6·4 BINOMIAL COEFFICIENTS AND IDENTITIES

The number of r-combinations from a set n elements is often denoted by $\binom{n}{r}$. This number is also called a binomial coefficients because these numbers occur as coefficients in the expansion of powers of binomial expression such as $(a+b)^n$.

THE BINOMIAL THEOREM

The binomial theorem gives the coefficients of the **expansion** of powers of binomial expressions. **The binomial expression** is simply the sum of two terms, such as $x+y$. (The terms can be products of constant and variables).

Let x and y be variables, and let n be a non-negative integer. Then

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{n-j} y^j$$
$$= \binom{n}{0} x^n + \binom{n}{1} x^{n-1} y + \cdots + \binom{n}{n-1} x y^{n-1} + \binom{n}{n} y^n$$

Binomial Expansion

The expansion of $(x+y)^3$ can be found using combinatorial reasoning instead of multiplying the three terms out.

$$(x+y)^3 = (x+y)(x+y)(x+y)$$

As $(x+y)^3$ is order of 3.

So, by using binomial theorem, we know that

Power of :
x starts from order 3 and decreases term by term upto 0

Power of :
y starts from 0 and increases term by term up to order 3.

For order three :

we have four terms including x, y as: (All combinations of x & y of order 3 appears in the expression)

$$x^3 y^0 \quad x^2 y^1 \quad x^1 y^2 \quad x^0 y^3$$

→ x Power : increases
→ y Power : decreases

$$(\ )x^3 + (\ )x^2 y + (\ )x y^2 + (\ )y^3$$

Now, we will guess the coefficients.

(order is same in each term)
x^3, $x^2 y^1$, $x^1 y^2$, y^3
$=3$, $(1+2=3)$, $1+2=3$, $=3$

Coefficients of x^3y^0: Choosing no y from 3 expressions i.e $\binom{3}{0}$ $\quad \overset{exp1}{(x+y)}\overset{exp2}{(x+y)}\overset{exp3}{(x+y)}$

(it can be treated by saying as: Choosing 3 xs out of 3 expressions i.e $\binom{3}{3}$) $\boxed{\binom{3}{0}=\binom{3}{3}}$

Coefficients of x^2y: Choosing 1 y out of 3 expressions i.e $\binom{3}{1}$ Also $\boxed{\binom{3}{1}=\binom{3}{2}}$
or
(Choosing 2 xs out of 3 expressions i.e $\binom{3}{2}$)

Coefficients of xy^2: Choosing 2 ys out of 3 expressions i.e $\binom{3}{2}$
or (Choosing 1 x out of 3 i.e $\binom{3}{1}$)

Coefficients of y^3: Choosing 3 ys out of 3 expression i.e $\binom{3}{3}$
or ...

So we can write it as:

$$(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$$

$\boxed{\text{generalization:}}$

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n-1}xy^{n-1} + \binom{n}{n}y^3$$

$$= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

Note :-
you can use constants, variables (or both) in binomial expression as

$$(3x-2y)$$

EXAMPLE1: Expand $(x+y)^4$ using binomial theorem.

$$(x+y)^4 = \sum_{j=0}^{n}\binom{n}{j}x^{n-j}y^j$$

$$= \binom{n}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

EXAMPLE 2 :

What is the Coefficient of $x^{12} y^{13}$ in the expansions of

(a) $(x+y)^{25}$ (b) $(2x-3y)^{25}$

Sol :

It would be really hard to actually expand these. This example will really demenstrate the power of BINOMIAL THEOREM.

a) By the binomial theorem, we have

$$(x+y)^{25} = \sum_{j=0}^{25} \binom{25}{j} x^{25-j} y^{j}$$

We need to find Coefficients of $x^{12} y^{13}$: (this is a valid term as $12+13=25$ meets the order)

$$(x+y)^{25} = \binom{25}{13} x^{25-13} y^{13}$$

$$= \frac{25!\, x^{12} y^{13}}{13!\,(25-13)!} = \frac{25!}{13!\cdot 12!} x^{12} y^{13}$$

$$= 5{,}200{,}300\, x^{12} y^{13}$$

$\boxed{\begin{array}{l}\text{We only need the term } x^{12} y^{13}\text{, so} \\ 25-j = 12 \\ \Rightarrow j = 25-12 = 13\end{array}}$

