Abstract Algebra : Ring Theory.

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Contents

0 symbols used

$_{s|t}$ $\rightarrow$ such that.

1 Basic Definitions

Ring R : is a set together with two binary operations $+$ and $\times$ (addition and multiplication) with properties :
- $(R, +)$ is an abelian group
- $\times$ is associative (i.e. $a \times (b \times c) = (a \times b) \times c$)
- $\times$ distributes over $+$ (i.e. $(a + b) \times c = (a \times c) + (b \times c)$).

if R is commutative w.r.t. $\times$ then R is a **Commutative ring**.

Additive identity in R is denoted by 0.

R is said to have identity (1) if there is a multiplicative identity in R
i.e. $\exists 1 \in R \; _{s|t} \; 1 \times a = a \times 1 = a \; \forall a \in R$.

now for $a \in R$ additive inverse of a is denoted by $-a$ and multiplicative inverse (if exists in R) by a^{-1}.

Division Ring (or skew field) D

is ring with identity 1, $1 \neq 0$ and for a element $a \in R$ not equal to 0 there exists $b \in R \; _{s|t} \; ab = ba = 1$. i.e. $\forall 0 \neq a \in R \; a^{-1} \in R$.

for a ring R is a non zero $a \in R$ is a **Zero Divisor** if there is non zero $b \in R$ such that $ab = 0$ or $ba = 0$.

for a ring R with 1 : $u \in R$ is a **unit** in R if it has multiplicative inverse in R

Integral domain I

is a commutative ring with identity $1 \neq 0$ having no zero divisors.

Field F

is a commutative ring with identity in which any non zero element is unit i.e. $0 \neq a \in F \implies a^{-1} \in F$. or a commutative Division ring.

Subring S of ring R is subgroup of R which is in itself a ring with same operations i.e. S is subring of R if $(S,+)$ is subgroup of $(R,+)$ and S is closed under $\times$.
(i.e. $a,b \in S \implies ab \in S$)

Center of a ring R is set $\{z \in R_s|_t\ zr = rz \forall r \in R\}$ i.e. all the elements that commute in R. (multiplicatively.)

$a \in R$ commutative ring is **nilpotent** if $a^m = 0$ for some $m \in \mathbb{Z}^+$

$a \in R$ commutative ring is **idempotent** if $a^2 = a$.

2 Properties of Rings

(instead of writing $a \times b$ we just write ab)

for any $a,b \in R$ (ring)
- $a0 = 0a = 0$.
- $(-a)b = a(-b) = -(ab)$.
- $(-a)(-b) = ab$
- if $1 \in R$ then $-a = (-1)a$.

An element cannot be both a zero divisor and a unit in R. (There can be elements that are neither)

if $ab = 0$ in a integral domain I with $a,b \in I$ then at least one of a or b is zero.

Cancellation Laws holds in any Integral domain. (note: the existence of multiplicative inverse is not needed here.)

Any Finite Integral Domain is a Field
(use bijective map $I \to I$ by $x \to ax$ for non-zero $a \in I$.)

if S,T subgrings of R then :
- $S \cap T$ is subring of R. (thus any arbitrary non-empty intersection of subrings is a subring.)
- S is subring of T is subring of R then S is subring of R.

Properties of Center of a Ring
- Center of a ring is a subring.
- Center of a Division ring is a Field.
- for fixed $a \in R$ set $C(a) = \{r \in R_s|_t\ ra = ar\}$ is a subring of R containing a.
- Center of $R = \bigcap_{a \in R} C(a)$.
- for any $a \in D$ division ring then $C(a)$ is a division ring.

if $x^2 = 1$ for some $x \in I$ integral domain then $x = \pm 1$ only.

if $x \in R$ commutative ring is nilpotent then
- x is either zero or zero divisor (use $x^m = x^{m-1}x = 0$)
- rx is nilpotent for any $r \in R$.
- if $1 \in R$ then $1+x$ is a unit in more general $u+x$ is a unit for any unit u and nilpotent x. (use $(1+x)(1-x+x^2+..+(-1)^{m-1}x^{m-1}) = 1+(-1)^{m-1}x^m = 1$.)

for rings R and S their direct product $R \times S$ is ring under corresponding component wise operation. (thus even any number of direct product of rings is a ring)

now if R,S are non zero fields then $R \times S$ is never a field. (as $(1,0)(0,1) = (0,0)$)

Characteristic $ch(R)$

Characteristic of a ring is a number n s.t $n1 = \overbrace{1+1+..+1}^{n \text{ times}} = 0$ if order of 1 is infinite then characteristic is said to be 0 (not ∞.)

every Integral domain has a Character 0 **or prime.** (if $q = mn$ is characteristic of I then $mn.1 = 0$ so either $m.1 = 0$ or $n.1 = 0$ a contradiction for minimality.)

3 Ring Homomorphism and Ideals

if R, S are two rings the a map $\phi : R \to S$ is a ring Homomorphism if it satisfies
- $\phi(a+b) = \phi(a) + \phi(b)$
- $\phi(ab) = \phi(a)\phi(b)$

- kernel of ϕ denoted by $\ker \phi$ is the set in R mapped to 0 in S
- image of ϕ and $\ker \phi$ are subrings.
- if $\alpha \in \ker \phi$ then $r\alpha \in \ker \phi \ \forall r \in R$ i.e. $r \ker \phi \subseteq \ker \phi$

Concept of Ideal

From the above point and to define the quotient operations as in group homomorphism i.e. if $I \subseteq R$ and to the quotient operations

$$(r+I)+(s+I) = (r+s)+I$$
$$(r+I)(s+I) = rs+I$$

to be well defined we need to have that replacement by any class representative gives same classes i.e. if $\alpha, \beta \in I$ then
$(r+\alpha)(s+\beta)+I = rs+I$
to achive this : we need $I \trianglelefteq R$ (w.r.t $+$) this can be satisfied by any subgroup of R as R is abelian in $+$, letting $r = s = 0$ we need I closed under multiplication so these two conditions boils down to I must be subring of R and also we need to have that I must be closed under left and right multiplication from any element in R i.e. $rI \subseteq I, Ir \subseteq I \ \forall r \in R$ this leads us to define Ideals

$I \subseteq R$ is a **left ideal** in R if I is subring of R and closed under left multiplications by elements of R, similarly $I \subseteq R$ is a **right ideal** in R if I is subring of R and closed under right multiplications by elements of R.
Finally $I \subseteq R$ is an **Ideal** of R if it is both left and right ideal of R

Ideal for Rings is 'similar' to Normal subgroups for Groups, Most of the following properties are Ring analogue of Group properties.

if I is an ideal of R ring then the quotient group (additive $\{r+I\}$) is a ring with binary operations as defined above. This Group R/I is called the **Quotient ring** of R by I.

