Matrix Properties

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Symbols used:

iff	$\rightarrow$	if and only if
Capital letters	$\rightarrow$	Matrices $A_{m\times n} = [a_{ij}]_{m\times n}$
A^T or A'	$\rightarrow$	Transpose of Matrix
$\bar{A}$	$\rightarrow$	Conjugate of Matrix
AB	$\rightarrow$	Matrix product
$\lvert A\rvert$ or $det(A)$	$\rightarrow$	Determinant of Matrix
$tr(A)$ or $trace(A)$	$\rightarrow$	trace of Matrix
A^*	$\rightarrow$	Conjugate transpose of Matrix
A^{-1}	$\rightarrow$	Inverse of Matrix
I	$\rightarrow$	Identity
$Im(A)$	$\rightarrow$	Image or range space of A
$rank(A)$ or $r(A)$	$\rightarrow$	Dimension of Range space of A
$ker(A)$	$\rightarrow$	Null space of A
$null(A)$	$\rightarrow$	Dimension of Null space of A
$\mathbb{F}$	$\rightarrow$	Field

1 Basic properties

- $A(BC) = (AB)C$
- $tr(AB) = tr(BA)$
- $(AB)' = B'A'$
- $(AB)^* = B^*A^*$
- if A is Hermitian then iA is skew-Hermitian and vise-versa.
- if A, B are symmetric, AB is symmetric iff $AB = BA$.
- $AA', A'A$ are always symmetric.
- For any Square Matrix A:
  - $A + A'$ is symmetric.
  - $A - A'$ is skew- symmetric.
  - $A + A^*$ is Hermitian.
  - $A - A^*$ is skew-Hermitian .
- By preceding point any Square matrix can be decomposed (by +) into symmetric - skew-symmetric or Hermitian- skew-Hermitian pair.
- $B'AB$ is symmetric or skew as is A
- B^*AB is hermitian or skew as is A
- Determinant is a Multiliear (row), Alternating and Normalized Function on Matrices.
- Determinant of upper or lower triangle or diagonal matrix is equal to product of diagonal elements.
- $|AB| = |A||B| = |BA|$
- $|A'| = |A|$
- $|A^*| = |\bar{A}|$
- A is invertible iff $|A| \neq 0$.
- $A^{-1} = \frac{adj(A)}{|A|}$ where $adj(A)$ is the transpose of co-factor matrix.
- $B^{-1} - A^{-1} = B^{-1}(A - B)A^{-1}$
- Cramer's rule for a system of linear equations $Ax = b$ where A is square and for $x = [x_1, x_2, .., x_n]^T$ we have $x_i = \frac{|A \leftarrow_i b|}{|A|}$ where $A \leftarrow_i b$ is obtained by replacing i^{th} column of A by b.
- $|adj(A)| = |A|^{n-1}$ where A is an $n \times n$ matrix
- $adj(A^*) = Adj(A)^*$
- $adj(A^{-1}) = adj(A)^{-1} = A/|A|$
- $adj(adj(A)) = |A|^{n-2}A$
- $adj(AB) = adj(B)adj(A)$ for non-singular matrices A, B.
- A is orthogonal if $A'A = I$
- A is orthogonal $\implies |A| = \pm 1 \implies$ invertible.

- A is unitary if $A^*A = I$
- if A, B are orthogonal then so are AB, BA. Similar result follows in unitary case also.
- $rank(A) = r$ iff all the $r+1$ order minors are zero i.e. if any one of r^{th} order minor is non zero then $rank(A) \geq r$.
- $rank(A) = rank(A') = rank(A^*)$
- Elementary transformation: exchange of rows, multiplication of row by non zero constant, addition of k multiple of a row to another row.
- Elementary transformations doesn't change the rank of a matrix.
- Every elementary transformation has a corresponding non singular matrix which when pre-multiplied to a given matrix gives the respective operation.
- Normal form of a matrix : (Echelon form) A matrix which can be partitioned into identity and null matrices where the identity is present in upper-left part.
- $\exists P, Q$ non-singular square matrices such that $N = PAQ$ where A is any matrix and N is its normal or Echelon form.
- $rank(AB) \leq min(\{rank(A), rank(B)\})$.
- $rank(A+B) \leq rank(A) + rank(B)$.
- **Sylvester inequality** :
for any matrices $A_{m\times k}, B_{k\times n}$

$$\begin{aligned} rank(AB) &= rank(B) - dim(Im(B) \cap ker(A)) \\ \text{so } rank(A) + Rank(B) - k &\leq rank(AB) \\ &\leq min(\{rank(A), Rank(B)\}). \end{aligned}$$

