Special examples

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yn37git.github.io/blog/2025/Short-Notes

1 Sequences and Series

1. Riemann-Zeta function $\zeta : \mathbb{C} \to \mathbb{C} + \infty$ defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

   (a) if $1 < Re(s) < \infty$ then the series converges uniformly and absolutely

   (b) clearly ζ is analytic for $Re(s) > 1$

   (c) **Euler's Product formula**:

$$\zeta(z) = \prod \left(1 - \frac{1}{p^s}\right)^{-1}$$

   where the product ranges over all primes p which implies $\zeta(s) \neq 0$ if $Re(s) > 1$. More generally we have

$$\zeta(s)(1-2^{-s})(1-3^{-s})\dots(1-p_N^{-s}) = \sum m^s = 1 + p_{N+1}^{-s}\dots$$

   where the R.H.S ranges for all $+ve$ integers that contain none of prime factors $2, 3, \dots, p_N$

   (d) now if $1 < s < \infty$ (i.e. s is real > 1) then

$$\zeta(s) = s \int_1^{\infty} \frac{[x]}{x^{s+1}} dx$$

   where $[x]$ is greatest integer $\leq x$

   (e) more generally if $Re(s) > 1$ then

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{x^s - 1}{e^x - 1} dx$$

   where $\Gamma(z)$ is defined by product representation for complex numbers.

2. $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

   (a) is convergent to $\ln(2)$

   (b) does not converge absolutely.

2 Functions in $\mathbb{R}$

1. Dirichlet Function $\delta(x)$

$$\delta(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

   (a) δ is not Riemann Integrable in any interval $[a, b]$

   (b) δ is Lebesgue Intergrable in $\mathbb{R}$ and has 0 integral value with usual lebesgue measure as the set for which δ is not zero is countable.

2. $f : \mathbb{R} \to \mathbb{R}$ such that for every rational $r = m/n$ where $n > 0$ and $m, n \in \mathbb{Z}$ with out any common divisors then $f(r) = f(m/n) = 1/n$, $x = 0$ take $n = 1$ i.e. $f(0) = 1$ and $f(x) = 0$ if x is irrational i.e.

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \end{cases}$$

   (a) $f(x)$ is continuous at every irrational number

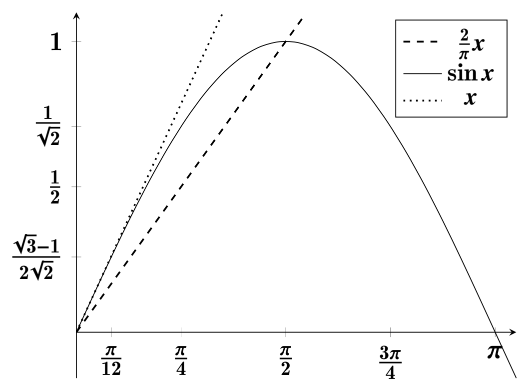
   (b) $f(x)$ is discontinuous at rational with simple discontinuities

3. $\sin(x)$

   (a) can be defined without geometric interpretation as $\sin x = (e^{ix} - e^{-ix})/2$ for real x

   (b) $\sin$ is continuous one-one function in domain $[-\pi, \pi]$ onto $[-1, 1]$ hence inverse $\sin^{-1}$ is defined in this area.

   (c) we see that $\frac{2}{\pi}x \leq \sin x \leq x$ holds $\forall x \in [0, \pi/2]$



4. Distance to a closed set function

   (a) if A is any closed set in $\mathbb{R}$ define $D_A(x) = \inf(d(x, a))$ for $a \in A$ and a metric d on $\mathbb{R}$ (usually the Euclidean metric).

   (b) for $x, y \in \mathbb{R}$ say $\inf(d(a, x))$ for $a \in A$ occurs at $p \in A$ and $\inf(d(a, y))$ for $a \in A$ occurs at $q \in A$ i.e. $|D_A(x) - D_A(y)| = |d(p, x) - d(q, y)|$ (this is possible since A is closed in $\mathbb{R}$) now as $d(q, y) \leq d(p, y)$ we have $|D_A(x) - D_A(y)| \leq |d(p, x) - d(p, y)| \leq |d(x, y)|$ as $d(p, x) \leq d(p, y) + d(x, y)$ so we get if $d(x, y) < \epsilon$ then $|D_A(x) - D_A(y)| < \epsilon$ thus D_A is uniformly continuous.

   (c) Thus there exist a uniformly continuous function of $\mathbb{R}$ that has zeroes exactly equal to a given closed set in $\mathbb{R}$ (namely D_A for a given closed set $A \subset \mathbb{R}$).