

Matrix Properties

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Symbols used:

iff	→	if and only if
Capital letters	→	Matrices $\mathbf{A}_{m \times n} = [a_{ij}]_{m \times n}$
\mathbf{A}^T or \mathbf{A}'	→	Transpose of Matrix
$\bar{\mathbf{A}}$	→	Conjugate of Matrix
\mathbf{AB}	→	Matrix product
$ \mathbf{A} $ or $\det(\mathbf{A})$	→	Determinant of Matrix
$\text{tr}(\mathbf{A})$ or $\text{trace}(\mathbf{A})$	→	trace of Matrix
\mathbf{A}^*	→	Conjugate transpose of Matrix
\mathbf{A}^{-1}	→	Inverse of Matrix
\mathbf{I}	→	Identity
$\text{im}(\mathbf{A})$	→	Image or range space of \mathbf{A}
$\text{rank}(\mathbf{A})$ or $r(\mathbf{A})$	→	Dimension of Range space of \mathbf{A}
$\text{ker}(\mathbf{A})$	→	Null space of \mathbf{A}
$\text{null}(\mathbf{A})$	→	Dimension off Null space of \mathbf{A}
\mathbb{F}	→	Field

1 Basic properties

- $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$
- $\text{tr}(\mathbf{AB}) = \text{tr}(\mathbf{BA})$
- $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}'$
- $(\mathbf{AB})^* = \mathbf{B}^*\mathbf{A}^*$
- if \mathbf{A} is Hermitian then $i\mathbf{A}$ is skew-Hermitian and vise-versa.
- if \mathbf{A}, \mathbf{B} are symmetric, \mathbf{AB} is symmetric iff $\mathbf{AB} = \mathbf{BA}$.
- $\mathbf{AA}', \mathbf{A}'\mathbf{A}$ are always symmetric.
- For any Square Matrix \mathbf{A} :
 - $\mathbf{A} + \mathbf{A}'$ is symmetric.
 - $\mathbf{A} - \mathbf{A}'$ is skew- symmetric.
 - $\mathbf{A} + \mathbf{A}^*$ is Hermitian.
 - $\mathbf{A} - \mathbf{A}^*$ is skew-Hermitian .

- By preceding point any Square matrix can be decomposed (by +) into symmetric - skew-symmetric or Hermitian- skew-Hermitian pair.
- $\mathbf{B}'\mathbf{AB}$ is symmetric or skew as is \mathbf{A}
- $\mathbf{B}^*\mathbf{AB}$ is hermitian or skew as is \mathbf{A}
- Determinant is a Multilinear (row), Alternating and Normalized Function on Matrices.
- Determinant of upper or lower triangle or diagonal matrix is equal to product of diagonal elements.
- $|\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = |\mathbf{BA}|$
- $|\mathbf{A}'| = |\mathbf{A}|$
- $|\mathbf{A}^*| = |\bar{\mathbf{A}}|$
- \mathbf{A} is invertible iff $|\mathbf{A}| \neq 0$.
- $\mathbf{A}^{-1} = \frac{\text{adj}(\mathbf{A})}{|\mathbf{A}|}$ where $\text{adj}(\mathbf{A})$ is the transpose of co-factor matrix.
- $\mathbf{B}^{-1} - \mathbf{A}^{-1} = \mathbf{B}^{-1}(\mathbf{A} - \mathbf{B})\mathbf{A}^{-1}$
- Cramer's rule for a system of linear equations $\mathbf{Ax} = \mathbf{b}$ where \mathbf{A} is square and for $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$ we have $x_i = \frac{|\mathbf{A} \leftarrow_i \mathbf{b}|}{|\mathbf{A}|}$ where $\mathbf{A} \leftarrow_i \mathbf{b}$ is obtained by replacing i^{th} column of \mathbf{A} by \mathbf{b} .
- $[\mathbf{A} \leftarrow_i \mathbf{b}]_{i=1..n} = \text{adj}(\mathbf{A})\mathbf{b}$
- $|\text{adj}(\mathbf{A})| = |\mathbf{A}|^{n-1}$ where \mathbf{A} is an $n \times n$ matrix
- $\text{adj}(\mathbf{A}^*) = \text{Adj}(\mathbf{A})^*$
- $\text{adj}(\mathbf{A}^{-1}) = \text{adj}(\mathbf{A})^{-1} = \mathbf{A}/|\mathbf{A}|$
- $\text{adj}(\text{adj}(\mathbf{A})) = |\mathbf{A}|^{n-2}\mathbf{A}$
- $\text{adj}(\mathbf{AB}) = \text{adj}(\mathbf{B})\text{adj}(\mathbf{A})$ for non-singular matrices \mathbf{A}, \mathbf{B} .
- \mathbf{A} is orthogonal if $\mathbf{A}'\mathbf{A} = \mathbf{I}$

• A is orthogonal $\implies |A| = \pm 1 \implies$ invertible.

