Linear System Theory
Problem Set 3
Linear Time-Varying Systems, Linear Time-Invariant Systems
Issue date: Oct. 21, 2019
Due date: Oct. 31, 2019

Exercise 1. (Linear Time-Varying Systems, [50 points in total])
Let $A_1(t), A_2(t),\ F(t) \in \mathbb{R}^{n\times n}$ be piecewise continuous matrix functions. Let Φ_i be the state transition matrix for $\dot{x}(t) = A_i(t)x(t)$, for $i = 1, 2$. Consider the matrix differential equation:

$$\dot{X}(t) = A_1(t)X(t) + X(t)A_2^T(t) + F(t),\ X(t_0) = X_0,$$

where $X(t) \in \mathbb{R}^{n\times n}$ for any $t \geq t_0$.

1. [20 **points**] Show that this is an affine time-varying system. (Hint: An affine time-varying system is a system of the form $\dot{x}(t) = A(t)x(t) + b(t)$, where $x(t)$ and $b(t)$ are vectors.)
2. [30 **points**] Assume that the solution of the above system can be written as:

$$X(t) = \Phi_1(t,t_0)X_0\Phi_2^T(t,t_0) + \int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)d\tau.$$

   Express the matrix $M(t,\tau)$ as a function of $\Phi_1(t,\tau)$, $F(t)$, and $\Phi_2(t,\tau)$. (Hint: $\Phi_1(t,\tau)$, $F(t)$, and $\Phi_2(t,\tau)$ may not all appear in the expression of $M(t,\tau)$.)

Solution 1.1
Let $x_{ij}, a_{ij}^{(1)}, a_{ij}^{(2)}, f_{ij}$ denote the i^{th} row and j^{th} column element of matrix X, A_1, A_2, F respectively. According to the matrix differential equation, x_{ij} can be expressed as

$$\begin{aligned} x_{ij} &= \sum_{k=1}^{n} a_{ik}^{(1)}x_{kj} + \sum_{k=1}^{n} x_{ik}a_{jk}^{(2)} + f_{ij} \\ &= \sum_{k=1}^{n}\left(a_{ik}^{(1)}x_{kj} + a_{jk}^{(2)}x_{ik}\right) + f_{ij} \end{aligned}$$

Define vector $v \in \mathbb{R}^{n^2}$ and let $v_{i+n(j-1)} = x_{ij}$. Plug v into the equation above:

$$v_{i+n(j-1)} = \sum_{k=1}^{n} a_{ik}^{(1)}v_{k+n(j-1)} + \sum_{k=1}^{n} v_{i+n(k-1)}a_{jk}^{(2)} + f_{ij} \tag{1}$$

Similarly, define vector b and let $b_{i+n(j-1)} = F_{ij}$. As a consequence, we can rewrite the system dynamics as

$$\dot{v}(t) = A(t)v(t) + b(t)\ ,$$

where $A(t)$ is uniquely defined by (1) and $b(t)$ defined by $F(t)$.
Thus, the system is an affine time-varying system.

Solution 1.2

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right] &= \int_{t_0}^{t}\frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_1(t,\tau)M(t,\tau)\right)\mathrm{d}\tau + \Phi_1(t,t)M(t,t)\cdot 1 \\ &= \int_{t_0}^{t} A_1(t)\Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + \int_{t_0}^{t}\Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \\ &= A_1(t)\int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + \int_{t_0}^{t}\Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \end{aligned}$$

$$
\begin{aligned}
\dot{X}(t) &= A_1(t)\Phi_1(t,t_0)X_0\Phi_2^T(t,t_0) + \Phi_1(t,t_0)X_0\Phi_2^T(t,t_0)A_2^T(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right] \\
&= A_1(t)X(t) - A_1(t)\int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + X(t)A_2^T(t) - \int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau A_2^T(t) \\
&\quad + A_1(t)\int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + \int_{t_0}^{t}\Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \\
&= A_1(t)X(t) + X(t)A_2^T(t) - \int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau A_2^T(t) + \int_{t_0}^{t}\Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t)
\end{aligned}
$$

From the matrix differential equation, we also know that

$$
\dot{X}(t) = A_1(t)X(t) + X(t)A_2^T(t) + F(t)
$$

Compare these two equations, it follows immediately

$$
F(t) = -\int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau A_2^T(t) + \int_{t_0}^{t}\Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t)
$$

Exercise 2. (Linear Time-Invariant Systems, [50 points in total])
Consider the following affine system:

$$\dot{x}(t) = A\,x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad y(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x(t).$$

where $A \in \mathbb{R}^{3\times 3}$. The matrix A has eigenvalues $\lambda_1 = -2$ with multiplicity 2, and $\lambda_2 = -1$ with multiplicity 1. The eigenvalue λ_1 has an eigenvector $v_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, and a generalized eigenvector $v_1' = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. The eigenvalue λ_2 has the eigenvector $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

1. [20 points] Find the matrix A.
2. [20 points] Calculate $\exp(At)$.
3. [10 points] Given $x(0) = [0\ 0\ 1]^\top$, compute $y(t)$.