# Matrix Properties

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yn37git.github.io/blog/2025/Short-Notes

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## Symbols used:

| | | |
|---|---|---|
| iff | $\rightarrow$ | if and only if |
| Capital letters | $\rightarrow$ | Matrices $A_{m\times n} = [a_{ij}]_{m\times n}$ |
| $A^T$ or $A'$ | $\rightarrow$ | Transpose of Matrix |
| $\bar{A}$ | $\rightarrow$ | Conjugate of Matrix |
| $AB$ | $\rightarrow$ | Matrix product |
| $\lvert A\rvert$ or $\det(A)$ | $\rightarrow$ | Determinant of Matrix |
| $\mathrm{tr}(A)$ or $\mathrm{trace}(A)$ | $\rightarrow$ | trace of Matrix |
| $A^*$ | $\rightarrow$ | Conjugate transpose of Matrix |
| $A^{-1}$ | $\rightarrow$ | Inverse of Matrix |
| $I$ | $\rightarrow$ | Identity |
| $\mathrm{Im}(A)$ | $\rightarrow$ | Image or range space of $A$ |
| $\mathrm{rank}(A)$ or $r(A)$ | $\rightarrow$ | Dimension of Range space of $A$ |
| $\ker(A)$ | $\rightarrow$ | Null space of $A$ |
| $\mathrm{null}(A)$ | $\rightarrow$ | Dimension of Null space of $A$ |
| $\mathbb{F}$ | $\rightarrow$ | Field |

## 1 Basic properties

- $A(BC) = (AB)C$
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$
- $(AB)' = B'A'$
- $(AB)^* = B^*A^*$
- if $A$ is Hermitian then $iA$ is skew-Hermitian and vise-versa.
- if $A, B$ are symmetric, $AB$ is symmetric iff $AB = BA$.
- $AA', A'A$ are always symmetric.
- For any Square Matrix $A$:
  - $A + A'$ is symmetric.
  - $A - A'$ is skew- symmetric.
  - $A + A^*$ is Hermitian.
  - $A - A^*$ is skew-Hermitian .
- By preceding point any Square matrix can be decomposed (by +) into symmetric - skew-symmetric or Hermitian- skew-Hermitian pair.
- $B'AB$ is symmetric or skew as is $A$
- $B^*AB$ is hermitian or skew as is $A$
- Determinant is a Multiliear (row), Alternating and Normalized Function on Matrices.
- Determinant of upper or lower triangle or diagonal matrix is equal to product of diagonal elements.
- $|AB| = |A||B| = |BA|$
- $|A'| = |A|$
- $|A^*| = |\bar{A}|$
- $A$ is invertible iff $|A| \neq 0$.

• $A^{-1} = \frac{adj(A)}{|A|}$ where $adj(A)$ is the transpose of co-factor matrix.

• $B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$

• Cramer's rule for a system of linear equations $Ax = b$ where $A$ is square and for $x = [x_1, x_2, .., x_n]^T$ we have $x_i = \frac{|A \leftarrow_i b|}{|A|}$ where $A \leftarrow_i b$ is obtained by replacing $i^{th}$ column of $A$ by $b$.

• $|adj(A)| = |A|^{n-1}$ where $A$ is an $n \times n$ matrix

• $adj(A^*) = Adj(A)^*$

• $adj(A^{-1}) = adj(A)^{-1} = A/|A|$

• $adj(adj(A)) = |A|^{n-2}A$

• $adj(AB) = adj(B)adj(A)$ for non-singular matrices $A, B$.

• $A$ is orthogonal if $A'A = I$

• $A$ is orthogonal $\implies |A| = \pm 1 \implies$ invertible.

• $A$ is unitary if $A^*A = I$

• if $A, B$ are orthogonal then so are $AB, BA$. Similar result follows in unitary case also.

• $rank(A) = r$ iff all the $r + 1$ order minors are zero i.e. if any one of $r^{th}$ order minor is non zero then $rank(A) \geq r$.

• $rank(A) = rank(A') = rank(A^*)$

• Elementary transformation: exchange of rows, multiplication of row by non zero constant, addition of k multiple of a row to another row.

• Elementary transformations doesn't change the rank of a matrix.

• Every elementary transformation has a corresponding non singular matrix which when pre-multiplied to a given matrix gives the respective operation.

• Normal form of a matrix : (Echelon form) A matrix which can be partitioned into identity and null matrices where the identity is present in upper-left part.

• $\exists P, Q$ non-singular square matrices such that $N = PAQ$ where $A$ is any matrix and $N$ is its normal or Echelon form.

• $rank(AB) \leq min(\{rank(A), rank(B)\})$.

• $rank(A + B) \leq rank(A) + rank(B)$.

