Polices & Value Functions
Lecture 3

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CSE4/510 Reinforcement Learning
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Overview

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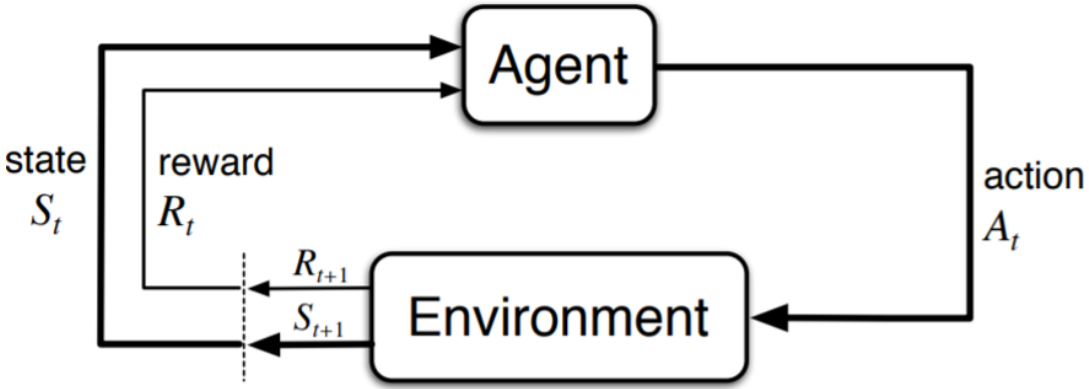
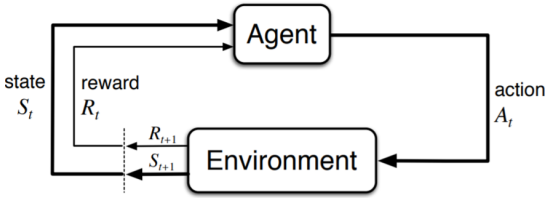
Recap: MDP



Recap: MDP



At each step t the agent:

- Receives state S_t / observation O_t and reward R_t
- Executes action A_t

The environment:

- Receives action A_t
- Emits state S_{t+1} / observation O_{t+1} and reward R_{t+1}

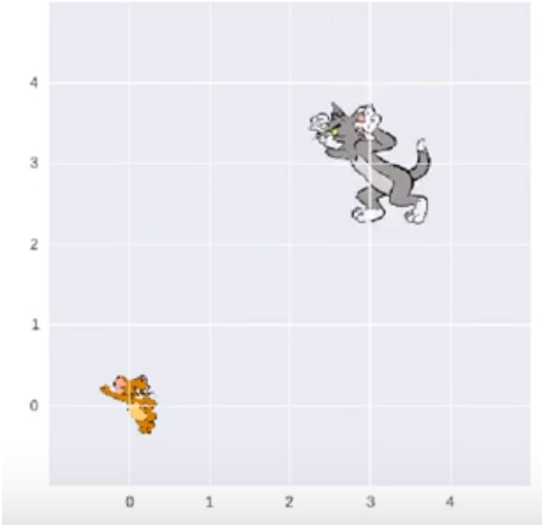
Recap: MDP Definitions

Definition

Markov decision process (MDP) defined by the tuple $\langle s, a, O, P, r, \rho_0, \gamma \rangle$, where

- $s \in S$ denotes states, describing all possible configurations;
- $a \in A$ denotes actions;
- $P : S \times A \times S \to \mathbb{R}$ is the states transition probability distribution;
- O is a set of observations;
- $r : S \to \mathbb{R}$ is the reward function;
- $\rho_0 : S \to [0, 1]$ is the distribution of the initial state s_0;
- $\gamma \in [0, 1]$ is a discount factor

Recap



Recap

Recap: Return G_t

Definition

The *return* G_t is the total discounted reward from time-step t.

$$G_t = R_t = r_{t+1} + \gamma r_{t+2} + \cdots = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1} \tag{1}$$

- γ is a discount factor ($\gamma \in [0, 1)$)
- r is the immediate reward, R is the cumulative reward
- The value of receiving reward R after $k + 1$ time-steps is $\gamma^k R$

Example: Return G_t

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$$G_t = R_{t+1} + \gamma R_{t+2} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \tag{2}$$

Suppose $\gamma = 0.5$ and the following sequence of rewards is received $R_1 = 1, R_2 = 8, R_3 = 4$, with $T = 3$. What is the return G_0, G_1, G_2, G_3?

