Homework 4

Vincent La
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1 Monday and Tuesday

3.11 (a) Show that e^z is entire by verifying the Cauchy-Riemann equations for its real and imaginary parts.

Proof. First, $e^z = e^x \cos(y) + ie^x \sin(y)$.
Thus,

$$u_x = e^x \cos(y) = v_y$$

and

$$u_y = -e^x \sin(y) = -v_x$$

Because e^z satisfies the Cauchy-Riemann equations for all x, y, it is analytic everywhere. Since $e^z = e^x \cos(y) + ie^x \sin(y)$ is the sum of the product of continuous functions, it is also continuous. Therefore, e^z is also differentiable everywhere (entire). □

(b) Prove

$$e^{z_1+z_2} = e^{z_1} e^{z_2}$$

3.13 Discuss the behavior of e^z as $z \to \infty$ along the various rays from the origin

Solution I will discuss eight rays. One for each direction of the x and y-axis, and one for each quadrant of the complex plane.

(a) First, fix $x = 0$ and let $y \to \infty$. Then we simply get

$$e^0 cos(y) + ie^0 \sin(y)$$

This function simply moves around the unit circle. The same could be said for $y \to -\infty$.

(b) Now, fix $y = 0$ and let $x \to \infty$. Then we get

$$e^x cos(0) + ie^x \sin(0) = e^x$$

This function simply diverges to infinity. On the other hand, if $x \to -\infty$, then $e^z \to 0$.

(c) Now, consider a ray shooting upwards through the first quadrant. Let x and y both go to infinity. Because of the behavior of e^x, e^z will grow in magnitude. However, because of the periodic nature and range of $cos(y)$ and $sin(y)$, this makes e^z spiral around the center. Thus, in this case e^z spirals out from the center. We can say the same about the case when $x \to \infty$ and $y \to -\infty$ (ray through the fourth quadrant).

(d) Lastly, let $x \to -\infty$ and $y \to \infty$ (ray through second quadrant). Because of the behavior of e^x, $e^z \to 0$. We can say the same in the case of $y \to -\infty$ (ray through the third quadrant).

2 Wednesday

4.2 Evaluate $\int_C f(z)dz$ where $f(z) = x^2 + iy^2$ and $z(t) = 2t + i2t$.

Solution First, $\dot{z}(t) = 2t + i2t$ so

$$\begin{aligned}
\int_a^b f(x(t), y(t)) \cdot \dot{z}(t)dt &= \int_0^1 f(t^2, t^2) \cdot (2t + i2t)dt \\
&= \int_0^1 (t^4 + it^4)(2t + i2t)dt \\
&= \int_0^1 2t^5 + i4t^5 + i^2t^5dt \\
&= 4i \int_0^1 t^5dt \\
&= 4i \cdot \frac{t^6}{6}|_0^1 \\
&= 4i
\end{aligned}$$

4.8 Show that $\int_C z^k dz = 0$ for any integer $k \neq -1$ and $C : z = Re^{i\theta}, 0 \leq \theta \leq 2\pi$.

(a) By showing that z^k is the derivative of a function analytic throughout C.

Proof. First, because z^k is a polynomial in z, it is by definition an analytic polynomial. Because it is analytic and continuous (because polynomials are continuous everywhere), it is therefore entire. We know by the Integral Theorem that z^k is therefore the derivative of an everywhere analytic function, which I will now proceed to specifically identify. Notice that

$$\frac{d}{dz}\frac{z^{k+1}}{k+1} = k + 1\frac{z^k}{k+1} = z^k$$

In other words, we z^k is the derivative of this function as long as $k \neq 1$.

Now, recall $C : z = Re^{i\theta} = (R\cos(\theta), iR\sin(\theta))$ is continuous and differentiable everywhere. Furthermore, $z'(t) = (-R\sin(\theta), iR\cos(\theta)$ is never 0 because sin and cos are never equal to zero at the same time. Therefore, C is a smooth curve, and because it is defined for $0 \leq \theta \leq 2\pi$, it is also closed.

In conclusion, because z^k is an entire function (and the derivative of an analytic function for $z \neq 1$) and C is a smooth, closed curve, then by the Closed Curve Theorem, $\int_C z^k dz = 0$ for $z \neq 1$. $\square$

(b) Directly, using the parameterization of C.

Proof. First, $z(\theta) = Re^{i\theta} = (R\cos(\theta), iR\sin(\theta)$, so,

$$\dot{z}(\theta) = -R\sin(\theta) + iR\cos(\theta)$$

Therefore,

$$\int_a^b f(z(\theta)) \cdot \dot{z}(\theta) d\theta = \int_0^{2\pi} (R\cos(\theta) + iR\sin(\theta))^k \cdot (-R\sin(\theta) + iR\cos(\theta)) d\theta$$

Now, let $u = R\cos(\theta) + iR\sin(\theta)$, implying that

$$du = (-R\sin(\theta) + iR\cos(\theta)) \cdot d\theta$$

Thus, our integral becomes

$$\begin{aligned}
\int_0^{2\pi} u^k du &= \frac{u^{k+1}}{k+1}|_0^{2\pi} \\
&= \frac{(R\cos(\theta) + iR\sin(\theta))^{k+1}}{k+1}|_0^{2\pi} \\
&= 0 & \text{Because cos and sin are periodic with period } 2\pi
\end{aligned}$$

$\square$