Assignment 1 - due March 30th

Answer 1. (a) We will prove that $U^*U = I$. We know U is diagonalizable, so U can be written as

$$U = \sum_{i=0}^{n} \lambda_i |v_i\rangle\langle v_i|$$

where all λ_i are eigenvalues and $|v_i\rangle$ are there respective eigenvectors. Then,

$$U^* = \sum_{i=0}^{n} \overline{\lambda_i} \langle v_i|v_i\rangle$$

Note that all eigenvectors are orthogonal, each means that, $\langle v_i|v_i\rangle|v_j\rangle\langle v_j| = 0$ when $i \neq j$, and $\langle v_i|v_i\rangle|v_j\rangle\langle v_j| = 1$ when $i = j$. So we have,

$$U^*U = \sum_{i=0}^{n} \overline{\lambda_i}\lambda_i$$

and by the hypothesis $\overline{\lambda_i}\lambda_i = |\lambda_i| = 1$. Therefore $U^*U = I$.

(b) Take λ a eigenvalue of U unitary, and $|v\rangle$ his associated eigenvector, so the equity bellow is true,

$$U|v\rangle = \lambda|v\rangle$$

Note that, $= \overline{\lambda}\langle v|$. Then, when we multiply the first equation for $\langle v|U^*$ on the left, we get

$$\langle v|U^*U|v\rangle = \langle v|U^*\lambda|v\rangle$$
$$\langle v|v\rangle = \langle v|\overline{\lambda}\lambda|v\rangle$$
$$\langle v|v\rangle = \overline{\lambda}\lambda\langle v|v\rangle$$

By the property of the inner product, we have that $\langle v|v\rangle \neq 0$, so we can divide the last equation on both sides for $\langle v|v\rangle$. And then,

$$1 = \overline{\lambda}\lambda = |\lambda|$$

(c) blank

Answer 2. Unfortunately I couldn't do the first part, but the second one is so simple that I feel bad not writing it. Assume $M|v\rangle = 0$, then

$$\langle v|M|v\rangle = \langle v|0 = 0$$

Sorry.

Answer 3. First, we will show that $A \otimes I$ and $I \otimes B$ are diagonalizable. Since A is diagonalizable, we can write A as $A = PDP^{-1}$, where D is a diagonal matrix and P is invertible. Then, by the properties of the kronecker product

$$A \otimes I = (PDP^{-1}) \otimes I = (P \otimes I)(D \otimes I)(P^{-1} \otimes I) = (P \otimes I)(D \otimes I)(P \otimes I)^{-1}$$

Is easy to see that, since D and I are diagonal $D \otimes I$ is also diagonal. So now we've found a diagonalizable representation of $A \otimes I$, with $P \otimes I$ invertible and $D \otimes I$ diagonal. By a analogous reasoning, is easy to see that $I \otimes B$ is also diagonalizable, just noticing that if $B = QEQ^{-1}$, $I \otimes E$ is diagonal. Let P be a invertible matrix, since I is the identity matrix, we can comute it in the product. So we have $P^{-1}IP = IP^{-1}P = I$. Now, to show that $A \otimes I + I \otimes B$ is diagonalizable, we need to find a matrix that diagonalizes it. Let $P \otimes Q$ be this candidate.

$$\begin{aligned}(P \otimes Q)^{-1}(A \otimes I + I \otimes B)(P \otimes Q) &= (P \otimes Q)^{-1}(A \otimes I)(P \otimes Q) + (P \otimes Q)^{-1}(I \otimes B)(P \otimes Q) \\ &= (P^{-1}AP) \otimes (Q^{-1}IQ) + (P^{-1}IP) \otimes (Q^{-1}BQ) \\ &= D \otimes I + I \otimes E\end{aligned}$$

Since $D \otimes I$ and $I \otimes E$ are diagonal, their sum is also diagonal. And then $A \otimes I + I \otimes B$ is diagonalizable. Now, we know D and E are the diagonal matrices with the eigenvalues for A and B respectivly. Note that $D \otimes I$ and $I \otimes E$ are diagonal matrices with the same eigenvalues of D and E, because the kronecker product, now the eigenvalues have multiplicity of the "size" of the square matrices. So if the eigenvalues of A are λ_i and the eigenvalues of B are δ_i, then the eigenvalues of $A \otimes I + I \otimes B$ are $\lambda_i + \delta_i$.

