

## Assignment 1 - due March 30th

**Answer 1.** (a) We will prove that  $U^*U = I$ . We know  $U$  is diagonalizable, so  $U$  can be written as

$$U = \sum_{i=0}^n \lambda_i |v_i\rangle \langle v_i|$$

where all  $\lambda_i$  are eigenvalues and  $|v_i\rangle$  are there respective eigenvectors. Then,

$$U^* = \sum_{i=0}^n \bar{\lambda}_i \langle v_i| v_i\rangle$$

Note that all eigenvectors are orthogonal, each means that,  $\langle v_i|v_i\rangle \langle v_j|v_j\rangle = 0$  when  $i \neq j$ , and  $\langle v_i|v_i\rangle \langle v_j|v_j\rangle = 1$  when  $i = j$ . So we have,

$$U^*U = \sum_{i=0}^n \bar{\lambda}_i \lambda_i$$

and by the hypothesis  $\bar{\lambda}_i \lambda_i = |\lambda_i| = 1$ . Therefore  $U^*U = I$ .

(b) Take  $\lambda$  a eigenvalue of  $U$  unitary, and  $|v\rangle$  his associated eigenvector, so the equity bellow is true,

$$U|v\rangle = \lambda|v\rangle$$

Note that,  $\langle v|v\rangle = 1$ . Then, when we multiply the first equation for  $\langle v|U^*$  on the left, we get

$$\langle v|U^*U|v\rangle = \langle v|U^*\lambda|v\rangle$$

$$\langle v|v\rangle = \langle v|\bar{\lambda}\lambda|v\rangle$$

$$\langle v|v\rangle = \bar{\lambda}\lambda\langle v|v\rangle$$

By the property of the inner product, we have that  $\langle v|v\rangle \neq 0$ , so we can divide the last equation on both sides for  $\langle v|v\rangle$ . And then,

$$1 = \bar{\lambda}\lambda = |\lambda|$$

(c) blank

**Answer 2.** blank

**Answer 3.** First, we will show that  $A \otimes I$  and  $I \otimes B$  are diagonalizable. Since  $A$  is diagonalizable, we can write  $A$  as  $A = PDP^{-1}$ , where  $D$  is a diagonal matrix and  $P$  is invertible. Then, by the properties of the kronecker product

$$A \otimes I = (PDP^{-1}) \otimes I = (P \otimes I)(D \otimes I)(P^{-1} \otimes I) = (P \otimes I)(D \otimes I)(P \otimes I)^{-1}$$

Is easy to see that, since  $D$  and  $I$  are diagonal  $D \otimes I$  is also diagonal. So now we've found a diagonalizable representation of  $A \otimes I$ , with  $P \otimes I$  invertible and  $D \otimes I$  diagonal. By a analogous reasoning, is easy to see that  $I \otimes B$  is also diagonalizable, just noticing that if  $B = QEQ^{-1}$ ,  $I \otimes E$  is diagonal. Let  $P$  be a invertible matrix, since  $I$  is the identity matrix, we can comute it in the product. So we have  $P^{-1}IP = IP^{-1}P = I$ . Now, to show that  $A \otimes I + I \otimes B$  is diagonalizable, we need to find a matrix that diagonalizes it. Let  $P \otimes Q$  be this candidate.

$$\begin{aligned} (P \otimes Q)^{-1}(A \otimes I + I \otimes B)(P \otimes Q) &= (P \otimes Q)^{-1}(A \otimes I)(P \otimes Q) + (P \otimes Q)^{-1}(I \otimes B)(P \otimes Q) \\ &= (P^{-1}AP) \otimes (Q^{-1}IQ) + (P^{-1}IP) \otimes (Q^{-1}BQ) \\ &= D \otimes I + I \otimes E \end{aligned}$$

Since  $D \otimes I$  and  $I \otimes E$  are diagonal, their sum is also diagonal. And then  $A \otimes I + I \otimes B$  is diagonalizable. Now, we know  $D$  and  $E$  are the diagonal matrixes with the eigenvalues for  $A$  and  $B$  respectively. Note that  $D \otimes I$  and  $I \otimes E$  are diagonal matrixes with the same eigenvalues of  $D$  and  $E$ , because the kronecker product, now the eigenvalues have multiplicity of the "size" of the square matrixes. So if the eigenvalues of  $A$  are  $\lambda_i$  and the eigenvalues of  $B$  are  $\delta_i$ , then the eigenvalues of  $A \otimes I + I \otimes B$  are  $\lambda_i + \delta_i$ .

