Homework 9

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1. Evaluate the improper integral
$$\int_0^\infty \frac{\cos ax}{x^2+1}dx$$

Solution First, consider the function e^{iax} and notice that
$$\begin{aligned} Re[e^{iax}] &= Re[e^0 \cos ax + ie^0 \sin ax] \\ &= \cos ax \end{aligned}$$

Thus, we can say that
$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos ax}{x^2+1}dx = 2\pi i Res(f) + Re\int_{C_R} \frac{\exp iaz}{z^2+1}dz$$

Furthermore, because $|e^{iz}| = e^{-y} \leq 1$ whenever z is on C_R, and $y \geq 0$, this implies
$$|Re\int_{C_R} \frac{\exp iz}{z^2+1}dz| \leq |\int_{C_R} \frac{\exp iz}{z^2+4}dz| \leq \frac{\pi R}{R^2+4}$$

As $R \to \infty$, this quotient disappears, so we are just left with $2\pi i Res(f)$.

Now, let us find $Res(f)$. The function $f(z) = \frac{\exp iaz}{z^2+1}$ has singularities whenever $z^2+1=0$, i.e. whenever $z = \pm i$. Here we are interested in the singularity $z = i$. Notice that
$$f(z) = \phi(z) \cdot \frac{1}{z-i} \quad \text{If } \phi(z) = \frac{e^{iaz}}{z+i}$$

At $z = i$, $\phi(z)$ is analytic, $p(z) = e^{iaz} \neq 0$, and $q(z) = z - i$ clearly has a zero. Thus, the residue of $f(z)$ at this $z = i$ is
$$\phi(i) = \frac{e^{i^2 a}}{2i} = \frac{e^{-a}}{2i}$$

Wrapping up from earlier, our integral is thus
$$2\pi i Res(f) = 2\pi i \cdot \phi(i) = \pi e^{-a}$$

2. Evaluate the improper integral
$$\int_0^\infty \frac{x \sin ax}{x^4+4}dx$$

Solution First, notice that consider the function e^{iax} and notice that

$$\begin{aligned} Im[e^{iax}] &= Im[e^0 \cos ax + ie^0 \sin ax] \\ &= \sin ax \end{aligned}$$

Thus, we can say that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\sin ax}{x^2+1} dx = Im[2\pi i Res(f) + \int_{C_R} \frac{\exp iaz}{z^2+1} dz]$$

Furthermore, because $|e^{iaz}| = e^{-ay} \leq 1$ whenever z is on C_R, and $y \geq 0$, this implies

$$|Im \int_{C_R} \frac{\exp iaz}{z^4+1} dz| \leq |\int_{C_R} \frac{\exp iaz}{z^4+1} dz| \leq \frac{\pi R}{R^4-1}$$

As $R \to \infty$, this quotient disappears, so we are just left with $2\pi i Res(f)$.

This function $f(z)$ has a singularity at whenever $z^4 + 4 = 0$, i.e. there is a singularity whenever

$$\begin{aligned} z^4 + 4 &= 0 \\ z^4 &= -4 \\ z^2 &= \sqrt{-4} = 2i \\ z &= \sqrt{2}i \end{aligned}$$

Finally, let us compute the residue of f at $z = \sqrt{2}i$. First, because

$$(z^2 - 2i)(z^2 + 2i) = z^4 + 2iz^2 - 2iz^2 - 4i^2 = z^4 + 4$$

we can write $f(z) = \frac{\phi(z)}{z^2+2i}$, where $\phi(z) = \frac{z \exp az}{z^2-2i}$. Furthermore, at $z = \sqrt{2}i$

- $p(z) = z \exp az = \sqrt{2}i \cdot \exp a \cdot \sqrt{2}i \neq 0$
- $q(z) = z^2 + 2i = (z - \sqrt{2}i)(z + \sqrt{2}i)$ has a zero
- $\phi(z)$ is analytic

Therefore, the residue there is

$$\phi(\sqrt{2}i) = \frac{\sqrt{2}i \cdot \exp(a \cdot \sqrt{2}i)}{(\sqrt{2}i)^2 - 2i} = \frac{\sqrt{2}i \cdot \exp(a \cdot \sqrt{2}i)}{-2-2i}$$

and our integral is

$$2\pi i \cdot Im[\phi(\sqrt{2}i)] = 2\pi i \cdot \frac{\sqrt{2}i \cdot \exp(a \cdot \sqrt{2}i)}{-2-2i}$$

3. Find the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5} dx$$

Solution First, notice that

$$f(x) = Re[\frac{(x+1)e^{ix}}{x^2+4x+5}]$$

implying that

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)dx = Re[2\pi i Res(\frac{(z+1)e^{iz}}{z^2+4z+5})]$$

Now, $f(z) = \frac{(z+1)\cos x}{z^2+4x5}$ has isolated singularities whenever $z^2+4x+5 = 0$, i.e. whenever

$$\begin{aligned} z &= \frac{-4 \pm \sqrt{4^2 - 4(1)(5)}}{2} \\ &= \frac{-4 \pm \sqrt{16-20}}{2} \\ &= \frac{-4 \pm \sqrt{-4}}{2} \\ &= -2 \pm i \end{aligned}$$

Only $z = -2+i$ lies in the region of interest. But before finding the residue there, apply the theorem about the residues of quotients. First, because

$$\begin{aligned} (z-(-2+i))(z-(-2-i)) &= (z+(2-i))(z+(2+i)) \\ &= z^2 + (2+i)z + (2-i)z + (2-i)(2+i) \\ &= z^2 + (2z+2z) + (iz-iz) + 4 + (2i-2i) - i^2 \\ &= z^2 + 4z + 5 \end{aligned}$$

we can therefore write

$$f(z) = \frac{\frac{(z+1)e^{iz}}{z-(-2-i)}}{z-(-2+i)} = \frac{\phi(z)}{z-(-2+i)}$$

Furthermore, we because $\phi(z)$ is analytic at $z = -2+i$, $p(-2+i) = [(-2+i)+1]e^{iz} \neq 0$, and $q(z) = z-(-2+i)$ clearly has a zero at $z = -2+i$, the residue of $f(z)$ at $-2+i$ is therefore

$$\begin{aligned} \phi(-2+i) &= \frac{((-2+i)+1)e^{iz}}{(-2+i)-(-2-i)} \\ &= \frac{(i-1)e^{i(i-2)}}{2i} \\ &= \frac{(2+2i)e^{-1-2i}}{4} \\ &= \frac{(1+i)e^{-1-2i}}{2} \end{aligned}$$

This implies that

$$P.V. \int_{\infty}^{\infty} \frac{(x+1)\cos x}{x^2+4x+5} dx = 2\pi i \cdot b_1 = \pi i \cdot (1+i)e^{-1-2i}$$