(use: for $Bx \neq 0$, $ABx = A(Bx) = 0$ iff $x \in Im(B) \cap ker(A)$ and that $dim(Im(B) \cap ker(A)) \leq null(A) = k - r(A)$ so $-dim(Im(B) \cap ker(A)) \geq r(A) - k$.)
- **Frobenius Inequality** :
for $A_{m\times k}, B_{k\times p}, C_{p\times n}$

$$rank(AB) + rank(BC) \leq rank(B) + rank(ABC).$$

- $rank(A) = rank(A^*A)$
- if all entries of A are real then $rank(A'A) = rank(A)$.
- if A is n-squared then :
  - $rank(A) = n \implies rank(adj(A)) = n$.
  - $rank(A) = n-1 \implies rank(adj(A)) = 1$.
  - $rank(A) < n-1 \implies rank(adj(A)) = 0$

i.e. $adj(A) \equiv 0$. (use minors and cofactor definition of $Adj(A)$.)
- $rank(A) \geq rank(A^2) \geq .. \geq rank(A^n) \geq ..$
- $null(A) \leq null(A^2) \leq .. \leq null(A^n) \leq ..$
- if $rank(A^m) = rank(A^{m+1})$ then
  - $rank(A^k) = rank(A^m) \quad \forall k \geq m$
  - $null(A^k) = null(A^m) \quad \forall k \geq m$
- Eigenvalues of Hermitian matrices are real.
(if λ is eigenvalue then $(Ax)^* = x^*A^* = x^*A = (\lambda x)^* = \bar{\lambda}x^*$ so $x^*A^*x = \lambda x^*x = \bar{\lambda}x^*x \implies \bar{\lambda} = \lambda$)
- Eigenvalues of Skew-Hermitian are purely imaginary or zero.
- If λ is Eigenvalue of Unitary matrix A then $|\lambda| = 1$
(if $Ux = \lambda x$ then $x^*U^*Ux = x^*Ix = x^*x$ but $(x^*U^*)(Ux) = \bar{\lambda}\lambda x^*x$.)
- Real Eigenvalues of Orthogonal Matrices are $1, -1$ only.
- Eigenvalues of A and A' are same.
- Eigenvalues of triangular, diagonal matrices are its diagonal elements.
- if λ is an eigenvalue of non-singular matrix A then
  - $\lambda \neq 0$
  - $\frac{1}{\lambda}$ is the eigenvalue of A^{-1}.
  - λ^k is the eigenvalue of A^k.
  - $\frac{|A|}{\lambda}$ is the eigenvalue of $adj(A)$.
- if $\{\lambda_i\}$ are eigenvalues of A then eigenvalues of $B = p(A)$ are of form $p(\lambda_i)$ only.
- For A_n with eigenvalues $\lambda_1, \lambda_2, .., \lambda_n$ $trace(A) = \sum_{i=1}^{n} \lambda_i$, $det(A) = \prod_{i=1}^{n} \lambda_i$ and $trace(adj(A)) = \sum_{i=1}^{n} \prod_{j \neq i}^{n} \lambda_i$.
- If $A = P^{-1}BP$ then A and B have same eigenvalues
- for square Matrices A, B eigenvalues of AB and BA are same.
(use if $ABx = \lambda x$ then $BA(Bx) = B(ABx) = \lambda Bx$ so λ is eigenvalue of BA also and vis-a-viz.)
- Geometric multiplicity (no of eigenvectors for an eigenvalue) $\leq$ Algebraic multiplic-

ity(order of eigenvalue in characteristic polynomial).

- $A = P^{-1}BP$ this Relation ARB (similarity) is equivalence, determinant invariant, eigenvalue invariant , trace invariant.
- A matrix is diagonalizable if it is similar to a diagonal matrix
- A matric is diagonalizable iff for each of its eigenvalue Geometric multiplicity = Algebraic multiplicity.
- square matix A is diagonalizable iff minimal polynomial of A splits into distinct linear factors in the given field i.e. minimal polynomial of A is separable and has only linear irreducible factors.
- A non-zero Nil-potent ($A^m = 0$) matrix has eigenvalues as zero only.
- A non-zero Nil-potent matrix is never Diagonalizable.