**Answer 4.** blank

**Answer 5.** (a) Assume  $P_1 + P_2$  is an orthogonal projection. We have

$$(P_1 + P_2)^2 = (P_1)^2 + 2P_1P_2 + (P_2)^2$$

Note that, we know from the classes that, if  $M$  is a orthogonal projection, then  $M^2 = M$ . Since  $P_1$  and  $P_2$  are orthogonal projections, we have

$$\begin{aligned} (P_1)^2 + 2P_1P_2 + (P_2)^2 &= P_1 + P_2 \\ P_1 + 2P_1P_2 + P_2 &= P_1 + P_2 \\ 2P_1P_2 &= 0 \\ P_1P_2 &= 0 \end{aligned}$$

Now assume that  $P_1P_2 = 0$ . To prove that  $P_1 + P_2$  is a orthogonal projection, we need to verify that  $(P_1 + P_2)^2 = P_1 + P_2$  and  $(P_1 + P_2)^* = P_1 + P_2$ . Since  $P_1$  and  $P_2$  are orthogonal projections, we have

$$\begin{aligned} (P_1 + P_2)^2 &= (P_1)^2 + 2P_1P_2 + (P_2)^2 \\ &= P_1 + 0 + P_2 \\ &= P_1 + P_2 \end{aligned}$$

and

$$(P_1 + P_2)^* = (P_1)^* + (P_2)^* = P_1 + P_2$$

- (b) We will use an analogous reasoning from the last item. Assume that  $(P_1 + \cdots + P_k)$  is an orthogonal projection. So is true that  $(P_1 + \cdots + P_k)^2 = (P_1 + \cdots + P_k)$ . But from the other hand, since for all  $i$ ,  $P_i$  is an orthogonal projection, we have

$$\begin{aligned} (P_1 + \cdots + P_k)^2 &= \sum_{i=1}^k (P_i)^2 + 2 \sum_{1 \leq i < j \leq k} P_i P_j \\ &= \sum_{i=1}^k P_i + 2 \sum_{1 \leq i < j \leq k} P_i P_j \end{aligned}$$

Then, because  $(P_1 + \cdots + P_k)$  is an orthogonal projection

$$\begin{aligned} \sum_{i=1}^k P_i + 2 \sum_{1 \leq i < j \leq k} P_i P_j &= \sum_{i=1}^k P_i \\ \sum_{1 \leq i < j \leq k} P_i P_j &= 0 \end{aligned}$$

Again, since for all  $i$ ,  $P_i$  is an orthogonal projection, we have that  $P_i P_j$  are positive semi-definite operators,  $P_i P_j \geq 0$ . Thus

$$\sum_{1 \leq i < j \leq k} P_i P_j = 0 \Rightarrow P_i P_j = 0 \quad \forall i \neq j$$

Now assume, that  $P_i P_j = 0 \quad \forall i \neq j$ . Similarly from the last item:

$$\begin{aligned} (P_1 + \cdots + P_k)^2 &= \sum_{i=1}^k (P_i)^2 + 2 \sum_{1 \leq i < j \leq k} P_i P_j \\ &= \sum_{i=1}^k P_i + 0 \\ &= (P_1 + \cdots + P_k) \end{aligned}$$

and also

$$\begin{aligned} (P_1 + \cdots + P_k)^* &= (P_1)^* + \cdots + (P_k)^* \\ &= P_1 + \cdots + P_k \end{aligned}$$

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**Answer 6.** bla bla bla