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**Linear System Theory**  
**Problem Set 1**  
**Linear Spaces, Linear Maps, and Representations**

**Issue date: September 19, 2019**  
**Due date: October 7, 2019**

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**Exercise 1. (Linear spaces [40 points])**

1. [18 **points**] Let  $S$  be a set, and  $F = \{f : S \rightarrow \mathbb{R}_+\}$  be the space of functions from  $S$  to the (strictly) positive reals. Let the operations  $\oplus : F \times F \rightarrow F$ ,  $\odot : \mathbb{R} \times F \rightarrow F$  be defined as follows:

$$\begin{aligned} [f_1 \oplus f_2](x) &= f_1(x)f_2(x) & \forall f_1, f_2 \in F, \forall x \in S \\ [\alpha \odot f](x) &= f(x)^\alpha & \forall \alpha \in \mathbb{R}, \forall f \in F, \forall x \in S \end{aligned}$$

- Show that  $(F, \mathbb{R}, \oplus, \odot)$  is a linear space.
  - Identify the zero-vector.
2. [12 **points**] Let  $S = \{a, b\}$ , and let

$$\begin{aligned} f_1(a) &= 2, & f_1(b) &= 1 \\ f_2(a) &= 1, & f_2(b) &= 3 \\ f_3(a) &= 4, & f_3(b) &= 1 \end{aligned}$$

Show that  $\{f_1, f_2\}$  are linearly independent and that  $\{f_1, f_3\}$  are linearly dependent.

3. [10 **points**] Let  $\varphi : F \rightarrow F$  be defined as follows:

$$[\varphi(f)](x) = \sqrt{f(x)} \quad \forall f \in F, \forall x \in S$$

Show that  $\varphi$  is a linear map over the space  $F$  on  $(F, \mathbb{R}, \oplus, \odot)$ .

### Solution 1.1

- **vector addition**

- **associative:**

$$\forall f_1, f_2, f_3 \in F, f_1 \oplus (f_2 \oplus f_3) = f_1 \oplus (f_2 f_3) = f_1 (f_2 f_3) = f_1 f_2 f_3 = (f_1 f_2) f_3 = (f_1 \oplus f_2) \oplus f_3$$

- **commutative:**

$$\forall f_1, f_2 \in F, f_1 \oplus f_2 = f_1 f_2 = f_2 f_1 = f_2 \oplus f_1$$

- **identity:**

$$\text{Define } f_0(x) = 1, \forall x \in S. F = \{f : S \rightarrow \mathbb{R}_+\} \Rightarrow f_0 \in F.$$

$$\forall f \in F, f \oplus f_0 = f \cdot 1 = f.$$

- **inverse:**

$$\forall f \in F, f(x) \in \mathbb{R}_+, f(x) > 0, \frac{1}{f} > 0, \frac{1}{f} \in F, f \oplus \frac{1}{f} = f(x) \cdot \frac{1}{f(x)} = 1$$

- **scalar multiplication**

- **associative:**

$$\forall a, b \in \mathbb{R}, \forall f \in F, a \odot (b \odot f) = a \odot f^b = (f^b)^a = f^{ab} = (a \cdot b) \odot f$$

- **identity:**

$$\forall f \in F, 1 \odot f = f^1 = f$$

- **distributive scalar multiplication**

$$- \forall a, b \in \mathbb{R}, \forall f \in F, (a + b) \odot f = f^{a+b} = f^a f^b = (f^a)(f^b) = (a \odot f) \oplus (b \odot f)$$

$$- \forall a \in \mathbb{R}, \forall f_1, f_2 \in F, a \odot (f_1 \oplus f_2) = a \odot (f_1 f_2) = (f_1 f_2)^a = f_1^a f_2^a = (a \odot f_1) \oplus (a \odot f_2)$$

Therefore,  $(F, \mathbb{R}, \oplus, \odot)$  is a linear space and the zero-vector is 1.

