Sequence and Series

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1 Trivial properties

• The below properties are for in general complete spaces. whose defining property is the following point

• Cauchy sequence $\iff$ Convergent sequence
(in general metric spaces $\mathbb{R}^n$ for $n \in \mathbb{N}$ are complete in particular $\mathbb{R}$ and $\mathbb{C}$ are complete).

• $a_n \to 0$ as $n \to \infty$ is a necessary condition for a series $\sum_{n=1}^{\infty} a_n$ to converge. (not sufficient eg: $\sum 1/n$ harmonic series)

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2 Tests for positive termed series

• Below tests apply for series whose general terms are positive only (i.e. ≥ 0)
(Note : it can also be used to check for absolute convergence as taking absolute value of each term results in terms ≥ 0)

• for rest of the notes let behaviour denote convergence and divergence simultaneously i.e. say $\{a_n\}$ follows behaviour of $\{b_n\}$ means that $\{a_n\}$ converges if $\{b_n\}$ converges and $\{a_n\}$ diverges if $\{b_n\}$ diverges.

• **Comparison test** : for series $\sum u_n, \sum v_n$ if $u_n \leq k \times v_n$ for $k > 0$ then u_n converges if v_n converges and v_n diverges if u_n diverges.

• **Limit form of comparison test** for series $\sum u_n, \sum v_n$ if

$$l = \lim_{n\to\infty} \frac{u_n}{v_n}.$$

then:

▪ if $l \neq 0$ then $\sum u_n$ follows behaviour of $\sum v_n$.
▪ if $l = 0$ then $\sum u_n$ converges if $\sum v_n$ converges.
(as $0 < u_m \leq v_m$ holds for sufficiently large m, and also if $\sum u_n$ diverges then $\sum v_n$ diverges).
▪ if $l = \infty$ $\sum u_n$ diverges if $\sum v_n$ diverges.
(as $0 < v_m \leq u_m$ holds like preceding point).

• **Cauchy's Condensation test** : if $f(n)$ is a monotone decreasing sequence of positive numbers (i.e. $f(n) > 0, f(k) \geq f(k+1)$ $\forall k \in \mathbb{N}$) then for $m \in \mathbb{N}$ $\sum f(n)$ and $\sum m^n f(m^n)$ have same behaviour. (Mostly used in the form $\sum 2^n f(2^n)$.)

• **Raabe's Test** : for series $\sum u_n$ of positive real numbers if $D_n = n\left(1 - \frac{u_n}{u_{n+1}}\right)$ and

$$D = \limsup D_n, \; d = \liminf D_n$$

then :
▪ if $D < 1$ series converges
▪ if $d > 1$ series diverges
▪ no conclusions if $d \leq 1 \leq D$

• **Integral test** : if $f(x) \geq 0$ in $[1, \infty)$ and is monotonically decreasing then $\sum_{n=1}^{\infty} f(n)$ and $\int_1^{\infty} f(x)dx$ follow same behaviour.

▪ **Intergral inequality** : if $\sum_{n=1}^{\infty} f(n)$ is as above and converges to s then the for partial sums $s_n = \sum_{k=1}^{n} f(k)$ we have

$$\int_{n+1}^{\infty} f(t)dt \leq s - s_n \leq \int_{n}^{\infty} f(t)dt$$

3 General tests

- **Ratio test** for series $\sum z_n$ with non zero terms $\in \mathbb{C}$ if $r_n = \left|\frac{z_{n+1}}{z_n}\right|$

$$r = \liminf r_n, \; R = \limsup r_n.$$

then :

- if $R < 1$ series converges absolutely
- if $r > 1$ series diverges
- no conclusion of behaviour if $r \leq 1 \geq R$

- **Root test** : for series $\sum z_n$ if

$$L = \limsup |z_n|^{1/n}$$

then :

- if $L < 1$ series converges absolutely.
- if $L > 1$ series diverges.
- if $L = 1$ no conclusion.

4 Miscellaneous series properties

- if $\sum(x_n + y_n)$ converges then both $\sum x_n$ and $\sum y_n$ converge or diverge (one cannot diverge and another converge).
- if $\sum a_n$ and $\sum b_n$ converge absolutely then $\sum c_n = \sum a_n b_n$ converges absolutely.
- restatement of above point : $a_n, b_n > 0$ and $\sum a_n, \sum b_n$ converge then $\sum a_n b_n$ converges
- if $a_n \geq 0$ and $\sum a_n$ converges then $\sum a_n^k$ for $k \geq 1$ converges (as $a_n \to 0$, for sufficiently large n we get $a_n < 1 \implies (a_n)^k \leq a_n$ and comparison test convergence follows).
- if $0 \leq a_n \to a$ then

