

# Linear Algebra

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## 0 Symbols and notations used

$A_{m \times n} \rightarrow m \times n$  matrix.  
 $A_n \rightarrow n \times n$  matrix.  
 $\sim \rightarrow$  the relation below  
 $A \sim B \implies A = P^{-1}AP$ .  
iff  $\rightarrow \iff$

1

## Basic Linear equations theory

Every  $A_{m \times n} = PR_{m \times n}$  for Row reduced Echelon form  $R$  and an invertible matrix  $P$  let this relation be denoted by  $A \text{ rrec } R$

if  $m < n$  then the homogeneous system  $A_{m \times n}X = 0$  has a non trivial solution  
i.e. if the number of equations is less than the number of variables then the Homogeneous System has a non trivial solution

### Inverse Properties

- $A_n$  has inverse  $A^{-1}$  iff  $AX = 0$  has only trivial solutions.
- $A$  is invertible iff  $A \text{ rrec } I$  (identity)
- if Elementary matrices are the corresponding matrices of elementary transforms (change of rows, addition of one row to another, multiplication of a row with an non zero constant) then  $A$  is invertible iff  $A$  is product of elementary matrices.

### Echelon Form

every  $A_{m \times n} = P_m R Q_n$  for  $P, Q$  invertible and  $R$  is such that it has an identity in

upper corner and all other entries zero i.e.

$$R = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \text{ for some identity } I_k.$$

### Consistency

System of linear equations :

$A_{m \times n}X_{n \times 1} = b_{1 \times m}$  for  $b \neq 0$  is consistent (has a solution) iff the row reduced Echelon form of augmented matrix  $[A : b]$  has same number of non zero rows as in row reduced echelon form of  $A$ .

2

## Vector Spaces

### Definition

$(V, \mathbb{F}, +)$  denoted by  $V(\mathbb{F})$  :  $V$  is vector space over Field  $\mathbb{F}$  if

- $(V, +)$  is a commutative group, for every  $\alpha, \beta \in \mathbb{F}$  and every  $a, b \in V$
- $1a = a$  where  $1 \in \mathbb{F}$  is multiplicative identity of  $\mathbb{F}$ .

$$\blacksquare (\alpha + \beta)a = \alpha a + \beta a$$

$$\blacksquare \alpha(a + b) = \alpha a + \alpha b$$

$$\blacksquare (\alpha\beta)a = \alpha(\beta a)$$

The elements of  $V$  are called **vectors** and elements of  $\mathbb{F}$  are called **scalars**

### Span

if  $K = \{v_1, v_2, \dots, v_n\} \subseteq V(\mathbb{F})$  then span of  $K$  is the set  $\{\sum \alpha_i v_i | v_i \in K, \alpha_i \in \mathbb{F}\}$  i.e. is all the formal sums from set  $K$  with  $\mathbb{F}$ . This is denoted by  $\text{span}(K)$ .

### Subspace

A subset  $S$  of vector space  $V(\mathbb{F})$  is a subspace if  $S(\mathbb{F})$  is a vector space by same operations as in  $V$

■ given any  $K \subseteq V(\mathbb{F})$   $\text{span}(K)$  is a subspace of  $V(\mathbb{F})$ .

■  $S$  is a subspace of  $V$  iff  $\alpha a + b \in S \forall a, b \in S$  and  $\alpha \in \mathbb{F}$  the underlying field of both spaces

■ Intersection of subspaces (arbitrary) is again a subspace i.e. if  $W_1, W_2$  are subspaces of  $V$  then  $W_1 \cap W_2$  is also a subspace of  $V$ .

■ Union of subspaces may not be a subspace

■ Union of two subspaces is a subspace iff one of them is contained in another i.e. for  $W_1, W_2$  subspaces of  $V$ ,  $W_1 \cup W_2$  is a subspace iff  $W_1 \subseteq W_2$  or  $W_2 \subseteq W_1$ .

