5·2: STRONG INDUCTION

Strong Induction is a form of mathematical induction where instead of assuming

$P(K)$ is true in our inductive hypothesis

we assume

$P(j)$ is true for $1 \leq j \leq K$

To prove that $P(n)$ is true for all positive integers n, where $P(n)$ is a propositional fuction, we Complete two steps:

BASIS :

We verify that the proposition $P(1)$ is true

INDUCTIVE STEP :

We show that the Conditional statement

$$[P(1) \wedge P(2) \wedge \cdots \wedge P(K)] \rightarrow P(K+1)$$

is true for all positive integers K.

i.e.,

If $P(1) \wedge P(2) \wedge \cdots P(K)$ ~~is true~~, **then**

$P(K+1)$

MEANS

We have $[P(1) \wedge P(2) \wedge \cdots P(K)]$ is true : Inductive Hypothesis

To Show :

$P(K+1)$ is true

EXAMPLE 1:

Suppose we can reach the first and second rungs of an infinite ladder and we know that if we can reach a rung, then we can reach two rungs higher.
Can we prove that we can reach every rung using principle of mathematical induction?
Can we prove that we can reach every rung using strong induction?

Sol:-
Let $P(n)$: We can reach every (i.e n) rung of an infinite Ladder

◉ Mathematical induction:

Basis :

$P(1)$ is true , $P(2)$ is true (given : we can reach 1st & 2nd rung)

Inductive Step:

Suppose we can reach K rung of ladder

To prove : we can reach $K+1$ rung of ladder

We donot know from the given information that we can reach $(K+1)$st rung from the Kth rung. (we only know if we can reach a rung, we can reach the rung two higher)

◉ Strong Induction:

Basis :

$P(1)$ and $P(2)$ are true.

Inductive Step :

Inductive Hypothesis:
Suppose that we can reach K steps through 1, 2, 3 upto K-rung.
i.e $P(1), P(2), \ldots P(K)$ are true i.e., $P(1) \wedge P(2) \wedge \ldots \wedge P(K)$.

To Show :

We can reach $(K+1)$st rung i.e $P(K+1)$ is true.

We have inductive hypothesis :

$P(1)$ is true : we can reach 1st rung ⎫ Basis
$P(2)$ is true : we can reach 2nd rung ⎭ (we already know)

$P(3)$ is true ; if we are in step one, you can reach two steps higher $1+2=3$

$P(4)$ is true : $P(4)$ is true as $P(4-2) = P(2)$ is true

⋮

$P(K)$ is true : as $P(K-2)$ is true

$P(1) \wedge P(2) \wedge \ldots \wedge P(K)$ is true.

∴ So we can reach $(K+1)$s rung from the $(K-1)$st rung, because we know from inductive hypothesis, we can reach $(K-1)$st rung, As we can reach $(K-1)$st rung, So according to statement we can reach two steps higher (i.e $K-1+2 = K+1$). So $P(K+1)$ is true.

EXAMPLE 2:

Show that if n is an integer greater than 1, then n canbe written as the product of primes.

Solution:

$P(n)$: If n is an integer > 1, then n canbe written as product of Primes

Base: True for $n=2$

$P(2)$ is true : 2 canbe written as the product of one prime

Inductive Step:

Inductive Hypothesis:

Supose that $P(j)$ is true for $2 \leq j \leq K$ (i.e. for all +ve integers j with $j \leq K$)
i.e.,

Assumption: j canbe written as the product of primes
(True for $P(2) \wedge P(3) \wedge \dots P(K)$)

Note:

Sometimes we will prove specific values are true for $2 \leq j \leq K$, and Sometimes just assume that $P(j)$ is true for $2 \leq j \leq K$.

Here we have assumed $P(2) \wedge P(3) \wedge \dots P(K)$ is true..

you can also prove it about the list of small values of n, as

$n=2$: $P(2)$ is true
$n=3$: True for $n=3$ (as 3 is prime)
$n=4$: True for $n=4$ (as 4 canbe written as $4 = 2 \cdot 2$, product of prims)
$n=5$: True for $n=5$ (Prime)
$n=6$: True for $n=6$ (as $6 = 2 \cdot 3$)
$n=7$: True for $n=7$ (Prime)
$n=8$: True for $n=8$ (as $8 = 2 \cdot 2 \cdot 2$)
$n=9$: True for $n=9$ (as $9 = 3 \cdot 3$)

So, we have seen here $P(j)$ is true for $2 \leq j \leq 9$ (Here $K=9$)
you can choose any K.

TO PROVE:

True for $P(K+1)$ i.e., $K+1$ canbe written as product of primes
(as shown above for K values)

We have an idea from the small list of values of n that

Either n is prime or

it is Composite an canbe written as number $\cdot$ other number

So, consider two cases here, Either $K+1$ is prime or $K+1$ is composite

To prove $K+1$ can be written as product of primes:

CASE 1: $K+1$ is prime (i.e. $K+1$ canbe written as product of one prime)

$P(K+1)$ holds : as we have assumed $P(2) \wedge P(3) \wedge \ldots \wedge P(K)$ is true

CASE 2: $K+1$ is composite

As $K+1$ is composite, so by the definition of composite, $K+1$ can be written as a product of factors **a** and **b**, where

$$2 \leq a \leq b < K+1 \quad : a \,\&\, b \text{ are factors}$$

By our inductive hypothesis:

We know that it's going to start, 2 is the smallest value that canbe written as product of primes and it goes all the way up to K (less than $K+1$) that can be written as product of primes.

