pag5.bmp

(4) a) Denote black and white with -1 and $+1$ respectively.

One-dimensional vector representation of pattern (a): $a = (-1, 1, -1, 1, -1, 1, -1, 1, -1)^T$

Hopfield: 9 neurons with bias $= 0$ (zero threshold)

w_{ij}: weight of link between neuron i and neuron j $(i, j = 1, 2, 3, ---, 9)$

Define weight matrix $W = (w_{ij})$: 9×9 matrix

Setting $W = a \cdot a^T - I$ yields a Hopfield that will recognize image (a).

This Hopfield will also recognize the negative of pattern (a): $(1, -1, 1, -1, 1, -1, 1, -1, 1)^T$

b) update rule:

- pick neuron i randomly
- if $\sum_j w_{ji} s_j >$ threshold then $s_i = +1$ else $s_i = -1$

Here: s_k is the state of neuron k

pag6.bmp

(4c) Cross-talk : network is not capable of distinguishing between two patterns because these patterns are too similar.

(4d) patterns (b) and (c) are orthogonal ~~and~~ (see below). therefore a Hopfield network will not suffer from crosstalk.

Two vectors are orthogonal $\Leftrightarrow$ inner product of the 2 vectors is zero.

$$(b, c) = \sum_{i=1}^{4} (b_i, c_i)$$ where b_i is 1×4 vector representing row i of pattern b

$$b_1 = c_1 \;, b_2 = c_2 \,, b_3 = -c_3 \,, b_4 = -c_4$$

$$\Rightarrow (b, c) = 4 + 4 + (-4) + (-4) = 0$$

pag7.bmp

(5) a) $\Gamma_1 = a = (2,0,0,0,2,0,0,0,2)^T$

$\Gamma_2 = b = (0,0,2,0,2,0,2,0,0)^T$

Average vector $\Psi = \frac{a+b}{2} = (1,0,1,0,2,0,1,0,1)^T$

$\phi_i = \Gamma_i - \Psi \Rightarrow \phi_1 = (1,0,-1,0,0,0,-1,0,1)^T$

$\phi_2 = (-1,0,1,0,0,0,1,0,-1)^T$

$A = \frac{1}{\sqrt{2}} (\phi_1 \;\; \phi_2)$: data matrix 9×2

Note: $A_1 = (\phi_1 \;\; \phi_2)$ is also OK. This yields other eigenvalues, but the same eigenvectors!

Covariance matrix $C = A \cdot A^T$: 9×9 matrix

Largest 2 eigenvalues of C are also eigenvalues of $D = A^T \cdot A$: 2×2 matrix.

$$D = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 \\ -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 1 & -1 \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

pag8.bmp

∴ Eigenvalues of D: $\begin{vmatrix} 2-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0$

$$\lambda^2 - 4\lambda + 4 - 4 = \lambda(\lambda - 4) = 0 \Rightarrow \lambda_1 = 4, \lambda_2 = 0$$

\# Eigenvector $\vec{v}$ of D corresponding to $\lambda_1 = 4$:

$$\begin{pmatrix} 2-4 & -2 \\ -2 & 2-4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Leftrightarrow v_1 = -v_2$$

Choose $\vec{v} = (1\ -1)^T$

\# Eigenvector $\vec{u}_1$ of C corresponding to $\lambda_1 = 4$

$$\vec{u}_1 = A \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} (2, 0, -2, 0, 0, 0, -2, 0, 2)^T$$

$$\|\vec{u}_1\| = \sqrt{\frac{1}{2}(4+4+4+4)} = \sqrt{8} = 2\sqrt{2}$$

\# Most important principal component $\vec{u}$: normalized eigenvector of C corresponding to largest eigenvalue

$$\vec{u} = \frac{\vec{u}_1}{\|\vec{u}_1\|} = \frac{\sqrt{2}\,(1, 0, -1, 0, 0, 0, -1, 0, 1)^T}{2\sqrt{2}} =$$

$$= \frac{1}{2} (1, 0, -1, 0, 0, 0, -1, 0, 1)^T$$

Note: * $-\vec{u}$ is also normalized eigenvector of C corresponding to $\lambda_1 = 4$

