

Complex Analysis

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1 Power Series

- $P(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 \dots = \sum_{n=0}^{\infty} a_n z^n.$

- If $P(z)$ converges at $z = a$ then it converges absolutely for all $|z| < |a|$.

- If $P(z)$ diverges at $z = d$ then it diverges absolutely for all $|d| < |z|$.

- If two power series $A(z) = \sum_{n=0}^{\infty} a_n z^n,$

$B(z) = \sum_{n=0}^{\infty} b_n z^n$ agree on an infinite sequence ($\neq 0$) converging to zero then they are same i.e. $a_i = b_i \forall i$.

- In general for $P_b(z) = \sum_{n=0}^{\infty} a_n (z - b)^n$ above holds as in displacement or translation of b to 0 i.e. $P_b(z) = P(w)$ for $w = z - b$.

- Radius of convergence of $P(z) = \sum_{n=0}^{\infty} a_n z^n$ is

R then:

- $R = \lim_{i \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|.$

- $R = \lim_{i \rightarrow \infty} \left| \frac{1}{n \sqrt[n]{|a_n|}} \right|.$

- $R = \lim_{i \rightarrow \infty} \left| \frac{1}{\limsup n \sqrt[n]{|a_n|}} \right|.$

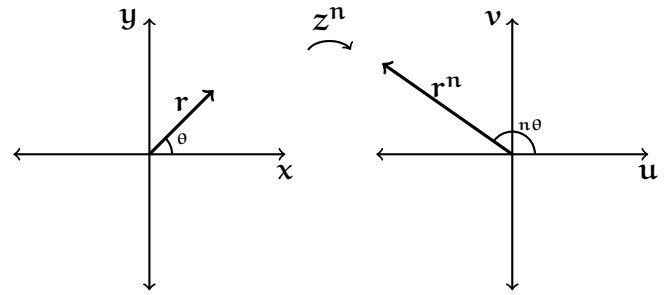
- Radius of convergence of the power series of $f(z)$ at k is equal to distance between k and closest singularity of $f(z)$ to k .

2 Transformations

2.1 Z^n .

- $w = z^n = r^n e^{in\theta}.$

- so from above each z is magnified $|z|^n$ times and rotated $n \arg(z)$ times in the plane i.e.



- Images of circles are circle (with expanded or contracted radius), lines are lines

- Most geometric shapes just expand/diminished (amplified) and gets rotated (twist)

2.2 e^z .

- $w = e^z = e^x e^{iy} = e^x \cos y + i e^x \sin y = u + iv.$

- $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$

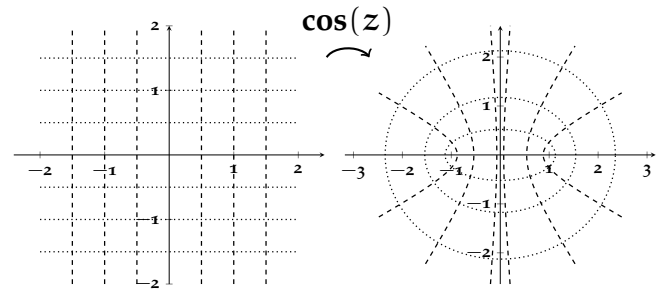
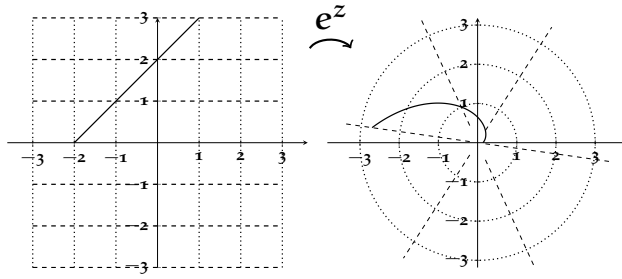
and radius of convergence $= \infty$.

- e^z takes all values in \mathbb{C} infinitely many times except zero i.e. $\text{range}(e^z) = \mathbb{C} - \{0\}$.

- if x is constant then $u^2 + v^2 = e^x = r \implies$ horizontal lines are mapped to circle.

- if y is constant then $\frac{v}{u} = \tan y$ or $v = cu \implies$ vertical lines are mapped to lines passing through origin (not including the origin).

- every other line is mapped to a spiral centred at origin (not including).



2.3 Trigonometric functions

$$\begin{aligned}\cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} \dots \\ &= \frac{e^{iz} + e^{-iz}}{2} \\ \sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \dots \\ &= \frac{e^{iz} - e^{-iz}}{2i}\end{aligned}$$

- $\cos(z - \pi/2) = \sin(z)$.
- $\cosh(z) = \cos(iz)$.
- $\sinh(z) = -i \sin(iz)$.
- so exploring only one of trigonometric functions namely $\cos z$ is sufficient
- now $\cos(x + iy) = \frac{e^{ix}e^{-y} + e^{-ix}e^y}{2}$
 $= \frac{e^y + e^{-y}}{2} \cos(x) - i \frac{e^y - e^{-y}}{2} \sin(x)$
 $= \cosh y \cos x - i \sinh y \sin x = u + iv$.
- for $z = x + iy$ and $w = \cos(z) = u + iv$ if $y = y_0$ is kept constant then
 $\frac{u^2}{\cosh^2 y_0} + \frac{v^2}{\sinh^2 y_0} = 1$.
- so every horizontal line is transformed to an ellipse with foci's ± 1 .
(as $a = \sinh y_0$ $b = \cosh y_0 \implies c = b^2 - a^2 = 1$ (unit from origin) so foci's = $(1, 0), (-1, 0)$.)
- similarly $x = x_0$ is kept constant then
 $\frac{u^2}{\cos^2 x_0} - \frac{v^2}{\sin^2 x_0} = 1$.
- so every vertical line is transformed to hyperbola with foci's ± 1 .

2.4 $\log(z)$.

- it denotes the inverse function of exponential
- $\log(re^{i\theta}) = \ln(r) + i\theta$.
- Clearly \log is a multifunction as $\log(re^{i\theta}) = \ln(r) + i(\theta + 2n\pi)$.
- properties of multifunctions:
 - a region in range where multifunction takes ordinary single value is called a branch.
 - typically branches are connected regions (simply or multiply)
 - q is branch point of multifunction if after a revolution around the point in domain the multifunction changes its values on the original observed point
 - q is algebraic branch point of $f(z)$ if $f(z)$ returns to original observed value after N revolutions around q , its order is $N - 1$, a simple branch point has order 1.
 - q is logarithmic branch point if order is ∞ i.e. the original value is not restored by any number of revolution around the point.
 - any curve drawn from branch point to ∞ is called a branch cut, typically is -ve real axis.
 - eg: $z^{\frac{m}{n}}$ is one- n multifunction has branch point o of order $n - 1$, z^τ for τ irrational has logarithmic branch point of o .
 - a function can have more than one branch point eg: $\sqrt{z^2 + 1} = \sqrt{(z - i)(z + i)}$ has $\pm i$ as simple branch points.
 - if a complex function or a branch of multifunction can be expressed as power series then the **radius of convergence** is distance to the nearest singularity or branch point.
- $\log(z)$ has logarithmic branch point at o .

