Common Power Series

Vincent La

August 23, 2017

1 Formulas

1.1 MacLaurin Series

$$f(z) = \sum_{n=0}^{n} a_n z^n \quad \text{where } a_n = \frac{f^{(n)}(0)}{n!}$$

2 Exponential Function

2.1 MacLaurin Series

The MacLaurin series for e^z is

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

which converges for $|z| < \infty$.

2.1.1 Formula for Series

Proof. Because $\frac{d}{dz}e^z = e^z$, $f^{(n)}(z) = e^z$ which implies $f^{(n)}(0) = 1$.

Continuing, this means

$$a_n = \frac{1}{n!}$$

so plugging into the general MacLaurin Series formula we get

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

□

2.1.2 Convergence

Proof. Let $z \in \mathbb{C}$ be arbitrary and recall that

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} = \lim_{n \to \infty} S_n$$

Expanding the partial sum on the RHS, we get

$$\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{z^0}{0!} + \frac{z^1}{1!} + \frac{z^2}{2!} + ... + \frac{z^n}{n!}$$

Now, let us show that the limit of the n^{th} term (denoted a_n) is zero.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{z^n}{n!} = \lim_{n\to\infty} \frac{z \cdot z \cdot z \cdot z \cdot ... \cdot z \cdot ... \cdot z}{0 \cdot 1 \cdot 2 \cdot ... \cdot z \cdot ... \cdot n}$$

Because $\lim_{n\to\infty} \frac{z}{n} = 0$, by the limit of products

$$\lim_{n\to\infty} a_n = 0$$

So, as $n \to \infty$, the n^{th} term of the partial sum is 0, while all the preceding terms are finite. Therefore, $\sum_{n=0}^{\infty} \frac{z^n}{n!}$ converges for all z. $\square$