(note: this is not the same in case of 3 subspaces : consider  $Z_2 \times Z_2$  ( $Z_2$ ) vector space here  $Z_2 \times Z_2 = \text{span}((0,1)) \cup \text{span}((1,0)) \cup \text{span}((1,1)).$ )

### Dependence

a set of vectors  $\{v_1, v_2, \dots, v_n\} \subseteq V(\mathbb{F})$  are called Linearly independent in  $V$  if  $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0 \implies$  all  $\alpha_i$ 's are 0 and no other choice is left. Other wise the subset is called linearly dependent

### Basis

a subset  $K$  of  $V$  is a spanning set of  $V$  if  $\text{span}(K) = V$ .

A Linearly independent spanning set of  $V(\mathbb{F})$  is called a Basis of  $V$ .

### Dimension

In a given vector space  $V(\mathbb{F})$ .

■ The number of elements in Basis is constant  $n \in \mathbb{Z}^+$ .

■ if a set contains more vectors than the Basis set of a vector space then it is linearly dependent.

■ if a linearly independent set contains exactly the same number of elements as a Basis

then it is also a Basis.

■ These above points leads us to the Definition : Number of elements  $n$  in The Basis set of  $V(\mathbb{F})$  is unique and is called the Dimension of  $V(\mathbb{F})$  denoted by  $\dim(V) = n$ .

if  $W_1, W_2 \subseteq V$  are subspaces then

■  $\dim(W_i) \leq V$ .

■ let  $W_1 + W_2 = \text{span}(W_1, W_2)$  then

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

(note: there cannot be a definite formula for  $\dim(\sum_{i=1}^n W_i)$  using dimensions of  $W_i$ 's and their counterparts (union, intersections) if  $n \geq 3$ .)

### Direct sum

Now if for two subspaces  $W_1, W_2$  of  $V$  if  $W_1 \cap W_2 = \emptyset$  we write their sum  $W_1 + W_2$  as  $W_1 \oplus W_2$

### Matrix Representation of vectors

Fix a basis  $\beta = \{b_1, b_2, \dots, b_n\}$  for a vector space  $V(\mathbb{F})$  then as  $B$  spans  $V$  every vector  $x \in V$  can be written as  $x = x_1 b_1 + x_2 b_2 + \dots + x_n b_n$  for  $x_i \in \mathbb{F}$  and  $b_i \in B$  and this representation is unique so each vector can be associated with a column matrix  $x_\beta = [x_1 \ x_2 \dots x_n]^T$

### Change of Basis Matrix

Given two basis  $\beta = \{b_1, b_2, \dots, b_n\}$ ,  $\beta' = \{b'_1, b'_2, \dots, b'_n\}$  for  $V$  Then one can change the representation of  $x \in V$  from  $[x]_\beta$  to  $[x]_{\beta'}$  by

$$[x]_{\beta'} = P[x]_\beta$$

where  $P_n$  is a invertible matrix given by if  $b_j = p_{1j} b'_1 + p_{2j} b'_2 + \dots + p_{nj} b'_n$  then  $[p_{1j} \ p_{2j} \dots p_{nj}]^T$  forms the  $j^{\text{th}}$  column of  $P$ .

## 3 Linear Transform

### Definition

a map  $T : V(\mathbb{F}) \rightarrow W(\mathbb{F})$  (between vector spaces with same underlying field) is called a linear transform if for every  $v, u \in V$  and  $\alpha \in \mathbb{F}$

- $T(v + u) = T(v) + T(u)$
- $T(\alpha v) = \alpha T(v)$

### Range and Null space

For a linear transform  $T : V \rightarrow W$ :

- Range Space of  $T$  denoted by  $R(T) \subseteq W$  is  $\{w | w = T(v) \text{ for some } v \in V\}$
- Null Space of  $T$  denoted by  $N(T) \subseteq V$  is  $\{v | T(v) = 0 \in W\}$
- Both of them are subspaces of the underlying space.
- $T$  is one-one iff  $N(T) = \{0\}$ .
- $T$  is onto if  $R(T) = W$
- if  $\dim(V) = \dim(W)$  and  $N(T) = \{0\}$  then  $T$  is onto thus  $T$  is bijective.

