Smoothed Particle Hydrodynamics

An Overview

Yogeshwaran R

FSI Lab, Department of Applied Mechanics
Indian Institute of Technology, Madras

5 May 2018

Outline

1. Introduction
2. Foundations of SPH
3. SPH Density Estimate
   - Kernel interpolation theory
   - Error calculation
4. From Density to Eqn. of motion
5. Derivatives
   - Paradox of SPH
6. Instabilities in SPH
   - Negative pressure and Tensile instability
   - Pairing instability
7. Artificial dissipation terms

Introduction

- Formulated by Lucy (1977) and Gingold and Monaghan (1977).
- Originally invented for astrophysical applications.
- Later, extended and applied to wide range of applications such as :
  - Dynamic Fluid flows
  - Explosions
  - Hydrodynamics with material strength etc.

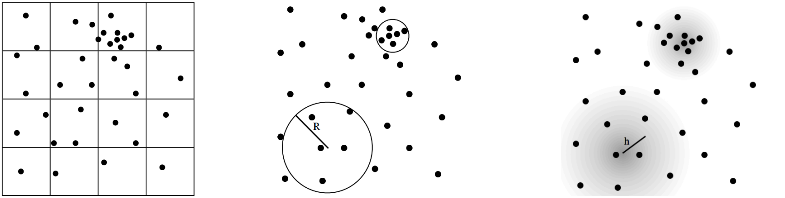
(Play video)

Foundations of SPH : Calculating Density

SPH :

"A meshfree lagrangian particle based method for solving equation of hydrodynamics"

How to compute density from an arbitrary distribution of point mass particles ?



(a) ρ=m/$\cancel{V}$

(b) based on local sampling

$$\rho(r) = \frac{\sum_{b=1}^{N_{neigh}} m_b}{\frac{4}{3}\pi R^3}$$

(c) SPH density estimator

$$\rho(r) = \sum_{b=1}^{N_{neigh}} m_b \, W(r-r_b,h)$$

FIGURE : Computing a continuous density field from a collection of point mass particles.

The SPH density estimate

The density is computed using a weighted summation over nearby particles, which is given by,

$$\rho(r) = \sum_{b=1}^{N_{neigh}} m_b W(r - r_b, h)$$

Accuracy of density estimate $\rightarrow$ depends on choice of weight function

A good density kernel should have following properties :

- W $\rightarrow$ '+'ve, decrease monotonically and have smooth derivatives.
- Symmetric w.r.t. to |r-r'| (i.e.,) W(r-r',h) $\equiv |W(|r' - r|, h)|$
- Compact support. $[\cancel{O(N^2)} \rightarrow O(N_{neigh}N) \Rightarrow$ Computationally efficient $]$

Basics :

$$A(r) = \int A(r') \underbrace{\delta(r - r')}_{\text{dirac delta fn.}} dr'$$

Two major approximations in SPH :

1) Kernel approximation 2) Particle approximation

1) Kernel approximation

$$A(r) = \int A(r') \, W(r - r', h) \, dr' + O(h^2)$$

2) Particle approximation

$$< A(r) > = \int \frac{A(r')}{\rho(r')} \, W(r - r', h) \, \rho(r') \, dr'$$

$$< A(r) > \approx \sum_{b=1}^{N_{neigh}} m_b \frac{A_b}{\rho_b} W(r - r_b, h)$$

Error

i) Error in Kernel approximation :

$$< A(r) >= \int A(r')\ W(r-r',h)\ d\text{\sout{V}}'$$

Expanding $\rho(r')$ in taylor's series about r,

$$< A(r) > = A(r) \underbrace{\int W(r-r',h)\ d\text{\sout{V}}'}_{\approx 1 \text{ (Normalization cond.)}} + \nabla A(r) \underbrace{\int (r'-r) W(r-r',h) d\text{\sout{V}}'}_{= 0 \text{ (symmetric)}} + O(h^2)$$

---

ii) Error in particle approximation :

$$< A_a >= \sum_b m_b \frac{A_b}{\rho_b} W_{ab}$$

Expanding A_b in taylor's series,

$$< A_a > = A_a \underbrace{\sum \frac{m_b}{\rho_b} W_{ab}}_{\approx 1} + \nabla A_a \underbrace{\sum_b \frac{m_b}{\rho_b}(r_b - r_a) W_{ab}}_{= 0 \text{ (anti-symm.)}} + O(h^2)$$

From Density to Hydrodynamics

$$L = \sum_b m_b \left[\frac{1}{2} v_b^2 - u_b(\rho_b, s_b) \right]$$ $\rightarrow$ Lagrangian

+

$$du = \frac{p}{\rho^2} d\rho$$ $\rightarrow$ 1^{st} Law of thermodynamics

+

$$\rho(r) = \sum_{b=1}^{N_{neigh}} m_b W(r - r_b, h)$$ $\rightarrow$ Density Sum

+

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \frac{\partial L}{\partial \mathbf{r}} = 0$$ $\rightarrow$ Euler-Lagrange equations

=

$$\boxed{\frac{d\mathbf{v}}{dt} = -\sum_j m_j \left(\frac{p_i}{\rho_i^2} + \frac{p_j}{\rho_j^2} \right) \nabla_i W_{ij}(h)}$$ $\Rightarrow$ Eqn. of motion

Generalised first derivative operators

Two basic types :

$$\textbf{(i)} \quad <\nabla A> \approx \sum_b \frac{m_b}{\rho_b}\frac{\phi_b}{\phi_a}(A_b - A_a)\nabla_a W_{ab}$$

$$\textbf{(ii)} \quad <\nabla A> \approx \sum_b \frac{m_b}{\rho_b}\left(\frac{\phi_b}{\phi_a}A_a + \frac{\phi_a}{\phi_b}A_b\right)\nabla_a W_{ab}$$

where ϕ is any arbitrary, differentiable scalar quantity defined on the particles.

