Linear System Theory

Solutions to Problem Set 3

Linear Time-Varying Systems, Linear Time-Invariant Systems

Issue date: Oct. 21, 2019.
Due date: Oct. 31, 2019

Solution 1. (Linear Time-Varying Systems, [50 points in total])

1. [20 **points**] Let

$$\tilde{x}(t) = \begin{bmatrix} x_{11}(t) \\ \vdots \\ x_{1n}(t) \\ x_{21}(t) \\ \vdots \\ x_{2n}(t) \\ x_{n1}(t) \\ \vdots \\ x_{nn}(t) \end{bmatrix} \in \mathbb{R}^{n^2} \quad \text{and} \quad \tilde{F}(t) = \begin{bmatrix} F_{11}(t) \\ \vdots \\ F_{1n}(t) \\ F_{21}(t) \\ \vdots \\ F_{2n}(t) \\ F_{n1}(t) \\ \vdots \\ F_{nn}(t) \end{bmatrix} \in \mathbb{R}^{n^2}.$$

That is, $\tilde{x}(t)$ ($\tilde{F}(t)$ resp.) is the vector made up of all the entries of the matrix $X(t)$ ($F(t)$ resp.).

The (i,j)-th entry of the matrix equation $\dot{X}(t) = A_1(t)X(t) + X(t)A_2^\top(t) + F(t)$ is

$$\dot{x}_{ij}(t) = \sum_{k=1}^{n} A_{1,ik}(t)x_{kj}(t) + \sum_{k=1}^{n} A_{2,jk}(t)x_{ik}(t) + F_{ij}(t),$$

where $A_{r,ij}(t)$ is the (i,j)-th entry of $A_r(t)$, for any $r \in \{1,2\}$ and any $i,j \in \{1,\dots,n\}$. For each pair (i,j), $x_{ij}(t)$ is an entry of the vector $\tilde{x}(t)$ and $\dot{x}_{ij}(t)$ is a linear combination of all the entries of $\tilde{x}(t)$, plus $F_{ij}(t)$. Therefore, for any $t \geq t_0$, there exists a matrix $A(t) \in \mathbb{R}^{n^2 \times n^2}$, as a function of $A_1(t)$ and $A_2(t)$'s entries, such that the original matrix equation can be equivalently written in the following vector form:

$$\dot{\tilde{x}}(t) = A(t)\tilde{x}(t) + \tilde{F}(t),$$

which, by definition, is an affine time-varying system.

2. [30 **points**] Since $X(t)$ is of the form

$$X(t) = \Phi_1(t,t_0)X_0\Phi_2^T(t,t_0) + \int_{t_0}^{t} \Phi_1(t,\tau)M(t,\tau)d\tau, \tag{1}$$

by computing the derivative with respect to t on both sides of the equation above and invoking Leibnitzs rule, we obtain

$$\begin{aligned}\frac{dX(t)}{dt} &= \frac{d\Phi_1(t,t_0)}{dt}X_0\Phi_2^\top(t,t_0) + \Phi_1(t,t_0)X_0\frac{d\Phi_2^\top(t,t_0)}{dt} + \Phi_1(t,t)M(t,t) \\ &\quad + \int_{t_0}^{t}\frac{\partial\Phi_1(t,\tau)}{\partial t}M(t,\tau)d\tau + \int_{t_0}^{t}\Phi_1(t,\tau)\frac{\partial M(t,\tau)}{\partial t}d\tau.\end{aligned}$$

In addition, since

$$\frac{\partial\Phi_1(t,\tau)}{\partial t} = A_1(t)\Phi_1(t,\tau) \quad \text{and} \quad \frac{\partial\Phi_2(t,\tau)}{\partial t} = A_2(t)\Phi_2(t,\tau)$$

for any t, $\tau \in \mathbb{R}$, we have

$$\begin{aligned}\frac{dX(t)}{dt} &= A_1(t)\Phi_1(t,t_0)X_0\Phi_2^\top(t,t_0) + \Phi_1(t,t_0)X_0\Phi_2^\top(t,t_0)A_2^\top(t) + \Phi_1(t,t)M(t,t) \\ &\quad + \int_{t_0}^{t}A_1(t)\Phi_1(t,\tau)M(t,\tau)d\tau + \int_{t_0}^{t}\Phi_1(t,\tau)\frac{\partial M(t,\tau)}{\partial t}d\tau. \qquad (2)\end{aligned}$$

Moreover, since $X(t)$ satisfies

$$\frac{dX(t)}{dt} = A_1(t)X(t) + X(t)A_2^\top(t) + F(t),$$

by substituting equation (1) into the right-hand side of the equation above, we obtain

$$\begin{aligned}\frac{dX(t)}{dt} &= A_1(t)\Phi_1(t,t_0)X_0\Phi_2^\top(t,t_0) + A_1(t)\int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)d\tau + F(t) \\ &\quad + \Phi_1(t,t_0)X_0\Phi_2^\top(t,t_0)A_2^\top(t) + \int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)A_2^\top(t)d\tau. \qquad (3)\end{aligned}$$

