6·2 : THE PIGEONHOLE PRINCIPLE :

If K is a positive integer and K+1 or more objects are placed into K boxes, then there is at least one box containing two or more of the objects.

EXAMPLE #1 :

Among the group of 367 people, There must be at least 2 with the same birthday.

Sol :

To make a challenge to pigeonhole principle, we may select all people in different date, even then

Sol :

$\#$ of pigeonholes = $\#$ of days in an year = K = 366 (incl. leap year)

$\#$ of pigeons = $\#$ of people = K+1 = 367

So, according to pigeonhole principle, there must be atleast 2 people with the same birthdate.

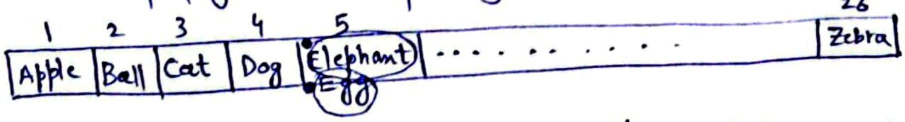
Example #2

In a group of 27 English words, there must be at least 2 that begin with the same letter.

Sol.

$\#$ of pigeonholes = $\#$ of English letters = K = 26

$\#$ of pigeons = $\#$ of English words = 27 = K+1



Even After taking worst case i.e., we may select all the words in separate pigeonhole.

There must be at least two English words that begin with Same letter.

Example # 3

How many Students must be in the class to guarantee that at least 2 students receive the Same Score. (Exam is graded on a scale 0 — 100 points)

Sol.

$\#$ of pigeonholes = K = 101

To guarantee that atleast 2 Students receive the Same score K+1 pigeons are required to guarantee, so

K+1 = 101 + 1 = 102 $\#$ of students in a class

PIGEONHOLE principle is a useful tool in many proofs, ~~Including~~ including proofs of surprising results such as that given in EXAMPLE 4 (Kind of real problems. The above three examples are simple case of pigionhole principle.

EXAMPLE#4 :

Show that for every n, there is a multiple of n that has only 0s and 1s in its decimal expansion.

3 x ? = (Decimal) Containing only 0's and 1's

Sol:

Let n be a positive integer. Suppose n=3

Consider n+1 integers (i.e 3+1=4 integers), ~~where~~ that have only 1's in its decimal expansion

the reason we choose more than one the number, because we want to apply pigeonhole principle

Four integers having only 1's :

1st. 1 one

2nd. 11 Eleven

3rd. 111 one hundred Eleven

4th. 1111 one thousand one hundred Eleven

List of 4 integers = {1 , 11 , 111 , 1111}

Every integer in the list when divided by n (i.e divided by 3), there are n possible remainders (i.e 3 possible remainders 0,1 & 2)

integers :	1	11	111	1111
remainders :	1	2	0	1

Same remainder

(we are guaranteed to have at least 2 remainder will be Same)

The basic idea is : Since, there are 4 integers then, if we divide each by 3, at least two of the remainders will be the Same and that Comes because of pigeonhole principle

Because there are n+1 integers in the list,
By pigeonhole principle,
There must be two integers (having only 1's in its decimal expansion) with the Same remainder when divided by n.

Multiple of n

The Larger of these integers less the Smaller one is a multiple of n, which has a decimal expansion Consisting of entirely of 0's and 1's.

Largest = 1111
Smalest = 1
(with Same remainder)

$$\begin{array}{r} 1111 \\ -\quad 1 \\ \hline 1110 \end{array}$$

(multiple of n i.e 3) mul. of

n=3

3x ? = 1110

3x370 = 1110

$$\begin{array}{r} 370 \\ 3\overline{)1110} \\ \underline{9} \\ 21 \\ \underline{21} \\ 0 \end{array}$$

Let $n = 4$

five integers $= n+1 = \{1, 11, 111, 1111, 11111\}$

remainder 1 3 3 3 3

largest and smallest with same remainder

$a = 11111$ $b = 11$

Largest − Smallest $= 11111 - 11 = 11100$ $n \mid a-b$ i.e $3 \mid a-b$

Let n is an integer, so, there is a multiple of n that has only 0's and 1's, in its decimal expansion.

$$4 * ? = 11100$$

$$4 * 2775 = 1110$$

$$\begin{array}{r} 2775 \\ 4 \overline{)11100} \\ 8 \\ \hline 31 \\ 28 \\ \hline 30 \\ 28 \\ \hline 20 \\ 20 \end{array}$$

GENERALIZED PIGEONHOLE PRINCIPLE

If N objects are placed in to K boxes, then there is atleast one box containing at least $\lceil \frac{N}{K} \rceil$ objects.

Note: A Common type of problem asks for the minimum # of objects such that at least r of these objects must be in the K boxes when these objects are distributed among the boxes.

Normal Pigeonhole principle:

It would just guarantee for at least 2 with the same pigeonhole.

Generalized Pigeonhole Principle:

Generalized pigeonhole principle tells us more than the normal pigeonhole principle.

In generalized principle, we cannot just guarantee atleast 2 with the same pigeonhole but we can guarantee atleast 3 or 4 or 5 with the same pigeonhole.

EXAMPLE 1:

a) What is the minimum number of people such that at least 9 of them have same month of birth.

