矩阵论及其应用
习题 1

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首先证明充分性:
利用反证法，设 $V_1 \cap V_2 = \{0\}$，若 $z \in V_1 + V_2$ 不能唯一地表示成 V_1 和 V_2 的向量的和，则必有 $x_1, x_2 \in V_1 \quad y_1, y_2 \in V_2$，且 $x_1 \neq x_2 \quad y_1 \neq y_2$，使得

$$z = x_1 + y_1 \qquad z = x_2 + y_2 \tag{1.1}$$

两式相减得到 $(x_1 - x_2) + (y_1 - y_2) = 0$，令 $w = x_1 - x_2 = -(y_1 - y_2) \neq 0$，而 $(x_1 - x_2) \in V_1, -(y_1 - y_2) \in V_2$，故 $w \in V_1 \cap V_2$，进而得到 $V_1 \cap V_2 \neq \{0\}$，与假设矛盾，充分性可证;
接着证明必要性:
再次利用反证法，假设 $V_1 \cap V_2 \neq \{0\}$，则 $\exists w \in V_1 \cap V_2$ 且 $w \neq 0$。
由 $w \in V_1, w \in V_2$ 可得

$$\begin{gathered} w + w = 2w \in V_1, w + w = 2w \in V_2 \\ 2w + w = 3w \in V_1, 2w + w = 3w \in V_2 \end{gathered} \tag{1.2}$$

由上可知 $4w = 2w + 2w = 3w + w \in V_1 + V_2$ 且 $2 \neq 3, 2 \neq 1$，所以其向量分解是不唯一的，从而得到 $V_1 + V_2$ 不是 V_1, V_2 的直接和空间，与假设矛盾，必要性可证

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设 $\varepsilon_1 = x^2 + x, \varepsilon_2 = x^2 - x, \varepsilon_3 = x + 1$，假设这三个向量线性相关，则必存在一组不全为 0 的实数 a, b, c 使得

$$a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_3 = (a + b)x^2 + (a - b + c)x + c = 0 \tag{2.1}$$

从而可以推出

$$\begin{cases} a + b = 0 \\ a - b + c = 0 \\ c = 0 \end{cases} \tag{2.2}$$

解得

$$a = b = c = 0 \tag{2.3}$$

故 $\varepsilon_1, \varepsilon_2, \varepsilon_3$ 线性无关，又有 $dim(p_2(x)) = 3$，故 $\varepsilon_1, \varepsilon_2, \varepsilon_3$ 是线性空间 $p_2(x)$ 的一组基。再令

$$\begin{aligned} 2x^2 + 7x + 3 &= \alpha_1\varepsilon_1 + \alpha_2\varepsilon_2 + \alpha_3\varepsilon_3 \\ &= (\alpha_1 + \alpha_2)x^2 + (\alpha_1 - \alpha_2 + \alpha_3)x + \alpha_3 \end{aligned} \tag{2.4}$$

于是有

$$\begin{cases} \alpha_1 + \alpha_2 = 2 \\ \alpha_1 - \alpha_2 + \alpha_3 = 7 \\ \alpha_3 = 3 \end{cases} \tag{2.5}$$

解得

$$\begin{cases} \alpha_1 = 3 \\ \alpha_2 = -1 \\ \alpha_3 = 3 \end{cases} \tag{2.6}$$

故其坐标为 $(3, -1, 3)^T$

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设 $k_1\beta_1 + k_2\beta_2 + k_3\beta_3 = 0$，即 $k_1(\alpha_1 - 2\alpha_2 + 3\alpha_3) + k_2(2\alpha_1 + 3\alpha_2 + 2\alpha_3) + k_3(4\alpha_1 + 13\alpha_2) = (k_1 + 2k_2 + 4k_3)\alpha_1 + (-2k_1 + 3k_2 + 13k_3)\alpha_2 + (3k_1 + 2k_2)\alpha_3 = 0$ 由于 $\alpha_1, \alpha_2, \alpha_3$ 是一组基底，故其线性无关，从而可得

$$\begin{cases} k_1 + 2k_2 + 4k_3 = 0 \\ -2k_1 + 3k_2 + 13k_3 = 0 \\ 3k_1 + 2k_2 = 0 \end{cases} \tag{3.1}$$

解得

$$\begin{cases} k_1 = 2k_3 \\ k_2 = -3k_3 \\ k_3 = k_3 \end{cases} \tag{3.2}$$

故 $2\beta_1 - 3\beta_2 + \beta_3 = 0$，$\beta_1, \beta_2, \beta_3$ 线性相关。又显见 β_1 与 β_2（或 β_1 与 β_3，β_2 与 β_3）线性无关，故 $dim(Span(\beta_1, \beta_2, \beta_3)) = 2$，基底由 β_1 与 β_2（或 β_1 与 β_3，β_2 与 β_3）组成

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由于

$$(e_3, e_2, e_1) = (e_1, e_2, e_3) \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = (e_1, e_2, e_3)B \tag{4.1}$$