• A is unitary if $A^*A = I$

• if A, B are orthogonal the so are AB, BA similar in hermitian case also.

• $\text{rank}(A) = r$ iff all the $r+1$ order minors are zero i.e. if any one of r order minor is non zero then $\text{rank}(A) \geq r$.

• $\text{rank}(A) = \text{rank}(A') = \text{rank}(A^*)$

• Elementary transformation: exchange of rows, multiplication of row by non zero constant, addition of k multiple of a row to another row.

• Elementary transformations doesn't change the rank of a matrix.

• Every elementary transformation has a corresponding non singular matrix which when pre-multiplied to a given matrix gives the respective operation.

• Normal form of a matrix: (row reduced Echelon form) A matrix which can be partitioned into identity and null matrices and the identity is present in upper-left part.

• $\exists P, Q$ non-singular square matrices such that $N = PAQ$ where A is any matrix and N is its normal or row-reduced Echelon form.

• $\text{rank}(AB) \leq \min(\{\text{rank}(A), \text{rank}(B)\})$.

• $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

• **Sylvester inequality :**

for any matrices $A_{m \times k}, B_{k \times n}$

$$\text{rank}(AB) = \text{rank}(B) - \dim(\text{Im}(B) \cap \ker(A))$$

$$\text{so } \text{rank}(A) + \text{Rank}(B) - k \leq \text{rank}(AB)$$

$$\leq \min(\{\text{rank}(A), \text{Rank}(B)\}).$$

(use: for $Bx \neq 0, ABx = A(Bx) = 0$ iff $x \in \text{Im}(B) \cap \ker(A)$ and that $\dim(\text{Im}(B) \cap \ker(A)) \leq \text{null}(A) = k - r(A)$ so $-\dim(\text{Im}(B) \cap \ker(A)) \geq r(A) - k$.)

• **Frobenius Inequality :**

for $A_{m \times k}, B_{k \times p}, C_{p \times n}$

$$\text{rank}(AB) + \text{rank}(BC) \leq \text{rank}(B) + \text{rank}(ABC).$$

• $\text{rank}(A) = \text{rank}(A^*A)$

• if all entries of A are real then $\text{rank}(A'A) = \text{rank}(A)$.

• if A is n -squared then :

■ $\text{rank}(A) = n \implies \text{rank}(\text{adj}(A)) = n$.

■ $\text{rank}(A) = n-1 \implies \text{rank}(\text{adj}(A)) = 1$.

■ $\text{rank}(A) < n-1 \implies \text{rank}(\text{adj}(A)) = 0$

i.e. $\text{adj}(A) \equiv 0$. (use minors and cofactor definition of $\text{Adj}(A)$.)

• $\text{rank}(A) \geq \text{rank}(A^2) \geq \dots \geq \text{rank}(A^n) \geq \dots$

• $\text{null}(A) \leq \text{null}(A^2) \leq \dots \leq \text{null}(A^n) \leq \dots$

• if $\text{rank}(A^m) = \text{rank}(A^{m+1})$ then

■ $\text{rank}(A^k) = \text{rank}(A^m) \quad \forall k \geq m$

■ $\text{null}(A^k) = \text{null}(A^m) \quad \forall k \geq m$

• Eigenvalues of Hermitian matrices are real.

(if λ is eigenvalue then $(Ax)^* = x^*A^* = x^*A = (\lambda x)^* = \bar{\lambda}x^*$ so $x^*A^*x = \lambda x^*x = \bar{\lambda}x^*x \implies \bar{\lambda} = \lambda$)

• Eigenvalues of Skew-Hermitian are purely imaginary or zero.

• If λ is Eigenvalue of Unitary matrix A then $|\lambda| = 1$

(if $Ux = \lambda x$ then $x^*U^*Ux = x^*Ix = x^*x$ but $(x^*U^*)(Ux) = \bar{\lambda}\lambda x^*x$.)

• Real Eigenvalues of Orthogonal Matrices are $1, -1$ only.

• Eigenvalues of A and A' are same.

• Eigenvalues of triangular, diagonal matrices are its diagonal elements.

• if λ is a eigenvalue of non-singular matrix A then

■ $\lambda \neq 0$

■ $\frac{1}{\lambda}$ is the eigenvalue of A^{-1} .

■ λ^k is the eigenvalue of A^k .

■ $\frac{|A|}{\lambda}$ is the eigenvalue of $\text{adj}(A)$.

• if $\{\lambda_i\}$ are eigenvalues of A then eigenvalues of $B = p(A)$ are of form $p(\lambda_i)$ only.

• For A_n with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$
 $\text{trace}(A) = \sum_{i=1}^n \lambda_i$, $\det(A) = \prod_{i=1}^n \lambda_i$ and
 $\text{trace}(\text{adj}(A)) = \sum_{i=1}^n \prod_{j \neq i} \lambda_j$.

