Phase Retrieval and Matrix Completion through Projection-Based Algorithms

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UNDER THE SUPERVISION OF PROF. TAMIR BENDORY

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Introduction

Overview and Motivation

Phase Retrieval: Reconstructing a signal from the magnitude of its Fourier transform where phase information is missing.

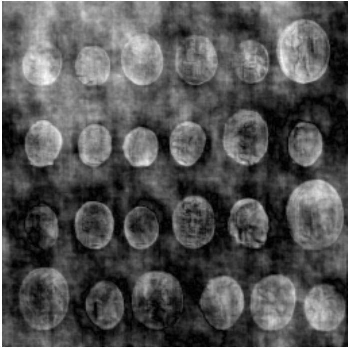
- Applications: optics, signal processing, and crystallography.
- Challenge: Magnitude defines energy distribution, phase defines spatial structure.

Key Formula: $F(u, v) = |F(u, v)|e^{i\phi(u,v)}$

Experiment: Phase Replacement

Objective: Demonstrate the role of phase.

(a) Original image.



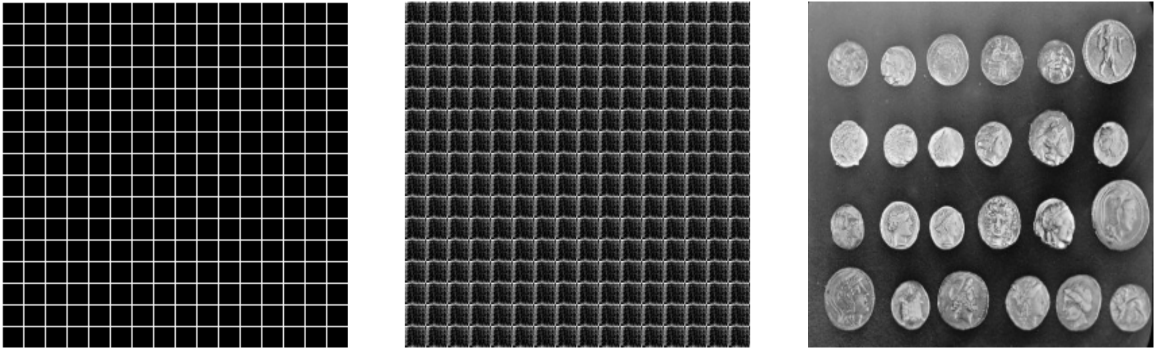
(b) Reconstruction using different phase.

(c) Second image (phase source).

Figure: Phase replacement experiment.

Periodic Patterns: Magnitude Dominance

Objective: Examine cases where magnitude dominates.



(a) Original grid image.

(b) Reconstruction (grid magnitude).

(c) Phase source image.

Figure: Grid example: Magnitude dominance.

Mathematical Problem and Set-Based Formulation

Phase Retrieval:

$$|Ax_0| = b, \tag{1}$$

- A: sensing matrix (e.g., Fourier transform matrix).
- b: magnitude measurements.

For $x \in \mathbb{C}^n$, let $y = Ax \in \mathbb{C}^m$.

Key Set Definitions:

$$\mathcal{B} = \{y \in \mathbb{C}^m : |y| = b\}, \quad \mathcal{A} \text{ encodes additional constraints.} \tag{2}$$

Research Scope and Objectives

Objective: Assess performance of projection-based algorithms across diverse scenarios.

1. **Random phase retrieval problem:**
   The matrix A is random, where every element is drawn i.i.d. from a normal distribution, real or complex.
2. **Crystallographic phase retrieval problem:**
   Recovering missing phase information from Fourier magnitude samples using a DFT matrix, with sparsity constraints.
3. **Matrix completion problem:**
   Reconstruct a matrix with missing elements using the rank r as a constraint.

Projection on Sets Method

Projection on Sets Method

Overview:

- Constraints are represented as projections onto defined spaces.
- Projections ensure the signal resides within spaces satisfying the constraints.
- The goal: Find the intersection point of the spaces $\mathcal{A}$ and $\mathcal{B}$.

Key Definitions:

- Projection of $y \in \mathbb{C}^n$ onto $\mathcal{A}$: $P_{\mathcal{A}}(y)$.
- Projection onto $\mathcal{B}$: $P_{\mathcal{B}}(y)$.
- Solution condition: $P_{\mathcal{A}}(y_0) = P_{\mathcal{B}}(y_0)$.

Set Projection Example Illustrated with Sudoku

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

(a) Row constraint

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

(b) Column constraint

5	3			7				
6			1	9	5			
	9	8					6	
8				6				3
4			8		3			1
7				2				6
	6					2	8	
			4	1	9			5
				8			7	9

(c) Block constraint

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

(d) Solution

Figure: Examples of Sudoku constraints: rows, columns, blocks, and the solution.

Random Phase Retrieval Problem

Magnitude Constraint: Project y onto the set $\mathcal{B} = \{y \in \mathbb{C}^m : |y| = b\}$ using:

$$P_{\mathcal{B}}(y) = b \odot \text{phase}(y),$$

where $\odot$ is the point-wise product and $\text{phase}(y)[i] = \frac{y[i]}{|y[i]|}$.
Signal Constraint: Projection onto the column space of A:

$$P_{\mathcal{A}}(y) = AA^{\dagger}y,$$

where $A^{\dagger}$ is the pseudo-inverse.

