Numerical Linear Algebra

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if a matrix is in triangular form one can easy calculate its inverse by making note that inverse of a triangular matrix is of the same triangular type i.e. for example A is upper triangular non-singular matrix then A^{-1} is also an upper triangular matrix.

LU Decomposition

■ From previous point we have if any non singular matrix A can be written as $A = LU$ for lower triangular L and upper triangular U then $A^{-1} = U^{-1}L^{-1}$ thus inverse can be easily calculated.

■ This Decomposition may not be unique

■ To decompose in a easy way we take diagonal elements of U or L as 1. (only in one of the factors) and compute the coefficients by writing $A = LU$ and solving some equations in a linear order.

■ Now in addition if principal minors (Δ_k) of matrix A are not zero then the above decomposition is unique.

Gauss elimination

if $A = [a_{ij}]$ be a $n \times n$ non singular matrix then for linear system $Ax = b$ then we can use elementary operations:
exchange of rows, addition of rows and multiplication by a non zero constant to a row to transform the linear system$A'x = b'$ such that $a'_{11} \neq 0$ and $a'_{i1} = 0$ for $i < 1$ and continuing this process to get for $i = 2, 3, .., n$ we get a system $Gx = \tilde{b}$ where G is upper triangular and has same solutions as original system.

Gauss-Jordan method

this method is similar to Gauss elimination but $Ax = b$ for non singular square A is transformed to $G_J x = \tilde{b}$ where G_J is diagonal i.e. for $A = [a_{ij}]$, a_{ii} is made non zero and all other a_{ij} is made zero with elementary transformations.

General Iterative methods

■ iterative methods can be generalised as $x^{(k)} = Tx^{(k-1)} + c$

■ this method converges to a unique solution for any initial approximation $x^{(0)}$ iff $(\iff)$ $\rho(T) < 1$ where $\rho(T) = \max(|\lambda|)$ for λ eigenvalue of T.

Jacobi's Method

■ if $Ax = b$ is a system such that for n-square $A = [a_{ij}]$ we have $a_{ii} \neq 0$ (if not is made by rearranging rows or equations if possible) then for $x = [x_i]^T$ we can transform $x_i = \left[\sum_{\substack{j=1 \\ j\neq i}}^{n} (-a_{ij}x_j/a_{ii}) + b_i/a_{ii}\right]$ from which we get the iterative method i.e. $x^{(0)}$ is initial approximation and for k^{th} approximation $x^{(k)}$ we have the iteration using $x^{(k-1)}$ given by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1 \\ j\neq i}}^{n} (-a_{ij}x_j^{(k-1)}) + b_i\right].$$

■ Now for matrix representation if $A = D + L + U$ where D is diagonal L is lower diagonal with diagonal entries 0 and U is upper diagonal with diagonal entries 0 then for Jacobi method we have

$$
\begin{aligned}
&(D+L+U)x = b \\
&\implies Dx* = -(L+U)x + b. \\
&\implies x = -D^{-1}(L+U)x + D^{-1}b. \\
&\text{i.e.} \quad x^{(k)} = -D^{-1}(L+U)x^{(k-1)} + D^{-1}b.
\end{aligned}
$$

so we get $T = -D^{-1}(L+U), c = D^{-1}b$ for general form.

Gauss-Seidel Method

■ This is similar to Gauss method but here we use the previous k^{th} iterated variables for the next k^{th} one i.e. for in $x_i^{(k)}$ iteration we can replace $x_j^{(k-1)}$ for $j < i$ with $x_j^{(k)}$ as these are already found i.e.

$x_i^{(k)} =$

$$
\frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + b_i\right].
$$

■ for matrix representation we rewrite the iterative formula as

$$
\sum_{j=1}^{i} a_{ij}x_j^{(k)} = -\sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + b_i
$$

similar to Jacobi's case if $A = D+L+U$ by above formula we have

$$
\begin{aligned}
&(D+L)x^{(k)} = -Ux^{(k-1)} + b. \\
&\text{i.e.} x^{(k)} = -(D+L)^{-1}Ux^{(k-1)} + (D+L)^{-1}b.
\end{aligned}
$$

so we get $T = -(D+L^{-1})U, c = (D+L)^{-1}b$.

for system $Ax = b$, $A = D+L+U$

■ if A is strictly diagonal then both Jacobi and Gauss-Seidel methods converge for every initial approximation $x^{(0)}$.

■ Gauss-Seidel method is twice as fast as Jacobi's method for convergence

now from general iterative methods we have

■ sufficient condition for convergence of Jacobi's method is that

$$
||T|| = ||-D^{-1}(L+U)|| < 1 \quad \text{i.e.}\rho(T) < 1.
$$

■ similarly sufficient condition for convergence of Gauss-Seidel method is that

$$
||T|| = ||-(D+L)^{-1}U|| < 1.
$$

■ Both these method also converge if $A = [a_{ij}]$ is such that

$\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \leqslant |a_{ii}|$ for $i = 1, 2, .., n$ and strict inequality holds for at least one i.