# 2.5 CARDINALITY OF SETS

We use the cardinality of **finite sets** to tell us when they have the same size or when one is bigger than the other.

Here, we extend this to Infinite Sets.

For Infinite sets, the definition of cardinality provides relative measure of the sizes of the two sets rather than a measure of the size of one particular set.

## COUNTABLE SET:

- A set that is either finite or has the same cardinality as the set of positive integers or

  An infinite set $S$ is called countably infinite, it is similar to natural number $N$.

- An infinite set is countable if and only if it is possible to list the elements of the set in a sequence (indexed by +ve integers) i.e it can be expressed in terms of sequence $a_1, a_2, a_3, \dots, a_n, \dots$

  where $f(x) = a_x$ i.e $f(1) = a_1$, $f(2) = a_2$, … $f(n) = a_n$, ….

Cardinality of countable infinite set $S$ can be denoted by $\aleph_0$ (called aleph null) i.e.,

$$|S| = \aleph_0$$

this set is the smallest set among the infinite sets.

Examples: $N$ (natural numbers), $W$ (whole numbers), $Z$ (Integers), $Q$ (Rational numbers) etc.

## UNCOUNTABLE SET:

A set is said to be uncountable if it is not countable i.e neither finite nor countably infinite.

e.g., $P(N)$: Power set of $N$, $R$: Real numbers, $Q^c$: Irrational numbers

# EXAMPLES OF COUNTABLE SETS



## Example #1

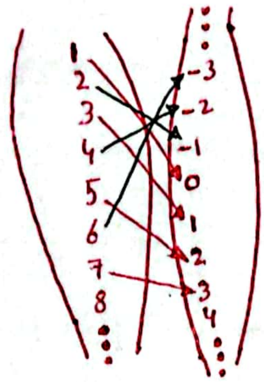
Show that the set of all integers is COUNTABLE.

**Sol:**

The set of Integers has the same size as the set of natural numbers.

> An infinite set is countable iff it is possible to list the elements in a sequence (indexed by positive integers).

We can list all integers in a sequence starting with 0 and alternating between positive and negative integers.




sequence:

$$\{a_1, a_2, a_3, a_4, \ldots\}$$

$$N = \{1, 2, 3, 4, 5, 6, \ldots\}$$
$$\downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow\ \downarrow$$
$$Z = \{0, -1, 1, -2, 2, -3, \ldots\}$$

we have one-to-one Correspondence between Integers & Natural Numbers.

So, we can say that the set of integers has the same size as the natural number i.e

$$|N| = |Z| = \aleph_0$$

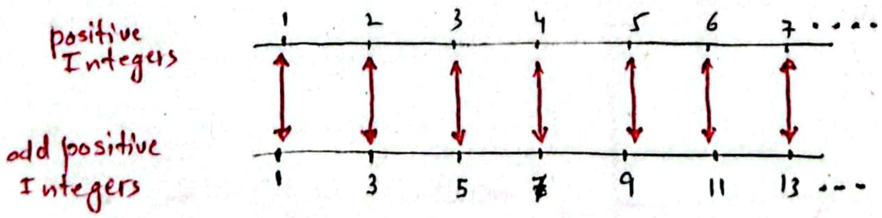
As, it is possible to list the elements of integer set in a sequence (indexed by positive integers). So Set of all integers is Countable.

## Example #2

Show that the set of odd positive integers is Countable.

**Sol,**

we can list all odd positive integers indexed by natural numbers (indexed by positive integers) as:




$$\text{Sequence}: \{a_1, a_2, a_3, \ldots\}$$

$$N = \{1, 2, 3, 4, 5, \ldots\}$$

$$\text{odd +ve Integers} = \{1, 3, 5, 7, 9, \ldots\}$$

So, it is Countable and we can write $|N| = |\text{odd +ve Integers}| = \aleph_0$

To show that odd positive integer Countable, we have to prove that there should be one-to-one Correspondence between odd positive Integers and positive Integers (natural numbers).

one-to-on Correspondence means $f$ is both
- one-to-one function &
- onto function

i.e $f$ is BIJECTIVE Function

Consider the function

$f(n) : 2n-1$ $\quad f: N \to$ odd +ve integers

To see $f$ is one-to-one function

Suppose $f(n) = f(m)$

$\Rightarrow 2n-1 = 2m-1$, so $n = m$

To see $f$ is onto

Suppose that $t$ is an odd +ve integer, then

$t$ is 1 less than an even integer $2K$ ($K$ is natural number)

Hence $t = 2K-1$

$= f(K)$ $\quad K$ is natural

So, there is one-to-one Correspondence between them.

