Linear System Theory
Solution Set

Yize Wang

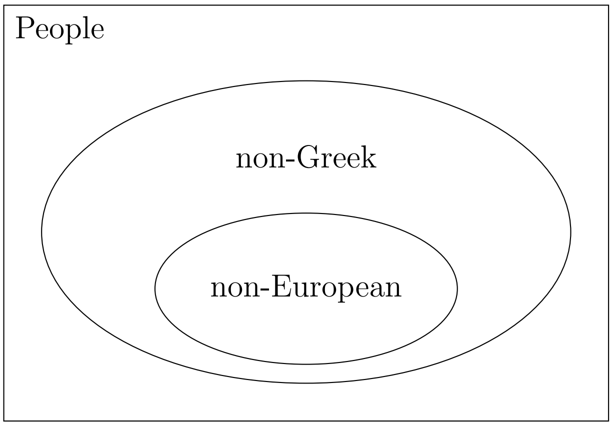
October 23, 2019

Contents

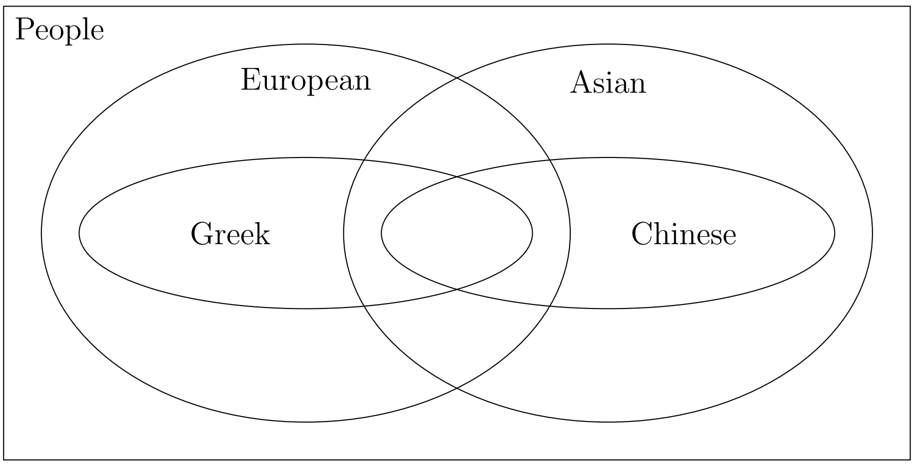
Chapter 1

Introduction

Exercise 1.1



Exercise 1.2



Exercise 1.3

statement p: $\sqrt{2} \in \mathbb{R}$
statement q: $\sqrt{2}$ is not rational
statement r: m and n have no common divisor

Exercise 1.4

$f \circ (g \circ h)(w) = f \circ g(h(w)) = f(g(h(w))) = (f \circ g)(h(w)) = (f \circ g) \circ h(w)$ ■

Exercise 1.5

identity map definition: $\forall x \in \mathcal{X}, 1_X(x) = x$
injective: $1_X(x_1) = 1_X(x_2) \Rightarrow x_1 = x_2$
surjective: $\forall y \in \mathcal{X}, \exists x = y \in \mathcal{X}, 1_X(x) = y$
injective $\wedge$ surjective $\Rightarrow$ bijective

■

Problem 1.1

Assume, for the sake of contradiction, $\exists n \in \mathbb{N}$, such that n is both odd and even. According to the definitions of even and odd numbers, we know that $\exists p \in \mathbb{N}, n = 2p + 1$ and $\exists q \in \mathbb{N}, n = 2q$. Thus, $2p + 1 = 2q$ and hence $q = p + 0.5$. Since $p \in \mathbb{N}$, $q = p + 0.5 \notin \mathbb{N}$, which leads to the contradiction with $q \in \mathbb{N}$. Therefore, n cannot be both odd and even.

■

Problem 1.2

1. f has a left inverse if and only if it is injective:

   - Suppose f has a left inverse g_L,
     $g_L : \mathcal{Y} \to \mathcal{X}$ such that $\forall x \in \mathcal{X}, g_L \circ f(x) = x$
     $f(x_1) = f(x_2) \Rightarrow g_L(f(x_1)) = g_L(f(x_2)) \Rightarrow x_1 = x_2 \Rightarrow$ injective
   - Suppose f is injective, $\forall y \in \textsc{Range}(f)$, $\exists! x \in \mathcal{X}, f(x) = y$.
     Construct $g(y)_L = \begin{cases} x & y \in \textsc{Range}(f) \\ 0 & otherwise \end{cases}$
     Then $g_L(y)$ defines the left inverse function of f.

