Complex Exponential Function

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1 Complex Exponential Function

The exponential function, when extended to the complex numbers, is defined as

$$e^z = e^x \cos(y) + ie^x \sin(y)$$

1.1 e^z is entire

- **Proposition 3.2** If f_x and f_y exist, are continuous in an neighborhood of z, and satisfying the Cauchy-Riemann equations, then f is differentiable there.
- **Definition:** f is entire if it is differentiable everywhere. In terms of the above proposition, that means its partial derivatives are continuous and satisfy the Cauchy-Riemann equations everywhere.

Proof. To prove that e^z is entire, we need to first show that its partial derivatives satisfy the Cauchy-Riemann Equations.

$$\begin{aligned} u_y &= -e^x \sin(y) \\ -v_x &= -e^x \sin(y) \\ \\ u_x &= e^x \cos(y) \\ v_y &= e^x \cos(y) \end{aligned}$$

Furthermore, we know that these partial derivatives are products of continuous functions, so they themselves are continuous. Because e^z has continuous partial derivatives everywhere satisfying the Cauchy-Riemann Equations, it is differentiable everywhere. Therefore, it is an entire function. $\square$

2 Complex Sine and Cosine

Let $z \in \mathbb{C}$

$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$$

2.1 Derivatives

$$\frac{d}{dz}\cos(z) = -\sin(z)$$

Proof.

$$\begin{aligned}
\frac{d}{dz}\cos(z) &= \frac{d}{dz}\frac{e^{iz} + e^{-iz}}{2} \\
&= \frac{ie^{iz} - ie^{-iz}}{2} \\
&= \frac{i(e^{iz} - e^{-iz})}{2} \cdot \frac{i}{i} \\
&= \frac{i^2(e^{iz} - e^{-iz})}{2i} \\
&= -(\frac{e^{iz} - e^{-iz}}{2i}) = -\sin(z)
\end{aligned}$$

□