Isomorphism Theorems for Ring

- **1st Isomorphism Theorems for Ring** :
if $\phi : R \to S$ is a ring homomorphism then $\ker \phi$ is an ideal of R, image of ϕ in S is a subring of S and $R/ker(\phi)$ is isomorphic (bijective ring homomorphism) to $\phi(R)$. let $\cong$ denote Ring isomorphism from here on so $R/ker(\phi) \cong \phi(R)$
if I is an ideal of R then the natural projection homomorphism $\pi_I : R \to R/I$ by $\pi_I(r) = r+I$ is ring homomorphism with kernel I i.e. ideal $\iff$ kernel.
- **2st Isomorphism Theorems for Ring** :
If A is subring of R and B ideal of R then $A+B = \{a+b | a \in A, b \in B\}$ is a subring of R, $A \cap B$ is an ideal of A and $(A+B)/B \cong A/(A \cap B)$.
- **3st Isomorphism Theorems for Ring** :
if I, J are ideals of R such that $I \subset J$ then J/I is an ideal of R/I and $(R/I)/(J/I) \cong R/J$
- **4st Isomorphism Theorems for Ring** :
If I is an Ideal of R ring then the map $A \to A/I$ is bijective between the set of subrings

A of R that contain I and set of subrings of R/I also A is an ideal of R containing I iff A/I is an ideal of R/I.

if $\phi : R \to S$ is homomorphism and $x \in R$ is nilpotent then $\phi(x)$ is nilpotent in S.

3.1 Properties of ideal

For the rest of this notes let every ring mentioned be a ring with identity unless specified.

for ideals I, J of R ring define $I + J = \{a + b | a \in I, b \in J\}$ and $IJ = \langle ab | a \in I, b \in J \rangle$ (i.e. the set generated by ab or set of finite sums of element of form ab) then
■ $I + J$ is the smallest ideal containing both I and J in R,
■ $IJ \subset I \cap J$ and both $IJ, I \cap J$ are ideals in R.
■ if $I + J = R$ then $IJ = I \cap J$.

Ideal Generated by $A \subseteq R$ denoted by (A) is the smallest ideal of R ring containing A.

define $RA = \{r_1a_1 + r_2a_2 + .. + r_na_n | r_i \in R, a_i \in A\}$ and $AR = \{a_1r_1 + a_2 + r_2a_2 + .. + a_nr_n | a_i \in A, r_i \in A\}$ and $RAR = \{r_1a_1r'_1 + r_2a_2r'_2 + .. + r_na_nr'_n | r_i, r'_i \in R, a_i \in A\}$

if R is commutative then $RAR = RA = AR = (A)$.

Principle Ideal

An ideal generated by a single element is called a Principle Ideal,
An ideal generated by a finite set is called **Finitely generated ideal.**

I is an ideal of R ring and $I = R$ iff I contains an unit of R.
So we get R is a field iff the only ideals of R are $\{0\}$ and R.

if F is field then any homomorphism from F is trivial ($\ker(\phi) = F$) or injective ($\ker(\phi) = 0$) i.e. **any non trivial homomorphism from a field is injective**

Maximal Ideal

M a proper ideal of R ring is called a Maximal Ideal of R if the only ideal containing M in R is R. i.e. no other proper ideals contains M in R.

M is maximal ideal of R ring iff R/M is a field.

for a Ring with identity every proper ideal is contained in a maximal ideal
(inclusion forms a partial ordering of proper ideals (non empty set) in the ring so a chain exist whose elements always contain the given proper ideal, now form an ideal by union of these ideals which is also proper (prove) thus a having upper bound and by Zorn's lemma a maximal element which is the maximal ideal.)

Prime ideal

An ideal P is Prime Ideal of R ring if $P \neq R$ and whenever $ab \in R$ the $a \in P$ or $b \in P$.
(this is sort of generalising 'prime' to a give ring R as in $\mathbb{Z}^+$ if p is a prime and $p|ab$ then $p|a$ or $p|b$.)

for R a commutative ring P is prime ideal in R iff R/P is an Integral domain.

from above point and similar point for maximal ideals we get :
in commutative ring R every maximal ideal is a prime ideal

similarly a commutative ring with identity is an integral domain iff $\{0\}$ is prime ideal in the ring.

in an Integral domain R $(a) = (b)$ for some $a, b \in R$ iff $a = ub$ for some unit $u \in R$.

if P a prime ideal of R commutative ring and P contains no zero divisors then R is an integral domain.

if $\phi : R \to S$ be a ring homomorphism between two commutative rings with identity:
■ for P prime ideal in S then $\phi^{-1}(P)$ is a prime ideal in R or $\phi^{-1}(P) = R$
■ if ϕ is **surjective** and M maximal ideal in S then $\phi^{-1}(M)$ is maximal in R.

In a finite commutative ring with identity every prime ideal is maximal ideal.
(use finite integral domain is a field.)

if R is commutative ring with property : for every $a \in R$ $\exists n \in \mathbb{Z}^+$ depending on $a \,{}_s|_t\, a^n = a$ then every prime ideal of R is maximal ideal.

Local Ring

A commutative ring R is a local ring if it has a Unique maximal Ideal.
■ If R is a local ring with maximal ideal M then every element of $R - M$ is a unit (precisely).
■ Conversely if R is commutative ring with 1 and if set of non units in R forms an ideal M then R is a local ring with unique maximal ideal M.
■ eg: R subring of Q in which denominators are odd forms a local ring i.e. $R = \{n/m | \ 2 \not| m, \ n, m \in \mathbb{Z}\}$ is local ring with unique maximal ideal is the principle generated by 2 which is a prime in R.

Nilradical

if R is a commutative ring the set of all nilpotent elements i.e.$\{x \in R | x^m = 0, \ m \in Z^+\}$ forms an Ideal called nilradical of R denoted by $\mathcal{N}(R)$. (use binomial thm : if $x^n = 0$ and $y^m = 0$ then for $k = 2\max(m, n)$ we have $(x + y)^k = 0$).
■ if $\mathcal{N}(R)$ is nilradical of R then the only nilpotent element of $R/\mathcal{N}(R)$ is zero i.e. $\mathcal{N}(R/\mathcal{N}(R)) = \{0\}$.

■ Prime Ideal of a commutative ring contains every nilpotent element i.e. nilradical of R commutative ring is contained in the intersection of all prime ideals of R. (more precisely nilradical of R is the intersection of all prime ideal in R.)

for a commutative ring R : R has exactly one prime ideal iff every element of R is either nilpotent of a unit iff $R/\mathcal{N}(R)$ is a field.

4 Ring of fractions

let R be a commutative ring and $\{0\} \neq D \subseteq R$ that doesn't contain 0, doesn't contain any zero divisors of R, closed under multiplication then there is a commutative ring Q with 1 such that R is a subring of Q and every element of D in Q has an inverse.
This ring Q has following additional properties :
■ every element of Q is of the form rd^{-1} for some $r \in R, d \in D$ in particular If $D = R - \{0\}$ then Q is a field.
■ Ring Q is the smallest ring containing R in which all elements of D are units
(i.e. if $\phi : R \to S$ is an injective homomorphism $\,{}_s|_t$ $\phi(d)$ is a unit in S then there is An isomorphic copy of Q in S.)
this Q is called the **Ring of Fractions.** of D w.r.t R.

Construction of Ring of fractions

let $\mathcal{F} = \{(r, d)\ {}_s|_t\ r \in R, d \in D\}$ and define relation $\sim$ on $\mathcal{F}$ by $(r, d) \sim (s, e)$ iff $re = sd$ this becomes an equivalence relation as d, e are not zero divisors, denote equivalence class of (r, d) by

$$\frac{r}{d} = \{(a, b)\ {}_s|_t\ a \in R, b \in D \text{ and } rb = ad\}$$

then Q becomes the set of equivalence classes under $\sim$
properties such as commutativity , $1 = \frac{d}{d}$, additive inverse of $\frac{a}{d}$ is $\frac{-a}{d}$, $d^{-1} = \frac{1}{d}\ \forall d \in D$ hold making this Q the ring of fractions.