In

$$2 \leq a \leq b < K+1 \quad \text{—— (1)}$$

the factors a and b are less than $K+1$ i.e these factors are within the range of K (i.e a and b are ~~are~~ equal to or less than K) and we know by the inductive hypothesis, factors 2, 3, 4, upto K canbe written as product of primes.

So, here in (1)

factors of $K+1$ are primes

As we know by the definition of composite,

$K+1$ canbe written as product of factors a, b

And by using the Inductive hypothesis, the above statement becomes

$K+1$ canbe written as product of primes

So, $P(K+1)$ holds. Now we can conclude That

$$P(n) \text{ is true} \quad \text{(Proved)}$$

WHY DID WE HAVE TO USE STRONG INDUCTION HERE?

We proved this by saying all of the values that are less than $K+1$ or all the values between 2 and K canbe written as product of primes. So, if we don't have this inductive hypothesis for the entire range of values, this proof would not infact work. (as here for $K+1$, we need previous two true results e.g., $P(K)$ and $P(K-1)$, we need two prime factors or two numbers that canbe written as product of primes. only $P(K)$ would not work

EXAMPLE 3(a):

Prove that every amount of postage of 12 cents or more can be formed using just 4-cent and 5-cent stamps.

Solution:

BASIS:

We can form postage of 12, 13, 14 and 15 cents using 4-cent and 5-cent stamps.
Proposition P(12), P(13), P(14), P(15) are true.

P(12) : true (as postage of 12 cents can be formed using three 4-cent stamps)
P(13) : true (as postage of 13 cents can be formed using two 4-cent & one 5-cent stamps)
P(14) : true (as 14 cents can be formed using one 4-cent & two 5-cent stamps)
P(15) : true (as postage of 15 cents can be formed using three 5-cent stamps)

$P(12) \wedge P(13) \wedge \cdots \wedge P(K)$	
12, 13, 14, 15	Basis
16	16−3 = 13 (we have)
17	17−3 = 14 (we have)
18	18−3 = 15
19	19−3 = 16

K steps

INDUCTIVE STEP:

Inductive Hypothesis:

Suppose P(j) is true for $12 \leq j \leq K$, where $K \geq 15$
i.e. We assume that
we can form postage of j cents using 4-cent and 5-cent stamps, where $12 \leq j \leq K$.

To show:

P(K+1) is true
i.e. We can form postage of K+1 cents using 4-cent & 5-cent stamps.

By inductive hypothesis, we have

P(K−3) is true (as we know P(12), P(13), upto P(K) are true
~~Also~~ i.e. True for P(30) if we take K=30
Also P(K−3) = P(15−3) = P(12) is true
Not need for P(14−3) = P(11) ≈ 12 is min. as given in question.)

If we can make postage of K−3 cents, then we can make postage of K+1 cents by adding a 4-cent stamp.

Why did we choose P(K−3)

If we took three away from each of these, we would get $K-3 \geq 12$, and we have already shown it for 12. So we are really taking it back down to that starting point. So we just going to assume that P(K−3) is true.

				K−3	K−2	K−1	K	K+1
12	13	14	15	16	17	18	19	20

we have P(K−3) = P(16) = 4444
P(K+1) = P(20) = 44444 is true

					K−3	K−2	K−1	K	K+1
12	13	14	15	16	17	18	19	20	21

we have P(K−3) = P(17) = 4445
P(K+1) = P(21) = 44454 : true

EXAMPLE 3(b) : USING WEAK INDUCTION ($P(K) \rightarrow P(K+1)$)

Postage of 12 cents or more by using just 4-Cent of 5-Cent Stamps.

Solution:

Let $P(n)$: Postage of n cents Canbe formed using 4-cent and 5-cent stamps if $n \geq 12$

BASIS:

$P(12)$ is true : as postage of 12 cents Canbe made using three 4-Cent Stamps.

INDUCTIVE STEP: $P(K) \rightarrow P(K+1)$

Inductive Hypothesis:

Suppose postage of K cents Canbe formed using 4-cent & 5-cent stamps.

To Prove:

postage of K+1 Cents Canbe formed using 4-Cent & 5-Cent Stamps.

Consider two Cases here :

Case 1: We have one or more 4-Cent stamps.
Case 2: We have not used a 4-Cent stamps.

12 : 444
13 : 445
15 : 555
16 : 4444
17 : 4445
18 : 4455
≡
25 : 55555
26 : 444455
27 : 444555
28 : 445555
≡
35 : 5555555
≡
45 : 555555555
≡

CASE 1: (having one or more 4-cent stamps)

By Inductive hypothesis, we have

Postage of K cents : formed using atleast on 4-Cent Stamp

K+1 Cents : - - 4 - - = - - (4+1) - - = - - 5 - -
(replace 4-Cent Stamp by 5-Cent Stamp) (Proved)

CASE 2: (having no 4-Cent Stamp)

Inductive hypothesis, we have

Postage of K cents : formed by only 5-Cent stamps

K+1 Cents : replace some 5-Cent with some 4-Cent Stamps (Proved)

K = 15 Cents : 555
K+1 = 16 Cents : 4444

K = 25 Cents : 55555
K+1 = 26 Cents : 444455

K = 35 Cents : 5555555
K+1 = 36 Cents : 444444444

K = 55 Cents : 55555555555
K+1 = 56 Cents : 444444445555