CAVEAT LECTOR:

THIS DOCUMENT CONTAINS ERRORS

Like, seriously, loads of them. No I do not know where they are. These attempts at solutions are in no way guaranteed to be accurate. Please do not read this and assume that you're totally correct/incorrect just because you agree/disagree with the stuff that's written in the big shiny LaTeX document. Rather, think as you read through it. Follow my logic, and if it doesn't make sense to you then ask me. Email me[1], grab me on Steam, Skype, League of Legends[2], facebook, bang on my front door repeatedly[3], just ask if something doesn't make sense, because I can and do make errors.

Of course, if you know what the error is and how to fix it, then feel free to write up an alternative solution of your own, or even an entire solution to a problem that I haven't yet attempted. I'll be happy to include it and give you credit for writing it. Either write it on paper, scan it to me, and email me[1], or for bonus points send me correct LaTeX that I can just copy paste in, or for super bonus points, fork me on github[4] and push a revision at me.

Right, now that that's all out of the way, let's do some maths.

[1] `ymbirtt@gmail.com`
[2] Ymbirtt on all
[3] Bring enough cake for me and my flatmates
[4] `https://github.com/Ymbirtt/Applied-Probability-II`

2011 paper

1. (a) Let X_i denote the number of descendents of child i from generation $j-1$. This gives us that

$$N_j = X_1 + \ldots + X_{N_j - 1}$$

Where K is a non-negative integer valued random variable, $\{Y_i\}$ is a sequence of independent identically distributed random variables, $Z = Y_1 + \cdots + Y_K$ is their sum, and $G_Z(s)$ is the probability generating function for Z, similarly for K and Y, we have that

$$G_Z(s) = G_N(G_Y(s))$$

In this question,

$$N_j = \sum_{i=1}^{N_j - 1} X_j$$

Each X_j is the number of offspring produced by a single parent, each distributed identically to N_1, so

$$G_j(s) = G_{j-1}(G_1(s))$$

i. If a generation has no children, then the next generation will also be childless, so,

$$N_j = 0 \implies N_{j+1} = 0$$

Eventual extinction corresponds to

$$\{\exists n \in \mathbb{N} \ st \ N_j = 0\} = \bigcup_{j=1}^{\infty} \{N_j = 0\}$$

By continuity of probability, for any increasing sequence of events such that $A_j \subseteq A_{j+1}$, we have that

$$\mathbb{P}(\bigcup_{j=1}^{\infty} A_j) = \lim_{j \to \infty} \mathbb{P}(A_j)$$

So, finally,

$$e = \mathbb{P}(\bigcup_{j=1}^{\infty} \{N_j = 0\}) = \lim_{j \to \infty} \mathbb{P}(N_j = 0) = \lim_{j \to \infty} e_j$$

$$e = \lim_{j \to \infty} e_j$$

ii. $G_X(s) = \sum_{x \in \mathcal{X}} \mathbb{P}(X = i) s^i$, so, $G_X(0) = \mathbb{P}(X = 0)$. We also have that $G_j(G_1(s)) = G_1(G_j(s))$.

$$\begin{aligned} e &= lim_{j \to \infty} G_j(0) \\ &= lim_{j \to \infty} G_1(G_{j-1}(0)) \quad \text{by continuity of G} \\ &= G_1(lim_{j \to \infty} G_{j-1}(0)) \\ &= G_1(e) \end{aligned}$$

So $e = G_1(e)$.

(b) i.

$$\mathbb{E}(X) = \frac{d}{ds} G_X(s)|_{s=0}$$

$$\begin{aligned}
G_j(s) &= G_{j-1}(G(s)) \\
\mathbb{E}(N_j) &= \frac{d}{ds} G_j(s)|_{s=0} \\
&= \frac{d}{ds} G_{j-1}(G_1(s))|_{s=0} \\
&= G_1'(s) G_{j-1}'(G_1(s))|_{s=0} \\
&= \mu G_{j-1}'(G_1(0)) \\
&= \mu G_{j-1}'(0) \quad \text{since } G_1(0) = \mathbb{P}(X_1 = 0) = 0 \\
&= \mu \mathbb{E}(N_{j-i})
\end{aligned}$$

So, $\mathbb{E}(N_1) = \mu$ and $\mathbb{E}(N_j) = \mu \mathbb{E}(N_{j-1})$. Induction will then give us that $\mathbb{E}(N_j) = \mu^j$.

2. (a) A continuous time counting process with stationary, independent increments, where the number of increments in time t follows a $Po(\lambda t)$ distribution.

