Homework 7

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1. Show that
$$Res_{z=\pi i}\frac{\exp zt}{\sinh z} + Res_{z=-\pi i}\frac{\exp zt}{\sinh z} = -2\cos \pi t$$

Proof. Let us tackle this problem by separating evaluating the numerators and denominators.

First, let us evaluate the denominators. In general, because $\sinh z = -i \sin iz$ it follows that $\frac{d}{dz}\sinh z = -i^2 \cdot \cos iz = \cos iz$ Therefore, the denominators are

$$\cos i(\pi i) = \cos \pi i^2 = \cos -\pi$$
$$\cos i(-\pi i) = \cos -\pi i^2 = \cos \pi$$

Both of these expressions are -1, so our residues share a common denominator.

Then, recall the identity $\exp z = \exp x \cos y + i \exp x \sin y$ where x and y are the real and imaginary parts of z respectively. Therefore,

$$\begin{aligned} \exp \pi it &= \exp 0 \cos \pi t + i \exp 0 \sin \pi t \\ &= \cos \pi t + i \sin \pi t \\ \exp -\pi it &= \exp 0 \cos -\pi t + i \exp 0 \sin -\pi t \\ &= \cos -\pi t + i \sin -\pi t \end{aligned}$$

Simplifying the original expression, we thus get

$$Res_{z=\pi i}\frac{\exp zt}{\sinh z} + Res_{z=-\pi i}\frac{\exp zt}{\sinh z} = \frac{(\cos \pi t + \cos -\pi t) + i(\sin \pi t + \sin -\pi t)}{-1}$$

Because cos is an even function, and sin is an odd function, this simplifies to $-2\cos \pi t$ as we set out to prove. □

2. Evaluate the integral
$$\int_C \frac{dz}{\sinh 2z}$$
about $C : |z| = 2$.

Solution First, notice that

$$\sinh 2z = i \sin i2z$$

is differentiable everywhere except for the singularities that occur whenever $\sin i2z = 0$. These occur when $z = \frac{i\pi n}{2}$ for $n \in \mathbb{Z}$. Within the circle $|z| = 2$, this gives us the isolated singular points $z = 0, \pm\pi i, \pm 2\pi i$. Now, applying the chain rule we get $\frac{d}{dz} \sinh 2z = 2 \cos i2z$. Therefore, our residues are

$$\frac{1}{\cos 0} + \frac{1}{\cos \pm 2\pi} + \frac{1}{\cos \pm 4\pi}$$

Now, recall that $\cos(0)$ is an even function with period 2π, so the above is really just,

$$5 \cdot \frac{1}{\cos 0} = 5$$

Applying Cauchy's Residue Theorem,

$$\int_C \frac{dz}{\sinh 2z} = 2\pi i \cdot \sum Res = 2\pi i \cdot 5 = 10\pi i$$