• If $A = P^{-1}BP$ the A and B have same eigenvalues

• for square Matrices A, B eigenvalues of AB and BA are same.

(use if $ABx = \lambda x$ then $BA(Bx) = B(ABx) = \lambda Bx$ so λ is eigenvalue of BA also and vis-a-viz.)

- Geometric multiplicity (no of eigenvectors for an eigenvalue) \leq Algebraic multiplicity (order of eigenvalue in characteristic polynomial).

- $A = P^{-1}BP$ this Relation ARB (similarity) is equivalence, determinant invariant, eigenvalue invariant, trace invariant.

- A matrix is diagonalizable if it is similar to a diagonal matrix

- A matrix is diagonalizable iff for each of its eigenvalue Geometric multiplicity = Algebraic multiplicity.

- square matrix A is diagonalizable iff minimal polynomial of A splits into distinct linear factors in the given field i.e. minimal polynomial of A is separable and has only linear irreducible factors.

- A non-zero Nil-potent ($A^m = 0$) matrix has eigenvalues as zero only.

- A non-zero Nil-potent is never Diagonalizable.

(if A is diagonalizable then $P^{-1}AP = D$ so $(P^{-1}AP)^m = P^{-1}A^mP = 0 = D^m$)

- **Schurs theorems:**

- Every Square matrix A is Unitarily similar to Upper triangular matrix whose diagonals are eigenvalues of A (complex values included).

- If $A \in M_n(\mathbb{R})$ and has only real eigenvalues then it is real orthogonally similar to real upper triangular matrix.

(say $\lambda_1, \lambda_2, \dots, \lambda_n$ are eigenvalues of $A_{n \times n}$ (with repeats) let x be normalised eigenvector of A to eigenvalue λ_1 then $x * x = 1$ $Ax = \lambda_1 x$ now from an orthonormal basis with x and let this matrix be $U_1 = [x \ u_2 \dots u_n]$ thus we have $U_1^* A U_1 = [\lambda_1, *; 0, A_1]$ for $A_{1 \times 1 \times n-1}$ and as U_1 is unitary we have eigenvalues of A_1 are $\lambda_2, \dots, \lambda_n$ only so let's commence the same procedure for $A_{1 \times 1 \times n-1}$ we get U_2 join this to form $V_2 = [1, 0; 0, U_2]$ then we get $(U_1 V_2)^* A U_1 V_2 = [\lambda_1, *, *; 0, \lambda_2, *; 0, 0, A_2]$ clearly $U_1 V_2$ was unitary so proceeding similarly we get the theorem)

- If $A \in M_n(\mathbb{R})$ has complex eigenvalues then it is similar to a matrix with diagonal blocks of 1-by-1 and 2-by-2 only (has upper triangular entries). Where 1-by-1 blocks are

real eigenvalue of A and 2-by-2 blocks are $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ for $a + ib$ eigenvalue.

(for $A_{n \times n}$ let $\lambda = a + ib$ and its eigenvector is $x = u + iv$ then prove $\bar{\lambda}, \bar{x}$ are eigenpairs so x, \bar{x} are linearly independent so are u, v and as $Au = au - bv, Av = bu + av$ and if $S = [u, v, S_1]_{n \times n}$ be made non singular thus $S^{-1}AS = [B, *; 0, A_1]$ for $B = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$)

- Every Symmetric matrix ($A \in M_n(\mathbb{R})$) is orthogonally similar to diagonal matrix (D) i.e. $D = P^T A P, P^T P = I$.

- Every Hermitian matrix (A) is unitarily similar to diagonal matrix (D) i.e. $D = P^* A P, P^* P = I$.

- A matrix A is normal iff $A^* A = A A^*$

- A matrix is Unitarily similar to diagonal matrix iff it is Normal.

- A triangular normal matrix is Diagonal also a block diagonal normal matrix has off diagonal blocks = 0.

- if A is normal then $p(A)$ (specially $A + \alpha I, \alpha \in \mathbb{C}$) is normal.

2 Quadratic Form

- $Q : \mathbb{F}^n \times \mathbb{F}^n \rightarrow \mathbb{F}$ given by $\sum_{i=0}^n \sum_{j=0}^n a_{ij} x_i x_j$

where $a_{ij} \in \mathbb{F}$ a field.

- It can be represented as $X'AX$ for $X = [x_1, x_2, \dots, x_n]^T$ and Symmetric matrix $A = [A]_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$

- Congruence relation (ARB) : if $A = P^T B P$ for some non-singular P, A, B square.

- Matrices congruent to Symmetric matrices are Symmetric.

- Quadratic forms are equivalent if the corresponding matrices are congruent.

- Congruent matrices or equivalent Forms have same Range.