Crystallographic Phase Retrieval Problem

Magnitude Constraint: Same as in the random phase retrieval problem.
Sparsity Constraint: Projection P_S retains the largest $|S|$ elements:

$$P_S(x)[i] = \begin{cases} x[i], & \text{if } i \text{ corresponds to one of the } |S| \text{ largest } |x|, \\ 0, & \text{otherwise.} \end{cases}$$

Matrix Completion Problem

Known Entries Constraint: Maintain fixed entries:

$$P_\Omega(X, M)[i,j] = \begin{cases} M[i,j], & (i,j) \in \Omega, \\ X[i,j], & (i,j) \notin \Omega. \end{cases}$$

Rank Constraint: Enforce rank-r using SVD:

$$P_r(M) = U\Sigma_r V^*,$$

where Σ_r retains the largest r singular values.

Projection-based algorithms

Reflections and Relaxation

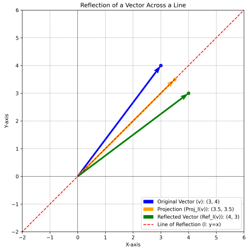
Reflection: For a set $\mathcal{L}$, the reflection across $\mathcal{L}$ is defined as:

$$\mathrm{Ref}_{\mathcal{L}}(y) = 2P_{\mathcal{L}}(y) - y. \tag{3}$$

Relaxation: Relaxation introduces flexibility by adjusting the influence of projections:

$$y^{(k+1)} = y^{(k)} + \beta\left(P_{\mathcal{L}}(y^{(k)}) - y^{(k)}\right). \tag{4}$$

Key Parameter: β, which controls the step size or influence.



Reflections and Relaxation

Reflection: For a set $\mathcal{L}$, the reflection across $\mathcal{L}$ is defined as:

$$\text{Ref}_{\mathcal{L}}(y) = 2P_{\mathcal{L}}(y) - y. \tag{5}$$

Relaxation: Relaxation introduces flexibility by adjusting the influence of projections:

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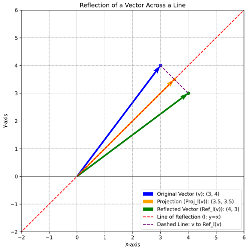
Key Parameter: β, which controls the step size or influence.



Alternating Projections (AP)

Algorithm:

$$y^{(k+1)} = P_{\mathcal{A}}\left(P_{\mathcal{B}}\left(y^{(k)}\right)\right), \tag{7}$$

where $y^{(k)}$ is the vector at the k-th iteration, $P_{\mathcal{A}}(y)$ and $P_{\mathcal{B}}(y)$ are the projection operators onto sets $\mathcal{A}$ and $\mathcal{B}$, respectively.

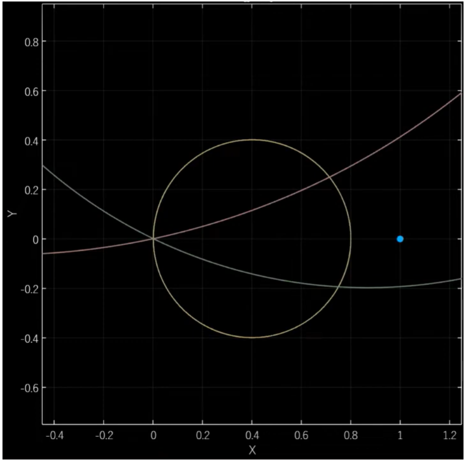
Description:

- Alternates projections onto sets $\mathcal{A}$ and $\mathcal{B}$.

Limitations:

- May stagnate if the process becomes stuck at a specific value.

Example: Circle Intersection in Alternating Projections (from [1])



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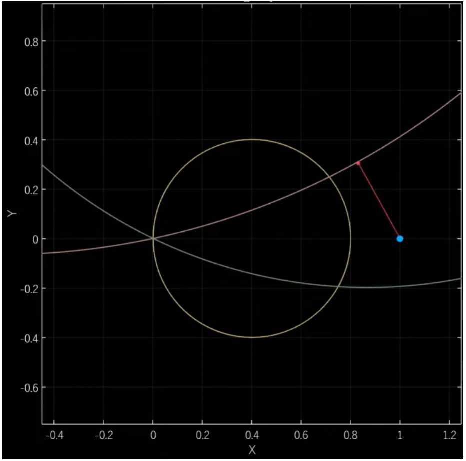

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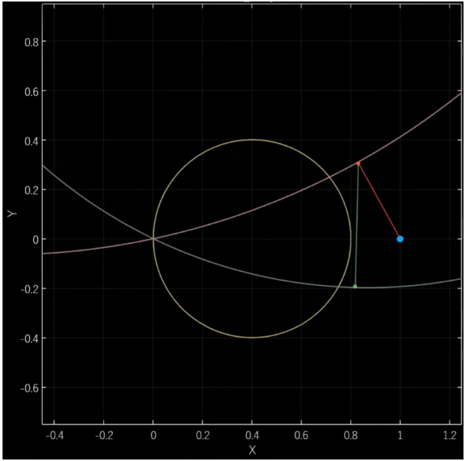

