Homework 3

Vincent La
Math 122A
July 15, 2017

1 Monday

Extra 5. Write down in terms of M-neighborhoods of ∞ what it means that $\lim_{z\to\infty} f(z) = \infty$.

Answer When we project the complex numbers on a sphere, we can see that as the magnitude of the number increases more and more they seem to converge to the north pole of the sphere.

Extra 6. Prove that $\lim_{z\to 0} \frac{1}{z} = \infty$ and that $\lim_{z\to 0} \frac{1}{z} = \infty$ and that $\lim_{z\to\infty} \frac{1}{z} = 0$.

(a) Prove that $\lim_{z\to 0} \frac{1}{z} = \infty$

Proof. Let $M > 0$, $z \in \mathbb{C}$ be arbitrary. Then pick $\delta = \frac{1}{M}$.
As a result, whenever $|z - 0| = |z| < \delta$, we get that

$$|z| < \frac{1}{M}$$

Now, multiply both sides by M, and we get

$$|z|M < 1$$

Divide both sides by $|z|$, and we get

$$M < \frac{1}{|z|} = f(z)$$

as we set out to prove. □

(b) Prove that $\lim_{z\to\infty} \frac{1}{z} = 0$

Proof. We want to show that for any $\epsilon > 0$, there is a $\delta > 0$ so that if $|z| > \delta$ then $|f(z) - 0| < \epsilon$. Let $\delta = \frac{1}{\epsilon} > 0$.

If $|z| > \delta$, then $|z| > \frac{1}{\epsilon}$. If we take the reciprocal of both sides, we get

$$f(z) = \frac{1}{|z|} < \epsilon$$

as we set out to prove. □

Extra 7. A complex values function $f : D \to \mathbb{C}$ is called bounded if there exists $M > 0$ such that $|f(z)| < M$ for all $z \in D$. Prove that $\{z_n\} \subseteq D$ such that $f(z_n) \to \infty$.

2 Tuesday

2.3 By comparing coefficients or by use of the Cauchy-Riemann equations, determine which of the following polynomials are analytic.

(a) $P(x+iy) = x^3 - 3xy^2 - x + i(3x^2y - y^3 - y)$

Analytic Here,

$$u(x,y) = x^3 - 3xy^2 - x$$

and

$$v(x,y) = 3x^2y - y^3 - y$$

. Therefore,

$$\begin{aligned} u_x &= 3x^2 - 3y^2 - 1 \\ v_y &= 3x^2 - 3y^2 - 1 \\ u_y &= -6xy \\ -v_x &= -6xy \end{aligned}$$

Because $u_x = v_y$ and $u_y = -v - x$, $P(x+iy)$ is analytic.

(b) $P(x+iy) = x^2 + iy^2$

Not Analytic Here, $u(x,y) = x^2$ and $v(x,y) = y^2$. Therefore,

$$\begin{aligned} u_x &= 2x \\ v_y &= 2y \end{aligned}$$

implying that $u_x \neq v_y$ or that $P(x+iy)$ is not analytic.

(c) $P(x+iy) = 2xy + i(y^2 - x^2)$

Analytic Here, $u(x,y) = 2xy$ and $v(x,y) = y^2 - x^2$. Therefore,

$$\begin{aligned} u_x &= 2y \\ v_y &= 2y \\ u_y &= 2x \\ -v_x = -(-2x) &= 2x \end{aligned}$$

Because $u_x = v_y$ and $u_y = -v_x$, $P(x+iy)$ is analytic.

2.4 Show that no nonconstant analytic polynomial can take imaginary values only.

Proof. Let us prove the contrapositive, that if f takes only imaginary values then f is either constant or not analytic.

If f only takes imaginary values then it has no real component, i.e. $u(x, y) = 0$. Thus, $u_x = 0$ and $u_y = 0$. If f is analytic, then necessarily $v_x = v_y = 0$. Because, only a constant function has these derivatives f is constant.

Now, if we relax the requirement that f satisfy the Cauchy-Riemann equations, then it is free to be non-constant. However, it will not be analytic, as we set out to prove. □

3 Wednesday and Thursday

3.2 Show $f(z) = x^2 + iy^2$ is differentiable at all points on the line $x = y$.

Proof. Let $z \in \mathbb{C}$ be arbitrary as long as $x = y$. Then, we want to show that f is differentiable, or in other words $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists.

