Linear System Theory

Problem Set 3
Linear Time-Varying Systems, Linear Time-Invariant Systems

Issue date: Oct. 21, 2019
Due date: Oct. 31, 2019

Exercise 1. (Linear Time-Varying Systems, [50 points in total])
Let $A_1(t), A_2(t)$, $F(t) \in \mathbb{R}^{n\times n}$ be piecewise continuous matrix functions. Let Φ_i be the state transition matrix for $\dot{x}(t) = A_i(t)x(t)$, for $i = 1, 2$. Consider the matrix differential equation:

$$\dot{X}(t) = A_1(t)X(t) + X(t)A_2^T(t) + F(t),\ X(t_0) = X_0,$$

where $X(t) \in \mathbb{R}^{n\times n}$ for any $t \geq t_0$.

1. [20 **points**] Show that this is an affine time-varying system. (Hint: An affine time-varying system is a system of the form $\dot{x}(t) = A(t)x(t) + b(t)$, where $x(t)$ and $b(t)$ are vectors.)

2. [30 **points**] Assume that the solution of the above system can be written as:

$$X(t) = \Phi_1(t, t_0)X_0\Phi_2^T(t, t_0) + \int_{t_0}^{t} \Phi_1(t, \tau)M(t, \tau)d\tau.$$

   Express the matrix $M(t, \tau)$ as a function of $\Phi_1(t, \tau)$, $F(t)$, and $\Phi_2(t, \tau)$. (Hint: $\Phi_1(t, \tau)$, $F(t)$, and $\Phi_2(t, \tau)$ may not all appear in the expression of $M(t, \tau)$.)

Solution 1.1

Let $x_{ij}, a_{ij}^{(1)}, a_{ij}^{(2)}, f_{ij}$ denote the i^{th} row and j^{th} column element of matrix X, A_1, A_2, F respectively. According to the matrix differential equation, x_{ij} can be expressed as

$$\begin{aligned}\dot{x}_{ij} &= \sum_{k=1}^{n} a_{ik}^{(1)}x_{kj} + \sum_{k=1}^{n} x_{ik}a_{jk}^{(2)} + f_{ij} \\ &= \sum_{k=1}^{n}\left(a_{ik}^{(1)}x_{kj} + a_{jk}^{(2)}x_{ik}\right) + f_{ij} \qquad (1)\end{aligned}$$

Let x_{ci} and f_{ci} denote the i^{th} column of matrix X and F respectively. Define vector $\tilde{x}, \tilde{f} \in \mathbb{R}^{n^2}$ by rearranging x_{ci} and f_{ci}:

$$\tilde{x} = \begin{bmatrix} x_{c1} \\ \vdots \\ x_{cn} \end{bmatrix}, \quad \tilde{f} = \begin{bmatrix} f_{c1} \\ \vdots \\ f_{cn} \end{bmatrix}.$$

Based on the equation (1), elements of $\dot{\tilde{x}}$ can be expressed as

$$\dot{\tilde{x}}_{i+n(j-1)} = \sum_{k=1}^{n} a_{ik}^{(1)}\tilde{x}_{k+n(j-1)} + \sum_{k=1}^{n} a_{jk}^{(2)}\tilde{x}_{i+n(k-1)} + f_{ij}\ , \qquad (2)$$

which defines an affine time-varying system. To express it more explicitly, we define augmented matrix $\tilde{A}_1$ and $\tilde{A}_2$ to facilitate further notation:

$$\tilde{A}_1 = \mathrm{diag}(A, \dots, A) = \begin{bmatrix} A & & \\ & \ddots & \\ & & A \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2},$$

$$\tilde{A}_2 = \begin{bmatrix} \mathrm{diag}(a_{11}^{(2)}) & \mathrm{diag}(a_{12}^{(2)}) & \cdots & \mathrm{diag}(a_{1n}^{(2)}) \\ \mathrm{diag}(a_{21}^{(2)}) & \mathrm{diag}(a_{22}^{(2)}) & \cdots & \mathrm{diag}(a_{2n}^{(2)}) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{diag}(a_{n1}^{(2)}) & \mathrm{diag}(a_{n2}^{(2)}) & \cdots & \mathrm{diag}(a_{nn}^{(2)}) \end{bmatrix} \in \mathbb{R}^{n^2 \times n^2},$$

where $\mathrm{diag}(a_{ij}^{(2)})$ denotes an $n \times n$ diagonal matrix whose diagonal elements are all $a_{ij}^{(2)}$. Then the system can be fully described by an affine time-varying system in the standard form:

$$\dot{\tilde{x}}(t) = \left(\tilde{A}_1(t) + \tilde{A}_2(t)\right) \tilde{x}(t) + \tilde{f}(t)$$

