Complex Analysis

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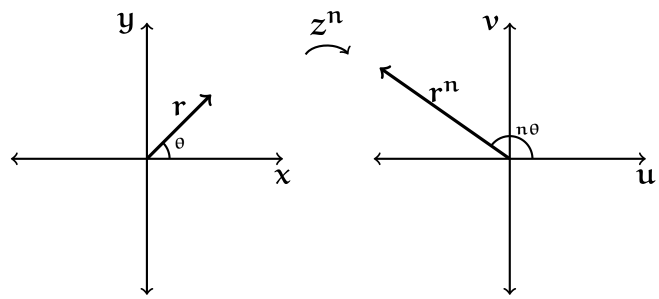
1 Power Series

- $P(z) = a_0 + a_1z + a_2z^2 + a_3z^3 \cdots = \sum_{n=0}^{\infty} a_nz^n$.
- If $P(z)$ converges at $z = a$ then it converges absolutely for all $|z| < |a|$.
- If $P(z)$ diverges at $z = d$ then it diverges absolutely for all $|d| < |z|$.
- If two power series $A(z) = \sum_{n=0}^{\infty} a_nz^n$, $B(z) = \sum_{n=0}^{\infty} b_nz^n$ agree on an infinite sequence ($\neq 0$) converging to zero then they are same i.e. $a_i = b_i\ \forall i$.
- In general for $P_b(z) \sum_{n=0}^{\infty} a_n(z-b)^n$ above holds as in displacement or translation of b to 0 i.e. $P_b(z) = P(w)$ for $w = z - b$.
- Radius of convergence of $P(z) = \sum_{n=0}^{\infty} a_nz^n$ is R then:
  - $R = \lim_{i\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$.
  - $R = \lim_{i\to\infty} \left|\frac{1}{\sqrt[n]{|a_n|}}\right|$.
  - $R = \lim_{i\to\infty} \left|\frac{1}{\limsup \sqrt[n]{|a_n|}}\right|$.
- Radius of convergence of the power series of $f(z)$ at k is equal to distance between k and closest singularity of $f(z)$ to k.

2 Transformations

2.1 Z^n.

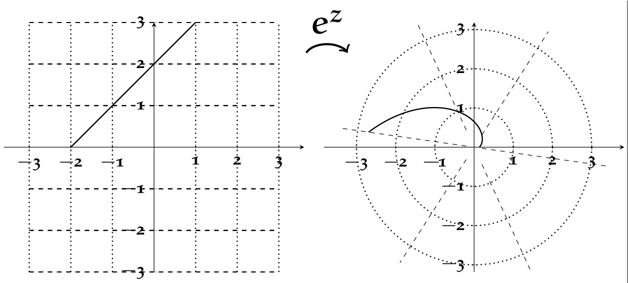
- $w = z^n = r^ne^{in\theta}$.
- so from above each z is magnified $|z|^n$ times and rotated $n\ arg(z)$ times in the plane i.e.

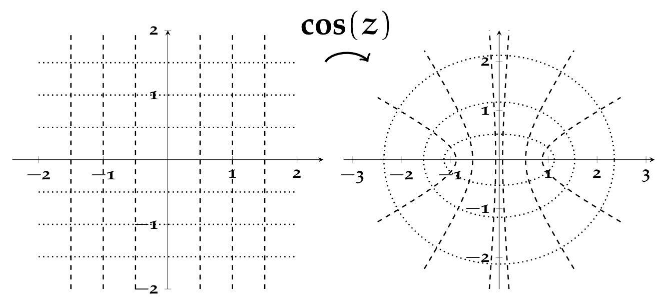


- Images of circles are circle (with expanded or contracted radius),lines are lines
- Most geometric shapes just expand/diminished (amplified) and gets rotated (twist)

2.2 e^z.

- $w = e^z = e^xe^{iy} = e^x\cos y + i\ e^x \sin y = u + iv$.
- $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots$ and radius of convergence $=\infty$.
- e^z takes all values in $\mathbb{C}$ infinitely many times except zero i.e. range(e^z)=$\mathbb{C} - \{0\}$.
- if x is constant then $u^2 + v^2 = e^x = r$ $\Longrightarrow$ horizontal lines are mapped to circle.
- if y is constant then $\frac{v}{u} = \tan y$ or $v = cu$ $\Longrightarrow$ vertical lines are mapped to lines passing through origin (not including the origin).
- every other line is mapped to a spiral centred at origin (not including).





2.3 Trigonometric functions

$$\begin{aligned}\cos(z) &= 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!}\dots\\ &= \frac{e^{iz}+e^{-iz}}{2}\\ \sin(z) &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!}\dots\\ &= \frac{e^{iz}-e^{-iz}}{2i}\end{aligned}$$

• $\cos(z-\pi/2) = \sin(z)$.

• $\cosh(z) = \cos(iz)$.

• $\sinh(z) = -i\sin(iz)$.

• so exploring only one of trigonometric functions namely $\cos z$ is sufficient

• now $\cos(x+iy) = \frac{e^{ix}e^{-y}+e^{-ix}e^{y}}{2}$
$= \frac{e^{y}+e^{-y}}{2}\cos(x) - i\frac{e^{y}-e^{-y}}{2}\sin(x)$
$= \cosh y\cos x - i\sinh y\sin x = u + iv$.

• for $z = x+iy$ and $w = \cos(z) = u+iv$ if $y = y_0$. is kept constant then
$\frac{u^2}{\cosh^2 y_0} + \frac{v^2}{\sinh^2 y_0} = 1$.

• so every horizontal line is transformed to an ellipse with foci's ± 1.
(as $a = \sinh y_0$ $b = \cosh y_0 \implies c = b^2 - a^2 = 1$ (unit from origin) so foci's = $(1,0), (-1,0)$.)

• similarly $x = x_0$. is kept constant then
$\frac{u^2}{\cos^2 x_0} - \frac{v^2}{\sin^2 x_0} = 1$.

• so every vertical line is transformed to hyperbola with foci's ± 1.

2.4 $\log(z)$.

• it denotes the inverse function of exponential

• $\log(re^{i\theta}) = \ln(r) + i\theta$.

• Clearly $\log$ is a multifunction as $\log(re^{i\theta}) = \ln(r) + i(\theta + 2n\pi)$.

• properties of multifunctions:

▪ a region in range where multifunction takes ordinary single value is called a branch.

▪ typically branches are connected regions (simply or multiply)

▪ q is branch point of multifunction if after a revolution around the point in domain the multifunction changes its values on the original observed point

▪ q is algebraic branch point of $f(z)$ if $f(z)$ returns to original observed value after N revolutions around q, its order is $N-1$, a simple branch point has order 1.

▪ q is logarithmic branch point if order is ∞ i.e. the original value is not restored by any number of revolution around the point.

▪ any curve drawn from branch point to ∞ is called a branch cut, typically is -ve real axis.

▪ eg: $z^{\frac{m}{n}}$ is one$-n$ mutifunction has branch point 0 of order $n-1$, z^{τ} for τ irrational has logarithmic branch point of 0.,

▪ a function can have more than one branch point eg: $\sqrt{z^2+1} = \sqrt{(z-i)(z+i)}$ has $\pm i$ as simple branch points.

