
Linear System Theory
Problem Set 1
Linear Spaces, Linear Maps, and Representations
Issue date: September 19, 2019
Due date: October 7, 2019

Exercise 1. (Linear spaces [40 points])

1. [18 **points**] Let S be a set, and $F = \{f : S \rightarrow \mathbb{R}_+\}$ be the space of functions from S to the (strictly) positive reals. Let the operations $\oplus : F \times F \rightarrow F$, $\odot : \mathbb{R} \times F \rightarrow F$ be defined as follows:

$$\begin{aligned} [f_1 \oplus f_2](x) &= f_1(x)f_2(x) & \forall f_1, f_2 \in F, \forall x \in S \\ [\alpha \odot f](x) &= f(x)^\alpha & \forall \alpha \in \mathbb{R}, \forall f \in F, \forall x \in S \end{aligned}$$

- Show that $(F, \mathbb{R}, \oplus, \odot)$ is a linear space.
 - Identify the zero-vector.
2. [12 **points**] Let $S = \{a, b\}$, and let

$$\begin{aligned} f_1(a) &= 2, & f_1(b) &= 1 \\ f_2(a) &= 1, & f_2(b) &= 3 \\ f_3(a) &= 4, & f_3(b) &= 1 \end{aligned}$$

Show that $\{f_1, f_2\}$ are linearly independent and that $\{f_1, f_3\}$ are linearly dependent.

3. [10 **points**] Let $\varphi : F \rightarrow F$ be defined as follows:

$$[\varphi(f)](x) = \sqrt{f(x)} \quad \forall f \in F, \forall x \in S$$

Show that φ is a linear map over the space F on $(F, \mathbb{R}, \oplus, \odot)$.

Solution 1.1

- **vector addition**

- **associative:**

$$\forall f_1, f_2, f_3 \in F, f_1 \oplus (f_2 \oplus f_3) = f_1 \oplus (f_2 f_3) = f_1 (f_2 f_3) = f_1 f_2 f_3 = (f_1 f_2) f_3 = (f_1 \oplus f_2) \oplus f_3$$

- **commutative:**

$$\forall f_1, f_2 \in F, f_1 \oplus f_2 = f_1 f_2 = f_2 f_1 = f_2 \oplus f_1$$

- **identity:**

$$\text{Define } f_0(x) = 1, \forall x \in S. F = \{f : S \rightarrow \mathbb{R}_+\} \Rightarrow f_0 \in F.$$

$$\forall f \in F, f \oplus f_0 = f \cdot 1 = f.$$

- **inverse:**

$$\forall f \in F, f(x) \in \mathbb{R}_+, f(x) > 0, \frac{1}{f} > 0, \frac{1}{f} \in F, f \oplus \frac{1}{f} = f(x) \cdot \frac{1}{f(x)} = 1$$

- **scalar multiplication**

- **associative:**

$$\forall a, b \in \mathbb{R}, \forall f \in F, a \odot (b \odot f) = a \odot f^b = (f^b)^a = f^{ab} = (a \cdot b) \odot f$$

- **identity:**

$$\forall f \in F, 1 \odot f = f^1 = f$$

- **distributive scalar multiplication**

$$- \forall a, b \in \mathbb{R}, \forall f \in F, (a + b) \odot f = f^{a+b} = f^a f^b = (f^a)(f^b) = (a \odot f) \oplus (b \odot f)$$

$$- \forall a \in \mathbb{R}, \forall f_1, f_2 \in F, a \odot (f_1 \oplus f_2) = a \odot (f_1 f_2) = (f_1 f_2)^a = f_1^a f_2^a = (a \odot f_1) \oplus (a \odot f_2)$$

Therefore, $(F, \mathbb{R}, \oplus, \odot)$ is a linear space and the zero-vector is 1.

