Summer Project 2022 Group 4(a)

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1 ODEs

1.1 4a

$$\frac{dx}{dt} = x\left(\frac{nx}{\theta + x} - \delta - \frac{x}{k}\right) - \frac{mxy}{ax + by + c}$$

$$\frac{dy}{dt} = \frac{emxy}{ax + by + c} - dy - hy^2$$

1.2 4b

$$\frac{dx}{dt} = x\left(\frac{nx}{\theta + x} - \delta - \frac{x}{k}\right) - \frac{mxy}{ax + by + c}$$

$$\frac{dy}{dt} = \frac{emxy}{ax + by + c} - dy$$

2 Equilibrium

2.1 4a

$E_0 = (0, 0)$

$E_1 = (0, -\frac{d}{h})$(Which may be excluded: See section 3)

$E_2 = \left(\frac{k\,n}{2} - \frac{\delta\,k}{2} - \frac{\theta}{2} - \frac{\sqrt{\delta^2\,k^2 - 2\,\delta\,k^2\,n - 2\,\delta\,k\,\theta + k^2\,n^2 - 2\,k\,n\,\theta + \theta^2}}{2}, 0\right)$

$E_3 = (\frac{k\,n}{2} - \frac{\delta\,k}{2} - \frac{\theta}{2} + \frac{\sqrt{\delta^2\,k^2 - 2\,\delta\,k^2\,n - 2\,\delta\,k\,\theta + k^2\,n^2 - 2\,k\,n\,\theta + \theta^2}}{2}, 0)$

Since it is quite confusing to output the later five points(E_4 to E_8) via long and verbose formulas, we provide the polynomial function from which the five solutions come:

$$P(x) = \sum_{i=0}^{7} a_i x^i$$

where the coefficients a_i's are:

