③ 3-dimensional test-function

$$f(X_1, X_2, X_3) = \sin(X_1) + A\sin^2(X_2) + B X_3^4 \sin(X_1)$$

X_1, X_2, X_3 iid $\mathcal{U}([-\pi, \pi])$

On pose
$$\begin{cases} a = \displaystyle\int_{-\pi}^{\pi} \sin^2(x_2)\frac{dx}{2\pi} = 1/2 \\ b = \displaystyle\int_{-\pi}^{\pi} x_3^4 \frac{dx_3}{2\pi} = \pi^4/5 \end{cases}$$

Calculer la décomposition de Sobol-Hoeffding de $f(X)$ et les indices de Sobol.

$$f(X_1, X_2, X_3) = f_0 + f_1(X_1) + f_2(X_2) + f_3(X_3) + f_{1,2}(X_1, X_2) + f_{1,3}(X_1, X_3) + f_{2,3}(X_2, X_3) + f_{1,2,3}(X_1, X_2, X_3)$$

$$f_0 = E[f(X)] = E[\sin(X_1)] + A\,E[\sin^2(X_2)] + B\,\underbrace{E[X_3^4\sin(X_1)]}_{= E[X_3^4]\,E[\sin(X_1)]}$$

$$= \underbrace{\int_{-\pi}^{\pi} \sin(x_1)\frac{dx_1}{2\pi}}_{= -\cos(\pi) + \cos(-\pi) = 0} + Aa + Bb \int_{-\pi}^{\pi} \sin(x_1)\, dx_1$$

fonction densité uniforme

$= -\cos(\pi) + \cos(-\pi)$

$= 0$

donc $f_0 = Aa$

$$f_1(X_1) = \mathbb{E}[f(X) \mid X_1] - f_0$$

$$\begin{aligned}
&= \mathbb{E}[\sin(X_1) \mid X_1] + \mathbb{E}[A \sin^2(X_2) \mid X_1] + \mathbb{E}[B X_3^4 \sin(X_1) \mid X_1] - f_0 \qquad X_2 \perp\!\!\!\perp X_1 \quad X_3 \perp\!\!\!\perp X_1 \\
&= \sin(X_1) + Aa + Bb \sin(X_1) - f_0 \\
&= \sin(X_1) + Bb \sin(X_1) = (1 + Bb) \sin(X_1)
\end{aligned}$$

$$f_2(X_2) = \mathbb{E}[f(X) \mid X_2] - f_0$$

$$\begin{aligned}
&= 0 + A \sin^2(X_2) + 0 - f_0 \\
&= A \sin^2(X_2) - Aa = A(\sin^2(X_2) - a)
\end{aligned}$$

$$f_3(X_3) = \mathbb{E}[f(X) \mid X_3] - f_0$$

$$\begin{aligned}
&= 0 + Aa + 0 - f_0 \\
&= 0
\end{aligned}$$

$$f_{1,3}(X_1, X_3) = \underbrace{\mathbb{E}[f(X) \mid X_1, X_3]} - \underbrace{f_1(X_1)} - \overbrace{f_3(X_3)}^{=0} - \underbrace{f_0}$$

$$\begin{aligned}
&= \underbrace{\cancel{\sin(X_1)} + \cancel{Aa} + B X_3^4 \sin(X_1)} - \underbrace{\cancel{\sin(X_1)} - Bb \sin(X_1)} - \underbrace{\cancel{Aa}} \\
&= B \sin(X_1) [X_3^4 - b].
\end{aligned}$$

Indices de Sobol $\quad S_i = \dfrac{\overbrace{Var(f_i(X_i))}^{= D_i}}{\underbrace{Var(f(X))}_{= D}}$

$S_3 = 0$ car $f_3(X_3) = 0$

$S_1 = \dfrac{D_1}{D}$ $\quad D_1 = Var(f_1(X_1)) = E[f_1(X_1)^2]$

car tous les termes sont centrés
car tous orthogonaux entre eux.

$$= (1 + Bb)^2 \underbrace{\int_{-\pi}^{\pi} \sin^2(x) \frac{dx}{2\pi}}_{= a}$$

$$= (1 + Bb)^2 a$$

$$D_{1,3} = var(f_{1,3}(X_1, X_3)) = E[f_{1,3}(X_1, X_3)^2]$$

$$= B^2 E[(X_3^4 - b)^2 \sin^2(X_1)]$$

$$= B^2 \underbrace{E[(X_3^4 - b)^2]}_{= \int_{-\pi}^{\pi} (x^4 - b)^2 \frac{dx}{2\pi}} \underbrace{E[\sin^2(X_1)]}_{= a}$$

$X_1 \perp\!\!\!\perp X_3$

$$\underline{D} = \operatorname{var}(f(X)) = \sum_{I \subseteq \{1,2,3\}} \operatorname{var}(f_I(X_I)) = D_1 + D_2 + D_{1,3}$$

$$\underline{D_2} = \operatorname{var}(f_2(X_2)) = E\left[f_2(X_2)^2\right]$$

$$= E\left[A^2(\sin^2(X_2) - a)^2\right]$$

$$= A^2 \underbrace{E\left[(\sin^2(X_2) - a)^2\right]}_{= \int_{-\pi}^{\pi} (\sin^2(x) - a)^2 \frac{dx}{2\pi}}$$