- $\text{Log}(z) = \ln|z| + i\text{Arg}(z)$ where the branch cut is -ve real axis and $-\pi < \text{Arg}(z) \leq \pi$ is called principle branch.

- continuity of $\text{Log}(z)$ breaks down at $\text{Arg}(z) = \pi$.

- $\text{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$ is a power series centered at 1 with radius of convergence 1 converges on this unit circle except for $z = -1$.

- other branches of $\log(z)$ can be explored by writing $\log(z) = \text{Log}(z) + 2n\pi i$.

- $z^k = e^{k\log(z)} = e^{2n\pi ki} e^{k\text{Log}(z)} = e^{2n\pi ki} [z^k]$ where $[z^k]$ denotes root in principle branch. thus

- $z^{p/q} = e^{p/q \cdot 2n\pi i} [z^{p/q}]$.

- Now for complex powers

$$\begin{aligned} [z^{a+ib}] &= e^{(a+ib)(\text{Log}(z))} \\ &= e^{(a+ib)(\ln(r)+i\theta)} \\ &= e^a \ln(r) e^{-b\theta} e^{i(a\theta+b\ln(r))} \end{aligned}$$

so $[z^{a+ib}] = |z|^a e^{-b \text{Arg}(z)} e^{i(a \text{Arg}(z) + b \ln|z|)}$

and $z^{a+ib} = e^{i2\pi na} e^{-2\pi nb} [z^{a+ib}]$

2.5 Geometric transforms

- translation by v : $J_v(z) = z + v$ translates $0 \rightarrow v$.

- rotation about origin by θ : $R_0^\theta(z) = e^{i\theta} z$.

- rotation about w by θ :

$$R_w^\theta = J_w \circ R_0^\theta \circ J_{-w}(z).$$

- Properties:

- $\{J_w\}$ forms a group under composition

- $R_w^\theta = J_v \circ R_0^\theta$ where $v = w(1 - e^{i\theta})$

. i.e. rotation about any point is equal to a rotation around origin proceeded by translation.

- $R_a^\theta \circ R_b^\phi = R_c^{\theta+\phi}$ where $c = \frac{ae^{i\phi}(1-e^{i\theta}) + b(1-e^{i\phi})}{(1-e^{i(\theta+\phi)})}$.

- if $\theta + \phi = 2n\pi$ then $R_a^\theta \circ R_b^\phi = J_v$ where $v = (b-a)(1-e^{i\phi})$.

- Reflection about a line $L_1 = \Re_{L_1}$.

- Reflection about real axis $\Re_{y=0} = \bar{z}$.

- Reflection about line $ax + by = c$ is

$$\Re_{ax+by=c} = \frac{(b-ia)\bar{z} + 2ic}{b+ia}.$$

(can be done by transforming line to real axis by translation and rotation then conjugation and followed by inverse back to same line transformation).

- Properties:

- If L_1 and L_2 intersect at O , and the angle from L_1 to L_2 is ϕ , then $\Re_{L_2} \circ \Re_{L_1}$ is a rotation of 2ϕ about O i.e. $R_O^{2\phi}$.

- If L_1 and L_2 are parallel, and v is the perpendicular vector to both lines, connecting L_1 to L_2 (i.e. distance vector), then $\Re_{L_2} \circ \Re_{L_1}$ is a translation of $2v$ i.e. J_{2v} .

2.6 $\frac{1}{z}$.

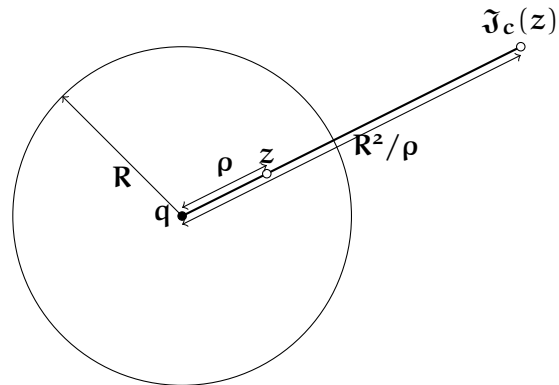
- before studying $\frac{1}{z}$ we can study inversion about a circle :

- $\mathfrak{I}_c(z)$ is the inversion of points in circle c centered at q with radius R i.e. it transforms interior of circle to exterior and points on circle remain fixed

- some defining properties of \mathfrak{I}_c (inversion of about circle c of radius R and centred at q) :

- $q \rightarrow \infty$.

- if z is at distance ρ from q then it is moved to distance R^2/ρ along same direction as z from q i.e.



$$\mathfrak{I}_c(z) = \frac{R^2}{\bar{z} - \bar{q}}$$

(as $\overline{(z - q)}(\mathfrak{I}_c(z) - q) = R^2$.)

- Properties of inversion (\mathfrak{I}_c centred q radius R):
 - inversion is involutory i.e. $\mathfrak{I}_c \circ \mathfrak{I}_c(z) = z$ or $\mathfrak{I}_c^2 = I$.
 - if $\tilde{a} = \mathfrak{I}_c(a)$ and $\tilde{b} = \mathfrak{I}_c(b)$ then $\triangle \tilde{a}q\tilde{b}$ is similar to $\triangle aqb$.
 - every line that does not pass through q is mapped to a circle passing through q .
 - as inversion is involutory it swaps the above point i.e. a circle passing through q is mapped to a line not passing through q .
 - A circle not passing through q is mapped to another circle not passing through q i.e. **inversion preserves circles**.
 - if a circle k cuts circle c at a and b at right angles i.e. k is orthogonal to c then k is mapped to itself i.e. **inversion maps orthogonal circles to c to itself**.
 - Inversion in a circle is anticonformal map
 - If a and b are symmetric with respect to circle k then their inversion images \tilde{a} and \tilde{b} are also symmetric with respect to the inversion image circle \tilde{k} of k .
 - i.e. Inversion maps any pair of orthogonal circles to another pair of orthogonal circles.
 - also if a and b are symmetric w.r.t line L_1 (i.e. are reflections) then their inversion images are also symmetric to the inversion line \tilde{L}_1 .
- now $\frac{1}{z} = \overline{\left(\frac{1}{\bar{z}}\right)}$ so $\frac{1}{z}$ is reflection of inversion centered at origin with unit radius on real axis, so all properties of inversion holds as reflection preserves shapes.
 - now as both inversion and conjugation are anticonformal implies $1/z$ is a conformal map
- define inverse point w.r.t. circle $C_{(z_0, R)} = \{z | |z - z_0| = R\}$ as a and a^* are inverse points w.r.t $C_{(z_0, R)}$ if $a \mapsto a^*$ under $\mathfrak{I}_{C_{(z_0, R)}}(z)$ i.e. if $a^* = z_0 + \frac{R^2}{\overline{a - z_0}}$ or $(a^* - z_0)\overline{(a - z_0)} = R^2$.