- Every Symmetric matrix is congruent to a diagonal matrix. (same as orthogonally diagonalizable)

- Every n -rowed real Symmetric matrix with rank r is congruent to a Diagonal matrix with

diagonal $[1, \dots, 1, -1, \dots, -1, 0, \dots, 0]$ with 1 appearing p times -1 appearing $r - p$ times and 0 $n - r$ times.

- Canonical Form of real Quadratic Form: for Q has matrix A and if $P'AP = \text{diag}[1, \dots, 1, -1, \dots, -1, 0, \dots, 0]$ then $X = PY$ which transforms Q to $y_1^2 + \dots + y_p^2 - y_{p+1}^2 - \dots - y_r^2$ for Real non singular matrix P .

- Number of positive terms in canonical form is **Index**, difference of positive and negative terms is **Signature**.

- Index and Signature are congruence invariant.

- Two real Quadratic forms (symmetric matrices) are orthogonally equivalent iff their matrices have same eigenvalues and multiplicities.

- A Quadratic Form Q is:

- positive definite if $Q(X) \geq 0$ and $Q(X) = 0 \iff X = 0$

- negative definite if $Q(X) \leq 0$ and $Q(X) = 0 \iff X = 0$

- positive semi-definite if $Q(X) \geq 0$
 - negative semi-definite if $Q(X) \leq 0$
 - or is indefinite

- if for a n dimensional Quadratic form Rank= r and Signature= s then it is :

- positive definite iff $s = r = n$.
 - negative definite iff $-s = r = n$.
 - positive semi-definite iff $s = r < n$.
 - negative semi-definite iff $-s = r < n$.
 - indefinite iff $|s| \neq r$

- Now as real Symmetric matrices are diagonalizable and have a canonical form we have:

- Index = number of positive eigenvalues.
 - Rank = number of non zero eigenvalues.
 - Signature = no of +ve - no of -ve eigenvalues.

- from above we have for a real Quadratic form Q with matrix A then Q is:

- positive definite iff all eigenvalues are positive or > 0 .

- negative definite iff all eigenvalues are positive or < 0 .

- positive semi-definite iff at-least one eigenvalues is 0 and others > 0 .

- negative semi-definite iff at-least one eigenvalues is 0 and others < 0 .

- indefinite iff eigenvalues are -ve as well as +ve.

- every real non-singular matrix $A = PS$ for P orthogonal S positive definite

$(S = Q'D_1Q, D_1 = \sqrt{\text{diagonalization}(A'A)}, P = AS')$

- Q with matrix A is positive definite iff all leading principal minors of A are positive.

- A matrix A is positive definite $\implies |A| > 0$

- A complex Quadratic form is hermitian if its corresponding matrix is hermitian.

- A Hermitian Form assumes only real values.

- if $\text{norm}(A) = \sum_{i,j} |[A]_{ij}|^2$ then $\text{norm}(A) = \text{trace}(A^*A)$.

3 Jordan Form

- **Canonical Form** : Given a equivalence relation on set of matrices, the main problem is to find whether A and B belong to same equivalence class. One classical way of doing this is choosing a set of representative matrices such that each matrix belong to only one class and distinct members are of different classes. Such a set of representatives is the Canonical Form of such relation.

- Jordan form is the canonical form for relation of Similarity.

- A matrix in Jordan form Consist of Jordan blocks $J_k(\lambda)$ which is a upper triangular matrix of size k -by- k with diagonal entries λ and super diagonal 1 and others 0 i.e.

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}_{k \times k}$$

- $J_k(0)^{k+n} = 0$ for $n \geq 0$ i.e. $J_k(0)$ is nilpotent matrix such that $J_k(0)^k = 0$.

- $\text{rank}(J_k(\mathbf{o})^l) = \max(k-l, \mathbf{o})$
- Convention: $\text{rank}(J_k(\mathbf{o})^0) = k$
- if $r_k(\mathbf{A}, \lambda) = \text{rank}(\mathbf{A} - \lambda \mathbf{I})^k$ and $w_k(\mathbf{A}, \lambda) = r_{k-1}(\mathbf{A}, \lambda) - r_k(\mathbf{A}, \lambda)$ then in Jordan Form of \mathbf{A} :

■ $w_k(\mathbf{A}, \lambda)$ = number of blocks with eigenvalue λ that has size at least k (use the fact for every Jordan block of λ , $\mathbf{A} - \lambda \mathbf{I}$ is Similar to Jordan form consisting of $J_k(\mathbf{o})$ Jordan block instead of λ so as we measure ranks each power decreases the rank of the block by one if the block size is greater than the power.)