$$s_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \to a$$

- for converse of above point if s_n converges and if for $a_n = s_n - s_{n-1}$, $\lim n a_n = 0$ then a_n converges
- similar to above point if $|n a_n| \leq M < \infty \; \forall n$ and $\lim s_n = s$ then $a_n \to s$
- if $0 < a_n \to a$ then

$$(a_1 . a_2 \ldots a_n)^{1/n} \to a$$

- if $\sum a_n$ converges then $\sum \frac{\sqrt{a_n}}{n}$ converges
- if $a_n > 0$ and $\sum a_n$ converges then $\sum \sqrt{a_n a_{n+1}}$ converges.
- Series $\sum_{n=0}^{\infty} \left(\frac{az+b}{cz+d}\right)^n$ for $|a| = |c| > 0$ converges whenever

$$\frac{|b|^2 - |d|^2}{2} < Re(z(c\bar{d} - a\bar{b})).$$

or in general if $|a| \neq |c|$, then converges whenever

$$\frac{(|a|^2 - |c|^2)|z|^2 + |b|^2 - |d|^2}{2} < Re(z(c\bar{d} - a\bar{b})).$$

- **Dirichlet's Test** :If $\left\{\sum_{k=1}^{n} a_k\right\}$ is a bounded sequence and $\{b_n\}$ is an null sequence ($b_n \to 0$ as $n \to \infty$) then $\sum_{n=1}^{\infty} a_n b_n$ converges.
- **Abel's Test :** if $\{x_n\}$ is convergent monotone sequence and series $\sum y_n$ is convergent then $\sum x_n y_n$ is convergent.
- if $a_n > 0$ and $\sum_{n=1}^{\infty} a_n$ diverges, $s_n = \sum_{k=1}^{n} a_k$ then
  - $\sum_{n=1}^{\infty} \frac{a_n}{s_n}$ diverges
  - $\sum_{n=1}^{\infty} \frac{a_n}{s_n^2}$ converges
- For any sequence $\{a_n\}$

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{1/n}$$

$$\leq \limsup |a_n|^{1/n} \leq \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

- if $\sum a_n$ converges and $\{b_n\}$ is monotonic and bounded then $\sum a_n b_n$ converges
- **Leibniz Theorem** : if $\{c_n\}$ is such that $c_n > 0$ and is monotonic decreasing to 0 (i.e. $c_{n+1} < c_n, \quad c_n \to 0$) then $\sum_{n=1}^{\infty} (-1)^{n+1} c_n$ converges.

- a series $\sum a_n$ is said to be absolutely convergent if $\sum |a_n|$ converges
- if a series is absolutely convergent the it is convergent.
- if $\sum_{n=0}^{\infty} a_n$ converges absolutely, $\sum_{n=0}^{\infty} a_n = A$, $\sum_{n=0}^{\infty} b_n = B$ and $c_n = \sum_{k=0}^{n} a_k b_{n-k}$ (Cauchy product) then $\sum_{n=0}^{\infty} c_n = AB$
- Cauchy product of two absolutely convergent series is absolutely convergent.
- if $\{k_n\}$ is a sequence in $\mathbb{N}$ such that every integer appears once and if $a'_n = a_{k_n}$ then a rearrangement of $\sum a_n$ is of type $\sum a'_n$
- **Riemann Rearrangement Theorem** : if series of real numbers $\sum a_n$ converges but not absolutely then for any $-\infty \geq \alpha \geq \beta \geq \infty$ series $\sum a_n$ can be rearranged to $\sum a'_n$ with partial sum s'_n such that

$$\liminf s'_n = \alpha \text{ and } \limsup s'_n = \beta$$

- for a given double sequence $\{a_{ij}\}$ for $i = 1, 2, \ldots, j = 1, 2, \ldots$ if $\sum_{j=1}^{\infty} |a_{ij}| = b_i$ and $\sum b_i$ converges then

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}$$

, same holds true i.e. summation can be changed if each of $a_{ij} \geq 0$ also.

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$$\lim_{n\to\infty} \sum_{r=\alpha}^{\beta} \frac{1}{n} f(\frac{r}{n}) = \int_a^b f(x)dx$$

where replace :

$$r/n \to x$$
$$1/n \to dx$$
$$a = \lim_{n\to\infty} \alpha/n$$
$$b = \lim_{n\to\infty} \beta/n$$

(to derive use simple notion of Riemann Integration: if f is integrable in $[a, b]$ then for every $\epsilon > 0$ $\left| \sum_{i=1}^{n} f(t_i)\Delta(x_i) - \int_a^b f(x)d(x) \right| < \epsilon$ holds for some partition $p([x_i, x_{i+1}]_1^{n-1})$ of $[a, b]$ and for any $t_i \in [x_i, x_{i+1}]$)