$f(n) = 2n-1$

$f(1) = 1$

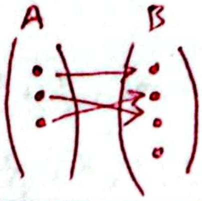
$f(2) = 3$

$f(3) = 5$

$\equiv$

**one-to-one function (Injective):**

Injective function does not map two distinct elements in A onto the same element of B.




**onto function (Surjective)**

where every element in B is of the form $f(a)$

where image is equal to Codomain.

**Bijective (Both one-to-one & onto)**

Every element in A is mapped to unique element in B and every element in B is of the form $f(a)$ for some elements in A

## Example # 3

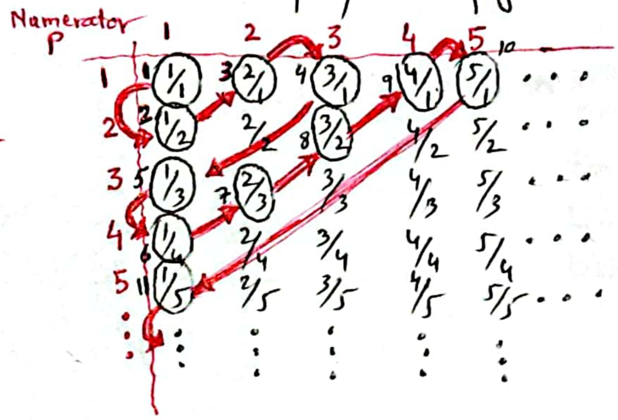
Show that the set of positive rational numbers is countable.

**Sol:-**

Surprisingly, set of positive rational numbers can be listed as a sequence of $r_1, r_2, r_3, \ldots, r_n, \ldots$

Here $\quad Q^+ = \dfrac{p}{q}$ , $p$ & $q$ are +ve integers

We can arrange the positive rational numbers by listing those with denomenator $q=1$ in the first row, those with denomenator $q=2$ in the second row and so, on as displayed in figure below:




Sequence:

first +ve rational $p/q$ (with $p+q=2$)
2nd +ve rational $p/q$ (with $p+q=3$) and so on

whenever we encounter a number $p/q$ that is already listed, we donot list it again (ignore it).

Sequence: $\{r_1, r_2, r_3, \ldots\}$

$$N\text{(natural)} : \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots\}$$
$$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$$
$$Q^+ : \{1, \tfrac{1}{2}, 2, 3, \tfrac{1}{3}, \tfrac{1}{4}, \tfrac{2}{3}, \tfrac{3}{2}, 4, 5, \ldots\}$$

So, we have shown that the set of positive rational numbers is COUNTABLE.

So, $|N| = |Q^+| = \aleph_0$

**NOTE:** If we find a pattern in a set and you should be able to tell, this one is first and the next one is second and so on, then this set must be countably infinite i.e you can list it with natural numbers

Example # 4

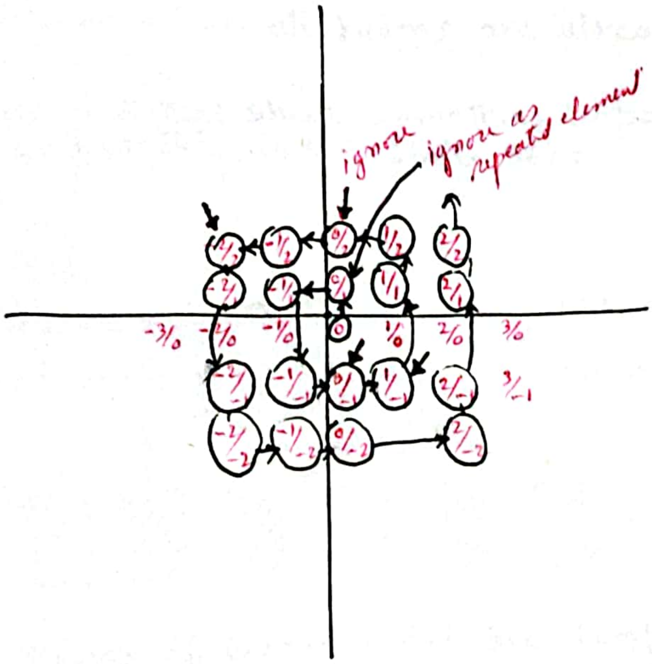
Show that the Set of rational numbers is Countable.