▪ if a complex function or a branch of multifunction can be expressed as power series then the **radius of convergence** is distance to the nearest singularity or branch point.

• $\log(z)$ has logarithmic branch point at 0.

• $\mathrm{Log}(z) = \ln|z| + i\mathrm{Arg}(z)$ where the branch cut is -ve real axis and $-\pi < \mathrm{Arg}(z) \leq \pi$ is called principle branch.

• continuity of $\mathrm{Log}(z)$ breaks down at $\mathrm{Arg}(z) = \pi$.

• $\mathrm{Log}(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$ is a power series centered at 1 with radius of convergence 1 converges on this unit circle except for $z = -1$.

• other branches of $\log(z)$ can be explored by writing $\log(z) = \mathrm{Log}(z) + 2n\pi i$.

• $z^k = e^{k\log(z)} = e^{2n\pi k i} e^{k\mathrm{Log}(z)} = e^{2n\pi k i}[z^b]$ where $[z^k]$ denotes root in principle branch. thus

▪ $z^{p/q} = e^{p/q\, 2n\pi i}[z^{p/q}]$.

▪ Now for complex powers

$$
\begin{aligned}
[z^{a+ib}] &= e^{(a+ib)(\mathrm{Log}(z))} \\
&= e^{(a+ib)(\ln(r)+i\theta)} \\
&= e^{a\ln(r)} e^{-b\theta} e^{i(a\theta + b\ln(r))}
\end{aligned}
$$

so $[z^{a+ib}] = |z|^a e^{-b\,\mathrm{Arg}(z)} e^{i(a\,\mathrm{Arg}(z) + b\ln|z|)}$

$$\text{and } z^{a+ib} = e^{i2\pi n a} e^{-2\pi n b}[z^{a+ib}]$$

2.5 Geometric transforms

• translation by v : $J_v(z) = z + v$ translates $0 \to v$.

• rotation about origin by θ : $R_0^\theta(z) = e^{i\theta}z$.

• rotation about w by θ. :
$R_w^\theta = J_w \circ R_0^\theta \circ J_{-w}(z)$.

• Properties:

▪ $\{J_w\}$ forms a group under composition

▪ $R_w^\theta = J_v \circ R_0^\theta$ where $v = w(1 - e^\theta)$
. i.e. rotation about any point is equal to a rotation around origin proceeded by translation.

▪ $R_a^\theta \circ R_b^\phi = R_c^{\theta+\phi}$ where $c = \frac{ae^{i\phi}(1-e^{i\theta}) + b(1-e^{i\phi})}{(1-e^{i(\theta+\phi)})}$.

▪ if $\theta + \phi = 2n\pi$ then $R_a^\theta \circ R_b^\phi = J_v$ where $v = (b-a)(1-e^{i\phi})$.

• Reflection about a line $L_1 = \mathfrak{R}_{L_1}$.

• Reflection about real axis $\mathfrak{R}_{y=0} = \bar{z}$.

• Reflection about line $ax + by = c$ is

$$\mathfrak{R}_{ax+by=c} = \frac{(b - ia)\bar{z} + 2ic}{b + ia}.$$

(can be done by transforming line to real axis by translation and rotation then conjugation and followed by inverse back to same line transformation).

• Properties:

▪ If $L1$ and $L2$ intersect at O, and the angle from $L1$ to $L2$ is ϕ., then $\mathfrak{R}_{L2} \circ \mathfrak{R}_{L1}$ is a rotation of 2ϕ about O i.e. $R_O^{2\phi}$.

▪ If $L1$ and $L2$ are parallel, and v is the perpendicular vector to both lines, connecting $L1$ to $L2$ (i.e. distance vector), then $\mathfrak{R}_{L2} \circ \mathfrak{R}_{L1}$ is a translation of $2v$ i.e. J_{2v}.

2.6 $\frac{1}{z}$.

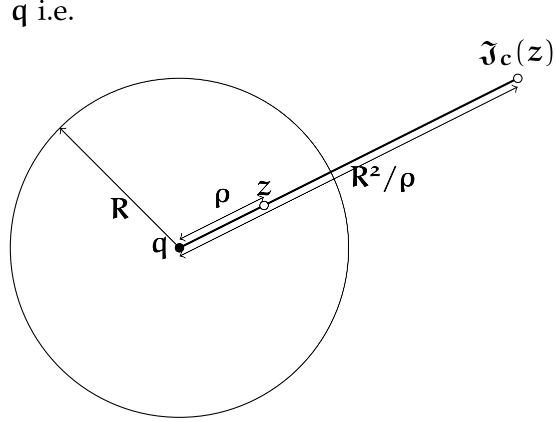
• before studying $\frac{1}{z}$ we can study inversion about a circle :

• $\mathfrak{J}_c(z)$ is the inversion of points in circle c centered at q with radius R i.e. it transforms interior of circle to exterior and points on circle remain fixed

• some defining properties of $\mathfrak{J}_c$ (inversion of about circle c of radius R and centred at q.) :

▪ $q \to \infty$.

▪ if z is at distance ρ from q then it is moved to distance R^2/ρ along same direction as z from q i.e.



▪

$$\mathfrak{J}_c(z) = \frac{R^2}{\bar{z} - \bar{q}}$$

(as $\overline{(z-q)}(\mathfrak{J}_c(z)-q) = R^2$.)

• Properties of inversion ($\mathfrak{J}_c$ centred q radius R.):

■ inversion is involutory i.e. $\mathfrak{J}_c \circ \mathfrak{J}_c(z) = z$ or $\mathfrak{J}_c^2 = I$.

■ if $\tilde{a} = \mathfrak{J}_c(a)$ and $\tilde{b} = \mathfrak{J}_c(b)$ then $\triangle \tilde{a} q \tilde{b}$ is similar to $\triangle aqb$.

■ every line that does not pass through q is mapped to a circle passing through q.

■ as inversion is involutory it swaps the above point i.e. a circle passing through q is mapped to a line not passing through q.

■ A circle not passing through q is mapped to another circle not passing through q i.e. **inversion preserves circles**.

■ if a circle k cuts circle c at a and b at right angles i.e. k is orthogonal to c then k is mapped to itself i.e. **inversion maps orthogonal circles to c to itself**.

■ Inversion in a circle is anticonformal map

■ If a and b are symmetric with respect to circle k then their inversion images $\tilde{a}$ and $\tilde{b}$ are also symmetric with respect to the inversion image circle $\tilde{k}$ of k.

■ i.e. Inversion maps any pair of orthogonal circles to another pair of orthogonal circles.

■ also if a and b are symmetric w.r.t line L_1 (i.e. are reflections) then their inversion images are also symmetric to the inversion line $\tilde{L_1}$.

• now $\frac{1}{z} = \overline{\left(\frac{1}{\bar{z}}\right)}$ so $\frac{1}{z}$ is reflection of inversion centered at origin with unit radius on real axis, so all properties of inversion holds as reflection preserves shapes.

