Exercise sheet Nr. 1

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(**G 1**)
Regard the following formal context K, given as a cross table:

	needs water to live	lives in water	lives on land	needs chlorophyll to produce food	two seed leaves	one seed leaf	can move around	has limbs	suckles its offspring
Leech	x	x					x		
Bream	x	x					x	x	
Frog	x	x	x				x	x	x
Spike-Weed	x	x		x		x			
Reed	x	x	x	x		x			
Bean	x		x	x	x				
Maize	x		x	x		x			

a) Specify the following sets:

1. {Bean}' = {needs water to live, lives on land, needs chlorophyll to produce food, two seed leaves}
2. {lives on land}' = {Frog, Reed, Bean, Maize}
3. {two seed leaves}" = {Bean}' = 1)
4. {Frog, Maize}' = {needs water to live, lives on land}
5. {needs clorophyll to produce food, can move around}' = {}
6. {lives in water, lives on land}' = {Frog, Reed}
7. {needs clorophyll to produce food, can move around}' = 5)

b) Extend K with both an object and an attribute

- $G \cup \{Lizard\},\ M \cup \{has\ scales\}$

	needs water to live	lives in water	lives on land	needs chlorophyll to produce food	two seed leaves	one seed leaf	can move around	has limbs	suckles its offspring	has scales
Leech	x	x					x			
Bream	x	x					x	x		x
Frog	x	x	x				x	x	x	
Spike-Weed	x	x		x		x				
Reed	x	x	x	x		x				
Bean	x		x	x	x					
Maize	x		x	x		x				
Lizard	x		x				x	x		x

(G 2)

Consider the formal context from Lecture 1. Use **ConExp** and **FCA Tools Bundle** to determine the set of concepts and to draw the concept lattices.

Using the FCA Tools Bundle from https://fca-tools-bundle.com/, the following concepts were generated:

Concept	Attribute
Leech	needs water to live
Leech	lives in water
Leech	can move around
Bream	needs water to live
Bream	lives in water
Bream	can move around
Bream	has limbs
Frog	needs water to live
Frog	lives in water
Frog	lives on land
Frog	can move around
Frog	has limbs
Frog	suckles its offspring
Spike-Weed	needs water to live
Spike-Weed	lives in water
Spike-Weed	needs chlorophyll to produce food
Spike-Weed	one seed leaf
Reed	needs water to live
Reed	lives in water
Reed	lives on land
Reed	needs chlorophyll to produce food
Reed	one seed leaf
Bean	needs water to live
Bean	lives on land
Bean	needs chlorophyll to produce food
Bean	two seed leaves
Maize	needs water to live
Maize	lives on land
Maize	can move around
Maize	has limbs

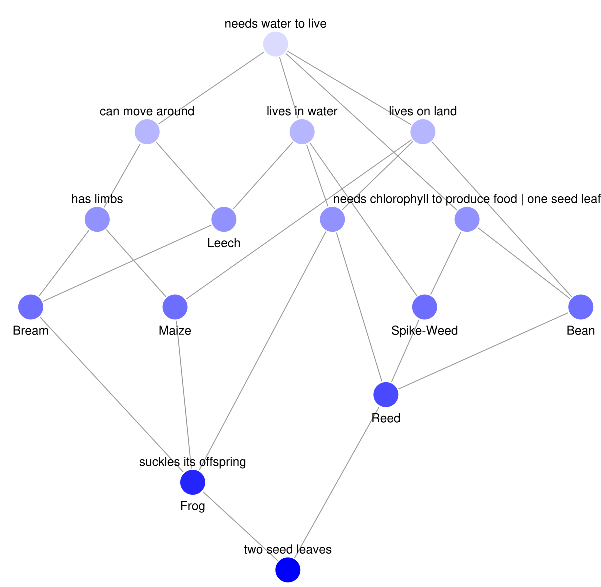

Figure 1: Concept lattice generated by FCA Tools Bundle

(**G 3**)

a) Recall: how is the derivation operator $(\cdot)'$ defined?

- The derivation operator can be applied on a set of objects or a set of attributes.
- When it is applied on a set of **objects**, the result is represented by the set of **attributes** all objects in the first set have in common.
- When it is applied on a set of **attributes**, the result is represented by the set of **objects** which have all the attributes from the original set.

b) Let $K = (G, M, I)$ be a formal context and let $A, B \subseteq G$. Prove the following statements:

1. $A \subseteq B$ implies $B' \subseteq A'$
   - if $A \subseteq B$, then for any attribute $m \in B'$, we know that $\forall g \in B : (g, m) \in I$
   - since $A \subseteq B$, every element in A is also in B
   - therefore, for any $g \in A$, we have $g \in B$, which means $(g, m) \in I$
   - this shows that $\forall g \in A : (g, m) \in I$, which is exactly the definition of $m \in A'$
   - hence, if $m \in B'$, then $m \in A'$, which proves that $B' \subseteq A'$
2. $A \subseteq A''$
   - for any $g \in A$, we need to show that $g \in A''$
   - by definition, $A'' = (A')'$, which means $A'' = \{g \in G \mid \forall m \in A' : (g, m) \in I\}$
   - for any $g \in A$ and any $m \in A'$, we know that $(g, m) \in I$ by the definition of A'
   - therefore, every $g \in A$ satisfies the condition for being in A''
   - hence, $A \subseteq A''$
3. $A' = A'''$
   - we'll prove both inclusions: $A' \subseteq A'''$ and $A''' \subseteq A'$
   - from (2), we know that $A' \subseteq A'''$ because we can substitute A' for A in the statement $A \subseteq A''$
   - for the other direction, let $m \in A'''$, then $\forall g \in A'' : (g, m) \in I$
   - from (2), we know $A \subseteq A''$, so for every $g \in A$, we have $g \in A''$
   - therefore, $(g, m) \in I$ for all $g \in A$
   - this means $m \in A'$ by definition
   - thus, $A''' \subseteq A'$
   - combining both inclusions, we have $A' = A'''$
4. For $C \subseteq G$ and $D \subseteq M$ holds: (C, D) is a formal concept if and only if there is some $E \subseteq G$ such that $C = E''$ and $D = E'$
   - first, we'll prove the forward direction: if (C, D) is a formal concept, then there exists $E \subseteq G$ such that $C = E''$ and $D = E'$
   - by definition, a formal concept (C, D) satisfies $C' = D$ and $D' = C$
   - let $E = C$, then $E' = C' = D$ and $E'' = D' = C$
   - therefore, there exists E (namely C itself) such that $C = E''$ and $D = E'$
   - for the reverse direction, assume there exists $E \subseteq G$ such that $C = E''$ and $D = E'$
   - we need to show that (C, D) is a formal concept, which means $C' = D$ and $D' = C$
   - since $D = E'$, we have $D' = (E')' = E''$ by definition
   - since $C = E''$, we have $D' = E'' = C$
   - now we need to show $C' = D$
   - since $C = E''$, by the property in (3), we have $C' = (E'')' = E'''$
   - again from (3), we know $E' = E'''$
   - therefore, $C' = E''' = E' = D$
   - thus, (C, D) is a formal concept if and only if there exists $E \subseteq G$ such that $C = E''$ and $D = E'$