Q may also be denoted by $D^{-1}R$ to emphasize the envolved R, D.
If R is integral domain and $D = R - \{0\}$ then $D^{-1}R$ is a field so is called **Field of Fractions** of R.
If R is an integral domain, Q its field of fractions then if any field F contains $R'\ {}_s|_t\ R' \cong R$ then the subfield generated by R' in F is isomorphic to Q.

This concept of integral domains and field of fractions are derived from observing $\mathbb{Z}, \mathbb{Q}$ this is generalized by
if F is Field then F contains a unique smallest subfield that is either isomorphic to Q or $\mathbb{Z}/p\mathbb{Z}$. (depending on its characteristic.)

5 Euclidean Domain.

Norm N

Norm N on R integral domain is a function from $R \to Z^+ \cup \{0\}$ with $N(0) = 0$.
if $N(a) > 0$ for $a \neq 0$ in R then N is called **positive norm .**

for R an integral domain is called an **Euclidean Domain** if there is a norm N on R such that for any $a, b \in R$ with $b \neq 0$ there exist elements $q, r \in R$ such that

$$a = qb + r \text{ with } r = 0 \text{ or } N(r) < N(b)$$

here q is called quotient and r the remainder of the division.

Euclidean Division algorithm is valid in an Euclidean Domain (i.e. it stops after finite steps.)

Every ideal in Euclidean Domain is principal (i.e. generated by single element that has the minimum norm.)

in a commutative ring R, $a, b \in R$ and $b \neq 0$
■ a is said to be multiple of b if there exist $x \in R\ {}_s|_t\ a = bx$. denoted by $b|a$
■ **Greatest common divisor** of $a, b \in R$ is a non zero $d \in R\ {}_s|_t\ d|a, d|b$ and if any other $d'|a, d'|b$ then $d'|d$
this is denoted by $(a, b) = d$.

Properties of Gcd

■ if $a, b \in R$ not zero , ideal generated by a, b is principal and equal to (d) then $(a, b) = d$ (literally $(a, b) = (d)$)
■ now for $(a, b) = d$ in R if I ideal generated by a, b then I is contained in (d) and any principle ideal (d') contains $I \implies (d) \subseteq (d')$
■ as $(d) = (d')$ then $d = ud'$ for some unit $u \in R$ Integral domain we have :
in an Integral domain R if d', d are both g.c.d of a, b then $d = ud'$ for some unit $u \in R$.
■ if R is an Euclidean domain if $(a, b) = d$ in R then $d = r_n$ the last remainder in euclidean algorithm applied to a, b in R and $d = xa + yb$ for $x, y \in R$.

Universal side divisor

$u \in R$ is Universal side divisor if for every $x \in R$ there is some $z \in R^* \cup \{0\}$ (set of units in $R + \{0\}$) such that $u|(x - z)$. i.e every x can be written as $x = qu + z$. for z unit or zero.

Test for not Euclidean domain

If R is an integral domain which is not a field , R is a Euclidean domain then there is a universal side divisor in R

(this point can be used to disprove a given ring is euclidean domain.)

6 Principal Ideal Domains

Principle ideal Domain is an Integral domain in which every ideal is principal.

Every Euclidean domain is a principal ideal domain.

if R is a P.I.D. (Principal Ideal Domains) , $(a, b) = d$ in R then $d = xa + by$ for $x, y \in R$ and d is unique upto multiplication by a unit in R.

Every non zero prime ideal of a P.I.D. is a maximal ideal

Dedekind-Hasse Norm

is positive norm on integral domain$R$ such that for every non zero $a, b \in R$ either $a \in (b)$ or there exist a non zero element of ideal (a, b) which has a norm strictly smaller than norm of b i.e. $\exists s, t \in R_{s|t}\, o < N(sa - tb) < N(b)$.

test for not a P.I.D

Integral domain R is a P.I.D. iff R has a Dedekind-Hasse Norm.

if R is an Integral domain in which every prime ideal is principal then R is a P.I.D.

Every Euclidean domain is a P.I.D as every ideal in Euclidean domain is principally generated by element of minimum norm.

7 Unique Factorization Domain

Irreducible and prime

For an Integral domain R :

■ $r \in R$ a non-zero non-unit element is called **irreducible** in R if whenever $r = ab$. with $a, b \in R$
then at least one of a, b is a unit in R (i.e. r cannot be factored into only non units)
otherwise r is said to be **reducible**.

■ non zero $p \in R$ is called **prime** in R if (p) is a prime ideal in R.
(i.e. if $ab \in (p)$ then $p|ab$ so $p|a$ or $p|b$ analogous to definition of 'primes' in $\mathbb{Z}$.)

■ two elements $a, b \in R$ are called **associates** in R if the differ by a unit in R i.e $a = ub$ for some unit $u \in R$

In an integral domain every prime is irreducible.

In a P.I.D. every non zero element is prime iff irreducible.

U.F.D. (Unique Factorization Domain) : is an integral domain in which every non zero element which is not a unit can be written as finite product of irreducibles and this decomposition is unique upto associates. (i.e. for every non zero non unit $r \in R$, $r = p_1 p_2 .. p_n$ for $p_i's$ irreducibles and if same $r = q_1 q_2 .. q_m$ for q_i irreducible then $m = n$ and we can rearrange these decompositions such that p_i, q_i are associates .)

In a U.I.D. every non zero element is prime iff irreducible.

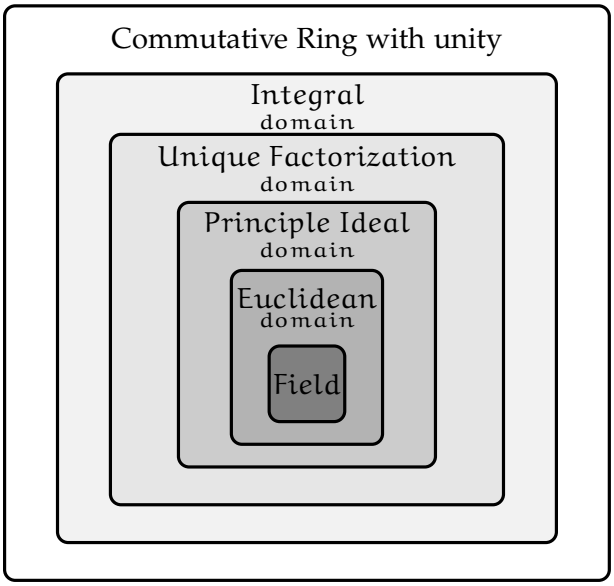
if $a, b \in R$ a U.I.D are such that $a = up_1^{e_1} p_2^{e_2} .. p_n^{e_n}$ and $b = vp_1^{f_1} p_2^{f_2} .. p_n^{f_n}$ for u, v units and $p_i's$ primes in R then

$$(a, b) = d = p_1^{\min(e_1, f_1)} p_2^{\min(e_2, f_2)} .. p_n^{\min(e_n, f_n)}.$$

Every P.I.D. is a U.I.D. in particularly every Euclidean Domain is a U.I.D. (prove that ascending chain of ideals $(i_1) \subset (i_2) \subset .. \subset R$ must stop in a P.I.D : R as $I = \cup_{j \in \mathbb{Z}^+}(i_j) = (a)$ and $a \in (i_n)$ for some n thus factorization of primes stop and can be proved unique up to associates by cancellation.)

