Homework 4

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1. In the w plane, integrate the Taylor series expansion

$$\frac{1}{w} = \sum_{n=0}^{\infty} (-1)^n (w-1)^n$$

where $|w-1| < 1$, along a contour interior to the circle of convergence from $w = 1$ to $w = z$ to obtain the representation

$$Log(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (z-1)^n$$

where $|z-1| < 1$.

Solution

$$\begin{aligned}
\int_w^z \frac{1}{w} &= \int_1^z \sum_{n=0}^{\infty} (-1)^n (w-1)^n \\
&= \sum_{n=0}^{\infty} (-1)^n \int u^n du && \text{Let } u = w-1,\ du = dw \\
&= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{u^{n+1}}{n+1} |_{w=1}^{w=z} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(w-1)^{n+1}}{n+1} |_{w=1}^{w=z} \\
&= \sum_{n=0}^{\infty} (-1)^n [\frac{(z-1)^{n+1}}{n+1} - \frac{(1-1)^{n+1}}{...}] \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(z-1)^{n+1}}{n+1} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(z-1)^n}{n}
\end{aligned}$$

2. Use the result of the previous exercise to show that if

$$f(z) = \frac{Log(z)}{z-1}$$

when $z \neq 1$, and $f(1) = 1$, then f is analytic throughout the domain $0 < |z| < \infty$, $-\pi < Argz < \pi$.

Proof. First, notice that dividing the power series representation for $Log(z)$

$$Log(z) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^n$$

by $z-1$, i.e.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(z-1)^{n-1}$$

converges to $f(z)$ for $z \neq 1$. Furthermore, when $z = 1$ we get

$$\frac{(-1)^2}{1} \cdot (1)^0 + ... \cdot 0 + ... \cdot 0$$

In other words, the power series converges to $f(1)$ when $z = 1$.

Now, $Log(z)$ is only continuous in $[-\pi, \pi]$. But otherwise, $f(z)$ is represented by a convergent power series for all $|z| > 0$, and is therefore entire. $\square$