08/29/17

Examples

1) $f(z) = \cot(z) = \dfrac{\cos(z)}{\sin(z)}$

$f(z) = \dfrac{p(z)}{q(z)}$,

$p(z) = \cos z$, $q(z) = \sin z$ both entire

$q(z) = 0 \longleftrightarrow \sin(z) = 0 \longleftrightarrow z = n\pi$,

$n = 0, \pm 1, \pm 2, \ldots$

Theorem:

- p, q analytic at z_0
- $p(z_0) \neq 0$
- $q(z_0) = 0$
- $q'(z_0) \neq 0$

$\Rightarrow z_0$ is a simple pole

$$\operatorname*{Res}_{z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

$p(z_n) = \pm 1 \neq 0$

$q'(z) = \cos(z)$, $q'(z_n) = \pm 1 \neq 0$

So $z_n = n\pi$, $n = 0, \pm 1, \ldots$ are simple poles

$$\operatorname*{Res}_{z_n} f = \frac{p(z_n)}{q'(z_n)} = \frac{\cos z_n}{\cos z_n} = 1$$

$\sinh(x) = -i\sin(ix)$
$\cosh(x) = \cos(ix)$
$\cosh(x)' = \sinh x$

7) $f(z) = \dfrac{\tanh(z)}{z^2} = \dfrac{\sinh(z)}{z^2\cosh(z)}$

Find $\operatorname{Res}_{z_0 = \frac{\pi i}{2}} f$

$p(z) = \sinh z$ analytic @ z_0
$q(z) = z^2 \cdot \cosh z$

$p\left(\frac{\pi i}{2}\right) = \sinh\left(\frac{\pi i}{2}\right) = i\sin\left(\frac{\pi i^2}{2}\right) = i\sin\left(-\frac{\pi}{2}\right) = -i \neq 0$

$q\left(\frac{\pi i}{2}\right) = \left(\frac{\pi i}{2}\right)^2 \cosh\frac{\pi i}{2} = -\frac{\pi^2}{4}\cos\frac{\pi i^2}{2}$

$= -\frac{\pi^2}{4}\cos\left(-\frac{\pi}{2}\right) = 0$

$q'(z) = 2z \cdot \cosh(z) + z^2 \cdot \sinh z$

$q'\left(\frac{\pi i}{2}\right) = \pi i\cosh\left(\frac{\pi i}{2}\right) - \frac{\pi^2}{4}\sinh\left(\frac{\pi i}{2}\right)$

$= \cancel{\pi i\cos\left(-\frac{\pi}{2}\right)}^{\,0} + \frac{\pi^2}{4} i\sin\left(-\frac{\pi}{2}\right)$

$= -\frac{\pi^2}{4} i \neq 0$

So, $z_0 = \frac{\pi i}{2}$ is a simple pole

$\operatorname{Res}_{\frac{\pi i}{2}} f = \dfrac{p\left(\frac{\pi i}{2}\right)}{q'\left(\frac{\pi i}{2}\right)} = \dfrac{i}{\frac{\pi^2}{4}i} = \dfrac{-4}{\pi^2}$

$$\left[\frac{f}{g}\right]' = \frac{f'g - g'f}{g^2}$$

$$\cancel{g'(z_0) = 0 + \frac{\cos(z_0)}{\sin^2(z_0)}}$$

2) Find $\displaystyle\int_{C:\,|z|=2} \tan(z)\,dz = 2\pi i \left(\sum_{z_i} \operatorname{Res}_{z_i} f \right)$