而

$$T(e_1,e_2,e_3)=(e_1,e_2,e_3)A \tag{4.2}$$

故

$$\begin{aligned}T(e_3,e_2,e_1)&=T(e_1,e_2,e_3)B\\&=(e_1,e_2,e_3)AB\\&=(e_3,e_2,e_1)B^{-1}AB\\&=(e_3,e_2,e_1)\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}A\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\\&=(e_3,e_2,e_1)\begin{bmatrix}a_{33}&a_{32}&a_{31}\\a_{23}&a_{22}&a_{21}\\a_{13}&a_{12}&a_{11}\end{bmatrix}\end{aligned} \tag{4.3}$$

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设 $x=(x_1,x_2,x_3)^T,y=(y_1,y_2,y_3)^T,\quad \lambda,\mu\in F$
对于 T_1:

$$T_1(\lambda x+\mu y)=\begin{bmatrix}\lambda x_1+\mu y_1+\lambda x_2+\mu y_2\\(\lambda x_1+\mu y_1)^2-(\lambda x_2+\mu y_2)^2\end{bmatrix} \tag{5.1}$$

$$\begin{aligned}\lambda T_1(x)+\mu T_1(y)&=\begin{bmatrix}\lambda x_1+\mu y_1+\lambda x_2+\mu y_2\\\lambda x_1^2+\mu y_1^2-(\lambda x_2^2+\mu y_2^2)\end{bmatrix}\\&\neq T_1(\lambda x+\mu y)\end{aligned} \tag{5.2}$$

故 $T_1:R^3\to R^2$ 不是线性映射
对于 T_2:

$$T_2(\lambda x+\mu y)=\begin{bmatrix}\lambda x_1+\mu y_1-(\lambda x_2+\mu y_2)\\\lambda x_2+\mu y_2+(\lambda x_3+\mu y_3)\end{bmatrix} \tag{5.3}$$

$$\begin{aligned}\lambda T_2(x)+\mu T_2(y)&=\begin{bmatrix}\lambda x_1+\mu y_1-(\lambda x_2+\mu y_2)\\\lambda x_2+\mu y_2+(\lambda x_3+\mu y_3)\end{bmatrix}\\&=T_2(\lambda x+\mu y)\end{aligned} \tag{5.4}$$

故 $T_2:R^3\to R^2$ 是线性映射

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(1) 设

$$X=\begin{bmatrix}x_1&x_2\\x_3&x_4\end{bmatrix},Y=\begin{bmatrix}y_1&y_2\\y_3&y_4\end{bmatrix}\in R^{2\times2} \tag{6.1}$$

对 $\forall\lambda,\mu\in F$，有

$$\begin{aligned}T(\lambda X+\mu Y)&=\begin{bmatrix}(\lambda x_1+\mu y_1)+(\lambda x_3+\mu y_3) & (\lambda x_2+\mu y_2)+(\lambda x_4+\mu y_4)\\2(\lambda x_1+\mu y_1)+2(\lambda x_3+\mu y_3) & 2(\lambda x_2+\mu y_2)+2(\lambda x_4+\mu y_4)\end{bmatrix}\\&=\begin{bmatrix}\lambda x_1+\lambda x_3 & \lambda x_2+\lambda x_4\\2\lambda x_1+2\lambda x_3 & 2\lambda x_2+2\lambda x_4\end{bmatrix}+\begin{bmatrix}\mu y_1+\mu y_3 & \mu y_2+\mu y_4\\2\mu y_1+2\mu y_3 & 2\mu y_2+2\mu y_4\end{bmatrix}\\&=\lambda\begin{bmatrix}x_1+x_3 & x_2+x_4\\2x_1+2x_3 & 2x_2+2x_4\end{bmatrix}+\mu\begin{bmatrix}y_1+y_3 & y_2+y_4\\2y_1+2y_3 & 2y_2+2y_4\end{bmatrix}\\&=\lambda T(X)+\mu T(Y)\end{aligned}\tag{6.2}$$

故 T 是 $R^{2\times 2}$ 上的线性变换

(2)

$$\begin{aligned}T(E_1)&=\begin{bmatrix}1&0\\2&0\end{bmatrix}=E_1-2E_3+2E_4\\T(E_2)&=\begin{bmatrix}1&1\\2&2\end{bmatrix}=-E_2+2E_4\\T(E_3)&=\begin{bmatrix}1&2\\2&4\end{bmatrix}=-E_1-2E_2+2E_3+2E_4\\T(E_4)&=\begin{bmatrix}2&2\\4&4\end{bmatrix}=-2E_2+4E_4\end{aligned}\tag{6.3}$$

故 $T(E_1,E_2,E_3,E_4)=(E_1,E_2,E_3,E_4)\begin{bmatrix}1&0&-1&0\\0&-1&-2&-2\\-2&0&2&0\\2&2&2&4\end{bmatrix}$

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(1)

$\left|B,AB,A^2B\right|=\begin{vmatrix}0&1&0\\1&-0.9&0.81\\1&0.5&0.25\end{vmatrix}=0.56\neq 0$

故方阵 $\left[B,AB,A^2B\right]$ 满秩，$rank(\left[B,AB,A^2B\right])=3$，矩阵对 (A,B) 可控

(2)

$\left|B,AB,A^2B\right|=\begin{vmatrix}1&0.5&0.19\\1&0.7&0.45\\0&0&0\end{vmatrix}=0$

故 B,AB,A^2B 这三个向量线性相关，$rank(\left[B,AB,A^2B\right])=2$，矩阵对 (A,B) 不可控