Assignment 1 - due March 31th

Solutions

Exercise 1. @

Recall that a complex square matrix U is unitary if $U^* \cdot U = I$.

(a) Let U be a complex matrix. It is diagonalizable, its eigenvectors are orthogonal, and its eigenvalues are complex numbers of absolute value equal to 1. Prove that U is unitary.

(b) Prove that the eigenvalues of unitary matrices are complex numbers of absolute value equal to 1.

(c) Prove that the eigenvectors of unitary matrices corresponding to distinct eigenvalues are orthogonal.

Solution:

For (a), we have $U = P^* D P$, with D diagonal with complex numbers of absolute value 1. Then

$$U^*U = (P^*D^*P)(P^*DP) = P^*D^*DP = P^*IP = I,$$

because the complex-conjugate of numbers in the unit circle are their inverses.

For (b), with $U|v\rangle = \lambda|v\rangle$ and assuming $\langle v|v\rangle = 1$, we have $\langle v|U|v\rangle = \lambda$, and taking the transpose conjugate on both sides, $\langle v|U^*|v\rangle = \overline{\lambda}$. On the other hand, $U^*U|v\rangle = \lambda U^*|v\rangle$ gives $\lambda^{-1}|v\rangle = U^*|v\rangle$, therefore $\lambda^{-1} = \langle v|U^*|v\rangle$. If $\overline{\lambda} = \lambda^{-1}$, λ is in the unit circle.

For (c), say $U|u\rangle = \theta|u\rangle$, and $U|v\rangle = \lambda|v\rangle$. Then

$$\theta^{-1}\lambda\langle u|v\rangle = \langle u|U^*U|v\rangle = \langle u|v\rangle,$$

thus, because $\lambda \neq \theta$, it must be that $\langle u|v\rangle = 0$.

Exercise 2. @

Assume M is positive semidefinite, that is, M is Hermitian and $\langle v|M|v\rangle \geq 0$ for all $|v\rangle$. Without assuming any other characterization (meaning: if you need some, prove it), show that $\langle v|M|v\rangle = 0$ if and only if $M|v\rangle = 0$.

Solution:

If $|v\rangle$ is eigenvector of M with eigenvalue λ, we have $0 \leq \langle v|M|v\rangle = \lambda$. So in the diagonalization $M = P^*DP$, D is a diagonal matrix of nonnegative real numbers. Hence it has a square root, say $\sqrt{D}$, whence

$$0 = \langle v|M|v\rangle = \langle v|P^*\sqrt{D}\sqrt{D}P|v\rangle = (\sqrt{D}P|v\rangle)^*(\sqrt{D}P|v\rangle),$$

and the norm of a vector is zero if and only the vector is 0, so

$$M|v\rangle = P^*\sqrt{D}\sqrt{D}P|v\rangle = P^*\sqrt{D}(\sqrt{D}P|v\rangle) = 0.$$

Exercise 3. @

Assume square matrices A and B are diagonalizable matrices. Show that $A \otimes I + I \otimes B$ is diagonalizable, and express its eigenvalues in terms of those of A and B. (Here I means the identity matrix of convenient size).

Solution:

If $A = P^{-1}DP$ and $B = Q^{-1}EQ$, note that

$$A \otimes I + I \otimes B = P^{-1}DP \otimes Q^{-1}Q + P^{-1}P \otimes Q^{-1}EQ = (P^{-1} \otimes Q^{-1})(D \otimes I + I \otimes E)(P \otimes Q).$$

This is precisely the diagonalization of $A \otimes I + I \otimes B$. The diagonal matrix $(D \otimes I + I \otimes E)$ contains all possible sums of an element in D and another in E, and these are the desired eigenvalues.

Exercise 4.

Let the matrices $R_x(\theta)$, $R_y(\theta)$ and $R_z(\theta)$ represent the rotations by an angle of θ in the Bloch sphere, about the corresponding axes (and following the rule of thumb direction). How do they look like? Explain well. Following, state which are their eigenvalues and eigenvectors.

Solution:

The first step here is to understand exactly what the question wants. A rotation in the tridimensional space is a real 3x3 matrix, but this is not what we want here. We want something that acts on qubit states and recover qubit states. So we want a 2x2 complex unitary matrix. Let's start observing some of the matrices we already know. Multiplying qubit states by the Pauli matrix X, results in what happening in the Bloch sphere? Well, looking at the general form of a trace 1 density matrix, we see

$$X\rho X^* = X\left(\frac{1}{2}(I + xX + yY + zZ)\right)X = \frac{1}{2}(I + xX - yY - zZ).$$

(here we noting the relations $XY = \mathrm{i}Z$, $XZ = -\mathrm{i}Y$, $YX = -\mathrm{i}Z$ and $ZX = \mathrm{i}Y$). Note also that any complex multiple of X would have given the same result!

Upon choosing different values of x, y and z, it is easy to convince yourself that we just witnessed a rotation of π degrees about the x axis.

So what is the natural guess now? Something that represents doing nothing at $\theta = 0$ and doing X at $\theta = \pi$? And that is related to rotations?

How about $R_x(\theta) = \cos(\theta/2)I + \sin(\theta/2)X$? Well, this is not unitary. Can we multiply X by something (which wouldn't change the rotation by π) so that this becomes unitary? Fortunately, yes. (the official solution starts now:)

Let $R_x(\theta) = \cos(\theta/2)I + \mathrm{i}\sin(\theta/2)X$. Then

$$R_x(\theta)\left(\frac{1}{2}(I + xX + yY + zZ)\right)R_x(\theta)^* =$$

$$= \frac{1}{2}(I + xX + (y\cos(\theta) + z\sin(\theta))Y + (-y\sin(\theta) + z\cos(\theta))Z).$$

which is almost what we wanted — the direction is wrong. Replacing θ by its negative, we arrive at

$$R_x(\theta) = \cos(\theta/2)I - \mathrm{i}\sin(\theta/2)X,$$

whose effect in (x, y, z) will be $(x, (y\cos(\theta) - z\sin(\theta)), (y\sin(\theta) + z\cos(\theta))$, exactly what a tridimensional rotation about the x axis does.