■ so $w_1(\mathbf{A}, \lambda) = n - r_1(\mathbf{A}, \lambda)$ = number of Jordan Blocks with eigenvalue λ = Geometric multiplicity of λ as eigenvalue of \mathbf{A}

■ $w_k(\mathbf{A}, \lambda) - w_{k+1}(\mathbf{A}, \lambda)$ = number of blocks of Size k

■ q : index of λ in \mathbf{A} = smallest integer such that $\text{rank}(\mathbf{A} - \lambda \mathbf{I})^{q+1} = \text{rank}(\mathbf{A} - \lambda \mathbf{I})^q = r_{q+1}(\mathbf{A}, \lambda) = r_q(\mathbf{A}, \lambda)$

■ $w_1(\mathbf{A}, \lambda) + w_2(\mathbf{A}, \lambda) + \dots + w_q(\mathbf{A}, \lambda)$ = Sum of dimensions (with repeat) all Jordan blocks in λ = Algebraic Multiplicity of λ as eigenvalue of \mathbf{A}

■ Weyr characteristic of $\mathbf{A} \in M_n$ associated with $\lambda \in \mathbb{C}$ is

$w(\mathbf{A}, \lambda) = (w_1(\mathbf{A}, \lambda), w_2(\mathbf{A}, \lambda), \dots, w_q(\mathbf{A}, \lambda))$

■ Segre characteristic of $\mathbf{A} \in M_n$ associated with $\lambda \in \mathbb{C}$ is

$s(\mathbf{A}, \lambda) = s_1(\mathbf{A}, \lambda) \geq s_2(\mathbf{A}, \lambda), \dots \geq s_{w_1}(\mathbf{A}, \lambda) > \mathbf{o}$ where s is sizes of Jordan Blocks in λ as they occur in Jordan form (non-increasing order)

■ for a given \mathbf{A}, λ eigenvalue, If we arrange $w(\mathbf{A}, \lambda)$ in dot form as rows (partitions: Ferrers diagram) then its columns are $s(\mathbf{A}, \lambda)$ and Vise-versa.

• for \mathbf{A}_n upper diagonal with $[A]_{ii} = \mathbf{1}$, $[A]_{i,i+1} \neq \mathbf{o}$ then \mathbf{A} is similar to $J_n(\mathbf{1})$

• if $\lambda = \mathbf{1}$ is the only eigenvalue of \mathbf{A} then \mathbf{A} is similar to \mathbf{A}^k

• in J Jordan form of \mathbf{A} :

■ Total No of Jordan blocks = Total no of independent eigenvectors.

■ No of Jordan blocks in λ = Dimension of eigenspace of λ

■ Sum of sizes of Jordan blocks in λ = Algebraic Multiplicity.

• If \mathbf{A}_n is non singular then \mathbf{A} is similar to \mathbf{A}^T . (use : for Jordan block $J_n = J_n(\lambda)$ and $\mathbf{B}_n = \mathbf{B}_{n \times n}$ reversal matrix (upside down identity) we have $J_n = \mathbf{B}_n J_n' \mathbf{B}_n$ as $\mathbf{B}_n^{-1} = \mathbf{B}_n$ we have $J_n \mathbf{R} J_n'$)

• If minimal polynomial of $\mathbf{A} = \prod_{i=1}^k (t - \lambda_i)^{r_i}$ then largest Jordan block of λ_i in JCF of \mathbf{A} is of size r_i .

4 Rational Form

• Jordan form of \mathbf{A}_n is possible iff The characteristics polynomial of \mathbf{A} splits completely to linear factors over \mathbb{F} (i.e. $(x - a_i)^{n_i}$, $a_i \in \mathbb{F}$), which may not be possible if there are irreducible polynomials of degree more than 1 in $\mathbb{F}[x]$, so to make canonical form under consideration of these Matrices we arrive at Rational form which uses the concept of Invariant subspaces, Cyclic subspaces and Primary Decomposition theorem.

• For given monic polynomial (characteristic/minimal) $p(x) = x^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ $a_i's \in \mathbb{F}$ of linear transform $T : V \rightarrow V$ if there exist x such that $T_x = \{x, T(x), T^2(x), \dots, T^{n-1}(x)\}$ is a linear independent set then The matrix of T with respect to T -cyclic basis T_x is Companion matrix which has same characteristic and minimal polynomial = $p(x)$ and is given by

$$C_A = \begin{bmatrix} \mathbf{o} & \dots & \mathbf{o} & -a_0 \\ \mathbf{1} & \mathbf{o} & \dots & -a_1 \\ \mathbf{o} & \mathbf{1} & \dots & -a_2 \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{o} & \dots & \mathbf{o} & \mathbf{1} & -a_{n-1} \end{bmatrix}$$

• If $p(x) = (p_1(x))^{n_1}(p_2(x))^{n_2} \dots (p_k(x))^{n_k}$ and $m(x) = (p_1(x))^{m_1}(p_2(x))^{m_2} \dots (p_k(x))^{m_k}$ are characteristics and minimal polynomial of linear transform $T : V \rightarrow V$ where $p_i's$ are irreducible in \mathbb{F} of degree d_i respectively then :

■ $K_{p_i} = \{x : (p_i(T))^k(x) = \mathbf{o}\}$ is T invariant Subspace of V

■ $K_{p_i} = \ker((p_i(T))^{m_i})$ (Null space) , $K_{p_i} \cap K_{p_j} = \{0\}$ for $i \neq j$

■ Every K_{p_i} has a union T -cyclic basis as a basis.