isolated pts. inside the region bounded by C

$$\tan(z) = \frac{\sin(z)}{\cos(z)}$$

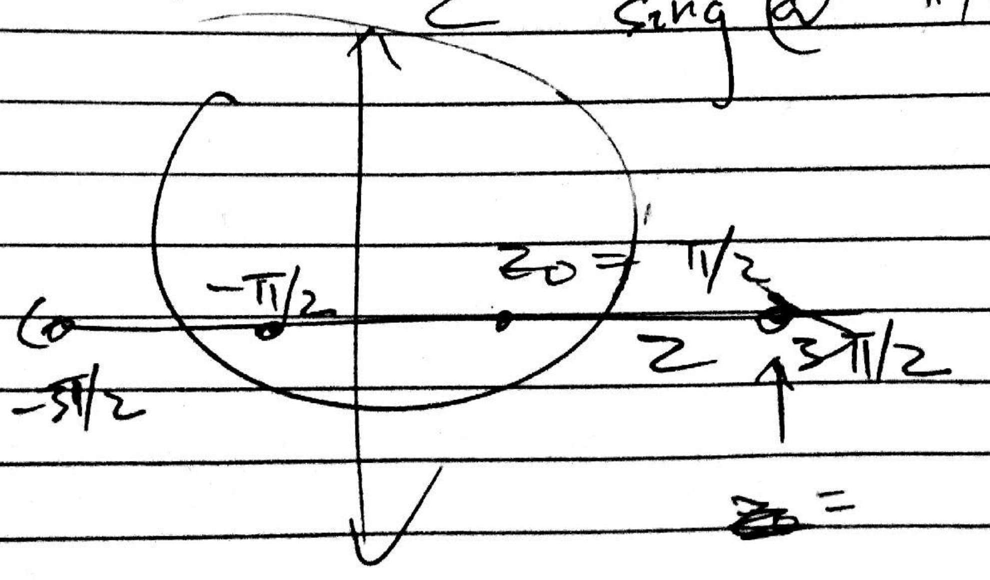
$\cos(z) = 0$ at $\pm\frac{\pi}{2}$ $\quad \frac{\pi}{2} + k\pi$

a) $\displaystyle\operatorname{Res}_{\pi/2} f = \frac{\sin(\pi/2)}{-\sin(\pi/2)} = -1$

b) $\displaystyle\operatorname{Res}_{-\pi/2} f = \frac{\sin(-\pi/2)}{-\sin(-\pi/2)} = -1$

So int $= \boxed{-4\pi i}$

sing @ $\pm\pi/2$



Appl. to physics: int. of RV func.

- p, q analytic at $\frac{\pi}{2}$

$p(\pi/2) = 1$, $q(\pi/2) = 0$, $q'(\frac{\pi}{2}) = -1$

$\pi/2$ is a simple pole, $\operatorname{Res}_{\pi/2} f = \frac{1}{-1} = -1$

- p, q analytic at $\pi/2$

$p(\frac{\pi}{2}$

$$\int_0^\infty f(x)\,dx = \lim_{R\to\infty}\int_0^R f(x)\,dx$$

$x \in \mathbb{R}$

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R_1\to\infty}\int_{-R_1}^{0} f(x)dx + \lim_{R_2\to\infty}\int_0^{R_2} f(x)dx$$

<u>Cauchy Principal Value</u> of $\int_{-\infty}^{\infty} f(x)\,dx$

is $\text{P.V.} \int_{\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx$

<u>attention</u>:

If $\int_{-\infty}^{\infty} f\,dx$ converges $\Longrightarrow$

Examples

3) $f(z) = \dfrac{z}{z^4+4}$

$p(z) = z$ $\qquad z = re^{i\theta}$

$q(z) = z^4 + 4$

$q(z) = 0 \iff \overbrace{z^4 = -4}\Rightarrow r^4 e^{i4\theta}$

$-4 = 4(\underbrace{\cos\pi + i\sin\pi}_{-1})$

$r = \sqrt{2}$

$\theta = \dfrac{\pi + 2k\pi}{4},$ $\qquad \Leftarrow \qquad$ r^4

$4\theta = \pi + 2k\pi,$

$k = 0, 1, \ldots$

$k = 0, 1, \ldots$

$\theta = \dfrac{\pi + 2k\pi}{4} = \dfrac{k\pi}{2} + \dfrac{\pi}{4}$

One of the solutions is

$z_0 = \sqrt{2}\, e^{i\pi/4} = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$= \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$

