Fractals and Dimension

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1 Introduction: The Cantor Set

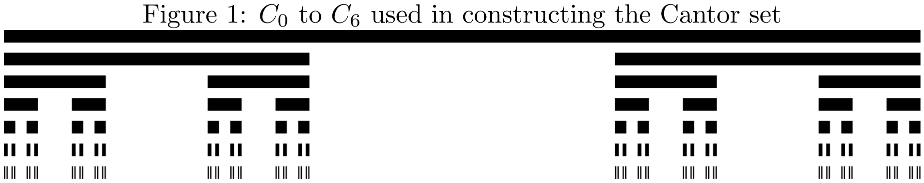
I will start by introducing a set from its analysis in the textbook [5].

1.1 Definition / Construction

Let C_0 be the closed interval $[0, 1]$. Each iteration, take out the open middle third from the interval. For example, $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ and $C_2 = ([0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}]) \cup ([\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1])$.

Name the Cantor set $C = \bigcap_{n=0}^{\infty} C_n$ **(figure 1)**

Figure 1: C_0 to C_6 used in constructing the Cantor set



1.2 Some Properties of the Cantor Set

1.2.1 Infinintely many points:

It is clear that $\forall n \in \mathbb{N}, \frac{1}{3^n} \in C$. So, the Cantor set has at least countably many points, and we will see that it has uncountably many.

1.2.2 Length Zero:

Without defining measure in detail, if $a < b$ then let the length of the interval $[a, b]$ to be $b - a$. From each C_{n-1} to C_n, 2^{n-1} intervals of size $(\frac{1}{3})^n$ are removed. The initial length is 1, so the final length is given by $1 - \frac{1}{3} - \frac{2}{9} - \frac{4}{27} - \cdots = 1 - \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1 - 1 = 0$.

1.2.3 Uncountable:

To see this, represent numbers in tetrinary; digits 0, 1, and 2. For example, 0.2 represents $\frac{2}{3}$, and 0.1201 represents $\frac{1}{3} + \frac{2}{9} + \frac{0}{27} + \frac{1}{81}$. Then, $x \in C <=> x = 0.a_1a_2a_3a_4\cdots$, where $\forall n \in \mathbb{N}, a_n = 0 \vee a_n = 2$.
Contrapositive. Let x = $0.a_1a_2a_3a_4\cdots$, where $\exists n \in \mathbb{N} \ni a_n = 1$. Then, if any other digit after n is not 0, then x is in an open middle third. Then, x is not in the Cantor set. If every digit after n IS 0, we can make an equivalent sequence for x being $0.a_1a_2\cdots a_{n-1}022222\cdots$, so that the sequence will converge to a number that is on the boundary of some closed interval in the construction.
Converse: If $x = 0.a_1a_2a_3a_4\cdots$, where $\forall n \in \mathbb{N}, a_n = 0 \vee a_n = 2$, then x is in each C_n, particularly in the closed interval corresponding to a_n. So, $x \in C$.
Now do the typical proof of uncountability that Canter did the real numbers. Suppose that you could enumerate all numbers in the Cantor set. Then, make a sequence of all numbers in the cantor set, say (x_n). Then, make a new number $y = 0.b_1b_2b_3b_4\cdots$ with the following rule: $b_n = 2$ if the corresponding digit in x_n is 0, and $b_n = 0$ if the corresponding digit in x_n is 2. This number is not in the original enumeration because it is distinct from every x_n. Also, this number is still in the Cantor set, as it still meets the criterion for being in C.
Note that the Cantor set being uncountable implies it contains irrational elements, because if it contained only rational elements then it would be countable (contradiction).

1.2.4 Compact:

Let x_n be a sequence in C. Then, x_n is also a sequence in $[0, 1]$, which itself is compact (Heine-Borel: it is closed and bounded). So there is some subsequence $x_{n_k} \to x \in [0, 1]$. Now, it remains to show that $x \in C$.
Proof by contradiction. Suppose that $x \notin C$. Then, x is in some open middle third of C_N for some $N \in \mathbb{N}$. By openness, there is some $\epsilon > 0$ so that $B(x; \epsilon) \subseteq$ that open middle third. In that case $x_{n_k} \in C_N => x_{n_k} \notin B(x; \epsilon)$. This diasterously fails the convergence, so a contradiction arises from letting $x \notin C$.

Conclude that $x \in C$. Hence, the Cantor set is compact. By Heine-Borel, it is closed and bounded as well.

1.2.5 It is also perfect and totally disconnected.

A set is perfect if it is closed and contains no isolated points; this is equivalent to saying the set is the set of all its limit points.
A set is E is totally disconnected if, given any two distinct points $x, y \in E$, there exist separated sets A and B with $x \in A$, $y \in B$, and $E = A \cup B$.
No point in the Cantor set is isolated, and for any two points there are two separated disjoint sets whose union is the Cantor set. The proofs are less interesting so I will omit them. For perfect, the principal argument is that the Cantor set is closed and any point is in all C_n so that it is contained in a sequence of decreasing intervals. For any ϵ it will have some n where some point $c \in C_n$ satisfies $\epsilon \geq \frac{1}{3^n} > |x - c|$. For totally disconnected, the same argument is done backwards. At some point, the closed intervals get too small to contain two distinct points of finite distance apart.

1.2.6 It is nowhere-dense in $[0, 1]$ and in $\mathbb{R}$

A set is nowhere-dense if its closure contains no nonempty open intervals.
The closure of C is C, because C is closed, as discussed. Without too much rigour, C contains no nonempty open intervals (epsilon balls) because any two points in the ball will be disconnected, which implies there is some point between them not in C. This contradicts the interval being an interval.
What this implies is that the complement of C in $[0, 1]$ and the complement of C in $\mathbb{R}$ is dense. This should be intuitively true, as the length of that complement in $[0, 1]$ is $1 - 0 = 1$, which is the length of the superset. Having a measure that isn't 0 doesn't mean a set can't be dense, since the rationals of length 0 are dense in the real number line of infinite length. However, if the measure of the set is that of the superset, then it is dense in that set: suppose that $[0, 1] - C$ wasn't dense in $[0, 1]$. Then for some $a < b \in [0, 1]$, there is an open interval (a, b) belonging entirely to C, so that the length of $[0, 1] - C$ is $\leq 1 - (b - a)$, a contradiction to its length being 1.

1.2.7 When added to itself, it is the interval $[0, 2]$

Define $S + T = \{s + t : s \in S, t \in T\}$. Then it is not hard to show that $[0, 1] + [0, 1] = [0, 2]$. If $x, y \in [0, 1]$ then $0 \leq x \leq 1, 0 \leq y \leq 1$, so that added together $0 \leq x + y \leq 2$. If $z \in [0, 2]$ then $0 \leq z \leq 2$ so that $0 \leq z/2 \leq 1$, so let $x, y = z/2$.
Now it may also be shown that $C + C = [0, 2]$. To prove this, it is easiest to complete the proof visually and inductively, and then summon the compactness of the Cantor set. Although more rigorous proof exists, it is boring and less instructional, so I'll do it this way. Firstly, $C + C \subseteq [0, 2]$, because $C \subseteq [0, 1]$ (I won't write it out in detail). For the reverse direction, lay out two C_ns on the