Sol:

Surprisingly, it is also Countable.

Rational numbers are dense in the sense that there are infinitely many rational numbers between any two natural numbers.

Rational numbers Canbe Listed as a sequence as displayed below as:




* whenever we encounter p/q that is already Listed, we donot List it again.

This mapping certainly maps every rational number to natural number because every rationtional number appear somewhere in the grid and the spiral hits every point in the grid (except along x-axis).

As,

Sequence = $\{r_1, r_2, r_3, \ldots\}$

$N = \{1, 2, 3, 4, 5, 6, \ldots\}$

$\updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow \quad \updownarrow$

$Q = \{0, -1, 1, \tfrac{1}{2}, -\tfrac{1}{2}, -2, \ldots\}$

So, Rational Numbers are Countable.

# HILBERT GRAND HOTEL : (Hypothetical Grand Hotel)

The famous mathematician David Hilbert inverted the notion of Grand Hotel, which has Countably infinite number of rooms, each occupied a guest.

- When a new guest arrives at a hotel with a finite # of rooms and all rooms are occupied, this guest can not be accomodated without evicting a current guest.

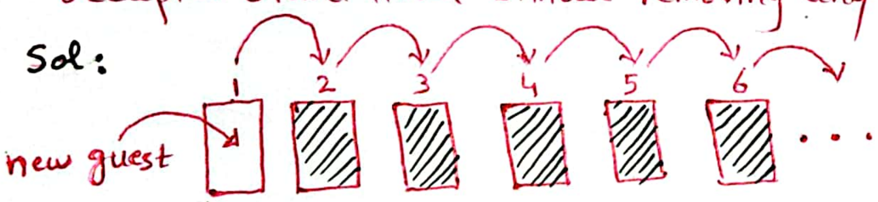
- However, we always accomodate a new guest at the Grand Hotel, even when all rooms are already occupied.

  (( A paradox that shows something impossible with finite sets may be possible with infinite sets ))

**Example #1**

How can we accomodate a new guest arriving at the fully occupied Grand Hotel without removing any of the current guest?

Sol:




$\infty + 1 = \infty$

As the rooms of Grand Hotel are computable, we can list them as Room1, Room2, .... and So on.

To accomodate a new guest,

Move the guest in room $n$ to $n+1$ i.e

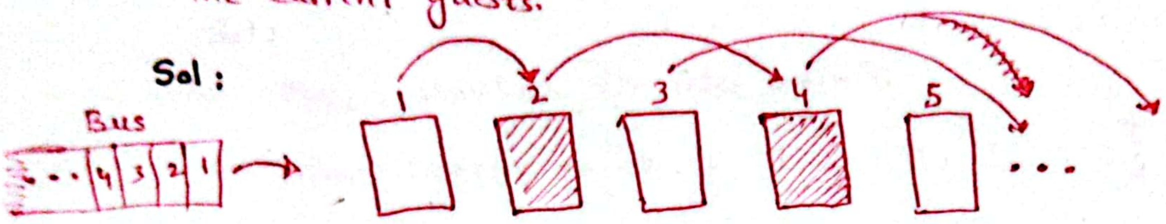
- guest currently in room 1 goes to room 2
- guest currently in room 2 goes to room 3
- and So on (show in figure above)

and put the new guest in room 1.

**Example #2** (Bus With Infinite Guests)

How Can we accomodate a bus with infinitely number of guests arriving at the fully occupied Grand Hotel without removing any of the current guests.

**Sol :**




To accomodate Countably infinitely many new guests,

Move the guest in room $n$ to room $2n$ i.e.