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Policies

Definition

A *policy* π is a disctribution over actions given states. It defines the agent's behaviour
It can be either deterministic or stochastic:

- Deterministic: $\pi(s) = a$
- Stochastic: $\pi(a|s) = \mathbb{P}_\pi[A = a|S = s]$

- A policy fully defines the behaviour of an agent
- MDP policies depend on the current state (not the history)

Notations: Transition Probability

$$\mathcal{P}^{a}_{s,s'} = P(S_{t+1} = s' | S_t = s, A_t = a)$$

$\mathcal{P}$ is the transition probability. If we start at state s and take action a and we end up in state s' with probability $\mathcal{P}^{a}_{ss'}$.

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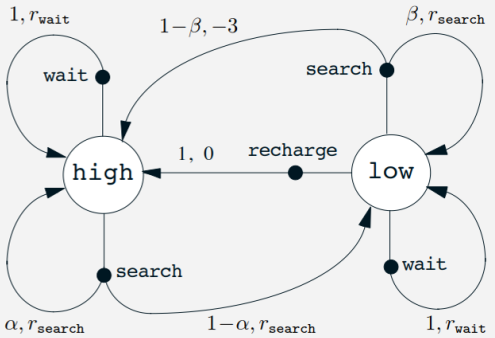
$$\mathcal{R}^{a}_{s,s'} = \mathbb{E}[R_{t+1} | S_t = s, S_{t+1} = s', A_t = a)$$

$\mathcal{R}^{a}_{ss'}$ is another way of writing the expected (or mean) reward that we receive when starting in state s, taking action a, and moving into state s'.

Example: Recycling Robot

s	a	s'	$p(s' \mid s, a)$	$r(s, a, s')$
high	search	high	α	$r_{\texttt{search}}$
high	search	low	$1-\alpha$	$r_{\texttt{search}}$
low	search	high	$1-\beta$	-3
low	search	low	β	$r_{\texttt{search}}$
high	wait	high	1	$r_{\texttt{wait}}$
high	wait	low	0	-
low	wait	high	0	-
low	wait	low	1	$r_{\texttt{wait}}$
low	recharge	high	1	0
low	recharge	low	0	-



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Reward and Return

RL agents learn to **maximize discounted cumulative future reward** (R).

Reward (r_{t+1}) and Return ($G_t = R_t$)

Cumulative reward:

$$R_t = r_{t+1} + r_{t+2} + r_{t+3} + r_{t+4} + \cdots = \sum_{k=0}^{\infty} r_{t+k+1}$$

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Discounted cumulative reward:

$$R_t = r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \gamma^3 r_{t+4} + \cdots = \sum_{k=0}^{\infty} \gamma^k r_{t+k+1}$$

where $0 \leq \gamma \leq 1$

Example: Reward (r_{t+1}) and Return ($G_t = R_t$)

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Example. Suppose $\gamma = 0.5$ and the following sequence of rewards is received $r_1 = 1, r_2 = 8, r_3 = 4$, with $T = 3$. What is the return R_0, R_1, R_2, R_3?

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Value Functions

There are two types of value functions:

- state value function $V(s)$

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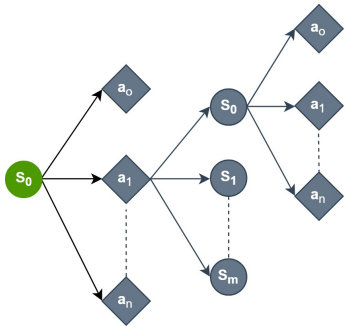
- state value function $V(s)$
- action value function $Q(s, a)$