$\mathbb{R} \times \mathbb{R}$ plane, i.e. $C_n \times C_n$.
Then if $x \in [0,2]$, a line $x_1 + x_2 = x$ that intersects $C_n \times C_n$ represents that at some point, $\exists x_1, x_2 \in C_n$ such that $x_1 + x_2 = x \in [0,2]$. In fact any line that does this will intersect C_n at least twice because of the interchangeability of x_1 and x_2, but the uniqueness does not matter here.
I claim that any line of the form $x \in [0,2]$, $x_1 + x_2 = x$ WILL intersect $C_n \times C_n$ $\forall n \in \mathbb{Z}_{\geq 0}$. $C_0 \times C_0$ is already known and is obvious. $C_1 \times C_1$ is represented in **figure 2**. This contains 4 subsquares. It is obvious visually, but it can be equivalently shown that $C_1 + C_1 = [0,2]$ by writing. Let us carry forth a modified induction. Let P(n) be the statment: Let $x \in [0,2]$. Then $x_{1n} + x_{2n} = x$ for some $x_{1n}, x_{2n} \in C_n$.
#1: P(0) and P(1) is true.
#2: Let P(n) be true for some $n \in \mathbb{Z}_{\geq 0}$. Then the line $x_{1n} + x_{2n} = x$ intersects $C_n \times C_n$. So the point (x_{1n}, x_{2n}) is in a subsquare of $C_n \times C_n$. Now zoom in to said subsquare, which is now analogous to $C_0 \times C_0$. When changing to $C_{n+1} \times C_{n+1}$, the operation is analogous to switching from the subsquare being $C_0 \times C_0$ to $C_1 \times C_1$. But we already know, the same line will intersect $C_1 \times C_1$, and equivalently will intersect $C_{n+1} \times C_{n+1}$. **Figure 3** can help visualize this. So by letting $x \in [0,2]$, we find $x_{1(n+1)} + x_{2(n+1)} = x$ for some $x_{1(n+1)}, x_{2(n+1)} \in C_{n+1}$. Hence, P(n+1) is true.
P(n) is true $\forall n \in \mathbb{Z}_{\geq 0}$. Now we have constructed some sequence (x_{1n}, x_{2n}). I will now summon the compactness of the Cantor set. Since $x_{1n}, x_{2n} \in C_n \subseteq C$ $\forall n \in \mathbb{Z}_{\geq 0}$, then x_{1n} has subsequence $x_{1n_k} \to x_1 \in C$, so that now x_{2n} has its own subsequence $x_{2n_{k_j}} \to x_2 \in C$. Take that final subsequence. Since $x_{1n} + x_{2n} \to x$ then any subsequence of it will converge to x as well, so by the ALT on the final subsequence, $x_{1n_{k_j}} + x_{2n_{k_j}} \to x$ but also $x_{1n_{k_j}} + x_{2n_{k_j}} \to x_1 + x_2$ so that $x_1 + x_2 = x$, and $x_1, x_2 \in C$.

Figure 2: $C_1 \times C_1$ to construct Cantor dust in 2D

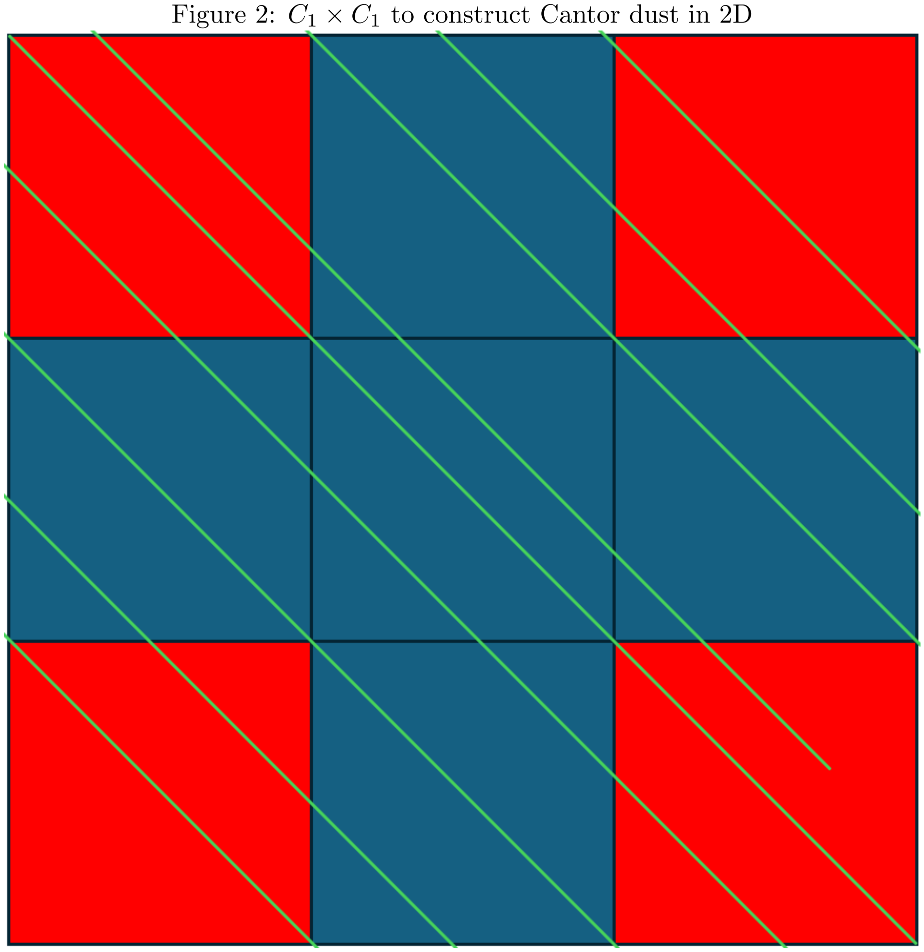
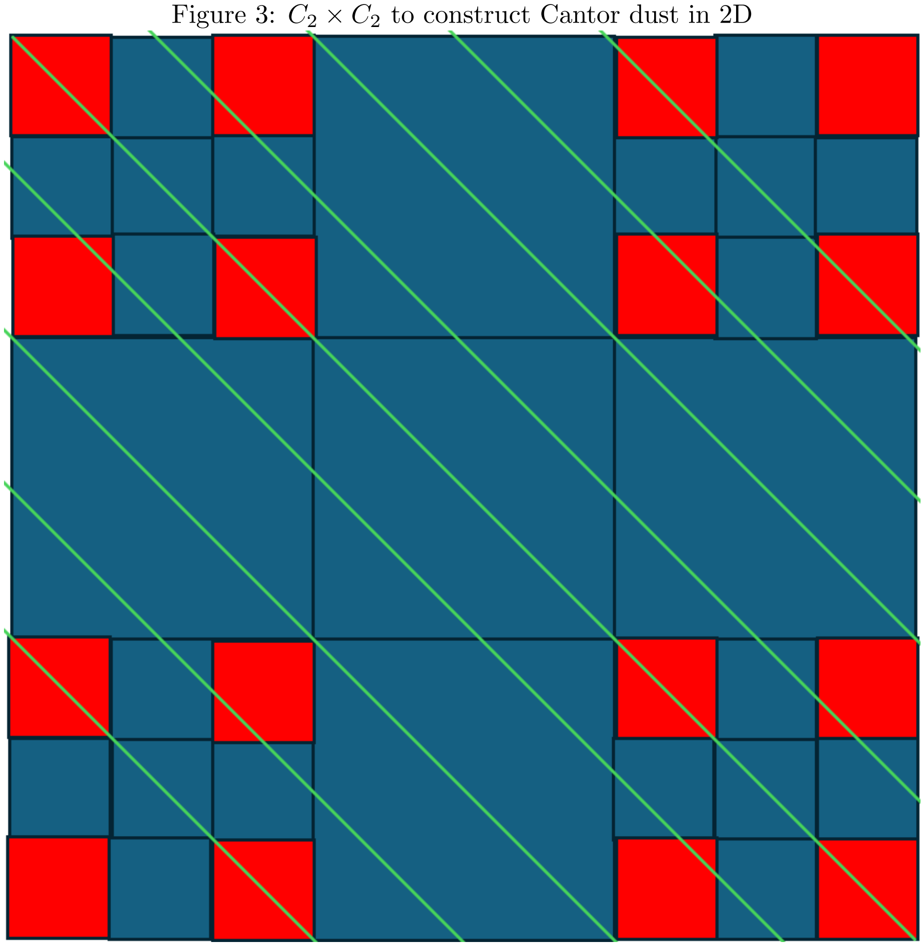


Figure 3: $C_2 \times C_2$ to construct Cantor dust in 2D



1.3 Operating on the Cantor Set: Dimension

The Cantor set is weird. It is much smaller than the real number line and all of its points are disconnected from each other, but no point is isolated. The set is still uncountable and cannot be represented as something smaller. It is much bigger than the rationals, and yet it is not dense. It is much smaller than $[0, 1]$, yet it is as sufficient as $[0, 1]$ to add to itself to get $[0, 2]$, so that it is somehow $[0, 1]$ with less junk for that task. So we are met with an object which does not work as expected, and eludes normal thought. On our lifelong pursuits to complete mathematics, we seek something that might quantify the properties of this set.

We conjecture that this object is not any object of normal dimension, and we want to find its dimension. For a self-similar shape, if you zoom into the figure, or alternatively if you multiply its scale (stretch it) by the same amount like α, then it will repeat itself α^d times where d is the dimension.

