向量的模与单位向量

$|\vec{a}|$　$|\vec{b}|$　$|\overrightarrow{AB}|\cdots$

$|\vec{a}|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$

$|\vec{b}|=\sqrt{3^2+2^2+1^2}=\sqrt{14}$

例 . 已知 $\boldsymbol{a}=(1,2,3)$，则 $|\boldsymbol{a}|=$________

$|\vec{a}|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$

例 . 求点 A(1,0,1)指向点 B(3, −2,2)方向的单位向量

即 求 $\overrightarrow{AB}$ 的单位向量

$$\overrightarrow{AB}^0=\frac{1}{|\overrightarrow{AB}|}\overrightarrow{AB}=\frac{1}{3}(2,-2,1)=\left(\frac{2}{3},-\frac{2}{3},\frac{1}{3}\right)$$

$$\overrightarrow{AB}=(3-1,-2-0,2-1)=(2,-2,1)$$

$$|\overrightarrow{AB}|=\sqrt{2^2+(-2)^2+1^2}=3$$

$\boldsymbol{a}\cdot\boldsymbol{b}$ 的计算与性质

例1. 已知 $\boldsymbol{a}=(1,2,3)$，$\boldsymbol{b}=(4,5,6)$，求 $\boldsymbol{a}\cdot\boldsymbol{b}$

$$
\begin{aligned}
\vec{a}\cdot\vec{b} &= 1\times4+2\times5+3\times6\\
&= 32
\end{aligned}
$$

例2. 已知 $|\boldsymbol{a}|=1$，$|\boldsymbol{b}|=3$，$\boldsymbol{a}+\boldsymbol{b}=(1,2,3)$，求 $\boldsymbol{a}\cdot\boldsymbol{b}$

$$|\vec{a}+\vec{b}|=\sqrt{1^2+2^2+3^2}=\sqrt{14}$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=|\vec{a}+\vec{b}|^2=\left(\sqrt{14}\right)^2=14$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}+\vec{b})=14$$

$$\Rightarrow \vec{a}\cdot(\vec{a}+\vec{b})+\vec{b}\cdot(\vec{a}+\vec{b})=14$$

$$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}=14$$

$$\Rightarrow \vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}=14$$

$$\Rightarrow \vec{a}\cdot\vec{a}+2\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{b}=14$$

$$\Rightarrow |\vec{a}|^2+2\vec{a}\cdot\vec{b}+|\vec{b}|^2=14$$

$$\Rightarrow 1^2+2\vec{a}\cdot\vec{b}+3^2=14 \Rightarrow \vec{a}\cdot\vec{b}=2$$

例3. 已知 $|\boldsymbol{a}|=2$，$|\boldsymbol{b}|=3$，且 $\boldsymbol{a}\cdot\boldsymbol{b}=3$，求 $\boldsymbol{a}$ 与 $\boldsymbol{b}$ 的夹角 θ

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow 3=2\times3\cos\theta$$

$$\Rightarrow \cos\theta=\frac{1}{2}$$

$\because 0°\le\theta\le180°$

$\therefore \theta=60°$

$\boldsymbol{a}\times\boldsymbol{b}$ 的计算与性质

例1. 已知 $\boldsymbol{a}=(1,2,3)$，$\boldsymbol{b}=(4,5,6)$，求 $\boldsymbol{c}=\boldsymbol{a}\times\boldsymbol{b}$

$$\vec{c}=\vec{a}\times\vec{b}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 2 & 3\\ 4 & 5 & 6\end{vmatrix}$$

$$=\vec{i}\begin{vmatrix}2 & 3\\ 5 & 6\end{vmatrix}-\vec{j}\begin{vmatrix}1 & 3\\ 4 & 6\end{vmatrix}+\vec{k}\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}$$

$$=\vec{i}(2\times 6-3\times 5)-\vec{j}(1\times 6-3\times 4)+\vec{k}(1\times 5-2\times 4)$$

$$=\vec{i}(-3)-\vec{j}(-6)+\vec{k}(-3)$$

$$=\vec{i}(-3)+\vec{j}(6)+\vec{k}(-3)$$

$$=(-3,\ 6,-3)$$

例2. 已知 $|\boldsymbol{a}|=2$，$|\boldsymbol{b}|=3$，且 $\boldsymbol{a}\cdot\boldsymbol{b}=3$，求 $|\boldsymbol{a}\times\boldsymbol{b}|$

$$|\vec{a}\times\vec{b}|=|\vec{a}||\vec{b}|\sin\theta \qquad \theta=60^\circ$$

$$=2\cdot 3\cdot\sin 60^\circ$$

$$=2\cdot 3\cdot\frac{\sqrt{3}}{2}$$

$$=3\sqrt{3}$$

已知 $|\boldsymbol{a}|=2$，$|\boldsymbol{b}|=3$，且 $\boldsymbol{a}\cdot\boldsymbol{b}=3$，求 $\boldsymbol{a}$ 与 $\boldsymbol{b}$ 的夹角 θ

$$\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$$

$$\Rightarrow\quad 3=2\times 3\cos\theta$$

$$\Rightarrow\cos\theta=\frac{1}{2}$$

$$\because 0^\circ\le\theta\le 180^\circ$$

$$\therefore\theta=60^\circ$$

向量垂直与平行

例1. 已知向量 $\boldsymbol{a} = x\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$， $\boldsymbol{b} = 4\boldsymbol{i} + x\boldsymbol{j} - 7\boldsymbol{k}$，则当 $x =$ _____ 时，$\boldsymbol{a}$ 垂直于 $\boldsymbol{b}$

$\because \ \vec{a} \perp \vec{b}$

$\therefore \ \vec{a} \cdot \vec{b} = 0$

$\therefore x \cdot 4 + 3x + 4 \times (-7) = 0$

$\Rightarrow \quad 7x - 28 = 0$

$\Rightarrow \quad x = 4$

例2. 已知 $\boldsymbol{a}$ 与 $\boldsymbol{b} = (2, -1, 3)$ 平行，且 $\boldsymbol{a} \cdot \boldsymbol{b} = 7$，求 $\boldsymbol{a}$

解法一：设 $\vec{a} = (x, y, z)$

$\because \vec{a} \parallel \vec{b}$

$\therefore \vec{a} \times \vec{b} = \vec{0}$

$\vec{i}(3y + z) - \vec{j}(3x - 2z) + \vec{k}(-x - 2y) = \vec{0}$

$$\therefore \begin{cases} 3y + z = 0 \\ 3x - 2z = 0 \\ -x - 2y = 0 \end{cases}$$

$\vec{a} \cdot \vec{b} = 7$

$x \cdot 2 + y \cdot (-1) + z \cdot 3 = 7$

$2x - y + 3z = 7$

$$\therefore \begin{cases} 3y + z = 0 & ① \\ 3x - 2z = 0 & ② \\ -x - 2y = 0 & ③ \\ 2x - y + 3z = 7 & ④ \end{cases}$$

由 ① 可以推出 $z = -3y$

由 ③ 可以推出 $x = -2y$

将 $z = -3y$， $x = -2y$ 代入④中可得

$2 \cdot (-2y) - y + 3 \cdot (-3y) = 7$

$-14y = 7$

$y = -\frac{1}{2}$

$z = -3y = -3 \cdot (-\frac{1}{2}) = \frac{3}{2}$

$x = -2y = -2 \cdot (-\frac{1}{2}) = 1$

$$\Rightarrow \begin{cases} x = 1 \\ y = -\frac{1}{2} \\ z = \frac{3}{2} \end{cases}$$

$\therefore \vec{a} = \left(1, -\frac{1}{2}, \frac{3}{2}\right)$

解法二：设 $\vec{a} = (x, y, z)$

$\because \vec{a} \parallel \vec{b}$

$$\therefore \begin{cases} a_x = kb_x \\ a_y = kb_y \quad , k \neq 0 \\ a_z = kb_z \end{cases}$$

$$\therefore \begin{cases} x = k \cdot 2 \\ y = k \cdot (-1) \ , k \neq 0 \\ z = k \cdot 3 \end{cases}$$

$$\text{即} \begin{cases} x = 2k \\ y = -k \quad , k \neq 0 \\ z = 3k \end{cases}$$

$\therefore \vec{a} = (2k, -k, 3k)$

$\vec{a} \cdot \vec{b} = 7$

$2k \cdot 2 + (-k) \cdot (-1) + 3k \cdot 3 = 7 \ \Rightarrow 4k + k + 9k = 7 \Rightarrow 14k = 7 \Rightarrow k = \frac{1}{2}$

$\therefore \vec{a} = \left(2 \times \frac{1}{2}, -\frac{1}{2}, 3 \times \frac{1}{2}\right) = \left(1, -\frac{1}{2}, \frac{3}{2}\right)$

向量共面与$(\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}$

例1. 已知 $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$ 共面，则 $[(\boldsymbol{a}+\boldsymbol{b})\times(\boldsymbol{b}+\boldsymbol{c})]\cdot(\boldsymbol{c}+\boldsymbol{a})=$______

$[(\vec{a}+\vec{b})\times(\vec{b}+\vec{c})]\cdot(\vec{c}+\vec{a})$

$=[\vec{a}\times(\vec{b}+\vec{c})+\vec{b}\times(\vec{b}+\vec{c})]\cdot(\vec{c}+\vec{a})$

$=(\vec{a}\times\vec{b}+\vec{a}\times\vec{c}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c})\cdot(\vec{c}+\vec{a})$

$=(\vec{a}\times\vec{b})\cdot(\vec{c}+\vec{a})+(\vec{a}\times\vec{c})\cdot(\vec{c}+\vec{a})+(\vec{b}\times\vec{b})\cdot(\vec{c}+\vec{a})+(\vec{b}\times\vec{c})\cdot(\vec{c}+\vec{a})$

