Ch 1: The Complex Numbers

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July 5, 2017

1 Introduction

A complex number $z \in \mathbb{C}$ can be written as $z = x + iy$ where x and y are real numbers.

- x is called the **real part** of z
- y is called the **imaginary part** of z

Reflecting this, sometimes this notation is used:

- $Re(z) = x$
- $Im(z) = y$
- $z = Re(z) + i \cdot Im(z)$

2 Complex Arithmetic

Let z_1 and z_2 be any complex numbers which we will denote as $z_1 := x_1 + iy_1$ and $z_2 := x_2 + iy_2$, where x_i, y_i are real numbers.

- **Addition** $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$
- **Multiplication**

$$\begin{aligned}
z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\
&= x_1x_2 + ix_1y_2 + ix_2y_1 + i^2y_1y_2 && \text{Apply distributive law} \\
&= x_1x_2 + ix_1y_2 + ix_2y_1 + (-1) \cdot y_1y_2 \\
&= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)
\end{aligned}$$

2.1 Square Roots of Complex Numbers

Every complex number $z = a + ib$ has two square roots s_1 and s_2. If we denote $s = x + iy$ then

$$x = \pm\sqrt{\frac{a}{2} + \frac{\sqrt{a^2 + b^2}}{2}}$$
$$y = \frac{b}{2x}$$

Proof. Let z be any complex number and write it as $a+ib$. Furthermore, denote its square root as $s = x + iy$. To find the square root, we simply have to solve $s^2 = z$ or equivalently $(x + iy)^2 = a + ib$.

Continuing, we have

$$\begin{aligned}
(x + iy)^2 &= a + ib \\
x^2 + 2ixy - y^2 &= a + ib \quad \text{Apply distributive rule to LHS} \\
(x^2 - y^2) + i(2xy) &= a + ib
\end{aligned}$$

By matching like terms (real part with real part, imaginary part with imaginary part), we get

$$\begin{aligned}
a &= x^2 - y^2 \\
b &= 2xy
\end{aligned}$$

Then, we want to find equations for our unknowns x, y in terms of our known variables a, b. First, $b = 2xy$ implies $y = \frac{b}{2x}$. Then, plugging this into the equation for a we get

$$\begin{aligned}
a &= x^2 - y^2 \\
&= x^2 - (\frac{b}{2x})^2 \\
&= x^2 - \frac{b^2}{4x^2} \\
4ax^2 &= 4x^4 - b^2 \qquad \text{Multiply both sides by } 4x^2 \\
4ax^2 - 4x^4 + b^2 = 0
\end{aligned}$$

Applying the usual quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

to the previous equation with $x = x^2, a = 4, b = -4a, c = -b^2$, we get

$$\begin{aligned} x^2 &= \frac{4a \pm \sqrt{4^2a^2 - 4 \cdot 4 \cdot (-b^2)}}{2 \cdot 4} \\ &= \frac{a}{2} + \frac{\sqrt{4^2}\sqrt{a^2+b^2}}{8} \\ &= \frac{a}{2} + \frac{\sqrt{a^2+b^2}}{2} \\ x &= \pm\sqrt{\frac{a}{2} + \sqrt{\frac{a^2+b^2}{2}}} \end{aligned}$$

$\square$

3 More Basic Definitions

- **Conjugate** The conjugate of $z = x + iy$ is the complex number $\bar{z} = x - iy$.
- **Multiplicative Inverse** The multiplicative inverse of z is the complex number $z^-1 = \frac{\bar{z}}{|z|^2}$

  One can remember this since:

$$\begin{aligned} \frac{1}{z} &= \frac{1}{x+iy} \\ &= \frac{1}{x+iy} \cdot \frac{(x-iy)}{(x-iy)} && \text{Multiply by complex conjugate} \\ &= \frac{x-iy}{x^2 - ixy + ixy - i^2y^2} \\ &= \frac{x-iy}{x^2+y^2} \\ &= \frac{\bar{z}}{|z|^2} \end{aligned}$$

4 Sequences

Definition: Sequence A sequence is an indexed list of complex numbers. As such, we can think of a sequence as a function $\mathbb{N} \to \mathbb{C}$, where we usually write z_k for $f(k)$.

Some notations include:

- $\{z_k\}_{k=1}^{\infty}$
- $\{z_k\}$
- $z_1, z_2, z_3, \ldots$
- $z_k = f(k)$

Motivation Sequences will allow us to understand limits and therefore derivatives

4.1 Convergence of a Sequence

Definition We say a sequence z_k **converges** to $z \in \mathbb{C}$ if the sequence of real numbers $|z_k - z|$ converges to 0. In other words,

$$\lim_{k \to \infty} |z_k - z| = 0$$

(We can think of $|z_k - z|$ as the distance between the values of the sequence and some number z).

Usually $|z_k - z|$ *converges to 0* is written as $z_k \to z$.

Definition: Epsilon-N Definition of Limit A sequence $\{z_k\}$ converges to $z \in \mathbb{C}$ if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for every $n > N$, $|z_n - z| < \epsilon$ for $n \geq N$.

In other words, for every positive ϵ–especially very small epsilons–we can find an N such that for every term past the N-th term, the distance between the values of the sequence and z are smaller than ϵ. Geometrically, this means if we surround z with a ball of radius ϵ, we can find some N where all for all $n > N$, z_n is within that ball.

(Continue later...)

5 Topology of Complex Numbers

Definition: Open Disc ...

Definition: Open Set $S \subset \mathbb{C}$ is called **open** if for every $z \in S$ there exists an $\epsilon > 0$ such that $D(z; \epsilon) \leq S$. In other words, for every point in an open set S, we can surround it with some ball of radius ϵ and have that ball be a subset of S.

Definition: Boundary A number $z \in \mathbb{C}$ is the the **boundary** of $S \subset \mathbb{C}$ if every disk $D(z, r)$ contains elements of both S and $\mathbb{C} \backslash S$.

In other words, we say z is part of the boundary of a set S if every time we try to draw a disc around it, we always get elements inside S and elements outside of S.

Definition: Closure The **closure** of $S \subset \mathbb{C}$ is the set $S = S \cup \delta S$.

The closure of a set is the set itself union its boundary.