if  $T, U$  are both linear transforms from  $V \rightarrow W$  and if both agree on a basis of  $V$  (i.e.  $T(b_i) = U(b_i) \forall i$  for some basis  $\beta = \{..., b_i, ...\}$  of  $V$ ) then both of them are same i.e.  $T \equiv U$ .

### Rank Nullity Theorem

for a linear transform  $T : V(\mathbb{F}) \rightarrow W(\mathbb{F})$  if  $\text{rank}(T) = \dim(R(T))$  and  $\text{nullity}(T) = \dim(N(T))$  then

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

(this is just an analogue of  $\mathbb{R}^{st}$  isomorphism theorems of Groups)

### Matrix of Linear Transform

Given a linear transform  $T : V \rightarrow W$ , basis  $\beta = \{b_1, b_2, \dots, b_n\}$  of  $V$  and basis  $\beta' = \{b'_1, b'_2, \dots, b'_m\}$  of  $W$  then we can write the linear transform in the corresponding matrix representation of vectors as

$$[T(x)]_{\beta'} = [T]_{\beta'}^{\beta} [x]_{\beta}$$

where  $[T]_{\beta'}^{\beta}$  is a  $m \times n$  matrix called Ma-

trix of linear transform of  $T$  and is given by if  $T(b_j) = t_{1j}b'_1 + t_{2j}b'_2 + \dots + t_{mj}b'_m$  then  $[t_{1j} \ t_{2j} \dots t_{mj}]^T$  forms the  $j^{\text{th}}$  column of  $[T]_{\beta'}^{\beta}$ .

### Change of Basis

if  $T : V \rightarrow V$  then  $[T]_{\beta}^{\beta}$  is simply written as  $[T]_{\beta}$  now if  $P$  is the change of basis matrix from basis  $\beta'$  to basis  $\beta$  of  $V$  i.e.  $[x]_{\beta} = P[x]_{\beta'}$  then

$$[T]_{\beta'} = P^{-1}[T]_{\beta}P$$

(This can be treated as the origin of 'similar' equivalence matrix relationship  $A \sim B \iff A = P^{-1}BP$ .)

### Isomorphism of Vector spaces

Two spaces  $V, W$  over same vector space  $\mathbb{F}$  are said to be isomorphic to each other if there exist an invertible linear transform  $T : V \rightarrow W$  (i.e.  $T$  is linear bijective map) and this is denoted by  $V \cong W$ .

■ if  $V(\mathbb{F})$  is of dimension  $n$  then  $V \cong \mathbb{F}^n = \{(\alpha_1, \alpha_2, \dots, \alpha_n) | \alpha_i \in \mathbb{F}\}$  i.e. set of  $n$  tuples of  $\mathbb{F}$  with component wise addition.

■ clearly  $V(\mathbb{F}) \cong W(\mathbb{F})$  iff  $\dim(W) = \dim(V)$ .

### Space of Linear Transform

Set of linear transforms

$L(V, W) = \{T | T : V \rightarrow W \text{ is linear transform}\}$  forms a commutative group under addition i.e.  $(T + U)(v) = T(v) + U(v)$  (as in  $W$ ) so it also forms a Vector space over  $\mathbb{F}$  (same field as in  $V$  and  $W$ .)

■ if  $\dim(V) = n$  and  $\dim(W) = m$  both finite then  $\dim(L(V, W)) = nm$

### Linear Functional

Linear transformation  $f : V(\mathbb{F}) \rightarrow \mathbb{F}$  is called a Linear Functional

■ This is possible as  $\mathbb{F}(\mathbb{F})$  is an one dimensional vector space.