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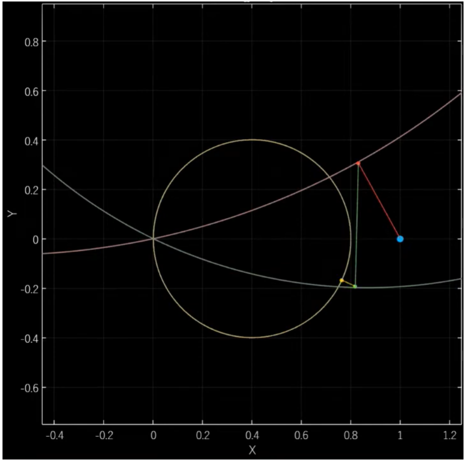
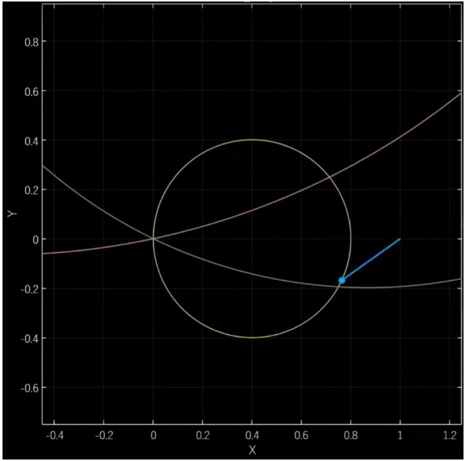
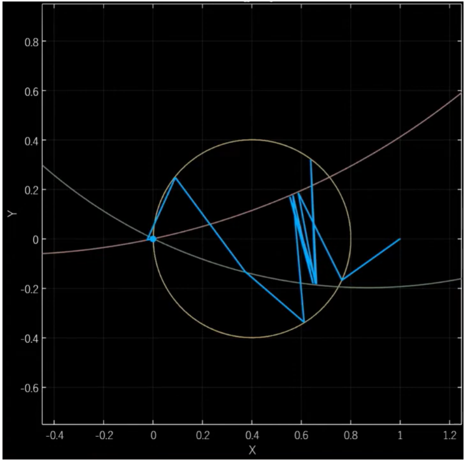


Example: Circle Intersection in Alternating Projections (from [1])



Example: Circle Intersection in Alternating Projections (from [1])



Relaxed Reflect Reflect (RRR)

Algorithm:

$$y^{(k+1)} = y^{(k)} + \gamma\Big(P_{\mathcal{A}}(2P_{\mathcal{B}}(y^{(k)}) - y^{(k)}) - P_{\mathcal{B}}(y^{(k)})\Big). \tag{8}$$

Advantages:

- Oscillatory behavior avoids stagnation.
- Converges robustly in complex cases.

Relaxed Averaged Alternating Reflections (RAAR)

Algorithm:

$$y^{(k+1)} = \beta\big(y^{(k)} + P_{\mathcal{A}}(2P_{\mathcal{B}}(y^{(k)}) - y^{(k)})\big) + (1 - 2\beta)P_{\mathcal{B}}(y^{(k)}). \tag{9}$$

Hybrid Input-Output (HIO)

Algorithm:

$$y^{(k+1)} = y^{(k)} + P_{\mathcal{A}}\big((1+\alpha)P_{\mathcal{B}}(y^{(k)}) - y^{(k)}\big) - \alpha P_{\mathcal{B}}(y^{(k)}). \tag{10}$$

Stopping Criteria

Condition for Convergence:

$$\|P_{\mathcal{A}}(y^{(k)}) - P_{\mathcal{B}}(y^{(k)})\| \leq \varepsilon, \tag{11}$$

where ε is the tolerance threshold.

In the crystallographic problem: the threshold is defined by:

$$\mu := \frac{\mathrm{I_S}}{\mathrm{I_F}}, \quad \mu > 0.95, \tag{12}$$

where I_S and I_F represent the power of the support region and the full image, respectively.

- **Practical Notes:** Smaller ε and larger μ ensure higher accuracy but may require more iterations.
- **Maximum Iterations:** In all experiments, a maximum number of iterations was defined. If the stopping criterion was not met within this limit, the solution was classified as "did not converge."

Theoretical Boundaries in Matrix Completion

Motivation: The Netflix Prize Model

- **User-Movie Matrix:** Rows represent users, columns represent movies, and cells contain user ratings (percentage values).
- **Challenge:** Predict missing ratings accurately while minimizing user input.
- **Key Insight:** The matrix rank r captures patterns in user preferences and movie characteristics.

	The Lion King	Avatar	Inception	Titanic	The Avengers
John	99%	85%	80%	-	75%
Emily	-	10%	-	5%	77%
Michael	99%	87%	12%	90%	80%
Sarah	91%	-	1%	97%	-

Figure: Netflix Prize Model: Predicting missing elements in a user-movie matrix.

Theoretical Boundaries: Maximum Deletable Elements

Claim: For an $n \times n$ matrix of rank r, it is possible to delete up to:

$$q = n^2 - (2nr - r^2) = (n - r)^2$$

elements while ensuring **unique** reconstruction, assuming the matrix rank is known.

Example: Best-Case Reconstruction Scenario

Example Matrix:

$$A = \begin{bmatrix} 1 & 1 & 1 & 6 & 3 \\ 1 & 2 & 3 & 12 & 6 \\ 1 & 3 & 4 & 16 & 8 \\ 9 & 18 & 24 & x & x \\ 3 & 6 & 8 & x & x \end{bmatrix}.$$

- $n = 5$, $r = 3$.
- $(n - r)^2 = 4$ deletable elements.
- x: Can be reconstructed by linear combinations of the first 3 columns.
- Ensures unique reconstruction under rank and deletable elements constraints.

