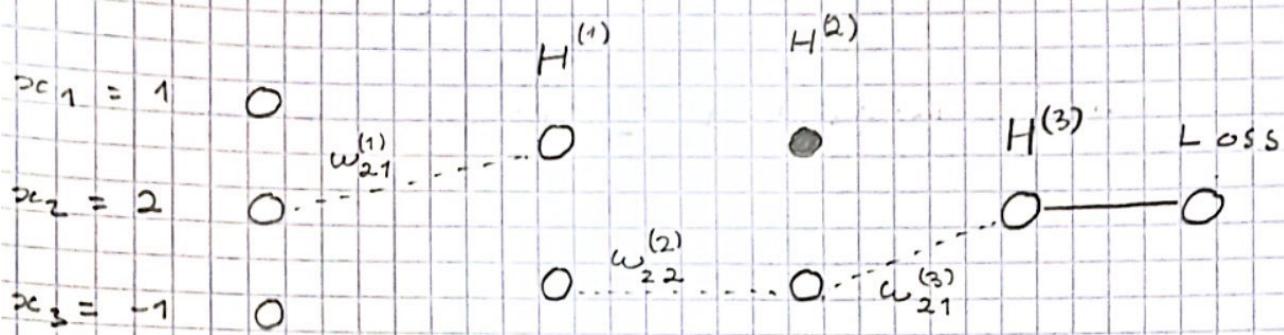


Question 1

Setup:

$$X = (1, 2, -1), Y = 0$$



Notation

Let $w_{jk}^{(L)}$ denote the weight from node j to node k , in to layer (L) . For example $w_{21}^{(1)}$ connects the second node of the input layer with the first node of Hidden layer 1. More examples on the graph.

Let $H_k^{(L)}$ denote the k -th node in the L -th hidden layer. For example, the filled-in node in black would be denoted as $H_1^{(2)}$. Finally, let every node in every layer have a bias $b_k^{(L)}$, where L denotes the hidden layer and k denotes the node in the hidden layer. For example, the bias of the filled-in black node would be $b_1^{(2)}$. Assuming all $b_k^{(L)} = 0$ for all L, k . The activation functions of $H^{(0)}$ and $H^{(2)}$ are all ReLU. All weights are initialized to 1.

$H^{(3)}$ is the output layer with the identity as the activation function. Assuming RSS loss.

Let $Z_k^{(L)}$ denote the input to ReLU for the k -th node in the L -th hidden layer such that $H_k^{(L)} = \text{ReLU}(Z_k^{(L)})$.

Forward Pass.

$$H_1^{(1)} = \text{ReLU}(\alpha_1 \cdot w_{11}^{(1)} + \alpha_2 \cdot w_{21}^{(1)} + \alpha_3 \cdot w_{31}^{(1)} + b_1^{(1)}) = \text{ReLU}(Z_1^{(1)})$$

$$H_1^{(1)} = \text{ReLU}(1 \cdot 1 + 2 \cdot 1 - 1 \cdot 1 + 0) = \text{ReLU}(2) = 2$$

$$H_2^{(1)} = \text{ReLU}(\alpha_1 \cdot w_{12}^{(1)} + \alpha_2 \cdot w_{22}^{(1)} + \alpha_3 \cdot w_{32}^{(1)} + b_2^{(1)}) = \text{ReLU}(Z_2^{(1)})$$

$$H_2^{(1)} = \text{ReLU}(1 \cdot 1 + 2 \cdot 1 - 1 \cdot 1 + 0) = \text{ReLU}(2) = 2$$

$$H_1^{(2)} = \text{ReLU}(H_1^{(1)} \cdot w_{11}^{(2)} + H_2^{(1)} \cdot w_{21}^{(2)} + b_1^{(2)}) = \text{ReLU}(Z_1^{(2)})$$

$$H_1^{(2)} = \text{ReLU}(2 \cdot 1 + 2 \cdot 1 + 0) = \text{ReLU}(4) = 4$$

$$H_2^{(2)} = \text{ReLU}(H_1^{(1)} \cdot w_{12}^{(2)} + H_2^{(1)} \cdot w_{22}^{(2)} + b_2^{(2)}) = \text{ReLU}(Z_2^{(2)})$$

