Homework 3

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1. Find the Laurent Series representation of

$$f(z) = z^2 \sin \frac{1}{z^2}$$

for the domain $0 < |z| < \infty$.

Solution First, consider the MacLaurin series for $\sin z$. Because it is analytic everywhere, its radius of convergence is all of $\mathbb{C}$

$$\sin z = \sum_{n=0}^{\infty} \frac{(-i)^n z^{2n+1}}{(2n+1)!}$$

This implies,

$$\begin{aligned}
\sin \frac{1}{z^2} &= \sum_{n=0}^{\infty} \frac{(-i)^n (\frac{1}{z^2})^{2n+1}}{(2n+1)!} \\
z^2 \sin \frac{1}{z^2} &= z^2 \sum_{n=0}^{\infty} \frac{(-i)^n (\frac{1}{z^2})^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n \cdot z^2 \cdot (\frac{1}{z^2})^{2n+1}}{(2n+1)!} \\
&= \sum_{n=0}^{\infty} \frac{(-i)^n (1)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-i)^n}{(2n+1)!}
\end{aligned}$$

2. Find the Laurent series representation of the function

$$f(z) = \frac{2}{(z+1)(z+3)}$$

In the annulus $1 < |z| < 3$ First, using partial fractions decomposition,

$$\begin{aligned}
f(z) &= \frac{1}{z+1} - \frac{1}{z+3} \\
&= \frac{1}{z} \frac{1}{1+\frac{1}{z}} - \frac{1}{z} \frac{1}{1+\frac{3}{z}}
\end{aligned}$$

Furthermore, the Taylor Series

$$f(z) = \frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

for $|z| < 1$. This does not converge for our current domain, however, because $|z| > 1 \implies |\frac{1}{z}| < 1$ we can use the change of variables $z = \frac{1}{z}$.

Thus,

$$\frac{1}{z} \cdot \frac{1}{1 + \frac{1}{z}} = \frac{1}{z} \cdot \sum_{n=0}^{\infty} (-1)^n z^{-n}$$

Moreover, we can substitute $z = \frac{3}{z}$ into the same Taylor Series above. From earlier, we had that $|\frac{1}{z}| < 1$, and because $|z| < 3$ implies $|\frac{1}{z}| > \frac{1}{3}$, we get that our representation works for $\frac{1}{3} < |\frac{1}{z}| < 1$, which is valid. In conclusion,

$$\begin{aligned} f(z) &= \frac{2}{(z+1)(z+3)} \\ &= \frac{1}{z} \cdot \sum_{n=0}^{\infty} (-1)^n z^{-n} + \frac{1}{z} \cdot \sum_{n=0}^{\infty} (-1)^n 3 \cdot z^{-n} \\ &= \frac{1}{z} \sum_{n=0}^{\infty} (-1)^n 4 \cdot z^{-n} \end{aligned}$$

In the annulus $1 < |z - 3| < 3$ ...

3. Find the Laurent Series representation of the function

$$f(z) = \frac{1 + 2z}{z^2 + z^3}$$

in the annulus $0 < |z| < 1$.

Solution

$$\begin{aligned} f(z) &= \frac{1 + z^2}{z^2 + z^3} \\ &= \frac{1}{z^2 + z^3} + \frac{2z}{z^2 + z^3} \\ &= \frac{1}{z^2} \frac{1}{1 + z} + \frac{2z}{z} \frac{1}{1 + z} \end{aligned}$$

We know that $\frac{1}{1-z}$ has the MacLaurin series representation $\sum_{n=0}^{\infty} z^n$ for $|z| < 1$. Furthermore, $\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$ if we use the change of variables $-z = z$, which leads to the radius of convergence $|-z| = |z| < 1$. Thus, we have

$$\begin{aligned} f(z) &= \frac{1}{z^2} \cdot \sum_{n=0}^{\infty} (-1)^n z^n + 2 \cdot \sum_{n=0}^{\infty} (-1)^n z^n \\ &= \frac{2}{z^2} \cdot \sum_{n=0}^{\infty} (-1)^n z^n \end{aligned}$$