(if A is diagonalizable then $P^{-1}AP = D$ so $(P^{-1}AP)^m = P^{-1}A^mP = 0 = D^m \implies D \equiv 0$ thus $A \equiv 0$)

- **Schurs theorems:**

▪ Every Square matrix A is Unitarily similar to Upper triangular matrix whose diagonals are eigenvalues of A (complex values included).

▪ If $A \in M_n(\mathbb{R})$ and has only real eigenvalues then it is real orthogonally similar to real upper triangular matrix.

(say $\lambda_1, \lambda_2, .., \lambda_n$ are eigenvalues of $A_{n\times n}$ (with repeats) let x be normalised eigenvector of A to eigenvalue λ_1 then $x^*x = 1$ and $Ax = \lambda_1 x$, now from an orthonormal basis with x and let this matrix be $U_1 = [x\ u_2..u_n]$ thus we have $U_1^*AU_1 = [\lambda_1, \star;\ 0, A_1]$ for $A_{1_{n-1\times n-1}}$ and as U_1 is unitary we have eigenvalues of A_1 are $\lambda_2, .., \lambda_n$ only so lets commence the same procedure for $A_{1_{n-1\times n-1}}$ we get U_2 join this to form $V_2 = [1, 0;\ 0, U_2]$ then we get $(U_1V_2)^*AU_1V_2 = [\lambda_1, \star, \star;\ 0, \lambda_2, \star;\ 0, 0, A_2]$ clearly U_1V_2 was unitary so proceeding similarly we get the theorem)

▪ If $A \in M_n(\mathbb{R})$ has complex eigenvalues then it is similar to a matrix with diagonal blocks of 1-by-1 and 2-by-2 only (has upper triangular entries). Where 1-by-1 blocks are real eigenvalue of A and 2-by-2 blocks are $[a \quad b; -b \quad a]$ for $a + ib$ eigenvalue.

(for $A_{n\times n}$ let $\lambda = a + ib$ and its eigenvector is $x = u + iv$ then prove $\bar{\lambda}, \bar{x}$ are eigenpairs so $x, \bar{x}$ are linearly independent so are u, v and as $Au = au - bv, Av = bu + av$ and if $S = [u, v, S_1]_{n\times n}$ be made non singular thus $S^{-1}AS = [B, \star;\ 0A_1]$ for $B = [a \quad b; -b \quad a]$.)

- **Every Symmetric matrix ($A \in M_n(\mathbb{R})$) is orthogonally similar to diagonal matrix (D) i.e. $D = P^TAP,\ P^TP = I$.**
- **Every Hermitian matrix (A) is unitarily similar to diagonal matrix (D) i.e. $D = P^*AP,\ P^*P = I$.**
- A matrix A is normal iff $A^*A = AA^*$
- **A matrix is Unitarily similar to diagonal matrix iff it is Normal.**
- A triangular normal matrix is Diagonal also a block diagonal normal matrix has off diagonal blocks $=0$.
- if A is normal then $p(A)$ (specially $A + aI$, $a \in \mathbb{C}$) is normal. In other words if A is diagonalisable then so is $P(A)$ (note: even zero matrix is considered as a diagonal matrix).

2 Quadratic Form

- $Q : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}$ given by $\displaystyle\sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}x_ix_j$ where $a_{ij} \in \mathbb{F}$ a field.
- It can be represented as $X'AX$ for $X = [x_1, x_2, .., x_n]^T$ and **Symmetric** matrix $A = [A]_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$
- Congruence relation (ARB) : if $A = P^TBP$ for some non-singular P, A, B square.
- Matrices congruent to Symmetric matrices are Symmetric.
- Quadratic forms are equivalent if the corresponding matrices are congruent.
- Congruent matrices or equivalent Forms have same Range.
- Every Symmetric matrix is congruent to a diagonal matrix. (same as orthogonally diagonalizable)
- Every n-rowed real Symmetric matrix with rank r is congruent to a Diagonal matrix with

diagonal $[1,..1,-1,..-1,0,..0]$ with 1 appearing p times -1 appearing $r-p$ times and 0 $n-r$ times.

- Canonical Form of real Quadratic Form: for Q has matrix A and if $P'AP$ = diag$[1,..1,-1,..-1,0,..0]$ then $X = PY$ which transforms Q to $y_1^2 + .. + y_p^2 - y_{p+1}^2 - .. - y_r^2$ for Real non singular matrix P.

- Number of positive terms in canonical form is **Index**, difference of positive and negative terms is **Signature**.

- Index and Signature are congruence invariant.

