

Storing a Sparse Table with $O(1)$ Worst Case Access Time

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Abstract. A data structure for representing a set of n items from a universe of m items, which uses space $n + o(n)$ and accommodates membership queries in constant time is described. Both the data structure and the query algorithm are easy to implement.

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1. Introduction

The following searching problem is considered. A set S of n distinct names from a universe U of m possible names ($m \geq n$) is to be stored in a memory T in a manner permitting efficient processing of membership queries of the form, “Is q in S , and if so, then where can it be found in T ?” We assume that the set S is static, so that our main concerns are the storage required for S and the time required for processing queries. Hashing schemes provide a solution to this problem utilizing $O(n)$ storage and permitting queries in $O(1)$ average time per query. In this paper, we concentrate on the worst case time required for a query, while retaining an $O(n)$ bound on storage. This question has been considered in various papers, for example, [1]–[4]. Yao [4] proposes an interesting complexity model for this problem. In Yao’s framework, S is a subset of $U = \{1, 2, \dots, m\}$ of cardinality n . The memory T stores an item from U in each of its cells, and these cells can be randomly accessed by address. A query for an element q in U is processed by probing a sequence of cells in T . This sequence of probes can be adaptive: The

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next cell to be probed is determined precisely by q and the contents of the cells previously probed. Query time is measured in terms of the number of cells probed, and storage is defined to be the size of T (total number of cells). Yao [4] shows that storage $n + 1$ and worst case query time 2 can be simultaneously attained provided that m grows at least exponentially in n ($m \geq e^{2n}$ suffices). Yao and Tarjan [3] show that $O(n)$ storage and worst case query time $O(\log m / \log n)$ can generally be attained. Therefore, if m is polynomially bounded in n , or grows at least exponentially in n , then linear storage and constant query time can be simultaneously achieved. However, as Yao [4] points out, there is an intermediate range for m , for example, $m = 2^{n^a}$, for which the possibility of linear storage and constant query time is not settled by the results quoted above.

In the next section we describe a storage technique achieving linear storage and constant worst case query time for all m and n . The query algorithm is especially easy to implement and the relative magnitudes of m and n play no role in the proofs. Section 3 discusses a general framework that motivates our construction. In Section 4 we describe a refinement that attains space $n + o(n)$, while retaining constant query time. Section 5 describes some variations of our method.

2. Basic Representation and Query Algorithm

In this section we illustrate the main idea behind our set representation method with a technique achieving linear storage and constant query time. Let $U = \{1, \dots, m\}$. To simplify the discussion we assume that $p = m + 1$ is a prime number. We use the notation $a \bmod b$ to denote the unique integer x , $1 \leq x \leq b$, such that $x \equiv a \pmod{b}$. We need the following lemma.

LEMMA 1. *Given $W \subseteq U$ with $|W| = r$, and given $k \in U$ and $s \geq r$, let $B(s, W, k, j) = |\{x \mid x \in W \text{ and } (kx \bmod p) \bmod s = j\}|$ for $1 \leq j \leq s$. In words, $B(s, W, k, j)$ is the number of times the value j is attained by the function $x \rightarrow (kx \bmod p) \bmod s$ when x is restricted to W . Then there exists a $k \in U$ such that*

$$\sum_{j=1}^s \binom{B(s, W, k, j)}{2} < \frac{r^2}{s}.$$

PROOF. We show that

$$\sum_{k=1}^{p-1} \sum_{j=1}^s \binom{B(s, W, k, j)}{2} < \frac{(p-1)r^2}{s} \quad (1)$$

from which the Lemma follows immediately. The sum in (1) is the number of pairs $(k, \{x, y\})$, with $x, y \in W$, $x \neq y$, $1 \leq k < p$, such that

$$(kx \bmod p) \bmod s = (ky \bmod p) \bmod s.$$

The contribution of $\{x, y\}$, $x \neq y$, to this quantity is at most the number of k such that

$$k(x - y) \bmod p \in \{s, 2s, 3s, \dots, p - s, p - 2s, p - 3s, \dots\}. \quad (2)$$

Because $x - y$ has a multiplicative inverse mod p , the number of k satisfying (2) is $\leq 2(p - 1)/s$. Summing over the $\binom{s}{2}$ possible choices for $\{x, y\}$, we conclude that the sum in (1) is indeed bounded by $(p - 1)r^2/s$, completing the proof. \square

COROLLARY 1. *There exists a $k \in U$ such that $\sum_{j=1}^r B(r, W, k, j)^2 < 3r$.*

PROOF. Combine Lemma 1 with the observation that $\sum_{j=1}^r B(r, W, k, j) = |W| = r$. \square

COROLLARY 2. *There exists a $k' \in U$, such that the mapping $x \rightarrow (k'x \bmod p) \bmod r^2$ is one-to-one when restricted to W .*