■  $\text{rank}(f) = 1$  or  $0$  so  $\text{Nullity}(f) = n -$

1 or  $n$  if  $\dim(V) = n < \infty$ .

■ **Dual space** of  $V$  denoted by  $V^* = L(V, \mathbb{F})$  is the set of all linear functionals on  $V$

■ clearly  $\dim(V^*) = \dim(V)$  if  $\dim(V)$  is finite

■ **Dual Basis** : for every basis  $\beta = \{b_1, b_2, \dots, b_n\}$  of  $V$  there exist a corresponding basis  $\beta^* = \{f_1, f_2, \dots, f_n\}$  of  $V^*$  such that  $f_i(b_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$  this  $\beta^*$  is called the dual basis of  $\beta$

■ if  $\{f_1, f_2, \dots\}$  is the dual basis of  $\{b_1, b_2, \dots\}$  and  $x \in V$  is represented as  $x = x_1 b_1 + x_2 b_2 + \dots + x_n b_n$  then  $x_i = f_i(x)$  i.e. the coordinate functions in representation is nothing but the dual functions, i.e.

$$x = \sum_{i=1}^n f_i(x) b_i.$$

■  $V \cong V^* \cong V^{**} = L(V^*, \mathbb{F})$  (note:  $\cong$  in  $V \cong V^{**}$  is nothing but functional evaluation at a point(vectors) only i.e. every element of  $V^{**}$  is of form  $\hat{x}$  for  $\hat{x}(\psi) = \psi(x)$  for some  $x \in V$ .)

#### Functional representation Theorem

if  $V$  is finite dimensional vector space,  $\beta = \{b_i\}$  is its basis and  $[x]_\beta = [x_1 \ x_2 \dots x_n]$  then every functional  $f$  is of form

$$f(x) = a_1 x_1 + a_2 x_2 + \dots + a_n x_n$$

in which  $a_i = f(b_i)$  are fixed but  $x_i$  varies on input representation  $x$ .

#### Annihilator

if  $A \subset V(\mathbb{F})$  be any subset of  $V$  then annihilators of  $A$  is the set of linear functionals  $A^0 = \{f \mid f(A) = 0, f \in V^*\} \subseteq V^*$

■ clearly  $A^0$  is a subspace of  $V^*$  for any subset  $A$  of  $V$

■ subspaces  $W_1 = W_2$  iff  $W_1^0 = W_2^0$

■  $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$ .

■ if  $W$  is subspace of  $V$  then

$$\dim(W) + \dim(W^0) = \dim(V).$$

■ if  $W$  is subspace of  $V$  then  $W \cong W^{00}$ .

#### Transpose of linear transform

if  $T : V \rightarrow W$  is linear transform then its transpose  $T^t : W^* \rightarrow V^*$  is a linear transform defined by the evaluation

$T^t(g(\cdot)) = g(T(\cdot))$  i.e. for  $g \in W^*$ ,  $T^t(g)$  is the functional  $f = g(T(\cdot)) \in V^*$

■  $[T^t]_{\gamma^*}^{\beta^*} = ([T]_{\beta}^{\gamma})^t$  i.e. the corresponding matrix of  $T^t$  in dual basis of  $\gamma$  in  $W$  and  $\beta$  in  $V$  is just the Transpose of the matrix of  $T$  in  $\beta$  and  $\gamma$ .

■ if  $W$  is finite dimensional then for linear  $T : V \rightarrow W$  we have

$$R(T^t) = (N(T))^0 \text{ and } N(T^t) = (R(T))^0$$

■  $T$  is 1-1 iff  $T^t$  is onto and  $T$  is onto iff  $T^t$  is 1-1.

■  $\text{Rank}(T^t) = \text{Rank}(T)$ .

if linear transform  $T \in L(V) = L(V, V)$  then it is called a linear operator.