- Two real Quadratic forms (symmetric matrices) are orthogonally equivalent iff their matrices have same eigenvalues and multiplicities.

- A Quadratic Form Q is:
  - positive definite if $Q(X) \geq 0$ and $Q(X) = 0 \iff X = 0$
  - negative definite if $Q(X) \leq 0$ and $Q(X) = 0 \iff X = 0$
  - positive semi-definite if $Q(X) \geq 0$
  - negative semi-definite if $Q(X) \leq 0$
  - or is indefinite
- if for a n dimensional Quadratic form Rank=r and Signature=s then it is :
  - positive definite iff $s = r = n$.
  - negative definite iff $-s = r = n$.
  - positive semi-definite iff $s = r < n$.
  - negative semi-definite iff $-s = r < n$.
  - indefinite iff $|s| \neq r$
- Now as real Symmetric matrices are diagionizable and have a canonical form we have:
  - Index = number of positive eigenvalues.
  - Rank = number of non zero eigenvalues.
  - Signature = no of +ve - no of -ve eigenvalues.
- from above we have for a real Quadratic form Q with matrix A then Q is:
  - positive definite iff all eigenvalues are positive or > 0.
  - negative definite iff all eigenvalues are negative or < 0.
  - positive semi-definite iff at-least one eigenvalues is 0 and others > 0.
  - negative semi-definite iff at-least one eigenvalues is 0 and others < 0.
  - indefinite iff eigenvalues are -ve as well as +ve.
  - every real non-singular matrix $A = PS$ for P orthogonal S positive definite $(S = Q'D_1Q, D_1 = \sqrt{diagonalization(A'A)}, P = AS')$
  - Q with matrix A is positive definite iff all leading principal minors of A are positive.
  - A matrix A is positive definite $\implies |A| > 0$
  - A complex Quadratic form is hermitian if its corresponding matrix is hermitian.
  - A Hermitian Form assumes only real values.
- if $norm(A) = \sum_{i,j} |[A]_{ij}|^2$ then $norm(A) = trace(A^*A)$.

3 Jordan Form

- **Canonical Form** : Given a equivalence relation on set of matrices, the main problem is to find whether A and B belong to same equivalence class. One classical way of doing this is choosing a set of representative matrices such that each matrix belong to only one class and distinct members are of different classes. Such a set of representatives is the Canonical Form of such relation.

- Jordan form is the canonical form for relation of Similarity.

- A matrix in Jordan form Consist of Jordan blocks $J_k(\lambda)$ which is a upper triangular matrix of size k-by-k with diagonal entries λ and super diagonal 1 and others 0 i.e.

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix}_{k \times k}$$

- $J_k(0)^{k+n} = 0$ for $n \geq 0$ i.e. $J_k(0)$ is nilpotent matrix such that $J_k(0)^k = 0$.

- $\mathrm{rank}(J_k(0)^l) = \max(k-l, 0)$
- Convention: $\mathrm{rank}(J_k(0)^0) = k$
- if $r_k(A,\lambda) = \mathrm{rank}(A-\lambda I)^k$ and $w_k(A,\lambda) = r_{k-1}(A,\lambda) - r_k(A,\lambda)$ then in Jordan Form of A :
  - $w_k(A,\lambda)$ = number of blocks with eigenvalue λ that has size at least k (use the fact for every Jordan block of λ, $A-\lambda I$ is Similar to Jordan form consisting of $J_k(0)$ Jordan block instead of λ so as we measure ranks each power decreases the rank of the block by one if the block size is greater than the power.)
  - so $w_1(A,\lambda) = n - r_1(A,\lambda)$ = number of Jordan Blocks with eigenvalue λ = Geometric multiplicity of of λ as eigenvalue of A
  - $w_k(A,\lambda) - w_{k+1}(A,\lambda)$ = number of blocks of Size k
  - q : index of λ in A = smallest integer such that $\mathrm{rank}(A-\lambda I)^{q+1} = \mathrm{rank}(A-\lambda I)^q = r_{q+1}(A,\lambda) = r_q(A,\lambda)$
  - $w_1(A,\lambda) + w_2(A,\lambda).. + w_q(A,\lambda)$ = Sum of dimensions all Jordan blocks in λ = Algebraic Multiplicity of λ as eigenvalue of A
  - Weyr characteristic of $A \in M_n$ associated with $\lambda \in \mathbb{C}$ is
    $w(A,\lambda) = (w_1(A,\lambda), w_2(A,\lambda).., w_q(A,\lambda))$
  - Segre characteristic of $A \in M_n$ associated with $\lambda \in \mathbb{C}$ is
    $s(A,\lambda) = s_1(A,\lambda) \geq s_2(A,\lambda), .. \geq s_{w_1}(A,\lambda) > 0$ where s is sizes of Jordan Blocks in λ as they occur in Jordan form (non-increasing order)
  - for a given A,λ eigenvalue, If we arrange $w(A,\lambda)$ in dot form as rows (partitions: Ferrers diagram) then its columns are $s(A,\lambda)$ and Vise-versa.
- for A_n upper diagonal with $[A]_{ii} = 1$, $[A]_{i,i+1} \neq 0$ then A is similar to $J_n(1)$
- if $\lambda = 1$ is the only eigenvalue of A then A is similar to A^k
- in J Jordan form of A:
  - Total No of Jordan blocks = Total no of independent eigenvectors.
  - No of Jordan blocks in λ = Dimension of eigenspace of λ
  - Sum of sizes of Jordan blocks in λ = Algebraic Multiplicity.
- If A_n is non singular then A is similar to A^T. (use : for Jordon block $J_n = J_n(\lambda)$ and $B_n = B_{n\times n}$ reversal matrix (upside down identity) we have $J_n = B_n J'_n B_n$ as $B_n^{-1} = B_n$ we have $J_n R J'_n$)
- If minimal polynomial of $A = \prod_{i=1}^{k}(t-\lambda_i)^{r_i}$ then largest Jordan block of λ_i in JCF of A is of size r_i.