● From above and Primary decomposition theorem we have: for a linear transformation $T : V \rightarrow V$ with matrix A has a basis in which A is similar to

$$\begin{bmatrix} C_1 & & & 0 \\ 0 & C_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & & C_k \end{bmatrix}$$

where C_i s are companion matrices related to minimal polynomial's irreducible terms.

● Dimension of $K_{p_i} = d_i n_i$ (d_i = degree of p_i , n_i = power of p_i in characteristic polynomial)

● $\dim(K_{p_i})$ = dimension of total blocks associated with p_i

● number of blocks associated with $p_i = r_i = \frac{1}{d_i}[\dim(V) - \text{rank}(p_i(A))]$

● number of blocks of size atleast $i - b_y - i = r_i = \frac{1}{d_i}[\text{rank}(p_i(A)^{i-1}) - \text{rank}(p_i(A)^i)]$

5 Mics Properties

● A has a block B_n in its block form iff it has an n dimensional invariant space associated.

● Λ_n is a block matrix in which $[\Lambda]_{i,j} = 0$ if $i \neq j$, $\Lambda_{ii} = \lambda_i I_{n_i}$ blocks and commutes with B iff B is a block Diagonal conformal with Λ i.e. iff

$$\Lambda = \begin{bmatrix} \lambda_1 I_{n_1} & & & 0 \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ 0 & & & \lambda_d I_{n_d} \end{bmatrix},$$

$$B = \begin{bmatrix} B_{n_1} & & & 0 \\ & B_{n_2} & & \\ & & \ddots & \\ 0 & & & B_{n_d} \end{bmatrix}$$

● Extremum of $X^T A X$ for constraint $X^T X = 1$ occurs in eigenvalues of A .

● From above Extremum of real Quadratic

Form $X^T A X$ with constraints $X^T X = 1$ is the largest eigenvalue of A vise-versa $\text{Max}\{X^T A X | A \text{ is symmetric, } X^T X = 1\}$ = largest eigenvalue of A .

● μ is a eigenvalue of $p(A)$ iff $\mu = p(\lambda)$ for an eigenvalue λ of A (where $p(\cdot)$ is a polynomial over \mathbb{F}).

● if λ is an eigenvalue of A then corresponding eigenvector are non-zero columns of $\text{adj}(A - \lambda I)$ (use full only if $\text{rank}(A - \lambda I) = n - 1$).

● Coefficients of Characteristic polynomial of A of degree n : $n \rightarrow 1, n-1 \rightarrow -\text{trace}(A)$, constant $\rightarrow (-1)^n \det(A)$.

● A, B are simultaneously Diagonalizable iff A, B commute i.e. if $D_1 = S^{-1} A S, D_2 = S^{-1} B S$ for same $S \iff AB = BA$. This even holds for a family of Diagonalizable matrices.

● for $A_{m \times n}$

$$\begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}^{-1} = \begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix}$$

● For $A_{m \times n} B_{n \times m}$ Eigenvalues of $AB =$ Eigenvalues of BA (including zero).

● Cauchy's Determinant Identity : $\det(A + xy^T) = \det(A) + y^T \text{adj}(A)x$ (so $|I + xy^*| = 1 + y^*x$)

● if $S = A + iB$ and non-singular then $\exists \tau \in \mathbb{R}$ such that $T = A + \tau B$ is non-singular.

(use that $p(t) = \det(A + tB)$ has at most n zeroes in complex plane so there is $\tau \in \mathbb{R}$ such that $p(\tau) \neq 0$)

● Every real Matrix A similar over \mathbb{C} to real matrix B is similar over \mathbb{R} . i.e. $0 \neq A, B \in M_n(\mathbb{R})$ if $S \in M_m(\mathbb{C})$ and $B = S^{-1} A S$ then $\exists T \in M_n(\mathbb{R})$ such that $B = T^{-1} A T$

● If A is diagonalizable i.e. $A = S^{-1} D S$ then $p(A) = S^{-1} p(D) S$ which makes evaluation of $p(A)$ easier.

● If A_n has distinct eigenvalues(diagonalizable) and Commutes with B then B is Diagonalizable (more precisely A_n, B are simultaneously diagonalizable) and $B = p(A)$

(use similarity, partition arguments and Lagrange interpolation poly which provides a polynomial map of n distinct reals to any n reals) for some polynomial

$p(t)$ of degree at most $n - 1$

- If B is Diagonalizable then B has a square-root i.e $\exists A | A^2 = B$.