Let us write $z = x_1 + iy_1$ and $h \in \mathbb{C}$ as $h = x_2 + iy_2$. Then, our limit becomes

$$\lim_{x_2+iy_2\to 0} \frac{f(x_1 + iy_1 + x_2 + iy_2) - f(x_1 + iy_1)}{x_2 + iy_2}$$

$$\lim_{x_2+iy_2\to 0} \frac{f[x_1 + x_2 + i(y_1 + y_2)] - f(x_1 + iy_1)}{x_2 + iy_2}$$

$$\lim_{x_2+ix_2\to 0} \frac{f[x_1 + x_2 + i(x_1 + x_2)] - f(x_1 + ix_1)}{x_2 + ix_2}$$

$$\lim_{x_2+ix_2\to 0} \frac{(x_1 + x_2)^2 + i(x_1 + x_2)^2 - (x_1^2 + ix_1^2)}{x_2 + ix_2}$$

$$\lim_{x_2+ix_2\to 0} \frac{x_1^2 + 2x_1x_2 + x_2^2 + ix_1^2 + 2ix_1x_2 + ix_2^2 - (x_1^2 + ix_1^2)}{x_2 + ix_2}$$

$$\lim_{x_2+ix_2\to 0} \frac{2x_1x_2 + x_2^2 + 2ix_1x_2 + ix_2^2}{x_2 + ix_2}$$

$$\lim_{x_2+ix_2\to 0} \frac{2x_1x_2(1 + i) + x_2^2(1 + i)}{x_2(1 + i)}$$

$$\lim_{x_2+ix_2\to 0} \frac{2x_1x_2 + x_2^2}{x_2}$$

$$\lim_{x_2+ix_2\to 0} 2x_1 + x_2 = 2x_1 + x_2$$

Therefore, our limit exists so f is differentiable.

Nowhere Analytic Show that f is nowhere analytic.

Using the Cauchy-Riemann Equations, we see that

$$u_x = 2y, v_y = 2y, u_y = 0, -v_x = 0$$

Therefore, f is analytic only on the line $x = y$. However, for f to be analytic for any z, it must be analytic in a neighborhood of z. However, for any z where f is analytic, its neighborhood contains an infinite number of points where $x \neq y$. Therefore, f is nowhere analytic. $\square$

3.5 Suppose f is analytic in a region and $f' \equiv 0$ there. Show that f is constant.

Proof. Let f be any function analytic in a region where $f'\ equiv0$ in that region. Because it is analytic, it satisfies the Cauchy-Riemann Equations, implying $u_x = v_y$ and $u_y = -v_x$. Because $f' \equiv 0$, it must mean that its partial derivatives either cancel each other out or are equal to 0. However, because f is analytic, the conditions imposed by the Cauchy-Riemann Equations imply $u_x = v_y = 0$ and $u_y = -v_x = 0$. Only a constant function has these derivatives, so f is constant. $\square$

3.9 Show that there are no analytic functions $f = u + iv$ with $u(x, y) = x^2 + y^2$.

Proof. Let f be an arbitrary function with $u(x, y) = x^2 + y^2$, and assume for a contradiction that it is analytic.

Because f is analytic, it satisfies the Cauchy-Riemann Equations. Because $u_x = 2x$ and $u_y = 2y$, this implies $v_y = 2x$ and $-v_x = 2y$. Now, notice that $v_y = 2x$ implies

$$\int v_y dy = \int 2x dy$$

$$v = 2xy + c(y)$$

On the other hand, $v_x = -2y$ implies

$$\int v_x dx = \int -2y dx$$

$$v = -2xy + c(x)$$

However, $v = 2xy + c(y)$ and $v = -2xy + c(x)$ cannot be true at the same time. Therefore, there are no analytic functions $f = u + iv$ with $u(x, y) = x^2 + y^2$. $\square$

3.10 Suppose f is an entire function of the form

$$f(x, y) = u(x) + iv(y)$$

Show that f is a linear polynomial.

Proof. Let f be any entire function of the form $u(x) + iv(y)$. If f is entire, it is analytic anywhere, so it constantly satisfies the Cauchy Riemann equations. Because $u_y = -v_x = 0$ for all x, y this imposes the constraint that $u_x = v_y$ for all x, y.

Suppose for a contradiction that $u(x), v(y)$ are not linear polynomials, i.e. they are polynomials with degree > 1. However, if we take partial derivatives u_x and v_y, then they will necessarily have some term involving x and y respectively. Now, take for example–$x = 0, y = 1$, then their partial derivatives evaluate to $u_x = 0, v_y = 1$. This contradicts the assumption f is entire, so this cannot be the case.

Now, we are left with the possibility that $u(x), v(y)$ are both linear polynomials. Write $u(x) = a + bx$ and $v(y) = c + dy$. Taking partial derivatives, we get $u_x = b$ and $v_y = d$. Thus, the Cauchy-Riemann Equations are satisfied if x and y have the same coefficients.

In conclusion, we have shown that any entire function of the form $u(x) + iv(y)$ has $u(x)$ and $v(y)$ as linear polynomials sharing the same coefficient on the first degree term. $\square$