4 Rational Form

- Jordan form of A_n is possible iff The characteristics polynomial of A splits completely to linear factors over $\mathbb{F}$ (i.e. $(x-a_i)^{n_i}$, $a_i \in \mathbb{F}$), which may not be possible if there are irreducible polynomials of degree more than 1 in $\mathbb{F}[x]$, so to make canonical form under consideration of these Matrices we arrive at Rational form which uses the concept of Invariant subspaces, Cyclic subspaces and Primary Decomposition theorem.
- For given monic polynomial (characteristic/minimal) $p(x) = x^n + a_{n-1}x^{n-1} + .. + a_1 x + a_0$ $a_i's \in \mathbb{F}$ of linear transform $T : V \to V$ if there exist x such that $T_x = \{x, T(x), T^2(x), .., T^{n-1}(x)\}$ is a linear independent set then The matrix of T with respect to $T-$cyclic basis T_x is Companion matrix which has same characteristic and minimal polynomial = $p(x)$ and is given by

$$C_A = \begin{bmatrix} 0 & .. & .. & 0 & -a_0 \\ 1 & 0 & .. & 0 & -a_1 \\ 0 & 1 & .. & 0 & -a_2 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & .. & 0 & 1 & -a_{n-1} \end{bmatrix}$$

- If $p(x) = (p_1(x))^{n_1}(p_2(x))^{n_2}..(p_k(x))^{n_k}$ and $m(x) = (p_1(x))^{m_1}(p_2(x))^{m_2}..(p_k(x))^{m_k}$ are characteristics and minimal polynomial of linear transform $T : V \to V$ where $p_i's$ are irreducible in $\mathbb{F}$ of degree d_i respectively then :
  - $K_{p_i} = \{x : (p_i(T))^k(x) = 0\}$ is T invariant Subspace of V
  - $K_{p_i} = \ker((p_i(T))^{m_i})$ (Null space) , $K_{p_i} \cap K_{p_j} = \{0\}$ for $i \neq j$

■ Every K_{p_i} has a union T—cyclic basis as a basis.

• From above and Primary decomposition theorem we have: for a linear transformation $T : V \to V$ with matrix A has a basis in which A is similar to

$$\begin{bmatrix} C_1 & & .. & 0 \\ 0 & C_2 & .. & 0 \\ \vdots & & \ddots & \vdots \\ 0 & & .. & C_k \end{bmatrix}$$

where $C_i s$ are companion matrices related to minimal polynomial's irreducible terms.

• Dimension of $K_{p_i} = d_i n_i$ (di = degree of p_i, n_i = power of p_i in characteristic polynomial)

• Dim(K_{p_i}) = dimension of total blocks associated with p_i

• number of blocks associated with p_i =r_1 = $\frac{1}{d_i}[\dim(V) - \mathrm{rank}(p_i(A))]$

• number of blocks of size atleast $i - by - i$ = $r_i = \frac{1}{d_i}[\mathrm{rank}(p_i(A)^{i-1}) - \mathrm{rank}(p_i(A)^i)]$

5 Mics Properties

• A has a block B_n in its block form iff it has an n dimensional invariant space associated.