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5 Some limits and theorems

- **L'Hospital Rule** : if f, g are real differentiable functions in (a, b) (for $-\infty \leq a < b \leq \infty$) such that $g'(x) \neq 0$ in (a, b) then as $x \to a$ $f(x) \to 0, g(x) \to 0$ or if $g(x) \to \pm\infty$ and if $\frac{f'(x)}{g'(x)} \to A$ then $\frac{f(x)}{g(x)} \to A$ (analogous result holds for $x \to b$) (is also true if f, g are complex valued and $f(x) \to 0, g(x) \to 0$)
- for $f, g : D \subset \mathbb{R} \to \mathbb{R}$, if $\lim_{x\to c} f(x) = 0$ and $g(x)$ is bounded in some deleted neighbourhood of c then $\lim_{x\to c} f(x)g(x) = 0$
- if $\lim_{x\to c} f(x) = l$ and g is continuous at l or in some set whose limit point is l then $\lim_{x\to c} g(f(x)) = \lim_{x\to l} g(x)$
- $\lim_{n\to\infty} \sum_{m=1}^{n} \frac{1}{m} - \ln n = \gamma$ a fixed number
- $\lim_{n\to\infty} z^n = 0$ if $|z| < 1$
- if $a > 1$ and $p(n)$ is a fixed polynomial in n then $\lim_{n\to\infty} \frac{a^n}{p(n)} = \pm\infty$ (depends on $p(n)$, precisely on coefficient of largest degree term).
- $\lim_{n\to\infty} n^{1/n} = 1$ in particular if $|z| \neq 0$ then $\lim_{n\to\infty} |z|^{1/n} = 1$
- $\lim_{n\to\infty} \left(1 + \frac{a}{n}\right)^n = e^a$
- for $\alpha \in \mathbb{R}, p > 0$ we have $\lim_{n\to\infty} \frac{n^\alpha}{(1+p)^n} = 0$
- if $\alpha, \beta > 0$ and $x \in \mathbb{R}$ then :
  - $\lim_{x\to\infty} \frac{(\ln(x))^\alpha}{x^\beta} = 0$
  - $\lim_{x\to\infty} \frac{x^\alpha}{e^{\beta x}} = 0$
- from some preceding points we get growth of $\ln(n)$ < growth of n < growth of $p(n)$ (for non constant $p(n)$.) < growth of a^n $(a > 1)$ < growth of $n!$.

• series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges for $p > 1$ and diverges for $p \leq 1$

• series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges for $p > 1$ and diverges for $p \leq 1$ this result can be continued to series like $\sum_{n=2}^{\infty} \frac{1}{n \ln n (\ln \ln n)^p}$, $\sum_{n=2}^{\infty} \frac{1}{n \ln n \ln \ln n (\ln \ln \ln n)^p}$ and so on.

• for series such as $\sum_{n=0}^{\infty} q^n z^{kn}$ for some $k \geq 0$ fixed then this series is equal to series $\sum_{n \geq 0} a_n z^n$ where

$$a_n = \begin{cases} q^{n/k} & \text{if } n = 0, k, 2k, 3k, \dots \\ 0 & \text{otherwise} \end{cases}$$

Thus $R = \limsup 1/|a_n|^{1/n} = q^{-1/k}$. for $\sum_{n=0}^{\infty} q^n z^{kn}$ series.

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6 Uniform Convergence

• define uniform norm for a function $f : A \subseteq \mathbb{R} \to \mathbb{R}$ as $||f||_A = \sup(|f(a)| \text{ for } a \in A)$

• A sequence of bounded functions $\{f_n\}$ in $\mathbb{R}$ converges uniformly to f in domain $A \subseteq R$ iff $||f_n - f||_A \to 0$ i.e. the uniform norm of $f_n - f$ converges to0.

• one way to find the uniform norm for a function is to differentiate it and find its maximum on domain.

• **Dinni's Theorem** : if $\{f_n\}$ is a monotone sequence of continuous functions on $[a, b]$ (closed and bounded) that converges to f which is continuous on $[a, b]$ then the convergence is uniform.

• if $f(x)$ is uniformly continuous on $\mathbb{R}$ and non zero at integer values then $\sum_{n=1}^{\infty} \frac{1}{f(n)}$ is never convergent (use $|f(x)| \leq A|x| + B$)

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References

[1] Rudin W.: Principles of Mathematical Analysis,McGraw-Hill,3,(1976).

[2] Ponnusamy S., Silverman H.: Complex Variables and Applications,Birkhauser,(2006).

[3] Robert G. Bartle, Donald R. Sherbert: Introduction to Real Analysis, Wiley publishers, 4, (2011).