$=(\vec{a}\times\vec{b})\cdot\vec{c}+\underbrace{(\vec{a}\times\vec{b})\cdot\vec{a}}_{0}+\underbrace{(\vec{a}\times\vec{c})\cdot\vec{c}}_{0}+\underbrace{(\vec{a}\times\vec{c})\cdot\vec{a}}_{0}+\underbrace{(\vec{b}\times\vec{b})\cdot\vec{c}}_{0}+\underbrace{(\vec{b}\times\vec{b})\cdot\vec{a}}_{0}+\underbrace{(\vec{b}\times\vec{c})\cdot\vec{c}}_{0}+(\vec{b}\times\vec{c})\cdot\vec{a}$

$=(\vec{a}\times\vec{b})\cdot\vec{c}+(\vec{b}\times\vec{c})\cdot\vec{a}$

$=(\vec{a}\times\vec{b})\cdot\vec{c}+[-(\vec{c}\times\vec{b})\cdot\vec{a}]$

$=(\vec{a}\times\vec{b})\cdot\vec{c}+\{-[-(\vec{a}\times\vec{b})\cdot\vec{c}]\}$

$=(\vec{a}\times\vec{b})\cdot\vec{c}+(\vec{a}\times\vec{b})\cdot\vec{c}$

$=2(\vec{a}\times\vec{b})\cdot\vec{c}$ $\quad\because\vec{a},\vec{b},\vec{c}$ 共面 $\Rightarrow(\vec{a}\times\vec{b})\cdot\vec{c}=0$

$=2\times0=0$

梯度、方向导数、散度、旋度

例1. 函数 $u = xy + \frac{z}{y}$ 的梯度 **grad** $u(x,y,z) =$_____

$$\text{grad}\ u(x,y,z) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$
$$= \left(y\ ,\ x - \frac{z}{y^2}\ ,\ \frac{1}{y}\right)$$

$$\frac{\partial u}{\partial x} = \frac{\partial\left(xy + \frac{z}{y}\right)}{\partial x} = \frac{\partial(xy)}{\partial x} + \frac{\partial\left(\frac{z}{y}\right)}{\partial x} = y + 0 = y$$

$$\frac{\partial u}{\partial z} = \frac{\partial\left(xy + \frac{z}{y}\right)}{\partial z} = \frac{\partial(xy)}{\partial z} + \frac{\partial\left(\frac{z}{y}\right)}{\partial z} = 0 + \frac{1}{y} = \frac{1}{y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial\left(xy + \frac{z}{y}\right)}{\partial y} = \frac{\partial(xy)}{\partial y} + \frac{\partial\left(\frac{z}{y}\right)}{\partial y} = x + \frac{\partial(z \cdot y^{-1})}{\partial y} = x + (-z \cdot y^{-2}) = x - \frac{z}{y^2}$$

例2. 函数 $u = \ln(x^2 + y^2 + z^2)$ 的梯度 **grad** $u(x,y,z) =$_____

$$\text{grad}\ u(x,y,z) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$
$$= \left(\frac{2x}{x^2+y^2+z^2}\ ,\ \frac{2y}{x^2+y^2+z^2}\ ,\ \frac{2z}{x^2+y^2+z^2}\right)$$

$$\frac{\partial u}{\partial x} = \frac{\partial[\ln(x^2+y^2+z^2)]}{\partial x} = \frac{1}{x^2+y^2+z^2} \cdot \frac{\partial(x^2+y^2+z^2)}{\partial x} = \frac{1}{x^2+y^2+z^2} \cdot (2x) = \frac{2x}{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial y} = \frac{\partial[\ln(x^2+y^2+z^2)]}{\partial y} = \frac{1}{x^2+y^2+z^2} \cdot \frac{\partial(x^2+y^2+z^2)}{\partial y} = \frac{1}{x^2+y^2+z^2} \cdot (2y) = \frac{2y}{x^2+y^2+z^2}$$

$$\frac{\partial u}{\partial z} = \frac{\partial[\ln(x^2+y^2+z^2)]}{\partial z} = \frac{1}{x^2+y^2+z^2} \cdot \frac{\partial(x^2+y^2+z^2)}{\partial z} = \frac{1}{x^2+y^2+z^2} \cdot (2z) = \frac{2z}{x^2+y^2+z^2}$$

例3. 函数 $u = xy + \frac{z}{y}$ 在点 (2,1,1) 处沿向量 $\overrightarrow{n} = (1,2,2)$ 的方向导数为_______

$$\left.\frac{\partial u}{\partial \overrightarrow{n}}\right|_{(x_0,y_0,z_0)} = \text{grad}\ u(x,y,z)|_{(x_0,y_0,z_0)} \cdot \overrightarrow{n}^0$$

$$\left.\frac{\partial u}{\partial \overrightarrow{n}}\right|_{(2,1,1)} = \text{grad}\ u(x,y,z)|_{(2,1,1)} \cdot \overrightarrow{n}^0$$
$$= \left.\left(y\ ,\ x - \frac{z}{y^2}\ ,\ \frac{1}{y}\right)\right|_{(2,1,1)} \cdot \overrightarrow{n}^0$$
$$= \left(1\ ,\ 2 - \frac{1}{1^2}\ ,\ \frac{1}{1}\right) \cdot \overrightarrow{n}^0$$
$$= (1,1,1) \cdot \overrightarrow{n}^0$$
$$= (1,1,1) \cdot \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$
$$= 1 \times \frac{1}{3} + 1 \times \frac{2}{3} + 1 \times \frac{2}{3}$$
$$= \frac{5}{3}$$

例1. 函数 $u = xy + \frac{z}{y}$ 的梯度 **grad** $u(x,y,z) =$_____

$$\text{grad}\ u(x,y,z) = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right)$$
$$= \left(y\ ,\ x - \frac{z}{y^2}\ ,\ \frac{1}{y}\right)$$

$$\overrightarrow{n}^0 = \frac{1}{|\overrightarrow{n}|}\overrightarrow{n} \qquad |\overrightarrow{n}| = \sqrt{1^2 + 2^2 + 2^2} = 3$$
$$= \frac{1}{3}(1,2,2)$$
$$= \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$$

例4. 已知 $\overrightarrow{A}=y\,\overrightarrow{i}+(x-\frac{z}{y^2})\overrightarrow{j}+\frac{1}{y}\overrightarrow{k}$，则 $\operatorname{div}\overrightarrow{A}=$______

$$\begin{aligned}\operatorname{div}\overrightarrow{A}&=\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}+\frac{\partial R}{\partial z}\\&=\frac{\partial y}{\partial x}+\frac{\partial\left(x-\frac{z}{y^2}\right)}{\partial y}+\frac{\partial\left(\frac{1}{y}\right)}{\partial z}\\&=\ 0\ +\ \frac{2z}{y^3}\ +\ 0\\&=\frac{2z}{y^3}\end{aligned}$$

$\frac{\partial y}{\partial x}=0$	$\begin{aligned}\frac{\partial\left(x-\frac{z}{y^2}\right)}{\partial y}&=\frac{\partial x}{\partial y}-\frac{\partial\left(\frac{z}{y^2}\right)}{\partial y}\\&=0-\frac{\partial(z\cdot y^{-2})}{\partial y}\\&=-(-2zy^{-3})\\&=\frac{2z}{y^3}\end{aligned}$	$\frac{\partial\left(\frac{1}{y}\right)}{\partial z}=0$

例5. 已知 $\overrightarrow{A}=y\,\overrightarrow{i}+(x-\frac{z}{y^2})\overrightarrow{j}+\frac{1}{y}\overrightarrow{k}$，则 **rot** $\overrightarrow{A}=$______

$$\operatorname{rot}\overrightarrow{A}=\begin{vmatrix}\overrightarrow{i}&\overrightarrow{j}&\overrightarrow{k}\\\frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\y&x-\frac{z}{y^2}&\frac{1}{y}\end{vmatrix}$$

$$\begin{aligned}&=\overrightarrow{i}\begin{vmatrix}\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\x-\frac{z}{y^2}&\frac{1}{y}\end{vmatrix}-\overrightarrow{j}\begin{vmatrix}\frac{\partial}{\partial x}&\frac{\partial}{\partial z}\\y&\frac{1}{y}\end{vmatrix}+\overrightarrow{k}\begin{vmatrix}\frac{\partial}{\partial x}&\frac{\partial}{\partial y}\\y&x-\frac{z}{y^2}\end{vmatrix}\\&=\overrightarrow{i}\left(\frac{d\left(\frac{1}{y}\right)}{dy}-\frac{\partial\left(x-\frac{z}{y^2}\right)}{\partial z}\right)-\overrightarrow{j}\left(\frac{\partial\left(\frac{1}{y}\right)}{\partial x}-\frac{\partial y}{\partial z}\right)+\overrightarrow{k}\left(\frac{\partial\left(x-\frac{z}{y^2}\right)}{\partial x}-\frac{dy}{dy}\right)\\&=\overrightarrow{i}\left(-\frac{1}{y^2}-\left(-\frac{1}{y^2}\right)\right)-\overrightarrow{j}(0-0)\ +\overrightarrow{k}(1-1)\\&=\overrightarrow{0}\end{aligned}$$

$\begin{aligned}\frac{d\left(\frac{1}{y}\right)}{dy}&=(y^{-1})'\\&=-y^{-2}\\&=-\frac{1}{y^2}\end{aligned}$	$\begin{aligned}\frac{\partial\left(x-\frac{z}{y^2}\right)}{\partial z}&=\frac{\partial x}{\partial z}-\frac{\partial\left(\frac{z}{y^2}\right)}{\partial z}\\&=0-\frac{1}{y^2}\\&=-\frac{1}{y^2}\end{aligned}$	$\frac{\partial\left(\frac{1}{y}\right)}{\partial x}=0$
$\frac{\partial y}{\partial z}=0$	$\begin{aligned}\frac{\partial\left(x-\frac{z}{y^2}\right)}{\partial x}&=\frac{dx}{dx}-\frac{\partial\left(\frac{z}{y^2}\right)}{\partial x}\\&=1-0\\&=1\end{aligned}$	$\frac{dy}{dy}=1$

