Linear System Theory

Problem Set 3
Linear Time-Varying Systems, Linear Time-Invariant Systems

Issue date: Oct. 21, 2019
Due date: Oct. 31, 2019

Exercise 1. (Linear Time-Varying Systems, [50 points in total])
Let $A_1(t), A_2(t), F(t) \in \mathbb{R}^{n\times n}$ be piecewise continuous matrix functions. Let Φ_i be the state transition matrix for $\dot{x}(t) = A_i(t)x(t)$, for $i = 1, 2$. Consider the matrix differential equation:

$$\dot{X}(t) = A_1(t)X(t) + X(t)A_2^T(t) + F(t),\ X(t_0) = X_0,$$

where $X(t) \in \mathbb{R}^{n\times n}$ for any $t \geq t_0$.

1. [20 **points**] Show that this is an affine time-varying system. (Hint: An affine time-varying system is a system of the form $\dot{x}(t) = A(t)x(t) + b(t)$, where $x(t)$ and $b(t)$ are vectors.)

2. [30 **points**] Assume that the solution of the above system can be written as:

$$X(t) = \Phi_1(t, t_0)X_0\Phi_2^T(t, t_0) + \int_{t_0}^{t} \Phi_1(t, \tau)M(t, \tau)d\tau.$$

   Express the matrix $M(t, \tau)$ as a function of $\Phi_1(t, \tau)$, $F(t)$, and $\Phi_2(t, \tau)$. (Hint: $\Phi_1(t, \tau)$, $F(t)$, and $\Phi_2(t, \tau)$ may not all appear in the expression of $M(t, \tau)$.)

Exercise 2. (Linear Time-Invariant Systems, [50 points in total])
Consider the following affine system:

$$\dot{x}(t) = A\,x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x(t).$$

where $A \in \mathbb{R}^{3\times3}$. The matrix A has eigenvalues $\lambda_1 = -2$ with multiplicity 2, and $\lambda_2 = -1$ with multiplicity 1. The eigenvalue λ_1 has an eigenvector $v_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and a generalized eigenvector $v_1' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

The eigenvalue λ_2 has the eigenvector $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

1. [20 points] Find the matrix A.
2. [20 points] Calculate $\exp(A\,t)$.
3. [10 points] Given $x(0) = [0\ 0\ 1]^\top$, compute $y(t)$.