

5.1: MATHEMATICAL INDUCTION

Principle of Mathematical Induction:

To prove that $P(n)$ is true for all positive integers n , where $P(n)$ is a propositional function, we complete two steps:

BASIS :

We verify that $P(1)$ is true.

INDUCTIVE STEP :

We show that the Conditional Statement

$P(k) \rightarrow P(k+1)$ is true
for all positive integers k .

SO, WE CAN SAY THAT

$P(n)$ is true for all positive integers n .

EXAMPLES OF PROOFS BY MATHEMATICAL INDUCTION

EXAMPLE 1:

Show that if n is positive integer, then

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Sol:

Let $P(n)$: sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Prove that $P(n)$ is true for all positive integers

BASIS:

$P(1)$ is true : True for $n=1$

$$1 = \frac{1(1+1)}{2} = \frac{2}{2} = 1 \quad P(1) \text{ holds}$$

INDUCTIVE STEP:

Suppose $P(n)$ is true for $n=k$

i.e.

$$\text{we have } 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$$

To Show: $P(n)$ is true for $n=k+1$

$$1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Consider the proposition for $n=k$ (we have)

$$1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$$

Add $(k+1)$ on both sides:

$$\begin{aligned} 1 + 2 + 3 + \dots + k + (k+1) &= \frac{k(k+1)}{2} + (k+1) \\ &= \frac{k(k+1) + 2(k+1)}{2} \end{aligned}$$

$$1 + 2 + 3 + \dots + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

So, $P(k+1)$ is true.

As $P(1)$ is true, and

$P(k) \rightarrow P(k+1)$ is true,

So,

$P(n)$ is true for all positive integers.

EXAMPLE 2 :

Use mathematical induction to show that

$$1 + 3 + 5 + \dots + 2n-1 = n^2 \text{ for odd +ve integer } n.$$

Sol :

Let $P(n)$: Sum of first n +ve odd integers is n^2

BASIS :

$P(1)$ is true as:

$$1 = (1)^2 = 1 \quad : (\text{Sum of the first one odd positive integer is } 1^2)$$

INDUCTIVE STEP :

Suppose $P(n)$ is true for K (i.e., $n=K$), so we have

$$1 + 3 + 5 + \dots + (2K-1) = K^2 \quad : \text{inductive hypothesis}$$

To show:

$P(n)$ is true for $n=K+1$ i.e.,

$$1 + 3 + 5 + \dots + (2K-1) \overset{\text{Add 2}}{+} (2K+1) = (K+1)^2$$

Consider the $P(K)$; Inductive hypothesis :

$$1 + 3 + 5 + \dots + (2K-1) = K^2$$

Add $(2K+1)$ on both sides, we have

$$\begin{aligned} 1 + 3 + 5 + \dots + (2K-1) + (2K+1) &= K^2 + (2K+1) \\ &= K^2 + 2K + 1 \\ &= (K+1)^2 \end{aligned}$$

i.e., $P(K+1)$ is true.

As $P(1)$ is true and $P(K) \rightarrow P(K+1)$ is true. So

$P(n)$ is true for odd positive integers.

EXAMPLE 3:

Use mathematical induction to show that

$$1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$$

Sol:

Let $P(n) : 1 + 2 + 2^2 + \dots + 2^n = 2^{n+1} - 1$ for all non-negative integers n .

BASIS:

$P(0)$ is true as :

$$1 = 2^{0+1} - 1 = 2 - 1 = 1$$

INDUCTIVE STEP :

Inductive hypothesis :

Suppose $P(n)$ is true for $n=k$ i.e. $P(k)$ is true. So we have

$$1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1$$

To Show :

True for $P(k+1)$ i.e.,

$$1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} = 2^{k+2} - 1$$

Consider the inductive hypothesis,

$$1 + 2 + 2^2 + \dots + 2^k = 2^{k+1} - 1$$

Add 2^{k+1} on both sides, we have

$$1 + 2 + 2^2 + \dots + 2^k + 2^{k+1} = 2^{k+1} - 1 + 2^{k+1}$$

$$= 2 \cdot 2^{k+1} - 1 \quad \text{as } (1 \cdot 2^{k+1} + 1 \cdot 2^{k+1} = 2 \cdot 2^{k+1})$$

$$= 2^{k+2} - 1$$

i.e., True for $P(k+1)$: (proved)

As $P(0)$ is true and $P(k) \rightarrow P(k+1)$ is true, So

$P(n)$ is true for all non-negative integers n .

