Linear Algebra

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0 Symbols and notations used

$A_{m\times n} \to m \times n$ matrix.
$A_n \to n \times n$ matrix.
$\sim \to$ the relation below
$A \sim B \implies A = P^{-1}AP.$
iff $\to \iff$

1 Basic Linear equations theory

Every $A_{m\times n} = PR_{m\times n}$ for Row reduced Echelon form R and an invertible matrix P let this relation be denoted by A rrec R

if $m < n$ then the homogeneous system $A_{m\times n}X = 0$ has a non trivial solution i.e. if the number of equations is less than the number of variables then the Homogeneous System has a non trivial solution

Inverse Properties

■ A_n has inverse A^{-1} iff $AX = 0$ has only trivial solutions.
■ A is invertible iff A rrec I (identity)
■ if Elementary matrices are the corresponding matrices of elementary transforms (change of rows, addition of one row to another, multiplication of a row with an non zero constant) then A is invertible iff A is product of elementary matrices.

Echelon Form

every $A_{m\times n} = P_mRQ_n$ for P, Q invertible and R is such that it has an identity in upper corner and all other entries zero i.e. $R = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$ for some identity I_k.

Consistency

System of linear equations :
$A_{m\times n}X_{n\times 1} = b_{1\times m}$ for $b \neq 0$ is consistent (has a solution) iff the row reduced Echelon form of augmented matrix $[A : b]$ has same number of non zero rows as in row reduced echelon form of A.

2 Vector Spaces

Definition

$(V, \mathbb{F}, +)$ denoted by $V(\mathbb{F})$: V is vector space over Field $\mathbb{F}$ if
■ $(V, +)$ is a commutative group,
for every $\alpha, \beta \in \mathbb{F}$ and every $a, b \in V$
■ $1a = a$ where $1 \in \mathbb{F}$ is multiplicative identity of $\mathbb{F}$.
■ $(\alpha + \beta)a = \alpha a + \beta a$
■ $\alpha(a + b) = \alpha a + \alpha b$
■ $(\alpha\beta)a = \alpha(\beta a)$
The elements of V are called **vectors** and elements of $\mathbb{F}$ are called **scalars**

Span

if $K = \{v_1, v_, .., v_n\} \subseteq V(\mathbb{F})$ then span of K is the set $\{\sum \alpha_i v_i | v_i \in K, \alpha_i \in \mathbb{F}\}$ i.e. is all the formal sums from set K with $\mathbb{F}$. This is denoted by $span(K)$.

Subspace

A subset S of vector space $V(\mathbb{F})$ is a subspace if $S(\mathbb{F})$ is a vector space by same operations as in V

■ given any $K \subseteq V(F)$ $span(K)$ is a subspace of $V(\mathbb{F})$.

■ S is a subspace of V iff $\alpha a + b \in S\ \forall a, b \in S$ and $\alpha \in \mathbb{F}$ the underlying field of both spaces

■ Intersection of subspaces (arbitrary) is again a subspace i.e. if W_1, W_2 are subspaces of V then $W_1 \cap W_2$ is also a subspace of V.

■ Union of subspaces may not be a subspace

■ Union of two subspaces is a subspace iff one of them is contained in another i.e. for W_1, W_2 subspaces of V, $W_1 \cup W_2$ is a subspace iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

(note: this is not the same in case of 3 subspaces : consider $Z_2 \times Z_2(Z_2)$ vector space here $Z_2 \times Z_2 = span((0,1)) \cup span((1,0)) \cup span((1,1))$.)

Dependence

a set of vectors $\{v_1, v_, .., v_n\} \subseteq V(\mathbb{F})$ are called Linearly independent in V if $\alpha_1 v_1 + \alpha_2 v_2 + .. + \alpha_n v_n = 0 \implies$ all $\alpha_i's$ are 0 and no other choice is left. Other wise the subset is called linearly dependent

Basis

a subset K of V is a spanning set of V if $span(K) = V$.

A Linearly independent spanning set of $V(\mathbb{F})$ is called a Basis of V.

Dimension

In a given vector space $V(\mathbb{F})$.

■ The number of elements in Basis is constant $n \in \mathbb{Z}^+$.

■ if a set contains more vectors than the Basis set of a vector space then it is linearly dependent.

■ if a linearly independent set contains exactly the same number of elements as a Basis then it is also a Basis.

