

Assignment 1 - due March 30th

Answer 1. bla bla bla Examples:

(a) We will prove that $U^*U = I$. We know U is diagonalizable, so U can be written as

$$U = \sum_{i=0}^n \lambda_i |v_i\rangle \langle v_i|$$

where all λ_i are eigenvalues and $|v_i\rangle$ are there respective eigenvectors. Then,

$$U^* = \sum_{i=0}^n \bar{\lambda}_i \langle v_i| v_i\rangle$$

Note that all eigenvectors are orthogonal, each means that, $\langle v_i| v_i\rangle \langle v_j| v_j\rangle = 0$ when $i \neq j$, and $\langle v_i| v_i\rangle \langle v_j| v_j\rangle = 1$ when $i = j$. So we have,

$$U^*U = \sum_{i=0}^n \bar{\lambda}_i \lambda_i$$

and by the hypothesis $\bar{\lambda}_i \lambda_i = |\lambda_i| = 1$. Therefore $U^*U = I$.

(b) Take λ a eigenvalue of U unitary, and $|v\rangle$ his associated eigenvector, so the equity bellow is true,

$$U|v\rangle = \lambda|v\rangle$$

Note that, $\bar{\lambda} \langle v|$. Then, when we multiply the first equation for $\langle v|U^*$ on the left, we get

$$\langle v|U^*U|v\rangle = \langle v|U^*\lambda|v\rangle$$

$$\langle v|v\rangle = \langle v|\bar{\lambda}\lambda|v\rangle$$

$$\langle v|v\rangle = \bar{\lambda}\lambda\langle v|v\rangle$$

By the property of the inner product, we have that $\langle v|v\rangle \neq 0$, so we can divide the last equation on both sides for $\langle v|v\rangle$. And then,

$$1 = \bar{\lambda}\lambda = |\lambda|$$

(c) $\langle u|M|v\rangle$

Answer 2. bla bla bla

Answer 3. First, we will show that $A \otimes I$ and $I \otimes B$ are diagonalizable. Since A is diagonalizable, we can write A as $A = PDP^{-1}$, where D is a diagonal matrix and P is invertible. Then, by the properties of the kronecker product

$$A \otimes I = (PDP^{-1}) \otimes I = (P \otimes I)(D \otimes I)(P^{-1} \otimes I) = (P \otimes I)(D \otimes I)(P \otimes I)^{-1}$$

Is easy to see that, since D and I are diagonal $D \otimes I$ is also diagonal. So now we've found a diagonalizable representation of $A \otimes I$, with $P \otimes I$ invertible and $D \otimes I$ diagonal. By a analogous reasoning, is easy to see that $I \otimes B$ is also diagonalizable, just noticing that if $B = QEQ^{-1}$, $I \otimes E$ is diagonal. Let P be a invertible matrix, since I is the identity matrix, we can comute it in the product. So we have $P^{-1}IP = IP^{-1}P = I$. Now, to show that $A \otimes I + I \otimes B$ is diagonalizable, we need to find a matrix that diagonalizes it. Let $P \otimes Q$ be this candidate.

$$\begin{aligned} (P \otimes Q)^{-1}(A \otimes I + I \otimes B)(P \otimes Q) &= (P \otimes Q)^{-1}(A \otimes I)(P \otimes Q) + (P \otimes Q)^{-1}(I \otimes B)(P \otimes Q) \\ &= (P^{-1}AP) \otimes (Q^{-1}IQ) + (P^{-1}IP) \otimes (Q^{-1}BQ) \\ &= D \otimes I + I \otimes E \end{aligned}$$

Since $D \otimes I$ and $I \otimes E$ are diagonal, their sum is also diagonal. And then $A \otimes I + I \otimes B$ is diagonalizable. Now, we know D and E are the diagonal matrixes with the eigenvalues for A and B respectively. Note that $D \otimes I$ and $I \otimes E$ are diagonal matrixes with the same eigenvalues of D and E , because the kronecker product, now the eigenvalues have multiplicity of the "size" of the square matrixes. So if the eigenvalues of A are λ_i and the eigenvalues of B are δ_i , then the eigenvalues of $A \otimes I + I \otimes B$ are $\lambda_i + \delta_i$.

Answer 4. bla bla bla

Answer 5. bla bla bla

Answer 6. bla bla bla

Answer 7. bla bla bla