It is not difficult to convince yourself that the same argument would hold for Y and Z, so

$$R_y(\theta) = \cos(\theta/2)I - \mathrm{i}\sin(\theta/2)Y \quad \text{and} \quad R_z(\theta) = \cos(\theta/2)I - \mathrm{i}\sin(\theta/2)Z.$$

((This last part of the solution was a bit funny because for some reason I was thinking about the eigenvalues of X, Y and Z, and not the rotaions depending on θ.))

These rotations move all states, except for two of them (the antipode points in the Bloch sphere in the axes they each fix). So the eigenvectors for $R_x(\theta)$ are the vectors spanning states corresponding to $\frac{1}{2}(I \pm X)$ (analogously for the others). The eigenvalues can be computed from the expressions above: in all cases, they sum to $2\cos(\theta/2)$ and their product is $\cos(\theta/2)^2 + \sin(\theta/2)^2 = 1$, so they must be $e^{\pm\mathrm{i}(\theta/2)}$.

Exercise 5.

Assume P_is are orthogonal projections.

(a) Show that $P_1 + P_2$ is an orthogonal projection if and only if $P_1P_2 = 0$ (no need to assume $P_2P_1 = 0$...)

(b) Show now that $P_1 + ... + P_k$ is an orthogonal projection if and only if $P_iP_j = 0$ for all $i \neq j$.

Solution:

There are two main ways to solve this question. The operational way, manipulating matrix products, and the geometric way. I will solve (a) in the operational way, and (b) in the geometric way. I left as a challenge to solve (b) in the operational way.

For (a), our main tool is the fact that $P_i^2 = P_i$, so we will have to explore a bit multiplying equalities by these matrices. Clearly $P_1 + P_2$ is an orthogonal projection if and only if $P_1P_2 + P_2P_1 = 0$, so we must show that the latter is equivalent to $P_1P_2 = 0$. One direction is quite clear: if $P_1P_2 = 0$, then $0 = (P_1P_2)^* = P_2^*P_1^* = P_2P_1$, thus $P_1P_2 + P_2P_1 = 0$.

If $P_1P_2 + P_2P_1 = 0$, then we may as well attempt multiplying by P_1, obtaining

$$P_1P_2 + P_1P_2P_1 = 0 \implies P_1P_2P_1 = -P_1P_2.$$

Now, let us conjugate transpose both sides, because one side is invariant, and the other is not!

$$P_1P_2P_1 = (P_1P_2P_1)^* = -(P_1P_2)^* = -P_2P_1,$$

thus $P_1P_2 = P_2P_1$, and if their sum is 0, then both are equal to 0.

For (b), one direction is quite immediate. For the other one, now we think what a projection does to a given vector. If Q is an orthogonal projection, then $0 \leq \langle v|Q|v\rangle \leq 1$ (at the same time this says that the projection doesn't reflect the vector negatively, and also that it doesn't stretch it at all...) So what can we do now? Assume $|v\rangle$ is a normalized eigenvector for P_1 with eigenvalue 1. Recalling the hypothesis that $\sum P_i$ is an orthogonal projection, we have

$$1 \geq \langle v|\sum P_i|v\rangle = \langle v|P_1|v\rangle + \langle v|\sum_{j>1} P_j|v\rangle \geq 1.$$

So equality holds throughout, forcing $\langle v|P_j|v\rangle = 0$ for all $j > 1$, that is, any eigenvector for P_1 with eigenvalue 1 is also an eigenvector for the other projections with eigenvalue 0. So let $\{|v_i\rangle\}$ be an orthogonal basis of eigenvector of P_1. It follows that

$$P_jP_1|v_i\rangle = 0$$

for all $j > 1$, either because $P_1|v_i\rangle = 0$ or because $P_j|v_i\rangle = 0$. Thus $P_jP_1 = 0$. This argument holds for all P_k instead of P_k, and the result follows.

Exercise 6.

Let ρ represent the state of qubits A and B (that is, ρ is a 4x4 density matrix). Assume ρ is a pure state. Show that the reduced state at A is pure if and only if ρ is not entangled. Think: does it make a big difference if instead of qubits A and B these are instead multi-qubit systems?

Solution:

If ρ is pure and not entangled, then $\rho = \alpha \otimes \beta$, with both α and β density matrices of rank 1. Its reduced state at A is simply $(\operatorname{tr} \beta) \cdot \alpha = \alpha$, which is therefore a pure state (because it is rank 1).

For the converse, assume ρ is pure and entangled. We use its Schmidt decomposition, obtaining

$$\rho = \lambda_1^2(\alpha_1 \otimes \beta_1) + \lambda_2^2(\alpha_2 \otimes \beta_2),$$

where the αs are both rank 1 and projecting onto orthogonal subspaces, the same for theβs. Both λs are nonzero, and $\lambda_1^2 + \lambda_2^2 = 1$. Tracing B out, we obtain

$$\rho_A = \lambda_1^2\alpha_1 + \lambda_2^2\alpha_2,$$

which is a mixed state of α_1 and α_2 (in particular, it is of rank 2 with eigenvalues λ_1^2 and λ_2^2).