• **Sylvester inequality** :
for any matrices $A_{m \times k}, B_{k \times n}$

$$\begin{aligned} rank(AB) &= rank(B) - dim(Im(B) \cap ker(A)) \\ \text{so } rank(A) + Rank(B) - k &\leq rank(AB) \\ &\leq min(\{rank(A), Rank(B)\}). \end{aligned}$$

(use: for $Bx \neq 0$, $ABx = A(Bx) = 0$ iff $x \in Im(B) \cap ker(A)$ and that $dim(Im(B) \cap ker(A)) \leq null(A) = k - r(A)$ so $-dim(Im(B) \cap ker(A)) \geq r(A) - k$. )

• **Frobenius Inequality** :
for $A_{m \times k}, B_{k \times p}, C_{p \times n}$

$$rank(AB) + rank(BC) \leq rank(B) + rank(ABC).$$

• $rank(A) = rank(A^*A)$

• if all entries of $A$ are real then $rank(A'A) = rank(A)$.

• if $A$ is n-squared then :
- $rank(A) = n \implies rank(adj(A)) = n$.
- $rank(A) = n - 1 \implies rank(adj(A)) = 1$.
- $rank(A) < n - 1 \implies rank(adj(A)) = 0$

i.e. $adj(A) \equiv 0$. (use minors and cofactor definition of $Adj(A)$.)

• $rank(A) \geq rank(A^2) \geq .. \geq rank(A^n) \geq ..$

• $null(A) \leq null(A^2) \leq .. \leq null(A^n) \leq ..$

• if $rank(A^m) = rank(A^{m+1})$ then
- $rank(A^k) = rank(A^m) \quad \forall k \geq m$
- $null(A^k) = null(A^m) \quad \forall k \geq m$

• Eigenvalues of Hermitian matrices are real.
( if $\lambda$ is eigenvalue then $(Ax)^* = x^*A^* = x^*A = (\lambda x)^* = \bar{\lambda}x^*$ so $x^*A^*x = \lambda x^*x = \bar{\lambda}x^*x \implies \bar{\lambda} = \lambda$)

• Eigenvalues of Skew-Hermitian are purely imaginary or zero.

• If $\lambda$ is Eigenvalue of Unitary matrix $A$ then $|\lambda| = 1$
(if $Ux = \lambda x$ then $x^*U^*Ux = x^*Ix = x^*x$ but $(x^*U^*)(Ux) = \bar{\lambda}\lambda x^*x$. )

• Real Eigenvalues of Orthogonal Matrices are $1, -1$ only.

• Eigenvalues of $A$ and $A'$ are same.

• Eigenvalues of triangular, diagonal matrices are its diagonal elements.

• if $\lambda$ is an eigenvalue of non-singular matrix $A$ then

▪ $\lambda \neq 0$

▪ $\frac{1}{\lambda}$ is the eigenvalue of $A^{-1}$.

▪ $\lambda^k$ is the eigenvalue of $A^k$.

▪ $\frac{|A|}{\lambda}$ is the eigenvalue of $adj(A)$.

• if $\{\lambda_i\}$ are eigenvalues of $A$ then eigenvalues of $B = p(A)$ are of form $p(\lambda_i)$ only.

• For $A_n$ with eigenvalues $\lambda_1, \lambda_2, .., \lambda_n$ $trace(A) = \sum_{i=1}^{n} \lambda_i$ , $det(A) = \prod_{i=1}^{n} \lambda_i$ and $trace(adj(A)) = \sum_{i=1}^{n} \prod_{j \neq i}^{n} \lambda_i$.

• If $A = P^{-1}BP$ then $A$ and $B$ have same eigenvalues

• for square Matrices $A, B$ eigenvalues of $AB$ and $BA$ are same.

( use if $ABx = \lambda x$ then $BA(Bx) = B(ABx) = \lambda Bx$ so $\lambda$ is eigenvalue of $BA$ also and vis-a-viz.)

• Geometric multiplicity (no of eigenvectors for an eigenvalue) $\leq$ Algebraic multiplicity(order of eigenvalue in characteristic polynomial).

• $A = P^{-1}BP$ this Relation $ARB$ (similarity) is equivalence, determinant invariant, eigenvalue invariant , trace invariant.

• A matrix is diagonalizable if it is similar to a diagonal matrix

• A matric is diagonalizable iff for each of its eigenvalue Geometric multiplicity = Algebraic multiplicity.

• square matix $A$ is diagonalizable iff minimal polynomial of $A$ splits into distinct linear factors in the given field i.e. minimal polynomial of $A$ is separable and has only linear irreducible factors.

• A non-zero Nil-potent ($A^m = 0$) matrix has eigenvalues as zero only.