2.7 Mobius Transforms

$$\begin{aligned} M(z) &= \frac{az + b}{cz + d} \\ &= \frac{a}{c} - \frac{ad - bc}{c^2} \left(\frac{1}{z + \frac{d}{c}} \right) \\ &= J_{a/c} \circ Az \circ \overline{\mathfrak{I}_u} \circ J_{d/c}(z) \end{aligned}$$

where $A = \frac{ad - bc}{-c^2}$, $u \equiv \{|z| = 1\}$.

- The only shape changing transformation in $M(z)$ is conjugate inversion, so all symmetries and properties of inversion follow to mobius transform.
- Properties
 - every mobius transform maps circles and straight lines onto circles and straight lines.
 - above point may not be same order i.e. some circles can be mapped to straight lines and vis-a-viz. namely a straight line or a circle maps onto a straight line if it passes through the point $z = -d/c$, and onto a circle if it does not (also lines not passing through $-d/c$).
 - mobius transform is conformal
 - more over mobius transforms are the only transforms that **map circles to circles**
 - To be specific A Mobius transformation maps an oriented circle C to an oriented circle \tilde{C} in such a way that the region to the left of C is mapped to the region to the left of \tilde{C} .
 - Symmetric principle: If two points are symmetric with respect to a circle i.e. inverse points w.r.t a circle, then their images under a Mobius transformation are symmetric with respect to the image circle. transformation are symmetric with respect to the image circle.
 - every mobius transform has only 2 fixed points
 - there exist a unique mobius transform sending any three points to any three points.
- the coefficients of a mobius transform $\{a, b, c, d\}$ are not unique as any $k \neq 0$. $\{ka, kb, kc, kd\}$ gives same mobius transform
- cross ration $[z, a, b, c] = \frac{(z-a)(b-c)}{(z-c)(b-a)}$.

- if p, q, r, s are mapped to $\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}$ by a Mobius Transformation iff

$$[p, q, r, s] = [\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}].$$

i.e. Mobius transforms are cross-ratio invariant.

- Unique Mobius transform $M(z) = w$ that transforms $a \rightarrow r, b \rightarrow s, c \rightarrow t$ is

$$[w, r, s, t] = [z, a, b, c]$$

or

$$\frac{(z-a)(b-c)}{(z-c)(b-a)} = \frac{(w-r)(s-t)}{(w-t)(s-r)}.$$

2.8 More on Mobius Transforms

- now as coefficients of mobius transform are not unique if $ad - bc = 1$ in $M(z)$ then

$$M(z) = \frac{ax+b}{cz+d} \longleftrightarrow [M] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

- Properties:

- $M_3 = M_1 \circ M_2(z)$ then $[M_3] = [M_2][M_1]$.
- if inverse of $M(z)$ is $M^{-1}(z)$ then $[M^{-1}] = [M]^{-1}$.
- identity transform $[I] = [1, 0; 0, 1]$.
- Thus $M(z)$ of form a group (for $ad - bc \neq 0, = 1$) as $SL(\mathbb{R}, 2)$ is a subgroup of $GL(\mathbb{R}, 2)$.
- Homogeneous coordinates $z = \frac{v_1}{v_2}$ for $v_i \in \mathbb{C}$.
- $[M]$ is a liner transform on homogeneous coordinates of z transforms to homogeneous coordinates of $M(z)$ i.e if $z = v_1/v_2$, $M(z) = w = \rho_1/\rho_2$. then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}.$$

(although homogeneous coordinates may not be unique but their ratios ought to be)

★ Properties:

- $z = (v_1/v_2)$ is a fixed point of $M(z)$ iff $[v_1 \ v_2]^T$ is an eigenvector of $[M]$.

3 Automorphisms, Conformality and map of unit disks

- any disk or half plain can be mapped to itself using mobius transform i.e. under specified mobius transforms say M_1 we can have $M_1(D) = D$ for a disk $D = \{z | |z - a| \leq r\}$ and for $M_2(\mathbb{H}) = \mathbb{H}$ for any half plane $\{z = x + iy | ax + by \geq c\}$. (note : this is mere a bijection with restrictions, not the identity map in disk or half plane.)

- more over the only conformal bijections (automorphisms) of disks \mapsto disks, half planes \mapsto half planes are **Mobius Transforms only**.

- Let C be a unit circle in \mathbb{C} and D be the unit disk it covers then

- mobius transform's are the only automorphisms conformal on this disk

- this mobius automorphism's have 3 degree of freedom (only 3 real numbers specify it)

- Now if two Mobius automorphisms M and N map two interior points to same image points i.e. the agree on two interior points then $M=N$ (as this takes 4 degree of freedom from both transforms)

- if D is centered at origin then these 3 degrees of freedom are a point in D ($a = (x + iy)$) that maps to origin and 1 is mapped to a point on C .

- as a is mapped to 0 , and mobius transform preserves symmetry b/w points and their images (inversion) we have the point $1/\bar{a}$ is mapped to ∞ (as C maps to itself, $a, 1/\bar{a}$ are symmetric w.r.t C so should be their images $0, \infty$).

- so now $a \mapsto 0 \implies M(z) = \frac{k(z-a)}{d}$, $1/\bar{a} \mapsto \infty \implies M(z) = k \frac{z-a}{\bar{a}z-1}$ and as $M(1) \in C \implies |M(1)| = 1 \implies k = e^{i\phi}$ so the automorphism of unit disk ($|z| \leq 1$) i.e. mobius transform is determined only by $a = x + iy \mapsto 0$ ($|a| < 1$) and $p \mapsto 1$ ($|p| = 1$) this is given by :

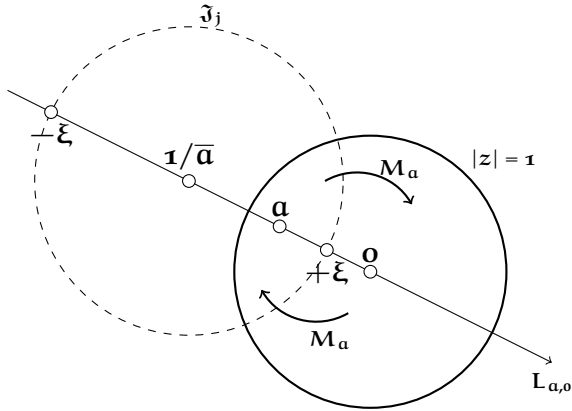
$$M_a^\phi(z) = e^{i\phi} \frac{z-a}{\bar{a}z-1}.$$

- now for

$$M(w) = \frac{pz + q}{\bar{q}z + \bar{p}}$$

for $|p| > |q|$ then $M(w)$ is an automorphism of unit disk (transform this to M_a^ϕ for $a = q/p$ and $e^{i\phi} = p/\bar{p}$.)