Solution 1.2

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right] &= \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}t}\left(\Phi_1(t,\tau)M(t,\tau)\right)\mathrm{d}\tau + \Phi_1(t,t)M(t,t)\cdot 1 \\ &= \int_{t_0}^{t} A_1(t)\Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \\ &= A_1(t)\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \end{aligned}$$

$$\begin{aligned} \dot{X}(t) &= A_1(t)\Phi_1(t,t_0)X_0\Phi_2^T(t,t_0) + \Phi_1(t,t_0)X_0\Phi_2^T(t,t_0)A_2^T(t) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right] \\ &= A_1(t)X(t) - A_1(t)\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + X(t)A_2^T(t) - \left(\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right)A_2^T(t) \\ &\quad + A_1(t)\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \\ &= A_1(t)X(t) + X(t)A_2^T(t) - \left(\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right)A_2^T(t) + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \end{aligned}$$

From the matrix differential equation, we also know that

$$\dot{X}(t) = A_1(t)X(t) + X(t)A_2^T(t) + F(t)$$

Compare these two equations, it follows immediately

$$F(t) = -\left(\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right)A_2^T(t) + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t)$$

Guess a solution for $M(t,\tau)$:

$$M(t,\tau) = F(\tau)\Phi_2^T(t,\tau)$$

The right-hand side would thus be

$$\begin{aligned} R &= -\left(\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right)A_2^T(t) + \int_{t_0}^{t} \Phi_1(t,\tau)\frac{\mathrm{d}}{\mathrm{d}t}M(t,\tau)\mathrm{d}\tau + M(t,t) \\ &= -\left(\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau\right)A_2^T(t) + \int_{t_0}^{t} \Phi_1(t,\tau)F(\tau)\Phi_2^T(t,\tau)A_2^T(t)\mathrm{d}\tau + F(t)\Phi_2^T(t,t) \\ &= F(t) \\ &= L \end{aligned}$$

Therefore, $X(t) = \Phi_1(t,t_0)X_0\Phi_2^T(t,t_0) + \int_{t_0}^{t} \Phi_1(t,\tau)F(\tau)\Phi_2^T(t,\tau)\mathrm{d}\tau$ is a solution to the differential equation. Next we prove the uniqueness of the solution.

Define $g(X) = \dot{X}$, $\forall X_1, X_2 \in \mathbb{R}^{n\times n}, \forall t \geq t_0$,

$$
\begin{aligned}
\|g(X_1) - g(X_2)\| &= \left\|\dot{X}_1 - \dot{X}_1\right\| \\
&= \left\|\left(A_1X_1 + X_1A_2^T + F\right) - \left(A_1X_2 + X_2A_2^T + F\right)\right\| \\
&= \left\|A_1X_1 - A_1X_2 + X_1A_2^T - X_2A_2^T\right\| \\
&= \left\|A_1(X_1 - X_2) + (X_1 - X_2)A_2^T\right\| \\
&\leq \|A_1(X_1 - X_2)\| + \left\|(X_1 - X_2)A_2^T\right\| \\
&\leq \|A_1\| \, \|X_1 - X_2\| + \|X_1 - X_2\| \left\|A_2^T\right\| \\
&= \left(\|A_1\| + \left\|A_2^T\right\|\right) \|X_1 - X_2\| \\
&:= K(t) \, \|X_1(t) - X_2(t)\|
\end{aligned}
$$

Therefore, $g(X)$ is globally Lipschitz with respect to X and piece-wise continuous with respect to t, which indicates the uniqueness of the solution to the matrix differential equation. Thus, the value of $\int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau$ is unique and $\forall t$

$$
\int_{t_0}^{t} \Phi_1(t,\tau)F(\tau)\Phi_2^T(t,\tau)\mathrm{d}\tau = \int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau \ .
$$

However, we cannot make sure the uniqueness of $M(t,\tau)$. For example, any $\tilde{M}(t,\tau)$ defined as

$$
\tilde{M}(t,\tau) = \begin{cases} F(\tau)\Phi_2^T(t,\tau) & t \notin \mathcal{D} \\ K & t \in \mathcal{D} \end{cases}
$$

where K is any finite real number, and $\mathcal{D}$ stands for the discontinuity set of $\tilde{M}(t,\tau)$. Then, $\tilde{M}(t,\tau)$ is also a valid solution to the matrix differential equation since

$$
\forall t \geq t_0, \int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)\mathrm{d}\tau = \int_{t_0}^{t} \Phi_1(t,\tau)\tilde{M}(t,\tau)\mathrm{d}\tau
$$