■ These above points leads us to the Definition : Number of elements n in The Basis set of $V(\mathbb{F})$ is unique and is called the Dimension of $V(\mathbb{F})$ denoted by $dim(V) = n$.

if $W_1, W_2 \subseteq V$ are subspaces then

■ $dim(W_i) \leq V$.

■ let $W_1 + W_2 = span(W_1, W_2)$ then

$$dim(W_1 + W_2) = dim(W_1) + dim(W_2) - dim(W_1 \cap W_2).$$

(note: there cannot be a definite formula for $dim(\sum_{i=1}^{n} W_i)$ using dimensions of $W_i's$ and their counterparts (union, intersections) if $n \geq 3$.)

Matrix Representation of vectors

Fix a basis $\beta = \{b_1, b_2, .., b_n\}$ for a vector space $V(\mathbb{F})$ then as B spans V every vector $x \in V$ can be written as $x = x_1 b_1 + x_2 b_2 + .. x_n b_n$ for $x_i \in \mathbb{F}$ and $b_i \in B$ and this representation is unique so each vector can be associated with a column matrix $x_\beta = [x_1\ x_2 .. x_n]^T$

Change of Basis Matrix

Given two basis $\beta = \{b_1, b_2, .., b_n\}$, $\beta' = \{b'_1, b'_2, .., b'_n\}$ for V Then one can change the representation of $x \in V$ from $[x]_\beta$ to $[x]_{\beta'}$ by

$$[x]_{\beta'} = P[x]_\beta$$

where P_n is a invertible matrix given by if $b_j = p_{1j} b'_1 + p_{2j} b'_2 + .. + p_{nj} b'_n$ then $[p_{1j}\ p_{2j} .. p_{nj}]^T$ forms the j^{th} column of P.

3 Linear Transform

Definition

a map $T : V(\mathbb{F}) \to W(\mathbb{F})$ (between vector spaces with same underlying field) is called a linear transform if for every $v, u \in V$ and $\alpha \in F$

- $T(v+u) = T(v) + T(u)$
- $T(\alpha v) = \alpha T(V)$

Range and Null space

For a linear transform $T : V \to W$:

- Range Space of T denoted by $R(T) \subseteq W$ is $\{w | w = T(v) \text{ for some } v \in V\}$
- Null Space of T denoted by $N(T) \subseteq V$ is $\{v | T(v) = 0 \in W\}$
- Both of them are subspaces of the underlying space.
- T is one-one iff $N(T) = \{0\}$.
- T is onto if $R(T) = W$
- if $\dim(V) = \dim(W)$ and $N(T) = \{0\}$ then T is onto thus T is bijective.

if T, U are both liner transforms from $V \to W$ and if both agree on a basis of V (i.e. $T(b_i) = U(b_i)\ \forall i$ for some basis $\beta = \{.., b_i, ..\}$ of V) then both of then are same i.e. $T \equiv U$.

Rank Nullity Theorem

for a linear transform $T : V(\mathbb{F}) \to W(\mathbb{F})$ if $\text{rank}(T) = \dim(R(T))$ and $\text{nullity}(T) = \dim(N(T))$ then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

(this is just an analogue of 1^{st} isomorphism theorems of Groups)

Matrix of Linear Transform

Given a linear transform $T : V \to W$, basis $\beta = \{b_1, b_2.., b_n\}$ of V and basis $\beta' = \{b'_1, b'_2, .., b'_m\}$ of W then we can write the liner transform in the corresponding matrix representation of vectors as

$$[T(x)]_{\beta'} = [T]_{\beta}^{\beta'} [x]_{\beta}$$

where $[T]_{\beta}^{\beta'}$ is a $m \times n$ matrix called Matrix of linear transform of T and is given by if $T(b_j) = t_{1j} b'_1 + t_{2j} b'_1 + .. + t_{mj} b'_m$ then $[t_{1j}\ t_{2j} .. t_{mj}]^T$ forms the j^{th} column of $[T]_{\beta'}^{\beta}$.

Change of Basis

if $T : V \to V$ then $[T]_{\beta}^{\beta}$ is simply written as $[T]_{\beta}$ now if P is the change of basis matrix from basis β' to basis β of V i.e. $[x]_{\beta} = P[x]_{\beta'}$ then

$$[T]_{\beta'} = P^{-1}[T]_{\beta} P$$

(This can be treated as the origin of 'similar' equivalence matrix relationship $A \sim B \iff A = P^{-1}BP$.)

