# Assignment 1 - due March 30th

**Answer 1.** (a) We will prove that $U^*U = I$. We know $U$ is diagonalizable, so $U$ can be written as

$$U = \sum_{i=0}^{n} \lambda_i |v_i\rangle\langle v_i|$$

where all $\lambda_i$ are eigenvalues and $|v_i\rangle$ are there respective eigenvectors. Then,

$$U^* = \sum_{i=0}^{n} \overline{\lambda_i} \langle v_i|v_i\rangle$$

Note that all eigenvectors are orthogonal, each means that, $\langle v_i|v_i\rangle|v_j\rangle\langle v_j| = 0$ when $i \neq j$, and $\langle v_i|v_i\rangle|v_j\rangle\langle v_j| = 1$ when $i = j$. So we have,

$$U^*U = \sum_{i=0}^{n} \overline{\lambda_i}\lambda_i$$

and by the hypothesis $\overline{\lambda_i}\lambda_i = |\lambda_i| = 1$. Therefore $U^*U = I$.

(b) Take $\lambda$ a eigenvalue of U unitary, and $|v\rangle$ his associated eigenvector, so the equity bellow is true,

$$U|v\rangle = \lambda|v\rangle$$

Note that, $= \overline{\lambda}\langle v|$. Then, when we multiply the first equation for $\langle v|U^*$ on the left, we get

$$\langle v|U^*U|v\rangle = \langle v|U^*\lambda|v\rangle$$
$$\langle v|v\rangle = \langle v|\overline{\lambda}\lambda|v\rangle$$
$$\langle v|v\rangle = \overline{\lambda}\lambda\langle v|v\rangle$$

By the property of the inner product, we have that $\langle v|v\rangle \neq 0$, so we can divide the last equation on both sides for $\langle v|v\rangle$. And then,

$$1 = \overline{\lambda}\lambda = |\lambda|$$

(c) blank

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**Answer 2.** Unfortunately I couldn't do the first part, but the second one is so simple that I feel bad not writing it. Assume $M|v\rangle = 0$, then

$$\langle v|M|v\rangle = \langle v|0 = 0$$

Sorry.

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**Answer 3.** First, we will show that $A \otimes I$ and $I \otimes B$ are diagonalizable. Since $A$ is diagonalizable, we can write $A$ as $A = PDP^{-1}$, where $D$ is a diagonal matrix and $P$ is invertible. Then, by the properties of the kronecker product

$$A \otimes I = (PDP^{-1}) \otimes I = (P \otimes I)(D \otimes I)(P^{-1} \otimes I) = (P \otimes I)(D \otimes I)(P \otimes I)^{-1}$$

Is easy to see that, since $D$ and $I$ are diagonal $D \otimes I$ is also diagonal. So now we've found a diagonalizable representation of $A \otimes I$ , with $P \otimes I$ invertible and $D \otimes I$ diagonal. By a analogous reasoning, is easy to see that $I \otimes B$ is also diagonalizable, just noticing that if $B = QEQ^{-1}$, $I \otimes E$ is diagonal. Let P be a invertible matrix, since $I$ is the identity matrix, we can comute it in the product. So we have $P^{-1}IP = IP^{-1}P = I$. Now, to show that $A \otimes I + I \otimes B$ is diagonalizable, we need to find a matrix that diagonalizes it. Let $P \otimes Q$ be this candidate.

$$\begin{aligned}
(P \otimes Q)^{-1}(A \otimes I + I \otimes B)(P \otimes Q) &= (P \otimes Q)^{-1}(A \otimes I)(P \otimes Q) + (P \otimes Q)^{-1}(I \otimes B)(P \otimes Q) \\
&= (P^{-1}AP) \otimes (Q^{-1}IQ) + (P^{-1}IP) \otimes (Q^{-1}BQ) \\
&= D \otimes I + I \otimes E
\end{aligned}$$