Fundamental theorem of Arithmetic

$\mathbb{Z}$ is U.F.D.



i.e. **Field** $\subset$ **Euclidean Domains** $\subset$ **P.I.D.** $\subset$ **U.F.D.** $\subset$ **Integral domains** $\subset$ **Commutative Rings with** 1

■ Subring of an Integral domain may not be an Integral domain (may not contain unity)
■ But if a Subring of Integral Domain contains unity then it is an Integral domain
Here define a **Subdomain** of a Ring is Subring which is an Integral domain
so any Subring of Integral domain containing unity is a Subdomain
■ Subrings and Subdomains of U.F.D maynot be U.F.D (eg: $Z[\sqrt{5}]$ subring of C but not an U.F.D)
■ Subrings and Subdomains of P.I.D maynot be P.I.D (eg: $Z[x]$ subring of $Q[x]$, $Z[x]$ is not a P.I.D as $\langle x, 2\rangle$ is not principle ideal)
■ Subrings and Subdomains of Euclidean Domains may not be Euclidean Domain (eg $Z[x] \subset Q[x]$)
■ Subrings and Subdomains of Fields maynot be fields (eg $Z \subset Q$)

Some examples

■ Euclidean domain but not a field :
$\mathbb{Z}$ with $\text{Norm}(a) = |a|$, $\mathbb{Q}[x]$ with $\text{Norm}(p(x)) = \text{degree}(p(x))$, generally any $F[x]$ for F field with $\text{Norm}(p(x)) = \text{degree}(p(x))$, $\mathbb{Q}[\sqrt{D}]$ with Norm $a + b\sqrt{D} = |a^2 - Db^2|$, $\mathbb{Z}[i]$ with Norm$a + b\sqrt{D} = a^2 + b^2$, etc
■ P.I.D but not a Euclidean domain :
$\mathbb{Z}[(1 + \sqrt{-19})/2]$
■ U.F.D but not a P.I.D :
$\mathbb{Z}[x]$ (as $(x, 2)$ is not principle)
■ Integral domain but not a U.F.D :
$\mathbb{Z}[\sqrt{-5}]$ (as $6 = 2.3 = (1 + \sqrt{-5})(1 - \sqrt{-5})$.),
■ Commutative ring with unity but not a integral domain:
Ring of functions on $\mathbb{R}$ (identity is the identity function, has zero divisors)

8 Quotient rings and its properties

■ if R is commutative ring then R/I is also a commutative ring (converse may not be true).
■ if R is commutative and M/I is an ideal in R/I iff M is an ideal containing I in R (restatement of 4^{th} isomorphism theorem).

8.1 Chinese Remainder Theorem (c.r.t)

proper ideals I, J of R ring are comaximal ideals if $I + J = R$

c.r.t

let $A_1, A_2, .., A_k$ be ideals in R then the map $R \to R/A_1 \times R/A_2 \times .. \times R/A_k$ defined by $r \to (r + A_1, r + A_2, .., r + A_k)$ is a ring homomorphism with kernel $A_1 \cap A_2 \cap .. \cap A_k$. and if for each $i, j \in \{1, 2, .., k\}$ with $i \neq j$ A_i, A_j are comaximal then this map is surjective and $A_1 \cap A_2 \cap .. \cap A_k = A_1 A_2 .. A_k$. so $R/A_1 \cap A_2 \cap .. \cap A_k \cong$
$R/A_1 A_2 .. A_k \cong R/A_1 \times R/A_2 \times .. \times R/A_k$.

Consequences of c.r.t

■ if $n = p_1^{a_1} p_2^{a_2} .. p_k^{a_k} \in \mathbb{Z}$ where $p_i's$ are prime in $\mathbb{Z}$, $a_i \in \mathbb{Z}^+$ then

$$\mathbb{Z}/n\mathbb{Z} \cong$$
$$(\mathbb{Z}/p_1^{a_1}\mathbb{Z}) \times (\mathbb{Z}/p_2^{a_2}\mathbb{Z}) \times .. \times (\mathbb{Z}/p_k^{a_k}\mathbb{Z}).$$
$$(\mathbb{Z}/n\mathbb{Z})^* \cong$$
$$(\mathbb{Z}/p_1^{a_1}\mathbb{Z})^* \times (\mathbb{Z}/p_2^{a_2}\mathbb{Z})^* \times .. \times (\mathbb{Z}/p_k^{a_k}\mathbb{Z})^*.$$

■ Chinese Remainder problem :
if $n_1, .., n_k$ are integers which are relatively prime i.e. $(n_i, n_j) = 1$ for $i \neq q$ and $a_1, .., a_k \in \mathbb{Z}$ then there is a solution to simultaneous congruences

$$x \equiv a_1 \pmod{n_1},$$
$$x \equiv a_2 \pmod{n_2}$$
$$\vdots$$
$$x \equiv a_k \pmod{n_k}.$$

${}_s|_t\, x \in \mathbb{Z}$ and is unique $\bmod n = n_1 n_2 .. n_k$ the solution given by :
let $n = n_1 n_2 .. n_k$, $n_i' = n/n_i$ and t_i be the inverse of n_i' $(\bmod n_i)$ then

$$x = a_1 t_1 n_1' + a_2 t_2 n_2' + .. + a_k t_k n_k' \pmod{n}$$

9 Quadratic Field and Quadratic Integer Ring

if $D \in \mathbb{Q}$ is such that $\sqrt{D} \notin \mathbb{Q}$ i.e. D is not a perfect square in $\mathbb{Q}$ then

$$\mathbb{Q}[\sqrt{D}] = \{a + b\sqrt{D} | a, b \in \mathbb{Q}\}$$

forms a Field called Quadratic Field. (more precisely a subfield of $\mathbb{C}$)($(a + b\sqrt{D})^{-1} = a - b\sqrt{D}/(a^2 - Db^2)$ this is possible as $a^2 - Db^2 \neq 0$ if any one of $a, b \neq 0$ as D is not a perfect square in $\mathbb{Q}$.)

if $D \in \mathbb{Q}$ and D' is the square free part of D i.e. $D = kD'$ no square divides D' and $k = b^2$ for some $b \in \mathbb{Q}$. then $\mathbb{Q}[\sqrt{D}] = \mathbb{Q}[\sqrt{D'}]$.

If D is square free in $\mathbb{Z}$ then

$$\mathbb{Z}[\sqrt{D}] = \{a + b\sqrt{D} | a, b \in \mathbb{Z}\}$$

forms a ring called Quadratic integer ring more precisely a subring of $\mathbb{Q}[\sqrt{D}]$.

if D square free in $\mathbb{Z}$ and $D \equiv 1 \pmod 4$ then

$$\mathbb{Z}[\sqrt{D}] \subset \mathbb{Z}[\frac{1 + \sqrt{D}}{2}] \subset \mathbb{Q}[\sqrt{D}]$$

i.e. $\mathbb{Z}[(1 + \sqrt{D})/2]$ is a slightly larger subring in $\mathbb{Q}[\sqrt{D}]$.