• A non-zero Nil-potent matrix is never Diagonalizable.

(if $A$ is diagonalizable then $P^{-1}AP = D$ so $(P^{-1}AP)^m = P^{-1}A^mP = 0 = D^m \implies D \equiv 0$ thus $A \equiv 0$ )

• **Schurs theorems:**

▪ Every Square matrix $A$ is Unitarily similar to Upper triangular matrix whose diagonals are eigenvalues of $A$ (complex values included).

▪ If $A \in M_n(\mathbb{R})$ and has only real eigenvalues then it is real orthogonally similar to real upper triangular matrix.

(say $\lambda_1, \lambda_2, .., \lambda_n$ are eigenvalues of $A_{n\times n}$ (with repeats) let $x$ be normalised eigenvector of $A$ to eigenvalue $\lambda_1$ then $x^*x = 1$ and $Ax = \lambda_1 x$, now from an orthonormal basis with $x$ and let this matrix be $U_1 = [x \; u_2..u_n]$ thus we have $U_1^*AU_1 = [\lambda_1, \star; \; 0, A_1]$ for $A_{1_{n-1\times n-1}}$ and as $U_1$ is unitary we have eigenvalues of $A_1$ are $\lambda_2, .., \lambda_n$ only so lets commence the same procedure for $A_{1_{n-1\times n-1}}$ we get $U_2$ join this to form $V_2 = [1, 0; \; 0, U_2]$ then we get $(U_1V_2)^*AU_1V_2 = [\lambda_1, \star, \star; \; 0, \lambda_2, \star; \; 0, 0, A_2]$ clearly $U_1V_2$ was unitary so proceeding similarly we get the theorem)

▪ If $A \in M_n(\mathbb{R})$ has complex eigenvalues then it is similar to a matrix with diagonal blocks of 1-by-1 and 2-by-2 only (has upper triangular entries). Where 1-by-1 blocks are real eigenvalue of $A$ and 2-by-2 blocks are $[a \quad b; -b \quad a]$ for $a + ib$ eigenvalue.

(for $A_{n\times n}$ let $\lambda = a + ib$ and its eigenvector is $x = u + iv$ then prove $\bar{\lambda}, \bar{x}$ are eigenpairs so $x, \bar{x}$ are linearly independent so are $u, v$ and as $Au = au - bv, Av = bu + av$ and if $S = [u, v, S_1]_{n\times n}$ be made non singular thus $S^{-1}AS = [B, \star; \; 0A_1]$ for $B = [a \quad b; -b \quad a]$. )

• **Every Symmetric matrix ($A \in M_n(\mathbb{R})$) is orthogonally similar to diagonal matrix $(D)$ i.e. $D = P^TAP, \; P^TP = I$.**

• **Every Hermitian matrix ($A$) is unitarily similar to diagonal matrix $(D)$ i.e. $D = P^*AP, \; P^*P = I$.**

• A matrix $A$ is normal iff $A^*A = AA^*$

• **A matrix is Unitarily similar to diagonal matrix iff it is Normal.**

• A triangular normal matrix is Diagonal also a block diagonal normal matrix has off diagonal blocks $=0$.

• if $A$ is normal then $p(A)$ (specially $A + aI$ , $a \in \mathbb{C}$) is normal. In other words if $A$ is diagonalisable then so is $P(A)$ ( note: even zero matrix is considered as a diagonal matrix).

## 2 Quadratic Form

• $Q : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ given by $\sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}x_i x_j$ where $a_{ij} \in \mathbb{F}$ a field.

• It can be represented as $X'AX$ for $X = [x_1, x_2, .., x_n]^T$ and **Symmetric** matrix $A = [A]_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$

• Congruence relation $(ARB)$ : if $A = P^TBP$ for some non-singular $P$, $A, B$ square.

• Matrices congruent to Symmetric matrices are Symmetric.

• Quadratic forms are equivalent if the corresponding matrices are congruent.

• Congruent matrices or equivalent Forms have same Range.

• Every Symmetric matrix is congruent to a diagonal matrix. (same as orthogonally diagonalizable)

• Every $n$-rowed real Symmetric matrix with rank $r$ is congruent to a Diagonal matrix with diagonal $[1, ..1, -1, .. -1, 0, ..0]$ with 1 appearing $p$ times -1 appearing $r - p$ times and 0 $n - r$ times.

• Canonical Form of real Quadratic Form: for $Q$ has matrix $A$ and if $P'AP = \text{diag}[1, ..1, -1, .. -1, 0, ..0]$ then $X = PY$ which transforms $Q$ to $y_1^2 + .. + y_p^2 - y_{p+1}^2 - .. - y_r^2$ for Real non singular matrix $P$.