State value function $V(s)$

Definition

State value function describes the value of a state when following a policy. It is the expected return when starting from state s acting according to our policy π:

$$V^{\pi}(s) = \mathbb{E}_{\pi}[R_t | S_t = s]$$

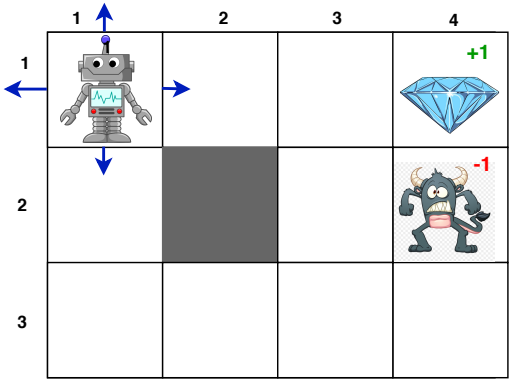


State value function

$V^{\pi}(s)$ can also be interpreted, as the **expected cumulative future discounted reward**, where

- "Expected" refers to the "expected value"
- "Cumulative" refers to the summation
- "Future" refers to the fact that it's an expected value of a future quantity with respect to the present quantity, i.e. $s_t = s$.
- "Discounted" refers to the "gamma" factor, which is a way to adjust the importance of how much we value rewards at future time steps, i.e. starting from $t + 1$.
- "Reward" refers to the main quantity of interested, i.e. the reward received from the environment.

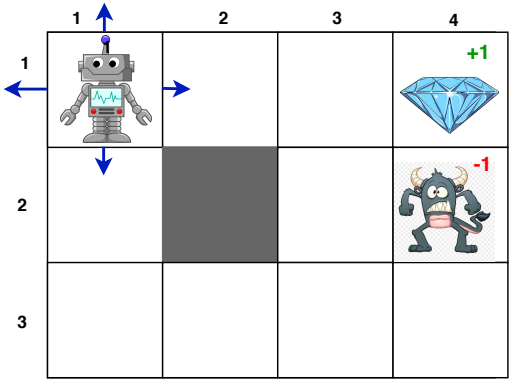
Example: State value function



Assume: actions deterministically successful. $\gamma = 1$. Agent is following an optimal policy.

$$V^*(1,4) =$$

Example: State value function



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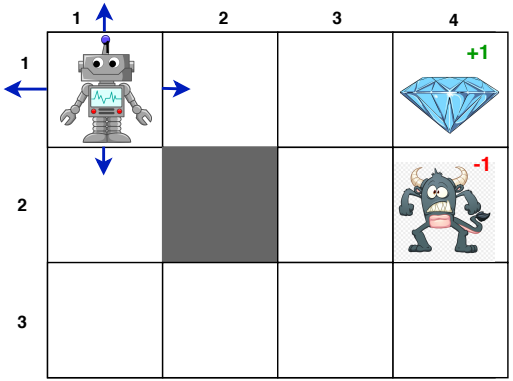
$$V^*(1,4) = 1$$

Example: State value function



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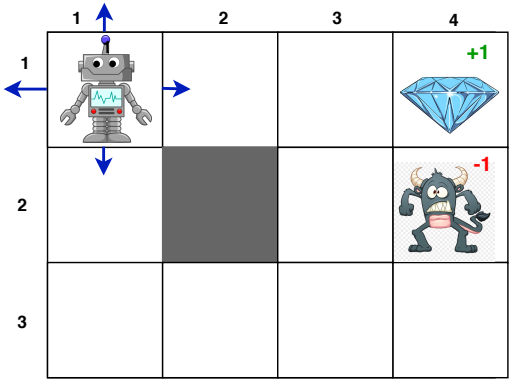
Example: State value function



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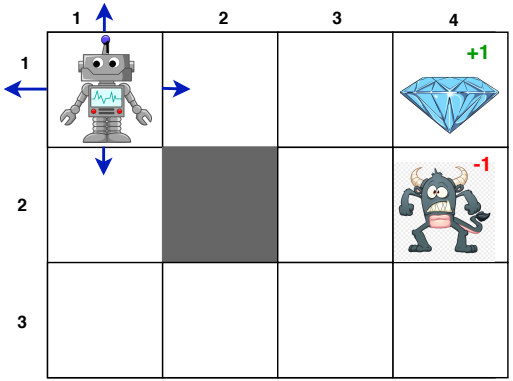