Define field norm $N(a + b\sqrt{D}) = a^2 - Db^2$ in $\mathbb{Z}[\sqrt{D}]$ clearly $N(\alpha\beta) = N(\alpha)N(\beta)$ and $N(\alpha) \in \mathbb{Z}$ only. (Generally **norm** is taken to be $|a^2 - Db^2|$ but field norm maps $\mathbb{Q}[\sqrt{D}] \to \mathbb{Q}$ which may be negative here it is restricted to $\mathbb{Z}[\sqrt{D}] \to \mathbb{Z}$).

thus $\alpha = a + b\sqrt{D}$ is a unit in $\mathbb{Z}[\sqrt{D}]$ iff $N(\alpha) = \pm 1$ (units in $\mathbb{Z}$)
iff $a^2 - Db^2 \in \{\pm 1\}$.

for $D \equiv 1 \pmod 4$, $\mathbb{Z}[\frac{1+\sqrt{D}}{2}]$, $w = \frac{1+\sqrt{D}}{2}$ and $\overline{w} = \frac{1-\sqrt{D}}{2}$ define the field norm as $N(a + bw) = (a + bw)(a + b\overline{w}) = a^2 + ab + \frac{1-D}{4}b^2$ same rule of units follow : $a + bw$ is unit iff $a^2 + ab + \frac{1-D}{4}b^2 = \pm 1$.

using the above defined field norm $N(a + b\sqrt{D}) = a^2 - Db^2$ or the other general **norm** we can use this norms property $N(ab) =$

$N(a)N(b)$ to check for irreducibility, reducibility and prime nature of an element in $Z[\sqrt{D}]$ like if $N(a) = \pm p$ for a prime p then a is irreducible in $\mathbb{Z}[\sqrt{D}]$

■ from this property we get if D is square free and $a, b \in Z[\sqrt{D}]$ are such that ab is a unit in $Z[\sqrt{D}]$ then both a and b are units in $Z[\sqrt{D}]$

■ $\mathbb{Z}[\sqrt{-D}]$ for square free $D > 3$ is not an U.F.D. (as $2, \sqrt{-n}, 1+\sqrt{-n}$ are irreducibles but if n is even then $(\sqrt{-n})^2 = -2n/2$ and $\sqrt{-n}|-n$ but $\sqrt{-n} \not| 2$ and $\sqrt{-n} \not| n/2$ thus $\sqrt{-n}$ is non-prime irreducible or if n is odd then $2|1+n$ and $1+n = (1+\sqrt{-n})(1-\sqrt{-n})$ thus $1+\sqrt{-n}|1+n = 2(1+n)/2$ but $1+\sqrt{-n} \not| 2$ and $\not|(1+n)/2$ thus $1+\sqrt{-n}$ is a non-prime irreducible)

Gaussian integer ring

■ Gaussian integer ring $\mathbb{Z}[\sqrt{-1}] = \mathbb{Z}[i]$ is an U.F.D more precisely a Euclidean domain.

■ Fermat's Theorem on sums of squares : prime p is the sum of two integer squares i.e. $p = a^2 + b^2$ for $a, b \in \mathbb{Z}$ iff $p = 2$ or $p = 1 \pmod 4$. Except for interchanging a and h or changing the signs of a and h, the representation of p as a sum of two squares is unique.

■ from above properties nd multiplicity of the norm we have Irreducible of $\mathbb{Z}[i]$ are of the form:

■ $1+i$ (which has norm 2)

■ the primes $p \in \mathbb{Z}$with $p \equiv 3 \pmod 4$(which have norm p^2)

■ $a+bi, a-bi$ the distinct irreducible factors of $p = a^2+b^2 = (a+bi)(a-bi)$ for the primes $p \in \mathbb{Z}$ with $p \equiv 1 \pmod 4$ (both of which have norm p).

10 Polynomial Rings

for any commutative ring R with identity we define R[x] as the ring of polynomial a set containing elements of type : $a_n x^n + a_{n-1}x^{n-1} + .. + a_1 x + a_0$ for $a_i \in R$, $n \geqslant 0$ and x a variable (simply denoted) is called the polynomial of x with coefficients in R, where n is degree a_n if $\neq 0$ is the leading coefficient.
This set is a ring with addition defined component wise and multiplication is done by defining $(ax^j)(bx^i) = abx^{i+j}$ distributing it over $+$.
i.e. if $a(x) = \sum_{i=1}^{n} a_i x^i$ and $b(x) = \sum_{j=1}^{m} b_j x^j$ then $a(x) + b(x) = \sum_{i=1}^{\max(m,n)} (a_i + b_i)x^i$ $a(x)b(x) = \sum_{i=1}^{m+n} (\sum_{j=0}^{i} a_i b_{i-j})x^i$
(we can write any number of terms in a given polynomial for these operations by assuming coefficients are 0)

if R is an **Integral domain** then

■ degree $a(x)b(x)$= degree $a(x)$ +degree $b(x)$

■ the only units of R[x] are the units of R

■ R[x] is an Integral domain.

(use fact that when polynomial with non zero leading coeffs are multiplied give a non zero leading coeff .)

$p(x)$ is zero divisor in R[x] iff $bp(x) = 0$ for some $b \in R$ (use fact that $g(x)p(x) = 0_{s|t}\, g(x)$ has minimal degree then the leading coeff of $g(x)p(x) = g_m p_n = 0$ so we have $p_n g(x)p(x)$ is also 0 but degree $p_n g(x) <$ degree $g(x)$ thus only possibility of $g(x) =$ constant $\in R$ is left out.)

if R is commutative $p(x) = a_n x^n + a_{n-1}x^{n-1} + .. + a_1 x + a_0$ is an element of R[x] then

■ $p(x)$ is nilpotent iff $a_n, a_{n-1}, .., a_1, a_0$ are nilpotent in R

(use induction: if $n = 0$ then clearly true, if $n = 1$ then $p(x) = a_1 x + a_0$ then any $p^k(x)$ has leading coeff a_1^k and constant term a_0^k so $p(x)^m = 0$ iff $a_1^m = 0, a_0^m =$

0 now for any n, $p(x) = x(a_n x^{n-1} + a_{n-1}x^{n-2} + .. + a_1) + a_0 = xq(x) + a_0$ by induction hypothesis $q^m(x) = 0$, if $a_0^n = 0$ let $k = max(n, m)$ now $(xq(x) + a_n)^{2k} = \sum_{i=0}^{2k} \binom{2k}{i} (xq(x))^i a_0^{2k-i}$ where the power of atleast one term $\geqslant k$ so $(xq(x) + a_n)^{2k} = 0$)

■ $p(x)$ is a unit in $R[x]$ iff a_0 is a unit in R and $a_n, a_{n-1}, .., a_1$ are nilpotent in R.