$$\begin{aligned}
a_0 =& - h\,c^2\,\delta^2\,k^3\,\theta^3 + b\,d\,c\,\delta^2\,k^3\,\theta^3 + d\,m\,c\,\delta\,k^3\,\theta^3 \\
a_1 =& - e\,b^2\,\delta^3\,k^3\,\theta^3 + 3\,d\,b\,c\,\delta^2\,k^3\,\theta^2 - 2\,d\,n\,b\,c\,\delta\,k^3\,\theta^2 + 2\,d\,b\,c\,\delta\,k^2\,\theta^3 - 2\,e\,b\,\delta^2\,k^3\,m\,\theta^3 + a\,d\,b\,\delta^2\,k^3\,\theta^3 \\
& - 3\,h\,c^2\,\delta^2\,k^3\,\theta^2 + 2\,h\,n\,c^2\,\delta\,k^3\,\theta^2 - 2\,h\,c^2\,\delta\,k^2\,\theta^3 - 2\,a\,h\,c\,\delta^2\,k^3\,\theta^3 + 3\,d\,c\,\delta\,k^3\,m\,\theta^2 \\
& - d\,n\,c\,k^3\,m\,\theta^2 + d\,c\,k^2\,m\,\theta^3 - e\,\delta\,k^3\,m^2\,\theta^3 + a\,d\,\delta\,k^3\,m\,\theta^3 \\
a_2 =& - h\,a^2\,\delta^2\,k^3\,\theta^3 + 3\,d\,a\,b\,\delta^2\,k^3\,\theta^2 - 2\,d\,a\,b\,\delta\,k^3\,n\,\theta^2 + 2\,d\,a\,b\,\delta\,k^2\,\theta^3 - 6\,h\,a\,c\,\delta^2\,k^3\,\theta^2 + 4\,h\,a\,c\,\delta\,k^3\,n\,\theta^2 \\
& - 4\,h\,a\,c\,\delta\,k^2\,\theta^3 + 3\,d\,a\,\delta\,k^3\,m\,\theta^2 - d\,a\,k^3\,m\,n\,\theta^2 + d\,a\,k^2\,m\,\theta^3 - 3\,e\,b^2\,\delta^3\,k^3\,\theta^2 + 3\,e\,b^2\,\delta^2\,k^3\,n\,\theta^2 \\
& - 3\,e\,b^2\,\delta^2\,k^2\,\theta^3 + 3\,d\,b\,c\,\delta^2\,k^3\,\theta - 4\,d\,b\,c\,\delta\,k^3\,n\,\theta + 6\,d\,b\,c\,\delta\,k^2\,\theta^2 + d\,b\,c\,k^3\,n^2\,\theta - 2\,d\,b\,c\,k^2\,n\,\theta^2 + d\,b\,c\,k\,\theta^3 \\
& - 6\,e\,b\,\delta^2\,k^3\,m\,\theta^2 + 4\,e\,b\,\delta\,k^3\,m\,n\,\theta^2 - 4\,e\,b\,\delta\,k^2\,m\,\theta^3 - 3\,h\,c^2\,\delta^2\,k^3\,\theta + 4\,h\,c^2\,\delta\,k^3\,n\,\theta - 6\,h\,c^2\,\delta\,k^2\,\theta^2 \\
& - h\,c^2\,k^3\,n^2\,\theta + 2\,h\,c^2\,k^2\,n\,\theta^2 - h\,c^2\,k\,\theta^3 + 3\,d\,c\,\delta\,k^3\,m\,\theta - 2\,d\,c\,k^3\,m\,n\,\theta + 3\,d\,c\,k^2\,m\,\theta^2 - 3\,e\,\delta\,k^3\,m^2\,\theta^2 \\
& + e\,k^3\,m^2\,n\,\theta^2 - e\,k^2\,m^2\,\theta^3 \\
a_3 =& - 3\,h\,a^2\,\delta^2\,k^3\,\theta^2 + 2\,h\,a^2\,\delta\,k^3\,n\,\theta^2 - 2\,h\,a^2\,\delta\,k^2\,\theta^3 + 3\,d\,a\,b\,\delta^2\,k^3\,\theta - 4\,d\,a\,b\,\delta\,k^3\,n\,\theta + 6\,d\,a\,b\,\delta\,k^2\,\theta^2 \\
& + d\,a\,b\,k^3\,n^2\,\theta - 2\,d\,a\,b\,k^2\,n\,\theta^2 + d\,a\,b\,k\,\theta^3 - 6\,h\,a\,c\,\delta^2\,k^3\,\theta + 8\,h\,a\,c\,\delta\,k^3\,n\,\theta - 12\,h\,a\,c\,\delta\,k^2\,\theta^2 - 2\,h\,a\,c\,k^3\,n^2\,\theta \\
& + 4\,h\,a\,c\,k^2\,n\,\theta^2 - 2\,h\,a\,c\,k\,\theta^3 + 3\,d\,a\,\delta\,k^3\,m\,\theta - 2\,d\,a\,k^3\,m\,n\,\theta + 3\,d\,a\,k^2\,m\,\theta^2 - 3\,e\,b^2\,\delta^3\,k^3\,\theta + 6\,e\,b^2\,\delta^2\,k^3\,n\,\theta \\
& - 9\,e\,b^2\,\delta^2\,k^2\,\theta^2 - 3\,e\,b^2\,\delta\,k^3\,n^2\,\theta + 6\,e\,b^2\,\delta\,k^2\,n\,\theta^2 - 3\,e\,b^2\,\delta\,k\,\theta^3 + d\,b\,c\,\delta^2\,k^3 - 2\,d\,b\,c\,\delta\,k^3\,n + 6\,d\,b\,c\,\delta\,k^2\,\theta \\
