Homework 10

Vincent La
Math 122B
September 11, 2017

1. Evaluate the improper integral

$$\int_0^\infty \frac{x^3 \sin x}{(x^2+1)(x^2+9)} dx$$

First, we need to evaluate

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{x^3 \sin x}{(x^2+1)(x^2+9)} dx + \int_{C_R} \frac{z^3 \exp iz}{(z^2+1)(z^2+9)} dz = 2\pi i Resf(z)$$

Furthermore, $f(z)$ has isolated singularities whenever $z^2+1=0$ and $z^2+9=0$ implying that $z=\pm i$ and $z=\pm 3i$. Thus, let $R>3$.

We are primarily concerned with the residues at $z=i$ and $z=3i$.

Residue at $z=i$ First, notice that we can write $f(z)$ as

$$f(z) = \frac{z^3 \exp iz}{(z^2+9)(z+i)} \cdot \frac{1}{z-i} = \frac{\phi(z)}{z-i}$$

Because, $\phi(i) = \frac{i^3 \exp i^2}{(i^2+9)(i+i)} \neq 0$, $q(i)=0$, and $q'(i)=1\neq 0$, $z=i$ is therefore a pole of order 1 at $z=i$ where

$$Resf(i) = \phi(i) = \frac{i^3 \cdot \exp i^2}{8 \cdot -1} = \frac{i \exp -1}{8}$$

Residue at $z=3i$ First, notice that we can write $f(z)$ as

$$f(z) = \frac{z^3 \exp iz}{(z+3i)(z^2+1)} \cdot \frac{1}{z-3i} = \frac{\phi(z)}{z-3i}$$

Because, $\phi(i) = \frac{(3i)^3 \exp 3i^2}{(3i+3i)((3i)^2+1)} \neq 0$, $q(i)=0$, and $q'(i)=1\neq 0$, $z=3i$ is therefore a pole of order 1 at $z=i$ where

$$Resf(i) = \phi(i) = \frac{(3i)^3 \exp 3i^2}{(3i+3i)((3i)^2+1)} = \frac{3^3 i^3 \exp -3}{6i \cdot (-8)} = \frac{27 \cdot -i \cdot \exp -3}{-48i} = \frac{9 \cdot \exp -3}{16}$$

Value of $\int f(z)$ Whenever $z \in C_R$, because $x \geq 0$ we get that

$$
\begin{aligned}
|Im \int \frac{z^3 e^{iz}}{(z^2+1)(z^2+9)}| &\leq \int |\frac{z^3 e^{iz}}{(z^2+1)(z^2+9)}| \\
&\leq |\frac{\pi \cdot R^4}{(R^2+1)(R^2+9)}| \\
&= \frac{R^4}{R^4+10R^2+9}
\end{aligned}
$$

Because the quotient dominates this fraction as $R \to \infty$, we can say that $\int f(z) \to 0$ as $R \to \infty$.

Conclusion By equating imaginary parts, we get

$$
\begin{aligned}
\lim_{R\to\infty} \int_{-R}^{R} \frac{x^3 \sin x}{(x^2+1)(x^2+9)} dx &= Im(2\pi i Res f(z)) - Im(\int_{C_R} \frac{z^3 \exp iz}{(z^2+1)(z^2+9)} dz) \\
&= Im(2\pi i Res f(z)) \\
&= 2\pi i \cdot \frac{\exp -1}{8}
\end{aligned}
$$

2.

$$
\int_0^\infty \frac{\cos(ax) - \cos(bx)}{x^2} dx = \frac{\pi}{2}(b-a)
$$

Solution ...