• Λ_n is a block matrix in which $[\Lambda]_{i,j} = 0$ if $i \neq j$, $\Lambda_{ii} = \lambda_i I_{n_i}$ blocks and commutes with B iff B is a block Diagonal conformal with Λ i.e. iff

$$\Lambda = \begin{bmatrix} \lambda_1 I_{n_1} & & & 0 \\ & \lambda_2 I_{n_2} & & \\ & & \ddots & \\ 0 & & & \lambda_d I_{n_d} \end{bmatrix},$$

$$B = \begin{bmatrix} B_{n_1} & & & 0 \\ & B_{n_2} & & \\ & & \ddots & \\ 0 & & & B_{n_d} \end{bmatrix}$$

• Extremum of $X^T AX$ for constraint $X^T X = 1$ occurs in eigenvalues of A.

• From above Extremum of real Quadratic Form $X^T AX$ with constraints $X^T X = 1$ is the largest eigenvalue of A vise-versa Max{$X^T AX$|A is symmetric, $X^T X = 1$} = largest eigenvalue of A.

• μ is a eigenvalue of $p(A)$ iff $\mu = p(\lambda)$ for an eigenvalue λ of A (where $p(.)$ is a polynomial over $\mathbb{F}$).

• if λ is an eigenvalue of A then corresponding eigenvector are non-zero columns of $\mathrm{adj}(A - \lambda I)$ (use full only if $\mathrm{rank}(A - \lambda I) = n - 1)$.

• Coefficients of Characteristic polynomial of A of degree n : $n \to 1, n-1 \to -\mathrm{trace}(A), \mathrm{constant} \to (-1)^n \det(A)$.

• A, B are simultaneously Diagonalizable iff A, B communicate i.e. if $D_1 = S^{-1}AS, D_2 = S^{-1}BS$ for same $S \iff AB = BA$. This even holds for a family of Diagonalizable matrices.

• for $A_{m \times n}$

$$\begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix}^{-1} = \begin{bmatrix} I_m & -A \\ 0 & I_n \end{bmatrix}$$

• For $A_{m\times n}B_{n\times m}$ Eigenvalues of AB = Eigenvalues of BA (including zero).

• Cauchy's Determinant Identity : $\det(A + xy^T) = \det(A) + y^T \mathrm{adj}(A)x$
(so $|I + xy^*| = 1 + y^* x$)

• if $S = A + iB$ and non-singular then $\exists \tau \in \mathbb{R}$ such that $T = A + \tau B$ is non-singular.
(use that $p(t) = \det(A + tB)$ has at most n zeroes in complex plane so there is $\tau \in \mathbb{R}$ such that $p(\tau) \neq 0$)

• Every real Matrix A similar over $\mathbb{C}$ to real matrix B is similar over $\mathbb{R}$. i.e. $0 \neq A, B \in M_n(R)$ if $S \in M_m(C)$ and $B = S^{-1}AS$ then $\exists T \in M_n(R)$ such that $B = T^{-1}AT$

• If A is diagonalizable i.e. $A = S^{-1}DS$ then $p(A) = S^{-1}p(D)S$ which makes evaluation of $p(A)$ easier.

• If A_n has distinct eigenvalues(diagonalizable) and Commutes with B then B is Diagonalizable (more precisely A_n, B are simultaneously diagonalizable) and $B = p(A)$
(use similarity, partition arguments and Lagrange interpolation poly which provides a polynomial map of n distinct reals to any n reals) for some polynomial $p(t)$ of degree at most $n - 1$

• If B is Diagonalizable then B has a squareroot i.e $\exists A | A^2 = B$.

- If A_n, B_n are similar so are $adj(A), adj(B)$.

- All Unitary Matrices Form a group in $GL(n, \mathbb{C})$ and compact in $\mathbb{C}^{n^2}$.

- Singular Value Decomposition: Every matrix $A_{m,n}$ can be written as $A = U_m S V_n$ where U, V are Unitary and S is the diagonal (with zero) entries that are eigenvalue of A^*A or AA^*.