As long as multiplication by a real number is defined, define $\alpha \cdot S$, where S is a set, to be $\alpha \cdot S = \{\alpha \cdot s : s \in S\}$. Then, say that in the multiplication of the set by $\alpha > 0$, we now have β copies of the original set S, perhaps offset by some amount. Then the dimension d is given by $\beta = \alpha^d$, or $d = \frac{ln\beta}{ln\alpha}$

Multiply the set $[0, 1]$ by 3. The new set is $3 \cdot [0, 1] = [0, 3] = [0, 1] \cup [1, 2] \cup [2, 3] = [0, 1] \cup ([0, 1] + \{1\}) \cup ([0, 1] + \{2\})$. These are 3 copies of $[0, 1]$ so we get $3 = 3^d$ or $d = 1$. So the line has dimension 1. Next try a square like $[0, 1] \times [0, 1]$. Multiply it by 3, and get $[0, 3] \times [0, 3]$ which can contain 9 of the squares like the original. Get that $3 = 3^d$ or $d = 2$. This is similar for the circle, triangle, and familiar shapes; they behave well here. For the cube, there are 27 copies of the original cube so $27 = 3^d$ so $d = 3$.

Now for the Cantor set. Multiply it by 3, and when making the construction observe that $3 \cdot C_1$ gives 2 C_0s. So we now have two Cantor sets that will be formed as the construction continues. In other words, $3 \cdot C = C \cup (C + \{2\})$, so $2 = 3^d$, so $d = \frac{ln2}{ln3} \approx 0.631$

I will extend this idea of dimension onto more sets that, when magnified, will become copies of themselves; these are called self-similar. This dimension has the chance of being silly for a variety of reasons, but for now let's just roll with it.

2 Making More Sets like Cantor on the Real Number Line

2.1 Construction

Start with $D_0 = [0,1]$. Let $a > 1$ and for each iteration, remove the open middle interval of diameter $\frac{1}{a}$ of the original interval, so that $D_1 = [0, \frac{1-\frac{1}{a}}{2}] \cup [1 - \frac{1-\frac{1}{a}}{2}, 1]$. Repeat. Then D $= \bigcap_{n=0}^{\infty} D_n$

For similar reasons as before, it should not be hard to convince oneself that this set D also has no length, is compact, uncountable, perfect, totally disconnected, and nowhere-dense in $[0,1]$ and $\mathbb{R}$.

2.2 Dimension

Make the set $\frac{2}{1-\frac{1}{a}} \cdot D$. Then set D_1 is now $[0,1] \cup [\frac{2}{1-\frac{1}{a}} - 1, \frac{2}{1-\frac{1}{a}}]$. Now there are two copies D_0s of their original size that exist here. We have that $2 = (\frac{2}{1-\frac{1}{a}})^d$, so $d = \frac{\ln 2}{\ln(\frac{2}{1-\frac{1}{a}})}$. If $a = 3$ then we recover the initial derivation, and if $a = 2$ then the dimension is exactly $\frac{1}{2}$. As $a \to \infty$, $d \to^- 1$, which makes sense, as you remove less and less of the set. And as $a \to^+ 1$, $d \to^+ 0$, which makes sense, as you remove more and more of the set.

2.3 Other possibilities for sets like Cantor

You may construct somewhat similar sets like above, but instead of the open middle interval, have two open middle intervals, or perhaps more. For example, $F_0 = [0,1]$, $F_1 = [0, \frac{1}{5}] \cup [\frac{2}{5}, \frac{3}{5}] \cup [\frac{4}{5}, 1]$, and so on. If you multiply the set by five, you get 3 copies, so that $d = \frac{\ln 3}{\ln 5} \approx 0.683$. This is different than simply letting $a = \frac{5}{2}$ for the construction of D, wherein $d = \frac{\ln 2}{\ln \frac{10}{3}} \approx 0.576$. In this scenario the amount taken away from each interval is the same, and yet the dimension is different. This shows that the shape of the construction and the self-similarity can also change the dimension, rather than just how much of the set is taken away each step.

3 Self-Repeating Fractals in the Plane and in the 3D Space

I will continue with the idea of dimension on other self-repeating objects, although be weary that this definition can be completely nonsensical if used wrongly, and I will explain why later. Let's call a set a fractal if its dimension is not an integer. So the Cantor set is a fractal, and so are the alternate sets described in the previous section. Here is an incomplete list of fractals.

3.1 Cantor dusts in 2D and 3D

3.1.1 2D cantor dust:

Consider the Cantor dust $C \times C$ described in 1.2.7 and constructed in **figure 2** and **figure 3**. When multiplying the set by 3, it will have 4 copies. So d = $\frac{\ln 4}{\ln 3} \approx 1.262$.

3.1.2 3D cantor dust:

Complete the same construction but in 3 dimensions, ie $C \times C \times C$, shown in **figure 4**. Multiply the set by 3, and it will have 8 copies. So d = $\frac{\ln 8}{\ln 3} \approx 1.893$. This is curious, as an object that clearly expands in 3 dimensions has a dimension less than 2, meaning it is somehow less than even the plane. You might conclude that this cantor dust can be projected onto the plane. This is true, and we can project it to an even "smaller" object (of less dimension), the Sierpinski Carpet.

3.2 Sierpinski Carpet

3.2.1 Construction and dimension

Complete the cantor dust but with an intersection instead of a union. Meaning that $S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in C \vee x_2 \in C\}$. This creates the Sierpinski Carpet **(figure 5)**. Multiplying this set by 3 yields 8 copies, so d = $\frac{\ln 8}{\ln 3} \approx 1.893$, exactly like the Cantor dust in 3D.

3.2.2 Projecting Cantor dust in 3D onto the Sierpinski carpet

The Cantor dust and the Sierpinski carpet have the same dimension. Perhaps this hints that they are similar objects? This would be surprising, as they look nothing alike. If we look at how the dimension is calculated, we find that multiplying each set by 3 yields 8 copies of the set. So relating each copy can help us find a relation between the two objects.
For each iteration of the Cantor dust in 3D, a cube turns into 8 cubes of $\frac{1}{3}$ size. And for each iteration of the Sierpinski carpet, a carpet turns into 8 carpets of $\frac{1}{3}$ size. We can understand that the Cantor dust and the Sierpinski carpet are compact, like the Cantor set, because they are made up of points whose coordinates are in the Cantor set or in the unit interval. Like in the Cantor

Figure 4: Cantor Dust in 3D [8]