Solution 1.2

- $\{f_1, f_2\}$

For the sake of contradiction, we assume $\{f_1, f_2\}$ are linearly dependent so that

$$\exists c_1, c_2 \in \mathbb{R} \text{ which are not both 0 such that } (c_1 \odot f_1(x)) \oplus (c_2 \odot f_2(x)) = 1$$

$$x = a: (c_1 \odot 2) \oplus (c_2 \odot 1) = 1 \Rightarrow 2^{c_1} \cdot 1^{c_2} = 2^{c_1} = 1 \Rightarrow c_1 = 0$$

$$x = b: (0 \odot 1) \oplus (c_2 \odot 3) = 1 \Rightarrow 2^0 \cdot 3^{c_2} = 3^{c_2} = 1 \Rightarrow c_2 = 0$$

$c_1 = c_2 = 0$ contradict our assumption $\Rightarrow \{f_1, f_2\}$ must be linearly independent.

- $\{f_1, f_3\}$

$\{f_1, f_3\}$ are linearly dependent, if we can find $c_1, c_2 \in \mathbb{R}$ that are not both 0 such that

$$(c_1 \odot f_1(x)) \oplus (c_2 \odot f_3(x)) = 1$$

$$x = a: (c_1 \odot 2) \oplus (c_2 \odot 4) = 1 \Rightarrow 2^{c_1} \cdot 4^{c_2} = 2^{c_1+2c_2} = 1$$

$$x = b: (c_1 \odot 1) \oplus (c_2 \odot 1) = 1 \Rightarrow 1^{c_1} \cdot 1^{c_2} = 1^{c_1+c_2} = 1$$

$1^{c_1+c_2} = 1$ implies c_1, c_2 can be any real number. We can choose $c_1 = 2, c_2 = -1$ to satisfy the two conditions above. Therefore, $\{f_1, f_3\}$ are linearly dependent.

Solution 1.3

$$\forall a_1, a_2 \in \mathbb{R}, \forall f_1, f_2 \in F,$$

$$\begin{aligned} \varphi((a_1 \odot f_1) \oplus (a_2 \odot f_2)) &= \varphi(f_1^{a_1} f_2^{a_2}) = \sqrt{f_1^{a_1} f_2^{a_2}} = \sqrt{f_1^{a_1}} \sqrt{f_2^{a_2}} \\ &= \sqrt{f_1^{a_1}} \sqrt{f_2^{a_2}} = (a_1 \odot \sqrt{f_1}) (a_2 \odot \sqrt{f_2}) \\ &= (a_1 \odot \varphi(f_1)) (a_2 \odot \varphi(f_2)) = (a_1 \odot \varphi(f_1)) \oplus (a_2 \odot \varphi(f_2)) \end{aligned}$$

Therefore, φ is a linear map over the space F on $(F, \mathbb{R}, \oplus, \odot)$.

Exercise 2. (Range and null space [40 points])

Let $(F, +, \cdot)$ be a field and consider the linear maps $\mathcal{A} : (F^n, F) \rightarrow (F^m, F)$ and $\mathcal{B} : (F^m, F) \rightarrow (F^p, F)$. Show, without using the matrix representation of linear maps, that:

1. [10 points] $0 \leq \dim(\text{RANGE}(\mathcal{A})) \leq \min\{m, n\}$.
2. [15 points] $\dim(\text{RANGE}(\mathcal{A})) + \dim(\text{RANGE}(\mathcal{B})) - m \leq \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A}))$
3. [15 points] $\dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq \min\{\dim(\text{RANGE}(\mathcal{A})), \dim(\text{RANGE}(\mathcal{B}))\}$.

Solution 2.1

The first inequality is trivial since the number of vectors of a basis should definitely non-negative, i.e., $0 \leq \dim(\text{RANGE}(\mathcal{A}))$. For the second inequality, we first prove a lemma.

Lemma 1 *If (V, F) has dimension n then any set of $n + 1$ or more vectors is linearly dependent.*

Proof 1 *For the sake of contradiction, we assume that there exists a set of $n + k$ ($k > 0, k \in \mathbb{N}$) linearly independent vectors $\{v_1, \dots, v_{n+k}\}$, then $\dim(\text{span}(\{v_1, \dots, v_{n+k}\})) = n + k$.*

Since $\{v_1, \dots, v_{n+k}\} \subseteq (V, F)$, $\text{span}(\{v_1, \dots, v_{n+k}\}) \subseteq (V, F)$, $n + k = \dim(\text{span}(\{v_1, \dots, v_{n+k}\})) \leq \dim((V, F)) = n$, which leads to contradiction.

