

## 5.1: MATHEMATICAL INDUCTION

Principle of Mathematical Induction:

To prove that  $P(n)$  is true for all positive integers  $n$ , where  $P(n)$  is a propositional function, we complete two steps:

BASIS :

We verify that  $P(1)$  is true.

INDUCTIVE STEP :

We show that the conditional statement

$P(k) \rightarrow P(k+1)$  is true

for all positive integers  $k$ .

SO, WE CAN SAY THAT

$P(n)$  is true for all positive integers  $n$ .

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## EXAMPLES OF PROOFS BY MATHEMATICAL INDUCTION

### EXAMPLE 1:

Show that if  $n$  is positive integer, then

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Sol:

Let  $P(n)$ : sum of the first  $n$  positive integers is  $\frac{n(n+1)}{2}$ .

Prove that  $P(n)$  is true for all positive integers

#### BASIS:

$P(1)$  is true : True for  $n=1$

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1 \quad P(1) \text{ holds}$$

#### INDUCTIVE STEP:

Suppose  $P(n)$  is true for  $n=k$

i.e.

$$\text{we have } 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$$

To Show:  $P(n)$  is true for  $n=k+1$

$$1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Consider the proposition for  $n=k$  (we have)

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$$

Add  $(k+1)$  on both sides:

$$\begin{aligned} 1 + 2 + 3 + \dots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{\cancel{k(k+1)} + 2(k+1)}{2} \end{aligned}$$

$$1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

So,  $P(k+1)$  is true.

As  $P(1)$  is true, and

$P(k) \rightarrow P(k+1)$  is true.

So,

$P(n)$  is true for all positive integers.

## EXAMPLE 2 :

use mathematical induction to show that

$$1 + 3 + 5 + \dots + (2n-1) = n^2 \text{ for odd tve integer } n.$$

Sol:

Let  $P(n)$  : sum of first  $n$  tve odd integers is  $n^2$

BASIS:

$P(1)$  is true as:

$$1 = (1)^2 = 1 \quad : (\text{sum of the first one odd positive integer is } 1^2)$$

INDUCTIVE STEP:

Suppose  $P(n)$  is true for  $K$  ( $n=k$ ), i.e., so we have

$$1 + 3 + 5 + \dots + (2k-1) = k^2 \quad : \text{inductive hypothesis}$$

To Show:

$P(n)$  is true for  $n=k+1$  i.e.,

$$1 + 3 + 5 + \dots + (2k-1) + (2k+1) = (k+1)^2$$

Consider the  $P(k)$ , inductive hypothesis :

$$1 + 3 + 5 + \dots + (2k-1) = k^2$$

Add  $(2k+1)$  on both sides, we have

$$\begin{aligned} 1 + 3 + 5 + \dots + (2k-1) + (2k+1) &= k^2 + (2k+1) \\ &= k^2 + 2k + 1 \\ &= (k+1)^2 \end{aligned}$$

i.e.,  $P(k+1)$  is true.

As  $P(1)$  is true and  $P(k) \rightarrow P(k+1)$  is true. So

$P(n)$  is true for odd positive integers.

### EXAMPLE 3:

use mathematical induction to show that

$$1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$$

Sol:

Let  $P(n) : 1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$  for all non-negative integers  $n$ .

BASIS:

$P(0)$  is true as :

$$1 = 2^0 - 1 = 2 - 1 = 1$$

INDUCTIVE STEP :

Inductive hypothesis :

Suppose  $P(n)$  is true for  $n=k$  i.e.  $P(k)$  is true. So we have

$$1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1$$

To Show :

True for  $P(k+1)$  i.e.,

$$1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} = 2^{k+2} - 1$$

Consider the inductive hypothesis,

$$1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1$$

Add  $2^{k+1}$  on both sides, we have

$$1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

$$= 2 \cdot 2^{k+1} - 1 \quad \text{as } (1 \cdot 2^{k+1} + 1 \cdot 2^{k+1} = 2 \cdot 2^{k+1})$$

$$= 2^{k+2} - 1$$

i.e., True for  $P(k+1)$  : (proved)

As  $P(0)$  is true and  $P(k) \rightarrow P(k+1)$  is true, so

$P(n)$  is true for all non-negative integers  $n$ .

### Example 4 :

Use mathematical induction to show the formula for the sum of a finite number of terms of a geometric progression:

$$\sum_{i=0}^n ar^i = a + ar + ar^2 + \dots + ar^n = \frac{ar^{n+1} - a}{r - 1} \quad \text{when } r \neq 1$$

where  $n$  is non-negative integer.

Sol:

Let

$P(n)$  : Sum of the first  $n+1$  terms of a geometric progression is  $\frac{ar^{n+1} - a}{r - 1}$

- { • non-negative integer
- Contain 0 as 1st element.
- Positive integer Starts at 1.