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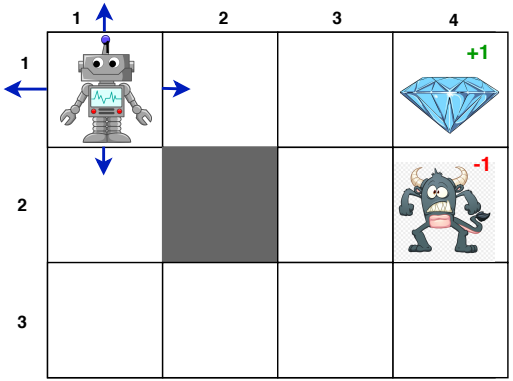

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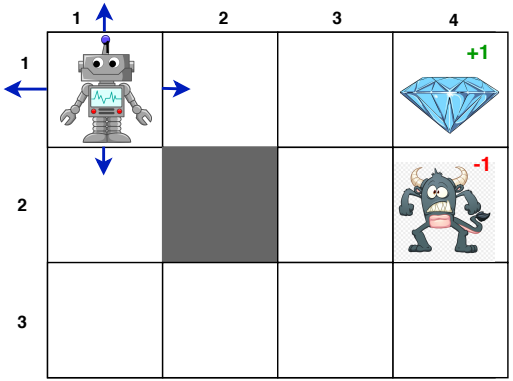

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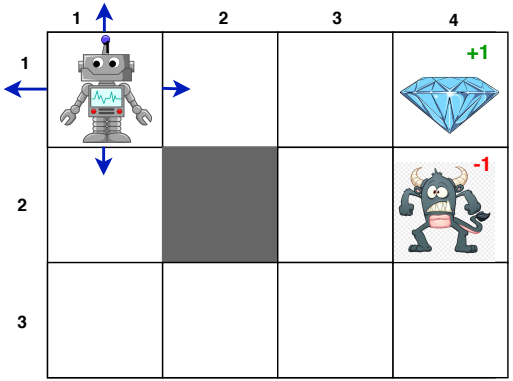

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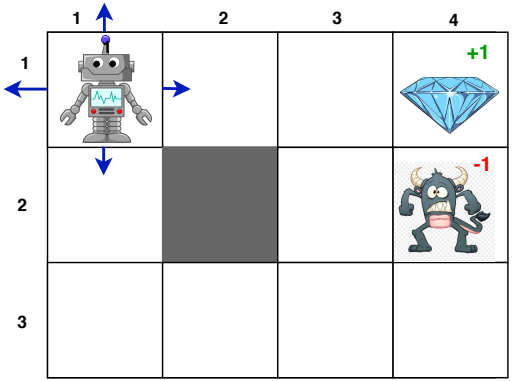

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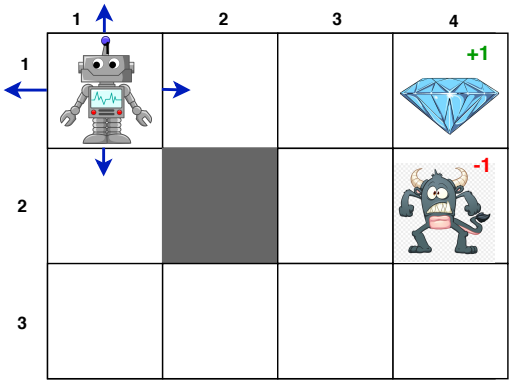

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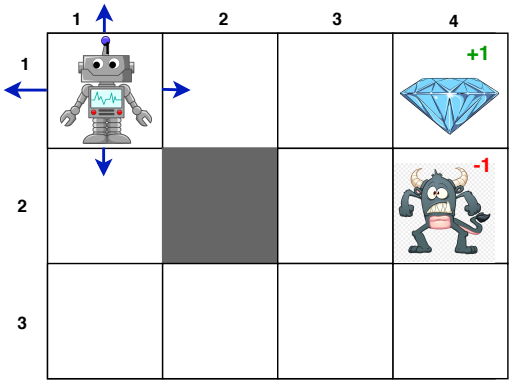