### Solution 1.2

- $\{f_1, f_2\}$

For the sake of contradiction, we assume  $\{f_1, f_2\}$  are linearly dependent so that

$$\exists c_1, c_2 \in \mathbb{R} \text{ which are not both 0 such that } (c_1 \odot f_1(x)) \oplus (c_2 \odot f_2(x)) = 1$$

$$x = a: (c_1 \odot 2) \oplus (c_2 \odot 1) = 1 \Rightarrow 2^{c_1} \cdot 1^{c_2} = 2^{c_1} = 1 \Rightarrow c_1 = 0$$

$$x = b: (0 \odot 1) \oplus (c_2 \odot 3) = 1 \Rightarrow 2^0 \cdot 3^{c_2} = 3^{c_2} = 1 \Rightarrow c_2 = 0$$

$c_1 = c_2 = 0$  contradict our assumption  $\Rightarrow \{f_1, f_2\}$  must be linearly independent.

- $\{f_1, f_3\}$

$\{f_1, f_3\}$  are linearly dependent, if we can find  $c_1, c_2 \in \mathbb{R}$  that are not both 0 such that

$$(c_1 \odot f_1(x)) \oplus (c_2 \odot f_3(x)) = 1$$

$$x = a: (c_1 \odot 2) \oplus (c_2 \odot 4) = 1 \Rightarrow 2^{c_1} \cdot 4^{c_2} = 2^{c_1+2c_2} = 1$$

$$x = b: (c_1 \odot 1) \oplus (c_2 \odot 1) = 1 \Rightarrow 1^{c_1} \cdot 1^{c_2} = 1^{c_1+c_2} = 1$$

$1^{c_1+c_2} = 1$  implies  $c_1, c_2$  can be any real number. We can choose  $c_1 = 2, c_2 = -1$  to satisfy the two conditions above. Therefore,  $\{f_1, f_3\}$  are linearly dependent.

### Solution 1.3

$$\forall a_1, a_2 \in \mathbb{R}, \forall f_1, f_2 \in F,$$

$$\begin{aligned} \varphi((a_1 \odot f_1) \oplus (a_2 \odot f_2)) &= \varphi(f_1^{a_1} f_2^{a_2}) = \sqrt{f_1^{a_1} f_2^{a_2}} = \sqrt{f_1^{a_1}} \sqrt{f_2^{a_2}} \\ &= \sqrt{f_1^{a_1}} \sqrt{f_2^{a_2}} = (a_1 \odot \sqrt{f_1}) (a_2 \odot \sqrt{f_2}) \\ &= (a_1 \odot \varphi(f_1)) (a_2 \odot \varphi(f_2)) = (a_1 \odot \varphi(f_1)) \oplus (a_2 \odot \varphi(f_2)) \end{aligned}$$

Therefore,  $\varphi$  is a linear map over the space  $F$  on  $(F, \mathbb{R}, \oplus, \odot)$ .

**Exercise 2. (Range and null space [40 points])**

Let  $(F, +, \cdot)$  be a field and consider the linear maps  $\mathcal{A} : (F^n, F) \rightarrow (F^m, F)$  and  $\mathcal{B} : (F^m, F) \rightarrow (F^p, F)$ . Show, without using the matrix representation of linear maps, that:

1. [10 points]  $0 \leq \dim(\text{RANGE}(\mathcal{A})) \leq \min\{m, n\}$ .
2. [15 points]  $\dim(\text{RANGE}(\mathcal{A})) + \dim(\text{RANGE}(\mathcal{B})) - m \leq \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A}))$
3. [15 points]  $\dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq \min\{\dim(\text{RANGE}(\mathcal{A})), \dim(\text{RANGE}(\mathcal{B}))\}$ .

**Solution 2.1**

The first inequality is trivial since the number of vectors of a basis should definitely non-negative, i.e.,  $0 \leq \dim(\text{RANGE}(\mathcal{A}))$ . For the second inequality, we first prove a lemma.

