Homework 6

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1 Homework

1. Write the principal part of the following functions at the singular point and determine whether that point is a pole, a removable singular point, or an essential singular point:

   (a) $\frac{z^2}{1+z}$

   Solution This function has a singular point at $z = -1$. First, we will find the Laurent series expansion. Notice that,

$$\begin{aligned} z^2 &= [(z+1)^2 - 2z - 1] \\ &= [(z+1)^2 - 2(z+1) + 1] \end{aligned}$$

   Therefore,

$$\begin{aligned} \frac{z^2}{1+z} &= z^2 \cdot \frac{1}{1+z} \\ &= [(z+1)^2 - 2(z+1) + 1] \cdot \frac{1}{1+z} \\ &= \frac{(z+1)^2 - 2(z+1) + 1}{1+z} \\ &= (z+1) - 2 + \frac{1}{1+z} \end{aligned}$$

   By inspection, the principal part of this function is

$$\frac{1}{1+z}$$

   so $z = -1$ is a pole of order 1.

   (b) $\frac{\sin z}{z}$

   Solution First, we will try to find a Laurent series representation for $f(z)$.

$$\begin{aligned} \frac{\sin z}{z} &= \frac{1}{z} \cdot \sin z \\ &= \frac{1}{z} \cdot \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1!)} && (|z| < \infty) \\ &= \frac{1}{z} \cdot [(-1)^0 \cdot \frac{z^1}{1!} + (-1)^1 \frac{z^3}{3!} + (-1)^2 \frac{z^5}{5!} + ...] && \text{Expanding the series} \end{aligned}$$

   By inspection, this Laurent Series has no terms with negative exponents and is really just a regular power series. Therefore, it is a removable singular point.

2. Show that the singular point of each function is a pole and determine its order m and find the corresponding residue:

   (a) $(\frac{z}{2z+1})^3$

   Solution First, this function has singularities whenever $(2z+1)^3 = 0$, implying $z = -\frac{1}{2}$ is a singular point.
   Now, notice that

$$\frac{z}{2z+1} = \frac{z}{2} \cdot \frac{1}{z + \frac{1}{2}}$$

   Furthermore, because $z = (z + \frac{1}{2}) - \frac{1}{2}$, the above is equivalent to

$$\begin{aligned} \frac{z}{2z+1} &= \frac{(z+\frac{1}{2}) - \frac{1}{2}}{2} \cdot \frac{1}{z+\frac{1}{2}} \\ &= \frac{1}{2} \cdot \frac{(z+\frac{1}{2}) - \frac{1}{2}}{z+\frac{1}{2}} \\ &= \frac{1}{2} \cdot [1 - \frac{0.5}{z+\frac{1}{2}}] \end{aligned}$$

   Finally, we get

$$\frac{1}{2} - \frac{0.25}{z+\frac{1}{2}}$$

   Therefore, because there is only $(z - z_0)^n$ term with an exponent of -1, this is a pole of order 1 with residue 0.25.

   (b) $\frac{\exp z}{z^2+\pi^2}$

   Solution Consider $z = \pi i$. At this point, the denominator $z^2 + \pi^2 = \pi^2 \cdot i^2 + \pi^2 = 0$, so πi is a singular point.

   First, let us find the Taylor Series of $\exp(z)$ about $z = \pi i$. Because $\frac{d}{dz}\exp(z) = \exp(z)$, and $\exp(\pi i) = -1$,

$$\begin{aligned} \frac{f^{(n)}(z_0)}{n!} &= \frac{\exp(\pi i)}{n!} \\ &= \frac{-1}{n!} \end{aligned}$$

   Thus, the Taylor Series of $\exp(z)$ centered at πi is

$$f(z) = \sum_{n=0}^{\infty} \frac{-1}{n!}(z - \pi i)^n$$

   which converges for all $z \in \mathbb{C}$.

Notice that

$$(z+\pi i)(z-\pi i) = (z^2 - z\pi i + z\pi i - \pi^2 i^2) = z^2 + \pi^2$$

Therefore,

$$\begin{aligned} \frac{\exp z}{z^2+\pi^2} &= \frac{1}{(z+\pi i)(z-\pi i)} \cdot \sum_{n=0}^{\infty} \frac{-1}{n!}(z-\pi i)^n \\ &= \frac{1}{z+\pi i} \cdot \sum_{n=0}^{\infty} \frac{-(z-\pi i)^{n-1}}{n!} \end{aligned}$$

And the completion of this solution has been left as an exercise for the reader.