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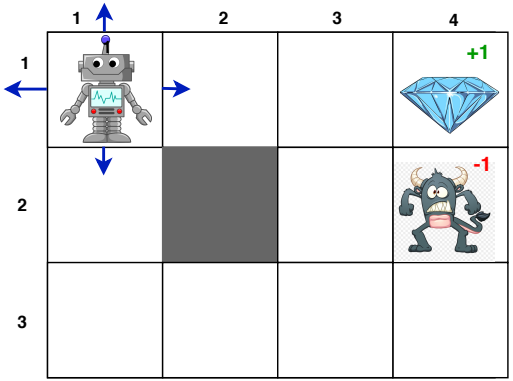

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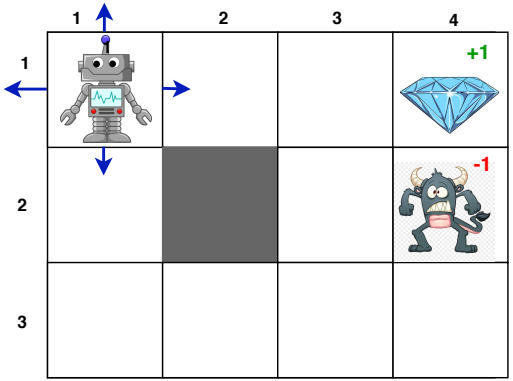
Example: State value function



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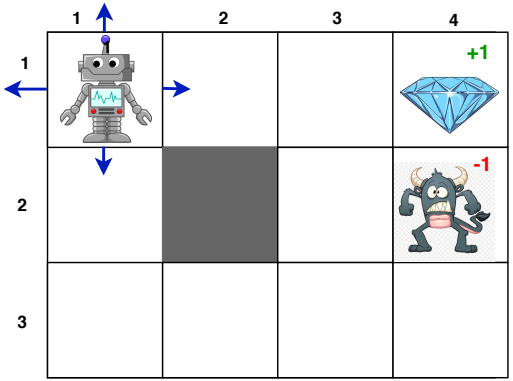

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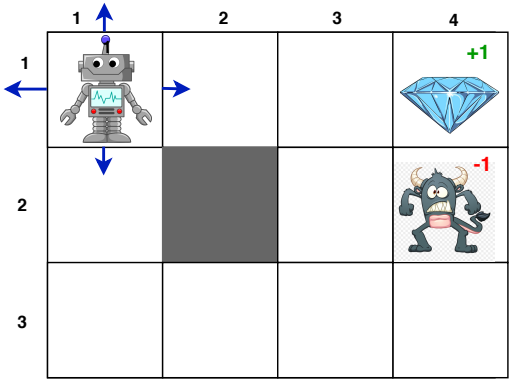
Example: State value function



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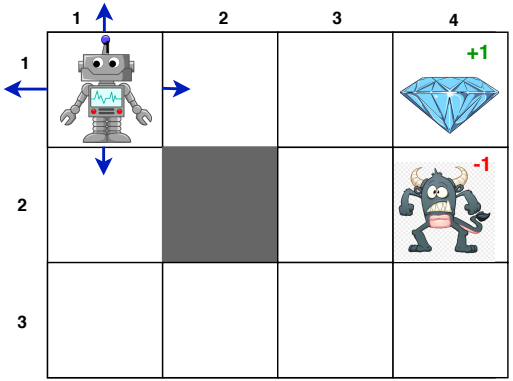

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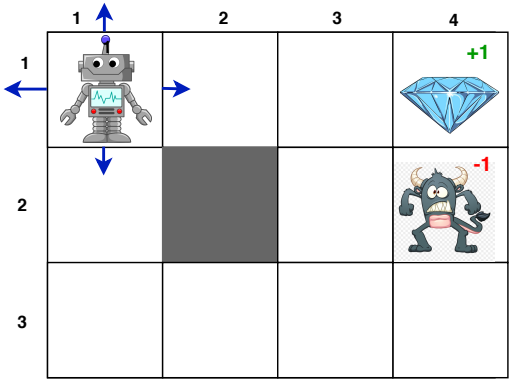