• Number of positive terms in canonical form is **Index**, difference of positive and negative terms is **Signature**.

• Index and Signature are congruence invariant.

• Two real Quadratic forms (symmetric matrices) are orthogonally equivalent iff their matrices have same eigenvalues and multiplicities.

• A Quadratic Form $Q$ is:
▪ positive definite if $Q(X) \geq 0$ and $Q(X) = 0 \iff X = 0$
▪ negative definite if $Q(X) \leq 0$ and $Q(X) = 0 \iff X = 0$
▪ positive semi-definite if $Q(X) \geq 0$
▪ negative semi-definite if $Q(X) \leq 0$
▪ or is indefinite

• if for a $n$ dimensional Quadratic form Rank=$r$ and Signature=$s$ then it is :
▪ positive definite iff $s = r = n$.
▪ negative definite iff $-s = r = n$.
▪ positive semi-definite iff $s = r < n$.
▪ negative semi-definite iff $-s = r < n$.
▪ indefinite iff $|s| \neq r$

• Now as real Symmetric matrices are diagionizable and have a canonical form we have:
▪ Index = number of positive eigenvalues.
▪ Rank = number of non zero eigenvalues.
▪ Signature = no of +ve - no of -ve eigenvalues.

• from above we have for a real Quadratic form $Q$ with matrix $A$ then $Q$ is:
▪ positive definite iff all eigenvalues are positive or $> 0$.
▪ negative definite iff all eigenvalues are negative or $< 0$.
▪ positive semi-definite iff at-least one eigenvalues is $0$ and others $> 0$.
▪ negative semi-definite iff at-least one eigenvalues is $0$ and others $< 0$.
▪ indefinite iff eigenvalues are -ve as well as +ve.
▪ every real non-singular matrix $A = PS$ for $P$ orthogonal $S$ positive definite ($S = Q'D_1Q, D_1 = \sqrt{\text{diagonalization}(A'A)}$, $P = AS'$)
▪ $Q$ with matrix $A$ is positive definite iff all leading principal minors of $A$ are positive.
▪ A matrix $A$ is positive definite $\implies |A| > 0$
▪ A complex Quadratic form is hermitian if its corresponding matrix is hermitian.
▪ A Hermitian Form assumes only real values.

• if $\text{norm}(A) = \sum_{i,j} |[A]_{ij}|^2$ then $\text{norm}(A) = \text{trace}(A^*A)$.

## 3 Jordan Form

• **Canonical Form** : Given a equivalence relation on set of matrices, the main problem is to find whether $A$ and $B$ belong to same equivalence class. One classical way of doing this is choosing a set of representative matrices such that each matrix belong to only one class and distinct members are of different classes. Such a set of representatives is the Canonical Form of such relation.

• Jordan form is the canonical form for relation of Similarity.

• A matrix in Jordan form Consist of Jordan blocks $J_k(\lambda)$ which is a upper triangular matrix of size k-by-k with diagonal entries $\lambda$ and super diagonal $1$ and others $0$ i.e.

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}_{k\times k}$$

• $J_k(0)^{k+n} = 0$ for $n \geq 0$ i.e. $J_k(0)$ is nilpotent matrix such that $J_k(0)^k = 0$.

• $rank(J_k(0)^l) = max(k-l, 0)$

• Convention: $rank(J_k(0)^0) = k$

• if $r_k(A,\lambda) = rank(A-\lambda I)^k$ and $w_k(A,\lambda) = r_{k-1}(A,\lambda) - r_k(A,\lambda)$ then in Jordan Form of $A$ :

▪ $w_k(A,\lambda)$ = number of blocks with eigenvalue $\lambda$ that has size at least $k$ (use the fact for every Jordan block of $\lambda$, $A-\lambda I$ is Similar to Jordan form consisting of $J_k(0)$ Jordan block instead of $\lambda$ so as we measure ranks each power decreases the rank of the block by one if the block size is greater than the power.)

▪ so $w_1(A,\lambda) = n - r_1(A,\lambda)$ = number of Jordan Blocks with eigenvalue $\lambda$ = Geometric multiplicity of of $\lambda$ as eigenvalue of $A$

▪ $w_k(A,\lambda) - w_{k+1}(A,\lambda)$ = number of blocks of Size $k$

▪ $q$ : index of $\lambda$ in $A$ = smallest integer such that $rank(A-\lambda I)^{q+1} = rank(A-\lambda I)^q = r_{q+1}(A,\lambda) = r_q(A,\lambda)$

▪ $w_1(A,\lambda) + w_2(A,\lambda) .. + w_q(A,\lambda)$ = Sum of dimensions all Jordan blocks in $\lambda$ = Algebraic Multiplicity of $\lambda$ as eigenvalue of $A$