(for one way use $p(x) = xq(x) + a_0$ = nilpotent + unit = unit , for other way : if $p(x) = xq(x) + a_0$ is a unit then $p^{-1}(x)(xq(x) + a_0) = 1$ so $xq(x)p^{-1}(x) = 1 - a_0 p^{-1}(x)$ equating for constant coefficient in $p^{-1}(x)$ we get $1 - a_0 b_0 = 0$ so a_0 is unit and we can transform $p(x)$ to $f(x) = b_0 p(x) = xg(x) + 1$ now $f^{-1}(x)$ exist and satisfies equation $f^{-1}(x) = 1 - xg(x)f^{-1}(x)$ from this equation and using recursive arguments we get $f^{-1}(x) = 1 - g(x)x + g^2(x)x^2 + .. + (-1)^n g^n(x)x^n + ..$ and goes on, now as degree of $f^{-1}(x)$ is finite we get $g^m(x) = 0$ for some $m \in \mathbb{Z}^+$ thus $g(x)$ is nilpotent and as $g(x) = b_0 q(x)$ we get $q(x)$ is nilpotent.)

Ideals in Polynomial rings

if I is an ideal in R then

■ $(I) = I[x]$ is an ideal in $R[x]$ and

■ $R[x]/(I) \cong (R/I)[x]$.

■ if I is prime ideal in R then $(I) = I[x]$ is prime ideal of $R[x]$.

■ if I is a maximal ideal in R then (I, x) i.e. ideal generated by I, x is maximal in $R[x]$

(note : if I is maximal in R then $I[x]$ may not be maximal in $R[x]$ for eg: $2\mathbb{Z}$ is maximal in R but $2\mathbb{Z}[x]$ is not maximal in $\mathbb{Z}[x]$ as $2\mathbb{Z}[x] \subset (x, 2) \subset \mathbb{Z}[x]$)

■ in $R[x]/(f(x))$ if $g(x)$ is a non-trivial factor ($\neq 0$ or $f(x)$) of $f(x)$ then $(\overline{g(x)}) = (g(x) + (f(x)))$ is a proper ideal of $R[x]/(f(x))$.

Characterisation of Poly rings

■ if F is a field then $F[x]$ is a Euclidean Domain (with norm = degree of the polynomial.) Hence $F[x]$ is P.I.D. and U.F.D.

■ if R is a commutative ring with identity then (x) is a prime ideal in $R[x]$ iff R is integral domain and (x) is maximal ideal in $R[x]$ iff R is a field. (use $R[x]/(x) \cong R$)

■ if R is commutative ring such that $R[x]$ is a P.I.D then R is a field.

Irreducibility

a polynomial $f(x)$ is irreducible in $R[x]$ if whenever $f(x)$ can be expressed as $f(x) = g(x)h(x)$ then either $g(x)$ or $h(x)$ is a unit in $R[x]$ (note: we don't say any thing about the degree of $g(x)$ or $h(x)$ some times they can be equal to $f(x)$ also for eg: $2x^2 + 4 = 2(x^2 + 2)$ in $\mathbb{Z}[x]$ this becomes reducible but is irreducible in $\mathbb{Q}[x]$)

Primitive Polynomial

a non-zero polynomial $a_n x^n + a_{n-1}x^{n-1} + .. + a_0$ is called a **primitive polynomial** if gcd of $a_i's$ are 1

■ product of two primitive polynomials is a primitive polynomial.

■ A polynomial in $\mathbb{Z}[x]$ is irreducible then it is primitive.

for $f(x) \in F[x]$ a polynomial ring generated by field F :

■ $\langle f(x) \rangle$ is a maximal ideal in $F[x]$ i.e. $F[x]/(f(x))$ is a field iff $f(x)$ is irreducible in $F[x]$.

■ $f(x)$ is irreducible in F then $\{a + \langle f(x) \rangle \ _s|_t \ a \in F\}$ is subfield in $F[x]/(f(x))$ that is isomorphic to F.

■ if degree $f(x) = n \geqslant 1$ and if bars on top denote the passage to $F[x]/(f(x))$ then for each $\overline{g(x)}$ there is a unique $g_0(x) \in F[x]$ with degree $< n \ _s|_t \ \overline{g(x)} = \overline{g_0(x)}$ i.e. $F[x]/(f(x))$ is n dimensional vector space with basis $\overline{1}, \overline{x}.. \overline{x^{n-1}}$ over F.

■ if F is a finite field of order q, degree $f(x) = n \geqslant 1$ then $F[x]/(f(x))$ has q^n elements.

if F is a finite field (or an infinite one) then there are infinitely many primes in $F[x]$.

Gauss Lemma

Let R be a U.F.D. , F its field of fractions and if $P(x)$ is reducible in $F[x]$ then $P(x)$ is reducible in $R[x]$

sort of converse of Gauss Lemma

Let R be a U.F.D. , F its field of fractions if $p(x) \in R[x]_{s}|_{t}$ and g.c.d of its coefficients is 1 i.e. $p(x)$ is primitive polynomial then $p(x)$ is irreducible in $R[x]$ iff $p(x)$ is irreducible in $F[x]$, in particular if $p(x)$ is monic polynomial irreducible in $R[x]$ then $p(x)$ is irreducible in $F[x]$.

from above point we get : Let R be a integral domain, F its field of fractions if $p(x) \in R[x]$ is a monic polynomial reducible in $F[x]_{s}|_{t}$ $p(x) = a(x)b(x)$ where $a(x), b(x)$ are monic and if $a(x) \notin R[x]$ then $R[x]$ **is not a U.F.D.**

R is a U.F.D. iff $R[x]$ is a U.F.D.

if $f(x) \in F[x]$ has $a_1, a_2, .. a_k$ as roots in F field then $f(x)$ has $(x-a_1)(x-a_2)..(x-a_k)$ as factors, in particular **a polynomial of degree n over F has at most n roots in F.**

every finite subgroup of multiplicative group of a field is cyclic , in particular $F^* = F - \{0\}$ for F field is a cyclic group (multiplicative).

(use fundamental theorem of finite abelian groups and last point to show subgroup is $\cong \mathbb{Z}/n_1\mathbb{Z} \times \mathbb{Z}/n_2\mathbb{Z} \times .. \mathbb{Z}/n_k\mathbb{Z}$ so has more than n_k roots for $x^{n_k} - 1$ if $k \geqslant 2$ as for each $d \| G|$ cyclic group there are exactly d elements of order dividing d in G, so $k = 1$ i.e. subgroup is $\cong \mathbb{Z}/n_1\mathbb{Z}$ only.)

eg : now as $\mathbb{Z}/p\mathbb{Z}$ is a field for prime p we get $(\mathbb{Z}/p\mathbb{Z})^*$ is cyclic group of order $p-1$ (multiplicative).

$\mathbb{Z}/p^\alpha\mathbb{Z}$ is cyclic group of order $p^{\alpha-1}(p-1)$ for all odd primes $p, \alpha \geqslant 1$

(use $(1+p)^{p^{n-1}} \equiv 1 \pmod{p^n}$ but $(1+p)^{p^{n-2}} \not\equiv 1 \pmod{p^n}$ so Sylow p subgroup is cyclic and homomorphism $\phi : (\mathbb{Z}/p^\alpha\mathbb{Z})^* \to (\mathbb{Z}/p\mathbb{Z})^*$ by $\phi(a) = a \pmod{p}$ then ϕ is serjective so any $p \neq q | p-1$ is Sylow q subgroup is mapped isomorphically to subgroup of $(\mathbb{Z}/p\mathbb{Z})^*$ which is cyclic so all Sylow subgroups of $(\mathbb{Z}/p^\alpha\mathbb{Z})^*$ are cyclic so by direct product and order deduction we have $(\mathbb{Z}/p^\alpha\mathbb{Z})^*$ is cyclic.)