(b)

$$\begin{aligned}
p_n(t+h) &= \mathbb{P}(N(t+n)=n|N(t)=n-1)\mathbb{P}(N(t)=n-1)+\mathbb{P}(N(t+n)=n|N(t)=n)\mathbb{P}(N(t)=n)+o(h)\\
&= (1-\lambda h+o(h))p_n(t)+(\lambda h+o(h))p_{n-1}(t)+o(h)\\
&= p_n(t)-\lambda h p_n(t)+\lambda h p_{n-1}(t)+o(h)\\
&\implies \frac{p_n(t+h)-p_n(t)}{h} = -\lambda(p_n(t)-p_{n-1}(t))\\
&\implies p_n'(t) = -\lambda(p_n(t)-p_{n-1}(t))
\end{aligned}$$

Where $N(t)\sim Po(\lambda t)$, $\mathbb{P}(N(t)=n)=\frac{e^{-\lambda t}(\lambda t)^n}{n!}$, so

$$\begin{aligned}
p_n'(t) &= \frac{d}{dt}\frac{e^{-\lambda t}(\lambda t)^n}{n!}\\
&= \frac{-\lambda e^{-\lambda t}(\lambda t)^n+n\lambda e^{-\lambda t}(\lambda t)^{n-1}}{n!}\\
&= -\lambda\left(\frac{e^{-\lambda t}(\lambda t)^n}{n!}-\frac{e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}\right)\\
&= -\lambda(p_n(t)-p_{n-1}(t))
\end{aligned}$$

And, where $n=0$, $p_{-1}(t)=0$, so

$$p_0'(t)=-\lambda p_0(t)$$

(c) Let $p_n(t)=\mathbb{P}(N(t)=n)$, as per usual.

$$\begin{aligned}
p_n(t+h) &= \mathbb{P}(N(t+h)=n|N(t)=n)\mathbb{P}(N(t)=n)+\mathbb{P}(N(t+h)=n|N(t)=n-1)\mathbb{P}(N(t)=n-1)+o(h)\\
&= \mathbb{P}(\text{no decay})\mathbb{P}(N(t)=n)+\mathbb{P}(\text{decay})\mathbb{P}(\text{not detected})\mathbb{P}(N(t)=n)\\
&\qquad +\mathbb{P}(\text{decay})\mathbb{P}(\text{detected})\mathbb{P}(N(t)=n-1)+o(h)\\
&= (1-\mu h+o(h))p_n(t)+(\mu h+o(h))(1-p)p_n(t)+(\mu h+o(h))(p)p_{n-1}(t)\\
&= p_n(t)-\mu hpp_n(t)+\mu hpp_{n-1}(t)+o(h)
\end{aligned}$$

If we let $\lambda=\mu p$, exactly the same argument as before holds, so $N(t)\sim Po(\mu pt)$, and $\mathbb{E}(N(t)) = \mu pt$ by properties of the poisson.

3. (a) i. A state, j, is transient $\iff$ there is a non-zero probability that, once we leave j, we never return.

   ii. A state, j, is recurrent $\iff$ it is not transient, ie when we leave state j, we return at some point in the future with probability 1. If $\sum_{n=0}^{\infty} p_{jj}(n) < \infty$, then the state j is transient, otherwise it is recurrent.

(b)

$$\begin{aligned} r_i &= \mathbb{P}(X_n = -a \wedge \forall k < n, X_k \neq b | X_0 = i) \\ &= \mathbb{P}(X_n = -a \wedge \forall k < n, X_k \neq b | X_1 = i+1 \wedge X_0 = i)\mathbb{P}(X_1 = i+1) \\ &\qquad + \mathbb{P}(X_n = -a \wedge \forall k < n, X_k \neq b | X_1 = i-1 \wedge X_0 = i)\mathbb{P}(X_1 = i-1) \\ &= \frac{r_{i+1} + r_{i-1}}{2} \quad \text{by the markov property} \end{aligned}$$

We also have that $r_{-a} = 1$, since the probability of being absorbed at $-a$ before b when we're already at $-a$ is 1, and similarly $r_b = 0$. We have a recurrence relation for r_i, so assume $r_i = \theta^i$, so

$$\begin{aligned} r_i &= \frac{r_{i+1} + r_{i-1}}{2} \\ \implies \theta^i &= \frac{\theta^{i+1} + \theta^{i-1}}{2} \\ \implies \theta^2 - 2\theta + 1 &= 0 \quad \text{Since } \theta = 0 \text{ is not a solution} \\ \implies \theta &= 1 \text{ twice} \end{aligned}$$

This gives us that $r_i = A\theta^i + Bi\theta^i = A + Bi$. Our values for r_{-a} and r_b give us that

$$\begin{aligned} A - aB &= 1 \\ A + bB &= 0 \\ \implies B &= \frac{-1}{a+b} \\ A &= \frac{b}{a+b} \\ \\ \implies r_i &= \frac{b-i}{a+b} \end{aligned}$$

Substituting in $i = 0$ gives the desired result.