& + d\,b\,c\,k^3\,n^2 - 4\,d\,b\,c\,k^2\,n\,\theta + 3\,d\,b\,c\,k\,\theta^2 - 6\,e\,b\,\delta^2\,k^3\,m\,\theta + 8\,e\,b\,\delta\,k^3\,m\,n\,\theta - 12\,e\,b\,\delta\,k^2\,m\,\theta^2 - 2\,e\,b\,k^3\,m\,n^2\,\theta \\
& + 4\,e\,b\,k^2\,m\,n\,\theta^2 - 2\,e\,b\,k\,m\,\theta^3 - h\,c^2\,\delta^2\,k^3 + 2\,h\,c^2\,\delta\,k^3\,n - 6\,h\,c^2\,\delta\,k^2\,\theta - h\,c^2\,k^3\,n^2 + 4\,h\,c^2\,k^2\,n\,\theta \\
& - 3\,h\,c^2\,k\,\theta^2 + d\,c\,\delta\,k^3\,m - d\,c\,k^3\,m\,n + 3\,d\,c\,k^2\,m\,\theta - 3\,e\,\delta\,k^3\,m^2\,\theta + 2\,e\,k^3\,m^2\,n\,\theta - 3\,e\,k^2\,m^2\,\theta^2 \\
a_4 =& - 3\,h\,a^2\,\delta^2\,k^3\,\theta + 4\,h\,a^2\,\delta\,k^3\,n\,\theta - 6\,h\,a^2\,\delta\,k^2\,\theta^2 - h\,a^2\,k^3\,n^2\,\theta + 2\,h\,a^2\,k^2\,n\,\theta^2 - h\,a^2\,k\,\theta^3 + d\,a\,b\,\delta^2\,k^3 \\
& - 2\,d\,a\,b\,\delta\,k^3\,n + 6\,d\,a\,b\,\delta\,k^2\,\theta + d\,a\,b\,k^3\,n^2 - 4\,d\,a\,b\,k^2\,n\,\theta + 3\,d\,a\,b\,k\,\theta^2 - 2\,h\,a\,c\,\delta^2\,k^3 + 4\,h\,a\,c\,\delta\,k^3\,n \\
& - 12\,h\,a\,c\,\delta\,k^2\,\theta - 2\,h\,a\,c\,k^3\,n^2 + 8\,h\,a\,c\,k^2\,n\,\theta - 6\,h\,a\,c\,k\,\theta^2 + d\,a\,\delta\,k^3\,m - d\,a\,k^3\,m\,n + 3\,d\,a\,k^2\,m\,\theta \\
& - e\,b^2\,\delta^3\,k^3 + 3\,e\,b^2\,\delta^2\,k^3\,n - 9\,e\,b^2\,\delta^2\,k^2\,\theta - 3\,e\,b^2\,\delta\,k^3\,n^2 + 12\,e\,b^2\,\delta\,k^2\,n\,\theta - 9\,e\,b^2\,\delta\,k\,\theta^2 + e\,b^2\,k^3\,n^3 \\
& - 3\,e\,b^2\,k^2\,n^2\,\theta + 3\,e\,b^2\,k\,n\,\theta^2 - e\,b^2\,\theta^3 + 2\,d\,b\,c\,\delta\,k^2 - 2\,d\,b\,c\,k^2\,n + 3\,d\,b\,c\,k\,\theta - 2\,e\,b\,\delta^2\,k^3\,m + 4\,e\,b\,\delta\,k^3\,m\,n \\
& - 12\,e\,b\,\delta\,k^2\,m\,\theta - 2\,e\,b\,k^3\,m\,n^2 + 8\,e\,b\,k^2\,m\,n\,\theta - 6\,e\,b\,k\,m\,\theta^2 - 2\,h\,c^2\,\delta\,k^2 + 2\,h\,c^2\,k^2\,n - 3\,h\,c^2\,k\,\theta \\
& + d\,c\,k^2\,m - e\,\delta\,k^3\,m^2 + e\,k^3\,m^2\,n - 3\,e\,k^2\,m^2\,\theta \\
a_5 =& - h\,a^2\,\delta^2\,k^3 + 2\,h\,a^2\,\delta\,k^3\,n - 6\,h\,a^2\,\delta\,k^2\,\theta - h\,a^2\,k^3\,n^2 + 4\,h\,a^2\,k^2\,n\,\theta - 3\,h\,a^2\,k\,\theta^2 + 2\,d\,a\,b\,\delta\,k^2 \\
& - 2\,d\,a\,b\,k^2\,n + 3\,d\,a\,b\,k\,\theta - 4\,h\,a\,c\,\delta\,k^2 + 4\,h\,a\,c\,k^2\,n - 6\,h\,a\,c\,k\,\theta + d\,a\,k^2\,m - 3\,e\,b^2\,\delta^2\,k^2 + 6\,e\,b^2\,\delta\,k^2\,n \\
& - 9\,e\,b^2\,\delta\,k\,\theta - 3\,e\,b^2\,k^2\,n^2 + 6\,e\,b^2\,k\,n\,\theta - 3\,e\,b^2\,\theta^2 + d\,b\,c\,k - 4\,e\,b\,\delta\,k^2\,m + 4\,e\,b\,k^2\,m\,n - 6\,e\,b\,k\,m\,\theta \\
& - h\,c^2\,k - e\,k^2\,m^2 \\
a_6 =& 3\,b^2\,e\,k\,n - 3\,b^2\,\delta\,e\,k - 3\,b^2\,e\,\theta - 3\,a^2\,h\,k\,\theta - 2\,a^2\,\delta\,h\,k^2 + 2\,a^2\,h\,k^2\,n + a\,b\,d\,k - 2\,a\,c\,h\,k - 2\,b\,e\,k\,m \\
a_7 =& - h\,k\,a^2 - e\,b^2
\end{aligned}$$

