Linear System Theory

Problem Set 3
Linear Time-Varying Systems, Linear Time-Invariant Systems

Issue date: Oct. 21, 2019
Due date: Oct. 31, 2019

Exercise 1. (Linear Time-Varying Systems, [50 points in total])
Let $A_1(t), A_2(t),\ F(t) \in \mathbb{R}^{n\times n}$ be piecewise continuous matrix functions. Let Φ_i be the state transition matrix for $\dot{x}(t) = A_i(t)x(t)$, for $i = 1, 2$. Consider the matrix differential equation:

$$\dot{X}(t) = A_1(t)X(t) + X(t)A_2^T(t) + F(t),\ X(t_0) = X_0,$$

where $X(t) \in \mathbb{R}^{n\times n}$ for any $t \geq t_0$.

1. [20 **points**] Show that this is an affine time-varying system. (Hint: An affine time-varying system is a system of the form $\dot{x}(t) = A(t)x(t) + b(t)$, where $x(t)$ and $b(t)$ are vectors.)
2. [30 **points**] Assume that the solution of the above system can be written as:

$$X(t) = \Phi_1(t, t_0)X_0\Phi_2^T(t, t_0) + \int_{t_0}^{t} \Phi_1(t, \tau)M(t, \tau)d\tau.$$

   Express the matrix $M(t, \tau)$ as a function of $\Phi_1(t, \tau)$, $F(t)$, and $\Phi_2(t, \tau)$. (Hint: $\Phi_1(t, \tau)$, $F(t)$, and $\Phi_2(t, \tau)$ may not all appear in the expression of $M(t, \tau)$.)

Solution 1.1

Let $x_{ij}, a_{ij}^{(1)}, a_{ij}^{(2)}, f_{ij}$ denote the i^{th} row and j^{th} column element of matrix X, A_1, A_2, F respectively. According to the matrix differential equation, x_{ij} can be expressed as

$$\begin{aligned}\dot{x}_{ij} &= \sum_{k=1}^{n} a_{ik}^{(1)} x_{kj} + \sum_{k=1}^{n} x_{ik} a_{jk}^{(2)} + f_{ij} \\ &= \sum_{k=1}^{n} \left(a_{ik}^{(1)} x_{kj} + a_{jk}^{(2)} x_{ik} \right) + f_{ij} \qquad (1)\end{aligned}$$

Let x_{ci} and f_{ci} denote the i^{th} column of matrix X and F respectively. Define vector $\tilde{x}, \tilde{f} \in \mathbb{R}^{n^2}$ by rearranging x_{ci} and f_{ci}:

$$\tilde{x} = \begin{bmatrix} x_{c1} \\ \vdots \\ x_{cn} \end{bmatrix}, \quad \tilde{f} = \begin{bmatrix} f_{c1} \\ \vdots \\ f_{cn} \end{bmatrix}$$

Based on the equation (1), elements of $\dot{\tilde{x}}$ can be expressed as

$$\dot{\tilde{x}}_{i+n(j-1)} = \sum_{k=1}^{n} a_{ik}^{(1)} \tilde{x}_{k+n(j-1)} + \sum_{k=1}^{n} a_{jk}^{(2)} \tilde{x}_{i+n(k-1)} + f_{ij}\ , \qquad (2)$$

which defines an affine time-varying system. To express it more explicitly, we define augmented matrix $\tilde{A}_1$ and $\tilde{A}_2$ to facilitate further notation:

$$\tilde{A}_1 = \operatorname{diag}(A, \ldots, A) = \begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2} ,$$

$$\tilde{A}_2 = \begin{bmatrix} \operatorname{diag}(a_{11}^{(2)}) & \operatorname{diag}(a_{12}^{(2)}) & \cdots & \operatorname{diag}(a_{1n}^{(2)}) \\ \operatorname{diag}(a_{21}^{(2)}) & \operatorname{diag}(a_{22}^{(2)}) & \cdots & \operatorname{diag}(a_{2n}^{(2)}) \\ \vdots & \vdots & \ddots & \vdots \\ \operatorname{diag}(a_{n1}^{(2)}) & \operatorname{diag}(a_{n2}^{(2)}) & \cdots & \operatorname{diag}(a_{nn}^{(2)}) \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2} ,$$

where $\operatorname{diag}(a_{ij}^{(2)})$ denotes an $n \times n$ diagonal matrix whose diagonal elements are all $a_{ij}^{(2)}$. Then the system can be fully described by an affine time-varying system in the standard form:

$$\begin{aligned} \dot{\tilde{x}}(t) &= \left(\tilde{A}_1(t) + \tilde{A}_2(t)\right) \tilde{x}(t) + \tilde{f}(t) \\ &:= A(t)\tilde{x}(t) + b(t) \end{aligned}$$