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$$V^*(1,4) = 1$$
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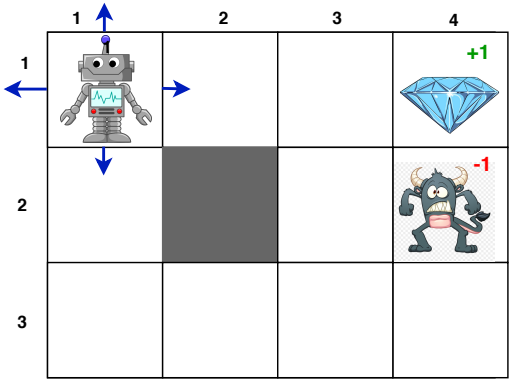
$$V^*(1,2) = 0.81$$

Example: State value function



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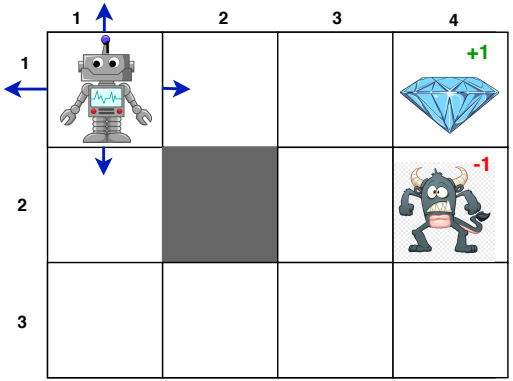

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$$V^*(1,4) = 1$$
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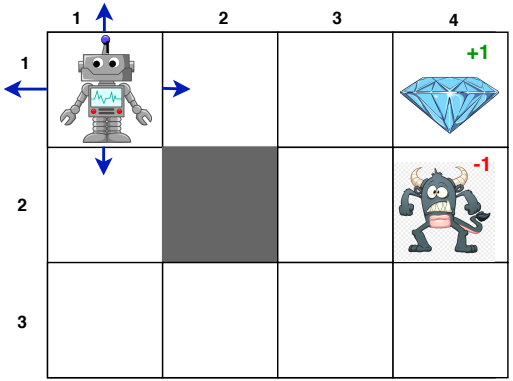
$$V^*(1,2) = 0.81$$
$$V^*(1,1) = 0.729$$

Example: State value function



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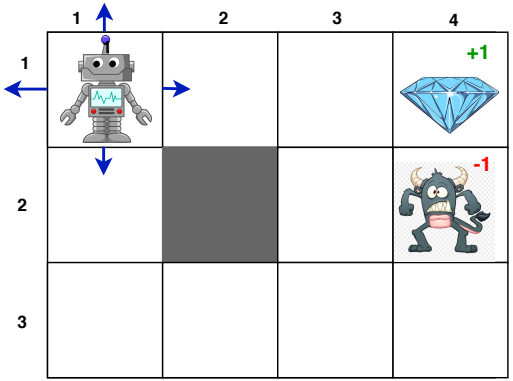
Example: State value function



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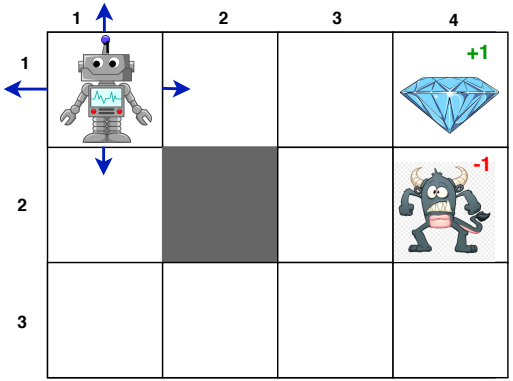

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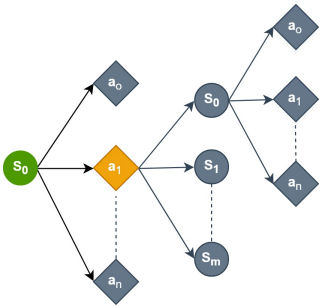
Action value function