$g(x) \in F[x]$ for a field F is such that $g(x) = f_1(x)^{n_1} f_2(x)^{n_2} .. f_k(x)^{n_k}$ be its factorization where $f_i(x)$ are distinct primes then

$$F[x]/(g(x)) \cong F[x]/(f_1(x)^{n_1}) \times F[x]/(f_2(x)^{n_2}) \times .. \times F[x]/(f_k(x)^{n_k}).$$

(use Chinese remainder theorem.)

If R is commutative, $f(x)$ is a monic polynomial of degree $n \geqslant 1$ and if bar denotes the passage to quotient ring $R[x]/(f(x))$ then

■ every element of $R[x]/(f(x))$ is of form $\overline{p(x)}$ for some polynomial $p(x) \in R[x]$ of degree less than n i.e.

$$R[x]/(f(x)) = \{\overline{a_0} + \overline{a_1 x} + .. + \overline{a_{n-1}x^{n-1}} | a_0, a_1, .., a_{n-1} \in R\}.$$

■ if $p(x)$ and $q(x)$ are distinct polynomial of $R[x]$ of degree less than n then $\overline{p(x)} \neq \overline{q(x)}$ in $R[x]/(f(x))$.

■ if $f(x) = a(x)b(x)$ for $a(x)$ and $b(x)$ degree less than n in $R[x]$ then $\overline{a(x)}, \overline{b(x)}$ are zero divisors in $R[x]/(f(x))$ i.e. if non-unit factors of $f(x)$ (of degree less than that of $f(x)$) in $R[x]$ are zero divisors in $R[x]/(f(x))$.

■ if $f(x) = x^n - a$ for some nilpotent element $a \in R$ then $\overline{x}$ is nilpotent in $R[x]/(f(x))$ (use: $\overline{x^n} = \overline{a}$ in $R[x]/(f(x))$).

■ for a prime p if $R = \mathbb{F}_p$ (field with p elements) and $f(x) = x^p - a$ for some $a \in R$ then $\overline{x - a}$ is nilpotent in $R[x]/(f(x))$.

10.1 Irreducibility Criterion and properties

■ for F is a field and $p(x) \in F[x]$ has a factor of degree one iff $p(x)$ has a root in F.

■ immediately from above point we get polynomial of degree two or three in $F[x]$ for F field is reducible iff it has roots in F.

Rational root Theorem

let $p(x) = a_n x^n + a_{n-1}x^{n-1} + .. + a_o$ be a polynomial with integer coefficients, if $r/s \in \mathbb{Q}$ in lowest form (i.e. $(r,s) = 1$) is a root of $p(x)$ then $r|a_o$ and $s|a_n$, in particular if $p(x)$ is monic with integer coefficients and $p(d) \neq o$ for all integer dividing the constant term of $p(x)$ then $p(x)$ has no root in $\mathbb{Q}$.

if I is a prime ideal of Integral Domain R, $p(x)$ a non constant monic polynomial in $R[x]_{s|t}$ its image in $(R/I)[x]$ cannot be factored into two polynomials of smaller degree in $(R/I)[x]$ then $p(x)$ is irreducible in $R[x]$. From this we get :

Mod p irreducibility test

For $f(x) \in \mathbb{Z}[x]$ with $deg(f(x)) \geqslant 1$, $\overline{f(x)} \in \mathbb{Z}_p[x]$ obtained from reducing coefficients of $f(x)$ modulo p for a prime $p \in \mathbb{Z}$ and if $\overline{f(x)}$ is irreducible in $\mathbb{Z}_p[x]$ and $deg(f(x)) = deg(\overline{f(x)})$ then $f(x)$ is irreducible in Q (**converse is not true** : i.e. if $\overline{f(x)}$ is reducible in $\mathbb{Z}_p$ then it may not be reducible in $\mathbb{Z}[x]$)

Eisenstein's Criterion

for P a prime ideal of integral domain R, $f(x) = x^n + a_{n-1}x^{n-1} + .. + a_1x + a_o \in R[x]_{s|t}$ $a_{n-1}, .. a_1, a_o$ are elements of P and a_o is not an element of P^2 then $f(x)$ is irreducible in $R[x]$. For eg :

- $f(x) = x^n + a_{n-1}x^{n-1} + .. + a_o \in \mathbb{Z}[x]_{s|t}$ $p|a_i$ but if $p^2 \not| a_o$ then $f(x)$ is irreducible in $\mathbb{Z}[x]$ which makes it irreducible in $\mathbb{Q}[x]$.
- $f(x) = a_n x^n + a_{n-1}x^{n-1} + .. + a_o \in \mathbb{Z}[x]_{s|t}$ $p|a_i$ **for** $o \leqslant i < n$ **but if** $p \not| a_n$, $p^2 \not| a_o$ **then** $f(x)$ **is irreducible in** $\mathbb{Z}[x]$ **which makes it irreducible in** $\mathbb{Q}[x]$**.** (write $D = \{o, a_n\}$ then the fraction ring $D^{-1}\mathbb{Z}$ has $p\mathbb{Z}$ as prime ideal and $g(x) = f(x)/a_n = x^n + \frac{1}{a_n}(a_{n-1}x^{n-1} + .. + a_o)$ in $D^{-1}\mathbb{Z}$ which satisfies original Eisenstiens criterion.)

for any field F and $o \neq a \in F$ then

- $af(x)$ is irreducible over F implies $f(x)$ is irreducible in F
- $f(ax)$ is irreducible over F implies $f(x)$ is irreducible in F
- $f(x + a)$ is irreducible over F implies $f(x)$ is irreducible in F

Cyclotomic polynomial : $\Phi_p(x) = \frac{x^p - 1}{x - 1}$ for a prime p is irreducible over $\mathbb{Q}$ (use $\Phi_p(x + 1)$ is irreducible by Eisenstien's criterion.)

10.2 Properties of usual polynomial rings

polynomial ring in $\mathbb{Z}$

every prime ideal of $\mathbb{Z}[x]$ is of form :

- (o)
- $(q(x))$ for $q(x)$ an irreducible polynomial in $\mathbb{Z}[x]$.
- (p) for p a prime in $\mathbb{Z}$.
- $(p, q(x))$ for p prime in $\mathbb{Z}$, $q(x)$ such that for $p(x) \equiv q(x) \pmod p$, $p(x)$ an irreducible polynomial in $\mathbb{Z}/p\mathbb{Z}[x] \cong \mathbb{Z}_p[x]$.

more over every maximal ideal of $\mathbb{Z}[p]$ is of the form $(p, q(x))$ as above also

$$Z[x]/(p, q(x)) \cong Z_p[x]/(q(x)) \cong \text{ some finite field of char } p.$$

10.3 Multivariable Polynomial Rings

For any ring R define inductively the polynomial ring in variables $x_1, x_2, .., x_n$ with coefficients in R denoted by $R[x_1, x_2, .., x_n]$ by

$$R[x_1, x_2, .., x_n] = R[x_1, x_2, .., x_{n-1}][x]$$

i.e. its elements are finite sum of non zero monomial terms like
$ax_1^{d_1}x_2^{d_2}..x_n^{d_n}$ for $a \in R, d_i's \geqslant o$.
where a monic term $x_1^{d_1}x_2^{d_2}..x_n^{d_n}$ is called **monomial**

d_i is degree of x_i, the sum $d = d_1 + d_2 + .. + d_n$ is called the degree of the term and the ordered n-tuple $(d_1, d_2, .. d_n)$ is called **multidegree** of the term.