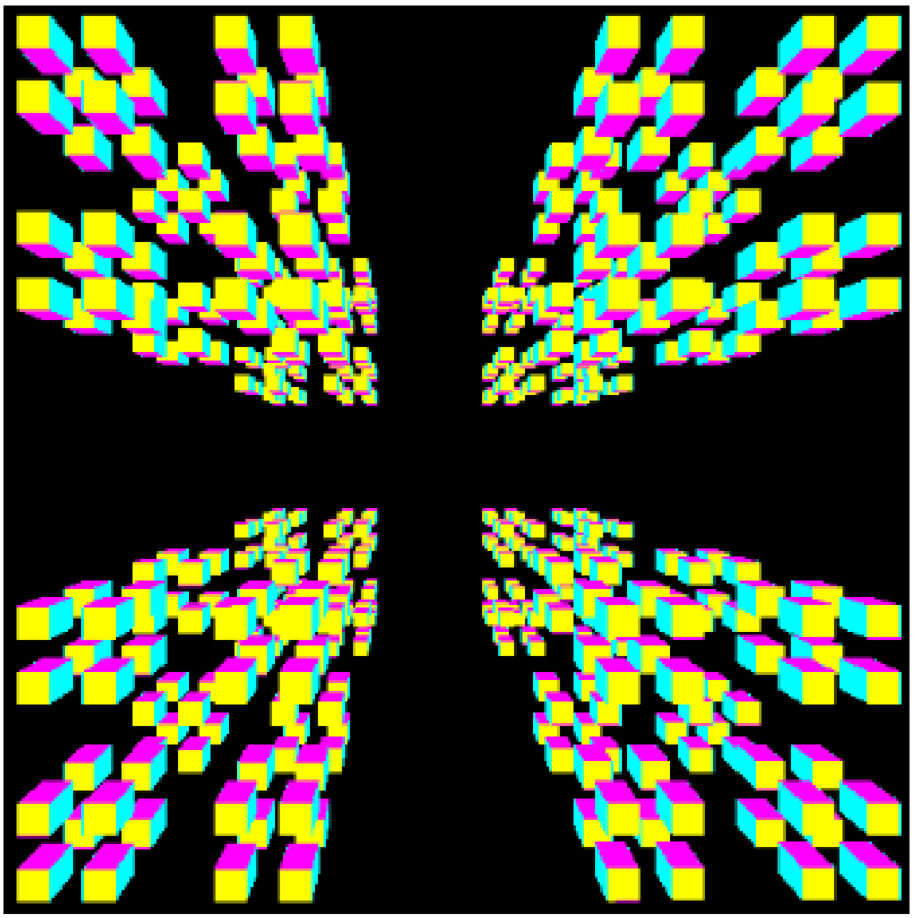
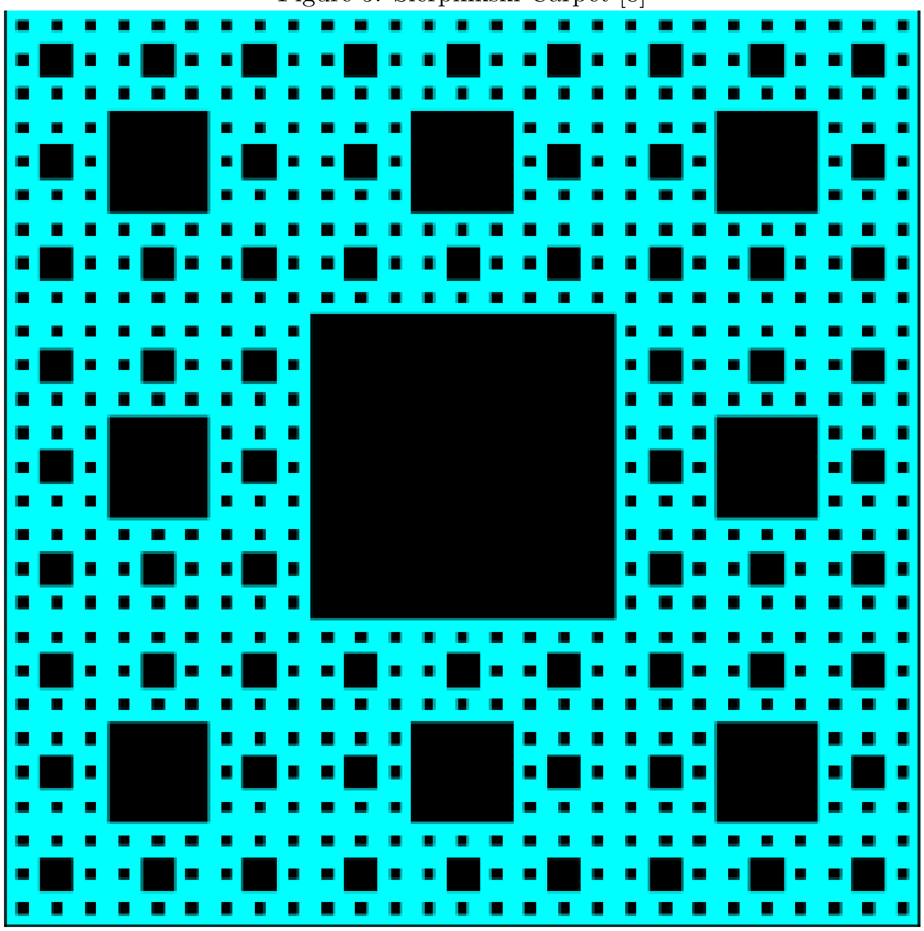


Figure 5: Sierpinkski Carpet [8]



set, we assert that an element is in the Cantor dust iff it can be written as a sequence $a_1, a_2, ...$ where a_n denotes which cube of the nth iteration the element is in. There are 8 possible cubes out of 27 for each a_n, so the sequence should be with each a_n be from 1 to 8 (if we number the first numbers 1-8 as the edge cubes).
We do a similar thing for the Sierpinski carpet. We assert that an element is in the Sierpinski carpet iff it can be written as a sequence $b_1, b_2, ...$ where b_n denotes which square of the nth iteration the element is in. There are 8 possible squares out of 9 for each a_n, so the sequence should be with each b_n from 1 to 8 (if we number the first numbers 1-8 as the outer squares).
We can project an element (a_n) from the dust to the carpet by letting its carpet sequence be $(b_n) = (a_n)$, and we can project an element $(b_n$ from the carpet to the dust by letting its dust sequence be $(a_n) = (b_n)$. So, you can form a bijection between the two objects. While a bijection doesn't signify the same dimension (there are curves from a 1D interval to a square), the construction should help show that the Cantor dust in 3D and the Sierpinski carpet are similar objects with the same self-similarity relation.

3.2.3 Menger sponge

Perform the similar construction of the carpet but in 3D **(figure 6)**. Multiplying this set by 3 yields $8 + 4 + 8 = 20$ copies, so d = $\frac{\ln 20}{\ln 3} \approx 2.727$.

3.3 Sierprinski Triangle and the Tetrix

3.3.1 Construction and dimension

There are many ways to construct the Sierprinsky Triangle. One way is to start with a line, and on each iteration replace it with 3 shorter segments of equal length at 120 degrees from each other **(figure 7)**. This creates a space-filling curve for the Sierprinsky Triangle. One can simply make the object by starting with an equilateral triangle and removing the open middle equilateral triangle for each subtriangle in the next iteration **(figure 8)**. Multiply the Sierprinsky Triangle by 2 and get 3 copies, so the dimension $d = \frac{\ln 3}{\ln 2} \approx 1.585$.

3.3.2 The Tetrix

Now make a similar thing in 3 dimensions **(figure 9)**. Start with an equilateral pyramid. Shrink to half size, and place 4 of them together: 3 at the bottom, 1 at the top. It follows that multiplying the tetrix by 2 will get 4 copies, so the dimension $d = \frac{\ln 4}{\ln 2} \approx 2$. If we interpret this in linear algebra dimension, we might conjecture that the tetrix can be projected onto a plane. And it can, in multiple ways. **Figure 10** and **figure 11** display 1 way to do it. The idea behind it is that you may color a square into 4 small squares of equal size, and repeat this indefinitely. There are 4 squares for each iteration, and there are 4 pyramids for each iteration of the tetrix; equating the sequences for these will give a bijection between the points in the tetrix and the plane, like the Cantor

Figure 6: Menger Sponge [8]