- clearly $M_a^\phi(z) = e^{i\phi} M_a^0(z)$ so is just rotation of $M_a^0 = M_a(z)$.
- properties of M_a .
- M_a is the only Mobius automorphism that swaps a and o (i.e. $M_a(a) = o, M_a(o) = a$.)
- now as an inversion about circle c maps circles orthogonal c to themselves (automorphism) thus automorphisms of unit circle can be viewed as inversions about circles orthogonal to unit circle to uncover this we break down that as $a \mapsto o$ and inversion circle is orthogonal to unit circle the center of inversion is on the line b/w a to o and as inversion is symmetric $1/\bar{a} \mapsto \infty$ we conclude that center of inversion is $1/\bar{a}$.
- as M_a is conformal the above inversion should be coupled with reflection (on line perhaps) to give the exact map, as this reflection leaves a, o fixed we conclude this is reflection about line a to o ($L_{a,o}$).
- thus $M_a = \mathfrak{R}_{L_{a,o}} \circ \mathfrak{I}_j$.
- thus fixed points ($\pm \xi$) of M_a is the intersection of $L_{a,o}$ and j .
- M_a is Involutory.



- if \mathbb{H}^\pm represents the upper or lower half

plane ($\text{Im}(z) > 0$ or < 0), $\delta = \Delta(o, 1)$ unit disk at origin and $\partial\Delta = \{|z| = 1\}$ then :

- for fixed $\beta \in \mathbb{C}, \theta \in \mathbb{R}$ if $\text{Im}(\beta) > 0$ then

$$w = f(z) = e^{i\theta} \frac{z - \beta}{z - \bar{\beta}}.$$

are the only conformal maps that maps

$\mathbb{H}^+ \mapsto \delta, \beta \mapsto o$ and real line $+\infty = \mathbb{R}_\infty \mapsto \partial\Delta$ (to see assume $|w| < 1 \iff |z - \bar{\beta}|^2 - |z - \beta|^2 > 0 \iff -2\text{Re}(z(\beta - \bar{\beta})) = 4(\text{Im}(z))(\text{Im}(\beta)) > 0$.)

- now if we use transform $R_0^\pi(z) = e^{i\pi}z = -z$ which rotates \mathbb{H}^+ to \mathbb{H}^- we get $g = f \circ \phi(Z)$.

$$g(z) = e^{i\theta} \frac{z - b}{z - \bar{b}}.$$

for $\text{Im}(b) < 0$, are the only conformal maps that map $\mathbb{H}^- \mapsto \delta, b \mapsto o$ and $\mathbb{R}_\infty \mapsto \partial\Delta$.

- similarly if $h(z) = f \circ R_0^{\pi/2}$

$$h(z) = e^{i\theta} \frac{z - \gamma}{z - \bar{\gamma}}.$$

for $\text{Re}(b) > 0$, are the only conformally maps that map Right half plane ($\text{Re}(z) > 0$) $\mapsto \delta, \gamma \mapsto o$.

- a Mobius transform $w = az + b/cz + d$ maps $\mathbb{H}^+ \mapsto \mathbb{H}^+$ iff $a, b, c, d \in \mathbb{R}, ad - bc > 0$ (i.e. automorphisms of \mathbb{H}^+ .)
- similar to above point a Mobius transform $w = az + b/cz + d$ maps $\mathbb{H}^- \mapsto \mathbb{H}^-$ iff $a, b, c, d \in \mathbb{R}, ad - bc < 0$ (i.e. automorphisms of \mathbb{H}^- .)

4 Stereographic projection

• To visually represent the whole complex plane and the point ∞ Riemann project the whole complex plane to a sphere : Riemann sphere (Σ) centered at origin a unit radius in 3 dimensions where the xy plane is \mathbb{C} .

• The point $N = (0, 0, 1)$ (north pole) maps to ∞ (in a pseudo sense) and every other point (z) is mapped to (\hat{z}) the point of intersection of the Riemann sphere and the line through N and the point.

- Properties:

- Unit circle $C = |z| = 1$ remains fixed
- interior of C is mapped to Southern hemisphere particularly $0 \mapsto (0, 0, -1) = S$ (south pole)
- exterior of C is mapped to Northern hemisphere
- A line in C is mapped to circle passing through N particularly the tangent of this circle at N is parallel to the line (in 3 dimensions)
- It is **conformal map** in accordance to an observer **from inside of Σ** .
- Stereographic projection is can be broken down as inversion in the plane through $\{N, z \mapsto \hat{z}\}$: if K is a circle centered at N of radius $\sqrt{2}$ in the plane where line through N and z passes then \hat{z} is the image $\mathfrak{I}_K(z)$ in this plane (this plane is considered as C for $\mathfrak{I}_K(z)$.)
- From above it is clear that Circles are mapped to circles in particular origin centered circles are mapped to horizontal circles (i.e circles in planes parallel to xy plane)
- Properties related to functions:
 - Complex conjugation in C . induces a reflection of the Riemann sphere in the vertical plane passing through the real axis.
 - Inversion of C in the unit circle induces a reflection of the Riemann sphere in its equatorial plane (i.e. Northern hemisphere \longleftrightarrow Southern Hemisphere).
 - The mapping $z \rightarrow (1/z)$ in C induces a rotation of the Riemann sphere about the real axis through an angle of π .
 - properties functions like conformality at ∞ can be checked through Stereographic projection.
- formulas of Projection
 - if $z \mapsto (X, Y, Z)$ then:

$$\mathbf{Z} = \frac{|z|^2 - 1}{|z|^2 + 1}, \quad \mathbf{X} + i\mathbf{Y} = \frac{2z}{1 + |z|^2} = \frac{2x + i2y}{1 + x^2 + y^2}$$
 - if $z \mapsto (\theta, \phi)$ for θ angle subtended around z axis in xy plane and ϕ angle subtended at center by N and \hat{z} then:
 - $z = \cot(\phi/2)e^{i\theta}$ or
 - $\theta = \text{Arg}(z), \quad \phi = 2 \cot^{-1}(|z|).$

5 Analyticity

- if $z(x + iy) \mapsto f(z) = w(u + iv)$ then $df = du + i dv$ $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$ and $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$ i.e.