$= 1 + i$

p, q analytic at $1+i$

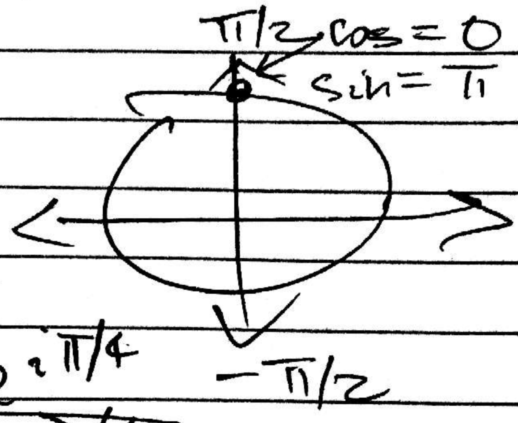
$p(1+i) = 1+i \neq 0$

$q(1+i) = 0$

$q'(z) = 4z^3,\quad q'(1+i) = 4(\sqrt{2}\, e^{i\frac{\pi}{4}})^3$

$= 4\left[(\sqrt{2})^3 e^{i3\pi/4}\right]$

$$= 8\sqrt{2}\, e^{i 3\pi/4}$$
$$= 8\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$
$$= 8\sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$
$$= -4 + 4i \neq 0$$



$$\text{So } \operatorname{Res}_{1+i} f = \frac{p(1+i)}{q'(1+i)}$$
$$= \frac{1+i}{-4+4i} = \frac{\sqrt{2}\, e^{i\pi/4}}{8\cancel{4}\sqrt{2}\, e^{i3\pi/4}}$$
$$= \frac{1}{8\cancel{4}} e^{i\left(\frac{\pi}{4} - \frac{3\pi}{4}\right)}$$
$$= \frac{1}{8} e^{i\left(-\frac{\pi}{2}\right)}$$
$$= \frac{1}{8}\left[\cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)\right]$$
$$= -i/8$$

Exercises

1) Show that 0 is a simple pole of $f(z) = \csc z = \frac{1}{\sin z}$ and find $\operatorname{Res}_0 f$

- $p(\overset{0}{\cancel{z}}) = 1 \neq 0$
- $q(\overset{0}{\cancel{z_0}}) = 0 = \sin(0)$
- $q'(z_0) = \cancel{\cos(0)}\ \cos(0)$

$q'(0) = 1 \neq 0$

$$\operatorname{Res}_0 f = \frac{p(0)}{q'(0)} = \frac{1}{1} = 1$$

P.V. $\int_{-\infty}^{\infty} f(x)dx$ exists and

$\int_{-\infty}^{\infty} f\,dx = \text{P.V.} \int_{-\infty}^{\infty} f\,dx$

(Converse is not true)

If P.V. $\int_{-\infty}^{\infty} f(x)\,dx$ exists, we do not know if $\int_{-\infty}^{\infty} f(x)\,dx$ converges

Proof Suppose $\int_{-\infty}^{\infty} f(x)\,dx$ converges

$$\begin{aligned}
\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx &= \lim_{R\to\infty} \int_{-R}^{R} f(x)dx \\
&= \lim_{R\to\infty} \left[\int_{-R}^{0} f(x)dx + \int_{0}^{R} f(x)dx \right] \\
&= \lim_{R\to\infty} \int_{-R}^{0} f(x)dx \\
&\quad + \lim_{R\to\infty} \int_{0}^{R} f(x)dx \text{ exist}
\end{aligned}$$

Example: $f(x) = x$

P.V. $\int_{-\infty}^{\infty} x\,dx = \lim_{R\to\infty} \int_{-R}^{R} x\,dx$

$$= \lim_{R \to \infty} \left[\frac{x^2}{2} \right]_{-R}^{R} = \lim_{R \to \infty} \left(\frac{R^2}{2} - \frac{R^2}{2} \right) = 0$$

$$\int_{-\infty}^{\infty} x\,dx = \lim_{R_1 \to \infty} \int_{-R_1}^{0} x\,dx + \lim_{R_2 \to \infty} \int_{0}^{R_2} x\,dx$$

$$= \lim_{R_1 \to \infty} \left[\frac{x^2}{2} \right]_{-R_1}^{0} + \lim_{R_2 \to \infty} \left[\frac{x^2}{2} \right]_{0}^{R_2}$$

$$= \lim_{R_1 \to \infty} \left(-\frac{R^2}{2} \right) + \lim_{R_2 \to \infty} \left(\frac{R^2}{2} \right)$$

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