■ now as both inversion and conjugation are anticonformal implies $1/z$ is a conformal map

• define inverse point w.r.t. circle $C_{(z_0,R)} = \{z||z-z_0| = R\}$ as a and a^* are inverse points w.r.t $C_{(z_0,R)}$ if $a \mapsto a^*$ under $\mathfrak{J}_{C_{(z_0,R)}}(z)$ i.e. if $a^* = z_0 + \frac{R^2}{\overline{a-z_0}}$ or $(a^*-z_0)\overline{(a-z_0)} = R^2$.

2.7 Mobius Transforms

$$\begin{aligned} M(z) &= \frac{az+b}{cz+d} \\ &= \frac{a}{c} - \frac{ad-bc}{c^2}\left(\frac{1}{z+\frac{d}{c}}\right) \\ &= J_{a/c} \circ Az \circ \overline{\mathfrak{J}_u} \circ J_{d/c}(z) \end{aligned}$$

where $A = \frac{ad-bc}{-c^2}, u \equiv \{|z| = 1\}$.

• The only shape changing transformation in $M(z)$ is conjugate inversion, so all symmetries and properties of inversion follow to mobius transform.

• Properties

■ every mobius transform maps circles and straight lines onto circles and straight lines.

■ above point may not be same order i.e. some circles can be mapped to straight lines and vis-a-viz. namely a straight line or a circle maps onto a straight line if it passes through the point $z = -d/c$, and onto a circle if it does not (also lines not passing trough $-d/c$.).

■ mobius transform is conformal

■ more over mobius transforms are the only transforms that **map circles to circles**

■ To be specific A Mobius transformation maps an oriented circle C to an oriented circle $\tilde{C}$ in such a way that the region to the left of C is mapped to the region to the left of $\tilde{C}$..

■ Symmetric principle: If two points are symmetric with respect to a circle i.e. inverse points w.r.t a circle, then their images under a Mobius transformation are symmetric with respect to the image circle. transformation are symmetric with respect to the image circle.

■ every mobius transform has only 2 fixed points

■ there exist a unique mobius transform sending any three points to any three points.

• the coefficients of a mobius transform $\{a, b, c, d\}$ are not unique as any $k \neq 0$. $\{ka, kb, kc, kd\}$ gives same mobius transform

• cross ration $[z, a, b, c] = \frac{(z-a)(b-c)}{(z-c)(b-a)}$.

• if p, q, r, s are mapped to $\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}$ by a Mobius Transformation iff

$$[p, q, r, s] = [\tilde{p}, \tilde{q}, \tilde{r}, \tilde{s}].$$

i.e. Mobius transforms are cross-ratio invariant.
• Unique Mobius transform $M(z) = w$ that transforms $a \to r, b \to s, c \to t$ is

$$[w, r, s, t] = [z, a, b, c]$$

or

$$\frac{(z-a)(b-c)}{(z-c)(b-a)} = \frac{(w-r)(s-t)}{(w-t)(s-r)}.$$

2.8 More on Mobius Transforms

• now as coefficients of mobius transform are not unique if $ad - bc = 1$ in $M(z)$ then

$$M(z) = \frac{ax+b}{cz+d} \longleftrightarrow [M] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

• Properties:
▪ $M_3 = M_1 \circ M_2(z)$ them $[M_3] = [M_2][M_1]$.
▪ if inverse of $M(z)$ is $M^{-1}(z)$ then $[M^{-1}] = [M]^{-1}$.
▪ identity transform $[I] = [1, 0; 0, 1]$.
▪ Thus $M(z)$ of form a group (for $ad - bc \neq 0, = 1$) as $\mathbb{SL}(\mathbb{R}, 2)$ is a subgroup of $\mathbb{GL}(\mathbb{R}, 2)$.
• Homogeneous coordinates $z = \frac{\nu_1}{\nu_2}$ for $\nu_i \in \mathbb{C}$.
• $[M]$ is a liner transform on homogeneous coordinates of z transforms to homogeneous coordinates of $M(z)$i.e if $z = \nu_1/\nu_2$, $M(z) = w = \rho_1/\rho_2$. then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \end{bmatrix}.$$

(although homogeneous coordinates may not be unique but their ratios ought to be)
⋆ Properties:
▪ $z = (\nu_1/\nu_2)$ is a fixed point of $M(z)$ iff $[\nu_1 \quad \nu_2]^T$ is an eigenvector of $[M]$.

3 Automorphisms, Conformality and map of unit disks

• any disk or half plain can be mapped to itself using mobius transform i.e. under specified mobius transforms say M_1 we can have $M_1(D) = D$ for a disk $D = \{z \| z - a| \leq r\}$ and for $M_2(\mathbb{H}) = \mathbb{H}$ for any half plane $\{z = x + iy | ax + by \geq c\}$.(note : this is mere a bijection with restrictions, not the identity map in disk or half plane.)
• more over the only conformal bijections (automorphisms) of disks $\mapsto$ disks, half planes $\mapsto$ half planes are **Mobius Transforms only.**
• Let C be a unit circle in $\mathbb{C}$ and D be the unit disk it covers then
• mobius transform's are the only automorphisms conformal on this disk
• this mobius automorphism's have 3 degree of freedom (only 3 real numbers specify it)
• Now if two Mobius automorphisms M and N map two interior points to same image points i.e. the agree on two interior points then M=N (as this takes 4 degree of freedom from both transforms)
• if D is centered at origin then these 3 degrees of freedom are a point in D ($a = (x + iy)$) that maps to origin and 1 is mapped to a point on C..
• as a is mapped to 0, and mobius transform preserves symmetry b/w points and their images (inversion) we have the point $1/\overline{a}$ is mapped to ∞ (as C maps to itself, $a, 1/\overline{a}$ are symmetric w.r.t C so should be their images $0, \infty$).
• so now $a \mapsto 0 \implies M(z) = \frac{k(z-a)}{d}$, $1/\overline{a} \mapsto \infty \implies M(z) = k\frac{z-a}{\overline{a}z-1}$ and as $M(1) \in C \implies |M(1)| = 1 \implies k = e^{i\phi}$ so the automorphism of unit disk ($|z| \leq 1$) i.e. mobius transform is determined only by $a = x + iy \mapsto 0$ ($|a| < 1$) and $p \mapsto 1$ ($|p| = 1$) this is given by :

$$M_a^{\phi}(z) = e^{i\phi}\frac{z-a}{\overline{a}z-1}.$$

• now for

$$M(w) = \frac{pz+q}{\overline{q}z+\overline{p}}$$

for $|p| > |q|$ then $M(w)$ is an automorphism of unit disk (transform this to M_a^{ϕ} for $a = q/p$ and $e^{i\phi} = p/\overline{p}$.)

• clearly $M_a^{\phi}(z) = e^{i\phi}M_a^0(z)$ so is just rotation of $M_a^0 = M_a(z)$.

• properties of M_a.

• M_a is he only Mobius automorphism that swaps a and 0 (i.e. $M_a(a) = 0, M_a(0) = a$.)