The $P(x)$ has a degree of 7, but we only have 5 solutions in $\mathbb{R}$, which means there must exist 2 roots in $(\mathbb{C}-\mathbb{R})$.

3 Analysis of Stability

3.1 Jacobian

$$J=\begin{pmatrix} \frac{n\,x}{\theta+x}-x\left(\frac{1}{k}-\frac{n}{\theta+x}+\frac{n\,x}{(\theta+x)^2}\right)-\frac{x}{k}-\frac{m\,y}{c+a\,x+b\,y}-\delta+\frac{a\,m\,x\,y}{(c+a\,x+b\,y)^2} & \frac{b\,m\,x\,y}{(c+a\,x+b\,y)^2}-\frac{m\,x}{c+a\,x+b\,y} \\ \frac{e\,m\,y}{c+a\,x+b\,y}-\frac{a\,e\,m\,x\,y}{(c+a\,x+b\,y)^2} & \frac{e\,m\,x}{c+a\,x+b\,y}-2\,h\,y-d-\frac{b\,e\,m\,x\,y}{(c+a\,x+b\,y)^2} \end{pmatrix}$$

$$J_0=\begin{pmatrix} -\delta & 0 \\ 0 & -d \end{pmatrix}$$

$$J_1=\begin{pmatrix} \frac{d\,m}{h\left(c-\frac{b\,d}{h}\right)}-\delta & 0 \\ -\frac{d\,e\,m}{h\left(c-\frac{b\,d}{h}\right)} & d \end{pmatrix}$$

$$J_2=\begin{pmatrix} j_{21} & \frac{m\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{c-a\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)} \\ 0 & -d-\frac{e\,m\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{c-a\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)} \end{pmatrix}$$

Where

$$j_{21}=\frac{\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}}{k}$$
$$-\left(\frac{n}{\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}}-\frac{1}{k}+\frac{n\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{\left(\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)^2}\right)$$
$$\cdot\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)-\delta-\frac{n\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}}$$

$$J_3=\begin{pmatrix} j_{31} & \frac{m\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{c-a\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)} \\ 0 & -d-\frac{e\,m\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{c-a\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)} \end{pmatrix}$$

Where

$$j_{31} = \frac{\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}}{k}$$
$$-\left(\frac{n}{\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}}-\frac{1}{k}+\frac{n\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{\left(\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)^2}\right)$$
$$\cdot\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)-\delta-\frac{n\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}}$$

Note that we do not simplify those expressions here by substitutions since we will *not* actually deal with the Jacobian matrices, but their eigenvalues, which can be obtained via Matlab.

3.2 eigenvalues

$$Ei_0 = \{-d, -\delta\}$$

$$Ei_1 = \left\{d, \frac{d\,m}{h\left(c-\frac{b\,d}{h}\right)}-\delta\right\}$$

$$Ei_2 = \{ei_1, ei_2\}$$

$$ei_1 = -d-\frac{e\,m\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{c-a\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}$$

$$ei_2 = \frac{\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}}{k}-$$
$$\left(\frac{n}{\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}}-\frac{1}{k}+\frac{n\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{\left(\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)^2}\right)\cdot$$
$$\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)-\delta-\frac{n\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}}$$

$$Ei_3 = \{ei_3, ei_4\}$$

$$ei_3 = -d-\frac{e\,m\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{c-a\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}$$

$$
\begin{aligned}
ei_4 = {} & \frac{\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}}{k} - \\
& \left(\frac{n}{\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}} - \frac{1}{k} + \frac{n\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{\left(\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)^2} \right) \cdot \\
& \left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right) - \delta - \frac{n\left(\frac{\theta}{2}+\frac{\delta\,k}{2}-\frac{k\,n}{2}-\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}\right)}{\frac{\theta}{2}-\frac{\delta\,k}{2}+\frac{k\,n}{2}+\frac{\sqrt{\delta^2\,k^2-2\,\delta\,k^2\,n-2\,\delta\,k\,\theta+k^2\,n^2-2\,k\,n\,\theta+\theta^2}}{2}}
\end{aligned}
$$

3.2.1 The behaviour of the system around $E_0(0,0)$

Let J_0 be the Jacobian matrix of the system 4a at equilibrium point E_0, then its eigenvalues are $-d$ and $-\delta$. Since both eigenvalues are strictly negative, we draw the following conclusion:

$E_0(0,0)$ *is always locally asymptotically stable.*

Then we discuss the global stability of E_0. Let $\mathbb{R}^2_{+\mathbb{X}} = [(x,y) \in \mathbb{R}^2 : x > 0, y \geq 0]$ and consider the Lyapunov function $L_0 : \mathbb{R}^2 \to \mathbb{R}$,

$$
V(x,y) = \frac{1}{2}(ex+y)^2 \tag{4a.0}
$$

Note that $V(0,0) = 0$ and $V(x,y) > 0$ in any neighbourhood of the point $E_0(0,0)$. Morever, we have:

$$
\begin{aligned}
\frac{dV}{dt} &= -(ex+y)(hy^2+dy-\frac{emxy}{ax+by+c}) - e(ex+y)\left(x(\delta+\frac{x}{k}-\frac{nx}{\theta+k})+\frac{mxy}{ax+by+c}\right) \\
&= -(ex+y)(hy^2+dy) - \frac{ex(ex+y)}{k(x+\theta)}\left(x^2+(\delta k+\theta-kn)x+\delta k\theta\right) \\
&\leq 0, \quad if\left(x^2+(\delta k+\theta-kn)x+\delta k\theta\right) \geq 0.
\end{aligned}
$$

Then we discuss the sign of the function $f(x) = \left(x^2+(\delta k+\theta-kn)x+\delta k\theta\right)$.
On the one hand, if $\delta k+\theta-kn \leq 0$, i.e., $k \geq \frac{\theta}{n-\delta}$, then $f(x) \geq 0$ will always hold since $f(0) = \delta k\theta$ is always greater than 0.
On the other hand, if $\delta k+\theta-kn \geq 0$, i.e., $n < \delta$ or $(n > \delta) \wedge (k < \frac{\theta}{n-\delta})$, then $f(x) \geq 0$ will always hold if the determinant of $f(x)$ such that $\Delta < 0$.

Therefore, the proof follows from the Lyapunov–LaSalle's invariance principle.

3.2.2 The behaviour of the system around $E_1(0,-\frac{d}{h})$

Let J_1 be the Jacobian matrix of the system 4a at equilibrium point E_1, then its eigenvalues are d and $\frac{d\,m}{h\left(c-\frac{b\,d}{h}\right)} - \delta$. Since there is an eigenvalue d which is always positive, the system is always unstable around E_1. More precisely, E_1 is a saddle point if $m < \frac{\delta\,h\left(c-\frac{b\,d}{h}\right)}{d}$; E_1 is a node if $m > \frac{\delta\,h\left(c-\frac{b\,d}{h}\right)}{d}$.

However, the equilibrium point E_1 is meaningless under the **practical biologic conditions** that both x and y must be positive or at least, greater or equal to 0, i.e., the number of predator or prey cannot be a negative value. Thus, we may exclude E_1.

3.2.3 The behaviour of the system around E_2 and E_3

To make the article more clear, some substitutions are introduced as follows:

$$\begin{aligned}
\Delta_0 &= \delta^2\,k^2 - 2\,\delta\,k^2\,n - 2\,\delta\,k\,\theta + k^2\,n^2 - 2\,k\,n\,\theta + \theta^2 \\
\Delta_1 &= \frac{\theta}{2} + \left(\frac{\delta\,k}{2} - \frac{k\,n}{2} + \frac{\sqrt{\Delta_0}}{2}\right) \\
\Delta_2 &= \frac{\theta}{2} - \left(\frac{\delta\,k}{2} - \frac{k\,n}{2} + \frac{\sqrt{\Delta_0}}{2}\right)
\end{aligned}$$

Thus, the eigenvalues of E_2 can be written as $Ei_2 = \{ei_1 = -d - \frac{em\Delta_1}{c-a\Delta_1}$, $ei_2 = \frac{\Delta_1}{k} - \left(\frac{n}{\Delta_2} - \frac{1}{k} + \frac{n\Delta_1}{\Delta_2^2}\right)\Delta_1 - \delta - \frac{n\Delta_1}{\Delta_2}\}$. We have the following conclusion:

*E_2 is locally **unstable**.*
Similarly, so as E_3.

Proof.

