Linear System Theory

Solutions to Problem Set 2
Normed Spaces, ODEs, and Linear Time-varying Systems

Issue date: October 7, 2019
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Solution 1. (Norms, [45 points])

1. [**15 points in total**] The ∞-norm of a vector $f(t)$ is given by:

$$\|f(t)\|_\infty = \max_{t\in[t_0,t_1]} \|f(t)\|_p$$

We have to verify the three defining properties of a norm.

(a) First, we show $\|f\|_\infty = 0 \iff f = \theta_n$, where θ_n is the zero vector in $\mathbb{R}^n$. Note that, $\|f\|_\infty = 0 \Rightarrow \|f(t)\|_p = 0$ for all t. This is only true if $f(t)$ is a vector of zeros for all t. On the other hand, the p-norm of $\theta_n(t) = 0$. Thus, $\|f\|_\infty = 0 \iff f(t) = \theta_n$ [5 **points**].

(b) Next, for $a \in \mathbb{R}$, $\|af\|_\infty = |a|\|f\|_\infty$:
$\|af\|_\infty = \max_{t\in[t_0,t_1]} \|af(t)\|_p = \max_{t\in[t_0,t_1]} |a|\|f(t)\|_p = |a| \max_{t\in[t_0,t_1]} \|f(t)\|_p = |a|\|f\|_\infty$, where the second equality is by property of $\|f(t)\|_p$ being a norm [5 **points**].

(c) Finally, we verify the triangle inequality, $\|f+g\|_\infty \le \|f\|_\infty + \|g\|_\infty$:
$\max_{t\in[t_0,t_1]} \|f(t)+g(t)\|_p \le \max_{t\in[t_0,t_1]} \{\|f(t)\|_p + \|g(t)\|_p\} \le \max_{t\in[t_0,t_1]} \|f(t)\|_p + \max_{t\in[t_0,t_1]} \|g(t)\|_p$ [5 **points**].

2. [**10 points in total**] Note that for any $x \in \mathbb{R}^m, x \neq \theta_m$,

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{m}\|x\|_\infty$$

From the above relation, we have $\dfrac{1}{\|x\|_\infty} \ge \dfrac{1}{\|x\|_2}$.

Therefore for $A \in \mathbb{R}^{m\times n}$ with $x \in \mathbb{R}^n$, $x \neq \theta_n$,

$$\|Ax\|_2 \le \sqrt{m}\|Ax\|_\infty \Rightarrow \frac{\|Ax\|_2}{\|x\|_2} \le \sqrt{m}\frac{\|Ax\|_\infty}{\|x\|_\infty}, \forall x \neq \theta_n \Rightarrow \|A\|_2 \le \sqrt{m}\|A\|_\infty [5\textbf{points}].$$

To prove the lower bound, reconsider the norm-equivalence relation for $x \in \mathbb{R}^n$

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$$

From the above relation, we have $\dfrac{1}{\sqrt{n}\|x\|_\infty} \leq \dfrac{1}{\|x\|_2}$.

Therefore for $A \in \mathbb{R}^{m\times n}$ with $x \in \mathbb{R}^n, x \neq \theta_n$,

$$\|Ax\|_\infty \leq \|Ax\|_2 \Rightarrow \frac{\|Ax\|_\infty}{\sqrt{n}\|x\|_\infty} \leq \frac{\|Ax\|_2}{\|x\|_2}, \forall x \neq \theta_n \Rightarrow \frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 [5\textbf{points}].$$

3. [20 **points in total**] Consider the functions

$$f_n : [0,1] \to \mathbb{R} \quad \text{s.t.} \quad f_n(x) = \begin{cases} n - n^2x & 0 \leq x \leq \frac{1}{n} \\ 0 & \text{elsewhere} \end{cases}$$

for $n \in \mathbb{N}$ (notice that all these functions are continuous on $[0,1]$) as the left-hand limit and the value of the function coincide for every f_n at $x = \frac{1}{n}$ [2 **points**].

The 1-norm of f_n for all n is:

$$\|f_n\|_1 = \int_0^{\frac{1}{n}} (n - n^2x)dx = \left[nx - \frac{n^2x^2}{2}\right]_0^{\frac{1}{n}} \equiv \frac{1}{2} \qquad [1\textbf{point}].$$

The 2-norm of f_n can be computed as:

$$\|f_n\|_2 = \sqrt{\int_0^{\frac{1}{n}} (n - n^2x)^2dx} = \sqrt{\left[n^2x - n^3x^2 + \frac{n^4x^3}{3}\right]_0^{\frac{1}{n}}} = \frac{\sqrt{n}}{\sqrt{3}} \qquad [1\textbf{point}].$$

The ∞-norm of f_n being the max value is n.

$$\|f_n\|_\infty = \max_{t\in[0,1]} |f_n(t)| = f_n(0) = n \qquad [1\textbf{point}].$$

(a) Consider the 1-norm and the 2-norm. For all $\alpha > 0$, if $n > \frac{3}{4\alpha^2}$, it holds

$$\alpha \|f_n\|_2 = \alpha\frac{\sqrt{n}}{\sqrt{3}} > \frac{1}{2} = \|f_n\|_1$$

In other words, there does not exist an α such that $\alpha \|f\|_2 \leq \|f\|_1$ for all $f \in C([0,1], \mathbb{R})$, hence the two norms are not equivalent [5 **points**].