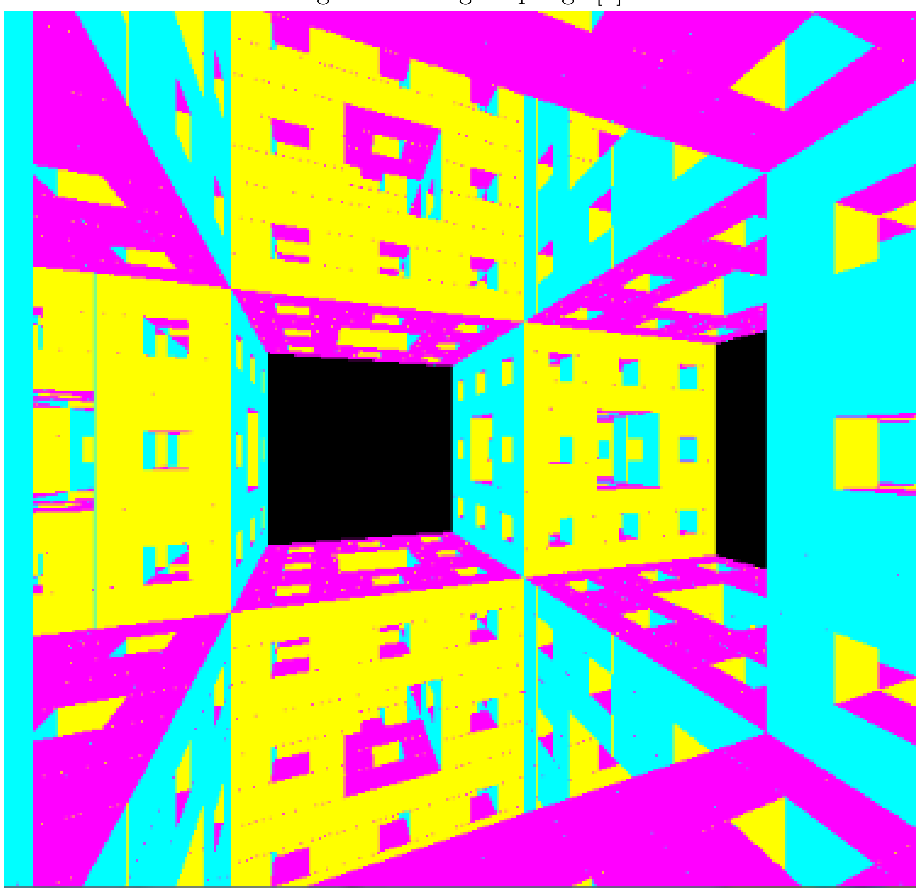


Figure 7: Sierprinsky Triangle Line Construction

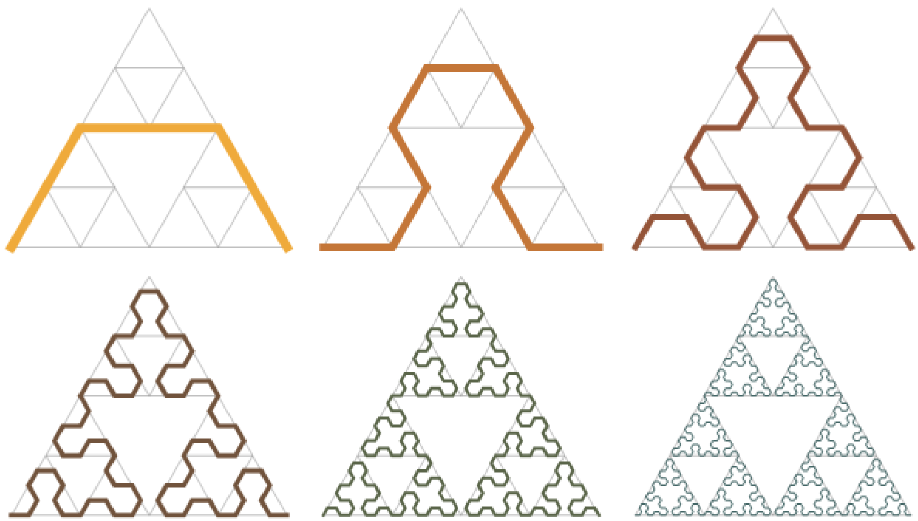


Figure 8: Sierprinsky Triangle

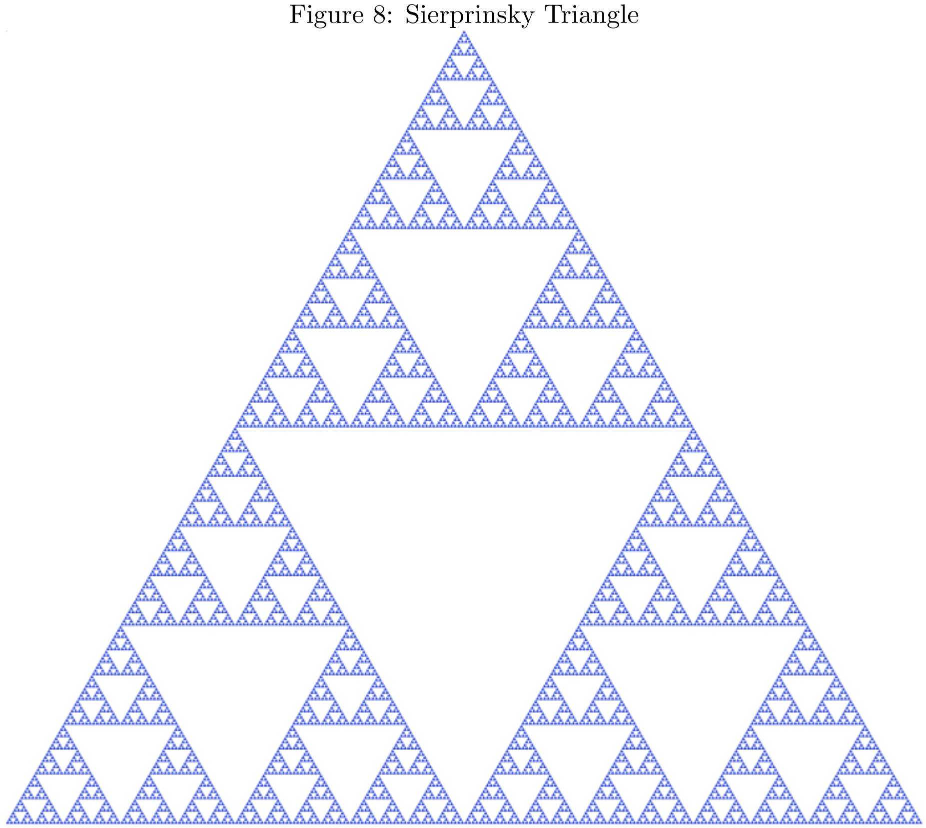
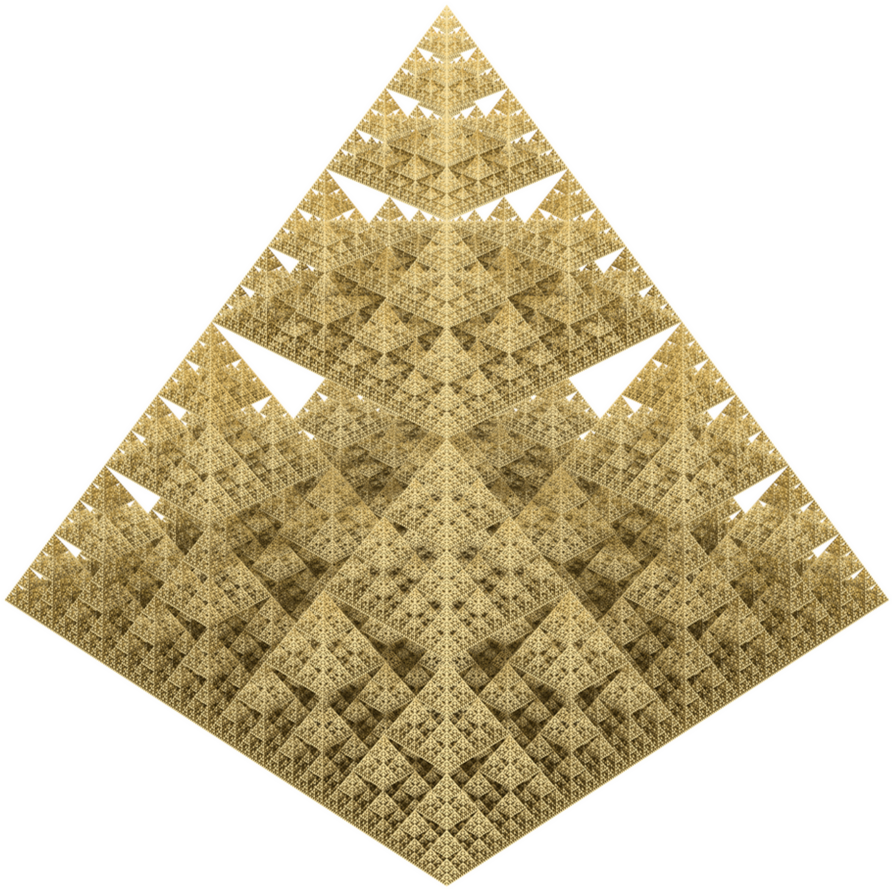


Figure 9: Tetrix (Sierpinski Pyramid)



dust to Sierprinski carpet equivalency. Knowing that its dimension is 2, the tetrix is not a fractal as defined.

3.4 Koch curves

3.4.1 Type 2 von Koch curve

Consider the construction of snaking a line into two squarical shapes, shown in **figure 12**. When multiplied by 4, the constructed fractal would have 8 identical copies: 1 on each end, and 3 on each squarical shape. So the dimension is $\frac{\ln 8}{\ln 4} = 1.5$.

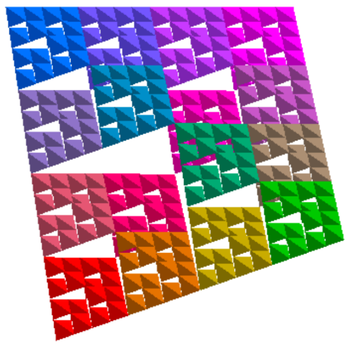


Figure 10: Rotate the tetrix.

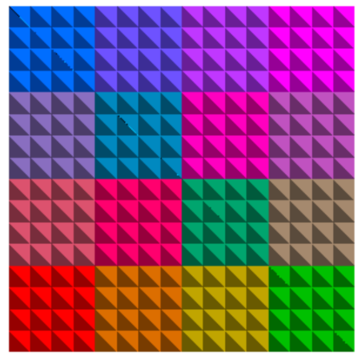


Figure 11: Get a bijection from the tetrix to a square.