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}.$$

- where the linear transform is the Jacobian matrix of f .
- now in C if $df(w) = f'(z)dz$ to be true $f'(z)$ should not depend on dz i.e. each infinitesimal vector dz at z should transform to dw at $w = f(z)$ by the same factor $f'(z)$ no matter the direction of dz .
- this condition tells us that dw is just the amplification and rotation or twist or together **amplitwist** of dz (as $f'(z) \in C \implies dw = f'(z)dz = r'e^{i\theta'}dz$.)
- now if f is differentiable at z then $f'(z)$ exist so the infinitesimal map at point z is an amplitwist.
- clearly amplitwist is conformal (as amplification and twist is)
- now for the converse if a map is conformal at z then it is not presupposed to be amplitwist at z as the amplification may vary but if we presuppose that the map is locally conformal at z (i.e in some whole neighborhood) then clearly the map is locally amplitwist at z (as infinitesimal Δ is mapped to similar infinitesimal Δ).
- By above we define **Analytic functions** : functions in C whose effect are locally (infinitesimal) an amplitwist or a function is analytic at z if it is differentiable at z and in a neighborhood of z . (as differentiable in neighborhood makes it locally conformal).
- Thus we have an **Analytic function is Conformal**.
- Geometric properties of Analytic function:
 - infinitesimal circles are mapped to infinitesimal circles
 - A mapping between spheres represents an analytic function iff it is conformal.

- Conformality of analytic functions breakdown near critical points ($f'(z) = 0$) and branch points.

- Geometric property of general transform on \mathbb{C} : as jacobian is a linear transform by singular value decomposition of 2×2 matrices we have the local linear transform by a complex mapping is a stretch in direction (\mathbf{d}), another stretch in direction perpendicular to in (\mathbf{d}^\perp), and finally a twist. in particular an infinitesimal circle is transformed to an ellipse (may not be conformal).

- **C-R equations :**

- now as f is analytic $\implies f'(z) \in \mathbb{C}$ so multiplying by Jacobian matrix is equivalent to a complex multiplication now as

$$(a + ib)(x + iy) = (ax - by) + i(bx + ay)$$

$$\longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax - by \\ bx + ay \end{bmatrix}$$

. we have $J = \begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. i.e.

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v.$$

$$i\partial_x f = \partial_y f.$$

which gives the Cartesian-Cartesian form(C-C) now in Polar-Cartesian (P-C) form we have $f(re^{i\theta}) = u + iv$ and C-R equations are

$$\partial_\theta v = r\partial_r u, \quad \partial_\theta u = -r\partial_r v.$$

$$\partial_\theta f = ir\partial_r f.$$

(P-P) form $f(re^{i\theta}) = Re^{i\Psi}$ C-R equations

$$\partial_\theta R = -rR\partial_r \Psi, \quad R\partial_\theta \Psi = r\partial_r R.$$

(C-P) form $f(x + iy) = Re^{i\Psi}$. C-R equations

$$\partial_x R = R\partial_y \Psi, \quad \partial_y R = -R\partial_x \Psi.$$

- General properties of Analytic functions:

- if f, g are analytic then $f + g, f \times g, f \circ g, f^{-1}$ are analytic when ever they are defined, in particular as f is amplitwist locally there is a 1-1 correspondence in a neighbourhood of non critical points to their images \implies that local inverse exists.

- if f is analytic in E then so is f' (i.e. f is infinitely differentiable in the defined region)

- every zero or an analytic point is isolated (generally p -point of f or pre-image of p in f doesn't have a limit point.)

- **Identity/Uniqueness Theorem:** restating the above we have, if $f(z)$ is analytic in D and if S set of zeroes has a limit point in D then $f(0) \equiv 0$ in D (in general if p -points of f has a limit point then $f(z) \equiv p$).

- Extending the above we get, if even an arbitrarily small segment of curve is crushed to a point by an analytic mapping, then its entire domain will be collapsed down to that point (i.e. the function is constant) (this property is known as **Rigidity**)

- from above if f, g analytic agree on a curve or more generally $\{a_n\} \mapsto a$ then $f \equiv g$.

- if some identity for f analytic holds when restricted to \mathbb{R} then it holds for entire \mathbb{C} . (eg: odd and evenness.)

6 Analytic continuation

- an analytic function or a power series can be extended (from defined) to other regions this is analytical so called Analytic continuation.

- Analytic continuation via reflection:

- if f is an generalization of a real function (defined on \mathbb{R}) and is known in upper or lower parts of real axis (in some region with some parts of \mathbb{R} as boundary) then it can be **analytically continued** by $f^*(z) = \overline{f(\bar{z})}$ in the other half part (reflection by \bar{z} part of region)(this holds by property of rigidity of analytic functions).

- In general if f maps a line (L) to another line (\hat{L}) then we can analytically continue one side of L to the other by using the fact that points symmetric in L map to points symmetric in \hat{L} .

- similarly if f maps a circle C to circle \hat{C} then mobius transforms can be used to translated these to symmetries i.e. $M : C \mapsto L, \hat{M} : \hat{C} \mapsto \hat{L}$ (as composition by mobius transfor

ehic are analytic doesnt change the analyticity of $f \mapsto \hat{M} \circ f \circ M^{-1}$.

• **Schwarzian Reflection:**

■ Given a sufficiently smooth curve K , it is possible to find an analytic function $S_K(z)$ such that $z \in K \implies S_K(z) = \bar{z}$ then

■ Schwarz function of $K = \tilde{z} = \Re_K(z) = \overline{S_K(z)}$.

■ clearly if $q \in K$ $\tilde{q} = \overline{S_K(q)} = \bar{q} = q$ i.e. remains unchanged.

■ Also as S_K just amplitwists infinitesimal disk at $q \in K$ to infinitesimal disk in $\bar{q} \in \bar{K}$ we observe that for $S_K|_{qp} \mapsto \bar{q}\bar{p}$ (for $p, q \in K$, qp infinitesimal) amplification = 1 and twist = -2ϕ where ϕ is the angle b/w tangent to K at q with horizontal

■ so from above we get if α is on infinitesimal circle passing through K then $\tilde{\alpha} = \Re_K(\alpha)$ is reflection along the tangent of K . i.e \Re_K near K is sort of like Reflection in K (pseudo).

■ \Re_K is anticonformal so $\Re_K \circ \Re_K$ is conformal so analytic (as amplification=1) and as $\Re_K \circ \Re_K$ maps infinitesimal areas around K to itself thus agrees with Identity so is Identity i.e. $\Re_K \circ \Re_K(z) = z$.