**Lemma 1** *If  $(V, F)$  has dimension  $n$  then any set of  $n + 1$  or more vectors is linearly dependent.*

**Proof 1** *For the sake of contradiction, we assume that there exists a set of  $n + k$  ( $k > 0, k \in \mathbb{N}$ ) linearly independent vectors  $\{v_1, \dots, v_{n+k}\}$ , then  $\dim(\text{span}(\{v_1, \dots, v_{n+k}\})) = n + k$ .*

*Since  $\{v_1, \dots, v_{n+k}\} \subseteq (V, F)$ ,  $\text{span}(\{v_1, \dots, v_{n+k}\}) \subseteq (V, F)$ ,  $n + k = \dim(\text{span}(\{v_1, \dots, v_{n+k}\})) \leq \dim((V, F)) = n$ , which leads to contradiction.*

Next, we prove  $\dim(\text{RANGE}(\mathcal{A})) \leq m$ . For the sake of contradiction, assume  $\dim(\text{RANGE}(\mathcal{A})) = p > m$ , and a basis of  $\text{RANGE}(\mathcal{A})$  is  $\{v_1, \dots, v_p\}$ , which are linearly independent.

The fact that linearly independent vectors  $\{v_1, \dots, v_p\} \subseteq \text{RANGE}(\mathcal{A}) \subseteq (V, F)$  and the dimension of  $(V, F)$  is  $m < p$  contradict the lemma. Therefore  $\text{RANGE}(\mathcal{A}) \leq m$ .

Finally, we prove  $\dim(\text{RANGE}(\mathcal{A})) \leq n$ .  $\forall v \in V, \exists u \in U, \mathcal{A}(u) = v$ . Let  $\{e_1, \dots, e_n\}$  be a basis of  $U$ .

$\forall u \in U, \exists a_1, \dots, a_n, u = a_1 e_1 + \dots + a_n e_n$ , where  $a_i$  are not all 0.

Therefore,  $\forall v \in V, v = \mathcal{A}(u) = \mathcal{A}(a_1 e_1 + \dots + a_n e_n) = a_1 \mathcal{A}(e_1) + \dots + a_n \mathcal{A}(e_n)$ . If  $\{\mathcal{A}(e_1), \dots, \mathcal{A}(e_n)\}$  are linearly independent, then  $\dim(\text{RANGE}(\mathcal{A})) = n$ , otherwise  $\dim(\text{RANGE}(\mathcal{A})) < n$ .

$$\begin{cases} \dim(\text{RANGE}(\mathcal{A})) \leq n \\ \dim(\text{RANGE}(\mathcal{A})) \leq m \end{cases} \Rightarrow \dim(\text{RANGE}(\mathcal{A})) \leq \min\{m, n\}, \text{ which completes the proof.}$$

**Solution 2.2**

$$\begin{aligned} & \text{RANGE}(\mathcal{A}) \cap \text{NULL}(\mathcal{B}) \subseteq \text{NULL}(\mathcal{B}) \\ \Leftrightarrow & \dim(\text{RANGE}(\mathcal{A}) \cap \text{NULL}(\mathcal{B})) \leq \dim(\text{NULL}(\mathcal{B})) \\ \Leftrightarrow & \dim(\text{RANGE}(\mathcal{A}) \cap \text{NULL}(\mathcal{B})) \leq \dim(\text{NULL}(\mathcal{B})) + \dim(\text{NULL}(\mathcal{A})) \\ \Leftrightarrow & \dim(\text{NULL}(\mathcal{B} \circ \mathcal{A})) \leq \dim(\text{NULL}(\mathcal{B})) + \dim(\text{NULL}(\mathcal{A})) \\ \Leftrightarrow & n - \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq m - \dim(\text{RANGE}(\mathcal{B})) + n - \dim(\text{RANGE}(\mathcal{A})) \\ \Leftrightarrow & \dim(\text{RANGE}(\mathcal{A})) + \dim(\text{RANGE}(\mathcal{B})) - m \leq \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \end{aligned}$$

**Solution 2.3**

$$\text{RANGE}(\mathcal{A}) \subseteq F^m \Rightarrow \text{RANGE}(\mathcal{B} \circ \mathcal{A}) \subseteq \text{RANGE}(\mathcal{B}) \Rightarrow \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq \dim(\text{RANGE}(\mathcal{B}))$$

Let  $\{v_1, \dots, v_r\}$  be a set of basis of  $\text{RANGE}(\mathcal{A})$ .

$$\text{RANGE}(\mathcal{B} \circ \mathcal{A}) = \{l \in F^p \mid \exists w \in \text{RANGE}(\mathcal{A}), \mathcal{B}(w) = l\}.$$

Since  $\text{RANGE}(\mathcal{A}) = \text{span}(\{v_1, \dots, v_r\})$ ,  $\forall w \in \text{RANGE}(\mathcal{A}), \exists c_1, \dots, c_r \in F, w = c_1 v_1 + \dots + c_r v_r$ .