Figure 12: von Koch curve of type 2 / Minowski Sausage [10]

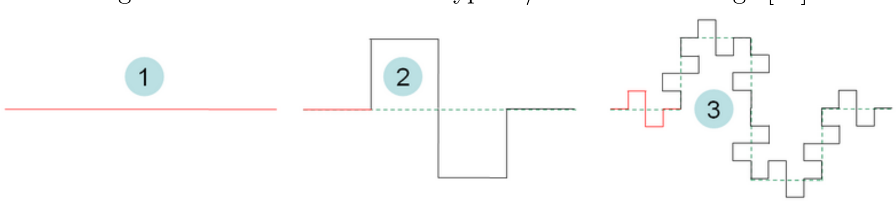
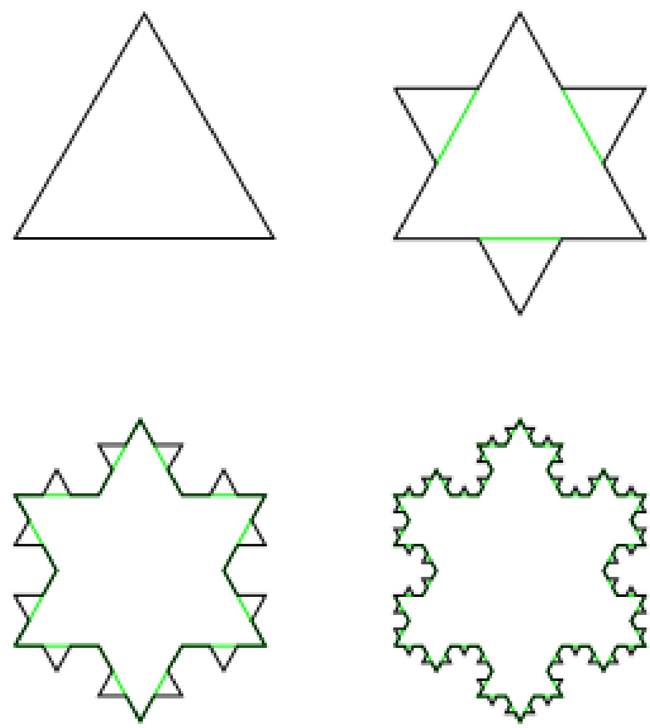


Figure 13: Koch snowflake



3.4.2 Koch curve / Koch snowflake

The construction can be started from a triangle (3 sides) or a line (1 side), either of which is fine. Start with your line. With each iteration, remove the middle third of the line and in place add an isosceles triangle with lengths a third of the line (**figure 13**). This is one of the earliest fractals to be constructed geometrically (in [13]), and demonstrates a shape of infinite perimeter (scaling with $(\frac{4}{3})^n$) and finite area. More importantly, the curve is continuous and has no tangents. Here tripling the size yields 4 copies (4 lines of original size), so that the dimension is $\frac{\ln 4}{\ln 3} \approx 1.262$.

3.5 Other

There are other self-repeating fractals, but I will not list them. They are more complex, with self-similarities that are much less obvious, making calculations less rudimentary to show.

4 Fractals are Not All Self-Repeating

For a finite set of points, the dimension will always be 0, as the number of points after being multiplied is the same, giving $\beta = 1$ so that $d = 0$. This is nice but somehow the dimension of a countably infinite number of points is less. For example, take the set $\mathbb{N}$. I will half this set. $\frac{1}{2} \cdot \mathbb{N} = \mathbb{N} \cup (\mathbb{N} + \{\frac{1}{2}\})$. In comparison to the original set, this can be said to be twice the original set, because for each natural number in the set there is a corresponding number a half to the left. This gives $\frac{1}{2} = 2^d$ so that $d = -1$. This is obviously purely nonsensical, and yet if you try to multiply the set by 2 you will find that it does shrink to half size as predicted. So something is definitely wrong about how we consider the self-similarity here. There is no end to any "copy," so forget about trying to find them. For example, $2 \cdot (\mathbb{R} \times \mathbb{R}) = \mathbb{R} \times \mathbb{R}$ so its dimension will be 0, but you'd be silly to say that the plane has a dimension of 0. With the current definition we may restrict ourselves only to objects which are bounded, and call the dimension of any set of a countable number of points to be 0. Even with this issue, there are still some objects which are not self similar. If I magnified a tree I would never be able to find another true copy of a tree in the tree. I cannot in good faith convince myself that the next branching off in the tree gives a perfect copy of the tree. In similar manner, sets like the Mandelbrot set will have apparent but imperfect self-similarity along certain paths, warranting the use of more sophisticated definitions of dimension.

4.1 Interpreting Dimension: Roughness

For self-similar objects, magnifying into the object (or increasing its size) will reveal more copies of the object the higher dimension it has. Therefore, for two objects observed at certain resolution, zooming into each objec will reveal more pertubations in the object with a higher dimension. Hence the high-dimension object can be said to be rougher. We can generalize this idea with multiple definitions of dimension.

4.2 Box-counting Dimension

4.2.1 Definition: cover

Let $S \subseteq X$, and let $O = \{O_j \subseteq X : j \in J\}$ be an arbitrarily (possibly uncountably) indexed collection of subsets of X. Then, O is a **cover** of S in X if $S \subseteq \bigcup\{O_j \subseteq X : j \in J\}$

4.2.2 Definition: $N(\epsilon)$

Let $S \subset \mathbb{R}^n$ be bounded. Tile $\mathbb{R}^n$ with lines, squares, cubes, or higher-dimensional counterparts (collectively I will call them "boxes") of side length ϵ. Since S is bounded, there are a finite number of boxes that cover S. Denote $N(\epsilon)$ as the minimal number of boxes required to make up a cover of S. In practice, this just requires counting all the boxes created such that a point from S is in each box.

4.2.3 Definition: Minkowski-Bouligand dimension [4]

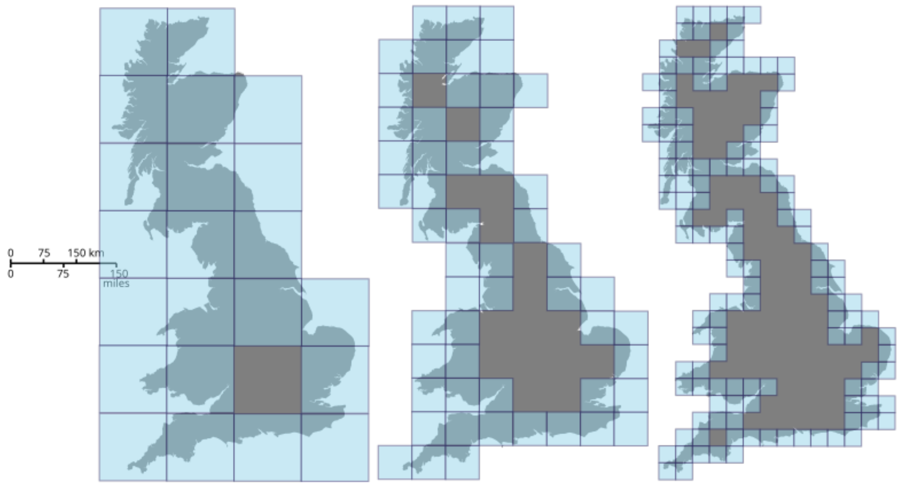
Let us approximate the size of an object by its number of boxes. Suppose that the object was magnified to $\frac{1}{\epsilon}$ original size, meaning that the resolution was changed; $\mathbb{R}^n$ was retiled with boxes of size ϵ. Then the approximated new size is $N(\epsilon)$. The dimension may be given by $(\frac{1}{\epsilon})^{d_\epsilon} = N(\epsilon)$, so that $d_\epsilon = \frac{\ln(N(\epsilon))}{\ln(\frac{1}{\epsilon})}$. Continue magnifying, so that $\dim_{box}(S) = lim_{\epsilon \to +0} d_\epsilon$, or alternatively if the limit does not exist, one can take the limsup and liminf for other upper and lower box counting dimension [4]. (in this case the limsup isn't of a sequence and can be defined as $\lim_{\epsilon \to +0} \sup\{d_\delta : 0 < \delta < \epsilon\}$, and similarly for the liminf.) A small note is that since S is bounded, so is ∂S, the boundary of S. So one can find the dimension of both a set and its boundary, and these are often but not always different.

4.2.4 Computing the box counting dimension of several irregular objects

Consider the coast of England shown in **figure 14**. Here ϵ decreases by $\frac{a}{2^n}$ by reducing the length of each box to a half each iteration. You can use the identity $d_\epsilon \ln(\frac{1}{\epsilon}) = \ln(N(\epsilon))$ so that computing $\dim_{box}$ means finding the slope, d_ϵ, of $\ln(N(\epsilon))$ WRT $\ln(\frac{1}{\epsilon})$ as ϵ decays. So when the graph of $\ln(N(\epsilon))$ WRT $\ln(\frac{1}{\epsilon})$ becomes consistently approximately linear (or else the resolution might be too low to pick up on the correct shape), one may approximate $\dim_{box}$ as the slope of the graph. This ends up being approximately 1.25. Of course, there are some problems to doing this. In real life the shape is finite, but we can't measure its true length. But at some point when ϵ is much less than the Planck length, the dimension will return to being 1.