Numerical Experiments

Methodology and Experiment Design

Methodology and Experiment Design

Setup:

- Random matrix $A \in \mathbb{C}^{m \times n}$ and vector $x \in \mathbb{C}^n$, with entries drawn from $\mathcal{N}(0,1)$ for both real and imaginary parts.
- Dimensions: $m = 25$, $n = 10$.
- Parameter: $\beta = 0.5$.

Initialization:

- Vector $y = Ax$ computed.
- Algorithms implemented: AP, RRR, RAAR, HIO.

Stopping Criterion:

- Convergence achieved if:
$$\|P_{\mathcal{A}}(y_0) - P_{\mathcal{B}}(y_0)\| \leq 10^{-6},$$
within a maximum of 100,000 iterations.
- Convergence is up to a global phase ambiguity.

Results: Graphical Representations

Vector Visualization:

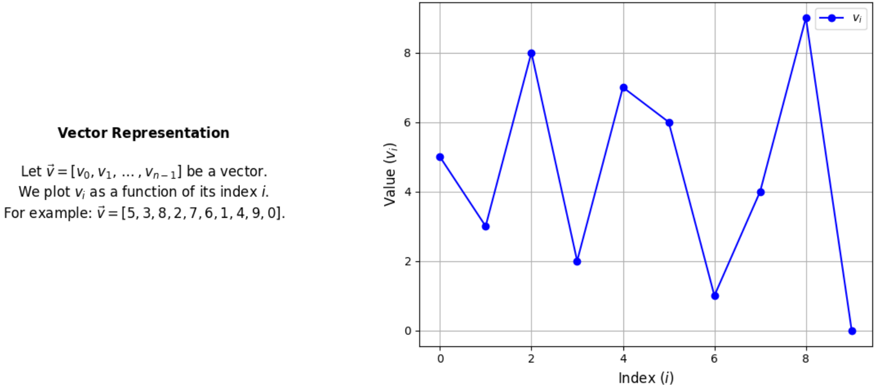


Figure: Graphical representation of a vector as a function of its index.

Results: Convergence Analysis

Performance of Algorithms:

- **Successful Algorithms:**
  - RRR: Converged in 493 iterations.
  - HIO: Converged in 726 iterations.
- **Unsuccessful Algorithms:**
  - AP and RAAR failed to satisfy the stopping criterion.

Parameters: $m = 25$, $n = 10$, $\beta = 0.5$.

Figures: Magnitude of Estimated Solution

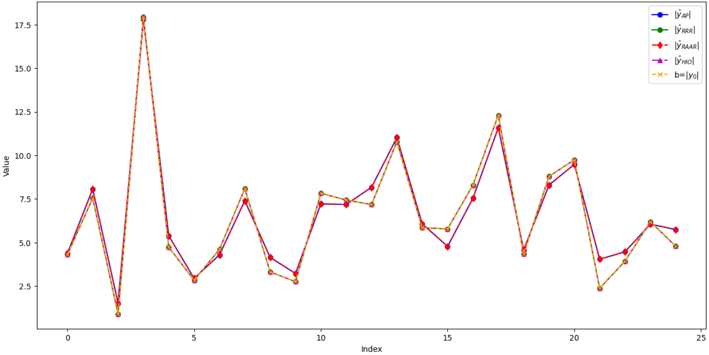


Figure: The magnitude of the estimated solution $\hat{y}$ after projection onto $\mathcal{A}$ (using $P_{\mathcal{A}}$) is compared to b, which represents the magnitude of the ground truth vector $y_0 = Ax_0$.

Statistical Analysis Overview

Statistical Analysis Overview

Objective: Evaluate the performance of four algorithms over 10,000 trials under the following parameters:

- $m = 25$, $n = 8$, $\beta = 0.5$
- Maximum iterations: 1000
- Tolerance: 10^{-4}

Key Metrics:

- Success rates (convergence percentages)
- Number of iterations required for convergence

Performance Distribution

Performance Distribution of Algorithms:

- RRR and HIO achieve substantially higher success rates compared to AP and RAAR.
- Minor differences observed between RRR and HIO in specific scenarios.

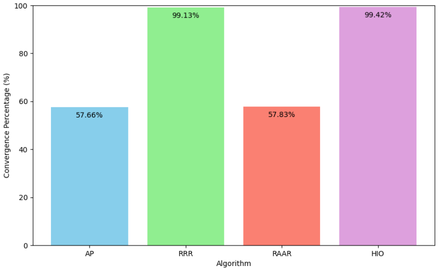


Figure: Performance distribution of 10,000 trials for the four algorithms, representing success rates (convergence percentages).

Iterations Required for Convergence

Convergence Behavior:

- Clear overlap between success rates and the number of iterations required for convergence.
- HIO converges slightly more often than RRR, with both outperforming AP and RAAR.
- Non-convergent trials are assigned the maximum number of iterations.

Figure: Iterations required for convergence for each algorithm.

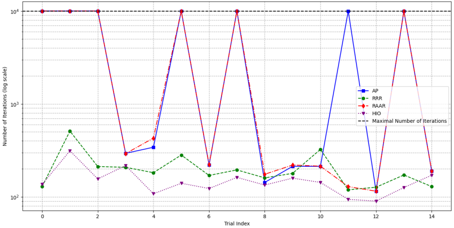


Figure: Zoomed-in view of convergence behavior for better visualization.