Various *'alternative formulations'* of SPH have been proposed based on the choice of ϕ

For example.,

Taking $\phi = \rho$ in (i)

$$\nabla A \approx \frac{1}{\rho_a}\sum_b m_b(A_b - A_a)\nabla_a W_{ab}$$

$\rightarrow$ **exact derivative**

Taking $\phi = \rho$ in (ii)

$$\nabla A \approx \rho_a \sum_b m_b\left(\frac{A_a}{\rho_a^2} + \frac{A_b}{\rho_b^2}\right)\nabla_a W_{ab}$$

$\rightarrow$ **exact conservation**

Which is preferable : exact derivative or exact conservation ?

Let us consider momentum eqn.,

$$\frac{d\mathbf{v}}{dt} = -\frac{\nabla p}{\rho}$$

Based on the *exact derivative approach* :

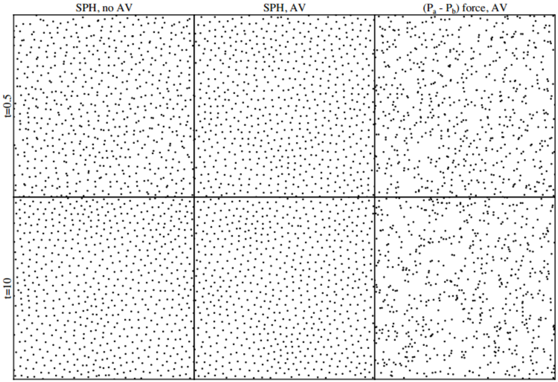
$$\frac{d\mathbf{v}_a}{dt} = \sum_b m_b \frac{p_a - p_b}{\rho_a \rho_b} \nabla_a W_{ab}$$

Based on the *exact conservation approach* :

$$\frac{d\mathbf{v}_a}{dt} = -\sum_b m_b \left(\frac{p_a}{\rho_a^2} + \frac{p_b}{\rho_b^2} \right) \nabla_a W_{ab}$$

For illustration, let us consider a distribution of particles in a closed box.

What happens if the pressure is constant ?



exact conservative

exact derivative

The exact conservative approach contains *"intrisic remeshing"* which cares about bad arrangement of particles.

Insensitive to particle arrangement leads to *interpolation error*.

Tensile instability

Consider a mom. eqn. with pressure gradient is of the form,

$$\frac{d\mathbf{v}}{dt} = -\sum_b m_b \left(\frac{p_a - p_0}{\rho_a^2} + \frac{p_b - p_0}{\rho_b^2} \right) \underbrace{\nabla_a W_{ab}}_{\text{'-' ve (for '+'ve definite kernels)}}$$

When p_0 > p, the pairwise force $\rightarrow$ '-'ve $\Rightarrow$ $\underbrace{\text{particles clump together unphysically}}_{\text{due to attractive force}}$

This is known as *"Tensile instability".*

Why one cannot simply use *'more neighbors?'*

- Consider a 2D setup with particles are initially placed on a close packed lattice.
- If we consider **M4 cubic spline kernel with a large ratio of smoothing length to particle**, what will happen ?

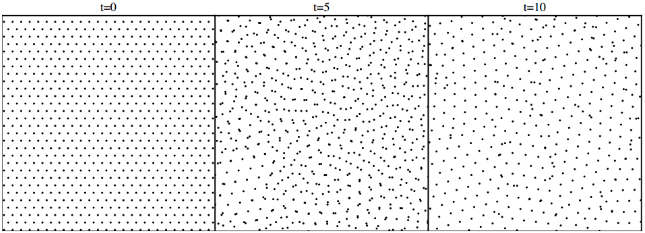
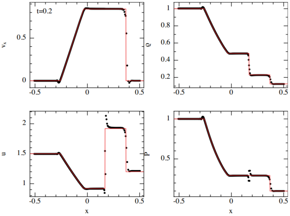
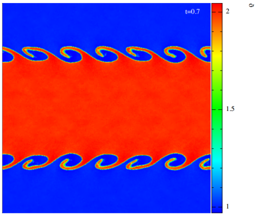


FIGURE : Pairing instability in action

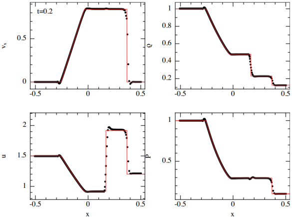


(a) 1D Sod Shock tube problem

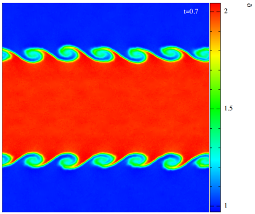


(b) 2D Kh instability

FIGURE : With artificial viscocity



(a) 1D Sod Shock tube problem



(b) 2D Kh instability

FIGURE : With artificial viscosity + artificial conductivity

Thank you ! ! !