(b) Next, consider the 2-norm and the ∞-norm. For all $\beta > 0$, if $n > \frac{\beta^2}{3}$, it holds

$$\beta \|f_n\|_2 = \beta\frac{\sqrt{n}}{\sqrt{3}} < n = \|f_n\|_\infty$$

In other words, there does not exist an β such that $\beta \|f\|_2 \geq \|f\|_\infty$ for all $f \in C([0,1], \mathbb{R})$, hence the two norms are not equivalent [5 **points**].

(c) Finally, consider the 1-norm and the ∞-norm. For all $\gamma > 0$, if $n > \frac{\gamma}{2}$, it holds

$$\gamma \|f_n\|_1 = \gamma\frac{1}{2} < n = \|f_n\|_\infty$$

In other words, there does not exist a γ such that $\gamma \|f\|_1 \geq \|f\|_\infty$ for all $f \in C([0,1], \mathbb{R})$, hence the two norms are not equivalent [5 **points**].

Solution 2. (Convergence and completeness [25 points in total])

1. [20 **points in total**]

   Given an arbitrary initial point x, consider the sequence of iterates $f^{[n]}(x) = f(f^{[n-1]}(x))$, where the first iterate is given by $f^{[0]}(x) = x$. To derive the proof, we first show that this iteration converges to an element in X. Then, it follows that this element is a fixed point of the function f. Finally, we show its uniqueness.

   For all $n \in \mathbb{N}$ and $x \in X$

   $$\left\| f^{[n+1]}(x) - f^{[n]}(x) \right\| \leq \alpha^n \left\| f(x) - x \right\|$$

   (this can be seen by inspection or proved easily by induction) [5 **points**]. By similar means,

   $$\begin{aligned} \left\| f^{[n+2]}(x) - f^{[n]}(x) \right\| &= \left\| f^{[n+2]}(x) - f^{[n+1]}(x) + f^{[n+1]}(x) - f^{[n]}(x) \right\| \\ &\leq \left\| f^{[n+2]}(x) - f^{[n+1]}(x) \right\| + \left\| f^{[n+1]}(x) - f^{[n]}(x) \right\| \\ &\leq \alpha^{n+1} \left\| f(x) - x \right\| + \alpha^n \left\| f(x) - x \right\| \\ &= (\alpha^{n+1} + \alpha^n) \left\| f(x) - x \right\|. \end{aligned}$$

   Iterating this procedure, we obtain

   $$\begin{aligned} \left\| f^{[n+m]}(x) - f^{[n]}(x) \right\| &\leq \left\| f^{[n+m]}(x) - f^{[n+m-1]}(x) \right\| + \cdots + \left\| f^{[n+1]}(x) - f^{[n]}(x) \right\| \\ &\leq (\alpha^{n+m-1} + \alpha^{n+m-2} + \cdots + \alpha^{n+1} + \alpha^n) \left\| f(x) - x \right\| \\ &= \alpha^n (\alpha^{m-1} + \cdots + \alpha^1 + \alpha^0) \left\| f(x) - x \right\| \\ &= \alpha^n \frac{1 - \alpha^m}{1 - \alpha} \left\| f(x) - x \right\| \\ &\leq \frac{\alpha^n}{1 - \alpha} \left\| f(x) - x \right\|. \end{aligned}$$

   Since the last term tends to 0 as $n \to \infty$, irrespective of both m and x, the sequence $f^{[n]}(x)$ is Cauchy. But since X is a Banach space, every Cauchy sequence in X converges to an element of X. Hence, there exists $\bar{x} \in X$ such that $f^{[n]}(x) \to \bar{x}$ [5 **points**].

   Now, for all n we have:

   $$\begin{aligned} \left\| f(\bar{x}) - \bar{x} \right\| &= \left\| f(\bar{x}) - f^{[n+1]}(x) + f^{[n+1]}(x) - \bar{x} \right\| \\ &\leq \left\| f(\bar{x}) - f^{[n+1]}(x) \right\| + \left\| f^{[n+1]}(x) - \bar{x} \right\| \\ &\leq \alpha \left\| \bar{x} - f^{[n]}(x) \right\| + \left\| f^{[n+1]}(x) - \bar{x} \right\|. \end{aligned}$$

   Since the last two terms tend to zero as $n \to \infty$, and since $\left\| f(\bar{x}) - \bar{x} \right\| \geq 0$, it must hold $\left\| f(\bar{x}) - \bar{x} \right\| = 0$, or $f(\bar{x}) = \bar{x}$ [5 **points**].

