Homework 2

Vincent La
Math 122A
July 10, 2017

Monday

1.22. Prove that a polygonally connected set is connected.

Proof.

Let S be any polygonally connected set and suppose for a contradiction that it is disconnected. Because S is disconnected, there are two open disjoint sets A and B such that $A \cup B$ contains S but neither set alone contains S.

On the other hand, because S is polygonally connected, for any points s_1, s_2 in S there is a finite union of polygonal lines connecting them. Let $s_1 \in S \cap A$, $s_2 \in S \cap B$ be arbitrary. Because A and B are disjoint, there is no line segment connecting any two points in A and B, i.e. there is no polygonal line between s_1, s_2. But this contradicts our assumption that S is polygonally connected. Thus, if S is polygonally connected, it must be that it is connected. □

Extra 3. Give an example of a set that is connected but not polygonally connected.

Answer Consider the set $S = \{z | Re(z) = x, Im(z) = e^x\}$. Clearly the set is connected since e^x is continuous everywhere. However, it is not polygonally connected because there are no straight line segments connecting any two points.

Wednesday

1.21. Show that

(a) $f(z) = \sum_{k=0}^{\infty} kz^k$ is continuous in $|z| < 1$

Proof. Using the M-Test, we can show that $g(z)$ converges to a continuous function in the right half-plane Re $z > 0$. Let $M_k =$ some sequence which I haven't found yet.

Continuity First, we need to show that f_k is continuous for $|z| < 1$. Because kz^k is a polynomial, and polynomials are continuous, f_k is continuous.
(Proof to be continued...)

□

(b) $g(z) = \sum_{k=1}^{\infty} \frac{1}{k^2+z}$ is continuous in the right half-plane Re $z > 0$

Proof. Using the M-Test, we can show that $g(z)$ converges to a continuous function in the right half-plane Re $z > 0$. Let $M_k = \cdot \frac{1}{k^2}$

Continuity First, we need to show that f_k is continuous for Re $z > 0$. In other words, let $\epsilon > 0$ and $\delta = \epsilon(k^2 + z)(k^2 + z_0) > 0$. Then, whenever $|z - z_0| < \delta$, we have

$$\begin{aligned} |f(z) - f(z_0)| &= |\frac{1}{k^2 + z} - \frac{1}{k^2 + z_0}| \\ &= |\frac{1}{k^2 + z} \cdot (\frac{k^2 + z_0}{k^2 + z_0}) - \frac{1}{k^2 + z_0} \cdot (\frac{k^2 + z}{k^2 + z})| \\ &= |\frac{k^2 + z_0 - k^2 + z}{(k^2 + z)(k^2 + z_0)}| \end{aligned}$$

Furthermore, by our assumption that $|z - z_0| = |-1(z - z_0)| = |z_0 - z| < \delta$, we have

$$\begin{aligned} |f(z) - f(z_0)| &< |\frac{\delta}{(k^2 + z)(k^2 + z_0)}| && \text{Because by assumption} \\ &= |\frac{\epsilon(k^2 + z)(k^2 + z_0)}{(k^2 + z)(k^2 + z_0)}| = \epsilon \end{aligned}$$

as required.

$\mathbf{|f_k(z)| \leq M_k}$ **throughout Re** $\mathbf{z > 0}$ Clearly,

$$f_k(z) = |\frac{1}{k^2 + z}| \leq |\frac{1}{k^2}| = M_k$$

$\mathbf{\sum_{k=1}^{\infty} M_k}$ **Converges** M_k is a p-series with $p > 2$. Therefore, it converges.

Because $g(z)$ passes the M-Test, it converges to a continuous function on Re $z > 0$, as we set out to prove.

$\square$