# Homework 8

Vincent La
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1. Evaluate the improper integral
$$\int_0^\infty \frac{dx}{(x^2+1)^2}$$

**Solution** We'll solve this by evaluating it over the complex axis of the real plane. First, notice that this is equivalent to the integral

$$\int_{-R}^{R} \frac{dx}{(x^2+1)^2} + \int_{C_R} \frac{dz}{(z^2+1)^2} = 2\pi i Res f(z)$$

Now, notice that our function has isolated singularities whenever $(z^2+1)^2 = 0 \implies z^2 = -1$ implying that $z = \pm i$. Therefore, we'll want to integrate over a semicircle of radius $R > 1$.

Moreover, whenever $z \in C_R$, this implies that $|z| = R$. Therefore,

$$(z^2+1)^2 \geq ||z|^2+1|^2 = |R^2+1|^2$$

This then implies

$$\int_{C_R} \frac{dz}{(z^2+1)^2} \leq \frac{1}{|R^2+1|^2} \cdot \int_{C_R} dz = \frac{\pi i}{(R^2+1)^2}$$

Since the right hand side goes to zero as $R \to \infty$, the entire complex integral disappears. Finally, we just need to calculate the residue of $f(z)$ in $C_R$ at $z = i$. Recall that the residue of quotients, and we get

$$\begin{aligned}
\frac{p(z_0)}{q'(z_0)} &= \frac{1}{[(z^2+1)^2]'} \\
&= \frac{1}{2(z^2+1) \cdot 2z} \\
&= 4z^3 + 4z
\end{aligned}$$

Because f is even,

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2} &= 2\pi i Res f(i) \\
&= 2\pi i \left(\frac{1}{4i^3+4}\right) \\
&= 2\pi i \cdot \frac{1}{4-4i}
\end{aligned}$$

2. Evaluate the improper integral

$$\int_0^\infty \frac{dx}{(x^2+1)(x^2+4)}$$

**Solution** We'll take roughly the same steps as above.

First, notice that this is equivalent to the integral

$$\int_{-R}^{R} \frac{dx}{(x^2+1)(x^2+4)} + \int_{C_R} \frac{dz}{(z^2+1)(z^2+4)} = 2\pi i Resf(z)$$

Using a similar argument to problem 1, we can show that the complex integral disappears. If I weren't so tired, I could probably come up with a better argument that this flimsy sentence.

Now, notice that the function

$$\frac{1}{(z^2+1)(z^2+4)}$$

has isolated singularities whenever $(z^2+1)(z^2+4) = 0$ or equivalently, whenever $z^4 + 5z^2 + 4 = 0$.

Applying the quadratic formula, this implies

$$\begin{aligned} z^2 &= \frac{-5+\sqrt{5^2-4\cdot 1\cdot 4}}{2} \\ &= \frac{-5+\sqrt{25-16}}{2} \\ &= \frac{-5+3}{2} \\ &= -4 \end{aligned}$$

Therefore, we have singularities whenever $z = 2i$. Finally, we just have to compute the residues and I can finally be done with this problem. But before doing that, notice that

$$\begin{aligned} \frac{d}{dz}(z^2+1)(z^2+4) &= 2z\cdot(z^2+4) - 2z\cdot(z^2+1) \\ 8z - 2z = 6z \end{aligned}$$

Furthermore, recall that the residue of a quotient implies that our residue is simply $\frac{1}{6z}$ evaluated at $z = 2i$. Therefore, our original integral is equal to $2\pi i \cdot \frac{\pi}{12i} = \frac{1}{6i}$.

3. Find the Cauchy principal value of

$$\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2x+2)}$$

**Solution** First, let us identify the singularities of

$$\frac{z}{(z^2+1)(z^2+2z+2)=0}$$

This function is singular whenever either $(z^2+1)=0$ or $z^2+2z+2=0$. For the first equation, this implies there are isolated singularities at $z=\pm i$. For the second, applying the quadratic equation and we get

$$\begin{aligned} z &= \frac{-2\pm\sqrt{2^2-4(1)(2)}}{2} \\ &= \frac{-2\pm\sqrt{4-8}}{2} \\ &= \frac{-2\pm 2i}{2} && = -1\pm 1i \end{aligned}$$

Because the principal value is simply

$$P.V. \int_{-\infty}^{\infty} f(x)dx = 2\pi i Res f(x)$$

and the residue of the quotient is simply the following quantity evaluated at $z=-1+i$

$$\begin{aligned} \frac{z}{(z^2+1)(z^2+2z+2)} &= \frac{z}{[(z^2+1)(z^2+2z+2)]']} \\ &= \frac{z}{2z(z^2+2z+2)-(2z+2)(z^2+1)} \\ &= 2z \end{aligned}$$

The principal value is $2\pi i\cdot 2\cdot(-1+i)=4(-1+i)$.