BASIS:

$P(0)$  is true as:

$$a = \frac{ar^{0+1} - a}{r - 1} = \frac{ar - a}{r - 1} = \frac{a(r-1)}{r-1} = a$$

INDUCTIVE STEP:

Inductive Hypothesis :

Suppose  $P(n)$  is true for  $n=k$  i.e  $P(k)$  is true, we have

$$a + ar + ar^2 + \dots + ar^k = \frac{ar^{k+1} - a}{r - 1}$$

To Show:

True for  $P(k+1)$  i.e.,

$$a + ar + ar^2 + \dots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r - 1} \quad (\text{To prove})$$

Consider the inductive hypothesis:

$$a + ar + ar^2 + \dots + ar^k = \frac{ar^{k+1} - a}{r - 1}$$

Add  $ar^{k+1}$  on both sides, we have

$$\begin{aligned} a + ar + ar^2 + \dots + ar^k + ar^{k+1} &= \frac{ar^{k+1} - a}{r - 1} + ar^{k+1} \\ &= \frac{ar^{k+1} - a + (r-1)ar^{k+1}}{r - 1} \\ &= \frac{ar^{k+2} - a}{r - 1} \quad (\text{proved}) \end{aligned}$$

As  $P(0)$  is true and  $P(k) \rightarrow P(k+1)$  is true  
So,  $P(n)$  is true.

## EXAMPLES OF PROVING INEQUALITIES:

### EXAMPLE 1 :

Use mathematical induction to show that

$n < 2^n$   
for all positive integers n.

Sol :

$$\text{Let } P(n) : n < 2^n$$

#### BASIS :

$P(1)$  is true as :

$$1 < 2^1$$

#### INDUCTIVE STEP :

Inductive Hypothesis :

Suppose  $P(n)$  is true for  $n=k$  i.e.,  $P(k)$  is true.

$$k < 2^k$$

To Show that

$$k+1 < 2^{k+1}$$

Consider the inductive hypothesis :

$$k < 2^k$$

Add 1 on both sides

$$k+1 < 2^k + 1$$

$$< 2^k + 2^k$$

• As  $1 \leq 2^k$  : good vs bad cheat  
(we adopted good cheat)

$$< 1 \cdot 2^k + 1 \cdot 2^k$$

• We can add anything in greater side of inequality (Inequality still holds)

$$< 2 \cdot 2^k$$

$$< 2^{k+1} \quad (\text{proved})$$

So,  $P(k+1)$  is true.

As  $P(1)$  is true and  $P(k) \rightarrow P(k+1)$  is true, so

$P(n)$  is true.

Example 2 :

Use mathematical induction to prove

$$2^n < n!$$

for every positive integer  $n$  with  $n \geq 4$ . (Note: this inequality is false for  $n=1, 2, 3$ )

Sol:-

Let  $P(n) : 2^n < n!$

BASIS :

$P(4)$  is true as :

$$2^4 < 4!$$

$$16 < 24 \quad (\text{as } 4! = 4 \times 3 \times 2 \times 1 = 24)$$

INDUCTIVE STEP :

Inductive Hypothesis :

Suppose  $P(n)$  is true for  $n=k$  i.e. true for  $P(k)$  as

$$2^k < k!$$

To show that

$P(k+1)$  is true (To prove)

i.e.  $2^{k+1} < (k+1)!$

Take the L.H.S of the  $(k+1)$  case

$$2^{k+1}$$

and we can write it as

$$2^{k+1} = 2 \cdot 2^k \quad \text{--- (1)}$$

From inductive hypothesis, we know that

$$2^k < k!$$

Place this result in eq(1) and focus on equality of (1). (will replace = by <

$$2^{k+1} = 2 \cdot \underline{\underline{2^k}}$$

$$< \underline{\underline{2}} \cdot k!$$

$$< (k+1) \cdot k!$$

$$< (k+1)!$$

proved { good cheat :

$2 < k+1$ , replace 2 by  $k+1$ , no effect on inequality

$$\boxed{6! = 6 \cdot 5! \\ (5+1)! = (5+1) \cdot 5!}$$

So,  $P(n)$  is true

### Example 3 :

Use mathematical induction to show (Inequality of Harmonic Numbers)

$$H_2^n \geq 1 + \frac{n}{2} \text{ whenever } n \text{ is non-negative}$$

Harmonic numbers  $H_j$ ,  $j=1, 2, 3, \dots$  are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{j} \quad (\text{For instance, } H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12})$$

Sol:

$$\text{Let } P(n) : H_{2^n} \geq 1 + \frac{n}{2}$$

BASIS:

$P(0)$  is true as

$$H_2^0 \geq 1 + \frac{0}{2}$$

$$H_1 \geq 1$$

$1 \geq 1$  holds as ( $H_j = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{j}$ , so  $H_1 = 1$ )

INDUCTIVE STEP:

Inductive Hypothesis:

Suppose  $P(n)$  is true for  $n=k$  i.e. true for  $P(k)$ , so we have

$$H_2^k \geq 1 + \frac{k}{2}$$

$$\text{or } 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k} \geq 1 + \frac{k}{2}$$

To Show:

$P(k+1)$  is true i.e.

$$H_2^{k+1} \geq 1 + \frac{k+1}{2}$$

Consider the L.H.S of  $P(k+1)$  i.e.