Definition

Action value function tells us the value of taking an action a in state s when following a certain policy π. It is the expected return given the state and action under π:

$$Q^{\pi}(s, a) = \mathbb{E}_{\pi}[R_t | S_t = s, A_t = a]$$



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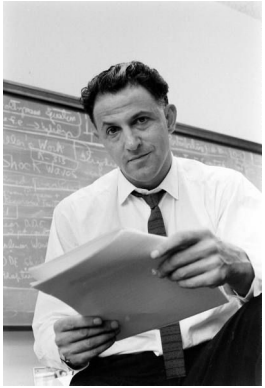
Bellman Equation

Richard Bellman was an American applied mathematician who derived the following equations which allow us to start solving these MDPs. The Bellman equations are ubiquitous in RL and are necessary to understand how RL algorithms work.



Bellman equation for the state value function

$$V^{\pi}(s) = \mathbb{E}_{\pi}[R_t | S_t = s]$$

Bellman equation for the state value function

$$
\begin{aligned}
V^{\pi}(s) &= \mathbb{E}_{\pi}[R_t | S_t = s] \\
&= \mathbb{E}_{\pi}[r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \gamma^3 r_{t+4} + \ldots | S_t = s]
\end{aligned}
$$

Bellman equation for the state value function

$$
\begin{aligned}
V^{\pi}(s) &= \mathbb{E}_{\pi}[R_t | S_t = s] \\
&= \mathbb{E}_{\pi}[r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \gamma^3 r_{t+4} + \ldots | S_t = s] \\
&= \mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} | S_t = s]
\end{aligned}
$$

Bellman equation for the state value function

$$
\begin{aligned}
V^{\pi}(s) &= \mathbb{E}_{\pi}[R_t | S_t = s] \\
&= \mathbb{E}_{\pi}[r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \gamma^3 r_{t+4} + \ldots | S_t = s] \\
&= \mathbb{E}_{\pi}[\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} | S_t = s] \\
&= \mathbb{E}_{\pi}[r_{t+1} + \gamma \sum_{k=0}^{\infty} \gamma^k r_{t+k+2} | S_t = s]
\end{aligned}
$$

Bellman equation for the state value function

The expectation here describes what we expect the return to be if we continue from state s following policy π. The expectation can be written explicitly by summing over all possible actions and all possible returned states.

$$\mathbb{E}_\pi[r_{t+1}|s_t = s] = \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} \mathcal{R}^a_{ss'}$$

$$\mathbb{E}_\pi[\gamma \sum_{k=0}^{\infty} \gamma^k r_{t+k+2}|S_t = s] = \sum_a \pi(s,a) \sum_{s'} \mathcal{P}^a_{ss'} \gamma \, \mathbb{E}_\pi[\sum_{k=0}^{\infty} \gamma^k r_{t+k+2}|S_{t+1} = s']$$

Bellman equation for the state value function

By distributing the expectation between these two parts, we can then manipulate our equation into the form:

$$V^{\pi}(s) = \sum_{a} \pi(s,a) \sum_{s'} \mathcal{P}^{a}_{ss'} \left[\mathcal{R}^{a}_{ss'} + \gamma \, \mathbb{E}_{\pi} \left[\sum_{k=0}^{\infty} \gamma^{k} r_{t+k+2} | s_{t+1} = s' \right] \right]$$

Bellman equation for the state value function

By distributing the expectation between these two parts, we can then manipulate our equation into the form:

$$
\begin{aligned}
V^{\pi}(s) &= \sum_{a} \pi(s,a) \sum_{s'} \mathcal{P}^{a}_{ss'} \left[\mathcal{R}^{a}_{ss'} + \gamma \, \mathbb{E}_{\pi} \left[\sum_{k=0}^{\infty} \gamma^{k} r_{t+k+2} | s_{t+1} = s' \right] \right] \\
&= \sum_{a} \pi(s,a) \sum_{s'} \mathcal{P}^{a}_{ss'} \left[\mathcal{R}^{a}_{ss'} + \gamma V^{\pi}(s') \right]
\end{aligned}
$$

Conclusion

- Bellman equations is that they let us express values of states as values of other states. This means that if we know the value of s_{t+1}, we can very easily calculate the value of s_t.
- Bellman equations is a foundation for iterative approaches to solve reinforcement learning task