The right-hand sides of equations (2) and (3) are equal if and only if the following equations hold:

$$\int_{t_0}^{t}\Phi_1(t,\tau)\frac{\partial M(t,\tau)}{\partial t}d\tau = \int_{t_0}^{t}\Phi_1(t,\tau)M(t,\tau)A_2^\top(t)d\tau, \qquad (4)$$

$$\phi_1(t,t)M(t,t) = F(t). \qquad (5)$$

If $M(t,\tau)$ satisfies that

$$\frac{\partial M(t,\tau)}{\partial t} = M(t,\tau)A_2^\top(t) \quad \text{and } M(t,t) = F(t),$$

then both equations (4) and (5) hold. Since $\Phi_2(t,\tau)$ has the property that $\partial\Phi_2^\top(t,\tau)/\partial t = \Phi_2^\top(t,\tau)A_2^\top(t)$, we guess that $M(t,\tau)$ contains the term $\Phi_2^\top(t,\tau)$. Let us try $M(t,\tau) = F(\tau)\Phi_2^\top(t,\tau)$. It turns out that

$$\begin{aligned}\frac{\partial M(t,\tau)}{\partial t} &= F(\tau)\frac{\partial\Phi_2^\top(t,\tau)}{\partial t} = F(\tau)\Phi_2^\top(t,\tau)A_2^\top(t) = M(t,\tau)A_2^\top(t), \\ M(t,t) &= F(t)\Phi_2^\top(t,t) = F(t),\end{aligned}$$

which in turn implies that this $M(t,\tau)$ satisfies equations (4) and (5). Therefore, $M(t,\tau) = F(\tau)\Phi_2^\top(t,\tau)$.

Solution 2. (Linear Time-Invariant Systems, [50 points in total])

1. [20 points] Using the information in the problem statement we get that the Jordan decomposition of A is:

$$J = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad T = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

and:

$$A = \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}}_{T} \underbrace{\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{J} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ -1 & 0 & 0 \\ -2 & 1 & -2 \end{bmatrix}}_{T^{-1}} = \begin{bmatrix} -2 & 0 & 0 \\ -4 & -1 & -2 \\ -1 & 0 & -2 \end{bmatrix}.$$

2. [20 points] Consider:

$$\exp(J\,t) = \begin{bmatrix} \exp(-2\,t) & t\,\exp(-2\,t) & 0 \\ 0 & \exp(-2\,t) & 0 \\ 0 & 0 & \exp(-t) \end{bmatrix}.$$

Hence:

$$\begin{aligned} \exp(A\,t) &= T\,\exp(J\,t)\,T^{-1} \\ &= \begin{bmatrix} \exp(-2\,t) & 0 & 0 \\ (2-2\,t)\,\exp(-2\,t) - 2\,\exp(-t) & \exp(-t) & -2\,\exp(-t) + 2\,\exp(-2\,t) \\ -t\,\exp(-2\,t) & 0 & \exp(-2\,t) \end{bmatrix}. \end{aligned}$$

3. [10 points] For any linear time-invariant system $\dot{x} = Ax + Bu$, the state-transition matrix $\Phi(t,\tau)$ is given by $\Phi(t,\tau) = \exp(A(t-\tau))$ for any t and τ. Since

$$x(t) = \Phi(t,0)x(0) + \int_0^t \Phi(t,\tau)Bu(\tau)d\tau,$$

$B = [0 \quad 1 \quad 0]^\top$, and $u(\tau) \equiv 1$, we have

$$x(t) = \exp(At)x(0) + \int_0^t \exp(A(t-\tau)) \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} d\tau$$

and

$$
\begin{aligned}
y(t) &= \begin{bmatrix}1 & 1 & 0\end{bmatrix} x(t) \\
&= \begin{bmatrix}1 & 1 & 0\end{bmatrix} \exp(At) \begin{bmatrix}0\\0\\1\end{bmatrix} + \begin{bmatrix}1 & 1 & 0\end{bmatrix} \exp(At) \int_0^t \exp -A\tau \begin{bmatrix}0 & 1 & 0\end{bmatrix} d\tau \\
&= \begin{bmatrix}1 & 1 & 0\end{bmatrix} \begin{bmatrix}0\\ -2\exp(-t) + 2\exp(-2t)\\ \exp(-2t)\end{bmatrix} + \begin{bmatrix}1 & 1 & 0\end{bmatrix} \exp(At) \int_0^t \begin{bmatrix}0\\ \exp(\tau)\\ 0\end{bmatrix} d\tau \\
&= -2\exp(-t) + 2\exp(-2t) + \begin{bmatrix}1 & 1 & 0\end{bmatrix} \exp(At) \begin{bmatrix}0\\ \exp(t) - 1\\ 0\end{bmatrix} \\
&= -2\exp(-t) + 2\exp(-2t) + \begin{bmatrix}1 & 1 & 0\end{bmatrix} \begin{bmatrix}0\\ 1 - \exp(-t)\\ 0\end{bmatrix} \\
&= 1 - 3\exp(-t) + 2\exp(-2t).
\end{aligned}
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