   To show the uniqueness of such "fixed point", suppose that $f(\bar{x}) = \bar{x}$ and $f(\bar{y}) = \bar{y}$. Then

   $$\left\| \bar{x} - \bar{y} \right\| = \left\| f(\bar{x}) - f(\bar{y}) \right\| \leq \alpha \left\| \bar{x} - \bar{y} \right\| \quad \Rightarrow \quad (1 - \alpha) \left\| \bar{x} - \bar{y} \right\| \leq 0.$$

   Since $0 \leq \alpha < 1$ and $\left\| \bar{x} - \bar{y} \right\| \geq 0$, the only possibility is $\left\| \bar{x} - \bar{y} \right\| = 0$, which implies $\bar{x} = \bar{y}$ [5 **points**].

2. [5 **points in total**] The induced norm of f is given by

$$\|f\| = \sup_{x \in X} \frac{\|f(x)\|}{\|x\|} = \sup_{x,y \in X} \frac{\|f(x-y)\|}{\|x-y\|} = \sup_{x,y \in X} \frac{\|f(x) - f(y)\|}{\|x-y\|} \le \alpha,$$

where $\alpha \in [0, 1)$. Hence, we can conclude that the induced norm of f is upper-bounded by α [5 **points**].

Solution 3. (Ordinary differential equations [30 points in total])

1. [12 **points in total**]

   (a) From the definition of the Lipschitz condition, we can easily verify that all differentiable functions with bounded derivatives are Lipschitz [3 **points**]. First, let us define $p\colon \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^2$ by:

   $$p(x,t) = \begin{bmatrix} -x_1(t) + e^t \cos(x_1(t) - x_2(t)) \\ -x_2(t) + \sin(x_1(t) - x_2(t)) \end{bmatrix}.$$

   We derive the Jacobian of p as follows

   $$\frac{\partial p(x,t)}{\partial x} = \begin{bmatrix} -1 - e^t \sin(x_1(t) - x_2(t)) & e^t \sin(x_1(t) - x_2(t)) \\ \cos(x_1(t) - x_2(t)) & -1 - \cos(x_1(t) - x_2(t)) \end{bmatrix}.$$

   It follows that

   $$\left\| \frac{\partial p(x,t)}{\partial x} \right\| \leq \max\left(|-1 - e^t| + |e^t|, 3\right) \leq 3e^t.$$

   Then, $k(t) = 3e^t$ is the Lipschitz constant of p at $t \in \mathbb{R}$, hence p is Lipschitz in x [6 **points**].

   (b) Since p is globally Lipschitz in x and continuous in t, this system admits a unique solution [3 **points**].

2. [18 **points in total**]

   (a) This system is not globally Lipschitz. First, let us define $p\colon \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^2$ by:

   $$p(x,t) = \begin{bmatrix} -3\sin(t)\, x_1(t) + x_1(t)\, x_2(t) \\ -2\, x_2(t) \end{bmatrix}.$$

   Let $\alpha \gg 0$ and $\hat{x} = \begin{bmatrix} \alpha \\ \alpha \end{bmatrix} \in \mathbb{R}^2$. Then:

   $$\begin{aligned} \|p(\hat{x},t)\|_\infty &= \left\| \begin{bmatrix} -3\sin(t)\,\alpha + \alpha^2 \\ -2\,\alpha(t) \end{bmatrix} \right\|_\infty \\ &= -3\,\sin(t)\,\alpha + \alpha^2 \\ &\geq -3\,\alpha + \alpha^2, \end{aligned}$$

   where the second equality follows since $\alpha \gg 0$.

   Now let us show that p is not globally Lipschitz. We want to show that for each $k > 0$ there exists a pair $x, y \in \mathbb{R}^2$ such that $\|p(x,t) - p(y,t)\|_\infty > k\, \|x - y\|_\infty$ [6 **points for the negation**]. Let $\alpha > k + 3$ such that $\alpha \gg 0$, then,

   $$\|p(x,t) - p(\mathbb{0},t)\|_\infty \geq \alpha^2 - 3\,\alpha > k\,\alpha = k\, \|x - 0\|_\infty,$$

   where $\mathbb{0}$ denotes the vector of zeros in $\mathbb{R}^2$. This concludes the existence of an x, y pair for each $k > 0$. Therefore, p is not globally Lipschitz in x [6 **points**].

(b) This system admits a unique solution. To see that, let $x(0) = \hat{x} \in \mathbb{R}^2$ be the initial condition, then $x_2(t) = e^{-2\,t}\,\hat{x}_2$ is unique [3 **points**]. Hence, we get

$$\dot{x}_1(t) = \bar{p}(x_1, t) = \left(-3\,\sin(t)\, + e^{-2\,t}\,\hat{x}_2\right)\, x_1(t).$$

Since $\bar{p}$ is globally Lipschitz in x_1 and continuous in t, this differential equation also admits a unique solution [3 **points**]. The unique solution is given by

$$x_1(t) = e^{3\,\cos(t) - \frac{1}{2}\,e^{-2\,t}\,\hat{x}_2}\,\hat{x}_1.$$