Chapter T6

Systems of Ordinary Differential Equations

T6.1. Linear Systems of Two Equations

T6.1.1. Systems of First-Order Equations

1. $x'_t = ax + by, \quad y'_t = cx + dy.$

System of two constant-coefficient first-order linear homogeneous differential equations.

Let us write out the characteristic equation

$$\lambda^2 - (a + d)\lambda + ad - bc = 0 \qquad (1)$$

and find its discriminant

$$D = (a - d)^2 + 4bc. \qquad (2)$$

1°. Case $ad - bc \neq 0$. The origin of coordinates $x = y = 0$ is the only one stationary point; it is

$$\begin{aligned} &\text{a node if } \ D = 0;\\ &\text{a node if } \ D > 0 \ \text{ and } \ ad - bc > 0;\\ &\text{a saddle if } \ D > 0 \ \text{ and } \ ad - bc < 0;\\ &\text{a focus if } \ D < 0 \ \text{ and } \ a + d \neq 0;\\ &\text{a center if } \ D < 0 \ \text{ and } \ a + d = 0. \end{aligned}$$

1.1. Suppose $D > 0$. The characteristic equation (1) has two distinct real roots, λ_1 and λ_2. The general solution of the original system of differential equations is expressed as

$$x = C_1 b e^{\lambda_1 t} + C_2 b e^{\lambda_2 t},$$
$$y = C_1(\lambda_1 - a)e^{\lambda_1 t} + C_2(\lambda_2 - a)e^{\lambda_2 t},$$

where C_1 and C_2 are arbitrary constants.

1.2. Suppose $D < 0$. The characteristic equation (1) has two complex conjugate roots, $\lambda_{1,2} = \sigma \pm i\beta$. The general solution of the original system of differential equations is given by

$$x = be^{\sigma t}\big[C_1 \sin(\beta t) + C_2 \cos(\beta t)\big],$$
$$y = e^{\sigma t}\big\{[(\sigma - a)C_1 - \beta C_2] \sin(\beta t) + [\beta C_1 + (\sigma - a)C_2 \cos(\beta t)\big],$$

where C_1 and C_2 are arbitrary constants.

1.3. Suppose $D = 0$ and $a \neq d$. The characteristic equation (1) has two equal real roots, $\lambda_1 = \lambda_2$. The general solution of the original system of differential equations is

$$x = 2b\left(C_1 + \frac{C_2}{a - d} + C_2 t\right)\exp\left(\frac{a + d}{2}t\right),$$
$$y = [(d - a)C_1 + C_2 + (d - a)C_2 t]\exp\left(\frac{a + d}{2}t\right),$$

where C_1 and C_2 are arbitrary constants. On substituting (2) into (1) and integrating, one arrives at the general solution of the original system in the form

$$x = C_3 t + t\int \frac{u(t)}{t^2}\,dt, \qquad y = C_4 t + t\int \frac{v(t)}{t^2}\,dt,$$

where C_3 and C_4 are arbitrary constants.

6. $\boldsymbol{x''_{tt} = f(t)(a_1x + b_1y), \qquad y''_{tt} = f(t)(a_2x + b_2y).}$

Let k_1 and k_2 be roots of the quadratic equation

$$k^2 - (a_1 + b_2)k + a_1b_2 - a_2b_1 = 0.$$

Then, on multiplying the equations of the system by appropriate constants and on adding them together, one can rewrite the system in the form of two independent equations:

$$z_1'' = k_1 f(t) z_1, \quad z_1 = a_2x + (k_1 - a_1)y;$$
$$z_2'' = k_2 f(t) z_2, \quad z_2 = a_2x + (k_2 - a_1)y.$$

Here, a prime stands for a derivative with respect to t.

7. $\boldsymbol{x''_{tt} = f(t)(a_1x'_t + b_1y'_t), \qquad y''_{tt} = f(t)(a_2x'_t + b_2y'_t).}$

Let k_1 and k_2 be roots of the quadratic equation

$$k^2 - (a_1 + b_2)k + a_1b_2 - a_2b_1 = 0.$$

Then, on multiplying the equations of the system by appropriate constants and on adding them together, one can reduce the system to two independent equations:

$$z_1'' = k_1 f(t) z_1', \quad z_1 = a_2x + (k_1 - a_1)y;$$
$$z_2'' = k_2 f(t) z_2', \quad z_2 = a_2x + (k_2 - a_1)y.$$

Integrating these equations and returning to the original variables, one arrives at a linear algebraic system for the unknowns x and y:

$$a_2x + (k_1 - a_1)y = C_1\int \exp\bigl[k_1F(t)\bigr]\,dt + C_2,$$
$$a_2x + (k_2 - a_1)y = C_3\int \exp\bigl[k_2F(t)\bigr]\,dt + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants and $F(t) = \int f(t)\,dt$.

8. $\boldsymbol{x''_{tt} = af(t)(ty'_t - y), \qquad y''_{tt} = bf(t)(tx'_t - x).}$

The transformation

$$u = tx_t - x, \qquad v = ty'_t - y \tag{1}$$

leads to a system of first-order equations:

$$u'_t = atf(t)v, \qquad v'_t = btf(t)u.$$