例6. 已知函数 $u=xy+\frac{z}{y}$，则 **rot (grad** u) = $\underline{\ \ \overrightarrow{0}\ \ \ \ }$

平面

例1. 求过 z 轴及点 (1,1,1) 的平面方程

z 轴上的点都在平面上 $\Longrightarrow$ (0,0,0)，(0,0,1)在平面上

设平面方程为 $Ax+By+Cz+D=0$

$$\begin{cases} A\cdot 1+B\cdot 1+C\cdot 1+D=0 \\ A\cdot 0+B\cdot 0+C\cdot 0+D=0 \\ A\cdot 0+B\cdot 0+C\cdot 1+D=0 \end{cases}$$

$$\Longrightarrow \begin{cases} A+B+C+D=0 \\ D=0 \\ C=0 \end{cases} \Longrightarrow \begin{cases} A+B=0 \\ D=0 \\ C=0 \end{cases} \Longrightarrow \begin{cases} A=-B \\ D=0 \\ C=0 \end{cases}$$

$$-Bx+By+0\cdot z+0=0$$

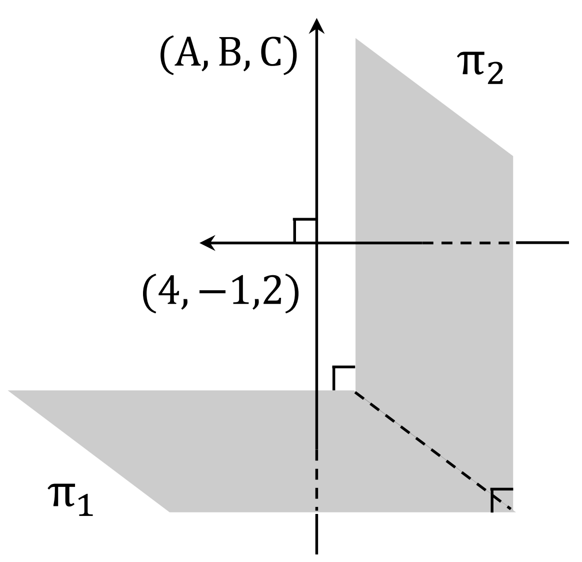
$$\Longrightarrow \quad -Bx+By=0$$

$$\Longrightarrow \quad -x+y=0$$

例2. 设平面π_1经过点(0,0,0)及点 $(6,-3,2)$，且与平面π_2 $4x-y+2z=8$垂直，求该平面方程

设平面方程为 $Ax+By+Cz+D=0$

$$\begin{cases} A\cdot 0+B\cdot 0+C\cdot 0+D=0 \\ A\cdot 6+B\cdot(-3)+C\cdot 2+D=0 \\ A\cdot 4+B\cdot(-1)+C\cdot 2=0 \end{cases}$$



由上图可知 $(A,B,C)\perp(4,-1,2)$

$\Longrightarrow (A,B,C)\cdot(4,-1,2)=0$

$\Longrightarrow\ A\cdot 4+B\cdot(-1)+C\cdot 2=0$

$$\Longrightarrow \begin{cases} D=0 \\ 6A-3B+2C=0 \\ 4A-B+2C=0 \end{cases} \Longrightarrow \begin{cases} D=0 \\ 6A-3B=-2C \\ 4A-B=-2C \end{cases} \Longrightarrow 6A-3B=4A-B \Longrightarrow 2A=2B$$

$$\Longrightarrow A=B \quad \Longrightarrow 6B-3B=-2C \Longrightarrow C=-\frac{3}{2}B \Longrightarrow \begin{cases} A=B \\ C=-\frac{3}{2}B \\ D=0 \end{cases}$$

$$Bx+By-\frac{3}{2}Bz+0=0$$

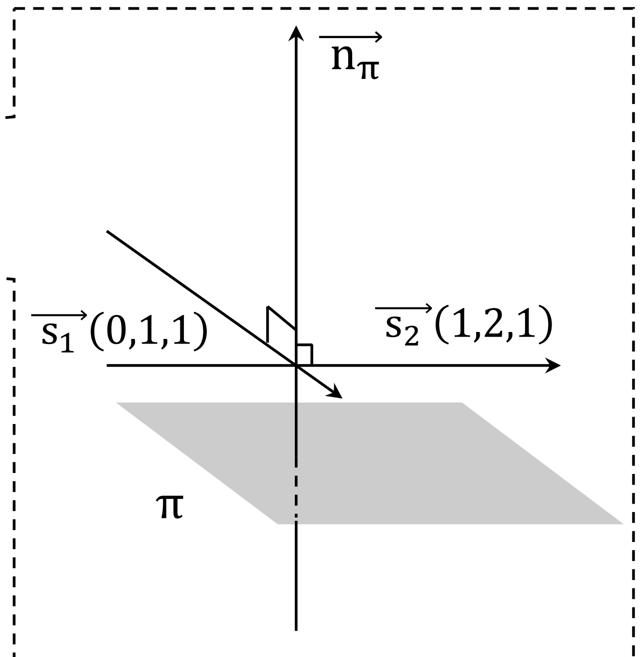
$$\Longrightarrow \quad x+y-\frac{3}{2}z=0$$

例3. 已知平面 π 与向量 $\boldsymbol{s}_1=(0,1,1)$ 和$\boldsymbol{s}_2=(1,2,1)$平行，且过(0,0,0)点，求平面方程

设平面方程为 $Ax+By+Cz+D=0$

$A\cdot 0+B\cdot 0+C\cdot 0+D=0 \Longrightarrow D=0$

$\overrightarrow{n_\pi}\perp\overrightarrow{s_1}$且 $\overrightarrow{n_\pi}\perp\overrightarrow{s_2}$



$$\Longrightarrow \overrightarrow{n_\pi}=\overrightarrow{s_1}\times\overrightarrow{s_2}$$

$$=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}=\vec{i}\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}-\vec{j}\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}+\vec{k}\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}$$

$$=\vec{i}(1\cdot 1-1\cdot 2)-\vec{j}(0\cdot 1-1\cdot 1)+\vec{k}(0\cdot 2-1\cdot 1)$$

$$=\vec{i}(-1)-\vec{j}(-1)+\vec{k}(-1)$$

$$=\vec{i}(-1)+\vec{j}(1)+\vec{k}(-1)$$

$$=(-1,1,-1)$$

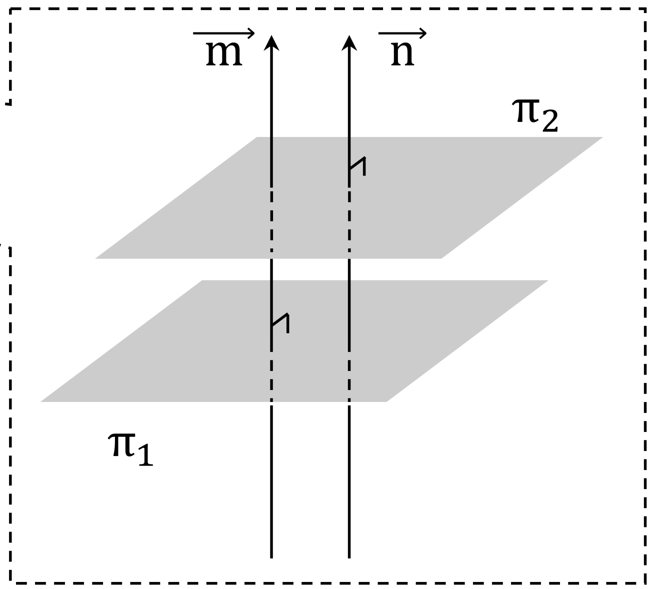
$$\Longrightarrow \ A=-1 \quad B=1 \quad C=-1$$

$$\Longrightarrow \quad -1\cdot x+1\cdot y+(-1)\cdot z+0=0$$

$$\Longrightarrow \quad -x+y-z=0$$

例4. 求过点 (3,2,0) 且平行于平面 π_2: $3x-4y+z=0$ 的平面π_1

$\overrightarrow{m} \parallel \overrightarrow{n}$



$\overrightarrow{m} = (A, B, C)$

$\overrightarrow{n} = (3, -4, 1)$

$$\begin{cases} A = 3k \\ B = -4k \quad , k \neq 0 \\ C = k \end{cases}$$

$\overrightarrow{m} = (3k, -4k, k)$

平面方程： $3k(x-3) - 4k(y-2) + k(z-0) = 0$

$\Rightarrow \quad 3(x-3) - 4(y-2) + (z-0) = 0$

$\Rightarrow \quad 3x - 4y + z - 1 = 0$

例5. 判断平面 π_1: $2x+4y+6z+8=0$ 与平面 π_2: $-6x-3y+4z+8=0$ 垂直还是平行

平面π_1的法向量 $\overrightarrow{n_{\pi_1}} = (2,4,6)$

平面π_2的法向量 $\overrightarrow{n_{\pi_2}} = (-6,-3,4)$

$\overrightarrow{n_{\pi_1}}$ 与 $\overrightarrow{n_{\pi_2}}$ 显然是不平行的

$(2,4,6) \cdot (-6,-3,4) = 2\times(-6) + 4\times(-3) + 6\times 4 = 0$

$\therefore \overrightarrow{n_{\pi_1}} \perp \overrightarrow{n_{\pi_2}}$

$\therefore \pi_1 \perp \pi_2$

直线

例1. 请找出直线 $L:\begin{cases}x+3y+2z+1=0\\2x-y-10z+3=0\end{cases}$ 的一个方向向量 $\overrightarrow{s}$

