M}(t, \tau)$ is also a valid solution to the matrix differential equation since

$$\forall t \geq t_0, \int_{t_0}^t \Phi_1(t, \tau)M(t, \tau)d\tau = \int_{t_0}^t \Phi_1(t, \tau)\tilde{M}(t, \tau)d\tau$$

Exercise 2. (Linear Time-Invariant Systems, [50 points in total])

Consider the following affine system:

$$\dot{x}(t) = Ax(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x(t).$$

where $A \in \mathbb{R}^{3 \times 3}$. The matrix A has eigenvalues $\lambda_1 = -2$ with multiplicity 2, and $\lambda_2 = -1$ with multiplicity 1. The eigenvalue λ_1 has an eigenvector $v_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and a generalized eigenvector $v'_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

The eigenvalue λ_2 has the eigenvector $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

1. [20 points] Find the matrix A .
2. [20 points] Calculate $\exp(At)$.
3. [10 points] Given $x(0) = [0 \ 0 \ 1]^\top$, compute $y(t)$.

Solution 2.1

$$\begin{aligned} Av_2 &= \lambda_2 v_2 \Rightarrow a_{12} = 0, a_{22} = -1, a_{23} = 0 \\ Av_1 &= \lambda_1 v_1 \Rightarrow a_{13} = 0, a_{23} = -2, a_{33} = -2 \\ (A - \lambda_1 \mathbb{I})v'_1 &= v_1 \Rightarrow a_{11} = -2, a_{21} = 4, a_{31} = -1 \\ A &= \begin{bmatrix} -2 & 0 & 0 \\ -4 & -1 & -2 \\ -1 & 0 & -2 \end{bmatrix} \end{aligned}$$

Solution 2.2

$$\begin{aligned} T^{-1} &= [v_1 \ v'_1 \ v_2] = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ -2 & 1 & -2 \end{bmatrix} \\ J_1 &= \begin{bmatrix} -2 & 1 \\ & -2 \end{bmatrix}, \quad J_2 = -1, \quad J = \begin{bmatrix} -2 & 1 & \\ & -2 & \\ & & -1 \end{bmatrix} \\ e^{J_1 t} &= \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix}, \quad e^{J_2 t} = e^{-t}, \quad e^{J t} = \begin{bmatrix} e^{-2t} & te^{-2t} & \\ & e^{-2t} & \\ & & e^{-t} \end{bmatrix} \\ e^{At} &= T^{-1} e^{J t} T = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 2e^{-2t} - 2e^{-t} - 2te^{-2t} & e^{-t} & 2e^{-2t} - 2e^{-t} \\ -te^{-2t} & 0 & e^{-2t} \end{bmatrix} \end{aligned}$$

Solution 2.3

$$\Phi(t, 0) = e^{At} = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 2e^{-2t} - 2e^{-t} - 2te^{-2t} & e^{-t} & 2e^{-2t} - 2e^{-t} \\ -te^{-2t} & 0 & e^{-2t} \end{bmatrix}, \quad B(t)u(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned} x(t) &= \Phi(t, 0)x_0 + \int_0^t \Phi(t, \tau)B(\tau)u(\tau)d\tau \\ &= \begin{bmatrix} 0 \\ 2e^{-2t} - 2e^{-t} \\ e^{-2t} \end{bmatrix} + \int_0^t \Phi(t, \tau) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} d\tau \\ &= \begin{bmatrix} 0 \\ 2e^{-2t} - 2e^{-t} \\ e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} 0 \\ e^{-t+\tau} \\ 0 \end{bmatrix} d\tau \\ &= \begin{bmatrix} 0 \\ 2e^{-2t} - 2e^{-t} \\ e^{-2t} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - e^{-t} \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 + 2e^{-2t} - 3e^{-t} \\ e^{-2t} \end{bmatrix} \\ y(t) &= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x(t) \\ &= 1 + 2e^{-2t} - 3e^{-t} \end{aligned}$$