PROOF. Choosing $s = r^2$, Lemma 1 provides a k' such that $B(r^2, W, k', j) \leq 1$ for all j . \square

Given $S \subseteq U$, $|S| = n$, our technique for representing the set S works as follows. The content k of cell $T[0]$ is used to partition S into n blocks W_j , $1 \leq j \leq n$, as determined by the value of the function $f(x) = (kx \bmod p) \bmod n$; pointers to corresponding blocks T_j in the memory T are provided in locations $T[j]$, $1 \leq j \leq n$. More specifically, a k is chosen satisfying Corollary 1 (with $W = S$ and $r = n$), so that $\sum |W_j|^2 < 3n$. The amount of space allocated to the block T_j for W_j is $|W_j|^2 + 2$. The subset W_j is resolved within this space by using the perfect hash function provided by Corollary 2 (setting $W = W_j$ and $r = |W_j|$). In the first location of T_j we store $|W_j|$, and in the second location we store the value k' provided by Corollary 2; each $x \in W_j$ is stored in location $[(k'x \bmod p) \bmod |W_j|^2] + 2$ of block T_j .

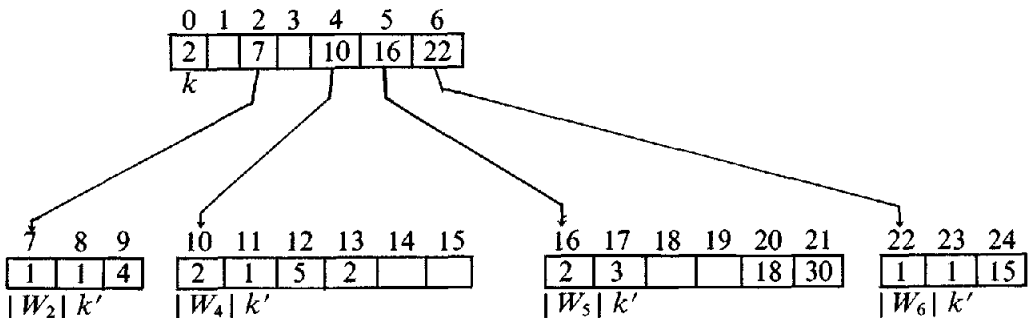
A membership query for q is executed as follows:

1. Set $k = T[0]$ and set $j = (kq \bmod p) \bmod n$.
2. Access in $T[j]$ the pointer to block T_j of T and use this pointer to access the quantities $|W_j|$ and k' in the first two locations of block T_j .
3. Access cell $((k'q \bmod p) \bmod |W_j|^2) + 2$ of block T_j ; q is in S if and only if q lies in this cell.

A query requires five probes, and our choice of k in Corollary 1 implies that the size of T is at most $6n$. An example is provided below.

Example

$$m = 30, \quad p = 31, \quad n = 6, \quad S = \{2, 4, 5, 15, 18, 30\}$$



A query for 30 is processed as follows:

1. $k = T[0] = 2$, $j = (30 \cdot 2 \bmod 31) \bmod 6 = 5$.
2. $T[5] = 16$, and from cells $T[16]$ and $T[17]$ we learn that block 5 has two elements and that $k' = 3$.
3. $(30 \cdot k' \bmod 31) \bmod 2^2 = 4$. Hence, we check the $4 + 2 = 6$ th cell of block 5 and find that 30 is indeed present.

The time required to construct the representation for S might be as bad as $O(mn)$ in the worst case; finding k may require testing many possible values before a suitable one is found. However, by increasing the size of T by a constant factor,

we can show that the representation can be constructed in random expected time $O(n)$, independently of m and S . Namely, we use the following variants of Corollaries 1 and 2.

COROLLARY 3. *For at least one-half of the values k in U ,*

$$\sum_{j=1}^r B(r, W, k, j)^2 < 5r.$$

PROOF. We use eq. (1) and the fact that at most one-half of the terms in a sequence can exceed twice the average value of the sequence to conclude that

$$\sum_{j=1}^r \binom{B(r, W, k, j)}{2} < 2r$$

for at least one-half of the values k in U , from which the corollary follows easily. \square

COROLLARY 4. *The mapping $x \rightarrow (k'x \bmod p) \bmod 2r^2$ is one-to-one when restricted to W for at least one-half of the values k' in U .*

PROOF. We set $s = 2r^2$ in eq. (1) and conclude that

$$\sum_{j=1}^{2r^2} \binom{B(2r^2, W, k', j)}{2} < 1$$

for at least one-half of the values k' in U , which implies the corollary. \square

Using Corollaries 3 and 4, we represent a set S of size n as before, except that now we allocate space $2 |W_j|^2 + 2$ in storing a block W_j of S . What we gain is the fact that the probability that a particular choice for k (or k') is suitable, exceeds $\frac{1}{2}$. The choices for k (or k') are selected at random until suitable values are found.