Since $D \otimes I$ and $I \otimes E$ are diagonal, their sum is also diagonal. And then $A \otimes I + I \otimes B$ is diagonalizable. Now, we know $D$ and $E$ are the diagonal matrixes with the eigenvalues for $A$ and $B$ respectivly. Note that $D \otimes I$ and $I \otimes E$ are diagonal matrixes with the same eigenvalues of $D$ and $E$, because the kronecker product, now the eigenvalues have multiplicity of the "size" of the square matrixes. So if the eigenvalues of $A$ are $\lambda_i$ and the eigenvalues of $B$ are $\delta_i$, then the eigenvalues of $A \otimes I + I \otimes B$ are $\lambda_i + \delta_i$.

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**Answer 4.** blank

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**Answer 5.** (a) Assume $P_1 + P_2$ is an orthogonal projection. We have

$$(P_1 + P_2)^2 = (P_1)^2 + 2P_1P_2 + (P_2)^2$$

Note that, we know from the classes that, if M is a orthogonal projection, then $M^2 = M$. Since $P_1$ and $P_2$ are orthogonal projections, we have

$$\begin{aligned}
(P_1)^2 + 2P_1P_2 + (P_2)^2 &= P_1 + P_2 \\
P_1 + 2P_1P_2 + P_2 &= P_1 + P_2 \\
2P_1P_2 &= 0 \\
P_1P_2 &= 0
\end{aligned}$$

Now assume that $P_1P_2 = 0$. To prove that $P_1 + P_2$ is a orthogonal projection, we need to verify that $(P_1 + P_2)^2 = P_1 + P_2$ and $(P_1 + P_2)^* = P_1 + P_2$. Since $P_1$ and $P_2$ are orthogonal projections, we have

$$\begin{aligned}
(P_1 + P_2)^2 &= (P_1)^2 + 2P_1P_2 + (P_2)^2 \\
&= P_1 + 0 + P_2 \\
&= P_1 + P_2
\end{aligned}$$

and

$$(P_1 + P_2)^* = (P_1)^* + (P_2)^* = P_1 + P_2$$

(b) We will use an analogous reasoning from the last item. Assume that $(P_1 + \cdots + P_k)$ is an orthogonal projection. So is true that $(P_1 + \cdots + P_k)^2 = (P_1 + \cdots + P_k)$. But from the other hand, since for all $i$, $P_i$ is an orthogonal projection, we have

$$\begin{aligned}(P_1 + \cdots + P_k)^2 &= \sum_{i=1}^{k}(P_i)^2 + 2\sum_{1\leq i<j\leq k} P_iP_j \\ &= \sum_{i=1}^{k} P_i + 2\sum_{1\leq i<j\leq k} P_iP_j\end{aligned}$$

Then, because $(P_1 + \cdots + P_k)$ is an orthogonal projection

$$\begin{aligned}\sum_{i=1}^{k} P_i + 2\sum_{1\leq i<j\leq k} P_iP_j &= \sum_{i=1}^{k} P_i \\ \sum_{1\leq i<j\leq k} P_iP_j &= 0\end{aligned}$$

Again, since for all $i$, $P_i$ is an orthogonal projection, we have that $P_iP_j$ are positive semi-definite operators, $P_iP_j \geq 0$. Thus

$$\sum_{1\leq i<j\leq k} P_iP_j = 0 \Rightarrow P_iP_j = 0 \quad \forall i \neq j$$

Now assume, that $P_iP_j = 0 \quad \forall i \neq j$. Similarly from the last item:

$$\begin{aligned}(P_1 + \cdots + P_k)^2 &= \sum_{i=1}^{k}(P_i)^2 + 2\sum_{1\leq i<j\leq k} P_iP_j \\ &= \sum_{i=1}^{k} P_i + 0 \\ &= (P_1 + \cdots + P_k)\end{aligned}$$

and also

$$\begin{aligned}(P_1 + \cdots + P_k)^* &= (P_1)^* + \cdots + (P_k)^* \\ &= P_1 + \cdots + P_k\end{aligned}$$

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**Answer 6.** blank