- If A_n, B_n are similar so are $\text{adj}(A), \text{adj}(B)$.

- All Unitary Matrices Form a group in $GL(n, \mathbb{C})$ and compact in \mathbb{C}^{n^2} .

- Singular Value Decomposition: Every matrix $A_{m,n}$ can be written as $A = U_m S V_n$ where U, V are Unitary and S is the diagonal (with zero) entries that are eigenvalue of A^*A or AA^* .

- Reversal Matrix B is matrix that is up-side-down of Identity and BA reverses row order of A , AB reverses column order of A And $B = B^* = B^{-1}$

- By Jordan Canonical form Every non-singular matrix is similar to its Transpose

- A is similar to \bar{A} iff A is Similar to a real matrix (Same condition for $A \sim A^*$)

- A is hermitian iff $\text{tr}(A^2) = \text{tr}(A^*A)$

- if A is hermitian then, $\forall x \in \mathbb{C}^n$:

- x^*Ax is positive iff all eigenvalues are positive

- x^*Ax is negative iff all eigenvalues are negative

- if eigenvalues are $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$ and subspaces $\{S\}$ of \mathbb{C}^n then $\lambda_1 = \min(\frac{x^*Ax}{x^*x}), \lambda_n = \max(\frac{x^*Ax}{x^*x})$,

$$\lambda_k = \min_{\{\dim(S)=k\}} \max_{0 \neq x \in S} \frac{x^*Ax}{x^*x}$$

$$= \max_{\{\dim(S)=n-(k+1)\}} \min_{0 \neq x \in S} \frac{x^*Ax}{x^*x}$$

- In general even if $A \in M_n$ is not hermitian with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ then

$$\min_{x \neq 0} \left| \frac{x^*Ax}{x^*x} \right| \leq |\lambda_i| \leq \max_{x \neq 0} \left| \frac{x^*Ax}{x^*x} \right|$$

(can be pure inequality also)

- Every Jordan matrix is similar to a complex symmetric matrix so **Every matrix is similar to a complex symmetric matrix**

6 Properties based on Matrix Norm

- A function $||| \cdot ||| : M_n \rightarrow \mathbb{R}$ is a matrix norm if:

1. $|||A||| \geq 0$ Non-negative

- 1a. $|||A||| = 0 \iff A = 0$ Positive

2. $|||cA||| = |c| |||A||| \quad \forall c \in \mathbb{C}$ Homogeneous

3. $|||A + B||| \leq |||A||| + |||B|||$ Triangular Inequality

4. $|||AB||| \leq |||A||| |||B|||$ Sub-multiplicativity

- Clearly $|||A^k||| \leq |||A|||^k$ now If $A^2 = A \implies |||A||| \geq 1$ in particular $|||I||| \geq 1$

- Some Matrix norms:

- l_1 norm : $|||A|||_1 = \sum_{i,j=1}^n |a_{ij}|$

- l_2 norm : $|||A|||_2 = |\text{tr}(A^*A)|^{1/2}$

$$= \sqrt{\sigma_1(A)^2 + \dots + \sigma_n(A)^2} = \left(\sum_{i,j=1}^n |a_{ij}|^2 \right)^{1/2}$$

- l_∞ norm : $|||A|||_\infty = \max_{1 \leq i,j \leq n} |a_{ij}|$

- max Column sum norm

$$|||A|||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^n |a_{ij}|$$

- max Row sum norm

$$|||A|||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^n |a_{ij}|$$

- Spectral norm $|||A|||_2 = \sigma_1(A)$ = Largest Singular Value of A

- **Matrix norm induced by vector norm** : if $|| \cdot ||$ is norm in \mathbb{C}^n then:

$$|||A||| = \max_{||x||=1} ||Ax|| = \max_{x \neq 0} \frac{||Ax||}{||x||}$$

$$= \max_{||x|| \leq 1} ||Ax|| = \max_{||x||_\alpha=1} \frac{||Ax||}{||x||}$$

(for any other norm $|| \cdot ||_\alpha$ in \mathbb{C}^n) is a Matrix norm with additional properties:

- $\|I\| = 1$
- $\|Ay\| \leq \|A\| \|y\|$
- For Any Matrix $A \in M_n(\mathbb{C})$ we have $|\lambda| \leq \rho(A) = \max(|\lambda_i|) \leq \|A\|$ and if A is non-singular then $\rho(A) \geq |\lambda| \geq 1/\|A\|$
- if there is Matrix norm such that $\|A\| < 1$ then $\lim_{k \rightarrow \infty} A^k = 0$
- from above we have $\lim_{k \rightarrow \infty} A^k = 0$ iff $\rho(A) < 1$
- For any given Matrix norm $\|\cdot\|$ we have $\rho(A) = \lim_{k \rightarrow \infty} \|A^k\|^{1/k}$
- Matrix power series $\sum_{k=0}^{\infty} a_k A^k$ converges if $\rho(A) \leq R$ where R is the radius of convergence of complex power series $\sum_{k=0}^{\infty} a_k z^k$ i.e. if $\exists \|\cdot\| : \|A\| < R$
- Matrix A is nonsingular if $\exists \|\cdot\| : \|I - A\| < 1$ and $A^{-1} = \sum_{k=0}^{\infty} (I - A)^k$
- From above we have if $A_n = [a_{ij}]$ and $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ i.e. absolute value of diagonal elements are greater than sum of absolute values of elements in corresponding rows (or columns) then A is non singular