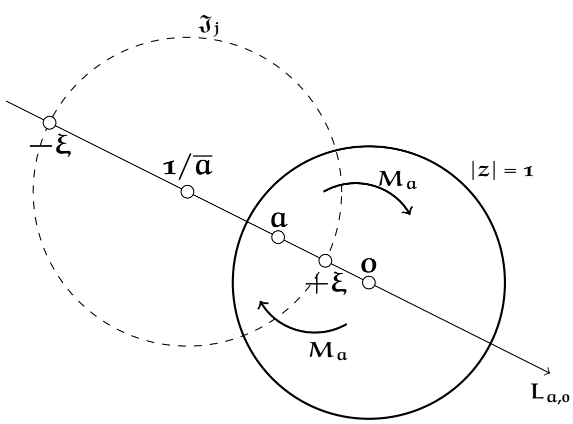
• now as an inversion about circle c maps circles orthogonal c to themselves (automorphism) thus automorphisms of unit circle can be viewed as inversions about circles orthogonal to unit circle to uncover this we break down that as $a \mapsto 0$ and inversion circle is orthogonal to unit circle the center of inversion is on the line b/w a to 0 and as inversion is symmetric $1/\overline{a} \mapsto \infty$ we conclude that center of inversion is $1/\overline{a}$.

• as M_a is conformal the above inversion should be coupled with reflection (on line perhaps) to give the exact map, as this reflection leaves $a, 0$ fixed we conclude this is reflection about line a to 0 ($L_{a,0}$.)

• thus $M_a = \mathfrak{R}_{L_{a,0}} \circ \mathfrak{J}_j$.

• thus fixed points ($\pm\xi$) of M_a is the intersection of $L_{a,0}$ and j.

• M_a is Involutory.



• if $\mathbb{H}^{\pm}$ represents the upper or lower half plane ($Im(z) > 0$ or < 0) , $\delta = \Delta(0,1)$ unit disk at origin and $\partial\Delta = \{|z| = 1\}$ then :

- for fixed $\beta \in \mathbb{C}, \theta \in \mathbb{R}$ if $Im(\beta) > 0$ then

$$w = f(z) = e^{i\theta}\frac{z-\beta}{z-\overline{\beta}}.$$

are the only conformal maps that maps $\mathbb{H}^+ \mapsto \delta$, $\beta \mapsto 0$ and real line+$\infty = \mathbb{R}_\infty \mapsto \partial\Delta$ (to see assume $|w| < 1 \iff |z-\overline{\beta}|^2 - |z-\beta|^2 > 0 \iff -2Re(z(\beta-\overline{\beta})) = 4(Im(z))(Im(\beta)) > 0$.)

• now if we use transform $R_0^{\pi}(z) = e^{i\pi}z = -z$ which rotates $\mathbb{H}^+$ to $\mathbb{H}^-$ we get $g = f \circ \phi(Z)$.

$$g(z) = e^{i\theta}\frac{z-b}{z-\overline{b}}.$$

for $Im(b) < 0$, are the only conformal maps that map $\mathbb{H}^- \mapsto \delta$, $b \mapsto 0$ and $\mathbb{R}_\infty \mapsto \partial\Delta$.

• similarly if $h(z) = f \circ R_0^{\pi/2}$

$$h(z) = e^{i\theta}\frac{z-\gamma}{z+\overline{\gamma}}.$$

for $Re(b) > 0$, are the only conformally maps that map Right half plane ($Re(z) > 0$) $\mapsto \delta$, $\gamma \mapsto 0$.

• a Mobius transform $w = az + b/cz + d$ maps $\mathbb{H}^+ \mapsto \mathbb{H}^+$ iff $a, b, c, d \in \mathbb{R}, ad - bc > 0$ (i.e. automorphisms of $\mathbb{H}^+$.)

• similar to above point a Mobius transform $w = az + b/cz + d$ maps $\mathbb{H}^- \mapsto \mathbb{H}^-$ iff $a, b, c, d \in \mathbb{R}, ad - bc < 0$ (i.e. automorphisms of $\mathbb{H}^-$.)

4 Stereographic projection

• To visually represent the whole complex plane and the point ∞ Riemann project the whole complex plane to a sphere : Riemann sphere (Σ) centered at origin a unit radius in 3 dimensions where the xy plane is $\mathbb{C}$.

• The point $N = (0, 0, 1)$ (north pole) maps to ∞ (in a pseudo sense) and every other point (z) is mapped to ($\hat{z}$)the point of intersection of the Riemann sphere and the line through N and the point.

• Properties:

■ Unit circle $\mathbf{C} = |z| = 1$ remains fixed
■ interior of $\mathbf{C}$ is mapped to Southern hemisphere particularly $0 \mapsto (0, 0, -1) = \mathbf{S}$.(south pole)
■ exterior of $\mathbf{C}$ is mapped to Northern hemisphere
■ A line in $\mathbb{C}$ is mapped to circle passing through $\mathbf{N}$ particularly the tangent of this circle at $\mathbf{N}$ is parallel to the line (in 3 dimensions)
■ It is **conformal map** in accordance to an observer **from inside of Σ**.
■ Stereographic projection is can be broken down as inversion in the plane through $\{\mathbf{N}, z \mapsto \hat{z}\}$: if $\mathbf{K}$ is a circle centered at $\mathbf{N}$ of radius $\sqrt{2}$ in the plane where line through $\mathbf{N}$ and z passes then $\hat{z}$ is the image $\mathfrak{J}_{\mathbf{K}}(z)$ in this plane (this plane is considered as $\mathbb{C}$ for $\mathfrak{J}_{\mathbf{K}}(z)$.)
■ From above it is clear that Circles are mapped to circles in particular origin centered circles are mapped to horizontal circles (i.e circles in planes parallel to xy. plane)
• Properties related to functions:
■ Complex conjugation in $\mathbb{C}$. induces a reflection of the Riemann sphere in the vertical plane passing through the real axis.
■ Inversion of $\mathbb{C}$ in the unit circle induces a reflection of the Riemann sphere in its equatorial plane (i.e. Northern hemisphere $\longleftrightarrow$ Southern Hemisphere).
■ The mapping $z \to (1/z)$ in $\mathbb{C}$ induces a rotation of the Riemann sphere about the real axis through an angle of π.
■ properties functions like conformality at ∞ can be checked through Stereographic projection.
• formulas of Projection
■ if $z \mapsto (X, Y, Z)$ then:
■ $Z = \frac{|z|^2 - 1}{|z|^2 + 1}$, $X + iY = \frac{2z}{1 + |z|^2} = \frac{2x + i2y}{1 + x^2 + y^2}$.
■ if $z \mapsto (\theta, \phi)$ for θ angle subtended around z axis in xy plane and ϕ angle subtended at center by $\mathbf{N}$ and $\hat{z}$ then:
■ $z = \cot(\phi/2) e^{i\theta}$ or
$\theta = \operatorname{Arg}(z), \quad \phi = 2\cot^{-1}(|z|)$.

5 Analyticity

• if $z(x + iy) \mapsto f(z) = w(u + iv)$ then $df = du + idv$ $du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$ and $dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$ i.e.