* Choosing $\vec{v} = (-1\ +1)^T$ leads to $-\vec{u}$

pag9.bmp

5b) Approximation $\hat{c}$ of (c) in the space formed by the principal component $\vec{u}$.

$$\hat{c} = \underset{\substack{\uparrow \\ \text{average vector}}}{\Psi} + w_c \cdot \vec{u} \quad \text{with } w_c = \vec{u}^T (\Gamma_3 - \Psi)$$

$\hat{c}$: 9×1 vector

#

$$\Gamma_3 = c = (0,2,0,0,2,0,0,2,0)^T$$

$$\Gamma_3 - \Psi = (-1,2,-1,0,0,0,-1,2,-1)^T$$

$$w_c = \frac{1}{2}(1,0,-1,0,0,0,-1,0,1)\begin{pmatrix}-1\\2\\-1\\0\\0\\0\\-1\\2\\-1\end{pmatrix} = \frac{1}{2}(-1+1+1-1) = 0$$

$$\hat{c} = \Psi = (1,0,1,0,2,0,1,0,1)^T$$

#

Error made by approximating (c) with $\hat{c}$:

$$\|c - \hat{c}\| = \|(-1,2,-1,0,0,0,-1,2,-1)^T\| =$$

$$= \sqrt{1+4+1+1+4+1} = \sqrt{12} = 2\sqrt{3}$$

pag10.bmp

5c In order to determine whether (a) or (b) is closest to (c) we need to calculate the distance of (c) to (a) and (b) within the face space.

\#

First calculate the approximation $\tilde{a}$ of image (a) in the face space.

$$\tilde{a} = \psi + w_a \cdot \vec{u} \quad \text{with } w_a = \vec{u}^T(a - \psi)$$

$$w_a = \tfrac{1}{2}(1,0,-1,0,0,0,-1,0,1) \cdot (1,0,-1,0,0,0,-1,0,1)^T = \tfrac{1}{2} \cdot 4 = +2$$

\#

Approximation $\hat{b}$ of image (b) in face space.

$$\hat{b} = \psi + w_b \cdot \vec{u} \quad \text{with } w_b = \vec{u}^T(b - \psi)$$

$$w_b = \tfrac{1}{2}(1,0,-1,0,0,0,-1,0,1) \cdot (-1,0,1,0,0,0,1,0,-1)^T = \tfrac{1}{2}(-4) = -2$$

\#

Mahalonabis distance of $\tilde{c}$ to $\tilde{a}$:

$$\frac{1}{\lambda_1} \cdot (w_a - w_c)^2 = \frac{1}{4} \cdot (2-0)^2 = +1$$

Mahalonabis distance of $\hat{c}$ to $\hat{b}$:

$$\frac{1}{\lambda_1}(w_b - w_c)^2 = \frac{1}{4}(-2-0)^2 = +1$$

Thus: in the face space image (c) is equally close to (a) and (b).

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5d) First calculate approximation $\tilde{d}$ of image (d) in the face space

$$\tilde{d} = \Psi + w_d \cdot \vec{u} \quad \text{with} \quad w_d = \vec{u}^T (d - \Psi)$$

$$w_d = \tfrac{1}{2}(1,0,-1,0,0,0,-1,0,1) \cdot (0,1,-1,2,0,2,-1,0,-1)^T =$$

$$= \tfrac{1}{2}(1+1-1) = \tfrac{1}{2}$$

#

Mahalanobis distance of $\tilde{d}$ to $\hat{a}$:

$$\frac{1}{\lambda_1}(w_a - w_d)^2 = \frac{1}{4}\left(2 - \frac{1}{2}\right)^2 = \frac{9}{16}$$

Mahalanobis distance of $\tilde{d}$ to $\hat{b}$:

$$\frac{1}{\lambda_1}(w_b - w_d)^2 = \frac{1}{4}\left(-2 - \frac{1}{2}\right)^2 = \frac{25}{16}$$

Thus: in the face space image (d) is closer to (a).