Midterm Practice

Vincent La
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1 Theorems

Residues at Poles Suppose that a function $f(z)$ can be written in the form

$$f(z) = \frac{\phi z}{z - z_0}$$

where $\phi(z)$ is analytic at z_0 and $\phi(z_0) \neq 0$. Then, $f(z)$ has a Laurent series representation

...

and its residue is given by

$$b_1 = \phi(z_0)$$

2 Homework

1. (a) Does the function $f(z) = \frac{e^z}{z}$ have a MacLaurin series representation?

   Solution Yes.
   First, recall that

   $$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

   which converges for all $z \in \mathbb{C}$. Therefore,

   $$\frac{e^z}{z} = \sum_{n=0}^{\infty} \frac{z^{n-1}}{n!}$$

   which converges for $0 < |z| < \infty$.

   (b) Given the Laurent series representation

   $$\frac{5z-2}{z(z-1)} = \frac{3}{z-1} + 2 - 2(z-1) + 2(z-1)^2 - 2(z-1)^3 + ...$$

   $|z-1| < 1$ determine whether the isolated singular point $z_0 = 1$ of $\frac{5z-2}{z(z-1)}$ is a pole of order m, a simple pole...

Answer The point $z_0 = 1$ is a simple pole (pole of order 1).

(c) Given the Laurent series expansion

$$f(z) = \frac{1}{z^2} + \frac{1}{z^2} + 1 + z + z^2 + z^3 + ...$$

which converges for $|z| < 1$, determine the residue of $f(z) = 0$.
The residue is 0, because there is no $\frac{b_1}{z^1}$ term.

(d) Does there exist a power series $\sum_{n=0} a_n z^n$ that converges at $z = 2 + 3i$ and diverges at $z = 3 - i$.

Answer Yes, and in fact there's an infinite number of them. Here, I will provide one example.
First, notice that

$$|2 + 3i|^2 = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$$
$$|3 - i|^2 = \sqrt{3^2 + (-1)^2} = \sqrt{9 + 1} = \sqrt{10}$$

Now, consider the power series

$$\sum_{n=0}^{\infty} (\frac{4}{z})^n$$

If $|z| = \sqrt{13}$, then $\frac{4}{z} < 1$ and this series converges. However, if $|z| = \sqrt{10}$, then $\frac{4}{z} > 1$ and this series diverges.

2. ...

3. (a) Give two Laurent series expansions in powers of z for the function

$$f(z) = \frac{1}{z^2(3 - z)}$$

and provide regions of validity.

Solution about $z = 0$ First, rewrite $f(z)$ as

$$f(z) = \frac{1}{z^2(3 - z)} = \frac{1}{3z^2} \cdot \frac{1}{1 - \frac{z}{3}}$$

Now, recall the MacLaurin series

$$\frac{1}{1 - z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

which converges for $|z| < 1$. Applying the change of variables $z = \frac{z}{3}$, we get

$$\frac{1}{1 - z/3} = \sum_{n=0}^{\infty} (-1)^n (\frac{z}{3})^n$$

which converges for $|\frac{z}{3}| < 1 \implies |z| < 3$.

Applying all of this, we get

$$\begin{aligned} f(z) &= \frac{1}{3z^2} \cdot \frac{1}{1 - \frac{z}{3}} \\ &= \frac{1}{3z^2} \cdot \sum_{n=0}^{\infty} (-1)^n (\frac{z}{3})^n \\ &= \frac{1}{3z^2} \cdot \sum_{n=0}^{\infty} (\frac{-1}{3})^n z^n \\ &= \frac{1}{3} \cdot \sum_{n=0}^{\infty} (\frac{-1}{3})^n z^{n-2} \qquad (0 < |z| < 3) \end{aligned}$$

4.

(a) Find the MacLaurin series representation of $\cos z = \frac{e^{iz} - e^{-iz}}{2}$

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n (z^{2n})}{(2^n)!}$$

Proof. First, calculate and evaluate the first few derivatives of $\cos z$ at $z = 0$.

$$\begin{aligned} f^{(0)}(0) &= \cos 0 = 1 \\ f^{(1)}(0) &= -\sin 0 = 0 \\ f^{(2)}(0) &= -\cos 0 = -1 \\ f^{(3)}(0) &= \sin 0 = 0 \\ f^{(4)}(0) &= \cos 0 = 1 \end{aligned}$$

As we can see, the derivatives of $\cos z$ follow a predictable pattern. Continuing, using the general formula for a MacLaurin series

$$f(z) = \sum a_n z^n$$

implies that

$$\cos z = \sum_{n \text{ odd}}^{\infty} 0 + \sum_{n \text{ even}}^{\infty} \frac{(-1)^{\frac{n}{2}}}{n!} \cdot (z^n)$$

Furthermore, if n is even, then there is some integer such that n is divisible by 2, i.e. $n = 2n$. Using this change of variables

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \cdot (z^{2n})$$

as we set out to prove. $\square$

(b) Using the MacLaurin series representation for the function $\cos z$, find the MacLaurin series representation for $\sin z$

6. Find the first three non-zero terms in the MacLaurin expansion of

$$f(z) = \int_0^z e^{s^2} ds$$

Solution First, recall the MacLaurin Series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Thus, letting $z = s^2$, we get

$$e^{s^2} = \sum_{n=0}^{\infty} \frac{s^{2n}}{n!}$$

which, like the original series, converges for all $|z| < \infty$.