## 4 Determinant

#### Motivation

for a finite dimensional space every linear transform in  $L(V)$  can be represented as a unique Matrix, but we need to 'uncover' this matrix to gain the properties of corresponding linear transform one such way is to create a Function from set of matrices to the underlying field with some properties which helps us with this 'gain'.

Some Properties needed for such a function are :

■ It must be a linear in terms of rows (or columns) of the matrix this is called  $n$ -linear.

■ It must be alternating i.e. if any 2 rows (or columns) are equal then it is zero.

■ its value on Identity should be 1.

Say we obtain a function  $D$  with this property for  $(n-1) \times (n-1)$  matrices then this

can be extend to  $n \times n$  by

$$E_j(A_n) = \sum_{i=1}^n (-1)^{i+j} a_{ij} D(A_{ij})$$

for fixed  $j \in \{1, 2, \dots, n\}$ , where  $a_{ij}$  is the  $i^{\text{th}}$  row  $j^{\text{th}}$  column entry of  $A$  and  $A_{ij}$  is the  $n-1 \times n-1$  matrix obtained from  $A_n$  by removing  $i^{\text{th}}$  row and  $j^{\text{th}}$  column.

#### Definition

From above points we get determinant for a  $n \times n$  matrix with entries from  $\mathbb{F}$  as  $D : \mathbb{F}^{n \times n} \rightarrow \mathbb{F}$  that is  $n$ -linear, Alternating and  $D(I) = 1$  is Defined by recursion from the above point or if  $(i_1, i_2, \dots, i_n)$  runs through all the possible permutations of  $n$  i.e  $n$ - tuple with elements from  $\{1, 2, \dots, n\}$  with out repetition then  $D(A = [a_{ij}]) = \sum_{(i_1, i_2, \dots, i_n)} (-1)^{i_1+i_2+\dots+i_n} a_{1i_1} a_{2i_2} \dots a_{ni_n}$

#### Additional Properties

■  $\det(A) = \det(B)$  if  $B$  is obtained by interchanging rows of  $A$

■  $\det \begin{bmatrix} A & B \\ 0 & C \end{bmatrix} = \det(A) \det(C).$

## 5 Diagonalizability

For linear operator  $T \in L(V)$  a vector  $\alpha \in V$  is called an eigenvector and  $\lambda$  called eigenvalue if  $T(\alpha) = \lambda\alpha$ . i.e.  $\alpha \in N(T - \lambda I)$

■ if  $A \in M_n(\mathbb{F})$  (all  $n \times n$  matrices with entries from  $\mathbb{F}$ ) then  $\lambda$  is an eigenvalue of  $A$  iff  $\det(A - \lambda I) = 0$ .

■ From above point we get all eigenvalues of  $A \in M_n(\mathbb{F})$  are the solutions of **Characteristic polynomial**  $f(t) = \det(A - tI)$ .

for a linear operator  $T$  on finite dimensional space  $V$

■ The polynomial  $p(T)$  such that  $p(T) \equiv 0$

i.e  $p(T)x = 0 \forall x \in V$  then  $p(T)$  is called the **annihilating polynomial** of  $T$

■ the set of all annihilating polynomials of  $T$  forms an ideal in  $\mathbb{F}[x]$  now as  $\mathbb{F}$  is a field it is also an euclidean domain so this ideal is principle thus is generated by a monic polynomial of minimum degree in it called the **minimal polynomial** of  $T$ .

**Algebraic Multiplicity** of an eigenvalue  $\lambda$  for a linear operator  $T$  is multiplicity of  $\lambda$  in the characteristic polynomial of  $T$ .

**Geometric multiplicity** of an eigenvalue  $\lambda$  for a linear operator  $T$  is the dimension of the nullspace of  $T - \lambda I$ .

A linear operator  $T$  on  $V$  is said to be Diagonalizable if there exist a basis of  $V$  containing only eigenvectors of  $T$ .

■  $T$  is diagonalisable iff every eigenvalue of  $T$  belongs to the underlying field and Algebraic multiplicity = Geometric multiplicity for every eigenvalue of  $T$ .