Solution 1.2

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right] &= \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_1(t,\tau)M(t,\tau)\right)\mathrm{d}\tau + \Phi_1(t,t)M(t,t) \cdot 1 \\ &= \int_{t_0}^{t} A_1(t)\Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \\ &= A_1(t)\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \end{aligned}$$

$$\begin{aligned} \dot{X}(t) &= A_1(t)\Phi_1(t,t_0)X_0\Phi_2^T(t,t_0) + \Phi_1(t,t_0)X_0\Phi_2^T(t,t_0)A_2^T(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right] \\ &= A_1(t)X(t) - A_1(t)\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + X(t)A_2^T(t) - \left(\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right)A_2^T(t) \\ &\quad + A_1(t)\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \\ &= A_1(t)X(t) + X(t)A_2^T(t) - \left(\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right)A_2^T(t) + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \end{aligned}$$

From the matrix differential equation, we also know that

$$\dot{X}(t) = A_1(t)X(t) + X(t)A_2^T(t) + F(t)$$

Compare these two equations, it follows immediately

$$F(t) = -\left(\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right)A_2^T(t) + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t)$$

Guess a solution for $M(t,\tau)$:

$$M(t,\tau) = F(\tau)\Phi_2^T(t,\tau)$$

The right-hand side would thus be

$$\begin{aligned} R &= -\left(\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right)A_2^T(t) + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \\ &= -\left(\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right)A_2^T(t) + \int_{t_0}^{t} \Phi_1(t,\tau)F(\tau)\Phi_2^T(t,\tau)A_2^T(t)\mathrm{d}\tau + F(t)\Phi_2^T(t,t) \\ &= F(t) \\ &= L \end{aligned}$$

Therefore, $X(t) = \Phi_1(t, t_0)X_0\Phi_2^T(t, t_0) + \int_{t_0}^{t} \Phi_1(t, \tau)F(\tau)\Phi_2^T(t, \tau)\mathrm{d}\tau$ is a solution to the differential equation. Next we prove the uniqueness of the solution.

Define $p(\tilde{x}(t), t) = \dot{\tilde{x}}(t)$, $\forall \tilde{x}_1, \tilde{x}_2 \in \mathbb{R}^n, \forall t \geq t_0$,

$$
\begin{aligned}
\|p(\tilde{x}_1(t), t) - p(\tilde{x}_2(t), t)\| &= \left\|\dot{\tilde{x}}_1(t) - \dot{\tilde{x}}_2(t)\right\| \\
&= \|(A(t)\tilde{x}_1(t) + b(t)) - (A(t)\tilde{x}_2(t) + b(t))\| \\
&= \|A(t)\left(\tilde{x}_1(t) - \tilde{x}_2(t)\right)\| \\
&\leq \|A(t)\| \, \|\tilde{x}_1(t) - \tilde{x}_2(t)\| \\
&:= K(t) \, \|\tilde{x}_1(t) - \tilde{x}_2(t)\|
\end{aligned}
$$

Therefore, $p(\tilde{x}(t), t)$ is globally Lipschitz with respect to $\tilde{x}(t)$ and piece-wise continuous with respect to t, which indicates the uniqueness of the solution to the matrix differential equation. Thus, the value of $\int_{t_0}^{t} \Phi_1(t, \tau)M(t, \tau)\mathrm{d}\tau$ is unique and $\forall t$

$$
\int_{t_0}^{t} \Phi_1(t, \tau)F(\tau)\Phi_2^T(t, \tau)\mathrm{d}\tau = \int_{t_0}^{t} \Phi_1(t, \tau)M(t, \tau)\mathrm{d}\tau \; .
$$