Example 4:

Use mathematical induction to show the formula for the sum of a finite number of terms of a geometric progression:

$$\sum_{i=0}^n ar^i = a + ar + ar^2 + \dots + ar^n = \frac{ar^{n+1} - a}{r-1} \quad \text{when } r \neq 1$$

where n is non-negative integer.

Sol:

Let

$P(n)$: Sum of the first $n+1$ terms of a geometric progression is $\frac{ar^{n+1} - a}{r-1}$

- non-negative integer
Contain 0 as 1st element
- Positive integer
Starts at 1.

BASIS:

$P(0)$ is true as:

$$a = \frac{ar^{0+1} - a}{r-1} = \frac{ar - a}{r-1} = \frac{a(r-1)}{r-1} = a$$

INDUCTIVE STEP:

Inductive Hypothesis:

Suppose $P(n)$ is true for $n=k$ i.e. $P(k)$ is true, we have

$$a + ar + ar^2 + \dots + ar^k = \frac{ar^{k+1} - a}{r-1}$$

To show:

True for $P(k+1)$ i.e.,

$$a + ar + ar^2 + \dots + ar^k + ar^{k+1} = \frac{ar^{k+2} - a}{r-1} \quad (\text{To prove})$$

Consider the inductive hypothesis:

$$a + ar + ar^2 + \dots + ar^k = \frac{ar^{k+1} - a}{r-1}$$

Add ar^{k+1} on both sides, we have

$$\begin{aligned} a + ar + ar^2 + \dots + ar^k + ar^{k+1} &= \frac{ar^{k+1} - a}{r-1} + ar^{k+1} \\ &= \frac{ar^{k+1} - a + (r-1)ar^{k+1}}{r-1} \\ &= \frac{\cancel{ar^{k+1}} - a + r \cdot \cancel{ar^{k+1}} - \cancel{ar^{k+1}}}{r-1} \\ &= \frac{ar^{k+2} - a}{r-1} \quad (\text{Proved}) \end{aligned}$$

As $P(0)$ is true and $P(k) \rightarrow P(k+1)$ is true
So, $P(n)$ is true.

EXAMPLES OF PROVING INEQUALITIES:

EXAMPLE 1:

Use mathematical induction to show that

$$n < 2^n$$

for all positive integers n .

Sol:

Let $P(n): n < 2^n$

BASIS:

$P(1)$ is true as:

$$1 < 2^1$$

INDUCTIVE STEP:

Inductive Hypothesis:

Suppose $P(n)$ is true for $n=k$ i.e., $P(k)$ is true.

$$k < 2^k$$

To Show that

$$k+1 < 2^{k+1}$$

Consider the inductive hypothesis:

$$k < 2^k$$

Add 1 on both sides

$$k+1 < 2^k + 1$$

$$< 2^k + 2^k$$

$$< 1 \cdot 2^k + 1 \cdot 2^k$$

$$< 2 \cdot 2^k$$

$$< 2^{k+1} \quad (\text{proved})$$

• As $1 \leq 2^k$: good vs bad cheat (we adopted good cheat)

• We Can add anything in greater side of inequality (Inequality still holds)

So, $P(k+1)$ is true.

As $P(1)$ is true and $P(k) \rightarrow P(k+1)$ is true, So $P(n)$ is true.

