Homework 1

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Monday

4. (a) Show $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

Proof. First, denote $z_k = x_k - iy_k$.
Then,

$$\begin{aligned}\overline{z_1} + \overline{z_2} &= (x_1 - iy_1) + (x_2 - iy_2)\\ &= (x_1 + x_2) - i(y_1 + y_2)\end{aligned}$$

Furthermore, because $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$, this implies $\overline{z_1 + z_2} = (x_1 + x_2) - i(y_1 + y_2)$.
Therefore, $\overline{z_1} + \overline{z_2} = \overline{z_1 + z_2}$. □

(b) Show $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$.

Proof. First, $\overline{z_1} \cdot \overline{z_2} = (x_1 - iy_1) \cdot (x_2 - iy_2)$. Simplifying, we get $x_1x_2 - ix_1y_2 - ix_2y_1 + i^2y_1y_2$ or equivalently $x_1x_2 - y_1y_2 - i(x_1y_2 + x_2y_1)$.

Continuing,

$$\begin{aligned}z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2)\\ &= x_1 \cdot x_2 + i \cdot x_1y_2 + i \cdot x_2y_1 + i^2y_1y_2\\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)\end{aligned}$$

This implies that $\overline{z_1 \cdot z_2} = (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1)$. Therefore, $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$ □

(c) Show that $\overline{P(z)} = P(\overline{z})$ where P is any polynomial with real coefficients.

Proof. Let $P(z)$ be any polynomial with real coefficients and write it as $P(z) = a_0 + a_1z + a_2z^2 + ... + a_nz^n$.
Then, we get

$$\begin{aligned}\overline{P(z)} &= \overline{a_0 + a_1z + a_2z^2 + ... + a_nz^n} &&\\ &= \overline{a_0} + \overline{a_1z} + \overline{a_2z^2} + \overline{...} + \overline{a_nz^n} && \text{By 4a}\\ &= \overline{a_0} + \overline{a_1}\overline{z} + \overline{a_2}\overline{z^2} + ... + \overline{a_n}\overline{z^n} && \text{By 4b}\\ &= a_0 + a_1\overline{z} + a_2\overline{z^2} + ... + a_n\overline{z^n} && \text{Since } a_i \in \mathbb{R},\ a_i = \overline{a_i}\\ &= a_0 + a_1\overline{z} + a_2\overline{z}^2 + ... + a_n\overline{z}^n && \overline{z^n} = \overline{z}^n \text{ by 4b}\end{aligned}$$

Trivially,

$$P(\overline{z}) = a_0 + a_1 \cdot \overline{z} + a_2 \cdot \overline{z}^2 + ... + a_n \cdot \overline{z}^n$$

Therefore, $\overline{P(z)} = P(\overline{z})$. □

(d) Show $\overline{\overline{z}} = z$.

Proof. If $\overline{z} = x - iy$, then $\overline{\overline{z}} = \overline{x - iy} = x - -(iy) = x + iy = z$. □

5. If P is a polynomial with real coefficients, we want to show that $P(z) = 0$ if and only if $P(\overline{z}) = 0$.

Proof. For one direction, assume that $P(z) = 0$ and try to prove that $P(\overline{z}) = 0$. First, if $P(z) = 0$ then $\overline{P(z)} = \overline{0} = 0$. Moreover, from 4c we know that $\overline{P(z)} = P(\overline{z})$. So if $\overline{P(z)} = 0$, then $P(\overline{z}) = 0$ as well.

Now, for the other direction assume that $P(\overline{z}) = 0$. With 4c, this implies $\overline{P(z)} = 0$. Because zero is its own complex conjugate, this implies $P(z) = 0$ and this completes the proof. □

12. Solve using polar coordinates.

(a) $z^6 = 1$

Solution First, notice that $z^6 = 1$ is equivalent to

$$\Pi_{i=1}^{n=6}|z|e^{i\theta} = |z|^6 e^{6_i\theta} = 1$$

This implies that z and $e^{6_i\theta}$ are both equal to 1. Because only $e^0 = 1$, this implies $6\theta = 0$ modulo 2π. Thus, this implies six different solutions:

- $6\theta = 0 \implies \theta = 0$
- $6\theta = 2\pi \implies \theta = \frac{1}{3}\pi$
- $6\theta = 4\pi \implies \theta = \frac{2}{3}\pi$
- $6\theta = 6\pi \implies \theta = \pi$
- $6\theta = 8\pi \implies \theta = \frac{4}{3}\pi$
- $6\theta = 10\pi \implies \theta = \frac{5}{3}\pi$

(b) $z^4 = -1$.