▪ Weyr characteristic of $A \in M_n$ associated with $\lambda \in \mathbb{C}$ is $w(A,\lambda) = (w_1(A,\lambda), w_2(A,\lambda) .., w_q(A,\lambda))$

▪ Segre characteristic of $A \in M_n$ associated with $\lambda \in \mathbb{C}$ is $s(A,\lambda) = s_1(A,\lambda) \geq s_2(A,\lambda), .. \geq s_{w_1}(A,\lambda) > 0$ where $s$ is sizes of Jordan Blocks in $\lambda$ as they occur in Jordan form (non-increasing order)

▪ for a given $A, \lambda$ eigenvalue, If we arrange $w(A,\lambda)$ in dot form as rows (partitions: Ferrers diagram) then its columns are $s(A,\lambda)$ and Vise-versa.

• for $A_n$ upper diagonal with $[A]_{ii} = 1$, $[A]_{i,i+1} \neq 0$ then $A$ is similar to $J_n(1)$

• if $\lambda = 1$ is the only eigenvalue of $A$ then $A$ is similar to $A^k$

• in $J$ Jordan form of $A$:

▪ Total No of Jordan blocks = Total no of independent eigenvectors.

▪ No of Jordan blocks in $\lambda$ = Dimension of eigenspace of $\lambda$

▪ Sum of sizes of Jordan blocks in $\lambda$ = Algebraic Multiplicity.

• If $A_n$ is non singular then $A$ is similar to $A^T$. (use : for Jordon block $J_n = J_n(\lambda)$ and $B_n = B_{n\times n}$ reversal matrix (upside down identity) we have $J_n = B_n J_n' B_n$ as $B_n^{-1} = B_n$ we have $J_n R J_n'$ )

• If minimal polynomial of $A = \prod_{i=1}^{k}(t-\lambda_i)^{r_i}$ then largest Jordan block of $\lambda_i$ in JCF of $A$ is of size $r_i$.

## 4 Rational Form

• Jordan form of $A_n$ is possible iff The characteristics polynomial of $A$ splits completely to linear factors over $\mathbb{F}$ (i.e. $(x-a_i)^{n_i}$, $a_i \in \mathbb{F}$), which may not be possible if there are irreducible polynomials of degree more than $1$ in $\mathbb{F}[x]$, so to make canonical form under consideration of these Matrices we arrive at Rational form which uses the concept of Invariant subspaces, Cyclic subspaces and Primary Decomposition theorem.

• For given monic polynomial (characteristic/minimal) $p(x) = x^n + a_{n-1}x^{n-1} + .. +$

$a_1x + a_0$ $a_i's \in \mathbb{F}$ of linear transform $T : V \to V$ if there exist $x$ such that $T_x = \{x, T(x), T^2(x), .., T^{n-1}(x)\}$ is a linear independent set then The matrix of $T$ with respect to $T-$cyclic basis $T_x$ is Companion matrix which has same characteristic and minimal polynomial = $p(x)$ and is given by

$$C_A = \begin{bmatrix} 0 & .. & .. & 0 & -a_0 \\ 1 & 0 & .. & 0 & -a_1 \\ 0 & 1 & .. & 0 & -a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & .. & 0 & 1 & -a_{n-1} \end{bmatrix}$$

- If $p(x) = (p_1(x))^{n_1}(p_2(x))^{n_2}..(p_k(x))^{n_k}$ and $m(x) = (p_1(x))^{m_1}(p_2(x))^{m_2}..(p_k(x))^{m_k}$ are characteristics and minimal polynomial of linear transform $T : V \to V$ where $p_i's$ are irreducible in $\mathbb{F}$ of degree $d_i$ respectively then :

  - $K_{p_i} = \{x : (p_i(T))^k(x) = 0\}$ is $T$ invariant Subspace of $V$
  - $K_{p_i} = ker((p_i(T))^{m_i})$ (Null space) , $K_{p_i} \cap K_{p_j} = \{0\}$ for $i \neq j$
  - Every $K_{p_i}$ has a union $T-$cyclic basis as a basis.

- From above and Primary decomposition theorem we have: for a linear transformation $T : V \to V$ with matrix $A$ has a basis in which $A$ is similar to

$$\begin{bmatrix} C_1 & & .. & 0 \\ 0 & C_2 & .. & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & .. & C_k \end{bmatrix}$$

where $C_is$ are companion matrices related to minimal polynomial's irreducible terms.