Exempl 2 :

Use mathematical induction to prove

$$2^n < n!$$

for every positive integer n with $n \geq 4$. Note: this inequality is false for $n=1, 2, 3$

Sol:-

Let $P(n) : 2^n < n!$

BASIS:

$P(4)$ is true as :

$$2^4 < 4!$$

$$16 < 24 \quad (\text{as } 4! = 4 \times 3 \times 2 \times 1 = 24)$$

INDUCTIVE STEP :

Inductive Hypothesis :

Suppose $P(n)$ is true for $n = k$ i.e true for $P(k)$ as

$$2^k < k!$$

To show that

$P(k+1)$ is true (To prove)

i.e $2^{k+1} < (k+1)!$

Take the L.H.S of the $(k+1)$ Case

$$2^{k+1}$$

and we can write it as

$$2^{k+1} = 2 \cdot 2^k \quad \text{--- ①}$$

From inductive hypothesis, we know that

$$2^k < k!$$

Place this result in eq ① and focus on equality of ①. (will replace = by <)

$$2^{k+1} = 2 \cdot \underline{2^k}$$

$$< \underline{2} \cdot k!$$

$$< (k+1) \cdot k!$$

$$< (k+1)! \quad \text{Proved}$$

{ good cheat : $2 < k+1$, replace 2 by $k+1$, no effect on inequality

$$\begin{aligned} 6! &= 6 \cdot 5! \\ (5+1)! &= (5+1) \cdot 5! \end{aligned}$$

So, $P(n)$ is true

Example 3:

Use mathematical induction to show (Inequality of Harmonic Numbers)

$$H_n \geq 1 + \frac{n}{2} \quad \text{whenever } n \text{ is non-negative}$$

Harmonic numbers H_j , $j=1, 2, 3, \dots$ are defined by

$$H_j = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{j} \quad \left(\text{For instance, } H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} = \frac{25}{12} \right)$$

Sol:

$$\text{Let } P(n): H_n \geq 1 + \frac{n}{2}$$

BASIS:

$P(0)$ is true as

$$H_0 \geq 1 + \frac{0}{2}$$

$$H_1 \geq 1$$

$$1 \geq 1 \text{ holds as } (H_j = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{j}, \text{ so } H_1 = 1)$$

INDUCTIVE STEP:

Inductive Hypothesis:

Suppose $P(n)$ is true for $n=k$ i.e. true for $P(k)$, so we have

$$H_k \geq 1 + \frac{k}{2}$$

$$\text{or } 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k} \geq 1 + \frac{k}{2}$$

To show:

$P(k+1)$ is true i.e.,

$$H_{k+1} \geq 1 + \frac{k+1}{2}$$

Consider the L.H.S of $P(k+1)$ i.e.

$$H_{k+1}$$

and it can be written as

$$H_{k+1} = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k} + \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \dots + \frac{1}{2 \cdot 2^k}$$

$$= H_k + \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \dots + \frac{1}{2^{k+1}} \quad \left(\text{By Def. as } H_k = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^k} \right)$$

$$\geq \left(1 + \frac{k}{2} \right) + \frac{1}{2^{k+1}} + \frac{1}{2^{k+2}} + \dots + \frac{1}{2^{k+1}} \quad \left(\text{using inductive hypothesis here and change } = \text{ with } \geq \right)$$

$$\geq \left(1 + \frac{k}{2} \right) + 2 \cdot \frac{1}{2^{k+1}}$$

$$\geq \left(1 + \frac{k}{2} \right) + \frac{1}{2} = 1 + \frac{k+1}{2}$$

$$\geq 1 + \frac{k+1}{2} \quad (\text{proved})$$

$$H_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

$$H_{4 \cdot 2} = H_8 = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}$$

$$H_{2^{2+1}} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \frac{1}{10} + \frac{1}{11} + \frac{1}{12} + \frac{1}{13} + \frac{1}{14} + \frac{1}{15} + \frac{1}{16}$$

$$\begin{aligned} \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} &= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \quad (\text{i.e. 2 terms 4 terms}) \\ &= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \quad \text{Every term is } > \frac{1}{8} \\ &\geq 4 \cdot \frac{1}{8} \quad \text{replace equal with } \geq \\ &\geq 2 \cdot \frac{1}{2^{2+1}} \quad \text{Small can be placed in smaller side} \end{aligned}$$

EXAMPLES OF PROVING DIVISIBILITY RESULTS

EXAMPLE 1:

Use mathematical induction to prove that

$n^3 - n$ is divisible by 3 whenever n is positive integer.