- guest currently in room 1 goes to room 2
- guest currently in room 2 goes to room 4
- guest currently in room 3 goes to room 6 and so on.

and put the new (infinite) guests in rooms $2n-1$ i.e

new guest 1 in room 1
new guest 2 in room 3
new guest 3 in room 5
and so on.

Example #3 ( Infinite Buses with Infinite guests)

How to accomodate infinite buses with infinitely number of guests arriving at fully occupied Grand Hotel.

Sol:

Prime Numbers are also infinite

Prime Numbers = { 2, 3, 5, 7, 11, 13, ---- }

To accomodate infinite buses with infinite guests,

For Shifting, take the prime number :

prime number

room #

Move the guest Currently in room 1 goes to $= 2^1 = 2$

Move the guest in room 2 goes to room $= 2^2 = 4$

Move the guest in room 3 goes to room $= 2^3 = 8$

Move " " " room 4 goes to room $= 2^4 = 16$

and so on.

and put the guest in Bus 1 (take next prime # for Bus 1) in room #s :

Seat #

guest of Bus 1 on seat Number 1 goes to room $= 3^1 = 3$

guest of Bus 1 on Seat 2 goes to room $= 3^2 = 9$

guest of Bus 1 on Seat 3 goes to room $= 3^3 = 27$ and So on.

put the passengers in Bus 2 (take the next prime # i.e 5 for Bus 2) in room #s.

guest of Bus 2 on Seat 1 goes to room $= 5^1 = 5$

" " " " Seat 2 goes to room $= 5^2 = 25$

and So on

⋮

Similarly for Bus 3, 4, and So on (as the prime # are also infinite). So we can accomodate Infinite buses.

Example #4 ( guest identified by rational numbers)

No answer to accomodate rational numbers at that time. German Mathematician proved that rational numbers Can be represented by natural numbers. Hilbert Grand Hotel is definitely Can accomodate all rational number's guests. In 1873, Cantor proved this is wrong that Hilbert Hotel Can accomodate all type of guests.

# AN UNCOUNTABLE SET

A set that is not Countable is called Uncountable set.

Examples are Real Numbers, Irrational Numbers etc.

## Example:

Show that the set of real numbers is an uncountable set.

## Sol:

We use a Contradiction here.
Suppose that the set of real numbers is Countable.
(i.e they can be listed)

Take a subset of R i.e $(0,1) \in \mathbb{R}$, open interval $(0,1)$.
(and we have to show that this subset of real numbers is Uncountable.)

First, we suppose this subset is Countable (a Contradiction).

As the subset of real numbers $(0,1)$ is Countable, so it can be listed in some order $r_1, r_2, r_3, \ldots$.

Let the decimal representation of real numbers between 0 & 1 be

$$r_1 = 0.d_{11}d_{12}d_{13}d_{14}\cdots$$
$$r_2 = 0.d_{21}d_{22}d_{23}d_{24}\cdots$$
$$r_3 = 0.d_{31}d_{32}d_{33}d_{34}\cdots$$
$$\vdots \qquad \vdots$$

Where $d_{ij} = \{0,1,2,3,\ldots,9\}$

Here we claim that we labelled all the elements in the interval 0 and 1.
(i.e this is the list of all elements between 0 & 1)

Now we form a new real number between 0 & 1 with decimal expansion

$$r = 0.b_1b_2b_3b_4\cdots$$

where decimal digits $b_1, b_2, b_3, \ldots$ are selected by the rule:

$$b_1 \neq d_{11}, \; b_2 \neq d_{22}, \; b_3 \neq d_{33} \text{ etc.} \quad (d_{11}, d_{22}, d_{33} : \text{diagonal})$$

i.e Choose $b_1$ different from $d_{11}$, $b_2$ different from $d_{22}$ and so on

As $b_1 \neq d_{11}$, so it confirms that $r$ cannot be equal to $r_1$
As $b_2 = d_{22}$, so it confirms that $r$ is not the same as $r_2$
and so on. and finally
we conclude that the real number $r$ is not in the list.

Because there is a real number $r = 0.b_1 b_2 b_3 \ldots$ between 0 and 1 that is not in the list, the assumption that all real numbers between 0 & 1 could be listed must be wrong.

Therefore, all the real numbers between 0 and 1 cannot be listed, so the set of real numbers between 0 and 1 is Uncountable.