Experiment Overview

Experiment Overview

Objective: Addressing a realistic case where x is sparse. Key setup details:

- x: Sparse vector with $S = 4$ non-zero entries.
- A: 40×40 Discrete Fourier Transform (DFT) matrix.
- Stopping condition: Ratio μ, defined by Equation (12).
- Thresholds: 0.95 and 0.999, illustrating trade-offs between accuracy and iterations.

Results for Threshold 0.95

Reconstruction Results:

- Sparse vector x reconstructed using the RRR algorithm.
- Threshold: $\mu = 0.95$.

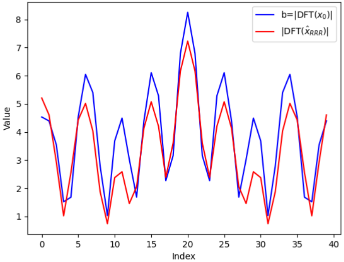


Figure: Reconstructed vector values for threshold 0.95 (magnitude).

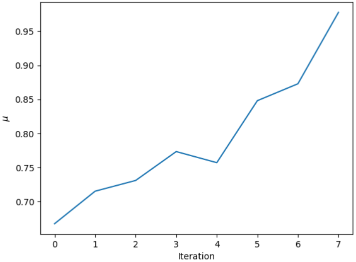


Figure: The ratio μ as a function of iterations for threshold 0.95.

Results for Threshold 0.999

Reconstruction Results:

- Threshold: $\mu = 0.999$.
- Higher threshold improves accuracy but requires more iterations.

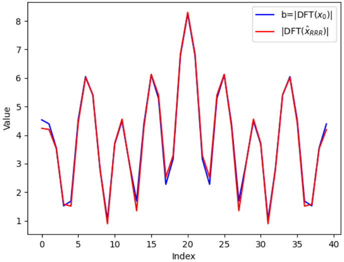


Figure: Reconstructed vector values for threshold 0.999 (magnitude).

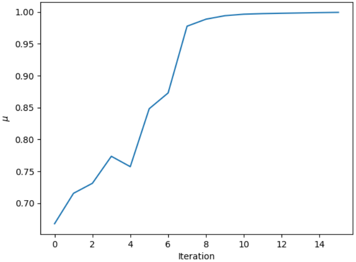


Figure: The ratio μ as a function of iterations for threshold 0.999.

Noise Analysis Overview

Noise Analysis Overview

Objective: Investigate the performance of the algorithm under noisy conditions. **Setup:**

- Gaussian noise with mean 0 and variance σ^2 added to the input vector x.
- Noise range: $\sigma \in [0, 10]$, sampled over 300 intervals.
- Noisy vector:
$$b_{\text{noisy}} = b + n, \quad n \sim \mathcal{N}(\mathbf{0}, \sigma^2 I),$$
where $\mathcal{N}(\mathbf{0}, \sigma^2 I)$ represents multivariate Gaussian noise.
- Parameters: $n = 40$, $S = 4$, maximum iterations $= 10{,}000$, threshold $= 0.95$.

Convergence as a Function of Noise

Results:

- Higher noise levels significantly increase the iterations required for convergence.
- Sparse regions in the figure indicate fewer successful convergences.
- Convergence nearly disappears for high σ, as the algorithm struggles to stabilize.

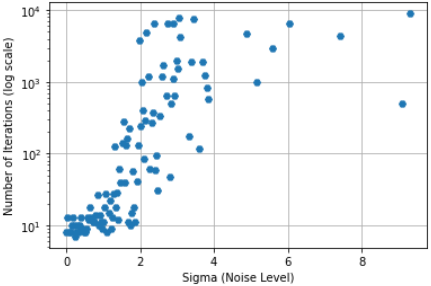


Figure: The number of iterations required for convergence as a function of σ. Higher noise levels significantly reduce convergence rates.

Statistical Experiment Overview

Statistical Experiment Overview

Objective: Analyze the performance of algorithms over a large number of experiments.
Setup:

- Number of experiments: 10,000.
- Matrix size: $n = 50$ (50×50 matrices).
- Sparse vector: $S = 5$ non-zero entries.
- Convergence threshold: 0.95.
- Maximum iterations: 10,000 (experiments not converging within this limit are marked as non-convergent).

Convergence Percentages

Results:

- RRR and HIO demonstrated significantly higher convergence rates compared to AP and RAAR.
- Robustness of RRR and HIO highlights their effectiveness in achieving the convergence threshold.

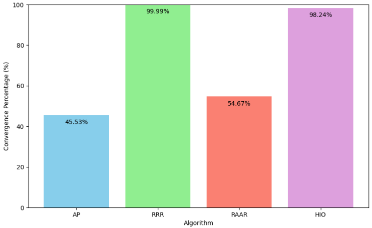


Figure: Convergence percentages for different algorithms over 10,000 experiments. Parameters used: $n = 50$, $S = 5$, maximum iterations $= 10{,}000$, $\beta = 0.5$, convergence threshold $= 0.95$.

Iterations Analysis

Analysis of Convergence Iterations:

- Iterations required for convergence vary significantly across algorithms.
- Non-convergent trials are assigned the maximum iteration count (10,000).

