Differential Geometry

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yn37git.github.io/blog/2025/Short-Notes

Contents

1 Introduction

1.1 Definitions

• **Euclidean space**: $\mathbb{R}^n$ metric space with norm: $||x|| = \sqrt{(x_1^2 + x_2^2 + x_3^2 \cdots + x_n^2)}$

now for $\mathbb{R}^3$ Euclidean space:

• **Scalar field** V assigns each point in $\mathbb{R}^3$ to a corresponding scalar

• **Vector field** $V : \mathbb{R}^3 \to \mathbb{R}^3$ assigns each point in $\mathbb{R}^3$ to a corresponding vector eg: natural frame fields: $U_1 = (1,0,0)_p, U_2 = (0,1,0)_p, U_3 = (0,0,1)_p$ Then every vector field $v(p) = \sum_{i=1}^3 v_i(p)U_i$ where v_i is scalar field

• **Tangent vector** V_p is a vector in V direction at point p i.e. $(v_1, v_2, v_3)_{(p_1,p_2,p_3)}$

1.2 Basics

• **Directional derivative** $v_p[f]$: for scalar field f directional derivative is the rate of its change at p in v direction so:

$$v_p[f] = \frac{d}{dt}(f(p+tv))|_{t=0}$$

here $p + tv$ is the line at p in v direction so at $t = 0$ line is at p hence the definition makes sense

• now if v_p is chosen as the vector from vector field V i.e. $V(p)_p$ then direction derivative in a way give change of scalar field with respect to (w.r.t) vector field at p in a sense it is like operating ***vector field on scalar field***

• if $v_p = (v_1, v_2, v_3)_{(p_1,p_2,p_3)}$ Then

$$v_p[f] = \sum_{i=1}^{3} v_i \frac{d}{dx_i}(f)(p)$$

• clearly directional derivative is linear and $v_p[fg] = v_p[f]g + fv_p[g]$ (Libnizian rule)

• **Curve** a : open interval of $\mathbb{R} \to \mathbb{R}^3$ and a is differentiable i.e. if $a(t) =$

$(a_1(t), a_2(t), a_3(t))$ then each $a_i(t)$ is differentiable real function
e.g.. straight line $a(t) = p + tV$

• $a'(t) = a'(t)_{a(t)}$ i.e a' is a tangent vector at a point in direction of rate change of a

• Re-parametrisation if I , J are open intervals in $\mathbb{R}$, $a : I \rightarrow \mathbb{R}^3$ is curve and $h : J \rightarrow I$ is a differentiable function then $b(s) = a(h(s))$ is a curve same as a but different velocity i.e.

$$b'(s) = \frac{dh}{ds}a'(h)$$

• Lemma $a'(t)[f] = \frac{d}{dt}(f(a))(t)$

• a curve a is **regular** if $a' \neq 0$

2 Forms

2.1 1-forms

• **1-form ϕ**: function from set of all tangent vector to $\mathbb{R}$ that is linear at each point i.e at p $\phi = \phi_p$ then $\phi_p(aV + bW) = a\phi_p(V) + b\phi_p(W)$

• so if $v_p = V(p)_p$ then 1-form acts on an vector field also converting it to a scalar in a way *vector field to scalar field*

• df : for a differentiable function define 1-form $df(v_p) = v_p[f]$

• now $dx_i(v_p) = v_i \; for \; i = 1, 2, 3$

• as 1-forms are linear at a point $\implies$ if $\psi(v_p) = f_1 dx_1 + f_2 dx_2 + f_3 dx_3(v_p) = f_1(p)dx_1(v) + f_2(p)dx_2(v) + f_3(p)dx_3(v)$ then ψ is a 1-form

• every 1-form $\phi = \sum f_i dx_i$ where $f_i = \phi(U_i)$

• so $df(v_p) = \sum \frac{\partial f}{\partial x_i}(p)dx_i(v)d$ Thus $df \equiv \sum \frac{\partial f}{\partial x_i}dx_i$

2.2 Differential forms

• if T_p is the vector space containing all tangent vectors at point p then 1-forms is a linear functional on this space

• going with the flow of 1-form we define other forms as linear in $T_p \times T_p$,$T_p \times T_p \times T_p$ etc.