$$H_2^{(2)} = \text{ReLU}(2 \cdot 1 + 2 \cdot 1 + 0) = \text{ReLU}(4) = 4$$

Output : $H^{(3)}$

$$H^{(3)} = H_1^{(2)} \cdot w_{11}^{(3)} + H_2^{(2)} \cdot w_{21}^{(3)} + b^{(3)}$$

$$H^{(3)} = 4 \cdot 1 + 4 \cdot 1 + 0 = 8$$

RSS Loss:

$$\text{Loss} = (Y - H^{(3)})^2$$

$$\text{Loss} = (0 - 8)^2 = 64$$

Backpropagation

For $b^{(3)}$:

$$\frac{\partial \text{loss}}{\partial b^{(3)}} = \frac{\partial \text{loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial b^{(3)}} = -2(Y - H^{(3)}) \cdot 1 = -2(0 - 8) \cdot 1 = 16$$

$$\frac{\partial \text{loss}}{\partial H^{(3)}} = 2(Y - H^{(2)})(-1) = -2(Y - H^{(2)}) = -2(0 - 8) = 16$$

$$\frac{\partial H^{(3)}}{\partial b^{(3)}} = 1$$

For $w_{11}^{(3)}$:

$$\frac{\partial \text{loss}}{\partial w_{11}^{(3)}} = \frac{\partial \text{loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial w_{11}^{(3)}} = 16 \cdot 4 = 64$$

$$\frac{\partial H^{(3)}}{\partial w_{11}^{(3)}} = H_1^{(2)} = 4$$

For $w_{21}^{(3)}$:

$$\frac{\partial \text{loss}}{\partial w_{21}^{(3)}} = \frac{\partial \text{loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial w_{21}^{(3)}} = 16 \cdot 4 = 64$$

$$\frac{\partial H^{(3)}}{\partial w_{21}^{(3)}} = H_2^{(2)} = 4$$

for $b_1^{(2)}$

$$\frac{\partial \text{loss}}{\partial b_1^{(2)}} = \frac{\partial \text{loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_1^{(2)}} \cdot \frac{\partial H_1^{(2)}}{\partial Z_1^{(2)}} \cdot \frac{\partial Z_1^{(2)}}{\partial b_1^{(2)}} = 16 \cdot 1 \cdot 1 \cdot 1 = 16$$

$$\frac{\partial H^{(3)}}{\partial H_1^{(2)}} = w_{11}^{(3)} = 1$$

$$\frac{\partial H_1^{(2)}}{\partial Z_1^{(2)}} = \frac{\partial \text{ReLU}(Z_1^{(2)})}{\partial Z_1^{(2)}} = 1$$

$Z_1^{(2)}$ is positive.

$$\frac{\partial Z_1^{(2)}}{\partial b_1^{(2)}} = 1$$

for $w_{11}^{(2)}$

$$\frac{\partial \text{loss}}{\partial w_{11}^{(2)}} = \frac{\partial \text{loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_1^{(2)}} \cdot \frac{\partial H_1^{(2)}}{\partial Z_1^{(2)}} \cdot \frac{\partial Z_1^{(2)}}{\partial w_{11}^{(2)}} = 16 \cdot 1 \cdot 1 \cdot 2 = 32$$

$$\frac{\partial Z_1^{(2)}}{\partial w_{11}^{(2)}} = H_1^{(1)} = 2$$

for $w_{21}^{(2)}$:

$$\frac{\partial \text{loss}}{\partial w_{21}^{(2)}} = \frac{\partial \text{loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_1^{(2)}} \cdot \frac{\partial H_1^{(2)}}{\partial Z_1^{(2)}} \cdot \frac{\partial Z_1^{(2)}}{\partial w_{21}^{(2)}} = 16 \cdot 1 \cdot 1 \cdot 2 = 32$$

$$\frac{\partial Z_1^{(2)}}{\partial w_{21}^{(2)}} = H_2^{(1)} = 2$$

For $b_2^{(2)}$

$$\frac{\partial \text{Loss}}{\partial b_2^{(2)}} = \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_2^{(2)}} \cdot \frac{\partial H_2^{(2)}}{\partial Z_2^{(2)}} \cdot \frac{\partial Z_2^{(2)}}{\partial b_2^{(2)}} = 16 \cdot 1 \cdot 1 \cdot 1 = 16$$