(c) Let $s_k = \mathbb{P}(X_n = 0 \wedge \forall k < nX_k \neq 0 \wedge \forall 1 < l < nX_l \neq K, -K | X_0 = k)$. s_k is the probability that we return to 0 before hitting either K or $-K$. We want to find s_0. From similar logic to before, we have that $s_0 = \frac{s_{-1}+s_1}{2}$.

If the process starts by moving upwards, so $Y_0 = 1$, we have a process similar to before, with $a = 0, b = K, X_0 = 1$, so

$$\begin{aligned} s_1 = r_1 &= \frac{b+1}{a+b} \\ &= \frac{K-1}{K} \end{aligned}$$

Similarly, if the process starts by moving downwards, $Y_0 = -1$, we have $a = K, b = 0, X_0 = -1$. r_{-1} will give the probability of being absorbed in state a before b, but we want the opposite of this, so we take

$$\begin{aligned} s_{-1} = 1 - r_{-1} &= 1 - \frac{1}{K} \\ &= \frac{K-1}{K} \end{aligned}$$

And, finally,

$$\begin{aligned}
s_0 &= \frac{1}{2}(s_{-1} + s_1) \\
&= \frac{1}{2}(\frac{K-1}{K} + \frac{K-1}{K}) \\
&= \frac{K-1}{K}
\end{aligned}$$

As required.

4. (a) i.

$$\forall n \in \mathbb{N}, \mathbb{E}(X_n) < \infty$$
$$\mathbb{E}(X_{n+1} - X_n | \underline{Y} = \underline{y}) = 0$$

Where $\underline{Y} = (Y_1, ..., Y_n)$ and $\underline{y} = (y_1, ..., y_n)$

ii.

$$\begin{aligned}
\mathbb{E}(2^{Y_1}) &= 2^2\frac{1}{7} + 2^{-1}\frac{6}{7} \\
&= \frac{8}{14} + \frac{6}{14} \\
&= 1
\end{aligned}$$

Well, that was hard...

$$X_n = 2^{S_n} = 2^{\sum_{i=1}^{n} Y_n} = \prod_{i=1}^{n} 2^{Y_n}$$

$$\begin{aligned}
\mathbb{E}(X_{n+1} - X_n | \underline{Y} = \underline{y}) &= \mathbb{E}(\prod_{i=1}^{n+1} 2^{Y_n} - \prod_{i=1}^{n} 2^{Y_n} | \underline{Y} = \underline{y}) \\
&= \mathbb{E}(2^{Y_{n+1}} \prod_{i=1}^{n} 2^{y_n} - \prod_{i=1}^{n} 2^{y_n}) \\
&= \prod_{i=1}^{n} 2^{y_n} \mathbb{E}(2^{Y_{n+1}} - 1) \\
&= \prod_{i=1}^{n} 2^{y_n} (\mathbb{E}(2^{Y_1}) - 1) \\
&= 0
\end{aligned}$$

So $\{X_n\}$ is a martingale wrt $\{Y_n\}$

(b) i.

$$\begin{aligned}
\mathbb{E}(\sum_{r=1}^{T} Y_r) &= \mathbb{E}(\mathbb{E}(\sum_{r=1}^{T} Y_r | T)) \\
&= \mathbb{E}(\mathbb{E}(Y_1 + Y_2 + \cdots + Y_T | T)) \\
&= \mathbb{E}(\mathbb{E}(TY_1 | T)) \quad \text{Since the } Y_i\text{s are all iidrvs} \\
&= \mathbb{E}(TY_1) \\
&= \mathbb{E}(T)\mathbb{E}(Y_1) \quad \text{Since } T \text{ and } Y_1 \text{ are independent}
\end{aligned}$$

ii. At time T, $S_T = -4$, so $\mathbb{E}(S_T) = -4$. We also have that $\mathbb{E}(Y_1) = \frac{-4}{7}$, so

$$\begin{aligned}
\mathbb{E}(S_T) &= \mathbb{E}(Y_1)\mathbb{E}(T) \\
\implies -4 &= \frac{-4\mathbb{E}(T)}{7} \\
\implies \mathbb{E}(T) &= 7
\end{aligned}$$

5. (a) i. A state, j, is recurrent if the probability of travelling from state j to j in some length of time is 1. j is positive recurrent if μ_{jj}, the expected time between returns, is finite.

ii. A stationary distribution is a vector, $\underline{\pi}$, satisfying:

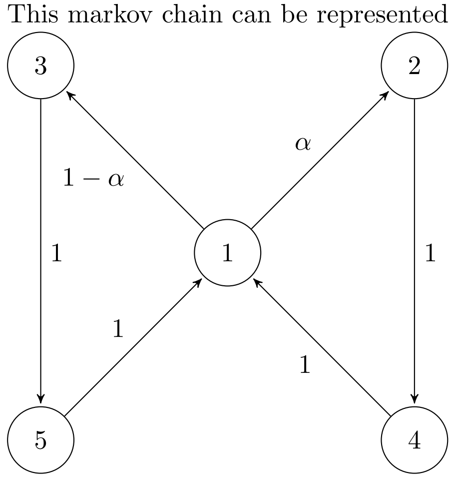
$$\begin{aligned}\underline{\pi}P &= \underline{\pi} \\ \wedge \sum_{s \in S} \pi_s &= 1\end{aligned}$$

iii. A state, j, has period

$$d_j = hcf\{n | p_{jj}(n) > 0\}$$

Where hcf denotes the highest common factor of a set.

(b) i. This markov chain can be represented by the following automaton:



Inspecting it, we see that $\forall i, j \in S$, $i \leftrightarrow j$ since there is a path between every pair of vertices, so the chain consists of a single closed communicating class.

ii. The chain consists of a single, irreducible, finite closed communicating class, so all states must be positive recurrent.

iii. We can see that there are exactly two cycles for state 1, namely $(1, 2, 4, 1)$ and $(1, 3, 5, 1)$. Both of these cycles are of order 3, so

$$\begin{aligned}d_1 &= hcf\{3, 6, 9, 12, \dots\} \\ &= 3\end{aligned}$$

Where $i \leftrightarrow j$, we have that $d_i = d_j$, so $\forall i, j \in S, d_i = d_j = d_1 = 3$, so

$$\forall i \in S, d_i = 3$$

iv. Suppose $\underline{\pi}P = \underline{\pi}$

$$\begin{aligned}\implies \pi_1 &= \pi_4 + \pi_5 & (1) \\ \pi_2 &= \alpha\pi_1 & (2) \\ \pi_3 &= (1-\alpha)\pi_1 & (3) \\ \pi_4 &= \pi_2 & (4) \\ \pi_5 &= \pi_3 & (5)\end{aligned}$$

Any choices of π_2 and π_3 would satisfy (4) and (5), so any choice of π_4 and π_5 would satisfy

(1). We can then reqrite everything as:

$$\begin{aligned}
\pi_1 &= \pi_1 \\
\pi_2 &= \alpha\pi_1 \\
\pi_3 &= (1-\alpha)\pi_1 \\
\pi_4 &= \alpha\pi_1 \\
\pi_5 &= (1-\alpha)\pi_1
\end{aligned}$$

We also require that $\sum_{i=1}^{5} \pi_i = 1$, so $3\pi_1 = 1$, and $\pi_1 = \frac{1}{3}$.

$$\underline{\pi} = \left(\frac{1}{3}, \frac{\alpha}{3}, \frac{1-\alpha}{3}, \frac{\alpha}{3}, \frac{1-\alpha}{3}\right) \text{ is a stationary distribution.}$$

v. No limiting distribution exists, since the chain is periodic and limiting distributions are only defined for aperiodic chains.

(c) Let $||\underline{x}||_1$ denote $\sum_{i\in\mathcal{X}} x_i$. Suppose $\underline{\mu}$ satisfies $\underline{\mu}Q = \underline{\mu} \wedge ||\underline{\mu}||_1 = 1$

$$\begin{aligned}
\iff \mu_i &= \sum_{j=1}^{n} q_{ij}\mu_j \\
&= \sum_{j=1}^{n} \delta p_{ij}\mu_j + (1-\delta)\mu_i \quad \text{ since } p_{ii} = 0 \\
\iff \delta\mu_i &= \sum_{j=1}^{n} \delta p_{ij}\mu_j \\
\iff \mu_i &= \sum_{j=1}^{n} p_{ij}\mu_j
\end{aligned}$$

So $\underline{\mu}$ satisfies $\underline{\mu}P = \underline{\mu} \wedge ||\underline{\mu}||_1 = 1$, and is therefore a stationary distribution for P. All of these steps will also work backwards, so stationary distributions for P are also stationary distributions for Q.

i. As before, each state intercommunicates, but now state 1 has cycle $(1,1)$ of order 1, so $d_1|1$ and $d_1|3$, hence $d_1 = 1$, since 1 and 3 are coprime. Similar logic to before will give us that $\forall i \in S, d_i = 1$, so the chane is aperiodic.

ii. Since the chain is aperiodic, a limiting distribution exists and is equal to the stationary distribution, namely:

$$\underline{\pi} = \left(\frac{1}{3}, \frac{\alpha}{3}, \frac{1-\alpha}{3}, \frac{\alpha}{3}, \frac{1-\alpha}{3}\right) \text{ is the limiting distribution.}$$