■ Now if K is a smooth enough curve to posses S_K and any analytical map f defined on a region bordering K such that $\hat{K} = f(K)$ also posses $S_{\hat{K}}$ then we can analytically continue f around K (reflection of region by K) by demanding points symmetric to K are mapped to points symmetric to \hat{K} by f and this analytic continuation is given by:

$$F = \Re_{\hat{K}} \circ f \circ \Re_K.$$

7 Complex Integration

• we define complex integration as the generalized Riemann Integration over a given path α to b or as contour integration

• clearly integration here depends on path

• complex integration can be visualized as weighted vector sum : if S is path from a to b and Δ_j 's are vector decomposition (partition

of S and linearly) that form S , $w_j = f(\text{mid } \Delta_j)$ i.e $f(\text{mid points of } \Delta_j)$ then we can generalize as

$$\int_S f(z) dz = \sum_{j \rightarrow \infty} w_j \Delta_j$$

• from above we get: if $|f| \leq M$ in image of K . then

$$\left| \int_S f(z) dz \right| \leq M \cdot \text{length of } K.$$

• **Winding number and properties :**

• winding number for a closed loop L and a point $a = v(L, a)$ is the number of revolutions $z - a$ makes as it traces L (where fixing a direction and counterclockwise revolution is +ve and clockwise is -ve by convention)

• A simple loop is a closed curve that doesnt intersect with itself

• now as a point moves from left to right if it crosses a boundary of the loop and the loops direction is downwards (upwards) the winding number increased (decreased) by 1 (here the first entry of the point to loop is made to be in loop moving in downwards direction).

• we define inside of a loop L to be regions (points) where $v[L, a] \neq 0$.

• Hopf's degree Theorem(stricted to \mathbb{C}): A loop K may be continuously deformed into another loop L , without ever crossing the point p , if and only if K and L have the same winding number round p .

• disp-point of a function f if set of pre-images of p in f contains d i.e. $d \in f^{-1}(p)$. (pre-image)

• **Argument-Principle** theorem: If $f(z)$ is analytic inside and on a simple loop Γ , and N is the number of p -points (counted with their multiplicities) inside Γ , then $N = v(f(\Gamma), p]$.

• if f analytic, $f(a) - p = 0$ and for $\Delta = z - a$ $f(a + \Delta) = p + \Omega(Z)\Delta^n$ (obtained by Taylor series) here algebraic multiplicity of a in f is n , for sufficiently small circle C_a around a that doesnt have any other p -points then

$$v(f(C_a), a) = n.$$

i.e. $f(C_a)$ loops around p exactly n times.

- now we define $v(a)$ for a continuous function h as : if $h(a) = p$, Γ_a is the loop having only a and no other p -points then topological multiplicity $v(a) = v(h(\Gamma_a), a)$.

- clearly as analytical maps are conformal we have $v(a)$ is always +ve ($\neq 0$.) for analytic functions

- $v(a) = \text{sign of } \det(J(a))$ where J is Jacobian

- **Topological Argument-Principle theorem**: for a continuous map h the total number of p -points inside Γ . (counted with their topological multiplicities) is equal to the winding number of $h(\Gamma)$ round p .

- **Darboux's Theorem** : If an analytic function h maps Γ onto $h(\Gamma)$ in one-to-one fashion, then it also maps the interior of Γ onto the interior of $h(\Gamma)$ in one-to-one fashion.

- **Rouche's Theorem** : for f, g analytic in and on Γ , If $|g(z)| < |f(z)|$ on Γ , then $(f + g)$ must have the same number of zeros inside Γ as f .

- **Brouwer's Fixed Point Theorem** : any continuous mapping of the disc to itself will have a fixed point.

In general there must be a fixed point if the disc is mapped into its interior and there are at most a finite number of fixed points. (now if the map is analytic then the number of fixed points inside the disk is only one).

- **If f is analytic inside and on a simple loop Γ then no point outside $f(\Gamma)$ can have a pre-image inside Γ .** (i.e interior of Γ maps to interior of $f(\Gamma)$.)

- **Maximum Modulus Theorem** : The maximum (minimum) of $|f(z)|$ on a region where f is analytic is always achieved by points on the boundary, never ones inside.

- **Schwarz's Lemma** : If an analytic mapping of the disc to itself leaves the center fixed, then either every interior point moves nearer to the center, or else the transformation is a simple rotation. (i.e. the map is contractive towards the center).

- **General Schwarz's Lemma** :

If $f : \Delta(\{ |z| < 1 \}) \mapsto \overline{\Delta}$ is analytic and has a zero of order n at origin then:

■

$$|f(z)| \leq |z|^n \quad \forall z \in \Delta.$$

■

$$|f^n(0)| \leq n!$$

■ Equality holds (any one) for any point in Δ other than 0 iff $f(z) = az^n, |a| = 1$.

- **Schwarz-Pick Lemma** : Unless an analytic mapping of the unit disc to itself is a automorphism the hyperbolic separation of every pair of interior points decreases.

i.e.

if f is analytic on Δ , $|f(z)| \leq 1 \forall z \in \Delta$ and $f(a) = b$ for some $a, b \in \Delta$, then

$$|f'(a)| \leq \frac{1 - |f(a)|^2}{1 - |a|^2}.$$

and for $a, a' \in \Delta$

$$\rho(f(a), f(a')) \leq \rho(a, a').$$

where $\rho(z, a) = |(z - a)/(\bar{a}z - 1)|$.

- **Liouville's Theorem** : An analytic mapping cannot compress the entire plane into a region lying inside a disc of finite radius without crushing it all the way down to a point, i.e. a bounded entire function is constant or bounded harmonic function is constant (by Taylor series)

- **Generalized Liouville's Theorem** : if f is an entire function such that $|f(z)| \leq M|z|^\alpha$ for all sufficiently large $|z|$ and $\alpha \geq 0$, $M > 0$ then f reduces to a polynomial of maximum degree n closest integer to α .

- **Generalized Argument-principle theorem** : Let f be analytic on a simple loop Γ and analytic inside except for a finite number of poles. If N and M are the number of interior p -points and poles, both counted with their multiplicities, then $v(f(\Gamma), p) = N - M$.

- for any closed loop L $\oint_L \frac{1}{z} dz = 2\pi i v(L, 0)$ in general

$$\oint_L \frac{1}{z - p} dz = 2\pi i v(L, p).$$

- now as $\text{Im}(\bar{a}\bar{b}) \equiv \mathbf{a} \times \mathbf{b}$ it gives $2 \times$ the area enclosed by triangle formed by sides \mathbf{a} and \mathbf{b} vectors so we have for a simple loop L :

$$\oint_L \bar{z} dz = 2i \times \text{area enclosed by } L.$$

for general loop L

$$\oint_L \bar{z} dz = 2i \times \sum_{\text{inside}} v_j A_j.$$

where A_j is the area enclosed by points which have $v_j = v(L, p) = \mathbf{a} \neq \mathbf{o}$ constant (i.e form a part of loop).