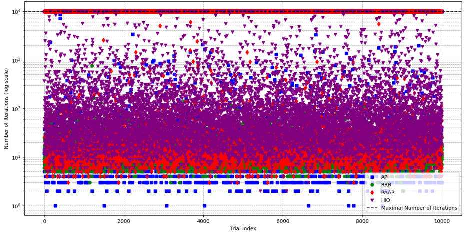


Figure: Iterations required for convergence for each algorithm.

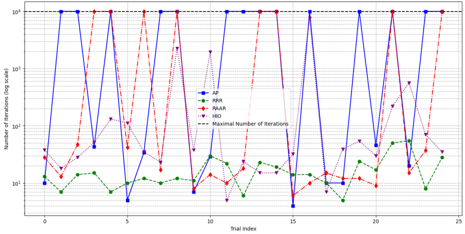


Figure: Zoomed-in view of convergence behavior for better visualization.

Matrix Completion Overview

Matrix Completion Overview

Objective: Reconstruct the ground-truth matrix using initial clues and rank information.

Setup:

- Ground-truth matrix: 11×11 matrix with rank 3.
- Randomly removed 25 elements to create the hint matrix.
- Inputs to the algorithm:
  - *Hint matrix*: Matrix with missing entries.
  - Rank information: rank $= 3$.

Example of Simulation

1.28	1.06	0.74	0.64	0.79	0.72	1.29	1.39	1.34	1.07	1.31
0.75	0.70	0.56	0.27	0.19	0.50	0.46	0.32	0.50	0.20	0.72
0.83	0.68	0.43	0.50	0.67	0.43	1.03	1.18	1.06	0.95	0.90
1.57	0.78	0.77	0.27	0.40	0.87	0.59	0.91	0.72	0.32	1.18
1.01	0.96	0.77	0.37	0.26	0.68	0.62	0.42	0.68	0.27	0.98
0.74	0.43	0.35	0.24	0.35	0.39	0.52	0.69	0.56	0.43	0.63
1.05	0.66	0.56	0.31	0.40	0.59	0.64	0.79	0.71	0.45	0.90
0.55	0.36	0.28	0.21	0.29	0.29	0.44	0.55	0.47	0.37	0.50
0.82	0.98	0.65	0.56	0.54	0.53	1.02	0.83	1.03	0.77	0.99
0.68	0.89	0.61	0.47	0.39	0.48	0.81	0.56	0.83	0.56	0.85
0.68	0.71	0.56	0.28	0.17	0.47	0.45	0.26	0.48	0.19	0.69

Figure: The ground-truth matrix (original matrix).

1.28	1.06	0.74	0.64	0.79	0.72	1.29	1.39	1.34	1.07	1.31
0.75	0.70	0.56	0.27				0.32	0.50		0.72
0.83	0.68	0.43		0.67	0.43	1.03	1.18		0.95	0.90
1.57	0.78	0.77	0.27		0.87	0.59	0.91	0.72	0.32	1.18
1.01		0.77	0.37		0.68	0.62	0.42	0.68	0.27	
0.74		0.35	0.24	0.35	0.39	0.52	0.69	0.56	0.43	
	0.66			0.40	0.59	0.64	0.79	0.71	0.45	0.90
0.55		0.28	0.21				0.55	0.47	0.37	0.50
0.82		0.65		0.54		1.02		1.03	0.77	0.99
	0.89	0.61	0.47	0.39	0.48	0.81	0.56	0.83	0.56	0.85
0.68	0.71	0.56	0.28	0.17		0.45	0.26	0.48	0.19	0.69

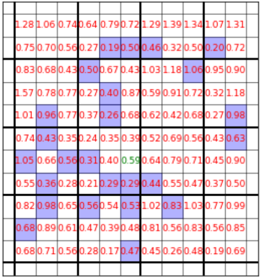
Figure: The hint matrix, with additional rank information (rank $= 3$).

Figure: Illustration of the original and hint matrices used in the simulation.

Initialization and Progress

Initialization:

- RRR algorithm begins with a random initialization.
- Progress of the algorithm visualized over iterations.
- Green elements indicate agreement with the ground-truth matrix; red elements indicate differences.



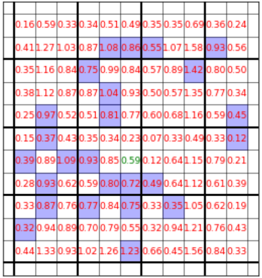


Figure: The ground-truth matrix (left) and initial random matrix (right).

Convergence Process: Iteration 1

Algorithm Progress:

- Visualization of the matrix at the first iteration.
- Red elements appear only in missing entries after enforcing the hint matrix projection.

1.28	1.06	0.74	0.64	0.79	0.72	1.29	1.39	1.34	1.07	1.31
0.75	0.70	0.56	0.27	0.19	0.50	0.46	0.32	0.50	0.20	0.72
0.83	0.68	0.43	0.50	0.67	0.43	1.03	1.18	1.06	0.95	0.90
1.57	0.78	0.77	0.27	0.40	0.87	0.59	0.91	0.72	0.32	1.18
1.01	0.96	0.77	0.37	0.26	0.68	0.62	0.42	0.68	0.27	0.98
0.74	0.43	0.35	0.24	0.35	0.39	0.52	0.69	0.56	0.43	0.63
1.05	0.66	0.56	0.31	0.40	0.59	0.64	0.79	0.71	0.45	0.90
0.55	0.36	0.28	0.21	0.29	0.29	0.44	0.55	0.47	0.37	0.50
0.82	0.98	0.65	0.56	0.54	0.53	1.02	0.83	1.03	0.77	0.99
0.68	0.89	0.61	0.47	0.39	0.48	0.81	0.56	0.83	0.56	0.85
0.68	0.71	0.56	0.28	0.17	0.47	0.45	0.26	0.48	0.19	0.69