One can consider the coastline as an example of a yet-formed fractal (in the non-self-similar sense) that, over infinite time and if space was truly continuous, will give a fractal due to the phenomena creating it (erosion, wave corrosion, etc); in this sense one may say that the phenomena creating the coastline may be modeled to roughen it to the degree of the calculated dimension. This is more sensical in the next example, where an imperfect self-similarity is created. Consider the cross sections of Green Broccoli or White Cauliflower. Somebody was either really bored or really interested, and decided to calculate dimensions for each. In this example a weak imperfect self-similarity may be apparent, being that the plants branch out to more branches of lesser size. So finding the dimension may help identify the degree to which the plant grows rough. In [9] the nerd in question describes using the box counting method on several vertical and horizontal cross sections of the broccolis and cauliflowers to find that the average dimension for a broccoli cross section is 1.78 ± 0.02 and that the average dimension for the cauliflower cross section is 1.88 ± 0.02. This should intuitively

Figure 14: Estimating the box-counting dimension of the coast (boundary) of England [11]

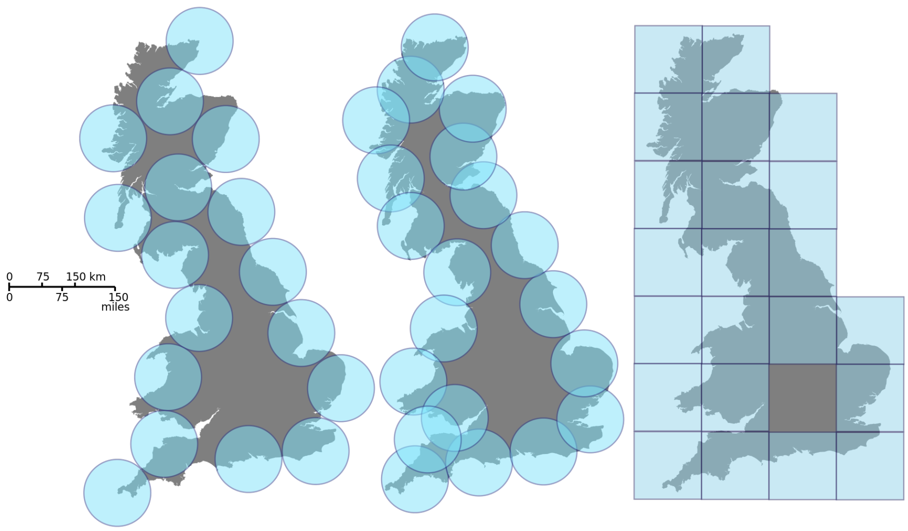


make sense, as a broccoli will end up thinner and with less branches than a cauliflower, which expands more quickly.

4.2.5 Alternative Definitions for box-counting dimension

Instead, let $N_{covering}(\epsilon)$ be the minimal number of open balls of radius ϵ required to cover S, with the requirement that each ball be centered at S. And, let $N_{packing}(\epsilon)$ be the maximum number of open balls of radius ϵ centered at S such that the balls are disjoint **(figure 15)**. These definitions generalize the box counting method to a general metric space (S, d), which is a set with a metric d (distance) that behaves as typically expected; and this is very good because a superset for S is not even required. Typically, these definitions give the same result.

Figure 15: Left: packing. Center: covering. Right: box covering. [12]



4.3 Hausdorff / Hausdorff-Besicovich Dimension

The Haudroff dimension will work for a general metric space, and its result can be infinite [1].

4.3.1 Definition: Hausdorff Measure [2]

Let (X, ρ) be a metric space, and for any $U \subseteq X$ define the diameter (traditionally) as $\text{diam}U := \sup\{\rho(x, y) : x, y \in U\}$ and $\text{diam}\phi := 0$.
We will do a similar construction to the ball covering. Have $\delta > 0$ be the maximal size of countably many sets U_i used to cover S. Approximating each set U_i as a ball in d dimensions, its characteristic volume should scale by $(\text{diam}U_i)^d$.
Formally, let $S \subseteq X$ and $\delta > 0$.
Then $H^d_\delta(S) = \inf\{\sum_{i=1}^\infty (\text{diam}U_i)^d : \bigcup_{i=1}^\infty U_i \supseteq S, \text{diam}U_i < \delta\}$. The smaller δ is, the less choice in the cover; $H^d_\delta(S)$ is non-decreasing with decreasing δ, and may become infinite. Finally, define the Hausdorff measure:
$H^d(S) = \sup_{\delta>0} H^d_\delta(S) = \lim_{\delta\to 0} H^d_\delta(S)$.
The Hausdorff measure is proportional to the typical lebesque measure (number of points for d=0, length for d=1, area for d=2, etc.) for integer d on nice sets (borel sets), because the diameter is scaled the same with the same d. To correct for true lebesque measure, simply scale the Haudroff measure by the size of a unit diameter "ball" in the d dimension.
The most important quality of the Hausdorff measure, other than the fact that it is a measure, is that on a dilation like $\alpha \cdot S$, $H^d(\alpha \cdot S) = \alpha^d H^d(S)$.

4.3.2 Definition: Hausdorff dimension

$\dim_H(S) := \inf\{d \geq 0 : H^d(S) = 0\}$ = (sometimes) $\sup\{d \geq 0 : H^d(S) = \infty\}$. One can think of the Hausdorff dimension as the passing point between when the object is negligible in $d > dim_H(S)$ dimensions to when it becomes too large in $d < dim_H(S)$ dimensions.

4.4 Properties of the Hausdorff vs the Box Counting Dimension [4] [1]

Firstly, the box counting dimension will not work well on unbounded sets, because $N(\epsilon)$ will simply end up infinite for any ϵ. However, the Hausdorff dimension works fine on unbounded sets, since measure may be infinite; the Hausdorff dimension is defined as the highest lower bound of the dimensions where the Hausdorff measure of the object is 0. Objects may be unbounded (like an infinite line in 3D space) but still have a Hausdorff dimension (1 in this example). For that reason the box dimension of a union of sets only works for finitely many sets, in which case the maximum dimension of the sets used is the dimension of the union. However, the Hausdorff dimension of the union of any countably many sets is the supremum of their dimensions.
Consider the set of rationals in $[0, 1]$. Because the set of rationals is dense, in any box in $\mathbb{R}$ there will be a rational number, so that $N(\epsilon) = \text{ceiling}(\frac{1}{\epsilon})$. This scales with $\frac{1}{\epsilon}$ so that the resulting box counting dimension is 1. Meanwhile, any countable set has a Hausdorff dimension of 0. This is because you simply need a countable cover of sets, with each set being a point from the original set wherein the diameter is 0. Hence the measure is 0 for any dimension, so the infimum taken will be 0.
$\dim_H(S) \leq \dim_{box,lower} \leq \dim_{box,upper}$ where the denoted lower and upper box dimensions are the liminf and limsup definitions of the box dimension (if the box dimension exists, it satisfies $\dim_{box,lower} \leq \dim_{box} \leq \dim_{box,upper}$.)
Some other niceties include:
$\dim_H(A \times B) \geq \dim_H(A) + \dim_H(B)$
$\dim_{box,upper}(A+B) \leq \dim_{box,upper}(A)+\dim_{box,upper}(B)$ (verify for Cantor set.)
While $\dim_H\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\} = 0$ (countable), $\dim_{box}\{0, 1, \frac{1}{2}, \frac{1}{3}, \dots\} = \frac{1}{2}$. One can verify this by using a similar idea to Cauchy condensation.

4.5 THEOREM: For self-similar shapes satisfying the open set condition (most of them), the Hausdorff dimension is the typical fractal dimension. [1]

4.5.1 (rough) Definition: contractive mapping

If (M, d) is a subspace, then a contractive mapping is a function $f : M \to M$ so that $\exists k \in [0, 1)$ so that $\forall x, y \in M, d(f(x), f(y)) \leq k \cdot d(x, y)$. Note that here the image of the function need not be the whole of M, and for us it *shouldn't* be the whole of M, as we will be shrinking the objects.