• **Wedge product** : it is a operation on two 1-forms defined by $dx_i \wedge dx_j(v) = dx_i(v)dx_j(v)$ and $dx_i \wedge dx_j = -dx_j \wedge dx_i$

• now other forms can be obtained by this wedge product i.e. 1-form $\wedge$ 1-form gives 2-form,
1-form $\wedge$ 2-form gives 3-form, etc

• so 1-form = $fdx + gdy + hdz$
2-form = $fdxdy + gdydz + hdxdz$
3-form = $fdxdydz$

• **Exterior derivative** : of 1-form ($\phi = \sum f_i dx_i$) = 2-form $d\phi = \sum df_i \wedge dx_i$ so exterior derivative can be used to convert 1-form to a 2-form, 2-form to 3-form . . . etc

• Theorem: for function f 1-forms ψ and ϕ then

1. $d(f\phi) = df \wedge \phi + fd\phi$
2. $d(\phi \wedge \psi) = d\phi \wedge \psi - \phi \wedge d\psi$

•

1. $df \leftrightarrow grad(f)$
2. if $\phi(1 - form) \leftrightarrow V$ then $d\phi \leftrightarrow curl(V)$
3. if $\eta(2 - form) \leftrightarrow V$ then $d\eta \leftrightarrow div(V)dxdydz$

3 Mapping

• **Mapping** $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $F(p) = (f_1(p), f_2(p), \ldots, f_n(p))$ then each f_i is differentiable real function

• **Tangent map** of F : $F*(v_p)$ is the initial velocity of curve $t \rightarrow F(p + tv)$ this sends tangent vectors in $\mathbb{R}^n$ to tangent vectors in $\mathbb{R}^m$

• $F*(v) = (v[f_1], v[f_2], \ldots, v[f_n])_{F(p)}$

• clearly tangent map is linear thus it is a linear transformation from from and to Tangent vector spaces

- F is regular iff $F*$ is one-one i.e. Jacobian matrix of has rank equal to domain space

4 Frame fields

- **frame**: a set of 3 unit vectors that are mutually perpendicular to each other in $\mathbb{R}^3$
- attitude matrix of a frame A: coordinate matrix of a frame (clearly it is orthogonal i.e $A.A^T = I$

4.1 Curves and Frame fields

- a curve a is said to have unit speed if $\|a'(t)\| = 1 \, \forall \, t$ in domain
- ***Theorem**: if a is a regular curve in $\mathbb{R}^3$ then there exist as reparametrisation b of a such that b is a unit speed curve (proof by inverse function theorem) now $b = a(s(t))$ which has unit length then $s(t)$ is the called arclength function of a as it converts $\|a'\|$ to one
- **Vector field on a curve** Y: (for a curve a) assigns a Tangent vector $Y(t)_{a(t)}$ for every point $a(t)$
- Y is parallel vector field to a id $\|Y(t)\| = 1 \, \forall \, t$

4.1.1 Franet fields

- if b is a unit speed curve then for b:
- $T = b'$ is called **Tangent vector field**, clearly $\|T\| = 1$ so T tells us the direction of change of b
- $T' = b''$ is called **Curvature vector field**, it measures how the curve is changing
- $N = \frac{T'}{\|T'\|}$ is called **Normal vector field**, clearly $\|N\| = 1$ so N measures the direction of change of b , clearly $\|B\| = 1$
- $B = T \times N$ is called **Binormal vector field**
- **Theorem**: for a unit curve b vector fields T, N, B form a frame at each point, this is called Frenet Frame field on b
- ***Curvature** k of a curve b at a point is $\|T'\|$ at that point, clearly there is a one-one correspondence between the curve 'turn rate' or 'bending rate' and curvature at the point
- **Torsion** τ of a curve b at a point is $-B'.N$ at that point, there is a one-one correspondence between the curve 'twist rate' or 'rotating rate' and Torsion at the point
- ***Theorem**

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

- $k = 0 \implies b$ is a straight line
- a curve a is plane curve if it lies entirely on a plane i.e. $\exists$ vectors p and q such that $((b(t) - p).q = 0 \, \forall \, t$
- Theorem: if $k > 0$, b is a plane curve iff $\tau = 0$ at every point
- Theorem: if $\tau = 0$, $k > 0$ and is constant then b is part of a circle of radius $\frac{1}{k}$

4.1.2 Arbitrary speed curves

- if $a(t)$ is a arbitrary speed curve (regular) then it can be reparametrised to unit speed curve $\overline{a}(s(t))$this concept is use for below and $v = \frac{ds}{dt}$ is speed of the curve as $b'(s) = (a(t(s)) = a'(t)\frac{dt}{ds} = 1$
- we define T, N, B, k, τ of $a(t)$ to be equivalent to that of $\overline{a}(s)$ i.e $T = \overline{T}(s), k = \overline{k}(s) \ldots$
- so now $T' = (\overline{T}(s))' = \overline{T}'(s)\frac{ds}{dt} = vT'$ and so on for others i.e. correct it by multiplying it with v
- **Theorem** same rule as above holds for franet frame also i.e