1.28	1.06	0.74	0.64	0.79	0.72	1.29	1.39	1.34	1.07	1.31
0.75	0.70	0.56	0.27	1.08	0.86	0.55	0.32	0.50	0.93	0.72
0.83	0.68	0.43	0.75	0.67	0.43	1.03	1.18	1.42	0.95	0.90
1.57	0.78	0.77	0.27	1.04	0.87	0.59	0.91	0.72	0.32	1.18
1.01	0.97	0.77	0.37	0.81	0.68	0.62	0.42	0.68	0.27	0.45
0.74	0.37	0.35	0.24	0.35	0.39	0.52	0.69	0.56	0.43	0.12
0.39	0.66	1.09	0.93	0.40	0.59	0.64	0.79	0.71	0.45	0.90
0.55	0.93	0.28	0.21	0.80	0.72	0.49	0.55	0.47	0.37	0.50
0.82	0.87	0.65	0.77	0.54	0.75	1.02	0.35	1.03	0.77	0.99
0.32	0.89	0.61	0.47	0.39	0.48	0.81	0.56	0.83	0.56	0.85
0.68	0.71	0.56	0.28	0.17	1.23	0.45	0.26	0.48	0.19	0.69

Figure: Iteration 1: Visualization of the matrix at the beginning of the RRR algorithm.

Convergence Process: Iteration 33

Algorithm Progress:

- Visualization of the matrix at the 33rd iteration.
- Many elements begin to align with the ground-truth matrix, turning green.

1.28	1.06	0.74	0.64	0.79	0.72	1.29	1.39	1.34	1.07	1.31
0.75	0.70	0.56	0.27	0.19	0.50	0.46	0.32	0.50	0.20	0.72
0.83	0.68	0.43	0.50	0.67	0.43	1.03	1.18	1.06	0.95	0.90
1.57	0.78	0.77	0.27	0.40	0.87	0.59	0.91	0.72	0.32	1.18
1.01	0.96	0.77	0.37	0.26	0.68	0.62	0.42	0.68	0.27	0.98
0.74	0.43	0.35	0.24	0.35	0.39	0.52	0.69	0.56	0.43	0.63
1.05	0.66	0.56	0.31	0.40	0.59	0.64	0.79	0.71	0.45	0.90
0.55	0.36	0.28	0.21	0.29	0.29	0.44	0.55	0.47	0.37	0.50
0.82	0.98	0.65	0.56	0.54	0.53	1.02	0.83	1.03	0.77	0.99
0.68	0.89	0.61	0.47	0.39	0.48	0.81	0.56	0.83	0.56	0.85
0.68	0.71	0.56	0.28	0.17	0.47	0.45	0.26	0.48	0.19	0.69

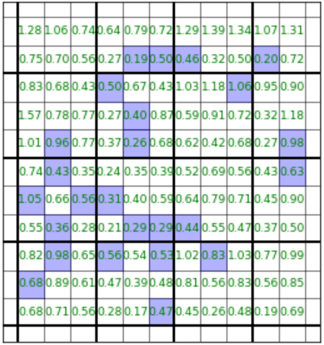
1.28	1.06	0.74	0.64	0.79	0.72	1.29	1.39	1.34	1.07	1.31
0.75	0.70	0.56	0.27	0.23	0.49	0.46	0.32	0.50	0.21	0.72
0.83	0.68	0.43	0.50	0.67	0.43	1.03	1.18	1.06	0.95	0.90
1.57	0.78	0.77	0.27	0.47	0.87	0.59	0.91	0.72	0.32	1.18
1.01	0.96	0.77	0.37	0.30	0.68	0.62	0.42	0.68	0.27	0.98
0.74	0.43	0.35	0.24	0.35	0.39	0.52	0.69	0.56	0.43	0.63
1.06	0.66	0.57	0.31	0.40	0.59	0.64	0.79	0.71	0.45	0.90
0.55	0.36	0.28	0.21	0.30	0.29	0.44	0.55	0.47	0.37	0.50
0.82	0.98	0.65	0.56	0.54	0.53	1.02	0.82	1.03	0.77	0.99
0.70	0.89	0.61	0.47	0.39	0.48	0.81	0.56	0.83	0.56	0.85
0.68	0.71	0.56	0.28	0.17	0.47	0.45	0.26	0.48	0.19	0.69

Figure: Iteration 33: Visualization of the matrix showing significant progress towards convergence.

Convergence Process: Final Iteration

Algorithm Progress:

- Visualization of the matrix at the final iteration.



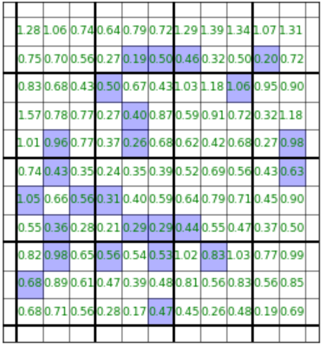


Figure: Final iteration: Visualization of the matrix after full convergence.