#### Cayley-Hamilton Theorem

if  $T$  is a linear operator on finite dimensional space  $V$  then characteristic polynomial of  $T$  divides minimal polynomial of  $T$  i.e. if  $f$  is characteristic polynomial of  $T$  then  $f(T) \equiv 0$ .

for a given eigenvalue  $\lambda$  of  $T \in L(V)$  the set of all eigenvectors corresponding to  $\lambda$  form a subspace of  $V$  this is called eigenspace of  $\lambda$ .

#### Invariant subspace

$W$  is an invariant subspace of  $T$  over  $V$  if  $T(W) \subseteq W$ .

Eigenspaces are invariant subspaces.

#### Diagonalizability test

$T$  is diagonalizable iff minimal polynomial of  $T$  ( $m_T(x)$ ) splits into distinct linear factors

in the underlying field  $\mathbb{F}$  i.e.

$T$  is diagonalizable  $\iff m_T(x) = (x - c_1)(x - c_2) \dots (x - c_n)$  for distinct  $c_i \in \mathbb{F}$

6

## Projections or Idempotent Operators

### Projections

$E : V(\mathbb{F}) \rightarrow V(\mathbb{F})$  (is a projection if  $E^2 = E$ )

■ if  $E$  is a projection then  $a \in R(T)$  iff  $E(a) = a$ .

if  $V$  is a finite dimensional vector space, say  $\{b_1, b_2, \dots, b_n\}$  is a given ordered basis then we can define projection operators  $E_i$  ( $i = 1, 2, \dots, n-1$ ) as follows: for  $x \in V$ ,  $x = \sum_{j=1}^n a_j b_j$  we have  $E_i(x) = \sum_{j=1}^i a_j b_j$  i.e. restriction of the element to a particular subspace. Here we get  $R(E_i) = \text{span}(\{b_1, \dots, b_i\})$  and  $N(E_i) = \text{span}(\{b_{i+1}, \dots, b_n\})$  (note:  $0$  and  $I$  are also projection operator so we can extend these definitions to include  $0$ -space and whole space.)

By intuition of above point we get

if vector space  $V = W_1 \oplus W_2 \oplus \dots \oplus W_n$  then there exists linear operators  $E_1, E_2, \dots, E_n$  such that

- Range of  $E_i = W_i$
- each  $E_i$  is a projection.
- $E_i E_j = 0$  for  $i \neq j$ .
- $I = E_1 + E_2 + \dots + E_n$

Conversely if above 4 points are satisfied for some set of linear operators  $\{E_i\}$  on finite dimensional vector space  $V$  then for  $W_i = R(E_i)$  we have  $V = W_1 \oplus W_2 \oplus \dots \oplus W_n$ .

if a linear operator  $T$  on  $V$  (finite dimensional) and if  $E$  the projection operator of subspace  $W \subseteq V$  (defining it can be done by using basis definition of the projections) then  $T$  commutes with  $E$  iff  $W$  is invariant on  $T$  i.e.

$$\text{for } E^2 = E \text{ and } R(E) = W$$

$$TE = ET \iff T(W) \subseteq W$$

### Diagonalizability and Projections

if a linear operator  $T$  on  $V$  is diagonalizable on  $V$  then for distinct eigenvalues  $\lambda_1, \lambda_2, \dots, \lambda_n$  of  $T$   $\exists$  projections  $E_1, E_2, \dots, E_n$  on  $V$  such that

- range of  $E_i =$  eigenspace of  $\lambda_i$  in  $V$ .
- $T = \lambda_1 E_1 + \lambda_2 E_2 + \dots + \lambda_n E_n$ .
- $E_i E_j = 0$  for  $i \neq j$ .
- $I = E_1 + E_2 + \dots + E_n$

**Conversely** if last 3 points are satisfied for any linear operator  $T$  and some set of projections  $\{E_i\}$  on finite dimensional vector space  $V$  then  $T$  is Diagonalisable.