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}.$$

• where the linear transform is the Jacobian matrix of f.
• now in $\mathbb{C}$ if $df(w) = f'(z)dz$ to be true $f'(z)$ should not depend on dz i.e. each infinitesimal vector dz at z should transform to dw at $w = f(z)$ by the same factor $f'(z)$ no matter the direction of dz..
• this condition tells us that dw is just the amplification and rotation or twist or together **amplitwist** of dz (as $f'(z) \in \mathbb{C} \implies dw = f'(z)dz = r'e^{i\theta'}dz$.)
• now if f is diffrentiable at z then $f'(z)$ exist so the infinitesimal map at point z is an amplitwist.
• clearly amplitwist is conformal (as amplification and twist is)
• now for the converse if a map is conformal at z then it is not presupposed to be amplitwist at z as the amplification may vary but if we presuppose that the map is locally conformal at z (i.e in some whole neighborhood) then clearly the map is locally amplitwist at z (as infinitesimal $\triangle$ is mapped to similar infinitesimal $\triangle$).
• By above we define **Analytic functions** : functions in $\mathbb{C}$ whose effect are locally (infinitesimal) an amplitwist or a function is analytic at z if it is diffrentiable at z and in a neighborhood of z. (as diffrentiable in neighborhood makes it locally conformal).
• Thus we have an **Analytic function is Conformal**.
• Geometric properties of Analytic function:
■ infinitesimal circles are mapped to infinitesimal circles
■ A mapping between spheres represents an analytic function iff it is conformal.

■ Conformality of analytic functions breakdown near critical points ($f'(z) = 0.$) and branch points.

• Geometric property of general transform on $\mathbb{C}$: as jacobian is a linear transform by singular value decomposition of 2×2 matrices we have the local linear transform by a complex mapping is a stretch in direction (d), another stretch in direction perpendicular to in $(d^{\perp})$. and finally a twist. in particular an infinitesimal circle is transformed to an ellipse (may not be conformal).

• **C-R equations** :

■ now as f is analytic $\implies f'(z) \in \mathbb{C}$ so multiplying by Jacobian matrix is equivalent to a complex multiplication now as

$$(a+ib)(x+iy) = (ax-by)+i(bx+ay)$$
$$\longleftrightarrow \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax-by \\ bx+ay \end{bmatrix}$$

. we have $J = \begin{bmatrix} \partial_x u & \partial_y u \\ \partial_x v & \partial_y v \end{bmatrix} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. i.e.

$$\partial_x u = \partial_y v, \quad \partial_y u = -\partial_x v.$$

$$i\partial_x f = \partial_y f.$$

which gives the Cartesian-Cartesian form(C-C) now in Polar-Cartesian (P-C) form we have $f(re^{i\theta}) = u + iv$ and C-R equations are

$$\partial_\theta v = r\partial_r u, \quad \partial_\theta u = -r\partial_r v.$$

$$\partial_\theta f = ir\partial_r f.$$

(P-P) form $f(re^{i\theta}) = Re^{i\Psi}$ C-R equations

$$\partial_\theta R = -rR\partial_r \Psi, \quad R\partial_\theta \Psi = r\partial_r R.$$

(C-P) form $f(x+iy) = Re^{i\Psi}$. C-R equations

$$\partial_x R = R\partial_y \Psi, \quad \partial_y R = -R\partial_x \Psi.$$

• General properties of Analytic functions:

■ if f, g are analytic then $f+g, f \times g, f \circ g, f^{-1}$ are analytic when ever they are defined, in particular as f is amplitwist locally there is a 1-1 correspondence in a neighbourhood of non critical points to their images $\implies$ that local inverse exists.

■ if f is analytic in E then so is f' (i.e. f is infinitely differentiable in the defined region)

■ every zero or an analytic point is isolated (generally p-point of f or pre-image of p in f doesn't have a limit point.)

■ **Identity/Uniqueness Theorem**: restating the above we have, if $f(z)$ is analytic in D and if S set of zeroes has a limit point in D then $f(0) \equiv 0$ in D (in general if p-points of f has a limit point then $f(z) \equiv p$).

■ Extending the above we get, if even an arbitrarily small segment of curve is crushed to a point by an analytic mapping, then its entire domain will be collapsed down to that point (i.e. the function is constant) (this property is known as **Rigidity**)

■ from above if f, g analytic agree on a curve or more generally $\{a_n\} \mapsto a$ then $f \equiv g$.

■ if some identity for f analytic holds when restricted to $\mathbb{R}$ then it holds for entire $\mathbb{C}$. (eg: odd and evenness.)

6 Analytic continuation

• an analytic function or a power series can be extended (from defined) to other regions this is analytical so called Analytic continuation.

• Analytic continuation via reflection:

■ if f is an generalization of a real function (defined on $\mathbb{R}$) and is known in upper or lower parts of real axis (in some region with some parts of $\mathbb{R}$ as boundary) then it can be **analytically continued by** $f*(z) = \overline{f(\overline{z})}$ in the other half part (reflection by $\overline{z}$ part of region)(this holds by property of rigidity of analytic functions).

■ In general if f maps a line (L) to another line ($\hat{L}$) then we can analytically continue one side of L to the other by using the fact that points symmetric in L map to points symmetric in $\hat{L}$.

■ similarly if f maps a circle C to circle $\hat{C}$ then mobius transforms can be used to translated these to symmetries i.e. $M : C \mapsto L$, $\hat{M} : \hat{C} \mapsto \hat{L}$ (as composition by mobius transfor ehic are analytic doesnt change the analyticity of $f \mapsto \hat{M} \circ f \circ M^{-1}$).

• **Schwarzian Reflection**:

▪ Given a sufficiently smooth curve K, it is possible to find an analytic function $S_K(z)$ such that $z \in K \implies S_K(z) = \bar{z}$ then

▪ Schwarz function of $K = \tilde{z} = \mathfrak{R}_K(z) = \overline{S_K(z)}$.

▪ clearly if $q \in K$ $\tilde{q} = \overline{S_K(q)} = \overline{\overline{q}} = q$ i.e. remains unchanged.

▪ Also as S_K just amplitwists infinitesimal disk at $q \in K$ to infinitesimal disk in $\overline{q} \in \overline{K}$ we observe that for $S_K|qp \mapsto \overline{qp}$(for $p, q \in K, qp$ infinitesimal) amplification = 1 and twist = -2ϕ where ϕ is the angle b/w tangent to K at q with horizontal

▪ so from above we get if a is on infinitesimal circle passing through K then $\tilde{a} = \mathfrak{R}_K(a)$ is reflection along the tangent of K. i.e $\mathfrak{R}_K$ near K is sort of like Reflection in K (pseudo).

▪ $\mathfrak{R}_K$ is anticonformal so $\mathfrak{R}_K \circ \mathfrak{R}_K$ is conformal so analytic (as amplification=1) and as $\mathfrak{R}_K \circ \mathfrak{R}_K$ maps infinitesimal areas around K to itself thus agrees with Identity so is Identity i.e. $\mathfrak{R}_K \circ \mathfrak{R}_K(z) = z$.