Convergence and Reconstruction Errors

Metrics Analyzed:

- Norm difference $|P_{\mathcal{B}}(y) - P_{\mathcal{A}}(y)|$ over iterations (stopping criterion).
- Norm difference between the estimated matrix and ground-truth matrix (evaluation-only metric).

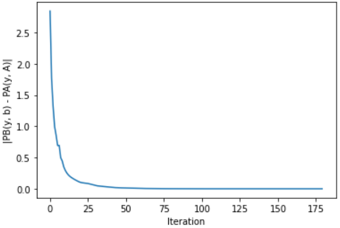


Figure: Norm difference $|P_{\mathcal{B}}(y) - P_{\mathcal{A}}(y)|$ as a function of iterations.

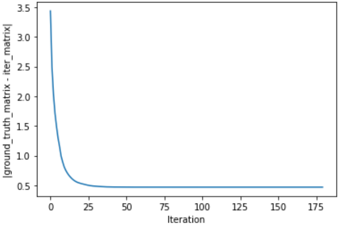


Figure: Norm difference between the estimate and ground-truth matrix.

Large-Scale Statistical Analysis Overview

Large-Scale Statistical Analysis Overview

Objective: Evaluate the performance of three matrix completion algorithms: AP, RRR, and RAAR. **Setup:**

- Matrix size: $n = 20$, rank: $r = 3$, missing elements: $q = 50$.
- Convergence criterion: Maximum 1000 iterations, tolerance 10^{-6}.
- Number of trials: 10,000 independent experiments.

Success Rates

Results:

- All three algorithms demonstrate high reliability, with success rates exceeding 97%.
- RRR achieved the highest success rate, underscoring its robustness.
- AP and RAAR offer viable alternatives with slightly lower success rates.

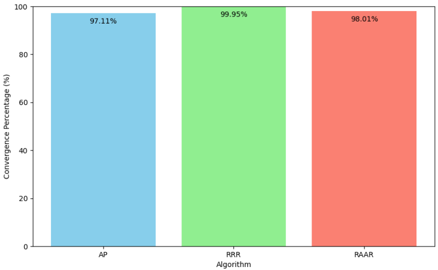


Figure: Convergence rates of the three algorithms over 10,000 trials: AP, RRR, and RAAR.

Iteration Analysis

Convergence Iterations:

- RRR achieved the highest convergence percentage but required more iterations on average.
- AP and RAAR often converged faster in trials where all algorithms succeeded.
- Non-convergent trials are assigned the maximum iteration count (1000).

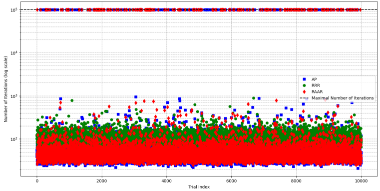


Figure: Iterations required for convergence for each algorithm.

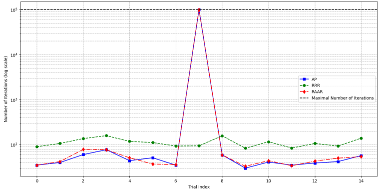


Figure: Zoomed-in view highlighting specific trials, such as Trial 7, where only RRR converged.

Effect of Increasing Missing Elements: Overview

Effect of Increasing Missing Elements: Overview

Objective: Investigate the effect of increasing the number of missing elements q on convergence behavior. **Setup:**

- q: Incrementally increased from 1 to $(n-r)^2$ (theoretical limit for matrix recovery).
- Maximum iterations: $100,000$.
- Cases considered:
  - **Case 1:** $n = 12$, $r = 3$.
  - **Case 2:** $n = 20$, $r = 5$.

Results for Increasing Missing Elements

Findings:

- Iteration counts increase with q for all algorithms, consistent with higher problem difficulty.
- RRR converges in significantly more cases, especially near the theoretical limit $(n-r)^2$.
- AP and RAAR show sparser convergence at higher q values, reflecting reduced reliability.

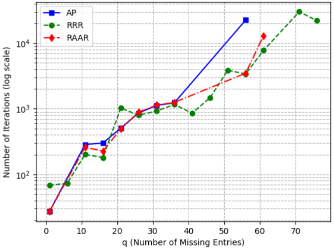


Figure: $n = 12$, $r = 3$.

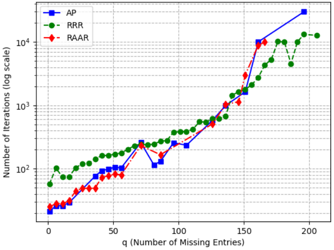


Figure: $n = 20$, $r = 5$.

Conclusion

Conclusion

Key Insights:

- **Random Matrices:**
  - AP and RAAR showed moderate success under standard conditions.
  - RRR and HIO outperformed, with RRR requiring fewer iterations and exhibiting superior robustness.
- **Crystallographic Phase Retrieval:**
  - Noise sensitivity was analyzed using the RRR algorithm, showing decreased convergence rates as noise increased.
  - RRR converged with the highest percentage, but was sometimes surpassed in terms of the number of iterations.
- **Matrix Completion:**
  - RRR emerged as the most reliable in handling high levels of missing data.
  - AP and RAAR often converged faster but were less consistent.

Future Directions

Potential Research Areas:

- Development of hybrid algorithms to balance efficiency and robustness.
- Exploring scalability to larger, more complex problems.
- Improving resilience to noise in real-world scenarios.

Goal: To optimize algorithmic performance across diverse use cases by addressing both theoretical and practical challenges.

Questions

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