### Primary Decomposition Theorem

for a Linear operator  $T$  on finite dimensional vector space  $V$  and if minimal polynomial of  $T = m_T(x) = P_1^{r_1}(x) P_2^{r_2}(x) \dots P_n^{r_n}(x)$  where  $P_i$  are distinct **primes**  $\mathbb{F}[x]$  then for  $W_i =$  Nullspace of  $P_i^{r_i}(T)$  we have

- $V = W_1 \oplus W_2 \oplus \dots \oplus W_n$ .
- $W_i$  is  $T$  invariant i.e.  $T(W_i) \subseteq W_i$ .
- for  $T_i$  restriction of  $T$  on subspace  $W_i$  has minimal polynomial  $P_i^{r_i}$ .

7

## Jordan Form

### Generalised eigenvectors

For a linear operator  $T$  on  $V$ , if  $\lambda$  is an eigenvalue of  $T$  then a vector  $v$  is such that  $(T - \lambda I)^k v = 0$  for some positive integer  $k$  is generalised eigenvector.

- The Subspace  $K_\lambda = \{v | (T - \lambda I)^k v = 0 \text{ for some +ve integer } k\}$  is called generalised eigenspace.

### properties of generalised eigenspaces

For a given linear operator let  $K_\lambda$  denote generalised eigenspace of  $T$  w.r.t (with respect to) eigenvalue  $\lambda$  of  $T$  then

- $K_\lambda$  is  $T$  invariant.

■ for eigenvalue  $\mu \neq \lambda$  of  $T$ :  $T - \mu I$  is one-one on  $K_\lambda$ .

■  $\dim(K_\lambda) = m_\lambda$  where  $m_\lambda =$  Algebraic multiplicity of  $\lambda$ .

■  $K_\lambda = N((T - \lambda I)^{m_\lambda})$  where  $m_\lambda =$  Algebraic multiplicity of  $\lambda$ .

■ if all of the eigenvalues of  $T$  belong to the underlying field then

$V = K_{\lambda_1} \oplus K_{\lambda_2} \oplus \dots \oplus K_{\lambda_n}$  where  $\lambda_1, \lambda_2, \dots, \lambda_n$  are distinct eigenvalues of  $T$ .

Cycle of generalised eigenvector : if  $v \in K_\lambda$  then the set  $\gamma = \{(T - \lambda I)^{k-1}v, (T - \lambda I)^{k-2}v, \dots, (T - \lambda I)v, v\}$ , where  $(T - \lambda I)^k v = 0$  and  $(T - \lambda I)^{k-1}v$  called as initial vector, forms a linearly independent set in  $K_\lambda$

■ if  $\gamma_1, \gamma_2, \dots, \gamma_l$  are cycle of generalised eigenvectors for a given eigenvalue  $\lambda$  such that for each  $\gamma_i$  initial vectors are distinct and linearly independent in  $K_\lambda$  then  $\gamma = \cup \gamma_i$  is a linearly independent set in  $K_\lambda$ .

#### existence Jordan canonical form

for any linear operator  $T \in L(V(F))$

■ every  $K_\lambda$  (generalised eigenspace) has a ordered basis constituting of cycle of generalised eigenvectors.

■ if characteristic polynomial of  $T$  completely splits into linear factors in  $F$  then there exist a basis of  $V$  containing only Cycle of generalised eigenvectors of  $T$ , this basis gives a unique characteristic to  $T$  which when viewed in matrix form of  $T$  gives raise to Jordan canonical form.

#### Consequences of Jordan Form

■ Two linear operators or square matrices (whose characteristics polynomial completely splits into linear factors in their under lying filed) are similar iff they have the same Jordan form representation.

■  $T \sim T^t$ .

### 8 Rational Form

### 9 Inner Product Spaces

### 10 Forms

### 11 Bilinear Forms