So, Coefficient of $x^{12}y^{13}$ is :

5,200,300

b) $(2x-3y)^{25}$

By the binomial theorem, we have

$$(2x+(-3y))^{25} = \sum_{j=0}^{25} \binom{25}{j} (2x)^{25-j} \cdot (-3y)^{j}$$

use $j=13$ to get Coefficient of $x^{12}y^{13}$

$$= \binom{25}{13} (2x)^{25-13} \cdot (-3y)^{13}$$

$$= \frac{25!}{13!\cdot 12!}\, 2^{12} \cdot x^{12} \cdot -3^{13} \cdot y^{13}$$

$$= -\frac{25!}{13!\cdot 12!} \cdot 2^{12} \cdot 3^{13}\, x^{12} y^{13}$$

coefficient

Corollary 1: Let n be a non-negative integer. Then

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof :-

Using binomial theorem:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

with $x = y = 1$

$$(1+1)^n = \sum_{k=0} \binom{n}{k} 1^{n-k} \cdot 1^k$$

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \quad \text{desired result.}$$

Corollary 2: Let n be a positive integer. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Proof: using binomial theorem with $x = 1$, $y = -1$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

$$(1-1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} (-1)^k$$

$$0^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

$$0 = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \quad \text{desired result.}$$

Corollary 3: Let n be a non-negative Integer. Then

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

Proof:

$$(1+2)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot 2^k$$

$$3^n = \sum_{k=0}^{n} 2^k \binom{n}{k} \quad \text{proved.}$$

Pascal's Identity and Triangle

The binomial Coefficients satisfy many different identities. We introduce one of the most important of these now.

PASCAL'S IDENTITY :

Let n and K be positive integers with $n \geq K$. Then

$$\binom{n+1}{K} = \binom{n}{K-1} + \binom{n}{K}$$

For example,

$$\underbrace{\overbrace{\binom{5}{3} + \binom{5}{4}}^{\text{same}}}_{\text{greater of the two i.e } 4} = ? \; \binom{5+1}{4} = \binom{6}{4}$$

This identity is useful and will help to construct Pascal's triangle. Pascal ~~identity~~ triangle is simply a triangle formed by taking binomial coefficients.

PASCAL'S TRIANGLE

Pascal's identity together with the initial Conditions

$$\binom{n}{0} = \binom{n}{n} = 1$$

for all integers n, canbe used to recursively define binomial Coefficients.

$$(x+y)^0 = \sum \binom{0}{0} = \binom{0}{0} \qquad \binom{0}{0}$$

$$(x+y)^1 = \sum_{j=0}^{1} \binom{1}{j} x^{1-j} y^j = \binom{1}{0} x + \binom{1}{1} y \qquad \binom{1}{0} \quad \binom{1}{1}$$

$$(x+y)^2 = \sum_{j=0}^{2} \binom{2}{j} x^{2-j} y^j = \binom{2}{0} x^2 + \binom{2}{1} xy + \binom{2}{2} y^2 \qquad \binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

There is only one way to select 0 out of anything.
$\binom{0}{0} = \binom{1}{0} = \binom{2}{0} = 1$

$$\binom{0}{0}$$

$$\binom{1}{0} \qquad \binom{1}{1}$$

$$\binom{2}{0} \qquad ? \qquad \binom{2}{2}$$

For given n, we are selecting all of them i.e., There is only one way to select everything.
$\Rightarrow \binom{1}{1} = \binom{2}{2} = \binom{3}{3} = 1$

What about inside terms? To Calculate inside, pascal identity is useful.

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

- - - - - - - -

$$\binom{1}{0} + \binom{1}{1} = \binom{1+1}{1} = \binom{2}{1}$$

1 $\quad n=0$

1 1 $\quad n=1$

1 2 1 $\quad n=2$

1 3 3 1 $\quad n=3$

1 4 6 4 1 $\quad n=4$

1 5 10 10 5 1

- - - - - - -

Ex: Find the binomial expansion of $(a+b)^4$ using pascal triangle.

Her $n=4$, So from pascal's Triangle,

1	4	6	4	1
a^4b^0	a^3b^1	a^2b^2	a^1b^3	a^0b^4

$$(a+b)^4 = a^4 + a^3b + a^2b^2 + ab^3 + b^4$$

Note : We Can use Pascal's triangle to directly write a binomial expression.