Residue Theorem

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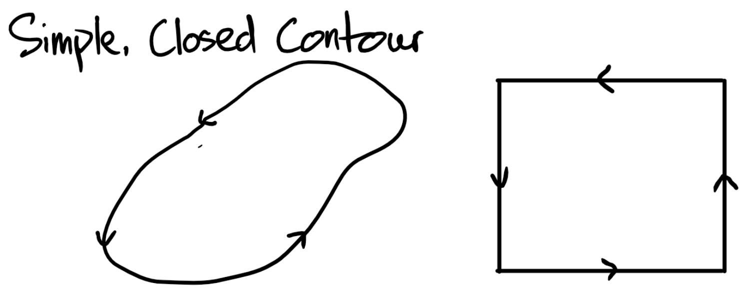
Abstract

1 Definitions

Simple, Closed Contour A simple closed contour is:

- A contour: An arc consisting of a finite union of smooth curves, connected end to end
- Simple: A curve that doesn't intersect itself
- Closed: Has the same starting and end point–i.e. it has no opening
- Examples: A circle, or the boundary of a square or triangle in a certain direction



2 Proof

2.1 Cauchy-Goursat Theorem

2.2 Residues

2.3 Residue Theorem

3 Example

Source: Complex Variables and Applications, pg. 159

Use the theorem to evaluate the integral

$$\int_C \frac{5z-2}{z(z-1)} dz$$

where C is the circle $|z| = 2$ described counterclockwise.

Does It Apply

- A circle is clearly a simple closed contour
- $f(z)$ can be written in terms of z, so it is analytic except for where we are dividing by zero

Singular Points This function has two isolated singularities at $z = 0$ and $z = 1$.

3.1 Solution

We're going to evaluate this integral by finding the two residues corresponding to our singular points.

First Residue First, recall the MacLaurin series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + ...$$

which is valid for $|z| < 1$.

This allows us to write the Laurent expansion as

$$\begin{aligned}
\frac{5z-2}{z(z-1)} &= \frac{5z-2}{z} \cdot (\frac{-1}{-1})(\frac{1}{z-1}) \\
&= \frac{5z-2}{z} \cdot \frac{-1}{1-z} \\
&= \frac{5z-2}{z} \cdot -\sum_{n=0}^{\infty} z^n && \text{Using MacLaurin series} \\
&= (5 - \frac{2}{z}) \cdot (-1 - z - z^2 - ...)
\end{aligned}$$

Now, because we're only interested in the b_1 term, I'm only going to foil the left-side quantity with the first three terms of the right-side quantity. By inspection, there will be no terms in the form $\frac{b_1}{z-z_0}$ once we foil past the first three RHS terms, so we can safely ignore the rest.

Expanding the terms as explained above, we get

$$\begin{aligned}
\frac{5z-2}{z(z-1)} &= (5 - \frac{2}{z}) \cdot (-1 - z - z^2 - ...) \\
&= -5 + 5z + \frac{2}{z} + \frac{2z}{z} + 5z^2 \\
&= (-5 + 2) + (5 - 2)z + \frac{2}{z} + ... \\
&= -\frac{2}{z} - 3 - 3z
\end{aligned}$$

This implies that $b_1 = -3$.

Second Residue We can't directly use the same MacLaurin series above because it's not valid at $z = 1$, so we'll perform the change of variables $z = z - 1$.

Thus, we get

$$\begin{aligned}
\frac{5z-2}{z(z-1)} &= \frac{5z-2}{z} \cdot \frac{-1}{1-z} && \text{Same as earlier} \\
&= \frac{5(z-1)-2}{z-1} \cdot \frac{-1}{1-(z-1)} && \text{Applying the change of variables}
\end{aligned}$$

which is valid for $0 < |z - 1| < 1$.