- Dimension of $K_{p_i} = d_in_i$ ($di$ = degree of $p_i$, $n_i$ = power of $p_i$ in characteristic polynomial)

- Dim($K_{p_i}$) = dimension of total blocks associated with $p_i$

- number of blocks associated with $p_i = r_1 = \frac{1}{d_i}[\dim(V) - rank(p_i(A))]$

- number of blocks of size atleast $i - by - i = r_i = \frac{1}{d_i}[rank(p_i(A)^{i-1}) - rank(p_i(A)^i)]$

# 5 Mics Properties

- $A$ has a block $B_n$ in its block form iff it has an $n$ dimensional invariant space associated.

- $\Lambda_n$ is a block matrix in which $[\Lambda]_{i,j} = 0$ if $i \neq j$, $\Lambda_{ii} = \lambda_i I_{n_i}$ blocks and commutes with $B$ iff $B$ is a block Diagonal conformal with $\Lambda$ i.e. iff

$$\Lambda = \begin{bmatrix} \lambda_1 I_{n_1} & & & 0 \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ 0 & & & \lambda_d I_{n_d} \end{bmatrix},$$

$$B = \begin{bmatrix} B_{n_1} & & & 0 \\ & B_{n_2} & & \\ & & \ddots & \\ 0 & & & B_{n_d} \end{bmatrix}$$

- Extremum of $X^TAX$ for constraint $X^TX = 1$ occurs in eigenvalues of $A$.

- From above Extremum of real Quadratic Form $X^TAX$ with constraints $X^TX = 1$ is the largest eigenvalue of $A$ vise-versa Max$\{X^TAX|A$ is symmetric, $X^TX = 1\}$ = largest eigenvalue of $A$.

- $\mu$ is a eigenvalue of $p(A)$ iff $\mu = p(\lambda)$ for an eigenvalue $\lambda$ of $A$ (where $p(.)$ is a polynomial over $\mathbb{F}$).

- if $\lambda$ is an eigenvalue of $A$ then corresponding eigenvector are non-zero columns of $adj(A - \lambda I)$ (use full only if $rank(A - \lambda I) = n - 1$).

- Coefficients of Characteristic polynomial of $A$ of degree n : $n \to 1, n-1 \to -trace(A), constant \to (-1)^n det(A)$.

- $A, B$ are simultaneously Diagonalizable iff $A, B$ communicate i.e. if $D_1 = S^{-1}AS, D_2 = S^{-1}BS$ for same $S \iff AB = BA$. This even holds for a family of Diagonalizable matrices.

- for $A_{m\times n}$
$$\begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}^{-1} = \begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix}$$

- For $A_{m\times n}B_{n\times m}$ Eigenvalues of $AB$ = Eigenvalues of $BA$ (including zero).

- Cauchy's Determinant Identity : $det(A +$

$xy^T) = det(A) + y^T adj(A)x$
(so $|I + xy^*| = 1 + y^*x$)

- if $S = A + iB$ and non-singular then $\exists \tau \in \mathbb{R}$ such that $T = A + \tau B$ is non-singular.
(use that $p(t) = det(A + tB)$ has at most $n$ zeroes in complex plane so there is $\tau \in \mathbb{R}$ such that $p(\tau) \neq 0$)
- Every real Matrix $A$ similar over $\mathbb{C}$ to real matrix $B$ is similar over $\mathbb{R}$. i.e. $0 \neq A, B \in M_n(R)$ if $S \in M_m(C)$ and $B = S^{-1}AS$ then $\exists T \in M_n(R)$ such that $B = T^{-1}AT$
- If $A$ is diagonalizable i.e. $A = S^{-1}DS$ then $p(A) = S^{-1}p(D)S$ which makes evaluation of $p(A)$ easier.
- If $A_n$ has distinct eigenvalues(diagonalizable) and Commutes with $B$ then $B$ is Diagonalizable (more precisely $A_n, B$ are simultaneously diagonalizable) and $B = p(A)$
(use similarity, partition arguments and Lagrange interpolation poly which provides a polynomial map of $n$ distinct reals to any $n$ reals ) for some polynomial $p(t)$ of degree at most $n - 1$
- If $B$ is Diagonalizable then $B$ has a square-root i.e $\exists A | A^2 = B$.
- If $A_n, B_n$ are similar so are $adj(A), adj(B)$.
- All Unitary Matrices Form a group in $GL(n, \mathbb{C})$ and compact in $\mathbb{C}^{n^2}$.
- Singular Value Decomposition: Every matrix $A_{m,n}$ can be written as $A = U_m S V_n$ where $U, V$ are Unitary and $S$ is the diagonal (with zero) entries that are eigenvalue of $A^*A$ or $AA^*$.
- Reversal Matrix $B$ is matrix that is up-side-down of Identity and $BA$ reverses row order of $A$, $AB$ reverses column order of $A$ And $B = B^* = B^{-1}$
- By Jordan Canonical form Every non-singular matrix is similar to its Transpose
- $A$ is similar to $\bar{A}$ iff $A$ is Similar to a real matrix (Same condition for $A \sim A^*$)
- $A$ is hermitian iff $tr(A^2) = tr(A^*A)$
- if $A$ is hermitian then, $\forall x \in \mathbb{C}^n$ :
  - $x^*Ax$ is positive iff all eigenvalues are positive
  - $x^*Ax$ is negative iff all eigenvalues are negative
  - if eigenvalues are$\lambda_1 \leq \lambda_2 \leq .. \lambda_n$ and subspaces $\{S\}$ of $\mathbb{C}^n$ then $\lambda_1 = min(\frac{x^*Ax}{x^*x}), \lambda_n = max(\frac{x^*Ax}{x^*x})$,