▪ Now if K is a smooth enough curve to posses S_K and any analytical map f defined on a region bordering K such that $\hat{K} = f(K)$ also posses $S_{\hat{K}}$ then we can analytically continue f around K (reflection of region by K) by demanding points symmetric to K are mapped to points symmetric to $\hat{K}$ by f and this analytic continuation is given by:

$$F = \mathfrak{R}_{\hat{K}} \circ f \circ \mathfrak{R}_K.$$

7 Complex Integration

• we define complex integration as the generalized Riemann Integration over a given path a to b or as contour integration

• clearly integration here depends on path

• complex integration can be visualized as weighted vector sum : if S is path from a to b and Δ_j 's are vector decomposition (partition of S and linearly) that form S , $w_j = f(\text{mid } \Delta_j)$ i.e f(mid points of Δ_j.) then we can generalize as

$$\int_S f(z)dz = \sum_{j \to \infty} w_j \Delta_j$$

• from above we get: if $|f| \leq M$ in image of K. then

$$\left| \int_S f(z)dz \right| \leq M.\text{length of } K.$$

• **Winding number and properties** :

• winding number for a closed loop L and a point $a = \nu(L, a)$ is the number of revolutions $z - a$ makes as it traces L (where we fixing a direction for counter-clockwise revolution is +ve and clockwise is -ve by convention)

• A simple loop is a closed curve that doesnt intersect with itself

• now as a point moves from left to right if it crosses a boundary of the loop and the loops direction is downwards (upwards) the winding number increased (decreases) by 1 (here the first entry of the point to loop is made to be in loop moving in downwards direction).

• we define inside of a loop L to be regions (points) where $\nu[L, a] \neq 0$.

• Hopf's degree Theorem(ristricted to $\mathbb{C}$): A loop K may be continuously deformed into another loop L, without ever crossing the point p, if and only if K and L have the same winding number round p.

• d is a p-point of a function f if set of pre-images of p in f contains d i.e. $d \in f^{-1}(p)$.(pre-image)

• **Argument-Principle** theorem: If $f(z)$ is analytic inside and on a simple loop Γ , and N is the number of p-points (counted with their multiplicities) inside Γ , then $N = \nu(f(\Gamma), p]$.

• if f analytic, $f(a) - p = 0$ and for $\Delta = z - a$ $f(a + \Delta) = p + \Omega(Z)\Delta^n$ (obtained by Taylor series) here algebraic multiplicity of a in f is n, for sufficiently small circle C_a around a that doesnt have any other p-points then

$$\nu(f(C_a), a) = n.$$

i.e. $f(C_a)$ loops around p exactly n times.

• now we define $\nu(a)$ for a continuous function h as : if $h(a) = p$, Γ_a is the loop having only a and no other p-points then topological multiplicity $\nu(a) = \nu(h(\Gamma_a), a)$.

• clearly as analytical maps are conformal we have $\nu(a)$ is always +ve ($\neq 0$.) for analytic functions

• $\nu(a)$ = sign of $det(J(a))$ where J is Jacobian

• **Topological Argument-Principle** theorem: for a continuous map h the total number of p-points inside Γ. (counted with their topological multiplicities) is equal to the winding number of $h(\Gamma)$ round p..

• **Darboux's Theorem** : If an analytic function h maps Γ onto $h(\Gamma)$ in one-to-one fashion, then it also maps the interior of Γ onto the interior of $h(\Gamma)$ in one-to-one fashion.

• **Rouche's Theorem** : for f, g analytic in and on Γ, If $|g(z)| < |f(z)|$ on Γ , then $(f+g)$ must have the same number of zeros inside Γ as f.

• **Brouwer's Fixed Point Theorem** : any continuous mapping of the disc to itself will have a fixed point.
In general there must be a fixed point if the disc is mapped into its interior and there are at most a finite number of fixed points. (now if the map is analytic then the number of fixed points inside the disk is only one).

• **If f is analytic inside and on a simple loop Γ then no point outside $f(\Gamma)$ can have a preimage inside Γ.**(i.e interior of Γ maps to interior of $f(\Gamma)$.)

• **Maximum Modulus Theorem** : The maximum (minimum respectively if $f(z) \neq 0$ inside the closed boundary) of $|f(z)|$ on a region where f is analytic is always achieved by points on the boundary, never ones inside.

• **Schwarz's Lemma** : If an analytic mapping of the disc to itself leaves the center fixed,then either every interior point moves nearer to the center, or else the transformation is a simple rotation. (i.e. them map is contractive towards the center).

• **General Schwarz's Lemma** :
If $f : \Delta(\{|z| < 1\}) \mapsto \overline{\Delta}$ is analytic and has a zero of order n at origin then:

■
$$|f(z)| \leq |z|^n \; \forall z \in \Delta.$$

■
$$|f^n(0)| \leq n!$$

■ if Equality holds (any one) for any point inside Δ other than 0 then $f(z) = az^n, |a| = 1$.

• modifying Schwarz's lemma we get for f analytic in $\Delta(a, R)$, $|f(z)| \leq M$ in $\Delta(a, R)$ and $f(a) = 0$ then (applying Schwarz's lemma for $g(z) = f(Rz + a)/M$ i.e. $z \to Rz + a$ for $|z| < 1$)

■
$$|f(z)| \leq \frac{M|z-a|}{R}$$
for every $z \in \Delta(a, R)$.

■
$$|f'(a)| \leq \frac{M}{R}.$$

■ and if equality holds for any two then $f = M\epsilon(z-a)/R$ for some $|\epsilon| = 1$.

• **Schwarz-Pick Lemma** : Unless an analytic mapping of the unit disc to itself is a automorphism the hyperbolic separation of every pair of interior points decreases.
i.e.
if f is analytic on Δ , $|f(z)| \leq 1 \forall z \in \Delta$ and $f(a) = b$ for some $a, b \in \Delta$, then
$$|f'(a)| \leq \frac{1-|f(a)|^2}{1-|a|^2}.$$
and for $a, a' \in \Delta$
$$\rho(f(a), f(a')) \leq \rho(a, a').$$
where $\rho(z, a) = |(z-a)/(\overline{a}z - 1)|$.

• **Liouville's Theorem** :An analytic mapping cannot compress the entire plane into a region lying inside a disc of finite radius without crushing it all the way down to a point, i.e. a bounded entire function is constant or bounded harmonic function is constant (by Taylor series)

• **Generalized Liouville's Theorem** : if f is an entire function such that $|f(z)| \leq M|z|^\alpha$ for all sufficiently large $|z|$ and $\alpha \geq 0$, $M > 0$ then f

reduces to a polynomial of maximum degree n closest integer to α.

• **Generalized Argument-principle theorem** :Let f be analytic on a simple loop Γ and analytic inside except for a finite number of poles. If N and M are the number of interior p-points and poles, both counted with their multiplicities, then $\nu(f(\Gamma), p) = N - M$.