$$\overrightarrow{s}=(1,3,2)\times(2,-1,-10)$$

$$=\begin{vmatrix}\overrightarrow{i}&\overrightarrow{j}&\overrightarrow{k}\\1&3&2\\2&-1&-10\end{vmatrix}$$

$$=\overrightarrow{i}\begin{vmatrix}3&2\\-1&-10\end{vmatrix}-\overrightarrow{j}\begin{vmatrix}1&2\\2&-10\end{vmatrix}+\overrightarrow{k}\begin{vmatrix}1&3\\2&-1\end{vmatrix}$$

$$=\overrightarrow{i}[3\times(-10)-2\times(-1)]-\overrightarrow{j}[1\times(-10)-2\times2]+\overrightarrow{k}[1\times(-1)-3\times2]$$

$$=-28\overrightarrow{i}+14\overrightarrow{j}+(-7)\overrightarrow{k}$$

$$=(-28,14,-7)$$

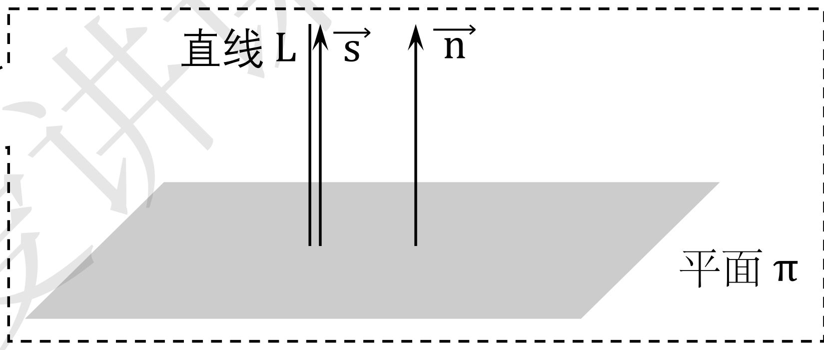
例2. 设有直线 $L:\begin{cases}x+3y+2z+1=0\\2x-y-10z+3=0\end{cases}$ 以及平面 $\pi:4x-2y+z-2=0$，则直线 L （ C ）

(A) 平行于 π　(B) 在 π 上　(C) 垂直于 π　(D) 与 π 斜交

直线 L 的方向向量 $\overrightarrow{s}=(-28,14,-7)$，详细计算过程见例1

平面 π 的法向量 $\overrightarrow{n}=(4,-2,1)$

$$\begin{cases}-28=-7\times4\\14=-7\times(-2)\\-7=-7\times1\end{cases}\Rightarrow\overrightarrow{s}\parallel\overrightarrow{n}\Rightarrow \text{直线 } L\perp\text{平面 }\pi$$



例3. 设有直线 $L_1:\frac{x-1}{-1}=\frac{y-5}{2}=\frac{z+8}{-1}$ 与 $L_2:\begin{cases}x-y=6\\2y+z=3\end{cases}$，则 L_1 与 L_2 的夹角为（ C ）

(A) $\frac{\pi}{6}$　(B) $\frac{\pi}{4}$　(C) $\frac{\pi}{3}$　(D) $\frac{\pi}{2}$

$L_1:\frac{x-1}{-1}=\frac{y-5}{2}=\frac{z-(-8)}{-1}\Rightarrow L_1$ 的方向向量 $\overrightarrow{s_1}=(-1,2,-1)$

$L_2:\begin{cases}x-y+0z-6=0\\0x+2y+z-3=0\end{cases}\Rightarrow L_2$ 的方向向量 $\overrightarrow{s_2}=(1,-1,0)\times(0,2,1)$

$$=\begin{vmatrix}\overrightarrow{i}&\overrightarrow{j}&\overrightarrow{k}\\1&-1&0\\0&2&1\end{vmatrix}$$

$$=\overrightarrow{i}\begin{vmatrix}-1&0\\2&1\end{vmatrix}-\overrightarrow{j}\begin{vmatrix}1&0\\0&1\end{vmatrix}+\overrightarrow{k}\begin{vmatrix}1&-1\\0&2\end{vmatrix}$$

$$=\overrightarrow{i}[(-1)\times1-0\times2]-\overrightarrow{j}(1\times1-0\times0)+\overrightarrow{k}[1\times2-(-1)\times0]$$

$$=-1\overrightarrow{i}+(-1)\overrightarrow{j}+2\overrightarrow{k}$$

$$=(-1,-1,2)$$

$|\overrightarrow{s_1}|=\sqrt{(-1)^2+2^2+(-1)^2}=\sqrt{6}$

$|\overrightarrow{s_2}|=\sqrt{(-1)^2+(-1)^2+2^2}=\sqrt{6}$

$\overrightarrow{s_1}\cdot\overrightarrow{s_2}=(-1)\times(-1)+2\times(-1)+(-1)\times2=-3$

$\overrightarrow{s_1}\cdot\overrightarrow{s_2}=|\overrightarrow{s_1}||\overrightarrow{s_2}|\cos\theta$

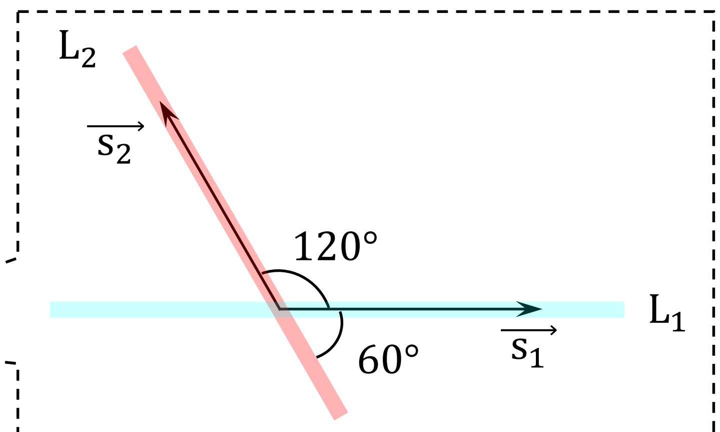
$\Rightarrow\quad -3=\sqrt{6}\times\sqrt{6}\cos\theta$

$\Rightarrow \cos\theta=-\frac{1}{2}$

$\because 0°\le\theta\le180°$

$\therefore \theta=120°$

$\therefore L_1$ 与 L_2 的夹角为 60°，即 $\frac{\pi}{3}$



例4. 求经过点$(-1,-4,3)$，且与下面两条直线 $L_1:\begin{cases}2x-4y+z-6=0\\x+3y+5=0\end{cases}$ 和 $L_2:\begin{cases}x=2+4t\\y=-1-t\\z=-3+2t\end{cases}$ 都垂直的直线 L

直线 L_1 的方向向量 $\overrightarrow{s_1}=(2,-4,1)\times(1,3,0)$

$$=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\2&-4&1\\1&3&0\end{vmatrix}$$

$$=\vec{i}\begin{vmatrix}-4&1\\3&0\end{vmatrix}-\vec{j}\begin{vmatrix}2&1\\1&0\end{vmatrix}+\vec{k}\begin{vmatrix}2&-4\\1&3\end{vmatrix}$$

$$=\vec{i}[(-4)\times0-1\times3]-\vec{j}(2\times0-1\times1)+\vec{k}[2\times3-(-4)\times1]$$

$$=-3\vec{i}+1\vec{j}+10\vec{k}$$

$$=(-3,1,10)$$

直线 L_2 的方向向量 $\overrightarrow{s_2}=(4,-1,2)$

直线 L_2 经过点 $(2,-1,-3)$

直线 L 同时垂直于 L_1 和 L_2

$\Longrightarrow$ $\vec{s}$ 同时垂直于 $\overrightarrow{s_1}$ 和 $\overrightarrow{s_2}$

$\Longrightarrow$ $\vec{s}=\overrightarrow{s_1}\times\overrightarrow{s_2}$

$$=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\-3&1&10\\4&-1&2\end{vmatrix}$$

$$=\vec{i}\begin{vmatrix}1&10\\-1&2\end{vmatrix}-\vec{j}\begin{vmatrix}-3&10\\4&2\end{vmatrix}+\vec{k}\begin{vmatrix}-3&1\\4&-1\end{vmatrix}$$

$$=\vec{i}[1\times2-10\times(-1)]-\vec{j}[(-3)\times2-10\times4]+\vec{k}[(-3)\times(-1)-1\times4]$$

$$=12\vec{i}+46\vec{j}+(-1)\vec{k}$$

$$=(12,46,-1)$$

∴ 直线 L 的方程为 $\frac{x-(-1)}{12}=\frac{y-(-4)}{46}=\frac{z-3}{-1}$

即 $\frac{x+1}{12}=\frac{y+4}{46}=\frac{z-3}{-1}$

例5. 请将直线 $\begin{cases}x+3y+2z+1=0\\2x-y-10z+3=0\end{cases}$ 转化为 $\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$ 的形式

$\vec{s}=(-28,14,-7)$，详细计算过程见例1

$\Longrightarrow l=-28\quad m=14\quad n=-7$

将 (x_0,y_0,z_0) 代入 原直线方程，有

$$\begin{cases}x_0+3y_0+2z_0+1=0\\2x_0-y_0-10z_0+3=0\end{cases}$$

令 $x_0=0$

则直线方程可变为:

$$\begin{cases}0+3y_0+2z_0+1=0\\2\cdot0-y_0-10z_0+3=0\end{cases}\Longrightarrow\begin{cases}3y_0+2z_0+1=0\\-y_0-10z_0+3=0\end{cases}\Longrightarrow\begin{cases}y_0=-\frac{4}{7}\\z_0=\frac{5}{14}\end{cases}$$