$$\lambda_k = \min_{\{dim(S)=k\}} \max_{0 \neq x \in S} \frac{x^*Ax}{x^*x}$$
$$= \max_{\{dim(S)=n-(k+1)\}} \min_{0 \neq x \in S} \frac{x^*Ax}{x^*x}$$

- In general even if $A \in M_n$ is not hermitian with eigenvalues $\lambda_1, \lambda_2 .., \lambda_n$ then $\min_{x \neq 0} \left| \frac{x^*Ax}{x^*x} \right| \leq |\lambda_i| \leq \max_{x \neq 0} \left| \frac{x^*Ax}{x^*x} \right|$
(can be pure inequality also)
- Every Jordan matrix is similar to a complex symmetric matrix so **Every matrix is similar to a complex symmetric matrix**

# 6 Properties based on Matrix Norm

- A function $|||\cdot||| : M_n \to \mathbb{R}$ is a matrix norm if:

1. $|||A||| \geq 0$ Non-negative

1a. $|||A||| = 0 \iff A = 0$ Positive

2. $|||cA||| = |c| \; |||A||| \quad \forall c \in \mathbb{C}$ Homogeneous

3. $|||A + B||| \leq |||A||| + |||B|||$ Triangular Inequality

4. $|||AB||| \leq |||A||| \; |||B|||$ Sub-multiplicativity

- Clearly $|||A^k||| \leq |||A|||^k$ now If $A^2 = A \implies |||A||| \geq 1$ in particular $|||I||| \geq 1$
- Some Matrix norms:
  - $l_1$ norm : $||A||_1 = \sum_{i,j=1}^{n} |a_{ij}|$
  - $l_2$ norm : $||A||_2 = |tr(A^*A)|$

$$= \sqrt{\sigma_1(A)^2 + .. + \sigma_n(A)^2} = \left( \sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

- $l_\infty$ norm : $||A||_\infty = \max_{1\le i,j\le n} |a_{ij}|$
- max Column sum norm

$|||A|||_1 = \max_{1\le j\le n} \sum_{i=1}^{n} |a_{ij}|$

- max Row sum norm

$|||A|||_\infty = \max_{1\le i\le n} \sum_{j=1}^{n} |a_{ij}|$

- Spectral norm $|||A|||_2 = \sigma_1(A)$ = Largest Singular Value of $A$

• **Matrix norm induced by vector norm** : if $||\cdot||$ is norm in $\mathbb{C}^n$ then:

$|||A||| = \max_{||x||=1} ||Ax|| = \max_{x\neq 0} \frac{||Ax||}{||x||}$

$= \max_{||x||\le 1} ||Ax|| = \max_{||x||_\alpha=1} \frac{||Ax||}{||x||}$

(for any other norm $||\cdot||_\alpha$ in $\mathbb{C}^n$) is a Matrix norm with additional properties:

- $|||I||| = 1$
- $||Ay|| \le |||A|||\ ||y||$

• For Any Matrix $A \in M_n(\mathbb{C})$ we have $|\lambda| \le \rho(A) = \max(|\lambda_i|) \le |||A|||$ and if $A$ is non-singular then $\rho(A) \ge |\lambda| \ge 1/|||A|||$

• if there is Matrix norm such that $|||A||| < 1$ then $\lim_{k\to\infty} A^k = 0$

• from above we have $\lim_{k\to\infty} A^k = 0$ iff $\rho(A) < 1$

• For any given Matrix norm $|||\cdot|||$ we have $\rho(A) = \lim_{k\to\infty} |||A^k|||^{1/k}$

• Matrix power series $\sum_{k=0}^{\infty} a_k A^k$ converges if $\rho(A) \le R$ where $R$ is the radius of convergence of complex power series $\sum_{k=0}^{\infty} a_k z^k$ i.e. if $\exists\ |||\cdot|||\ :\ |||A||| < R$

