Homework 5

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1 Monday and Tuesday

4.3 Evaluate $\int_C f$ where $f(z) = x^2 + iy^2$ as in Example 1, but where C is given by $z(t) = t^2 + it^2$, $0 \le t \le 1$.

Solution This implies $\dot{z}(t) = \cos(t) - i\sin(t)$, so

$$\int_a^b f(z(t))\dot{z}(t)dt = \int_0^{2\pi} (\frac{\sin(t)}{\sin^2(t) + \cos^2(t)} - i\frac{\cos(t)}{\sin^2(t) + \cos^2(t)})(\cos(t) - i\sin(t)dt$$

This in turn simplifies to

$$\begin{aligned} \int_0^{2\pi} (\sin(t) - i\cos(t))(\cos(t) - i\sin(t))dt &= \int_0^{2\pi} \sin^2(t) + \cos^2(t)dt \\ &= -\int_0^{2\pi} 1dt \\ &= -2\pi \end{aligned}$$

This is unlike Example 2, where the answer was $2\pi i$. Now, line integrals should be independent of parameterization, but this parameterization is actually the same as the one in Example 2 but moving in the opposite direction.

4.5 Use the Fundamental Theorem of Calculus (Proposition 4.12) to prove that if F is analytic on a region and $F'(z) = 0$ then F is constant.

Proof. Let F be any analytic function where $F'(z) = 0$. Let the bounds of integration $a, b \in \mathbb{Z}$ be arbitrary. First, the Fundamental Theorem of Calculus states that

$$0 = F'(z) = \int_C f(z)dz = F(z(b))] - F(z(a))$$

Thus,

$$0 = \int 0dz = F(z(b)) - F(z(a))$$

This is only true if $a = b$ or F is constant. However, since a, b arbitrary, it must be that F is constant. □

2 Wednesday and Thursday

1. Prove that $\lim_{n\to\infty} \xi^n = 0$ whenever $|\xi| < 1$. Conversely, prove that if $|\xi| \geq 1$ then $\{\xi^n\}$ is a divergent sequence.

   (a) $\lim_{n\to\infty} \xi^n = 0$ whenever $|\xi| < 1$

   Proof. Let $\epsilon > 0$ be arbitrary and let $\delta = \frac{1}{\epsilon}$. Then, show that whenever $|n| > \delta$ that $|f(n) - 0| < \epsilon$. Notice, $n > \delta = \frac{1}{\epsilon}$ implies that

   $$|\xi^n| < |\xi^{\frac{1}{\epsilon}}| < 1$$

   because $|\xi| < 1$.

   This further implies that $|\xi^{\frac{1}{\epsilon}}|^{\epsilon} < \epsilon$, or $|\xi^n| < |\xi^1| < \epsilon$ as we set out to prove. □

   (b)

2.1 Extra Problems

5.1 Find the powers series expansion of $f(z) = z^2$ around $z = 2$.

In general, a Taylor Series about a function is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n$$

Filling the terms in, here it is $4 + 4(z-2) + (z-2)^2$.

5.2 Find power series expansion for e^z about any point a

5.3 Show that an odd entire function only has odd terms in its power series expansion about $z = 0$.

Proof. (Insert proof that f even implies f' odd and that f odd implies f' even here).

If an odd function is entire, it has a power series expansion which we can write

$$\frac{f^{(0)}(0)}{0!}z^0 + \frac{f^{(1)}(0)}{1!}z^1 + \frac{f^{(2)}(0)}{2!}z^2 + \frac{f^{(3)}(0)}{3!}z^3 + ...$$

Now, suppose that if the n^{th} term is odd, then the $n + 1^{th}$ term will also be odd and induct on $n \in \mathbb{Z}^+$. □