To make the equilibrium point have a practical meaning, we assume that $\frac{k\,n}{2} - \frac{\delta\,k}{2} - \frac{\theta}{2} - \frac{\sqrt{\delta^2\,k^2 - 2\,\delta\,k^2\,n - 2\,\delta\,k\,\theta + k^2\,n^2 - 2\,k\,n\,\theta + \theta^2}}{2} > 0$, which implies that $\frac{kn}{2} > \frac{\delta k + \theta + \sqrt{\Delta_0}}{2}$. Thus, we have $\Delta_1 < \left(\frac{\theta}{2} + \frac{\delta k}{2} + \frac{\sqrt{\Delta_0}}{2}\right) - \frac{\delta k + \theta + \sqrt{\Delta_0}}{2} = 0$; $\Delta_2 > \left(\frac{\theta}{2} - \frac{\delta k}{2} - \frac{\sqrt{\Delta_0}}{2}\right) + \frac{\delta k + \theta + \sqrt{\Delta_0}}{2} = \theta > 0$.

For all $\Delta_0 \geq 0$, then $\Delta_1, \Delta_2 \in \mathbb{R}$, which implies that $ei_1, ei_2 \in \mathbb{R}$. Therefore, E_2 is locally stable if $ei_1 < 0$ and $ei_2 < 0$. For ei_1, $ei_1 < 0 \Leftrightarrow m < \frac{-d(c-a\Delta_1)}{e\Delta_1}$. For ei_2, we rewrite it into $ei_2 = -n\left(\frac{\Delta_1}{\Delta_2}\right)^2 - 2n\left(\frac{\Delta_1}{\Delta_2}\right) + \frac{2\Delta_1}{k} - \delta$. Let $t = \frac{\Delta_1}{\Delta_2}$ and $ei_2 = f(t) = -n^2 t - 2nt + \frac{2\Delta_1}{k} - \delta$ such that $t < 0$. Since $-n < 0$ and $-\frac{-2n}{2(-n)} = -1 < 0$, to make $ei_2 < 0$, we need to make the determinant of $f(t)$ less than 0, i.e., $\Delta = (-2n)^2 - 4(-n)(\frac{2\Delta_1}{k} - \delta) < 0 \Leftrightarrow 4n(n + \frac{2\Delta_1}{k} - \delta) < 0$. However, $n + \frac{2\Delta_1}{k} - \delta = n + (\frac{\theta}{k} + \delta - n + \frac{\sqrt{\Delta_0}}{k}) - \delta = \frac{\theta}{k} + \frac{\sqrt{\Delta_0}}{k} > 0$. That is, the determinant cannot be negative, so, the two eigenvalues cannot always be both negative.

By the homogeneity of the two equilibrium points, E_3 is always unstable as well.

4 Numerical Solutions

We produced the numerical solutions of the point E_2 as well as E_3 which are remain fruitless. We use the Matlab script with the parameters introduced in the article 'Mainul MB 2011'. The following figure shows the result of E_2.

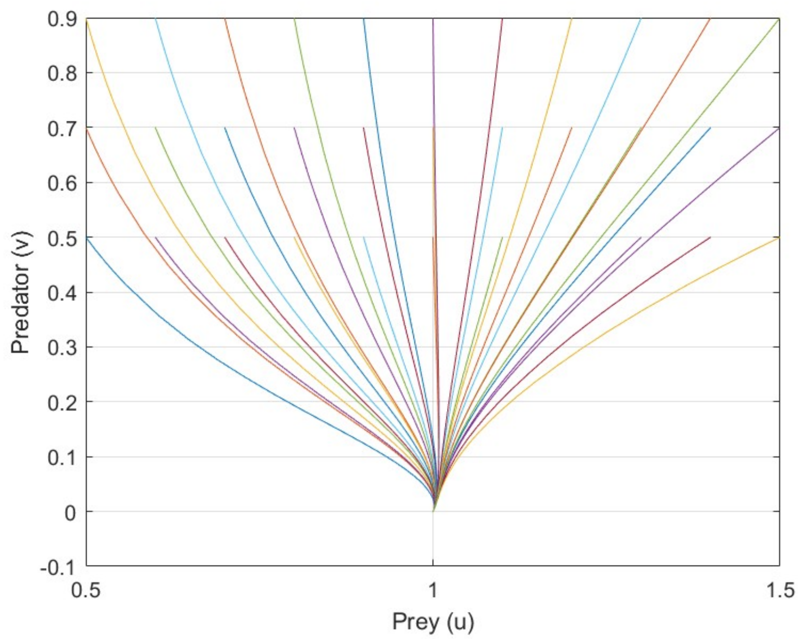

Figure 1: Numerical Solution of E_2

In this graph, its Prey(u) is our x, its Predator(v) is our y in the ODEs. The figure show that the solutions converge to the point $(1, 0)$.