则 $(0,-\frac{4}{7},\frac{5}{14})$ 为直线上的一点

$$\frac{x-0}{-28}=\frac{y-\left(-\frac{4}{7}\right)}{14}=\frac{z-\frac{5}{14}}{-7}\Longrightarrow(-7)\cdot\frac{x}{4\times(-7)}=(-7)\cdot\frac{y+\frac{4}{7}}{-2\times(-7)}=(-7)\cdot\frac{z-\frac{5}{14}}{1\times(-7)}\Longrightarrow\frac{x}{4}=\frac{y+\frac{4}{7}}{-2}=\frac{z-\frac{5}{14}}{1}$$

平面束方程

例1. 求过点 (3,2,0) 和直线 $x+1=y-3=\frac{z}{2}$ 的平面方程

直线方程可整理为 $\begin{cases} x+1=y-3 \\ y-3=\frac{z}{2} \end{cases}$

$$\Longrightarrow \begin{cases} 1x-1y+0z+4=0 \\ 0x+2y-1z-6=0 \end{cases}$$

经过该直线的平面方程为:

$$(1+\lambda\cdot 0)x+(-1+\lambda\cdot 2)y+[0+\lambda\cdot(-1)]z+[4+\lambda\cdot(-6)]=0$$

$$\Longrightarrow \quad x+(2\lambda-1)y-\lambda z-6\lambda+4=0$$

将点(3,2,0)代入上述方程，则有 $3+(2\lambda-1)\cdot 2-\lambda\cdot 0-6\lambda+4=0$

$$\Longrightarrow \quad -2\lambda+5=0$$

$$\Longrightarrow \quad \lambda=\frac{5}{2}$$

∴ 所求平面方程为:

$$x+\left(2\cdot\frac{5}{2}-1\right)y-\frac{5}{2}z-6\cdot\frac{5}{2}+4=0$$

$$\Longrightarrow \quad x+4y-\frac{5}{2}z-11=0$$

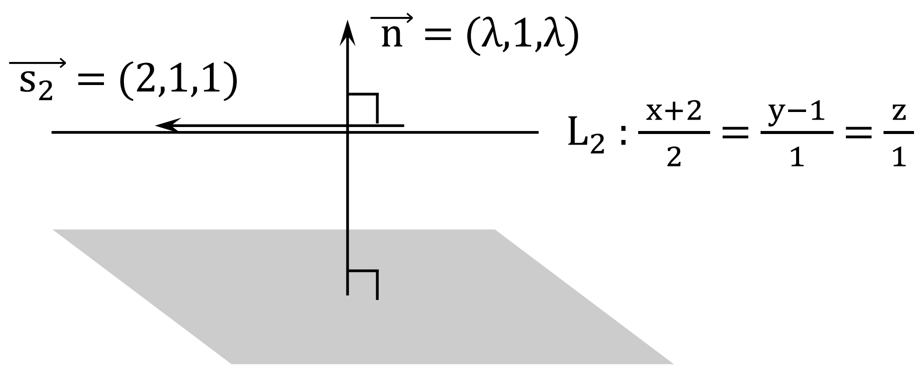
$$\Longrightarrow \quad 2x+8y-5z-22=0$$

例2. 已知两条直线 $L_1:\frac{x-1}{1}=\frac{y-2}{0}=\frac{z-3}{-1}$，$L_2:\frac{x+2}{2}=\frac{y-1}{1}=\frac{z}{1}$，求过 L_1 且平行于 L_2 的平面方程

直线 L_1 方程可整理为 $\begin{cases} \frac{x-1}{1}=\frac{y-2}{0} \\ \frac{x-1}{1}=\frac{z-3}{-1} \end{cases}$

$$\Longrightarrow \begin{cases} (x-1)\cdot 0=(y-2)\cdot 1 \\ (x-1)\cdot(-1)=(z-3)\cdot 1 \end{cases}$$

$$\Longrightarrow \begin{cases} 0x+1y+0z-2=0 \\ 1x+0y+1z-4=0 \end{cases}$$



经过直线 L_1 的平面方程为:

$$(0+\lambda\cdot 1)x+(1+\lambda\cdot 0)y+(0+\lambda\cdot 1)z+[-2+\lambda\cdot(-4)]=0$$

$$\Longrightarrow \quad \lambda x+y+\lambda z-4\lambda-2=0$$

$$\overrightarrow{s_2}\cdot\overrightarrow{n}=0$$

$$\Longrightarrow \quad 2\lambda+1\times 1+1\lambda=0$$

$$\Longrightarrow \quad 3\lambda+1=0$$

$$\Longrightarrow \quad \lambda=-\frac{1}{3}$$

∴ 所求平面方程为:

$$-\frac{1}{3}x+y+\left(-\frac{1}{3}\right)z-4\cdot\left(-\frac{1}{3}\right)-2=0$$

$$\Longrightarrow -\frac{1}{3}x+y-\frac{1}{3}z-\frac{2}{3}=0$$

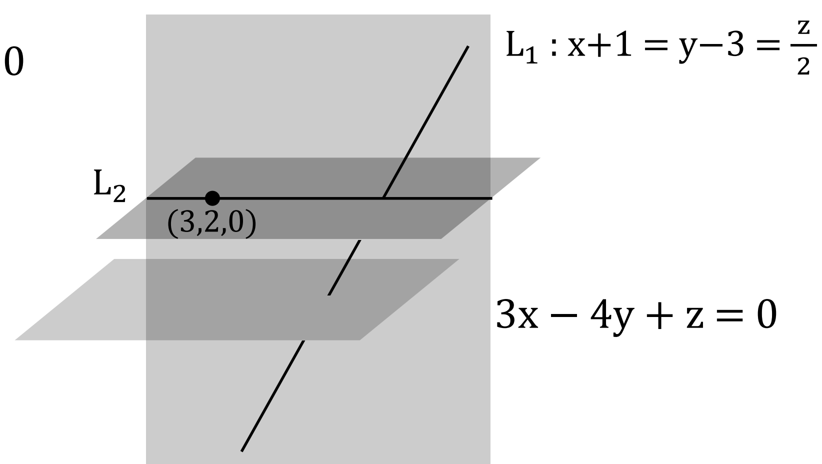
$$\Longrightarrow x-3y+z+2=0$$

例3. 求过点 (3,2,0) 且 平行于平面 $3x-4y+z=0$，又与直线 $L_1: x+1=y-3=\frac{z}{2}$ 相交的直线方程 L_2

由本课例1可知，过点 (3,2,0) 和直线 $x+1=y-3=\frac{z}{2}$ 的平面为 $2x+8y-5z-22=0$

由《平面》那课里的例4可知，过点 (3,2,0) 且平行于平面 $3x-4y+z=0$ 的平面为

$$3x-4y+z-1=0$$



∴ L_2的方程为 $\begin{cases} 2x+8y-5z-22=0 \\ 3x-4y+z-1=0 \end{cases}$

点到平面的距离、点到直线的距离

例1. 求点 (2,1,0) 到平面 3x+4y+5z=0 的距离 d

$$3x+4y+5z+0=0$$

$$d=\frac{|Ax_0+By_0+Cz_0+D|}{\sqrt{A^2+B^2+C^2}}=\frac{|3\times2+4\times1+5\times0+0|}{\sqrt{3^2+4^2+5^2}}=\frac{|6+4+0+0|}{\sqrt{50}}=\frac{10}{5\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

例2. 求点 $\left(1,\frac{3}{7},\frac{19}{14}\right)$ 到直线 L: $\begin{cases}x+3y+2z+1=0\\2x-y-10z+3=0\end{cases}$ 的距离 d

将直线 $\begin{cases}x+3y+2z+1=0\\2x-y-10z+3=0\end{cases}$ 转化为 $\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{z-z_0}{n}$ 的形式

结果为 $\frac{x}{4}=\frac{y+\frac{4}{7}}{-2}=\frac{z-\frac{5}{14}}{1}$

$\vec{s}=(-28,14,-7)$，详细计算过程见《直线》那课里的例1

$\Longrightarrow l=-28 \quad m=14 \quad n=-7$

将 (x_0,y_0,z_0) 代入 原直线方程，有

$$\begin{cases}x_0+3y_0+2z_0+1=0\\2x_0-y_0-10z_0+3=0\end{cases}$$

令 $x_0=0$

则直线方程可变为:

$$\begin{cases}0+3y_0+2z_0+1=0\\2\cdot0-y_0-10z_0+3=0\end{cases}\Longrightarrow\begin{cases}3y_0+2z_0+1=0\\-y_0-10z_0+3=0\end{cases}\Longrightarrow\begin{cases}y_0=-\frac{4}{7}\\z_0=\frac{5}{14}\end{cases}$$

则 $\left(0,-\frac{4}{7},\frac{5}{14}\right)$ 为直线上的一点

$$\frac{x-0}{-28}=\frac{y-\left(-\frac{4}{7}\right)}{14}=\frac{z-\frac{5}{14}}{-7}\Longrightarrow(-7)\cdot\frac{x}{4\times(-7)}=(-7)\cdot\frac{y+\frac{4}{7}}{-2\times(-7)}=(-7)\cdot\frac{z-\frac{5}{14}}{1\times(-7)}$$

$$\Longrightarrow\frac{x}{4}=\frac{y+\frac{4}{7}}{-2}=\frac{z-\frac{5}{14}}{1}$$

$$\Longrightarrow\frac{x-0}{4}=\frac{y-\left(-\frac{4}{7}\right)}{-2}=\frac{z-\frac{5}{14}}{1}$$

$$d=\frac{|(a-x_0,b-y_0,c-z_0)\times(l,m,n)|}{\sqrt{l^2+m^2+n^2}}$$

$$=\frac{\left|\left(1-0,\frac{3}{7}-\left(-\frac{4}{7}\right),\frac{19}{14}-\frac{5}{14}\right)\times(4,-2,1)\right|}{\sqrt{4^2+(-2)^2+1^2}}$$

$$=\frac{|(1,1,1)\times(4,-2,1)|}{\sqrt{21}}$$

$$=\frac{|(3,3,-6)|}{\sqrt{21}}$$

$$(1,1,1)\times(4,-2,1)=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\1&1&1\\4&-2&1\end{vmatrix}=\vec{i}\begin{vmatrix}1&1\\-2&1\end{vmatrix}-\vec{j}\begin{vmatrix}1&1\\4&1\end{vmatrix}+\vec{k}\begin{vmatrix}1&1\\4&-2\end{vmatrix}$$

$$=\vec{i}[1\cdot1-(-2\cdot1)]-\vec{j}(1\cdot1-4\cdot1)+\vec{k}[1\cdot(-2)-4\cdot1]$$

$$=\vec{i}(3)-\vec{j}(-3)+\vec{k}(-6)$$

$$=\vec{i}(3)+\vec{j}(3)+\vec{k}(-6)$$

$$=(3,3,-6)$$

$$=\frac{\sqrt{3^2+3^2+(-6)^2}}{\sqrt{21}}$$

$$=\frac{\sqrt{54}}{\sqrt{21}}$$

$$=\frac{3\sqrt{14}}{7}$$

求曲线在某点处的切线与法平面

例1. 求曲线$\begin{cases} x(t) = t \\ y(t) = -t^2 \\ z(t) = t^3 \end{cases}$在 $t = 1$处的切线方程与法平面方程