• for any closed loop L $\oint_L \frac{1}{z} dz = 2\pi i \nu(L, 0)$ in general

$$\oint_L \frac{1}{z-p} dz = 2\pi i \nu(L, p).$$

• now as $Im(a\bar{b}) \equiv a \times b$ it gives $2\times$ the area enclosed by triangle formed by sides a and b vectors so we have for a simple loop L:

$$\oint_L \bar{z} dz = 2i \times \text{area enclosed by } L.$$

for general loop L

$$\oint_L \bar{z} dz = 2i \times \sum_{\text{inside}} \nu_j A_j.$$

where A_j is the area enclosed by points which have $\nu_j = \nu(L, p) = a \neq 0$ constant (i.e form a part of loop).

• **Cauchy's Theorem** :If an analytic mapping has no singularities "inside" a loop, its integral round the loop vanishes (i.e. = 0).

• from above we get in integral of analytic functions are **path independent**.

• **Morera's Theorem** : If all the loop integrals of f are known to vanish in a region then f is analytic in that region.

• if $m \neq -1$ then

$$\int_A^B z^m dz = \frac{1}{m+1}(B^{m+1} - A^{m+1})$$

• clearly from above we have

$$\oint z^m dz = 0 \text{ if m } \neq -1.$$

• **Deformation Theorem** : If a contour sweeps only through analytic points as it is deformed, the value of the integral does not change.

• **Cauchy's formula** : if $f(z)$ is analytic inside a simple loop L then

$$f^n(a) = \frac{n!}{2\pi i} \oint_L \frac{f(z)}{(z-a)^{n+1}} dz.$$

• **General Cauchy's theorem** : if L is not simple then

$$\nu(L, a) f^n(a) = \frac{n!}{2\pi i} \oint_L \frac{f(z)}{(z-a)^{n+1}} dz.$$

• **Taylor Series** : If $f(z)$ is analytic, and a is neither a singularity nor a branch point, then $f(z)$ may be expressed as the following power series, which converges to $f(z)$ within the disc whose radius is the distance from a to the nearest singularity or branch point:

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n. \text{ where}$$

$$c_n = \frac{f^n(a)}{n!} = \frac{1}{2\pi i} \oint_L \frac{f(z)}{(z-a)^{n+1}} dz.$$

• **Laurent Series** : if f is analytic inside an annulus centered at a then f an be expressed as the following series (for any simple loop K inside the annulus)

$$f(z) = \sum_{-\infty}^{\infty} a_n (z-a)^n. \text{ where}$$

$$a_n = \frac{1}{2\pi i} \oint_L \frac{f(z)}{(z-a)^{n+1}} dz.$$

• **General Residue Theorem** : from Laurent series and integral of z^m we have if f is analytic then for a loop L containing only isolated singularities $\{a_k\}$ of f, we have:

$$\oint_L f(z) dz = 2\pi i \sum_k \nu[L, a_k] Res(f, a_k).$$

where $Res(f, a_i) = a_{-1}$ or coefficient of $1/(z - a_i)$ when f is written as Laurent series centered at a_i containing no other singularity.

• if a is a pole of f of order m.
(i.e. $\lim_{z\to a}(z-a)^m f(z) = c$ defined) then $\mathrm{Res}(f(z), a)$

$$= \lim_{z\mapsto a} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z).$$

• if P/Q has a simple pole (order 1) at a then

$$\mathrm{Res}\left(\frac{P}{Q}(z), a\right) = \frac{P(a)}{Q'(a)}.$$

• Gauss mean value theorem : for a harmonic function ϕ ($\partial_x^2\phi + \partial_y^2\phi = 0$) the mean value of ϕ on a circle is equat to the vale of function at center of the circle i.e.
if $f(z)$ is analytic then

$$\frac{1}{2\pi}\int_0^{2\pi} f(a + re^{i\theta})d\theta = f(a)$$

• Residue at infinity : for analytic f we have

$$\mathrm{Res}(f(z), \infty) = -\mathrm{Res}\left(\frac{f(1/z)}{z^2}, 0\right).$$

$= \frac{1}{2\pi i}\oint_{C^-} f(z)dz = -a_{-1}$, where C^- is a circle oriented $-$vely covering all singularities ($\neq\infty$) of $f(z)$.

• **Extended Residue theorem**: for analytic f we have

$$\mathrm{Res}\left(\frac{f(1/z)}{z^2}, 0\right) = \sum_k \mathrm{Res}(f, a_k)$$

where $a_k \neq \infty$ also if simple loop γ includes all finite singularities of $f(z)$ then

$$\oint_\gamma f(z)dz = 2\pi i\, \mathrm{Res}\left(\frac{f(1/z)}{z^2}, 0\right).$$

• **Argument-Principle theorem (integral form)** : if $f(z)$ is a meromorphic function in domain $D \subseteq \mathbb{C}$, has finitely many zeroes and poles in D, C is any simple loop in D such that no pole or zero lie 'on' C then

$$\oint_C \frac{f'(z)}{f(z)}dz = 2\pi i(N - P).$$

where N and P denote the number of zeroes and poles of f inside C (counted with their multiplicities and order).

• **General Rouche's Theorem** : for f, g analytic in and on C with finite number of poles and zeroes inside the Domain covering C, If $|g(z)| < |f(z)|$ on C , then

$$N_{f+g} - P_{f+g} = N_f - P_f$$

where N_h, P_h denote the number of zeroes and poles of h inside C (counted with their multiplicities and order).

• Alternative form of Rouche's Theorem : if same conditions as above hold for $g - f(z), f(z)$ and $|g(z) - f(z)| < |f(z)|$ then

$$N_g - P_g = N_f - P_f.$$

(can used for calculating the number of zeroes of polynomial in a give loop)

• Application of Rouche's Theorem to polynomials

▪ eg: consider the polynomial $g(z) = z^6 - 5z^4 + 7$

⋆ now $|g(z) - 7| \le |z|^6 + 5|z|^4 \le 7$ if $|z| \le 1$(as $6 + 1 \le 7$) thus $g(z)$ has same number of zeroes as $f(z)7$ in $|z| \le 1$ i.e. $g(z)$ has no zeroes inside $|z| \le 1$.

⋆ similarly if $f(z) = -5z^4$ we have $|g(z) - f(z)| \le |z|^6 + 7 \le 5|z|^4$ if $|z| \le 2$ (as $2^6 + 7 = 71 \le 5.2^4 = 80$) thus $g(z)$ has 4 zeroes in $|z| \le 2$.

⋆ similarly if $f(z) = z^6$ we have $|g(z) - f(z)| \le 5|z|^4 + 7 \le |z|^6$ if $|z| \le 3$ (as $5.3^4 + 7 = 412 \le 3^6 = 729$) thus all zeroes of $g(z)$ lie inside $|z| \le 3$.

8 Mics Properties

• A real valued function of a complex variable $f : \mathbb{C} \mapsto \mathbb{C}$ has derivative zero or non existent i.e if f is analytic the is a constant.

• for an analytic function in domain D if one of : $|f|, \mathrm{Re}(f), \mathrm{Im}(f), \mathrm{Arg}(f)$ is constant in D then f is constant.