• Matrix $A$ is nonsingular if $\exists |||\cdot|||\ |\ |||I - A||| < 1$ and $A^{-1} = \sum_{k=0}^{\infty} (I - A)^k$

• From above we have if $A_n = [a_{ij}]$ and $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ i.e. absolute value of diagonal elements are greater than sum of absolute values of elements in corresponding rows (or columns) then **$A$ is non singular**

# 7 Properties associated to Quadratic forms

• $A_n$ if Hermitian iff :

- $x^*Ax$ if real for all $x \in \mathbb{C}^n$
- $A$ is normal and all its eigenvalues are real
- $S^*AS$ is Hermitian $\forall S \in M_n$

• from above $A$ is +ve (-ve) semi-definite ($x^*AX \ge 0$ or $\le 0$) $\implies$ $A$ is hermitian

• if $A$ is +ve definite (-ve) then $A^*, A^{-1}, A^T, \bar{A}$ are all +ve definite (-ve).

• every Diagonal entry of +ve (-ve) definite (semi) Matrix are +ve(non -ve, -ve) only.

• A positive semi-definite matrix is positive definite iff it is non-singular

• for $A_n = [a_{ij}]$ a +ve (-ve) semi-definite matrix if $a_{kk} = 0$ then $a_{ik} = a_{ki} = 0\ \forall i \in \{1, 2, .., n\}$ i.e. if diagonal entry is 0 then that row and column are 0.

• $A$ is positive semi definite iff $A = B^*B$ for some $B$

• $A_n$ is positive definite iff $\det(p_k) > 0\ \forall 1 \le k \le n$ where $p_k$ is the $k \times k$ principle matrix partitioned in $A$ (along the diagonal).

# 8 Other Important Theorems

• Gersgorin Theorem: for a matrix $A_n = [a_{ij}]$

- A Gersgorin Disk of $A$ =

$\{z \in \mathbb{C} : |z - a_{ii}| \le R_i'(A) = \sum_{j\neq i} |a_{ij}|\}$ for $i = 1, 2, .., n$

- Eigenvalues of $A$ are all in the union of Gersgorin Discs of $A$ i.e.

$\{\lambda_i\} \in G(A) = \bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - a_{ii}| \le R_i'(A)\}$

- if $G(A)$ forms a disjoint set $G_k(A)$ which is union of $k$ discs then $G_k(A)$ contains exactly $k$ eigenvalues (counted according to algebraic multiplicity).
- The above statements remain true even if radius of the discs are $C_j' = \sum_{i\neq j} |a_{ij}|$ as $A^T$ has same eigenvalues.
- from above we have

$\rho(A) \le \min\left\{\max_i \sum_{j=1}^{n} |a_{ij}|, \max_j \sum_{i=1}^{n} |a_{ij}|\right\}$

■ if $p_1, p_2, ..., p_n$ are positive real numbers then
$\{\lambda_i\} \in \bigcup_{i=1}^{n}\{z \in \mathbb{C} : |z - a_{ii}| \leq \frac{1}{p_i} \sum_{j \neq i} p_j |a_{ij}|\}$
or
$\{\lambda_i\} \in \bigcup_{i=1}^{n}\{z \in \mathbb{C} : |z - a_{jj}| \leq p_j \sum_{i \neq j} \frac{1}{p_i} |a_{ij}|\}$
as similar matrices have same eigenvalues

• $A$ is Diagonally dominant if $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ and strictly diagonally dominant if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$

• if $A$ is strictly diagonally dominant then : $A$ is non-singular, if $a_{ii} > 0$ $\forall i = 1, 2, ..., n$ then every eigenvalue of $A$ has a positive real part, and if $A$ is hermitian and $a_{ii} > 0$ $\forall i = 1, 2, ..., n$ then $A$ is positive definite.

• $A_n$ has nonzero diagonal entries, is diagonally dominant and $|a_{ii}| > R_i'$ for atleast $n - 1$ values of $i$ then $A$ is non singular.

• If every entry of $A$ is non zero, $A$ is diagonally dominant and $|a_{kk}| > R_k'$ for any $k$ then $A$ is non singular

• if $A_n$ has the property that $\forall p, q \in \{1, 2, .., n\}$ $\exists$ sequence of distinct integers $p = k_1, k_2, .., k_m = q$ such that $a_{k_1 k_2}, a_{k_2 k_3}, ..a_{k_{m-1} k_m}$ are non zero, $A$ is diagonally dominant and $|a_{kk}| > R_k'$ for any $k$ then $A$ is non singular

• The above property states that if $A$ is a probability/stochastic matrix then for each node in directed graph of $A$ is strongly connected (for each pair of nodes there is a finite length directed path to them or the stochastic matrix has only one class and all states are communicating)

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