另一种问法. 求曲线$\begin{cases} x(t) = t \\ y(t) = -t^2 \\ z(t) = t^3 \end{cases}$在 $(1, -1,1)$ 处的切线方程与法平面方程

① $\begin{cases} x'(t) = t' = 1 \\ y'(t) = (-t^2)' = -2t \\ z'(t) = (t^3)' = 3t^2 \end{cases}$ $\begin{cases} x'(1) = 1 \\ y'(1) = -2 \\ z'(1) = 3 \end{cases}$ $\begin{cases} x(1) = 1 \\ y(1) = -1 \\ z(1) = 1 \end{cases}$

② 切线方程：$\frac{x-x(1)}{x'(1)} = \frac{y-y(1)}{y'(1)} = \frac{z-z(1)}{z'(1)}$

$$\frac{x-1}{1} = \frac{y-(-1)}{-2} = \frac{z-1}{3}$$

$$\Rightarrow \frac{x-1}{1} = \frac{y+1}{-2} = \frac{z-1}{3}$$

法平面方程 : $x'(1)[x - x(1)] + y'(1)[y - y(1)] + z'(1)[z - z(1)] = 0$

$$1 \cdot (x - 1) - 2[y - (-1)] + 3(z - 1) = 0$$

$$\Rightarrow \quad x - 2y + 3z - 6 = 0$$

例2. 在曲线$\begin{cases} x(t) = t \\ y(t) = -t^2 \\ z(t) = t^3 \end{cases}$的所有切线中，与平面 π: $x+2y+z = 4$平行的切线有几条

① $\begin{cases} x'(t) = 1 \\ y'(t) = -2t \\ z'(t) = 3t^2 \end{cases}$

② 切线方程：$\frac{x-x(t_0)}{x'(t_0)} = \frac{y-y(t_0)}{y'(t_0)} = \frac{z-z(t_0)}{z'(t_0)}$

$$\Rightarrow \frac{x-t_0}{1} = \frac{y-(-t_0^2)}{-2t_0} = \frac{z-t_0^3}{3t_0^2}$$

切线的方向向量$\vec{s}=(1, -2t_0, 3{t_0}^2)$

平面的法向量$\vec{n}=(1,2,1)$

$$\vec{s} \perp \vec{n} \ \Rightarrow \ \vec{s} \cdot \vec{n} = 0$$

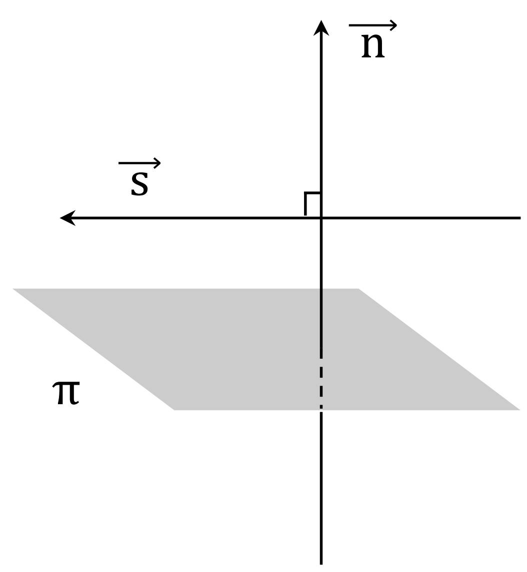
$$(1, -2t_0, 3{t_0}^2) \cdot (1,2,1) = 0$$

$$1 \times 1 - 2t_0 \cdot 2 + 3{t_0}^2 \cdot 1 = 0$$

$$(3t_0 - 1)(t_0 - 1) = 0$$

$$\Rightarrow \quad t_{01} = \frac{1}{3} \text{、} t_{02} = 1$$

∴ 符合要求的切线有两条



例3. 求曲线$\begin{cases}x^2+y^2+z^2=6\\ z=x^2+y^2\end{cases}$ 在 $(-1,1,2)$ 处的切线方程与法平面方程

本题即：求曲线$\begin{cases}x(t)=t\\ x^2(t)+y^2(t)+z^2(t)=6\\ z(t)=x^2(t)+y^2(t)\end{cases}$ 在 $(-1,1,2)$ 处的切线方程与法平面方程

$$\Longrightarrow\begin{cases}x(t)=t\\ t^2+y^2(t)+z^2(t)=6\\ z(t)=t^2+y^2(t)\end{cases}$$

$$\xRightarrow{\text{两边同时求导}}\begin{cases}x'(t)=(t)'\\ [t^2+y^2(t)+z^2(t)]'=(6)'\\ z'(t)=[t^2+y^2(t)]'\end{cases}$$

$$\Longrightarrow\begin{cases}x'(t)=1 & ①\\ 2t+2\cdot y(t)\cdot y'(t)+2\cdot z(t)\cdot z'(t)=0 & ②\\ z'(t)=2t+2\cdot y(t)\cdot y'(t) & ③\end{cases}$$

$$\Longrightarrow\begin{cases}x'(t)=1\\ y'(t)=-\frac{t}{y(t)}\\ z'(t)=0\end{cases}$$

② + ③ 得 $2t+2\cdot y(t)\cdot y'(t)+2\cdot z(t)\cdot z'(t)+z'(t)=0+2t+2\cdot y(t)\cdot y'(t)$

$\Longrightarrow [2z(t)+1]\cdot z'(t)=0$

$\Longrightarrow z'(t)=0$

根据题干可知，$z(t)=x^2(t)+y^2(t)\geq 0$

$\therefore 2z(t)+1>0$ $\therefore [2z(t)+1]\cdot z'(t)=0$ 说明 $z'(t)=0$

将 $z'(t)=0$ 代入② 或 ③ ,可以求得 $y'(t)=-\frac{t}{y(t)}$

在 $(-1,1,2)$ 处$\Longrightarrow\begin{cases}x(t_0)=-1\\ y(t_0)=1\\ z(t_0)=2\end{cases}\Longrightarrow t_0=-1\Longrightarrow\begin{cases}x'(-1)=1\\ y'(-1)=1\\ z'(-1)=0\end{cases}$

切线方程：$\frac{x-x(-1)}{x'(-1)}=\frac{y-y(-1)}{y'(-1)}=\frac{z-z(-1)}{z'(-1)}\Longrightarrow\frac{x-(-1)}{1}=\frac{y-1}{1}=\frac{z-2}{0}\Longrightarrow\frac{x+1}{1}=\frac{y-1}{1}=\frac{z-2}{0}$

法平面方程：$x'(-1)[x-x(-1)]+y'(-1)[y-y(-1)]+z'(-1)[z-z(-1)]=0$

$\Longrightarrow 1\cdot[x-(-1)]+1\cdot(y-1)+0\cdot(z-2)=0$

$\Longrightarrow x+1+y-1=0$

$\Longrightarrow x+y=0$

曲面在某点处的法向量、切平面、法线

例1. 设 $\boldsymbol{n}$ 是曲面 $2x^2+3y^2+z^2=6$ 在点 $P(1,1,1)$ 处的指向外侧的法向量，求 $\boldsymbol{n}$

① 曲面方程可变为：$2x^2+3y^2+z^2-6=0$

设 $F=2x^2+3y^2+z^2-6$

② $F'_x=\frac{\partial F}{\partial x}=\frac{\partial(2x^2+3y^2+z^2-6)}{\partial x}=2\cdot 2x=4x$

$F'_y=\frac{\partial F}{\partial y}=\frac{\partial(2x^2+3y^2+z^2-6)}{\partial y}=2\cdot 3y=6y$

$F'_z=\frac{\partial F}{\partial z}=\frac{\partial(2x^2+3y^2+z^2-6)}{\partial z}=2z$

$\Rightarrow\ F'_x(1,1,1)=4\times1=4$

$\Rightarrow\ F'_y(1,1,1)=6\times1=6$

$\Rightarrow\ F'_z(1,1,1)=2\times1=2$

③ $\overrightarrow{n}=(4,6,2)$

例1改编. 求曲面 $2x^2+3y^2+z^2=6$ 在点 $P(1,1,1)$ 处的切平面与法线

曲面 $2x^2+3y^2+z^2=6$ 在点 $P(1,1,1)$ 处的法向量为 $\overrightarrow{n}=(4,6,2)$

∴ 切平面方程为 $4(x-x_0)+6(y-y_0)+2(z-z_0)=0$

$\Rightarrow 4(x-1)+6(y-1)+2(z-1)=0$

$\Rightarrow\ 2x+3y+z-6=0$

法线方程为 $\frac{x-x_0}{4}=\frac{y-y_0}{6}=\frac{z-z_0}{2}$

$\Rightarrow\ \frac{x-1}{4}=\frac{y-1}{6}=\frac{z-1}{2}$

$\Rightarrow 2\cdot\frac{x-1}{2\times2}=2\cdot\frac{y-1}{3\times2}=2\cdot\frac{z-1}{1\times2}$