- **Cauchy's Theorem** : If an analytic mapping has no singularities "inside" a loop, its integral round the loop vanishes (i.e. = 0).
- from above we get in integral of analytic functions are **path independent**.
- **Morera's Theorem** : If all the loop integrals of f are known to vanish in a region then f is analytic in that region.
- if $m \neq -1$ then

$$\int_A^B z^m dz = \frac{1}{m+1} (B^{m+1} - A^{m+1})$$

- clearly from above we have

$$\oint z^m dz = 0 \text{ if } m \neq -1.$$

- **Deformation Theorem** : If a contour sweeps only through analytic points as it is deformed, the value of the integral does not change.
- **Cauchy's formula** : if $f(z)$ is analytic inside a simple loop L then

$$f^n(a) = \frac{n!}{2\pi i} \oint_L \frac{f(z)}{(z-a)^{n+1}} dz.$$

- **General Cauchy's theorem** : if L is not simple then

$$v(L, a) f^n(a) = \frac{n!}{2\pi i} \oint_L \frac{f(z)}{(z-a)^{n+1}} dz.$$

- **Taylor Series** : If $f(z)$ is analytic, and a is neither a singularity nor a branch point, then $f(z)$ may be expressed as the following power series, which converges to $f(z)$ within the disc whose radius is the distance from a to the nearest singularity or branch point:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n. \text{ where}$$

$$c_n = \frac{f^n(a)}{n!} = \frac{1}{2\pi i} \oint_L \frac{f(z)}{(z-a)^{n+1}} dz.$$

- **Laurent Series** : if f is analytic inside an annulus centered at a then f can be expressed as the following series (for any simple loop K inside the annulus)

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n. \text{ where}$$

$$a_n = \frac{1}{2\pi i} \oint_L \frac{f(z)}{(z-a)^{n+1}} dz.$$

- **General Residue Theorem** : from Laurent series and integral of z^m we have if f is analytic then for a loop L containing only isolated singularities $\{a_k\}$ of f , we have:

$$\oint_L f(z) dz = 2\pi i \sum_k v[L, a_k] \text{Res}(f, a_k).$$

where $\text{Res}(f, a_i) = a_{-1}$ or coefficient of $1/(z-a_i)$ when f is written as Laurent series centered at a_i containing no other singularity.

- if a is a pole of f of order m . (i.e. $\lim_{z \rightarrow a} (z-a)^m f(z) = c$ defined) then $\text{Res}(f(z), a)$

$$= \lim_{z \rightarrow a} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z).$$

- if P/Q has a simple pole (order 1) at a then

$$\text{Res}\left(\frac{P}{Q}(z), a\right) = \frac{P(a)}{Q'(a)}.$$

- **Gauss mean value theorem** : for a harmonic function ϕ ($\partial_x^2 \phi + \partial_y^2 \phi = 0$) the mean value of ϕ on a circle is equal to the value of function at

center of the circle i.e.
if $f(z)$ is analytic then

$$\frac{1}{2\pi} \int_0^{2\pi} f(a + re^{i\theta}) d\theta = f(a)$$

- Residue at infinity : for analytic f we have

$$\text{Res}(f(z), \infty) = -\text{Res}\left(\frac{f(1/z)}{z^2}, 0\right).$$

$= \frac{1}{2\pi i} \oint_{C^-} f(z) dz = -a_{-1}$, where C^- is a circle oriented $-$ vely covering all singularities ($\neq \infty$) of $f(z)$.

- **Extended Residue theorem:** for analytic f we have

$$\text{Res}\left(\frac{f(1/z)}{z^2}, 0\right) = \sum_k \text{Res}(f, a_k)$$

where $a_k \neq \infty$ also if simple loop γ includes all finite singularities of $f(z)$ then

$$\oint_{\gamma} f(z) dz = 2\pi i \text{Res}\left(\frac{f(1/z)}{z^2}, 0\right).$$

- **Argument-Principle theorem (integral form)** : if $f(z)$ is a meromorphic function in domain $D \subseteq \mathbb{C}$, has finitely many zeroes and poles in D , C is any simple loop in D such that no pole or zero lie 'on' C then

$$\oint_C \frac{f'(z)}{f(z)} dz = 2\pi i (N - P).$$

where N and P denote the number of zeroes and poles of f inside C (counted with their multiplicities and order).

- **General Rouché's Theorem** : for f, g analytic in and on C with finite number of poles and zeroes inside the Domain covering C , If $|g(z)| < |f(z)|$ on C , then

$$N_{f+g} - P_{f+g} = N_f - P_f$$

where N_h, P_h denote the number of zeroes and poles of h inside C (counted with their multiplicities and order).

- Alternative form of Rouché's Theorem : if same conditions as above hold for $g - f(z)$, $f(z)$ and $|g(z) - f(z)| < |f(z)|$ then

$$N_g - P_g = N_f - P_f.$$

(can used for calculating the number of zeroes of polynomial in a give loop)

- Application of Rouché's Theorem to polynomials

■ eg: consider the polynomial $g(z) = z^6 - 5z^4 + 7$

★ now $|g(z) - 7| \leq |z|^6 + 5|z|^4 \leq 7$ if $|z| \leq 1$ (as $6 + 1 \leq 7$) thus $g(z)$ has same number of zeroes as $f(z) = 7$ in $|z| \leq 1$ i.e. $g(z)$ has no zeroes inside $|z| \leq 1$.

★ similarly if $f(z) = -5z^4$ we have $|g(z) - f(z)| \leq |z|^6 + 7 \leq 5|z|^4$ if $|z| \leq 2$ (as $2^6 + 7 = 71 \leq 5 \cdot 2^4 = 80$) thus $g(z)$ has 4 zeroes in $|z| \leq 2$.

★ similarly if $f(z) = z^6$ we have $|g(z) - f(z)| \leq 5|z|^4 + 7 \leq |z|^6$ if $|z| \leq 3$ (as $5 \cdot 3^4 + 7 = 412 \leq 3^6 = 729$) thus all zeroes of $g(z)$ lie inside $|z| \leq 3$.

8 Mics Properties

- A real valued function of a complex variable $f : \mathbb{C} \mapsto \mathbb{C}$ has derivative zero or non existent i.e if f is analytic the is a constant.

- for an analytic function in domain D if one of : $|f|, \text{Re}(f), \text{Im}(f), \text{Arg}(f)$ is constant in D then f is constant.

- **Harmonic functions:**

■ $\phi(x, y)$ a real valued function is harmonic iff $\nabla^2 \phi = 0$.

■ real and imaginary parts of analytical function's are harmonic (in the defined "Domain" (a connected open set)) (converse is not true).

■ $f(z)$ is analytic in Domain D iff real and imaginary parts of both $f(z)$ and $zf(z)$ are harmonic.

■ if ϕ is a harmonic function in a Domain the $f = \phi_x - i\phi_y$ is analytic in the domain.