Now,

$$\begin{aligned} \int_0^z e^{s^2} ds &= \int_0^z \sum_{n=0}^{\infty} \frac{s^{2n}}{n!} ds \\ &= \sum_{n=0}^{\infty} \frac{\int_0^z s^{2n} ds}{n!} && \text{Integrate term by term} \\ &= \sum_{n=0}^{\infty} \frac{s^{2n+1}}{(2n+1) \cdot n!} \end{aligned}$$

Therefore, the first three non-zero terms are

$$\frac{s^{2(0)+1}}{(2 \cdot 0 + 1) \cdot 0!} + \frac{s^{2(0)+1}}{(2 \cdot 1 + 1) \cdot 1!} + \frac{s^{2(0)+1}}{(2 \cdot 2 + 1) \cdot 2!}$$

8. Find residue at $z = 0$ of $\frac{1}{z^2+z^3}$.

First, recall the MacLaurin series

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

which converges for $|z| < 1$. Thus,

$$\begin{aligned} \frac{1}{z^2} \cdot \frac{1}{1+z} &= \sum_{n=0}^{\infty} (-1)^n z^{n-2} && (|z| < 1) \\ &= (-1)^0 z^{-2} + (-1)^1 z^{-1} + (-1)^2 z^0 + ... \\ &= z^{-2} - \mathbf{z^{-1}} + ... \end{aligned}$$

Therefore, $Res_{z=0} f(z) = 1$.

9. (a) Use Cauchy's Residue Theorem to evaluate the integral around the circle $|z| = 3$ with a positive orientation fo $z^3 \cdot \exp \frac{1}{z^2}$

Solution To begin, we have an isolated singularity at $z = 0$. Then, recall the MacLaurin Series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Using the change of variables $z = \frac{1}{z^2}$, we get

$$\begin{aligned} z^3 \cdot \exp \frac{1}{z^2} &= z^3 \sum_{n=0}^{\infty} \frac{(\frac{1}{z^2})^n}{n!} \\ &= z^3 \sum_{n=0}^{\infty} \frac{z^{-2n}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{z^{-2n+3}}{n!} \end{aligned}$$

Expanding this series, we get

$$\frac{z^3}{0!} + \frac{z^{-2+3}}{1!} + \frac{\mathbf{z^{-4+3}}}{\mathbf{2!}} + \ldots$$

Therefore our residue is $2! = 2$.

(b) ...

10. Write the principal part of the following functions at their singular point and determine whether that point is a pole, a removable singular point ,or an essential singular point:

(a) $z \exp \frac{1}{z^3}$

Solution First, notice that we have a singular point at $z = 0$. Then, recall the MacLaurin Series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Using the change of variables $z = \frac{1}{z^3}$, we get

$$\begin{aligned} z \exp \frac{1}{z^3} &= z \cdot \sum_{n=0}^{\infty} \frac{(\frac{1}{z^3})^n}{n!} \\ &= z \cdot \sum_{n=0}^{\infty} \frac{z^{-3n}}{n!} \\ &= \sum_{n=0}^{\infty} \frac{z^{-3n+1}}{n!} \end{aligned}$$

Expanding the series, we get

$$\frac{z^1}{0!} + \frac{z^{-3+1}}{1!} + \frac{z^{-6+1}}{2!} + ...$$

Because this series has an infinite number of negative terms, it is an essential singular point.

(b) $\frac{z^3}{1+z}$

First, notice that $f(z)$ has a singularity when $z + 1 = 0 \implies z = -1$. Then, recall the MacLaurin Series

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

which converges for $|z| < 1$. Therefore,

$$\begin{aligned} z^3 \cdot \frac{1}{1+z} &= z^3 \cdot \sum_{n=0}^{\infty} (-1) z^n \quad (|z| < 1) \\ &= \sum_{n=0}^{\infty} (-1) z^{n+3} \end{aligned}$$

By looking at the exponents on z within the sum, we can tell that this series has an infinite number of terms with positive exponents but none with negative ones. Therefore, $z = -1$ is a removal singular point.

11. Show that the singular point of each function is a pole and determine its order m and the corresponding residue

(a) $\frac{z^2+2}{z^2-1}$

Solution First, notice that this function has two singular points that occur when

$$\begin{aligned} z^2 - 1 &= 0 \\ z^2 &= 1 \\ z &= \pm\sqrt{1} \\ z &= \pm 1 \end{aligned}$$

Also, I will be using the fact that

$$(z-1)(z+1) = z^2 - 1$$

i. $\mathbf{z = 1}$ First, rewrite $f(z)$ as follows

$$\begin{aligned} \frac{z^2+2}{z^2-1} &= \frac{z^2+2}{(z-1)(z+1)} \\ &= \frac{z^2+2}{(z-1)(z+1)} \cdot \frac{\frac{1}{z+1}}{\frac{1}{z+1}} \\ &= \frac{\frac{\mathbf{z^2+2}}{\mathbf{z+1}}}{z-1} \end{aligned}$$

Now, call the bolded part $\phi(z)$. Furthermore, notice that $\phi(z)$ is analytic at $z = 1$ and that

$$\phi(1) = \frac{1^2 + 2}{1 + 1} = \frac{3}{2} \neq 0$$

Therefore, this point is a simple pole with residue $\frac{3}{2}$.

ii. $\mathbf{z = -1}$

(b) $(\frac{z}{3z+5})^3$

12. Evaluate the following integrals