- Reversal Matrix B is matrix that is up-side-down of Identity and BA reverses row order of A, AB reverses column order of A And $B = B^* = B^{-1}$

- By Jordan Canonical form Every non-singular matrix is similar to its Transpose

- A is similar to $\bar{A}$ iff A is Similar to a real matrix (Same condition for $A \sim A^*$)

- A is hermitian iff $tr(A^2) = tr(A^*A)$

- if A is hermitian then, $\forall x \in \mathbb{C}^n$:

▪ x^*Ax is positive iff all eigenvalues are positive

▪ x^*Ax is negative iff all eigenvalues are negative

▪ if eigenvalues are $\lambda_1 \leq \lambda_2 \leq ..\lambda_n$ and subspaces $\{S\}$ of $\mathbb{C}^n$ then $\lambda_1 = min(\frac{x^*Ax}{x^*x}), \lambda_n = max(\frac{x^*Ax}{x^*x})$,

$$\lambda_k = \min_{\{dim(S)=k\}} \max_{0 \neq x \in S} \frac{x^*Ax}{x^*x}$$
$$= \max_{\{dim(S)=n-(k+1)\}} \min_{0 \neq x \in S} \frac{x^*Ax}{x^*x}$$

- In general even if $A \in M_n$ is not hermitian with eigenvalues $\lambda_1, \lambda_2.., \lambda_n$ then
$$\min_{x \neq 0} \left| \frac{x^*Ax}{x^*x} \right| \leq |\lambda_i| \leq \max_{x \neq 0} \left| \frac{x^*Ax}{x^*x} \right|$$
(can be pure inequality also)

- Every Jordan matrix is similar to a complex symmetric matrix so **Every matrix is similar to a complex symmetric matrix**

6 Properties based on Matrix Norm

- A function $|||\cdot||| : M_n \to \mathbb{R}$ is a matrix norm if:

1. $|||A||| \geq 0$ Non-negative

1a. $|||A||| = 0 \iff A = 0$ Positive

2. $|||cA||| = |c|\ |||A||| \quad \forall c \in \mathbb{C}$ Homogeneous

3. $|||A + B||| \leq |||A||| + |||B|||$ Triangular Inequality

4. $|||AB||| \leq |||A|||\ |||B|||$ Sub-multiplicativity

- Clearly $|||A^k||| \leq |||A|||^k$ now If $A^2 = A \implies |||A||| \geq 1$ in particular $|||I||| \geq 1$
- Some Matrix norms:

▪ l_1 norm : $||A||_1 = \sum_{i,j=1}^{n} |a_{ij}|$

▪ l_2 norm : $||A||_2 = |tr(A^*A)|$
$$= \sqrt{\sigma_1(A)^2 + .. + \sigma_n(A)^2} = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2}$$

▪ l_∞ norm : $||A||_\infty = \max_{1 \leq i,j \leq n} |a_{ij}|$

▪ max Column sum norm
$$|||A|||_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{n} |a_{ij}|$$

▪ max Row sum norm
$$|||A|||_\infty = \max_{1 \leq i \leq n} \sum_{j=1}^{n} |a_{ij}|$$

▪ Spectral norm $|||A|||_2 = \sigma_1(A)$ = Largest Singular Value of A

- **Matrix norm induced by vector norm** : if $||\cdot||$ is norm in $\mathbb{C}^n$ then:
$$|||A||| = \max_{||x||=1} ||Ax|| = \max_{x \neq 0} \frac{||Ax||}{||x||}$$
$$= \max_{||x|| \leq 1} ||Ax|| = \max_{||x||_\alpha = 1} \frac{||Ax||}{||x||}$$
(for any other norm $||\cdot||_\alpha$ in $\mathbb{C}^n$) is a Matrix norm with additional properties:

- $|||I||| = 1$
- $||Ay|| \leq |||A|||\, ||y||$

• For Any Matrix $A \in M_n(\mathbb{C})$ we have $|\lambda| \leq \rho(A) = \max(|\lambda_i|) \leq |||A|||$ and if A is non-singular then $\rho(A) \geq |\lambda| \geq 1/|||A|||$

• if there is Matrix norm such that $|||A||| < 1$ then $\lim\limits_{k\to\infty} A^k = 0$

• from above we have $\lim\limits_{k\to\infty} A^k = 0$ iff $\rho(A) < 1$

• For any given Matrix norm $|||\cdot|||$ we have $\rho(A) = \lim\limits_{k\to\infty} |||A^k|||^{1/k}$

• Matrix power series $\sum\limits_{k=0}^{\infty} a_k A^k$ converges if $\rho(A) \leq R$ where R is the radius of convergence of complex power series $\sum\limits_{k=0}^{\infty} a_k z^k$ i.e. if $\exists\, |||\cdot||| \,:\, |||A||| < R$