Next, we prove $\dim(\text{RANGE}(\mathcal{A})) \leq m$. For the sake of contradiction, assume $\dim(\text{RANGE}(\mathcal{A})) = p > m$, and a basis of $\text{RANGE}(\mathcal{A})$ is $\{v_1, \dots, v_p\}$, which are linearly independent.

The fact that linearly independent vectors $\{v_1, \dots, v_p\} \subseteq \text{RANGE}(\mathcal{A}) \subseteq (V, F)$ and the dimension of (V, F) is $m < p$ contradict the lemma. Therefore $\text{RANGE}(\mathcal{A}) \leq m$.

Finally, we prove $\dim(\text{RANGE}(\mathcal{A})) \leq n$. $\forall v \in V, \exists u \in U, \mathcal{A}(u) = v$. Let $\{e_1, \dots, e_n\}$ be a basis of U .

$\forall u \in U, \exists a_1, \dots, a_n, u = a_1 e_1 + \dots + a_n e_n$, where a_i are not all 0.

Therefore, $\forall v \in V, v = \mathcal{A}(u) = \mathcal{A}(a_1 e_1 + \dots + a_n e_n) = a_1 \mathcal{A}(e_1) + \dots + a_n \mathcal{A}(e_n)$. If $\{\mathcal{A}(e_1), \dots, \mathcal{A}(e_n)\}$ are linearly independent, then $\dim(\text{RANGE}(\mathcal{A})) = n$, otherwise $\dim(\text{RANGE}(\mathcal{A})) < n$.

$$\begin{cases} \dim(\text{RANGE}(\mathcal{A})) \leq n \\ \dim(\text{RANGE}(\mathcal{A})) \leq m \end{cases} \Rightarrow \dim(\text{RANGE}(\mathcal{A})) \leq \min\{m, n\}, \text{ which completes the proof.}$$

Solution 2.2

$$\begin{aligned} & \text{RANGE}(\mathcal{A}) \cap \text{NULL}(\mathcal{B}) \subseteq \text{NULL}(\mathcal{B}) \\ \Leftrightarrow & \dim(\text{RANGE}(\mathcal{A}) \cap \text{NULL}(\mathcal{B})) \leq \dim(\text{NULL}(\mathcal{B})) \\ \Leftrightarrow & \dim(\text{RANGE}(\mathcal{A}) \cap \text{NULL}(\mathcal{B})) \leq \dim(\text{NULL}(\mathcal{B})) + \dim(\text{NULL}(\mathcal{A})) \\ \Leftrightarrow & \dim(\text{NULL}(\mathcal{B} \circ \mathcal{A})) \leq \dim(\text{NULL}(\mathcal{B})) + \dim(\text{NULL}(\mathcal{A})) \\ \Leftrightarrow & n - \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq m - \dim(\text{RANGE}(\mathcal{B})) + n - \dim(\text{RANGE}(\mathcal{A})) \\ \Leftrightarrow & \dim(\text{RANGE}(\mathcal{A})) + \dim(\text{RANGE}(\mathcal{B})) - m \leq \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \end{aligned}$$

Solution 2.3

$$\text{RANGE}(\mathcal{A}) \subseteq F^m \Rightarrow \text{RANGE}(\mathcal{B} \circ \mathcal{A}) \subseteq \text{RANGE}(\mathcal{B}) \Rightarrow \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq \dim(\text{RANGE}(\mathcal{B}))$$

Let $\{v_1, \dots, v_r\}$ be a set of basis of $\text{RANGE}(\mathcal{A})$.

$$\text{RANGE}(\mathcal{B} \circ \mathcal{A}) = \{l \in F^p \mid \exists w \in \text{RANGE}(\mathcal{A}), \mathcal{B}(w) = l\}.$$

Since $\text{RANGE}(\mathcal{A}) = \text{span}(\{v_1, \dots, v_r\})$, $\forall w \in \text{RANGE}(\mathcal{A}), \exists c_1, \dots, c_r \in F, w = c_1 v_1 + \dots + c_r v_r$.