Solution Using polar coordinates, this is equivalent to

$$|z|^4[\cos(4\theta) + i\sin(4\theta)] = -1$$

Because $|z|^4$ can only ever be a positive quantity, this implies $z = 1$ while $\cos(4\theta) + i\sin(4\theta) = -1$. Furthermore, recall that

$$\cos(\theta) + i\sin(\theta) = -1 + 0 = -1$$

when $\theta = \pi$.
Therefore, $4 \cdot \pi = \theta$ modulo 2π. This implies four solutions:

- $4\theta = \pi \implies \theta = \frac{\pi}{4}$
- $4\theta = 3\pi \implies \theta = \frac{3\pi}{4}$
- $4\theta = 5\pi \implies \theta = \frac{5\pi}{4}$
- $4\theta = 7\pi \implies \theta = \frac{7\pi}{4}$

13. We want to show that the n-th roots of 1 (aside from 1) satisfy the cyclotomic equation $z^{n-1} + z^{n-2} + ... + z + 1 = 0$.

    Proof. Let z be the n-th root of unity where $n > 1$, implying that $z^n = 1$. Obviously, it follows $z^n - 1 = 0$.

    Using the identity $z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + ... + 1)$, this implies the right hand side must be zero as well. But because $z - 1$ is a non-zero quantity, it must be that

    $$z^{n-1} + z^{n-2} + ... + 1 = 0$$

    as we set out to prove. □

Wednesday

Lemma The sequence $\{i^k\}$ takes on values of either $i, -1, -i$, or 1.

Proof. First, let $k \in \mathbb{N}$ be arbitrary and consider the following exhaustive cases.

Case 1: k modulo 4 = 0 This implies that k is a multiple of 4. Then, notice that

$$i^k = (i^4)^{k/4}$$

Because k is a multiple of 4, $k/4$ is an integer. Therefore, i^n is simply a repeated multiplication of i^4. Because $i^4 = 1$, $i^n = 1$ also. □

Case 2: k modulo 4 = 1 This implies that $k - 1$ is a multiple of 4. Therefore,

$$\begin{aligned} i^k &= i^{k-1} \cdot i \\ &= 1 \cdot i \qquad i^{k-1} = 1 \text{ by Case 1} \end{aligned}$$

Case 3: k modulo 4 = 2 This implies that $k - 2$ is a multiple of 4. Therefore,

$$\begin{aligned} i^k &= i^{k-2} \cdot i^2 \\ &= 1 \cdot i^2 = -1 \end{aligned}$$

Case 4: k modulo 4 = 3 This implies that $k-3$ is a multiple of 4. Therefore,

$$\begin{aligned} i^k &= i^{k-3} \cdot i^3 \\ &= 1 \cdot i^3 = -i \end{aligned}$$

1. Prove that $\{i^k\}$ is not a Cauchy sequence.

   Proof. Assume (for contradiction) that $\{i^k\}$ is a Cauchy sequence. This implies that for any $\epsilon > 0$, e.g. $\epsilon_1 = \frac{1}{2}$, we can find an N such that for all $n, m > N$ where $|a_n - a_m| < \epsilon_1 = \frac{1}{2}$.

   But as shown previously, for any N there are some $n, m > N$ where $a_n = 1$ and $a_n = -1$. Because, $|1 - (-1)| = 2 \not< \frac{1}{2}$, this contradicts our previous assumption that $\{i^k\}$ is Cauchy. □

2. Prove that if $\{z_n\}$ converges to z, and z_n is an element of the unit circle S^1 for all n, then z is also in S^1.

   Proof. Assume that $\{z_n\}$ converges to z, where $z_n \in S^1$ for all n. Then suppose for a contradiction that z is not in S^1. First, because $\{z_n\}$ converges to z then for any $\epsilon > 0$ there is some N such that for all $n > N, |z_n - z| < \epsilon$. Moreover, because z_n is on the unit circle while z is not, $|z_n - z|$ is always some positive quantity, call it ϵ_1. Now suppose we pick some point on the unit circle that minimizes ϵ_1 and call it ϵ_L. Again, for the same reasoning $\epsilon_L > 0$.

   However, recall that because $\{z_n\} \to z$, then for any $\epsilon > 0$, e.g. $\frac{\epsilon_L}{2}$, there is some z_n such that $|z_n - z| < \frac{\epsilon_L}{2}$. Because $\frac{\epsilon_L}{2}$ is less than the minimum distance between any point on the unit circle and z, it it implies z_n is not on the unit circle–contradicting our previous assumption that $z_n \in S^1$ for all n.

   Therefore, it must be that if $\{z_n\}$ converges to z, and z_n is an element of the unit circle S^1 for all n, then z is also in S^1. □