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \mathbf{v} \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

- for a reggular curve a
  1. $T = a'/\|a'\|$
  2. $k = \|a' \times a''\|/\|a'\|^3$
  3. $B = a' \times a''/\|a' \times a''\|$

4. $\tau = (a' \times a'').a'''/||a' \times a''||^2$

4.2 *Covariant derivative

• ***Covariant derivative**: of vector field W w.r.t v_p = $\nabla_v W = W'(p + tv)|_{t=0}$ i.e. it gives initial rate of change of $W(p)$ as it moves in v direction

• if $W = (w_1, w_2, w_3)$ then $\nabla_v W = \sum vv[w_i]U_i(p)$

• clealy his opeation is linear and obeys Libnizian rule

• now if $v_p = V(p)_p$ then covatiant derivative is like operating a **vector field on a vector field**

4.3 Frame fields

• **Frame fields**: Vector Fields E_1, E_2, E_3 in $\mathbb{R}^3$ constitute a frame field if $E_i.E_j = \delta_{ij}$ at each point eg: spherical frame fields, cylindrical frame fields

5 Transforms

• Isometry F: $\mathbb{R}^3 \to \mathbb{R}^3$ such that $d(F(p), F(q)) = d(p, q) \, \forall \, p, q$

• eg: Translation: $T_a(p) = p + a$ for fixed a, Rotation :$R_{xy\theta}(p_1, p_2, p_3) = (p_1 cos(\theta) - p_2 sin(\theta), p_1 sin(\theta) + p_2 cos(\theta), p_3)$

• Orthogonal Transformation C : $\mathbb{R}^3 \to \mathbb{R}^3$ such that $C(p).C(q) = p.q$ and is linear
eg: Rotation

• Lemma: if C is an orthogonal transformation then C is an isometry

• Lemma: if F is an isometry and $F(0) = 0$ then F is an orthogonal transformation

6 *Surfaces

• Coordinate patch x: $D \to \mathbb{R}^3$ (D is any open set in $\mathbb{R}^2$ that is one-one and regular (i.e. $x*$ is also one-one)

• ***Proper patch** x: a coordinate patch with $x^{-1} : x(D) \to D$ is continuous

• ***Surface** in $\mathbb{R}^3$ is a subset M such that for each point p of M there exist a proper patch in M whose image contains a neighborhood of p in M

• clearly if $x(u, v) = (u, v, f(u, v))$ where f is real differentiable function then x is a patch , this type of patch is called **Monge patch**

• A surface which is proper patch in its self is called a **Simple surface**

• ***Theorem**: $M : g(x, y, z) = c$ is a surface iff $dg \neq 0 \, \forall \, p \epsilon M$
(proof by implicit function theorem)

• patch computation: M is a surface iff M is one-one and Jacobian matrix of M has rank 2

• partial velocity functions: $x_u = \frac{\partial x}{\partial u} = (\frac{\partial x_1}{\partial u}, \frac{\partial x_2}{\partial u}, \frac{\partial x_3}{\partial u})$, $x_v = \frac{\partial x}{\partial v} = (\frac{\partial x_1}{\partial v}, \frac{\partial x_2}{\partial v}, \frac{\partial x_3}{\partial v})$ these essentially give tangent vectors in u an v directions at a point in x

• Tangent vector to a plane M v_p: if $p \epsilon M$ and v is initial velocity of some curve in M (i.e. a curve that is on the surface itself)

• ***Lemma**: if $x(u_0, v_0) = p$ and v_p is tangent vector to x iff v_p can be expressed as linear combination of $x_u(u_0, v_0)$ and $x_v(u_0, v_0)$

• Euclidean vector field Z: is a vector field defined for all points on a surface M in $\mathbb{R}^3$ and assigns $Z(p)_p$ tangent vector to p (basically a tangent vector map defined on a surface)

• Tangent vector field on M V: a euclidean vector field on M for which $V(p)_p$ is tangent to M

• Normal vector field on M N: a euclidean vector field on M for which $N(p)_p$ is orthogonal to tangent plane of M at p ($T_p(M)$)

• clearly for $M : g = c$ the gradient(g) vector field forms a normal vector field