$$H_2^{k+1}$$

and it can be written as

$$\begin{aligned}
 H_2^{k+1} &= \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k}}_{\text{L.H.S. of } P(k)} + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2 \cdot 2^k} \\
 &= H_2^k + \frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^{k+1}} \quad (\text{By Def. } H_2^k = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k}) \\
 &\geq \left(1 + \frac{k}{2}\right) + \underbrace{\frac{1}{2^k+1} + \frac{1}{2^k+2} + \dots + \frac{1}{2^{k+1}}}_{\text{using inductive hypothesis here}} \quad (\text{and change } = \text{with } \geq) \\
 &\geq \left(1 + \frac{k}{2}\right) + 2^k \cdot \frac{1}{2^{k+1}} \\
 &\geq \left(1 + \frac{k}{2}\right) + \frac{1}{2} \approx 1 + \frac{k+1}{2} \\
 &\geq 1 + \frac{k+1}{2} \quad (\text{proved})
 \end{aligned}$$

$$\begin{aligned}
 H_4 &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \\
 H_{4 \cdot 2} &= H_8 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+2} + \frac{1}{4+3} + \frac{1}{4+4} \\
 H_2^{2+1} &= 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+2} + \frac{1}{4+3} + \frac{1}{2 \cdot 2^2}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} &= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8+1} \quad \text{i.e. 2 terms} \\
 &= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \quad \text{4 terms} \\
 &\geq \frac{1}{5} \cdot \frac{1}{8} \quad \text{Every term is } > \frac{1}{8} \\
 &\geq 2 \cdot \frac{1}{2^2+1} \quad \text{replace equal with } \geq \\
 &\geq 2 \cdot \frac{1}{2^2+1} \quad \text{Small can be placed in smaller side}
 \end{aligned}$$

## EXAMPLES OF PROVING DIVISIBILITY RESULTS

### EXAMPLE 1:

Use mathematical induction to prove that

$n^3 - n$  is divisible by 3 whenever  $n$  is positive integer.

Solution:

Let  $P(n)$ :  $n^3 - n$  is divisible by 3.

BASIS:

$P(1)$  is true as

$$n^3 - n = 1^3 - 1 = 0 \text{ is divisible by 3.}$$

INDUCTIVE STEP:

Inductive Hypothesis:

Suppose  $K^3 - K$  is divisible by 3. (i.e.,  $P(K)$  is true)

To show:

$$(K+1)^3 - (K+1) \text{ is divisible by 3. (To prove, } P(K+1) \text{ is true)}$$

Consider the  $P(K+1)$  step:

$$(K+1)^3 - (K+1)$$

and we have to show that it is divisible by 3 as:

$$\begin{aligned} (K+1)^3 - (K+1) &= (K^3 + 3K^2 + 3K + 1) - (K+1) \\ &= K^3 + 3K^2 + 3K + 1 - K - 1 \\ &= (K^3 - K) + 3(K^2 + K) \end{aligned}$$

(no simplification of  $3K - K$  is required. Because we have to find inductive hypothesis here)

$K^3 - K$  is divisible by 3 : inductive hypothesis

$3(K^2 + K)$  is divisible by 3 : as it is 3 times an integer

So,  $(K^3 - K) + 3(K^2 + K)$  is divisible by 3 :  $\left( \begin{array}{l} a \text{ is divisible by } j \Rightarrow j | a+b \\ b \text{ is divisible by } j \Rightarrow j | a, j | b \\ j | a+b \end{array} \right)$

So, we can say that

$(K+1)^3 - (K+1)$  is divisible by 3. (proved i.e. true for  $P(K+1)$ )

As  $P(1)$  is true and  $P(K) \rightarrow P(K+1)$  is true, so

$P(n)$  is true for all positive integers.

### Example 2 :

Use mathematical induction to prove that

$7^{n+2} + 8^{2n+1}$  is divisible by 57  
for every non-negative integer n.

Sol:

Let  $P(n)$ :  $7^{n+2} + 8^{2n+1}$  is divisible by 57

BASIS:

$P(0)$  is true as  $7^{0+2} + 8^{2 \cdot 0+1} = 57$  is divisible by 57

INDUCTIVE STEP:

Inductive Hypothesis:

Suppose  $P(k)$  is true i.e

$7^{k+2} + 8^{2k+1}$  is divisible by 57

To Show:  $P(k+1)$  is true

$7^{(k+1)+2} + 8^{2(k+1)+1}$  is divisible by 57

Consider the  $P(k+1)$  step as:

$$\begin{aligned} 7^{k+3} + 8^{2k+3} &= 7 \cdot 7^k + 8 \cdot 8^k \\ &= 7 \cdot 7^k + 64 \cdot 8^k \\ &= 7 \cdot 7^k + 7 \cdot 8^k + 57 \cdot 8^k \\ &= 7 \cdot \underbrace{(7^k + 8^k)}_{\text{divisible by 57}} + \underbrace{57 \cdot 8^k}_{\text{divisible by 57}} \end{aligned}$$

So,  $7^{k+3} + 8^{2k+3}$  is divisible by 57

$\Rightarrow P(k+1)$  is true

So,

$P(n)$  is true for every non-negative integer n.