Isomorphism of Vector spaces

Two spaces V, W over same vector space $\mathbb{F}$ are said to be isomorphic to each other if there exist an invertible linear transform $T : V \to W$ (i.e. T is linear bijective map) and this is denoted by $V \cong W$.

- if $V(\mathbb{F})$ is of dimension n then $V \cong \mathbb{F}^n = \{(\alpha_1, \alpha_2, .. \alpha_n) | \alpha_i \in \mathbb{F}\}$ i.e. set of n tuples of $\mathbb{F}$ with component wise addition.
- clearly $V(\mathbb{F}) \cong W(\mathbb{F})$ iff $\dim(W) = \dim(V)$.

Space of Linear Transform

Set of linear transforms $L(V, W) = \{T | T : V \to W \text{ is linear transform}\}$ forms a commutative group under addition i.e. $(T + U)(v) = T(v) + U(v)$ (as in W) so it also forms a Vector space over $\mathbb{F}$ (same field as in V and W.)

- if $\dim(V) = n$ and $\dim(W) = m$ both finite then $\dim(L(V, W)) = nm$

Linear Functional

Linear transformation $f : V(\mathbb{F}) \to \mathbb{F}$ is called a Linear Functional

- This is possible as $\mathbb{F}(\mathbb{F})$ is an one dimensional vector space.
- $\text{rank}(f) = 1$ or 0 so $\text{Nullity}(f) = n - 1$ or n if $\dim(V) = n < \infty$.
- **Dual space** of V denoted by $V^* = L(V, \mathbb{F})$ is the set of all linear functionals on V

■ clearly $\dim(V^*) = \dim(V)$ if $\dim(V)$ is finite

■ **Dual Basis :** for every basis $\beta = \{b_1, b_2, .., b_n\}$ of V there exist a corresponding basis $\beta^* = \{f_1, f_2, .., f_n\}$ of V^* such that $f_i(b_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$ this β^* is called the dual basis of β

■ if $\{.., f_i, ..\}$ is the dual basis of $\{.., b_i, ..\}$ and $x \in V$ is represented as $x = x_1 b_1 + x_2 b_2 + .. + x_n b_n$ then $x_i = f_i(x)$ i.e. the coordinate functions in representation is nothing but the dual functions, i.e.
$x = \sum_{i=1}^{n} f_i(x) b_i$.

■ $V \cong V^* \cong V^{**} = L(V^*, \mathbb{F})$ (note: $\cong$ in $V \cong V^{**}$ is nothing but functional evaluation at a point(vectors) only i.e. every element of V^{**} is of form $\hat{x}$ for $\hat{x}(\psi) = \psi(x)$ for some $x \in V$.)

Functional representation Theorem

if V is finite dimensional vector space, $\beta = \{b_i\}$ is its basis and $[x]_\beta = [x_1 \ x_2 .. x_n]$ then every functional f is of form

$$f(x) = a_1 x_1 + a_2 x_2 + .. + a_n x_n$$

in which $a_i = f(b_i)$. are fixed but x_i varies on input representation x.

Annihilator

if $A \subset V(\mathbb{F})$ be any subset of V then annihilators of A is the set of linear functionals $A^o = \{f | f(A) = 0, f \in V^*\} \subseteq V^*$

■ clearly A^o is a subspace of V^* for any subset A of V

■ subspaces $W_1 = W_1$ iff $W_1^o = W_2^o$

■ $(W_1 + W_2)^o = W_1^o \cap W_2^o$.

■ if W is subspace of V then

$$\dim(W) + \dim(W^o) = \dim(V).$$

■ if W is subspace of V then $W \cong W^{oo}$.

Transpose of linear transform

if $T : V \to W$ is linear transform then its transpose $T^t : W^* \to V^*$ is a linear transform defined by the evaluation
$T^t(g(.)) = g(T(.))$ i.e. for $g \in W^*$, $T^t(g)$ is the functional $f = g(T(.)) \in V^*$

■ $[T^t]_{\gamma^*}^{\beta^*} = ([T]_\beta^\gamma)^t$ i.e. the corresponding matrix of T^t in dual basis of γ in W and β in V is just the Transpose of the matrix of T in β and γ.