Sol:-

$r = 9$ (at least pegeons in one box)

$K = 12$ (months i.e # of pigionholes)

$N = \#$ of people (min. # of people)

$$r-1 < \frac{N}{K} \leq r$$

$$8 < \frac{N}{12} \leq 9$$

$$8 * 12 < N \leq 9 * 12$$

$$96 < N \leq 108$$

1	2	3	4	...	11	12
8	8	8	8	...	8	9

$11 \times 8 = 88$ $\quad$ 9

$\boxed{88 + 9 = 97}$

Range of N:

the range of people such that atleast 9 of them have same birth month is greater than 96 and less than or equal to 108.

$$N = K(r-1) + 1$$

$$= 12(9-1) + 1 = 12 * 8 + 1 = 97$$

Smallest integer N

$\frac{N}{K} > r-1$

or $N > K(r-1)$

or $N = K(r-1)+1$

EXAMPLE #1

b) Among 100 people, what is the least #, who were born in same month.

Sol: $N = 100$ (# of pigeons)

$K = 12$ (# of pigeonholes)

$r = ?$ (at least r ~~have same~~ were born in same month)

$$\left\lceil \frac{N}{K} \right\rceil = r \qquad \left\lceil \frac{100}{12} \right\rceil = \lceil 8.33 \rceil = 9$$

1	2	3	3	4		9	10	11	12
8	8	8	8	8	..	9	9	9	9

$8 \times 8 = 64$ $\qquad$ $9 \times 4 = 36$

$64 + 36 = 100$

EXAMPLE #2:

What is the minimum # of students required in a discrete structure class that at least 6 will receive the same grade, if there are possible 5 grades, A, B, C, D, F.

Sol: $r = 6$ (at least pigeons in one box)

$K = 5$ (# of pigeonholes)

# of pigeons $= N = ?$ (minimum N)

$$r - 1 < \frac{N}{K} \leq r$$

$$5 < \frac{N}{5} \leq 6$$

$$5 * 5 < N \leq 6 * 5$$

$$\underset{\text{min}}{+1:}\ 25 < N \leq 30\ \text{max}$$

A	B	C	D	F	
5	5	5	5	6	≈ 26 (min)

6	6	6	6	6	= 30 (max)

Range of N (i.e range of students):

greater than 25

Less than or equal to 30

Smallest N:

$$\frac{N}{K} > r - 1 \implies N = K(r-1) + 1$$

$$N = 5 * 5 + 1 = 26 \quad \text{min. \# of students}$$

$$\boxed{\begin{aligned} N &= K(r-1) + (r-1) \quad \text{max. \# of Students} \\ &= 5*5 + 5 = 25 + 5 = 30 \end{aligned}}$$

EXAMPLE # 3

a) How many cards must be selected from a deck of 52 cards to guarantee that at least 3 cards of the same suit are chosen?

b) How many must be selected to guarantee that at least 3 hearts are selected.

Sol:-

a) ~~# of p~~

1	2	3	4
Heart	spades	Clubs	Diamond

$K = 4$ suits (# of pigeonholes)

$r = 3$ (at least 3 cards of the same suit)

$N = ?$ (# of pigeons)

$$2 < \frac{N}{4} \leq 3$$

$$2*4 < N \leq 3*4$$

$$8 < N \leq 12$$

The smallest # of cards

$$N = K(r-1)+1$$

$$N = 4(2)+1 = 9$$

b) We donot use generalized pigeonhole principle here to answer this, because we want to make sure that there are 3 hearts, not just 3 cards of one suit

We may select all clubs, diamonds and spades before any heart i.e., 39 cards.

The next 3 cards will be all hearts, so we may need to select $39+3=42$ cards to get 3 hearts.

Example # 4

What is the minimum # of students in the class such that

a) There must be atleast 2 with same birthday (day of week).

b) " " " " " 3 " " " " " " ".

Sol:

a) $K = 7$ (Days of week i-e pigeonholes)

$r = 2$ $\quad (r-1) < \frac{N}{K} \leq r$

$N = ?$

$$1 < \frac{N}{7} \leq 2$$

$$7 < \underline{N} \leq 14$$

Smallest

$$N = K(r-1)+1$$

$$N = 7*1+1 = 8$$

1	2	3	4	5	6	7
•	•	•	•	•	•	••
1	1	1	1	1	1	2

$\left\lceil \frac{8 \to 14}{7} \right\rceil = 2$

$\left\lceil \frac{8}{7} \right\rceil = 2$

$\left\lceil \frac{9}{7} \right\rceil = 2$

$\vdots$

$\left\lceil \frac{14}{7} \right\rceil = 2$

b) $K = 7$

$r = 3$

$$(r-1) < \frac{N}{K} \leq r$$

$$2 < \frac{N}{7} \leq 3$$

$$14 < N \leq 21$$

$$N = K(r-1)+1$$

$$N = 7(2)+1 = 15$$

$\left\lceil \frac{15 \text{ to } 21}{7} \right\rceil = 3$

$\left\lceil \frac{15}{7} \right\rceil = 3$

$\left\lceil \frac{16}{7} \right\rceil = 3$

$\vdots$

$\left\lceil \frac{21}{7} \right\rceil = 3$