$$\frac{\partial H^{(3)}}{\partial H_2^{(2)}} = w_{21}^{(3)} = 1 \quad , \quad \frac{\partial Z_2^{(2)}}{\partial b_2^{(2)}} = 1$$

$$\frac{\partial H_2^{(2)}}{\partial Z_2^{(2)}} = 1$$

for $w_{12}^{(2)}$:

$$\frac{\partial \text{Loss}}{\partial w_{12}^{(2)}} = \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_2^{(2)}} \cdot \frac{\partial H_2^{(2)}}{\partial Z_2^{(2)}} \cdot \frac{\partial Z_2^{(2)}}{\partial w_{12}^{(2)}} = 16 \cdot 1 \cdot 1 \cdot 2 = 32$$

$$\frac{\partial Z_2^{(2)}}{\partial w_{12}^{(2)}} = H_1^{(1)} = 2$$

for $w_{22}^{(2)}$

$$\frac{\partial \text{Loss}}{\partial w_{22}^{(2)}} = \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_2^{(2)}} \cdot \frac{\partial H_2^{(2)}}{\partial Z_2^{(2)}} \cdot \frac{\partial Z_2^{(2)}}{\partial w_{22}^{(2)}} = 16 \cdot 1 \cdot 1 \cdot 2 = 32$$

$$\frac{\partial Z_2^{(2)}}{\partial w_{22}^{(2)}} = H_2^{(1)} = 2$$

for $b_1^{(1)}$

$$\frac{\partial \text{Loss}}{\partial b_1^{(1)}} = \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_1^{(2)}} \cdot \frac{\partial H_1^{(2)}}{\partial Z_1^{(2)}} \cdot \frac{\partial Z_1^{(2)}}{\partial H_1^{(1)}} \cdot \frac{\partial H_1^{(1)}}{\partial Z_1^{(1)}} \cdot \frac{\partial Z_1^{(1)}}{\partial b_1^{(1)}}$$

$$+ \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_2^{(2)}} \cdot \frac{\partial H_2^{(2)}}{\partial Z_2^{(2)}} \cdot \frac{\partial Z_2^{(2)}}{\partial H_1^{(1)}} \cdot \frac{\partial H_1^{(1)}}{\partial Z_1^{(1)}} \cdot \frac{\partial Z_1^{(1)}}{\partial b_1^{(1)}}$$

$$= 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 + 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 32$$

$$\frac{\partial Z_1^{(2)}}{\partial H_1^{(1)}} = w_{11}^{(2)} = 1 \quad , \quad \frac{\partial Z_1^{(1)}}{\partial b_1^{(1)}} = 1$$

$$\frac{\partial H_1^{(1)}}{\partial Z_1^{(1)}} = \frac{\partial \text{ReLU}(Z_1^{(1)})}{\partial Z_1^{(1)}} = 1$$

for $w_{11}^{(1)}$

$$\frac{\partial \text{Loss}}{\partial w_{11}^{(1)}} = \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_1^{(2)}} \cdot \frac{\partial H_1^{(2)}}{\partial Z_1^{(2)}} \cdot \frac{\partial Z_1^{(2)}}{\partial H_1^{(1)}} \cdot \frac{\partial H_1^{(1)}}{\partial Z_1^{(1)}} \cdot \frac{\partial Z_1^{(1)}}{\partial w_{11}^{(1)}}$$

$$+ \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_2^{(2)}} \cdot \frac{\partial H_2^{(2)}}{\partial Z_2^{(2)}} \cdot \frac{\partial Z_2^{(2)}}{\partial H_1^{(1)}} \cdot \frac{\partial H_1^{(1)}}{\partial Z_1^{(1)}} \cdot \frac{\partial Z_1^{(1)}}{\partial w_{11}^{(1)}}$$

$$= 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 + 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 32$$

$$\frac{\partial Z_1^{(1)}}{\partial w_{11}^{(1)}} = x_1 = 1$$

for $w_{21}^{(1)}$

$$\begin{aligned}\frac{\partial \text{LOSS}}{\partial w_{21}^{(1)}} &= \frac{\partial \text{LOSS}}{\partial H_1^{(3)}} \cdot \frac{\partial H_1^{(3)}}{\partial Z_1^{(2)}} \cdot \frac{\partial Z_1^{(2)}}{\partial H_1^{(1)}} \cdot \frac{\partial H_1^{(1)}}{\partial Z_1^{(1)}} \cdot \frac{\partial Z_1^{(1)}}{\partial w_{21}^{(1)}} \\ &\quad + \frac{\partial \text{LOSS}}{\partial H_2^{(3)}} \cdot \frac{\partial H_2^{(3)}}{\partial Z_2^{(2)}} \cdot \frac{\partial Z_2^{(2)}}{\partial H_1^{(1)}} \cdot \frac{\partial H_1^{(1)}}{\partial Z_1^{(1)}} \cdot \frac{\partial Z_1^{(1)}}{\partial w_{21}^{(1)}} \\ &= 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 + 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 = 64\end{aligned}$$