2. f has a right inverse if and only if it is surjective:

- Suppose f has a right inverse g_R, $\forall y \in \mathcal{Y}, f \circ g(y) = y$.
  $\forall y \in \mathcal{Y}, \exists x = g(y) \in \mathcal{X},$ such that $f(x) = y \Rightarrow$ surjective
- Suppose f is surjective, $\forall y \in \mathcal{Y}, \exists x \in \mathcal{X}, f(x) = y$.
  Let $g_R(y) = x$. If multiple x exist, choose one of them. $f(g_R(y)) = f(x) = y = 1_{\mathcal{Y}}(y)$
  Therefore $g_R(y)$ defines the right inverse function of f.

3. f is invertible if and only if it is bijective:

   - Suppose f is invertible, there exist left and right inverse of f. Then f is both injective and surjective, which implies f is bijective.
   - Suppose f is bijective, we only need to prove $g = g_L = g_R$.
     $g_L(y) = g_L \circ 1_{\mathcal{Y}}(y) = g_L(1_{\mathcal{Y}}(y)) = g_L(f \circ g_R(y)) = (g_L \circ f) \circ g_R(y) = 1_{\mathcal{X}} \circ g_R(y) = g_R(y)$

$\blacksquare$

Chapter 2

Introduction to Algebra

Exercise 2.1

Exercise 2.2

Chapter 3

Introduction to Analysis

Chapter 4

Time Varying Linear Systems: Solutions

Tutorial Exercise 4.1

Given the system dynamics $\dot{x}(t) = A(t)x(t)$ and let $\Phi(t, t_0)$ be the transition matrix. Show that

$$\frac{\partial}{\partial t}\Phi(t_0, t) = -\Phi(t_0, t)A(t)\ .$$

Tutorial Exercise Solution 4.1

$$\begin{aligned}
\Phi(t, t_0)\Phi(t_0, t) &= \mathbb{I} \\
\frac{\partial}{\partial t}\Phi(t, t_0)\Phi(t_0, t) + \Phi(t, t_0)\frac{\partial}{\partial t}\Phi(t_0, t) &= 0 \\
A\Phi(t, t_0)\Phi(t_0, t) + \Phi(t, t_0)\frac{\partial}{\partial t}\Phi(t_0, t) &= 0 \\
\Phi(t, t_0)\frac{\partial}{\partial t}\Phi(t_0, t) &= -A \\
\frac{\partial}{\partial t}\Phi(t_0, t) &= -\Phi(t_0, t)A(t)
\end{aligned}$$

Tutorial Exercise 4.2

Given the system dynamics $\dot{x}(t) = A(t)x(t)$ and let $\Phi(t, t_0)$ be the transition matrix. Prove that if $\mathcal{X}_0 \subseteq \mathbb{R}^n$ is convex, then $\forall t \geq 0$, the set $\mathcal{X}(t) = \{s(t, t_0, x_0, 0) : x \in \mathcal{X}_0\}$ is convex.

Tutorial Exercise Solution 4.2

$$
\begin{aligned}
&\forall s_1, s_2 \in \mathcal{X}, \exists x_0^1, x_0^2 \in \mathcal{X}_0, \text{such that } s_1 = s(t, t_0, x_0^1, 0), s_2 = s(t, t_0, x_0^2, 0) \\
&\forall \lambda \in [0,1], \lambda s_1 + (1-\lambda)s_2 = \lambda s(t, t_0, x_0^1, 0) + (1-\lambda)s(t, t_0, x_0^2, 0) \\
&\qquad\qquad\qquad\qquad\qquad\quad = s(t, t_0, \lambda x_0^1 + (1-\lambda)x_0^2, 0) \\
&\mathcal{X}_0 \text{ is convex} \Rightarrow \lambda x_0^1 + (1-\lambda)x_0^2 \in \mathcal{X}_0 \Rightarrow \lambda s_1 + (1-\lambda)s_2 \in \mathcal{X}
\end{aligned}
$$

Therefore, $\mathcal{X}(t)$ is a convex set.

Chapter 5

First Meeting Discussion

Discussion

- For the first phase, we meet several times to modify the style. For the next phase, I discuss PhD students to make sure solutions are correct and concise.
- For version 1.0, I will simply copy the exercise statements. But many of them heavily depend on the context. After finishing the main part, I will come back to this. Or we can simply include only solution and exercise index.
- It is better to have a label for each exercise so that you can freely add or delete them.
- In the homework, commands of LaTeXand TeXare mixed.
- Maybe add a Theorem collection or index.
- In Page 71, I think it's better to write B instead of $B(\tau)$ because it is not the standard form of convolution, although they lead to the same result.
- In Page 71, I think you mean "In some cases this can be done directly from the **finite** series".
- In Page 74, may be use symmetric instead of diagnoal?
- Is Definitioin 5.1 rigorous?

Questions

- Am I going to have a position in the office?
- You corrected typos with an additional document instead of fixing them in the lecture notes. Is there something wrong with the source file?
- What is the difference between "linear in u and x" and "linear jointly in u and x"?
- When does the system diverge exponentially?
- Is the script your work or student's work?
- What does $\{u_j\}_{j=1}^n \xrightarrow{A\in F^{m\times n}} \{v_i\}_{i=1}^m$ mean?