Homework 2

Vincent La
Math 122B
August 10, 2017

1. Expand $f(z) = \sin z$ into a Taylor series about the point $z_0 = \frac{\pi}{6}$.

   Solution

$$\begin{aligned}
\sin z &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} && \text{In general} \\
&= \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{\pi}{6})^{2n+1}}{(2n+1)!} \\
&= \frac{(-1)^0(\frac{\pi}{6})}{1!} - \frac{(-1)(\frac{\pi}{6})^{2+1}}{3!} + \frac{1(\frac{\pi}{6})^5}{5!} + ... \\
&= \frac{\pi}{6} - \frac{\frac{\pi}{6}^3}{3!} + \frac{\frac{\pi}{6}^5}{5!} + ...
\end{aligned}$$

2. Expand $f(z) = \log 1 + z$ into a MacLaurin series.

   Solution Because the derivatives of $\log 1 + z$ are $f^0 = \log 1 + z$, $f^1 = \frac{1}{1+z}$, $f^2 = \frac{-1}{(1+z)^2}$, ... the MacLaurin series expansion (Taylor series about $z_0 = 0$) is

$$\begin{aligned}
\log(1 + z_0) \cdot ... + \frac{\frac{1}{1+z_0}}{1!} \cdot z + \frac{\frac{-1}{(1+z_0)^2}}{2!} \cdot z^2 &= \log(1) \cdot ... + \frac{\frac{1}{1+0}}{1} \cdot z - \frac{\frac{-1}{(1+0)^2}}{2!} \cdot z^2 + ... \\
&= 0 + z - \frac{z^2}{2!} + ...
\end{aligned}$$

3. Derive

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}$$

   which converges for $|z - i| < \sqrt{2}$.

   Proof. First, we know that

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

   with radius of convergence $|z| < 1$. Now, because

$$\frac{1}{1-z} = \frac{1}{1-i} \cdot \frac{1}{1 - \frac{z-i}{1-i}}$$

if we change $z = \frac{z-i}{1-i}$ we get

$$\begin{aligned}\frac{1}{1-z} &= \frac{1}{1-i} \cdot \sum_{n=0}^{\infty} (\frac{z-i}{1-i})^n \\ &= \sum_{n=0}^{\infty} \frac{(z-i)^n}{(1-i)^{n+1}}\end{aligned}$$

Furthermore, by the same change of variables

$$\begin{aligned}|z| < 1 \implies & |\frac{z-i}{1-i}| < 1 \\ & |z-i| < |1-i| \\ & |z-i| < \sqrt{1^2 + (-1)^2} \\ & |z-i| < \sqrt{2}\end{aligned}$$

as we set out to prove. □

4. Using $\cos z = -\sin z - \frac{\pi}{2}$, expand $\cos z$ into a Taylor series about the point $z_0 = \frac{\pi}{2}$.

Solution First, let's compute the first few derivatives of $\cos z$.

$$\begin{aligned}f^{(0)} &= \cos z = -\sin z - \frac{\pi}{2} \\ f^{(1)} &= -\sin z = -\sin z \\ f^{(2)} &= -\cos z = +\sin z - \frac{\pi}{2} \\ f^{(3)} &= \sin z = \sin z\end{aligned}$$

Because $\cos z$ repeats this same pattern for every consecutive four orders of derivatives, its Taylor Series about $\frac{\pi}{2}$ has terms which cancel each other out so $\cos \frac{\pi}{2} = 0$ as expected.