4.5.2 (rough) Definition: open set condition (OSC)

Suppose $\psi_i : \mathbb{R}^n \to \mathbb{R}^n, i = 1, \ldots, m$ is a finite sequence of contractive mappings ($0 \leq k < 1$ for each). Then this list of contractive mappings has the **open set condition** if
There is an open set $V \subseteq R$ with a compact closure such that $\bigcup_{i=1}^m \psi_i(V) \subseteq V$, with the important condition that all the $\psi_i(V)$ are pairwise disjoint.
The idea behind this is that the contractions do not create a shape that overlaps a lot. The shape created might be overlapping at the boundary, but the Hausdorff measure of such boundary would not be signficant. For example, in the Sierpinski triangle, the Sierpinski triangles created by contracting the overall figure do overlap each at one of their edges, but the contractions defined to create the Sierpinski triangle will meet the OSC.

4.5.3 Isometries and dilations

When making self-similar recursion, we used the idea that each copy must preserve the original shape of the overall object. An isometry is a perfect distance-preserving map, like a translation or a rotation (or both together). Ie, if f is the isometry then $d(x, y) = d(f(x), f(y)) \forall x, y$ (for this application both the domain and codomain is $\mathbb{R}^n$ so the metric is the same).
A dilation is a mapping from a set to itself so that $d(x, y) = \alpha \cdot d(f(x), f(y)) \forall x, y$ for some $\alpha > 0$. For this application we will have $\alpha < 1$ so that the object is shrunk.

4.5.4 Formulation of the theorem:

Let $\psi_i : \mathbb{R}^n \to \mathbb{R}^n, i = 1, \ldots, m$ be a finite sequence of contractive mappings that are also similitudes, ie strictly compositions of isometries and dilations (rotated and/or/both moved shrunk copies of the original shape, with $\alpha_i < 1$ being the dilation factor for each ψ_i). Let those contractions together meet the OSC. Then the unique fixed point $S = \bigcup_{i=1}^m \psi_i(S)$ is a set whose Hausdorff dimension is s where s is the unique solution of $\sum_{i=1}^m \alpha_i^s = 1$. And, a set A is self-similar for a list of contractive mappins as described above iff $H^s(\psi_i(A) \cap \psi_j(A)) = 0 \forall i \neq j$ (ie, the overlap between each copy is negligible); in this case S is self-similar.

4.5.5 (rough) Proof: [7]

Without many of the conditions present in the proof, let us simplify the problem and assume that S is self-similar with $S = \bigcup_{i=1}^m \psi_i(S)$ for the $\psi_i : \mathbb{R}^n \to \mathbb{R}^n, i = 1, \ldots, m$ similitudes that meet the OSC, describing its self similarity with corresponding α_i dilation factor for each. And suppose that d is the Hausdorff dimension of the object, and that the object is finite in said dimension: $0 < H^d(S) < \infty$. Then we may compute the dimension of said object.
Since H^d is a measure and $H^d(\psi_i(S)) = \alpha_i^d H^d(S)$ for each of the dilations of factor α_i, then $H^d(S) = H^d(\bigcup_{i=1}^m \psi_i(S)) = \sum_{i=1}^m H^d(\psi_i(S)) = \sum_{i=1}^m \alpha_i^d H^d(S)$. Divide both sides by $H^d(S)$ and get

$1 = \sum_{i=1}^{m} \alpha_i^d$
But, if $\alpha_i = \alpha$ is the same for each similtude, we have that $1 = m\alpha^d$ where me is the number of copies. We have $m = (\frac{1}{\alpha})^d$, which recovers the original definition of dimension presented. Notice that also if the α_is are different then this equation can give a simple(esque) way to calculate the dimension if the self-similarity produces copies which are not the same scale.

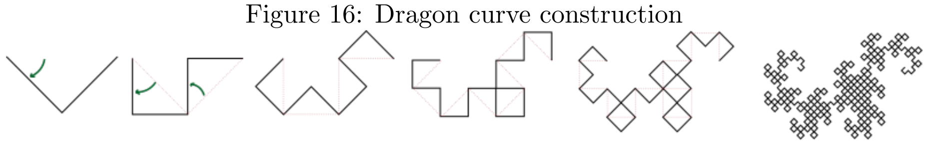
4.5.6 Use of the theorem to compute dimensions of self-similar shapes

Since the computation and formula is already provided, I will do one example; the Hausdorff dimension of examples already presented will be obvious from there. Take the Cantor set C. $\psi_1 : \mathbb{R} \to \mathbb{R}$ is given by $\psi_1(A) = \frac{1}{3} \cdot A$. And, $\psi_2 : \mathbb{R} \to \mathbb{R}$ is given by $\psi_2(A) = (\frac{1}{3} \cdot A) + \{\frac{2}{3}\}$. For each of these two ($m = 2$), $\alpha_i = \alpha = \frac{1}{3}$. C is the set for which $\bigcup_{i=1}^{m} \psi_i(C) = C$, so its dimension is given by $2 = (\frac{1}{1/3})^d$, whose solution is $d = \frac{\ln 2}{\ln 3} \approx 0.631$.

4.5.7 Modified Cantor Set With an Uncentered Open Interval Removed

Let us make a Cantor set, but place the open interval being removed somewhere else at each step. We can make a construction with α_1 being the ratio of the size of the left closed interval to the interval, and α_2 being the ratio of the size of the right closed interval to the interval. With both $\alpha_1, \alpha_2 \geq 0$ and $\alpha + \beta < 1$, we have two contractive mappings. So $\sum_{i=1}^{m} \alpha_i^d = \alpha_1^d + \alpha_2^d = 1$, which may be solved numerically.
If we let $\alpha_1 = \frac{1}{4}, \alpha_2 = \frac{1}{2}$ (by removing the second open quarter, for example), then $(\frac{1}{4})^d + (\frac{1}{2})^d = 1$, $1 + 2^d = 2^{2d}$. Substitute $d = \log_2 x$, so now the equation is $1 + x = x^2$. This is solved explicitly with the quadratic formula: the result is the reciprocal golden ratio, $x = \frac{1+\sqrt{5}}{2}$. Now we have $d = \log_2(\frac{1+\sqrt{5}}{2}) \approx 0.694$
Of course, you may revise the construction to remove multiple open intervals each step (thus having 3 or more contraction mappings), in which the same equation $\sum_{i=1}^{m} \alpha_i^d = 1$ may be used. It is highly computational and less interesting to carry this out.

Figure 16: Dragon curve construction



4.5.8 Heighway's Dragon

The dimension d of a Heighway dragon's boundary is ≈ 1.524, which is given by the equations $x = 2^{d/2}$ and $x^3 - x^2 - 2 = 0$ from relations of its self-similarity. The curve is created by folding a paper indefinitely **(figure 16)**. [6] uses the formula $1 = \sum_{i=1}^{m} \alpha_i^d$ but on infinite contractions $m = \infty$ that don't overlap to find the Hausdorff dimension d. It also provides a more satisfying finite method that produces two contractions of size $2^{-3/2}$ and a single contraction of size $2^{-1/2}$, so that $2 \cdot (2^{-3/2})^d + (2^{-1/2})^d = 1$, which produces the same result.
(Note: the Hausdorff dimension of the dragon curve is 2; it is made up of solid regions of squares. But the Hausdorff dimension of its boundary is that described above, ≈ 1.524.)

4.6 Extras: examples of the Hausdorff dimensions of irregular shapes of dubious self-similarity

For this section you can refer to the list of fractals by Hausdorff dimension on wikipedia [3]. A bunch of smart people figured this out. I will just list some results anticlimactically; these will be interesting to some people, without explanation.

4.6.1 Space filling curves

Space-filling curves, created by homeomorphisms (continuous bijective mappings that preserve topology) from the unit interval $[0, 1]$ to their respective set, will have a Hausdorff dimension of 1 for $[0, 1]$ but a different Hausdorff dimension of the respective image. (For example, the Peano curve can fill the $[0, 1] \times [0, 1]$ plane from the unit interval $[0, 1]$. This curve has a dimension of 2.) This has to do with how the objects are interlayed; while the unit interval is in a straight line, the space filling curve generated will have different distance between points. And, since the space filling curve literally ends up being the space it fills, it is the same object and has the same dimension, otherwise should I say the Hausdorff dimension of a square is 1? The dimension seemingly has more to do with how the shape is layed out, rather than a notion of its relative size and bijectivity to other sets.
In previous examples I have showed that dimension might correspond to bijection between objects, but it is apparent that this is not really correct; having a different dimension does not prohibit bijectivity between objects, and saying that having the same dimension is the inhibitor of bijectivity is short sighted.

4.6.2 Random Motion

Brownian motion, the random motion of objects suspended in a medium, is conjectured to have a (line) trajectory with a Hausdorff dimension of 2.

4.6.3 Mandelbrot set

The Hausdorff dimension of it and its boundary is 2.

5 Finish.

Thanks for reading.

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