$$\frac{\partial Z_1^{(1)}}{\partial w_{21}^{(1)}} = \alpha_2 = 2$$

for $w_{31}^{(1)}$:

$$\begin{aligned}\frac{\partial \text{LOSS}}{\partial w_{31}^{(1)}} &= \frac{\partial \text{LOSS}}{\partial H_1^{(3)}} \cdot \frac{\partial H_1^{(3)}}{\partial H_1^{(2)}} \cdot \frac{\partial H_1^{(2)}}{\partial Z_1^{(2)}} \cdot \frac{\partial Z_1^{(2)}}{\partial H_1^{(1)}} \cdot \frac{\partial H_1^{(1)}}{\partial Z_1^{(1)}} \cdot \frac{\partial Z_1^{(1)}}{\partial w_{31}^{(1)}} \\ &\quad + \frac{\partial \text{LOSS}}{\partial H_2^{(3)}} \cdot \frac{\partial H_2^{(3)}}{\partial H_2^{(2)}} \cdot \frac{\partial H_2^{(2)}}{\partial Z_2^{(2)}} \cdot \frac{\partial Z_2^{(2)}}{\partial H_1^{(1)}} \cdot \frac{\partial H_1^{(1)}}{\partial Z_1^{(1)}} \cdot \frac{\partial Z_1^{(1)}}{\partial w_{31}^{(1)}} \\ &= 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot (-1) + 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot (-1) = -32\end{aligned}$$

$$\frac{\partial Z_1^{(1)}}{\partial w_{31}^{(1)}} = \alpha_3 = -1$$

for $b_2^{(1)}$

$$\frac{\partial \text{Loss}}{\partial b_2^{(1)}} = \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_1^{(2)}} \cdot \frac{\partial H_1^{(2)}}{\partial Z_1^{(2)}} \cdot \frac{\partial Z_1^{(2)}}{\partial H_2^{(1)}} \cdot \frac{\partial H_2^{(1)}}{\partial Z_2^{(1)}} \cdot \frac{\partial Z_2^{(1)}}{\partial b_2^{(1)}}$$

$$+ \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_2^{(1)}} \cdot \frac{\partial H_2^{(1)}}{\partial Z_2^{(2)}} \cdot \frac{\partial Z_2^{(2)}}{\partial H_2^{(1)}} \cdot \frac{\partial H_2^{(1)}}{\partial Z_2^{(1)}} \cdot \frac{\partial Z_2^{(1)}}{\partial b_2^{(1)}}$$

$$= 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 + 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 32$$

$$\frac{\partial Z_2^{(1)}}{\partial b_2^{(1)}} = 1$$

For $\omega_{12}^{(1)}$

$$\frac{\partial \text{Loss}}{\partial \omega_{12}^{(1)}} = \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_1^{(2)}} \cdot \frac{\partial H_1^{(2)}}{\partial Z_1^{(2)}} \cdot \frac{\partial Z_1^{(2)}}{\partial H_2^{(1)}} \cdot \frac{\partial H_2^{(1)}}{\partial Z_2^{(1)}} \cdot \frac{\partial Z_2^{(1)}}{\partial \omega_{12}^{(1)}}$$

$$+ \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(3)}}{\partial H_2^{(1)}} \cdot \frac{\partial H_2^{(1)}}{\partial Z_2^{(2)}} \cdot \frac{\partial Z_2^{(2)}}{\partial H_2^{(1)}} \cdot \frac{\partial H_2^{(1)}}{\partial Z_2^{(1)}} \cdot \frac{\partial Z_2^{(1)}}{\partial \omega_{12}^{(1)}}$$

$$= 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 + 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 32$$