■ if W is finite dimensional then for linear $T : V \to W$ we have
$R(T^t) = (N(T))^o$ and $N(T^t) = (R(T))^o$

■ T is $1-1$ iff T^t is onto and T is onto iff T^t is $1-1$.

■ $\mathrm{Rank}(T^t) = \mathrm{Rank}(T)$.

if linear transform $T \in L(V) = L(V, V)$ then it is called a linear operator.

4 Determinant

Motivation

for a finite dimensional space every linear transform in $L(V)$ can be represented as a unique Matrix, but we need to 'uncover' this matrix to gain the properties of corresponding linear transform one such way is to create a Function from set of matrices to the underlying field with some properties which helps us with this 'gain'.

Some Properties needed for such a function are :

■ It must be a linear in terms of rows (or columns) of the matrix this is called n-linear.

■ It must be alternating i.e. if any 2 rows (or columns) are equal then it is zero.

■ its vale on Identity should be 1.

Say we obtain a function D with this property for $(n-1) \times (n-1)$ matrices then this can be extend to $n \times n$ by

$$E_j(A_n) = \sum_{i=1}^{n} a_{ij} D(A_{ij})$$

for fixed $j \in \{1, 2, .., n\}$, where $a_i j$ is the i^{th} row j^{th} column entry of A and $A_i j$ is the $n - 1 \times n - 1$ matrix obtained from A_n by removing i^{th} row and j^{th} column.

Definition

From above points we get determinant for a $n \times n$ matrix with entries from $\mathbb{F}$ as $D : \mathbb{F}^{n \times n} \to \mathbb{F}$ that is n-linear, Alternating and $D(I) = 1$ is Defined by recursion from the above point or if $(i_1, i_2, .., i_n)$ runs trough all the possible permutations of n i.e n- tuple with elements from $\{1, 2, .., n\}$ with out repetition then $D(A = [a_{ij}]) = \sum_{(i_1, i_2, .., i_n)} (-1)^{i_1 + i_2 + .. + i_n} a_{1i_1} a_{2i_2} .. a_{ni_n}$

Additional Properties

■ $\det(A) = \det(B)$ if B is obtained by interchanging rows of A

■ $\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det(A)\det(C)$.

5 Canonical Forms

5.1 Digonalization

For linear operator $T \in L(V)$ a vector $\alpha \in V$ is called an eigenvector and λ called eigenvalue if $T(\alpha) = \lambda\alpha$. i.e. $\alpha \in N(A - \lambda I)$

■ if $A \in M_n(\mathbb{F})$ (all $n \times n$ matrices with entries from $\mathbb{F}$) then λ is an eigenvalue og A iff $\det(A - \lambda I) = 0$.

■ From above point we get all eigenvalues of $A \in M_n(\mathbb{F})$ are the solutions of **Characteristic polynomial** $f(t) = \det(A - tI)$.

for a linear operator T on finite dimensional space V

■ The polynomial $p(T)$ such that $p(T) \equiv 0$ i.e $p(T)x = 0 \; \forall x \in V$ then $p(T)$ is called the **annihilating polynomial** of T

■ the set of all annihilating polynomials of T forms an ideal in $\mathbb{F}[x]$ now as $\mathbb{F}$ is a field it is also an euclidean domain so this ideal is principle thus is generated by a monic polynomial of minimum degree in it called the **minimal polynomial** of T.

Cayley-Hamilton Theorem

if T is a linear operator on finite dimensional space V then characteristic polynomial of T divides minimal polynomial of T i.e. if f is characteristic polynomial of T then $f(T) \equiv 0$.

for a given eigenvalue λ of $T \in L(V)$ the set of all eigenvectors corresponding to λ form a subspace of V this is called eigenspace of λ.

Invariant subspace

W is an invariant subspace of T over V if $T(W) \subseteq W$.

Eigenspaces are invariant subspaces.

Diagonalizability test

$T \in L(V)$ is diagonalisable if there exist an ordered basis $\beta = \{b_1, b_2, .., b_n\}$ of V such that each of the vector in β is an eigenvector of T.

■ T is diagonalizable iff characteristic polynomial of T splits in the underlying field and for each eigenvalue λ of T the multiplicity (in characteristic polynomial) equals $n - \operatorname{rank}(T - \lambda I)$.

6 Inner Product Spaces

7 Forms

8 Bilinear Forms