By modifying our methods slightly, we can guarantee a worst case construction time of $O(n^3 \log m)$.

LEMMA 2. *There exists a prime $q < n^2 \log m$ that does not divide any of the elements in S , and that separates these elements into distinct residue classes mod q .*

PROOF. For $S = \{x_1, \dots, x_n\}$ let $t = \prod_{i < j} (x_i - x_j) \prod_i x_i$. Clearly, $\log |t| \leq (\frac{n+1}{2}) \log m$. Since the prime number theorem gives $\log(\prod_{q < x, q \text{ prime}} q) = x + o(x)$, we conclude that some prime $q < n^2 \log m$ cannot divide t . This prime q satisfies the lemma. \square

We proceed as follows. If $m < n^2 \log n$, then $O(nm) = O(n^3 \log m)$. If $m \geq n^2 \log n$, then in time $O(nq)$ we produce a prime q satisfying Lemma 2 and store it in location $T[-1]$. The remainder of T is specified as before, except that the location where $x \in S$ gets stored is determined by using the hash value $x \bmod q$ in place of x , in effect replacing U with a smaller universe, $U' = \{1, \dots, q-1\}$, with $q \leq n^2 \log m$. The total construction time is bounded by $O(nq) = O(n^3 \log m)$.

3. Discussion

The scheme described above can be couched in the following general framework. The value k in $T[0]$ induces a coloring of U with n colors, namely $x \rightarrow (kx \bmod p) \bmod n$. Yao's two-probe method is likewise based on an indexed family $\chi = \{C_k\}$, $|\chi| \leq m$, of n -colorings having the property that for each $S \subseteq U$, $|S| =$

n , there exists a C_k in χ that is one-to-one when restricted to S . Yao refers to such a family χ as a separating system. With Yao's method, the set S is stored in T by placing k in $T[0]$, to invoke the coloring C_k , and placing $x \in S$ in $T[j]$ where j is the color of x under C_k . This approach works provided that such a χ exists. The restriction $|\chi| \leq m$ arises from the fact that its elements are indexed by the permissible range of $T[0]$. A simple counting argument shows that at least $\binom{m}{n}/(m/n)^n$ colorings are required for a separating system, from which we deduce that $m \geq n^n/n!$. R. Graham uses a probabilistic argument to show that if $m \geq n^{n+2}/n! \approx e^n$ then a separating system χ exists.

To extend Yao's method when $m = \exp(o(n))$, we resign ourselves to the fact that collisions are inevitable under the coloring induced by $T[0]$. Referring to the monochromatic blocks of S as bins, we attempt to use secondary colorings to separate the elements within bins. If a bin size b is sufficiently small; that is, $b \leq \log m$, then that bin can be resolved by choosing a b -coloring from a family χ' that comprises a separating system for subsets of size b .

Now a probabilistic argument shows that for all $m \geq n$, there exists a family of n -colorings χ , $|\chi| \leq m$, such that for each $S \subseteq U$, $|S| = n$, there exists a coloring $C \in \chi$ that partitions S into bins of size $< \log n \leq \log m$. Therefore, we conclude from this reasoning that there exist table storage schemes under Yao's model with $O(1)$ query time and $O(n)$ storage. However, we have not been able to explicitly construct a class of storage schemes for all $m \geq n$ along these lines. We refer to storage schemes of this kind, where bin sizes are uniformly bounded by $\log m$, as L^∞ schemes.

Returning again to Yao's two-probe method, we consider the possibility of utilizing more table space, in effect using t -colorings with $t \geq n$ to completely resolve the elements of an n set S . Again, using counting and probabilistic arguments, we can show that a family χ of t -colorings exists, $|\chi| \leq m$, that resolves all S of size n , provided that m is roughly at least $\exp(n^2/t)$, which is roughly best possible. Therefore, by choosing $t = n^2$, we remove any constraint on m .

Although using n^2 colors, or equivalently space n^2 is very inefficient in terms of our original problem, it is reasonable to use b^2 colors to resolve bins of size b , provided that $\sum b^2$ is small. Probabilistic arguments show, in fact, that almost all families of n -colorings χ with $|\chi| = m$ achieve $\sum b^2 = O(n)$ for every S of size n , for all $m \geq n$. This provides another class of linear space, constant query time table storage schemes, which we refer to as L^2 schemes. Contrary to the difficulty we have in constructing explicit L^∞ schemes, the construction in Section 2 provides an explicit class of L^2 schemes.