As, Any set with uncountable subset is Uncountable, So

Set of real numbers is uncountable.

As an example,

Suppose
$$r_1 = 0.23794102\ldots \qquad d_{11} = 2$$
$$r_2 = 0.44590138\ldots \qquad d_{22} = 4$$
$$r_3 = 0.09118764\ldots \qquad d_{33} = 1$$
$$r_4 = 0.80553900\ldots \qquad d_{44} = 5$$
$$\vdots \qquad\qquad\qquad \vdots$$

we can choose $r = 0.4936\ldots$ (by choosing $b_1 \neq d_{11}, b_2 \neq d_{22}$ etc.)

this numble must not be in the list of all real numbers between 0 and 1.

So $(0,1) \in R$, but is not in the list i.e cannot label all the elements between 0 and 1, So $(0,1) \in R$ is not Countable.

As $(0,1) \subset R$

and any set with uncountable sub is Uncountable

$\Rightarrow$ Real number Set is not Countable.

THEOREM:

If A and B are Countable sets, then $A \cup B$ is also Countable.

Proof:

Suppose A and B are Countable and we assume that A & B are disjoint. Furthermore, if one of the two sets is Countably infinite and other finite. we can assume that B is the one that is finite.

There are three Cases to consider:

(i) A and B are both finite,

(ii) A is infinite and B is finite,

(iii) A and B are both Countably infinite.

(i) when A and B are finite, $A \cup B$ is also finite, so $A \cup B$ is Countable

(ii) Because A is Countably infite, its elements canbe liste in an infinite Sequence

$$A \sim a_1, a_2, a_3, \ldots \quad \text{and}$$

(finite) $B \sim$ ~~finite~~ $b_1, b_2, \ldots b_m$ for some +ve Integer m

then $A \cup B \sim b_1, b_2, b_3, \ldots b_m, a_1, a_2, \ldots\ldots$
(Canbe labelled in sequence)

So $A \cup B$ is Countably infinite.

(iii) As A and B are both Countably infinite

i.e $A \sim a_1, a_2, a_3, \ldots$
$B \sim b_1, b_2, b_3, \ldots$

and $A \cup B$ Can be written as (it Canbe lited)

$$A \cup B = a_1, b_1, a_2, b_2, \ldots \quad \text{(alternatiterm Sequence)}$$

So $A \cup B$ is Countably infinite.

So, we Can Conclude that

$A \cup B$ is Countable

## Natural Numbers (or Positive Integers), Integers, Rational Numbers

We have shown that the set of integers Z and the set of rational numbers Q have same cardinality as there is one-to-one correspondence betweet these and the set of natural numbers and we can denote their cardinality by $\aleph_0$ (aleph naut).

$$|Z^+| = |Z| = |Q| = \aleph_0 \quad \left(\text{The smallest infinite set among the infinite sets}\right)$$

## Real Number Set

We have shown, the set of real numbers is uncountable and its cardinality ~~canbe~~ is denoted as by c.

$$|R| = c$$

It canbe shown that

the power set of natural numbers and the set of real numbers have the same cardinality.

i.e $$|P(Z^+)| = |R| = c$$

Also, we know that

cardinality of a set is always less than the cardinality of its power set.

$$|Z^+| < |P(Z^+)|$$

or $$\aleph_0 < 2^{\aleph_0}$$

The Continuum Hypothesis asserts that

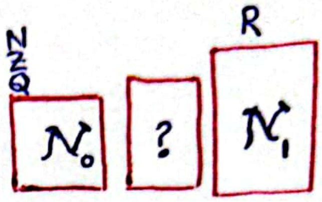
there is no set Larger than natural number and less than Real numbers.

If $\aleph_0$ is the smallest infinite set, then form of the infinite sequence should be $\aleph_0 < \aleph_1 < \aleph_2 \cdots$

If we assume that continuum hypothesis true, then it would follow that $|R| = \aleph_1$ $(\aleph_0 < \aleph_1)$: no other infinite symbol

The Continuum hypothesis is still an open question.

The Continuum hypothesis proposed by Cantor in 1877, but unsuccessfull to prove it.




It canbe neither proved nor disproved.

(treated as PARADOX).

Sofar,

no one has been able to prove an ~~number~~ infinite set that is greater than natural number and less than real numbers.

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