$\Rightarrow\ \frac{x-1}{2}=\frac{y-1}{3}=\frac{z-1}{1}$

例2. 设点 P 为椭球面 $S: x^2+y^2+z^2-yz=1$ 上的动点，已知椭球面 S 在点 P 处的切平面与 xOy 面垂直，求点 P 的轨迹 C

设点 P 坐标为 (x_0,y_0,z_0)

① 曲面方程可变为：$x^2+y^2+z^2-yz-1=0$

设 $F=x^2+y^2+z^2-yz-1$

② $F'_x=\frac{\partial F}{\partial x}=\frac{\partial(x^2+y^2+z^2-yz-1)}{\partial x}=2x$

$F'_y=\frac{\partial F}{\partial y}=\frac{\partial(x^2+y^2+z^2-yz-1)}{\partial y}=2y-z$

$F'_z=\frac{\partial F}{\partial z}=\frac{\partial(x^2+y^2+z^2-yz-1)}{\partial z}=2z-y$

$\Rightarrow\ F'_x(x_0,y_0,z_0)=2x_0$

$\Rightarrow\ F'_y(x_0,y_0,z_0)=2y_0-z_0$

$\Rightarrow\ F'_z(x_0,y_0,z_0)=2z_0-y_0$

③ 切平面方程为 $2x_0(x-x_0)+(2y_0-z_0)(y-y_0)+(2z_0-y_0)(z-z_0)=0$

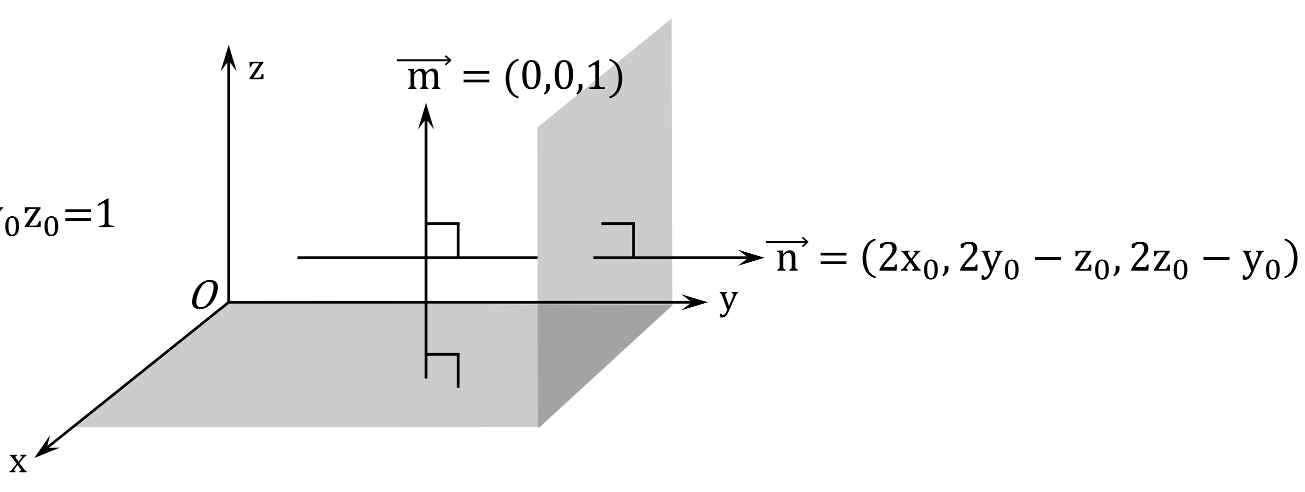
$\overrightarrow{n}\cdot\overrightarrow{m}=0$

$\Rightarrow 2x_0\cdot0+(2y_0-z_0)\cdot0+(2z_0-y_0)\cdot1=0$

$\Rightarrow 2z_0-y_0=0$

$\Rightarrow$ 点 P (x_0,y_0,z_0)要满足 $2z_0-y_0=0$ 和 $x_0{}^2+y_0{}^2+z_0{}^2-y_0z_0=1$

$\Rightarrow$ 点 P 的轨迹为$\begin{cases}x^2+y^2+z^2-yz=1\\2z-y=0\end{cases}$



例3. 曲面 $z=x^2+y^2$ 与平面 $2x+4y-z=0$ 平行的切平面的方程是________

设切点坐标为 (x_0,y_0,z_0)

① 曲面方程可变为：$x^2+y^2-z=0$

设 $F=x^2+y^2-z$

② $F'_x=\frac{\partial F}{\partial x}=\frac{\partial(x^2+y^2-z)}{\partial x}=2x$ | $F'_y=\frac{\partial F}{\partial y}=\frac{\partial(x^2+y^2-z)}{\partial y}=2y$ | $F'_z=\frac{\partial F}{\partial z}=\frac{\partial(x^2+y^2-z)}{\partial z}=-1$

$\Longrightarrow\ F'_x(x_0,y_0,z_0)=2x_0$

$\Longrightarrow\ F'_y(x_0,y_0,z_0)=2y_0$

$\Longrightarrow\ F'_z(x_0,y_0,z_0)=-1$

③ 切平面方程为 $2x_0(x-x_0)+2y_0(y-y_0)-(z-z_0)=0$

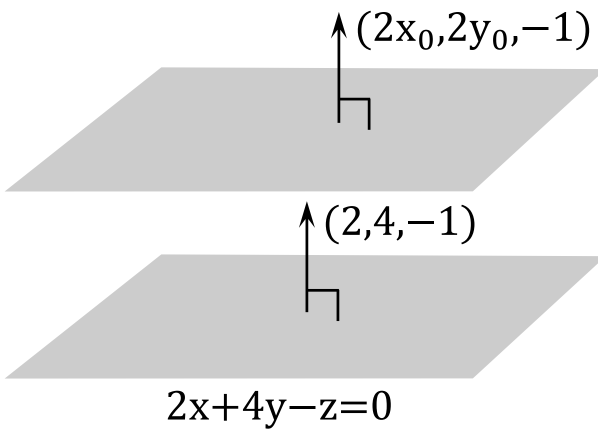
$\begin{cases}2y_0=4\\2x_0=2\end{cases}\Longrightarrow\begin{cases}y_0=2\\x_0=1\end{cases}$

∴ 切平面方程为 $2\cdot1(x-1)+2\cdot2(y-2)-(z-z_0)=0$

$\Longrightarrow 2x+4y-z+z_0-10=0$

∵ 切点 (x_0,y_0,z_0) 在曲面 $z=x^2+y^2$ 上 ∴ $z_0={x_0}^2+{y_0}^2=1^2+2^2=5$

∴ 切平面方程为 $2x+4y-z-5=0$



例4. 过点 (1,0,0)、(0,1,0)，且与曲面 $z=x^2+y^2$相切的平面为 (B)

(A) z=0 与 x+y−z−1=0 (B) z=0与2x+2y−z−2=0

(C) x=y与x+y−z−1=0 (D) x=y与2x+2y−z−2=0

设切点坐标为 (x_0,y_0,z_0)

① 曲面方程可变为：$x^2+y^2-z=0$

设 $F=x^2+y^2-z$

② $F'_x=\frac{\partial F}{\partial x}=\frac{\partial(x^2+y^2-z)}{\partial x}=2x$ | $F'_y=\frac{\partial F}{\partial y}=\frac{\partial(x^2+y^2-z)}{\partial y}=2y$ | $F'_z=\frac{\partial F}{\partial z}=\frac{\partial(x^2+y^2-z)}{\partial z}=-1$

$\Longrightarrow\ F'_x(x_0,y_0,z_0)=2x_0$

$\Longrightarrow\ F'_y(x_0,y_0,z_0)=2y_0$

$\Longrightarrow\ F'_z(x_0,y_0,z_0)=-1$

③ 切平面方程为 $2x_0(x-x_0)+2y_0(y-y_0)-(z-z_0)=0$

∵ 切平面过点 (1,0,0)、(0,1,0)，切点 (x_0,y_0,z_0) 在曲面 $z=x^2+y^2$ 上

$\therefore\begin{cases}2x_0-2{x_0}^2-2{y_0}^2+z_0=0\\-2{x_0}^2+2y_0-2{y_0}^2+z_0=0\\z_0={x_0}^2+{y_0}^2\end{cases}\Longrightarrow\begin{cases}x_0=0\\y_0=0\\z_0=0\end{cases}$ 或 $\begin{cases}x_0=1\\y_0=1\\z_0=2\end{cases}$

第一个式子减第二个式子：$y_0=x_0$

将 $y_0=x_0$ 代入第三个式子：$z_0=2{x_0}^2$

将 $y_0=x_0$、$z_0=2{x_0}^2$ 代入第一个式子：

$x_0-{x_0}^2=0\Longrightarrow x_0=0$ 或 1

当 $x_0=0$ 时，$y_0=x_0=0$、$z_0=2{x_0}^2=0$

当 $x_0=1$ 时，$y_0=x_0=1$、$z_0=2{x_0}^2=2$

∴ 切平面方程为

$2\cdot0(x-0)+2\cdot0(y-0)-(z-0)=0$ 与 $2\cdot1(x-1)+2\cdot1(y-1)-(z-2)=0$

$\Longrightarrow z=0$ 与 $\Longrightarrow 2x+2y-z-2=0$

例5. 过直线 $\frac{x-1}{-1}=\frac{y}{1}=\frac{z}{0}$，且与曲面 $z=x^2+y^2$相切的平面为 (B)