7 Properties associated to Quadratic forms

- A_n is Hermitian iff :
 - $x^* A x$ is real for all $x \in \mathbb{C}^n$
 - A is normal and all its eigenvalues are real
 - $S^* A S$ is Hermitian $\forall S \in M_n$
- from above A is +ve (-ve) semi-definite ($x^* A x \geq 0$ or ≤ 0) $\implies A$ is hermitian
- if A is +ve definite (-ve) then $A^*, A^{-1}, A^T, \bar{A}$ are all +ve definite (-ve).
- every Diagonal entry of +ve (-ve) definite (semi) Matrix are +ve(non -ve, -ve) only.
- A positive semi-definite matrix is positive definite iff it is non-singular

- for $A_n = [a_{ij}]$ a +ve (-ve) semi-definite matrix if $a_{kk} = 0$ then $a_{ik} = a_{ki} = 0 \forall i \in \{1, 2, \dots, n\}$ i.e. if diagonal entry is 0 then that row and column are 0.
- A is positive semi definite iff $A = B^* B$ for some B
- A_n is positive definite iff $\det(p_k) > 0 \forall 1 \leq k \leq n$ where p_k is the $k \times k$ principle matrix partitioned in A (along the diagonal).

8 Other Important Theorems

- Gersgorin Theorem: for a matrix $A_n = [a_{ij}]$
 - A Gersgorin Disk of $A = \{z \in \mathbb{C} : |z - a_{ii}| \leq R'_i(A) = \sum_{j \neq i} |a_{ij}|\}$ for $i = 1, 2, \dots, n$
 - Eigenvalues of A are all in the union of Gersgorin Discs of A i.e. $\{\lambda_i\} \in G(A) = \bigcup_{i=1}^n \{z \in \mathbb{C} : |z - a_{ii}| \leq R'_i(A)\}$
 - if $G(A)$ forms a disjoint set $G_k(A)$ which is union of k discs then $G_k(A)$ contains exactly k eigenvalues (counted according to algebraic multiplicity).
 - The above statements remain true even if radius of the discs are $C'_j = \sum_{i \neq j} |a_{ij}|$ as A^T has same eigenvalues.
 - from above we have $\rho(A) \leq \min \left\{ \max_i \sum_{j=1}^n |a_{ij}|, \max_j \sum_{i=1}^n |a_{ij}| \right\}$
 - if p_1, p_2, \dots, p_n are positive real numbers then $\{\lambda_i\} \in \bigcup_{i=1}^n \{z \in \mathbb{C} : |z - a_{ii}| \leq \frac{1}{p_i} \sum_{j \neq i} p_j |a_{ij}|\}$ or $\{\lambda_i\} \in \bigcup_{i=1}^n \{z \in \mathbb{C} : |z - a_{jj}| \leq p_j \sum_{i \neq j} \frac{1}{p_i} |a_{ij}|\}$ as similar matrices have same eigenvalues
- A is Diagonally dominant if $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ and strictly diagonally dominant if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$
- if A is strictly diagonally dominant then : A is non-singular, if $a_{ii} > 0 \forall i = 1, 2, \dots, n$ then every eigenvalue of A has a positive real part, and if A is hermitian and $a_{ii} > 0 \forall i = 1, 2, \dots, n$ then A is positive definite.
- A_n has nonzero diagonal entries, is diagonally dominant and $|a_{ii}| > R'_i$ for atleast $n - 1$

values of i then A is non singular.

- If every entry of A is non zero, A is diagonally dominant and $|a_{kk}| > R'_k$ for any k then A is non singular
- if A_n has the property that $\forall p, q \in \{1, 2, \dots, n\} \exists$ sequence of distinct integers $p = k_1, k_2, \dots, k_m = q$ such that $a_{k_1 k_2}, a_{k_2 k_3}, \dots, a_{k_{m-1} k_m}$ are non zero, A is diagonally dominant and $|a_{kk}| > R'_k$ for any k then A is non singular
- The above property states that if A is a probability/stochastic matrix then for each node in directed graph of A is strongly connected (for each pair of nodes there is a finite length directed path to them or the stochastic matrix has

only one class and all states are communicating)

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