$$\frac{\partial Z_2^{(1)}}{\partial \omega_{12}^{(1)}} = x_1 = 1$$

For $w_{22}^{(1)}$

$$\frac{\partial \text{Loss}}{\partial w_{22}^{(1)}} = \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(2)}}{\partial H_1^{(2)}} \cdot \frac{\partial H_1^{(2)}}{\partial Z_1^{(2)}} \cdot \frac{\partial Z_1^{(2)}}{\partial H_2^{(1)}} \cdot \frac{\partial H_2^{(1)}}{\partial Z_2^{(1)}} \cdot \frac{\partial Z_2^{(1)}}{\partial w_{22}^{(1)}}$$

$$+ \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(2)}}{\partial H_2^{(2)}} \cdot \frac{\partial H_2^{(2)}}{\partial Z_2^{(2)}} \cdot \frac{\partial Z_2^{(2)}}{\partial H_2^{(1)}} \cdot \frac{\partial H_2^{(1)}}{\partial Z_2^{(1)}} \cdot \frac{\partial Z_2^{(1)}}{\partial w_{22}^{(1)}}$$

$$= 16 \cdot 1 \cdot 1 \cdot 1 \cdot 2 + 16 \cdot 1 \cdot 1 \cdot 1 \cdot 2 = 64$$

$$\frac{\partial Z_2^{(1)}}{\partial w_{22}^{(1)}} = \gamma_2 = 2$$

for $w_{32}^{(1)}$

$$\frac{\partial \text{Loss}}{\partial w_{32}^{(1)}} = \frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(2)}}{\partial H_1^{(2)}} \cdot \frac{\partial H_1^{(2)}}{\partial Z_1^{(2)}} \cdot \frac{\partial Z_1^{(2)}}{\partial H_2^{(1)}} \cdot \frac{\partial H_2^{(1)}}{\partial Z_2^{(1)}} \cdot \frac{\partial Z_2^{(1)}}{\partial w_{32}^{(1)}}$$

+

$$\frac{\partial \text{Loss}}{\partial H^{(3)}} \cdot \frac{\partial H^{(2)}}{\partial H_2^{(2)}} \cdot \frac{\partial H_2^{(2)}}{\partial Z_2^{(2)}} \cdot \frac{\partial Z_2^{(2)}}{\partial H_2^{(1)}} \cdot \frac{\partial H_2^{(1)}}{\partial Z_2^{(1)}} \cdot \frac{\partial Z_2^{(1)}}{\partial w_{32}^{(1)}}$$

$$= 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot (-1) + 16 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot (-1) = -32$$

$$\frac{\partial Z_2^{(1)}}{\partial w_{32}^{(1)}} = \gamma_3 = -1$$

In summary - Back propagation

$$\frac{\partial \text{Loss}}{\partial b^{(3)}} = 16$$

$$\frac{\partial \text{Loss}}{\partial w_{11}^{(3)}} = 64$$

H⁽³⁾

$$\frac{\partial \text{Loss}}{\partial w_{21}^{(3)}} = 64$$

$$\frac{\partial \text{Loss}}{\partial b_1^{(2)}} = 16$$

$$\frac{\partial \text{Loss}}{\partial w_{11}^{(2)}} = 32$$

$$\frac{\partial \text{Loss}}{\partial w_{21}^{(2)}} = 32$$

H⁽²⁾

$$\frac{\partial \text{Loss}}{\partial b_2^{(2)}} = 16$$

$$\frac{\partial \text{Loss}}{\partial w_{12}^{(2)}} = 32$$

$$\frac{\partial \text{Loss}}{\partial w_{22}^{(2)}} = 32$$

H⁽¹⁾

$$\frac{\partial \text{Loss}}{\partial b_1^{(1)}} = 32$$

$$\frac{\partial \text{Loss}}{\partial w_{11}^{(1)}} = 32$$

$$\frac{\partial \text{Loss}}{\partial w_{21}^{(1)}} = 64$$

$$\frac{\partial \text{Loss}}{\partial w_{31}^{(1)}} = -32$$

$$\frac{\partial \text{Loss}}{\partial b_2^{(1)}} = 32$$

$$\frac{\partial \text{Loss}}{\partial w_{12}^{(1)}} = 32$$

$$\frac{\partial \text{Loss}}{\partial w_{22}^{(1)}} = 64$$

$$\frac{\partial \text{Loss}}{\partial w_{32}^{(1)}} = -32$$