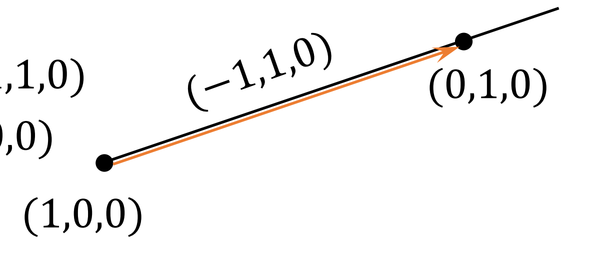
(A) z=0 与 x+y−z−1=0 (B) z=0与2x+2y−z−2=0

(C) x=y与x+y−z−1=0 (D) x=y与2x+2y−z−2=0

经过转化之后本题与例4是一样的，∴ 选(B)

方向向量为 (−1,1,0)

直线经过点 (1,0,0)



线绕坐标轴旋转所形成的曲面的方程

例1. 椭球面 S_1 是椭圆 $\frac{x^2}{4}+\frac{y^2}{3}=1$ 绕 x 轴旋转而成，求椭球面 S_1 的方程

① 原曲线方程补全为 $\begin{cases}\frac{x^2}{4}+\frac{y^2}{3}=1\\ z=0\end{cases}$

② $\begin{cases}y^2=3\left(1-\frac{x^2}{4}\right)=3-\frac{3x^2}{4}\\ z^2=0\end{cases}$

$y^2+z^2=3-\frac{3x^2}{4}+0$

$\Rightarrow y^2+z^2=3-\frac{3x^2}{4}$

$\Rightarrow \frac{x^2}{4}+\frac{y^2}{3}+\frac{z^2}{3}=1$

∴答案为 $\frac{x^2}{4}+\frac{y^2}{3}+\frac{z^2}{3}=1$

②结果中转轴未知数的取值范围	原方程中转轴未知数的取值范围
$\frac{x^2}{4}=1-\frac{y^2}{3}-\frac{z^2}{3}\Rightarrow x^2=4-\frac{4y^2}{3}-\frac{4z^2}{3}$ $\Rightarrow x^2\le 4$ $\Rightarrow -2\le x\le 2$	$\frac{x^2}{4}=1-\frac{y^2}{3}\Rightarrow x^2=4-\frac{4y^2}{3}$ $\Rightarrow x^2\le 4$ $\Rightarrow -2\le x\le 2$

例2. 求曲线 $\begin{cases}x=2\\ y^2+z^2=2\end{cases}$ 绕 y轴旋转一周所成曲面的方程

① 曲线方程 $\begin{cases}x=2\\ y^2+z^2=2\end{cases}$

② $\begin{cases}x^2=4\\ z^2=2-y^2\end{cases}$

$x^2+z^2=4+2-y^2$

$\Rightarrow x^2+y^2+z^2=6\ \ (-\sqrt{2}\le y\le\sqrt{2})$

③ ∴ 答案为 $x^2+y^2+z^2=6\ \ (-\sqrt{2}\le y\le\sqrt{2})$

②结果中转轴未知数的取值范围	原方程中转轴未知数的取值范围
$y^2=6-x^2-z^2$ $\Rightarrow y^2\le 6$ $\Rightarrow -\sqrt{6}\le y\le\sqrt{6}$	$y^2=2-z^2$ $\Rightarrow y^2\le 2$ $\Rightarrow -\sqrt{2}\le y\le\sqrt{2}$ 小的范围

例3. 求直线 $\begin{cases}x=3-t\\ y=-1+t\\ z=1+2t\end{cases}$ 绕 y 轴旋转一周所成曲面的方程

① 直线方程 $\begin{cases}x=3-t & (1)\\ y=-1+t & (2)\\ z=1+2t & (3)\end{cases}$

② 令 (1) +(2)

$x+y=3-t-1+t\Rightarrow x+y=2\Rightarrow x=2-y$

令2*(2) −(3)

$2y-z=-2+2t-1-2t$

$\Rightarrow 2y-z=-3\Rightarrow z=2y+3$

$\begin{cases}x^2=y^2-4y+4\\ z^2=4y^2+12y+9\end{cases}$

$x^2+z^2=y^2-4y+4+4y^2+12y+9$

$\Rightarrow x^2+z^2=5y^2+8y+13$

$\Rightarrow x^2-5y^2+z^2-8y-13=0$

∴ 答案为 $x^2-5y^2+z^2-8y-13=0$

②结果中转轴未知数的取值范围	原方程中转轴未知数的取值范围
$-5y^2-8y=13-x^2-z^2$ $\Rightarrow -5y^2-8y\le 13$ $\Rightarrow y^2+\frac{8}{5}y+\frac{13}{5}\ge 0$ $\Rightarrow y^2+2\cdot\frac{4}{5}y+\left(\frac{4}{5}\right)^2+\frac{49}{25}\ge 0$ $\Rightarrow \left(y+\frac{4}{5}\right)^2+\frac{49}{25}\ge 0$ $\Rightarrow y\in R$	t 为任意值 $\Rightarrow y\in R$

曲线在坐标平面上的投影柱面的方程

例1. 求曲线$\begin{cases} x^2+y^2+z^2=2 \\ z=x^2+y^2 \end{cases}$ 在 xOz 坐标面上的投影柱面

① $y^2=2-x^2-z^2$

$y^2=z-x^2$

$\therefore 2-x^2-z^2=z-x^2$

$\Rightarrow 2-z^2=z$

$\Rightarrow z^2+z-2=0$

$\Rightarrow (z-1)(z+2)=0$

在原曲线方程里有$z=x^2+y^2$

$\therefore z\geq 0$

$\therefore z+2>0$

$\Rightarrow z=1$

在①方程里:$\begin{cases} x\in R \\ z=1 \end{cases}$

单分析 $x^2+y^2+z^2=2 \Rightarrow \begin{cases} -\sqrt{2}\leq x\leq \sqrt{2} \\ -\sqrt{2}\leq z\leq \sqrt{2} \end{cases}$

$x^2+y^2+z^2=2$

$\Rightarrow x^2=2-y^2-z^2\leq 2$

$\Rightarrow -\sqrt{2}\leq x\leq \sqrt{2}$

$x^2+y^2+z^2=2$

$\Rightarrow z^2=2-x^2-y^2\leq 2$

$\Rightarrow -\sqrt{2}\leq z\leq \sqrt{2}$

单分析 $z=x^2+y^2 \quad \Rightarrow \begin{cases} x\in R \\ z\geq 0 \end{cases}$

两式合在一起分析 $\quad \Rightarrow \begin{cases} x\in R \\ z=1 \end{cases}$

将 $z=x^2+y^2$ 代入 $x^2+y^2+z^2=2$:

$\Rightarrow x^2+y^2+(x^2+y^2)^2=2$

$\because a^2+a-2=(a-1)(a+2)$

$\Rightarrow (x^2+y^2)^2+x^2+y^2-2=0$

$\Rightarrow (x^2+y^2-1)(x^2+y^2+2)=0$

$\Rightarrow x^2+y^2-1=0 \Rightarrow x^2=1-y^2 \begin{cases} -1\leq x\leq 1 \\ z\in R \end{cases}$

$x^2=1-y^2$

$\Rightarrow x^2\leq 1$

$\Rightarrow -1\leq x\leq 1$

综上,在曲线方程里:$\begin{cases} -1\leq x\leq 1 \\ z=1 \end{cases}$

② 所求投影柱面方程为 $z=1\ (-1\leq x\leq 1)$

例2. 设薄片型物体 S 是圆锥面 $z=\sqrt{x^2+y^2}$ 被柱面 $z^2=2x$割下的有限部分。记圆锥面与柱面的交线为 C，求 C 在 xOy 坐标面上的投影柱面

① C 的方程是$\begin{cases} z=\sqrt{x^2+y^2} \\ z^2=2x \end{cases}$

$\therefore x^2+y^2=2x$

$\Rightarrow x^2-2x=-y^2$

$\Rightarrow x^2-2x+1=-y^2+1$

$\Rightarrow (x-1)^2=-y^2+1\le 1$

$\Rightarrow -1\le x-1\le 1$

$\Rightarrow -1+1\le x-1+1\le 1+1$

$\Rightarrow 0\le x\le 2$

$x^2+y^2=2x$

$\Rightarrow y^2=2x-x^2$

$\Rightarrow y^2=2x-x^2-1+1$

$\Rightarrow y^2=-(x^2-2x+1)+1$

$\Rightarrow y^2=-(x-1)^2+1\le 1$

$\Rightarrow -1\le y\le 1$

在①方程里:$\begin{cases} 0\le x\le 2 \\ -1\le y\le 1 \end{cases}$

单分析 $z=\sqrt{x^2+y^2}$ $\begin{cases} x\in R \\ y\in R \end{cases}$

单分析 $z^2=2x$ $\begin{cases} x\ge 0 \\ y\in R \end{cases}$

$$\boxed{\begin{gathered} z^2=2x \\ \Rightarrow x=\frac{z^2}{2}\ge 0 \end{gathered}}$$

两式合在一起分析
$x^2+y^2=2x$ $\begin{cases} 0\le x\le 2 \\ -1\le y\le 1 \end{cases}$

综上，在曲线 C 里:$\begin{cases} 0\le x\le 2 \\ -1\le y\le 1 \end{cases}$

② 所求投影柱面方程为 $x^2+y^2=2x$

曲线在坐标平面上的投影曲线的方程

例1. 设薄片型物体 S 是圆锥面 $z=\sqrt{x^2+y^2}$ 被柱面 $z^2=2x$割下的有限部分。记圆锥面与柱面的交线为 C，求 C 在 xOy 坐标面上的投影曲线

在上一课里已求得曲线在 xOy 坐标面上的投影柱面为 $x^2+y^2=2x$

∴ 曲线在 xOy 坐标面上的投影曲线 $\begin{cases} x^2+y^2=2x \\ z=0 \end{cases}$