4. Refinement

In this section we show how to reduce the storage utilization to $n + o(n)$ while retaining constant query time. First, we provide a sketch. Our data structure in Section 2 involves an initial partition of S into n blocks, followed by resolutions of these blocks at the second level of the data structure. Our refinement involves an initial partition of S into a larger number of blocks, $g(n)$ (to be specified below), of which, obviously, at most n are nonempty. Those blocks that have more than one element are resolved at the second level as before. However, there will be very few blocks with more than one element; and moreover, the total space required to resolve them is only $o(n)$. The element of a singleton block is directly stored in the initial level of the data structure. To reduce the space requirement for the initial level of the data structure from $g(n)$ to $n + o(n)$, we use an auxiliary data structure (to be described).

Choosing $W = S$, $s = g(n)$, and $r = n$ in Lemma 1, we find that for some $k \in U$,

$$\sum_{j=1}^{g(n)} \binom{B(g(n), S, k, j)}{2} = O\left(\frac{n^2}{g(n)}\right). \quad (3)$$

Since $x^2 = O\left(\binom{x}{2}\right)$ for $x \geq 2$, eq. (3) implies that

$$\sum' B(g(n), S, k, j)^2 = O\left(\frac{n^2}{g(n)}\right) \quad (4)$$

where \sum' denotes the sum over all j such that $1 \leq j \leq g(n)$ and $B(g(n), S, k, j) \geq 2$. The set S is partitioned into blocks as determined by the values of the function $f(x) = (kx \bmod p) \bmod g(n)$. Since $g(n)$ will be chosen so that $\lim n/g(n) = 0$, eq. (4) implies that the total space required to resolve those blocks having two or more elements (using the method in Section 2) is $o(n)$.

In processing a membership query for q , we first determine the number $j = (kq \bmod p) \bmod g(n)$ of the block W_j of the partition of S to which q must belong if q belongs to S . At most n of these blocks are nonempty. With each nonempty block W_j we associate a cell of T in which we store either (a) the single item of W_j in the event that $|W_j| = 1$, or (b) a pointer to the second level of our data structure where W_j is resolved if $|W_j| \geq 2$. We also use a tag bit to indicate which of (a) or (b) applies. (These tag bits can be packed into $O(n/\log m) = o(n)$ words.) This approach requires an auxiliary data structure to determine whether a block W_j is nonempty, and to find the cell and tag bit associated with W_j when W_j is nonempty. The design of this auxiliary data structure is a slight modification of a similar construction due to Tarjan and Yao [3]. The cells associated with nonempty W_j are arranged consecutively with increasing j . Let T' designate the portion of T in which these cells are located. We partition the interval $I = [1, g(n)]$ into $n^2/g(n)$ subintervals of size $(g(n)/n)^2$. With each of the $n^2/g(n)$ subintervals σ of I , we associate a base address $B[\sigma]$, which is the address of the location immediately preceding the cells in T' associated with nonempty W_j , $j \in \sigma$. These base addresses are stored in a table of size $n^2/g(n) = o(n)$. A second table $A[j]$, $j \in I$, is used to store offsets: $A[j] = 0$ if $W_j = \emptyset$, otherwise $B[\sigma] + A[j]$ is the address in T' associated with W_j for $j \in \sigma$. Since $A[j]$ assumes at most $(g(n)/n)^2 + 1$ possible values, the entire table $A[j]$, $j \in I$ can be packed into $O(g(n)\log(g(n)/n)/\log n)$ cells of T . Picking $g(n) = n(\log n)^{1/2}$ the resulting space requirement for the $A[j]$ table is $o(n)$, and so the total space requirement for our data structure is $n + o(n)$.

The remarks at the end of Section 2 concerning the time required to construct the representation for S carry over and apply here.

5. Variations

The results presented here remain valid if we substitute the mapping $x \rightarrow l(kx \bmod p) \cdot s/p$ in place of $(kx \bmod p) \bmod s$. Presumably, many other suitable mappings can be found. Another mapping that may be of interest, particularly if the multiplication of large numbers is considered objectionable, is the following. Assume that U is the set of d dimensional s -ary vectors where s is a prime. Given two vectors $\mathbf{k} = (k_1, \dots, k_d)$ and $\mathbf{x} = (x_1, \dots, x_d)$ in U , we let $\mathbf{k} \cdot \mathbf{x}$ denote the inner product: $\mathbf{k} \cdot \mathbf{x} = \sum k_i x_i \bmod s$. Then the analog to Lemma 1 holds for the mapping $\mathbf{x} \rightarrow \mathbf{k} \cdot \mathbf{x}$. This mapping avoids multiplication by large numbers and has the further advantage that $\mathbf{k} \cdot \mathbf{x}$ can be computed more rapidly for "short" \mathbf{x} (\mathbf{x} with small Hamming weight). Our data structure, however, requires a variety of such mappings

(s is a bounded parameter), which in turn requires that it be easy to convert between different representations (having differing values of s) of the elements in U . We would also like the Hamming weight to be roughly preserved in switching between representations. An obvious way to accomplish this is to use a block code approach to these representations, of which the binary coded decimal is an example.

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