• Matrix A is nonsingular if $\exists |||\cdot|||\ |\ |||I - A||| < 1$ and $A^{-1} = \sum\limits_{k=0}^{\infty} (I - A)^k$

• From above we have if $A_n = [a_{ij}]$ and $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ i.e. absolute value of diagonal elements are greater than sum of absolute values of elements in corresponding rows (or columns) then A **is non singular**

7 Properties associated to Quadratic forms

• A_n if Hermitian iff :
- x^*Ax if real for all $x \in \mathbb{C}^n$
- A is normal and all its eigenvalues are real
- S^*AS is Hermitian $\forall S \in M_n$

• from above A is +ve (-ve) semi-definite ($x^*AX \geq 0$ or ≤ 0) $\implies$ A is hermitian

• if A is +ve definite (-ve) then $A^*, A^{-1}, A^T, \bar{A}$ are all +ve definite (-ve).

• every Diagonal entry of +ve (-ve) definite (semi) Matrix are +ve(non -ve, -ve) only.

• A positive semi-definite matrix is positive definite iff it is non-singular

• for $A_n = [a_{ij}]$ a +ve (-ve) semi-definite matrix if $a_{kk} = 0$ then $a_{ik} = a_{ki} = 0\ \forall i \in \{1, 2, .., n\}$ i.e. if diagonal entry is 0 then that row and column are 0.

• A is positive semi definite iff $A = B^*B$ for some B

• A_n is positive definite iff $\det(p_k) > 0\ \forall 1 \leq k \leq n$ where p_k is the $k \times k$ principle matrix partitioned in A (along the diagonal).

8 Other Important Theorems

• Gersgorin Theorem: for a matrix $A_n = [a_{ij}]$
- A Gersgorin Disk of $A =$ $\{z \in \mathbb{C} : |z - a_{ii}| \leq R'_i(A) = \sum_{j\neq i} |a_{ij}|\}$ for $i = 1, 2, .., n$
- Eigenvalues of A are all in the union of Gersgorin Discs of A i.e. $\{\lambda_i\} \in G(A) = \bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - a_{ii}| \leq R'_i(A)\}$
- if $G(A)$ forms a disjoint set $G_k(A)$ which is union of k discs then $G_k(A)$ contains exactly k eigenvalues (counted according to algebraic multiplicity).
- The above statements remain true even if radius of the discs are $C'_j = \sum_{i\neq j} |a_{ij}|$ as A^T has same eigenvalues.
- from above we have $\rho(A) \leq \min\left\{\max_i \sum_{j=1}^{n} |a_{ij}|, \max_j \sum_{i=1}^{n} |a_{ij}|\right\}$
- if $p_1, p_2, ..., p_n$ are positive real numbers then $\{\lambda_i\} \in \bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - a_{ii}| \leq \frac{1}{p_i} \sum_{j\neq i} p_j |a_{ij}|\}$ or $\{\lambda_i\} \in \bigcup_{i=1}^{n} \{z \in \mathbb{C} : |z - a_{jj}| \leq p_j \sum_{i\neq j} \frac{1}{p_i} |a_{ij}|\}$ as similar matrices have same eigenvalues

• A is Diagonally dominant if $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$ and strictly diagonally dominant if $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$

• if A is strictly diagonally dominant then : A is non-singular, if $a_{ii} > 0\ \forall i = 1, 2, ..., n$ then every eigenvalue of A has a positive real part, and if A is hermitian and $a_{ii} > 0\ \forall i = 1, 2, ..., n$ then A is positive definite.

• A_n has nonzero diagonal entries, is diagonally dominant and $|a_{ii}| > R'_i$ for atleast $n - 1$

values of i then A is non singular.

• If every entry of A is non zero, A is diagonally dominant and $|a_{kk}| > R'_k$ for any k then A is non singular

• if A_n has the property that $\forall p, q \in \{1, 2, .., n\}$ $\exists$ sequence of distinct integers $p = k_1, k_2, .., k_m = q$ such that $a_{k_1k_2}, a_{k_2k_3}, ..a_{k_{m-1}k_m}$ are non zero, A is diagonally dominant and $|a_{kk}| > R'_k$ for any k then A is non singular

• The above property states that if A is a probability/stochastic matrix then for each node in directed graph of A is strongly connected (for each pair of nodes there is a finite length directed path to them or the stochastic matrix has only one class and all states are communicating)

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