However, we cannot make sure the uniqueness of $M(t, \tau)$. For example, any $\tilde{M}(t, \tau)$ defined as

$$
\tilde{M}(t, \tau) = \begin{cases} F(\tau)\Phi_2^T(t, \tau) & t \notin \mathcal{D} \\ K & t \in \mathcal{D} \end{cases}
$$

where K is any finite real number, and $\mathcal{D}$ stands for the finite discontinuity set of $\tilde{M}(t, \tau)$. Then, $\tilde{M}(t, \tau)$ is also a valid solution to the matrix differential equation since

$$
\forall t \geq t_0, \int_{t_0}^{t} \Phi_1(t, \tau)M(t, \tau)\mathrm{d}\tau = \int_{t_0}^{t} \Phi_1(t, \tau)\tilde{M}(t, \tau)\mathrm{d}\tau
$$

Exercise 2. (Linear Time-Invariant Systems, [50 points in total])
Consider the following affine system:

$$\dot{x}(t) = A\,x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x(t).$$

where $A \in \mathbb{R}^{3\times 3}$. The matrix A has eigenvalues $\lambda_1 = -2$ with multiplicity 2, and $\lambda_2 = -1$ with multiplicity 1. The eigenvalue λ_1 has an eigenvector $v_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and a generalized eigenvector $v_1' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

The eigenvalue λ_2 has the eigenvector $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

1. [20 points] Find the matrix A.
2. [20 points] Calculate $\exp(At)$.
3. [10 points] Given $x(0) = [0\ 0\ 1]^\top$, compute $y(t)$.

Solution 2.1

$$\begin{aligned}
Av_2 = \lambda_2 v_2 &\Rightarrow a_{12} = 0, a_{22} = -1, a_{23} = 0 \\
Av_1 = \lambda_1 v_1 &\Rightarrow a_{13} = 0, a_{23} = -2, a_{33} = -2 \\
(A - \lambda_1 \mathbb{I})\, v_1' = v_1 &\Rightarrow a_{11} = -2, a_{21} = 4, a_{31} = -1 \\
A &= \begin{bmatrix} -2 & 0 & 0 \\ -4 & -1 & -2 \\ -1 & 0 & -2 \end{bmatrix}
\end{aligned}$$

Solution 2.2

$$\begin{aligned}
T^{-1} &= \begin{bmatrix} v_1 & v_1' & v_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ -2 & 1 & -2 \end{bmatrix} \\
J_1 &= \begin{bmatrix} -2 & 1 \\ & -2 \end{bmatrix}, \quad J_2 = -1, \quad J = \begin{bmatrix} -2 & 1 & \\ & -2 & \\ & & -1 \end{bmatrix} \\
e^{J_1 t} &= \begin{bmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{bmatrix}, \quad e^{J_2 t} = e^{-t}, \quad e^{Jt} = \begin{bmatrix} e^{-2t} & te^{-2t} & \\ & e^{-2t} & \\ & & e^{-t} \end{bmatrix} \\
e^{At} &= T^{-1} e^{Jt} T = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 2e^{-2t} - 2e^{-t} - 2te^{-2t} & e^{-t} & 2e^{-2t} - 2e^{-t} \\ -te^{-2t} & 0 & e^{-2t} \end{bmatrix}
\end{aligned}$$

Solution 2.3

$$\Phi(t,0) = e^{At} = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 2e^{-2t} - 2e^{-t} - 2te^{-2t} & e^{-t} & 2e^{-2t} - 2e^{-t} \\ -te^{-2t} & 0 & e^{-2t} \end{bmatrix}, \quad B(t)u(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
x(t) &= \Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)\mathrm{d}\tau \\
&= \begin{bmatrix} 0 \\ 2e^{-2t} - 2e^{-t} \\ e^{-2t} \end{bmatrix} + \int_0^t \Phi(t,\tau) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathrm{d}\tau \\
&= \begin{bmatrix} 0 \\ 2e^{-2t} - 2e^{-t} \\ e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} 0 \\ e^{-t+\tau} \\ 0 \end{bmatrix} \mathrm{d}\tau \\
&= \begin{bmatrix} 0 \\ 2e^{-2t} - 2e^{-t} \\ e^{-2t} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - e^{-t} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 + 2e^{-2t} - 3e^{-t} \\ e^{-2t} \end{bmatrix} \\
y(t) &= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x(t) \\
&= 1 + 2e^{-2t} - 3e^{-t}
\end{aligned}$$