Solution:

Let $P(n) : n^3 - n$ is divisible by 3

BASIS:

$P(1)$ is true as

$$n^3 - n = 1^3 - 1 = 0 \text{ is divisible by 3.}$$

INDUCTIVE STEP:

Inductive Hypothesis:

Suppose $K^3 - K$ is divisible by 3. (i.e., $P(K)$ is true)

To show:

$(K+1)^3 - (K+1)$ is divisible by 3 (To prove, $P(K+1)$ is true)

Consider the $P(K+1)$ step:

$$(K+1)^3 - (K+1)$$

and we have to show that it is divisible by 3 as:

$$(K+1)^3 - (K+1) = (K^3 + 3K^2 + 3K + 1) - (K+1)$$

$$= K^3 + 3K^2 + 3K + 1 - K - 1$$

(no simplification of $3K - K$ is required. Because we have to find inductive hypothesis here)

$$= (K^3 - K) + 3(K^2 + K)$$

$K^3 - K$ is divisible by 3 : inductive hypothesis

$3(K^2 + K)$ is divisible by 3 : as it is 3 times an integer

So, $(K^3 - K) + 3(K^2 + K)$ is divisible by 3 : $\left(\begin{array}{l} a \text{ is divisible by } j \\ b \text{ is divisible by } j \end{array} \Rightarrow \begin{array}{l} j|a+b \\ j|a, j|b \\ j|a+b \end{array} \right)$

So, we can say that

$(K+1)^3 - (K+1)$ is divisible by 3 (proved i.e. true for $P(K+1)$)

As $P(1)$ is true and $P(K) \rightarrow P(K+1)$ is true, So

$P(n)$ is true for all positive integers.

Example 2 :

Use mathematical induction to prove that

$7^{n+2} + 8^{2n+1}$ is divisible by 57
for every non-negative integer n .

Sol :

Let $P(n) : 7^{n+2} + 8^{2n+1}$ is divisible by 57

BASIS :

$P(0)$ is true as

$$7^{0+2} + 8^{2 \cdot 0 + 1} = 57 \text{ is divisible by } 57$$

INDUCTIVE STEP :

Inductive ~~step~~ Hypothesis :

Suppose $P(k)$ is true i.e

$$7^{k+2} + 8^{2k+1} \text{ is divisible by } 57$$

To Show: $P(k+1)$ is true

$$7^{(k+1)+2} + 8^{2(k+1)+1} \text{ is divisible by } 57$$

Consider the $P(k+1)$ step as:

$$\begin{aligned} 7^{k+3} + 8^{2k+3} &= 7 \cdot 7^{k+2} + 8 \cdot 8^{2k+1} \\ &= 7 \cdot 7^{k+2} + 64 \cdot 8^{2k+1} \\ &= 7 \cdot 7^{k+2} + (7 + 57) \cdot 8^{2k+1} \\ &= 7 \cdot 7^{k+2} + 7 \cdot 8^{2k+1} + 57 \cdot 8^{2k+1} \\ &= 7 \cdot (7^{k+2} + 8^{2k+1}) + 57 \cdot 8^{2k+1} \end{aligned}$$

divisible by 57 divisible by 57

divisible by 57

So, $7^{k+3} + 8^{2k+3}$ is divisible by 57

$\Rightarrow P(k+1)$ is true

So,

$P(n)$ is true for every non-negative integer n .

$$\begin{aligned} j | a &\Rightarrow j | at \\ j | b &\Rightarrow j | b2 \\ j | a+b \end{aligned}$$