11 Ring of functions and evaluation maps

If A is a ring and X a non empty subset of A then R the set of all functions from $X \to A$ forms a Ring with usual point wise addition and multiplication of functions (note multiplication is not composition of functions).

■ This ring R is commutative iff A is commutative.

■ R has identity iff A has Identity (if so then constant function mapping to identity is identity in R.)

familiar examples and their properties:

■ Consider the set of all functions from $\mathbb{R} \to \mathbb{R}$ with compact support (i.e. $f(x) \neq 0$ only in a compact set of $\mathbb{R}$) then this set forms a commutative Ring with no identity.

■ if R is ring of all functions from $[0,1] \to \mathbb{R}$ then units in R are functions that are not zero at any point, and if $f \in R$ is not a unit and not zero then it is a zero divisor as $g(x)$ defined by $g(x) = 1$ at point where $f(x) = 0$ and $g(x) = 0$ at points $f(x) \neq 0$ are such that $f(x)g(x) \equiv 0$.

■ Similarly if R is ring of **continuous** functions from $[0,1] \to \mathbb{R}$ then units are same as presiding point but not true for zero divisors, functions with countably many zeros are neither units nor zero divisors and R also has zero divisors (like continuous functions with zero on a closed interval in $[0,1]$.)

If R is a ring of functions from a non-empty set X to a **field** F then R contains no nonzero nilpotent elements.

Evaluation map

for X non-empty subset of ring A let R be ring of functions from $X \to A$ for each $c \in X$ fixed the evaluation map $E_c : R \to A$ defined by $E_c(f) = f(c)$ is a surjective ring homomorphism (if $a \in A$ then constant function $f(x) = a$ is in preimage of a i.e. $f \in E_c^{-1}(a)$) with a kernel $M_c = \{f \in R | f(c) = 0\}$ thus $R/ker E_c \cong A$.

■ thus the set $M_c = \{f \in R | f(c) = 0, c \in A\}$ is an ideal in R.

Examples and their properties :

■ consider R ring of all functions from $[0,1] \to \mathbb{R}$ then for $c \in [0,1]$ let M_c be the kernel of evaluation at c then M_c is generated by $g(x)$ defined by 0 at c and 1 elsewhere, (as $f.g = f$ for all $f \in M_c$) thus M_c is principle ideal in R. (more precisely M_c is generated by any function with zero at c and nonzero elsewhere.)

■ now as $R/M_c \cong \mathbb{R}$ so we get M_c is a maximal ideal in R as $\mathbb{R}$ is a field. i.e. any ring of functions to a field has kernel of evaluations as maximal ideals (thus prime ideals also).

■ if R' is ring of **continuous** functions from $[0,1] \to \mathbb{R}$ then

■ M is any maximal ideal in R' then $M = M_c$ for some $c \in [0,1]$ i.e. M is maximal ideal in R' iff $M = M_c$.

■ if $a \neq b$ in $[0,1]$ then $M_a \neq M_b$ in R'

■ M_c is not principle and not even finitely generated.

(note: compactness of $[0,1]$ plays a major role in proofs of preceding points, if this is taken out then we get the following exception)

■ Consider R" ring of continuous functions from $\mathbb{R} \to \mathbb{R}$ here

■ I a collection of continuous functions with compact support forms an Ideal that is not prime in R".
■ if M is a maximal ideal of R" containing I then $M \neq M_c$ for any $c \in \mathbb{R}$.

■ now if $I_{a,b}$ is a subset of R′ ring of continuous functions from $[0,1] \to \mathbb{R}$ with such that $I = \{f \in R | f(a) = f(b) = 0$ for some $a, b \in [0,1]$ and $a \neq b\}$ then I is not a prime ideal.

12 Matrix Rings

for any non trivial ($\neq \{0\}$) ring R let $M_n(R) = [a_{ij}]$ be set of all $n \times n$ matrices with entries a_{ij} from R with component wise addition and matrix multiplication this $M_n(R)$ forms ring with properties as follows :
■ $M_n(R)$a non commutative ring whenever $R \neq \{0\}$ and $n \geqslant 2$
■ $M_n(R)$ contains a zero divisor whenever $n \geqslant 2$
■ The set of scalar matrices ($a_{ii} = a \forall i$ $a_{ij} = 0$ if $i \neq j$.) in $M_n(R)$ forms a subring isomorphic to R.
■ center of $M_n(R)$ is the set of scalar matrices.
■ if S is a subring of R then $M_n(S)$ is subring of $M_n(R)$

if $M_n(R)$ for $n \geqslant 2$ is a matrix ring of R a commutative ring with identity then consider the set C_j ($j \in \{1, 2, .., n\}$) of matrices with arbitrary entries in j^{th} column and 0 in all other columns then
■ C_j is a group under matrix addition
■ C_j is closed only under left multiplication.
■ And $TC_j = C_j$ for any $T \in M_n(R)$ thus C_j is left ideal that is not a right ideal in $M_n(R)$ Similarly one can construct R_j with 0 entries in rows except j^{th} one, then R_j is a right ideal that is not a left ideal in R.

All rings with unity of order p and p^2 are commutative the ring $\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}_{s|t} a, b, c \in F_p \text{ field of order } p \right\}$ is a non-commutative ring with unity of order p^3

13 Group Rings

for a commutative ring R with identity $1 \neq 0$ and $G = \{g_1, g_2, .., g_n\}$ be a finite group with group operations written multiplicatively then
RG a group ring is defined to be set of formal sums $a_1g_1 + a_2g_2 + .. + a_ng_n$ for $a_i \in R$ if g_1 is identity then a_1g_1 is simply written as a_1
with addition defined component wise and multiplication defined by $(ag_j)(bg_i) = abg_k$ for $g_ig_j = g_k$ in G and obeying distribution w.r.t. + i.e. if $\alpha = \sum_{i=1}^{n} a_ig_i, \beta = \sum_{j=1}^{n} b_jg_j$ then $\alpha + \beta = \sum_{i=1}^{n}(a_i + b_i)g_i$. and
$\alpha\beta = \sum_{k=i}^{n}(\sum_{g_jg_i=g_k} a_ia_j)g_k$. then these operations make RG a ring with following properties

G $\subset$ RG is subgroup of units of RG (note $1g_1 = g_1 \in RG$)

if $|G| > 1$ then RG has a zero divisor (if $g^m = 1$ in G then $(1-g)(1+g+..+g^{m-1}) = 1 - g^m = 0$)

if S is a subring of R then SG is subring of RG.

If $\mathcal{K} = \{k_1, k_2, .., k_n\}$ is one of the conjugacy classes of group G then
$K = k_1 + k_2 + .. + k_n$ is in center of RG
(as $g^{-1}Kg = K \forall g \in G \implies agK = Kag$)

13 References

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