• **Harmonic functions:**

■ $\phi(x, y)$ a real valued function is harmonic iff $\nabla^2\phi = 0$.

■ real and imaginary parts of analytical function's are harmonic (in the defined "Domain"(a connected open set)) (converse is not true).

■ $f(z)$ is analytic in Domain D iff real and imaginary parts of both $f(z)$ and $zf(z)$ are harmonic.

■ if ϕ is a harmonic function in a Domain the $f = \phi_x - i\phi_y$ is analytic in the domain.

■ Harmonic conjugate of harmonic function ϕ is another harmonic function ψ such that $f = \phi + i\psi$ (i.e ψ is the imaginary part od analytic function whose real part is ϕ).

■ if ϕ is harmonic in a simply connected region then it has a harmonic conjugate in this region.

• if f is analytic in a simply connected region Ω and $f(z) \neq 0$ in Ω then $\exists h$ analytic in Ω such that

$$e^{h(z)} = f(z).$$

($h'(z) = f'(z)/f(z)$ claim $f.e^{-h(z)} = c = e^k$ prove by differentiating) (domain can be whole $\mathbb{C}$).

• if f satisfies the above conditions then $\exists g$ analytic in Ω such that $g^2(z) = f(z)$ in Ω (choose $g(z) = e^{h(z)/2}$).

• **Cauchy's Inequality** : if f is analytic in an open disk centered at a of radius $R = \Delta(a, R) = |z - a| < R$ and $|f(z)| \leq M$ on boundary $\overline{\Delta(a, r)}$ for $0 < r < R, \zeta \in \partial\Delta(a, r)$ we have

$$|f^k(a)| \leq \frac{M.k!}{r^k}.$$

(use estimation of Cauchy integral).

• for an open set D if $f_n : D \mapsto \mathbb{C}$ are analytic for each n and if $f_n \mapsto f$ uniformly on each compact subset of D then f is analytic and more over $f_n^k \mapsto f^k$ uniformly in the compact subsets, the same is true for series also if all conditions hold.

• every zero of an analytical function is isolated.

• from above we have if a_n are the zeros of analytical map f, $a_n \mapsto a \in \mathbb{C}$ then $f \equiv 0$.

• in general if if q_n are p-points of analytical map f, $q_n \mapsto q \in \mathbb{C}$ then $f \equiv p$ (use $h(q_n) = f(q_n) - p = 0$.)

• also if f, g analytic in Domain D , $f - g$ has set S of zeroes that has a limit point then $f \equiv g$ in D (in general if $f - g$ has set Q of p-points that has a limit point then $f(z) = g(z) + p$.)

• four distinct points in $\mathbb{C}_\infty$ all lie on a circle or line iff their cross ratio is real.

• a singularity at z_0 of $f(z)$ is removable if f can be defined at z_0 so that it is analytic at z_0.

• **Riemann's Removable Singularity theorem**: if f has an isolated singularity at z_0 then z_0 is removable iff one of the below holds.

■ f is bounded in deleted neighborhood of z_0.

■ $\lim\limits_{z \mapsto z_0} f(z)$. exists

■ $\lim\limits_{z \mapsto z_0} (z - z_0)f(z) = 0$.

• **Picard's Little Theorem** : every non constant entire function only omits at most one value from this we get if a entire function omits two value then it is a constant.

• **Picard's Great theorem** : if z_0 is the essential singularity of $f(z)$ analytic in then $\Delta(z_0, r) - z_0$ then $\mathbb{C} - f(\Delta(z_0, r) - z_0)$ is a singleton set.

• **Picards little theorem for meromorphic functions**: A meromorphic function omits three distinct values then it is a constant.

• if f is an even anlytic function (i.e. f(-z)=f(z)) then for z_0 isolated singularity of f $Res(f(z), z_0) = 0$. (there are no odd power terms in Laurent series expansion).

• if analytic function f is such that $f(z) = f(z + z_1) = f(z + z_2)$ (doubly periodic)and if $z_1/z_2 \notin \mathbb{R}$ then f is a constant (as z_1, z_2 will be linearly independent).

• if $p(z)$ is a polynomial of degree $n \geq 1$ then every zero of $p'(z)$: (z'_k)lies in the complex hull of zeroes of $p(z)$: (z_k) i.e $z'_k = \sum_{k=1}^{n} \lambda_k z_k$, for $\sum_{k=1}^{n} \lambda_k = 1$.

• if f is analytic in $|z| < M$ iff $\overline{f(\overline{z})}$ is also analytic in $|z| < M$ (as amplitwistness of $f(z)$ doesnt change).

• if $p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1}z^{n-1} + z^n$, simple loop C covers all zeroes of $p(z)$ then

$$\oint_C \frac{zf'(z)}{f(z)} = -2\pi i a_{n-1}.$$

$$\oint_C \frac{z^2 f'(z)}{f(z)} = 2\pi i(a_{n-1}^2 - 2a_{n-2}).$$

• z_1, z_2 and z_3 are vertices of equilateral triangle iff

$$\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0.$$

i.e.

$$z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1.$$

• z_1, z_2 and z_3 iff

$$z_3 = t(z_1) + (1 - t)z_2 \text{ for } t \in \mathbb{R}$$

(i.e equation of line in $2D$.)

• if analytic function $f(z)$ is real on real line and purely imaginary on imaginary axis then $f(-z) = -f(z)$ i.e. f is odd.

• for $f(z)$ analytic in Domain D then:

▪ if f is even i.e. $f(z) = f(-z)$ then $\exists\ g(z)$ analytic in D such that $f(z) = g(z^2)$.

▪ if f is odd i.e. $-f(z) = f(-z)$ then $\exists\ g(z)$ analytic in D such that $f(z) = zg(z^2)$.

▪ Every meromorphic function in $\mathbb{C}$ can be represented as quotient of two entire functions.

▪ **Open mapping Theorem** : if $f(z)$ is a non constant analytic function in Domain D then it is open mapping i.e. $f(O)$ is open for every open set $O \in \mathbb{C}$.

• Clearly if f is analytic in D a Domain (open connected set) then $f(D)$ is also a Domain.

• **Hurwitz's Theorem** : if $\{f_n\}$ are non vanishing ($\neq 0$) in a Domain D and converges uniformly to f on every compact subset of D then either f has no zeroes or $f \equiv 0$.

• **Local mapping theorem** : if f is analytic at a the there exist a neighborhood of a where f is one-one iff $f'(a) \neq 0$. or
if f is univalent and analytic in a Domain D then $f'(z) \neq 0$ in D.

• if f is meromorphic at pole a and is one-one in neighborhood of a iff a is a simple pole.

• from above if f is meromorphic and univalent in D then f has only simple poles in D.

• for f analytic at ∞ is univalent at ∞(in its nbd) iff $Res(f, \infty) \neq 0$.

• **Riemann mapping theorem** : every simply connected domain which is a proper subset of $\mathbb{C}$ is Conformally equivalent to a unit disk i.e.
if Ω is a simply Connected open set then there exist a function f analytic in Ω such that $f(\Omega) = \Delta$.

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