■ Harmonic conjugate of harmonic function ϕ is another harmonic function ψ such that

$f = \phi + i\psi$ (i.e ψ is the imaginary part of analytic function whose real part is ϕ).

- if ϕ is harmonic in a simply connected region then it has a harmonic conjugate in this region.

- if f is analytic in a simply connected region Ω and $f(z) \neq 0$ in Ω then $\exists h$ analytic in Ω such that

$$e^{h(z)} = f(z).$$

($h'(z) = f'(z)/f(z)$ claim $f \cdot e^{-h(z)} = c = e^k$ prove by differentiating) (domain can be whole \mathbb{C}).

- if f satisfies the above conditions then $\exists g$ analytic in Ω such that $g^2(z) = f(z)$ in Ω (choose $g(z) = e^{h(z)/2}$).

- **Cauchy's Inequality** : if f is analytic in an open disk centered at a of radius $R = \Delta(a, R) = \{z - a \mid |z - a| < R \text{ and } |f(z)| \leq M \text{ on boundary } \Delta(a, r) \text{ for } 0 < r < R \text{ then}$

$$|f^k(a)| \leq \frac{M \cdot k!}{r^k}.$$

- for an open set D if $f_n : D \mapsto \mathbb{C}$ are analytic for each n and if $f_n \mapsto f$ uniformly on each compact subset of D then f is analytic and more over $f_n^k \mapsto f^k$ uniformly in the compact subsets, the same is true for series also if all conditions hold.

- every zero of an analytical function is isolated.

- from above we have if a_n are the zeros of analytical map f , $a_n \mapsto a \in \mathbb{C}$ then $f \equiv 0$.

- in general if q_n are p -points of analytical map f , $q_n \mapsto q \in \mathbb{C}$ then $f \equiv p$ (use $h(q_n) = f(q_n) - p = 0$.)

- also if f, g analytic in Domain D , $f - g$ has set S of zeroes that has a limit point then $f \equiv g$ in D (in general if $f - g$ has set Q of p -points that has a limit point then $f(z) = g(z) + p$.)

- four distinct points in \mathbb{C}_∞ all lie on a circle or line iff their cross ratio is real.

- a singularity at z_0 of $f(z)$ is removable if f can be defined at z_0 so that it is analytic at z_0 .

- **Riemann's Removable Singularity theorem**: if f has an isolated singularity at z_0 then z_0 is removable iff one of the below holds.

- f is bounded in deleted neighborhood of z_0 .

- $\lim_{z \mapsto z_0} f(z)$ exists

- $\lim_{z \mapsto z_0} (z - z_0)f(z) = 0$.

- **Picard's Little Theorem** : every non constant entire function only omits at most one value from this we get if a entire function omits two value then it is a constant.

- **Picard's Great theorem** : if z_0 is the essential singularity of $f(z)$ analytic in then $\Delta(z_0, r) - z_0$ then $\mathbb{C} - f(\Delta(z_0, r) - z_0)$ is a singleton set.

- **Picards little theorem for meromorphic functions**: A meromorphic function omits three distinct values then it is a constant.

- if f is an even analytic function (i.e. $f(-z) = f(z)$) then for z_0 isolated singularity of f $\text{Res}(f(z), z_0) = 0$. (there are no odd power terms in Laurent series expansion).

- if analytic function f is such that $f(z) = f(z + z_1) = f(z + z_2)$ (doubly periodic) and if $z_1/z_2 \notin \mathbb{R}$ then f is a constant (as z_1, z_2 will be linearly independent).

- if $p(z)$ is a polynomial of degree $n \geq 1$ then every zero of $p'(z) : (z'_k)$ lies in the complex hull of zeroes of $p(z) : (z_k)$ i.e $z'_k = \sum_{k=1}^n \lambda_k z_k$, for $\sum_{k=1}^n \lambda_k = 1$.

- if f is analytic in $|z| < M$ iff $\overline{f(\bar{z})}$ is also analytic in $|z| < M$ (as amplitwistness of $f(z)$ doesn't change).

- if $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_{n-1} z^{n-1} + z^n$, simple loop C covers all zeroes of $p(z)$ then

$$\oint_C \frac{zf'(z)}{f(z)} = -2\pi i a_{n-1}.$$

$$\oint_C \frac{z^2 f'(z)}{f(z)} = 2\pi i (a_{n-1}^2 - 2a_{n-2}).$$

- z_1, z_2 and z_3 are vertices of equilateral triangle iff

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0.$$

i.e.

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

- z_1, z_2 and z_3 iff

$$z_3 = t(z_1) + (1 - t)z_2 \text{ for } t \in \mathbb{R}$$

(i.e equation of line in $2\mathbb{D}$.)

- if analytic function $f(z)$ is real on real line and purely imaginary on imaginary axis then $f(-z) = -f(z)$ i.e. f is odd.
- for $f(z)$ analytic in Domain \mathbb{D} then:
 - if f is even i.e. $f(z) = f(-z)$ then $\exists g(z)$ analytic in \mathbb{D} such that $f(z) = g(z^2)$.
 - if f is odd i.e. $-f(z) = f(-z)$ then $\exists g(z)$ analytic in \mathbb{D} such that $f(z) = zg(z^2)$.
 - Every meromorphic function in \mathbb{C} can be represented as quotient of two entire functions.
 - **Open mapping Theorem** : if $f(z)$ is a non constant analytic function in Domain \mathbb{D} then it is open mapping i.e. $f(\mathbb{O})$ is open for every open set $\mathbb{O} \in \mathbb{C}$.
- Clearly if f is analytic in \mathbb{D} a Domain (open connected set) then $f(\mathbb{D})$ is also a Domain.
- **Hurwitz's Theorem** : if $\{f_n\}$ are non vanishing ($\neq 0$) in a Domain \mathbb{D} and converges uniformly to f on every compact subset of \mathbb{D} then either f has no zeroes or $f \equiv 0$.

- **Local mapping theorem** : if f is analytic at a then there exist a neighborhood of a where f is one-one iff $f'(a) \neq 0$. or

if f is univalent and analytic in a Domain \mathbb{D} then $f'(z) \neq 0$ in \mathbb{D} .

- if f is meromorphic at pole a and is one-one in neighborhood of a iff a is a simple pole.

- from above if f is meromorphic and univalent in \mathbb{D} then f has only simple poles in \mathbb{D} .

- for f analytic at ∞ is univalent at ∞ (in its nbd) iff $\text{Res}(f, \infty) \neq 0$.

- **Riemann mapping theorem** : every simply connected domain which is a proper subset of \mathbb{C} is Conformally equivalent to a unit disk i.e. if Ω is a simply Connected open set then there exist a function f analytic in Ω such that $f(\Omega) = \Delta$.

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