Exercise 2. (Linear Time-Invariant Systems, [50 points in total])
Consider the following affine system:

$$\dot{x}(t) = A\,x(t) + \begin{bmatrix}0\\1\\0\end{bmatrix} \qquad y(t) = \begin{bmatrix}1 & 1 & 0\end{bmatrix}\,x(t).$$

where $A \in \mathbb{R}^{3\times 3}$. The matrix A has eigenvalues $\lambda_1 = -2$ with multiplicity 2, and $\lambda_2 = -1$ with multiplicity 1. The eigenvalue λ_1 has an eigenvector $v_1 = \begin{bmatrix}0\\2\\1\end{bmatrix}$, and a generalized eigenvector $v_1' = \begin{bmatrix}-1\\0\\1\end{bmatrix}$. The eigenvalue λ_2 has the eigenvector $v_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$.

1. [20 points] Find the matrix A.
2. [20 points] Calculate $\exp(At)$.
3. [10 points] Given $x(0) = [0\ 0\ 1]^\top$, compute $y(t)$.

Solution 2.1

$$\begin{aligned}
Av_2 = \lambda_2 v_2 &\Rightarrow a_{12} = 0, a_{22} = -1, a_{23} = 0\\
Av_1 = \lambda_1 v_1 &\Rightarrow a_{13} = 0, a_{23} = -2, a_{33} = -2\\
(A - \lambda_1 \mathbb{I})\, v_1' = v_1 &\Rightarrow a_{11} = -2, a_{21} = 4, a_{31} = -1\\
A &= \begin{bmatrix}-2 & 0 & 0\\ -4 & -1 & -2\\ -1 & 0 & -2\end{bmatrix}
\end{aligned}$$

Solution 2.2

$$T^{-1} = \begin{bmatrix}v_1 & v_1' & v_2\end{bmatrix} = \begin{bmatrix}0 & -1 & 0\\ 2 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}, \quad T = \begin{bmatrix}1 & 0 & 1\\ -1 & 0 & 0\\ -2 & 1 & -2\end{bmatrix}$$

$$J_1 = \begin{bmatrix}-2 & 1\\ & -2\end{bmatrix}, \quad J_2 = -1, \quad J = \begin{bmatrix}-2 & 1 & \\ & -2 & \\ & & -1\end{bmatrix}$$

$$e^{J_1 t} = \begin{bmatrix}e^{-2t} & te^{-2t}\\ 0 & e^{-2t}\end{bmatrix}, \quad e^{J_2 t} = e^{-t}, \quad e^{Jt} = \begin{bmatrix}e^{-2t} & te^{-2t} & \\ & e^{-2t} & \\ & & e^{-t}\end{bmatrix}$$

$$e^{At} = T^{-1} e^{Jt} T = \begin{bmatrix}e^{-2t} & 0 & 0\\ 2e^{-2t} - 2e^{-t} - 2te^{-2t} & e^{-t} & 2e^{-2t} - 2e^{-t}\\ -te^{-2t} & 0 & e^{-2t}\end{bmatrix}$$

Solution 2.3

$$\Phi(t,0) = e^{At} = \begin{bmatrix} e^{-2t} & 0 & 0 \\ 2e^{-2t} - 2e^{-t} - 2te^{-2t} & e^{-t} & 2e^{-2t} - 2e^{-t} \\ -te^{-2t} & 0 & e^{-2t} \end{bmatrix}, \quad B(t)u(t) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{aligned}
x(t) &= \Phi(t,0)x_0 + \int_0^t \Phi(t,\tau)B(\tau)u(\tau)\mathrm{d}\tau \\
&= \begin{bmatrix} 0 \\ 2e^{-2t} - 2e^{-t} \\ e^{-2t} \end{bmatrix} + \int_0^t \Phi(t,\tau) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \mathrm{d}\tau \\
&= \begin{bmatrix} 0 \\ 2e^{-2t} - 2e^{-t} \\ e^{-2t} \end{bmatrix} + \int_0^t \begin{bmatrix} 0 \\ e^{-t+\tau} \\ 0 \end{bmatrix} \mathrm{d}\tau \\
&= \begin{bmatrix} 0 \\ 2e^{-2t} - 2e^{-t} \\ e^{-2t} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 - e^{-t} \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 1 + 2e^{-2t} - 3e^{-t} \\ e^{-2t} \end{bmatrix} \\
y(t) &= \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x(t) \\
&= 1 + 2e^{-2t} - 3e^{-t}
\end{aligned}$$