Due to the linear map  $\mathcal{B}$ ,  $\mathcal{B}(w) = c_1 \mathcal{B}(v_1) + \dots + c_r \mathcal{B}(v_r)$ .

$$\forall l \in F^p, \exists w \in \text{RANGE}(\mathcal{A}), \exists c_1, \dots, c_r, l = \mathcal{B}(w) = \mathcal{B}(c_1 v_1 + \dots + c_r v_r) = c_1 \mathcal{B}(v_1) + \dots + c_r \mathcal{B}(v_r).$$

If  $\{\mathcal{B}(v_1), \dots, \mathcal{B}(v_r)\}$  are linearly independent,  $\dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) = r = \dim(\text{RANGE}(\mathcal{A}))$ ,

otherwise  $\dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) < r = \dim(\text{RANGE}(\mathcal{A}))$ .

$$\begin{cases} \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq \dim(\text{RANGE}(\mathcal{A})) \\ \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq \dim(\text{RANGE}(\mathcal{B})) \end{cases} \Rightarrow \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq \min\{\dim(\text{RANGE}(\mathcal{A})), \dim(\text{RANGE}(\mathcal{B}))\}$$

**Exercise 3. (Linear maps and matrix representations [20 points])**

Consider a linear map  $\mathcal{A} : (U, F) \rightarrow (U, F)$  where  $U$  has finite dimension  $n$ .

1. [10 points] Assume there exists a basis  $\nu_i$ ,  $i = 1, \dots, n$  for  $U$  such that  $\mathcal{A}(\nu_n) = \lambda\nu_n$  and  $\mathcal{A}(\nu_i) = \lambda\nu_i + \nu_{i+1}$ ,  $i = 1, \dots, n-1$ . Derive the representation of  $\mathcal{A}$  with respect to this basis.
2. [10 points] Assume there exists a vector  $b \in U$  such that the set  $\{b, \mathcal{A}(b), \mathcal{A} \circ \mathcal{A}(b), \dots, \mathcal{A}^{n-1}(b)\}$  is linearly independent. Derive the representation of  $\mathcal{A}$  with respect to this basis.

**Solution 3.1**

$$\begin{aligned}
 \mathcal{A}(\nu_1) &= \lambda\nu_1 + \nu_2 \\
 \mathcal{A}(\nu_2) &= \lambda\nu_2 + \nu_3 \\
 &\vdots \\
 \mathcal{A}(\nu_{n-1}) &= \lambda\nu_{n-1} + \nu_n \\
 \mathcal{A}(\nu_n) &= \lambda\nu_n
 \end{aligned}$$

$$a_{ij} = \begin{cases} \lambda & i = j \\ 1 & i = j + 1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow A = \begin{bmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & 1 & \ddots & & \\ & & \ddots & \lambda & \\ & & & 1 & \lambda \end{bmatrix}$$

**Solution 3.2**

$\mathcal{A} : (U, F) \rightarrow (U, F) \Rightarrow \mathcal{A}^n(b) \in U$ .

Since the set  $\{b, \mathcal{A}(b), \mathcal{A} \circ \mathcal{A}(b), \dots, \mathcal{A}^{n-1}(b)\}$  is linearly independent and  $\dim(U) = n$ ,  $\exists c_1, \dots, c_n \in F$ , which are not all zero, such that  $\mathcal{A}^n(b) = c_1b + \dots + c_n\mathcal{A}^{n-1}(b)$ .

$$a_{ij} = \begin{cases} 1 & i = j + 1 \\ c_j & j = n \\ 0 & \text{otherwise} \end{cases} \Rightarrow A = \begin{bmatrix} 1 & & & & c_1 \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & \vdots \\ & & & & c_n \end{bmatrix}$$