Due to the linear map \mathcal{B} , $\mathcal{B}(w) = c_1 \mathcal{B}(v_1) + \dots + c_r \mathcal{B}(v_r)$.

$$\forall l \in F^p, \exists w \in \text{RANGE}(\mathcal{A}), \exists c_1, \dots, c_r, l = \mathcal{B}(w) = \mathcal{B}(c_1 v_1 + \dots + c_r v_r) = c_1 \mathcal{B}(v_1) + \dots + c_r \mathcal{B}(v_r).$$

If $\{\mathcal{B}(v_1), \dots, \mathcal{B}(v_r)\}$ are linearly independent, $\dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) = r = \dim(\text{RANGE}(\mathcal{A}))$,

otherwise $\dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) < r = \dim(\text{RANGE}(\mathcal{A}))$.

$$\begin{cases} \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq \dim(\text{RANGE}(\mathcal{A})) \\ \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq \dim(\text{RANGE}(\mathcal{B})) \end{cases} \Rightarrow \dim(\text{RANGE}(\mathcal{B} \circ \mathcal{A})) \leq \min\{\dim(\text{RANGE}(\mathcal{A})), \dim(\text{RANGE}(\mathcal{B}))\}$$

Exercise 3. (Linear maps and matrix representations [20 points])

Consider a linear map $\mathcal{A} : (U, F) \rightarrow (U, F)$ where U has finite dimension n .

1. [10 points] Assume there exists a basis $\nu_i, i = 1, \dots, n$ for U such that $\mathcal{A}(\nu_n) = \lambda\nu_n$ and $\mathcal{A}(\nu_i) = \lambda\nu_i + \nu_{i+1}, i = 1, \dots, n-1$. Derive the representation of \mathcal{A} with respect to this basis.
2. [10 points] Assume there exists a vector $b \in U$ such that the set $\{b, \mathcal{A}(b), \mathcal{A} \circ \mathcal{A}(b), \dots, \mathcal{A}^{n-1}(b)\}$ is linearly independent. Derive the representation of \mathcal{A} with respect to this basis.

Solution 3.1

$$\begin{aligned}
 \mathcal{A}(\nu_1) &= \lambda\nu_1 + \nu_2 \\
 \mathcal{A}(\nu_2) &= \lambda\nu_2 + \nu_3 \\
 &\vdots \\
 \mathcal{A}(\nu_{n-1}) &= \lambda\nu_{n-1} + \nu_n \\
 \mathcal{A}(\nu_n) &= \lambda\nu_n
 \end{aligned}$$

$$a_{ij} = \begin{cases} \lambda & i = j \\ 1 & i = j + 1 \\ 0 & \text{otherwise} \end{cases} \Rightarrow A = \begin{bmatrix} \lambda & & & & \\ 1 & \lambda & & & \\ & 1 & \ddots & & \\ & & \ddots & \lambda & \\ & & & 1 & \lambda \end{bmatrix}$$

Solution 3.2

$\mathcal{A} : (U, F) \rightarrow (U, F) \Rightarrow \mathcal{A}^n(b) \in U$.

Since the set $\{b, \mathcal{A}(b), \mathcal{A} \circ \mathcal{A}(b), \dots, \mathcal{A}^{n-1}(b)\}$ is linearly independent and $\dim(U) = n$, $\exists c_1, \dots, c_n \in F$, which are not all zero, such that $\mathcal{A}^n(b) = c_1b + \dots + c_n\mathcal{A}^{n-1}(b)$.

$$a_{ij} = \begin{cases} 1 & i = j + 1 \\ c_j & j = n \\ 0 & \text{otherwise} \end{cases} \Rightarrow A = \begin{bmatrix} 1 & & & & c_1 \\ & 1 & & & \vdots \\ & & \ddots & & \vdots \\ & & & 1 & \vdots \\ & & & & c_n \end{bmatrix}$$