Answer 4. We know that a given qubit state is given by a vector $|v\rangle = a|0\rangle + b|1\rangle$. We also know that this state can be seen as a point in the Bloch Sphere. Since $|v\rangle$ can be written as $|v\rangle = a|0\rangle + b|1\rangle$, and we got a condition for the values of a and b, given by $|a|^2 + |b|^2 = 1$, we can also represent the coeficients of $|v\rangle$ as

$$a = \cos\frac{\theta}{2}, \qquad b = \sin\frac{\theta}{2}$$

And then we can rewrite the former representation as

$$|v\rangle = e^{i\gamma}(\cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle)$$

The first factor $e^{i\gamma}$ is important for observable experiments (when comparing multiple qubits). So, θ and ϕ determine the position of a point in the Bloch sphere, and then represent the state of a qubit. Remember the Pauli matrices X, Y and Z. When we exponentiate this matrices we gain operators that rotate the position of the qubit in the Bloch sphere about the x, y and z axes.

$$
\begin{aligned}
R_x(\theta) &= e^{-i\theta X/2} \\
&= \cos\frac{\theta}{2} I - i\sin\frac{\theta}{2} X \\
&= \begin{pmatrix} \cos\frac{\theta}{2} & 0 \\ 0 & \cos\frac{\theta}{2} \end{pmatrix} + \begin{pmatrix} 0 & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & 0 \end{pmatrix} \\
&= \begin{pmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}
\end{aligned}
$$

The same is done for the other rotations.

$$
\begin{aligned}
R_y(\theta) &= \begin{pmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \\
R_z(\theta) &= \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}
\end{aligned}
$$

For this question I look it up in the Nielsen's book, pages 15 and 174.

Answer 5. (a) Assume $P_1 + P_2$ is an orthogonal projection. We have

$$(P_1 + P_2)^2 = (P_1)^2 + 2P_1P_2 + (P_2)^2$$

Note that, we know from the classes that, if M is a orthogonal projection, then $M^2 = M$. Since P_1 and P_2 are orthogonal projections, we have

$$
\begin{aligned}
(P_1)^2 + 2P_1P_2 + (P_2)^2 &= P_1 + P_2 \\
P_1 + 2P_1P_2 + P_2 &= P_1 + P_2 \\
2P_1P_2 &= 0 \\
P_1P_2 &= 0
\end{aligned}
$$

Now assume that $P_1P_2 = 0$. To prove that $P_1 + P_2$ is a orthogonal projection, we need to verify that $(P_1 + P_2)^2 = P_1 + P_2$ and $(P_1 + P_2)^* = P_1 + P_2$. Since P_1 and P_2 are orthogonal projections, we have

$$
\begin{aligned}
(P_1 + P_2)^2 &= (P_1)^2 + 2P_1P_2 + (P_2)^2 \\
&= P_1 + 0 + P_2 \\
&= P_1 + P_2
\end{aligned}
$$

and

$$(P_1 + P_2)^* = (P_1)^* + (P_2)^* = P_1 + P_2$$

(b) We will use an analogous reasoning from the last item. Assume that $(P_1 + \cdots + P_k)$ is an orthogonal projection. So is true that $(P_1 + \cdots + P_k)^2 = (P_1 + \cdots + P_k)$. But from the other hand, since for all i, P_i is an orthogonal projection, we have

$$\begin{aligned}(P_1 + \cdots + P_k)^2 &= \sum_{i=1}^{k} (P_i)^2 + 2 \sum_{1 \leq i < j \leq k} P_i P_j \\ &= \sum_{i=1}^{k} P_i + 2 \sum_{1 \leq i < j \leq k} P_i P_j\end{aligned}$$

Then, because $(P_1 + \cdots + P_k)$ is an orthogonal projection

$$\begin{aligned}\sum_{i=1}^{k} P_i + 2 \sum_{1 \leq i < j \leq k} P_i P_j &= \sum_{i=1}^{k} P_i \\ \sum_{1 \leq i < j \leq k} P_i P_j &= 0\end{aligned}$$

Again, since for all i, P_i is an orthogonal projection, we have that $P_i P_j$ are positive semi-definite operators, $P_i P_j \geq 0$. Thus

$$\sum_{1 \leq i < j \leq k} P_i P_j = 0 \Rightarrow P_i P_j = 0 \quad \forall i \neq j$$

Now assume, that $P_i P_j = 0 \quad \forall i \neq j$. Similarly from the last item:

$$\begin{aligned}(P_1 + \cdots + P_k)^2 &= \sum_{i=1}^{k} (P_i)^2 + 2 \sum_{1 \leq i < j \leq k} P_i P_j \\ &= \sum_{i=1}^{k} P_i + 0 \\ &= (P_1 + \cdots + P_k)\end{aligned}$$

and also

$$\begin{aligned}(P_1 + \cdots + P_k)^* &= (P_1)^* + \cdots + (P_k)^* \\ &= P_1 + \cdots + P_k\end{aligned}$$

Answer 6. blank