• **Manifold*** (M,P): in n dimensions, M is a set with P being a collection of abstract patches (functions $D \to M$ that is 1-1 where D is a open set of $\mathbb{R}^2$) which satisfy:

1. The covering property : The images of patches in P cover M
2. The smooth overlay property : for any patches x, y in P functions $y^{-1}x, x^{-1}y$ are euclidean differentiable (differentiable in euclidean space) and defined are on open sets of $\mathbb{R}^n$
3. The Hausdorff property : for any $p \neq q$ in M there are disjoint patches x and y in P with $p \epsilon x$ and $q \epsilon y$

• clearly manifold generalizes the concept of surface (surface in $\mathbb{R}^3$ is just 2-D manifold: (surface point set, set of patches that cover it))

7 *Curvature

• ***Shape operator** S: for a surface M and p on it and V_p tangent to it we have $S_p(v) = -\nabla_p U$ where U is the unit normal vector field on neighbourhood of p in M
clearly as U is unit normal to tangent plane at p $\nabla_p U$ tells us how U changes in v direction i.e. how tangent plane is changing (in directions) giving us a local picture of how M itself is changing at p

• Lemma: shape operator is a liner operator i.e. $S_p : T_p(M) \to T_p(M)$

• * **Normal curvature** $k(u) = S(u).u$ where u is the unit vector tangent to M at $p \epsilon M$

• lemma: for a curve a in M and unit normal vector U at a point in a $a''U = S(a')a'$
from for a given curve on a surface with given velocity then its acceleration in normal direction is entirely defined by the surface

• from above lemma if we define $u = a'(0)$ (initial velocity) then $k(u) = s(u).u = s(a').a' = a''U = k(0)N(0)U(p)$ (k is curvature of a curver)
$= k(0).cos(\eta)$ (since N and U are both unit vectors) so now if we orient a or rather take a to be in plane determined by $U(p)$ and $u = a'$ only then $\eta = 0$ *or* π only thus gives geometrical meaning to normal curvature

• **Principle curvatures** k_1 and k_2: the maximum and the minimum values of $k(u)$ of M at a point p and the directions in which they occur is the principal directions

• Umbilic point p: of M if umbilical if $k(u)$ is constant in all directions at p

• * **Theorem**: now as shape operator is linear operator it can be expressed in matrix form for this : if p is not an umbilical point then:

1. Principal directions (of k_1 and k_2) are orthogonal
2. These directions are eigenvectors of S_p with k_1 and k_2 as eigenvalues

• * **Gaussian curvature** K: at a point p is equal to $det(S_p)$ thus is a function on M

• * **Mean curvature** H : at a point p is equal to $1/2 \; trace(S_p)$

• **Lemma** : $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$

• Theorem: if v and w are linearly independent tangent vectors at a point p of M then:

$$S(v) \times S(w) = K(p) v \times W$$
$$S(v) \times w + v \times S(w) = 2H(p) v \times w$$

this can be use to formulate formulas for K and H
• Corollary $k_1, k_2 = H \pm \sqrt{H^2 - K}$

7.1 Curvature computation

• For a surface M if

$$E = x_u.x_u \quad F = x_u.x_v \quad G = x_v.x$$
$$U = \frac{x_u \times x_v}{\|x_u \times x_v\|}$$
$$l = U.x_{uu} \quad m = U.x_{uv} \quad n = U.x_{vv}$$

then

$$K = \frac{ln - m^2}{EG - F^2}$$
$$H = \frac{Gl + En - 2FM}{2(EG - F^2)}$$

8 Tensors

8.1 Definitions

- Einstein summation convection $\sum_{i=1}^{n} a_i x^i = a_i x^i$ i.e. summation symbol is just removed (here dimension of the space should be known (n))
- Dummy index: any index which is repeated in a given term and which can be replaced by other index without changing the expression
- Free index: index occurring only once in any given term
- Kronecker delta:

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

- Contra-variant Vectors: if A_i in X coordinate system are transformed to $\overline{A_i}$ in Y coordinate system by rule:

$$\overline{A_i} = \frac{\partial \overline{x^j}}{\partial x_i} A_j$$

- Covariant Vectors: if A_i in X coordinate system are transformed to $\overline{A_i}$ in Y coordinate system by rule:

$$\overline{A_i} = \frac{\partial x^j}{\partial \overline{x_i}} A_j$$

References

[1] Barrett O'Neill: Elementary Differential Geometry,Elsevier Academic press,2,2006.