Poles, Removable and Essential Singular Points

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1 Definitions

Principal Part The terms in the Laurent series expansion of a function that contain negative exponents

Pole A singular point that contains only a finite number of terms with negative exponents

Essential Singular Point A singular point with an infinite number of terms with negative exponents

Removable Singular Point A singular point whose Laurent expansion contains only terms with positive exponents. All removable singular points have a residue of 0.

2 Exercises

Source: Complex Variables and Applications, Section 56

2. Show that the singular point of each of the following functions is a pole. Determine the order m of that pole and the corresponding residue B.

   (a)

   (b) $\frac{1-\exp 2z}{z^4}$

   Solution Recall the MacLaurin series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} (|z| < 1)$$

Then, write

$$\begin{aligned}\frac{1-\exp 2z}{z^4} &= \frac{1}{z^4} - \frac{\exp 2z}{z^4} \\ &= \frac{1}{z^4} - \sum_{n=0}^{\infty} \frac{\frac{(2z)^n}{n!}}{z^4} \\ &= \frac{1}{z^4} - \sum_{n=0}^{\infty} \frac{(2z)^n}{n! \cdot z^4} \\ &= \frac{1}{z^4} - \sum_{n=0}^{\infty} \frac{2^n z^n}{n! \cdot z^4} \\ &= \frac{1}{z^4} - \sum_{n=0}^{\infty} \frac{2^n z^{n-4}}{n!}\end{aligned}$$

Expanding the series, we get

$$\begin{aligned}\frac{1-\exp 2z}{z^4} &= \frac{1}{z^4} - \sum_{n=0}^{\infty} \frac{2^n z^{n-4}}{n!} \\ &= \frac{1}{z^4} - [\frac{2^0 z^{-4}}{0!} + \frac{2^1 z^{-3}}{1!} + \frac{2^2 z^{-2}}{2!} + \boldsymbol{\frac{2^3 z^{-1}}{3!}} + ...]\end{aligned}$$

Therefore, the b_1 term of the Laurent expansion (and therefore our residue) is $-\frac{2^3}{3!} = -\frac{8}{6} = -\frac{4}{3}$.

Answers (Provided by the book)

i. $m = 1, B = \frac{-1}{2}$,
ii. $m = 3, B = \frac{-4}{3}$
iii. $m = 2, B = 2e^2$