Phase shift analysis of $p - {}^{16}O$ elastic scattering at Astrophysical Energies using phase function method

Report

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by

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CENTRAL UNIVERSITY OF HIMACHAL PRADESH

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It is certified that the work contained in the report entitled "Phase shift analysis of $P-^{16}O$ elastic scattering at Astrophysical Energies using phase function method" has been done under the supervision of Prof. O.S.K.S. Sastri . It is also certified that the work has not been submitted to any other institute for any degree or diploma and whenever I have used materials from other sources, I have given due credit to them by giving their details in the references.

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Thank You
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Abstract

In this work, we have studied the s-wave scattering phase shifts of $p-^{16}$O for lab energies below 2 MeV using Phase Function Method. The nuclear interaction has been modeled using Morse potential and electromagnetic screened Coulomb interaction by atomic Hulthén. The phase equation for the s-wave has been solved numerically using Runge- Kutta 4^{th} order method. The model parameters are optimized by minimizing the mean squared error (MSE) between the expected data and the computed scattering phase shifts using the Variational Monte Carlo (VMC) technique. The simulated phase shifts for $^2S_{1/2}$ are found to be in good agreement with those expected, with an MSE of 1.1.

keywords:$p-^{16}$O, elastic scattering, phase function method, scattering phase shifts, Morse potential, atomic Hulthén potential, Variational Monte carlo (VMC)

Contents

1 Introduction

Since the Coulomb interaction has an infinitely long range, one of the most challenging issues for theoretical physicists is to accurately comprehend scattering between a charged projectile and a charged target [1]. In the past few decades, a number of theoretical models and research have been put forth to address low- and intermediate-energy proton-nucleus scattering.Two potentials are involved in proton-nucleus scattering: an electromagnetic potential that manages charges, and a nuclear potential that originates from the proton [2]. Understanding the interaction between nucleon-nucleon, nucleon-nucleus, and nucleus-nucleus is fundamental to determining nuclear structure for nuclei A < 20.

The neutron-proton interaction is modeled by Yukawa [3] using a meson exchange mechanism to explain the nature of nuclear force, which is fundamentally different from the known gravitational and electromagnetic forces. This was followed by various formalism that involve one pion exchange [4], two pion exchange [5], and multiple meson exchanges [6]. The current high precision potentials to understand neutron-proton (n-p) and proton-proton (p-p) interaction have been reviewed by Naghdi [7]. Recently, Sastri *et.al.* [8] have given an alternative approach based on the combination of the phase function method and Variational Monte Carlo (VMC) algorithm for optimization of model parameters to construct inverse potentials for both n-p and p-p systems. While the Morse function has been utilized for modeling the nuclear interaction, an $erf()$ based Coulomb ansatz was chosen to represent the screened Coulomb interaction. This methodology was extended to nucleon-nucleus interaction such as neutron-deuteron [9] and proton-deuteron [10] , neutron-alpha, proton-alpha [11] systems. Similarly, the nucleus-nucleus interaction was also studied for $\alpha - \alpha$ [12, 13],$\alpha-^3$He [14] and $\alpha-^3$H [15] systems. All of the above systems involve nuclei only up to A $= 4$. This methodology has also been extended to study the scattering of $\alpha -^{12} C$ [16]. Recently, atomic Hulthén potential has been utilized for modeling the electromagnetic interaction for studying the charged particle scattering [17].

In this work the study involves scattering of proton-^{16}O system where one can consider ^{16}O to be consisting of a cluster of 4 alpha particles. The $O^{16}(p,\gamma)F^{16}$ reaction holds significant importance in both stellar energy production and nuclear physics radiative capture processes. The specific characteristics of the $^2S_{1/2}$ phase shifts are vital factors influencing the thermo-nuclear processes involved in radiative capture at astrophysical energies [2]. Nuclear Astrophysics is the name given to the field of nuclear physics, which plays a crucial role in astrophysics. Nuclear physics provides the foundation for many observable features. With the help of astrophysical reactions, we can understand the birth and death of stars, the energy that powers the stars and the formation of the elements and their nuclear interaction properties.

This reaction has been obtained both experimentally [23] and theoretically [2] by many researchers. While scattering cross section data is what is available from experiments, only wave function from Schrodinger equation is within the reach of theoretical physicists. Hence, understanding the scattering phenomenon, phase wave analysis generally requires three steps:

- Obtain the scattering phase shifts for various ℓ- channels from the experimentally observed scattering cross section at various lab energies [22].
- The scattering phase shifts (SPS) for a selected interaction model can be theoretically determined from the wave function obtained as the solution of the time-independent Schrodinger equation.
- The model parameters are adjusted to match the determined SPS from both the procedures.

While Laha *et.al* [2], use the Jost function approach based on wave function to obtain the SPS, Sastri *et.al.* [18] have utilized the phase function method which only needs the interaction potential as input for determining the SPS.
In this work our objective is to study $p-^{16}$O scattering for elastic energies in the range from 0.1 to 2 MeV by utilizing phase function method. We choose Morse potential as model for nuclear interaction and atomic Hulthén potential for modeling the screened Coulomb interaction.

2 Methodology

In order to study $p-^{16}$O using PFM, one need to numerically solve the phase equation. So, we develop the theoretical and computational framework using modeling theory [19] and simulation methodology [8].

Theoretical model for $p-^{16}$O scattering :

2.1 Description Stage:

2.1.1 System description

- **Type of System :** Here, we are considering charged particle scatterings. The charged projectile is accelerated to an energy $E_{\ell ab}$ and scattering of a charged nucleus target.
- **Composition :** Proton(p) is the projectile and ^{16}O nucleus is the target.
- **Variables:**

System variables	Proton (p)	^{16}O
Mass (MeV/c^2)	938.272	14902.973
Charge (Z)	e	8e
Spin	0	$\frac{1}{2}^+$
$E_{\ell ab}$	[0.38-1.99]MeV	Rest

2.1.2 Interaction Description

- **Type of Interaction :** Elastic scattering of a proton beam on ^{16}O target. In this elastic scattering process two type of interaction occurs:
  - Attraction between p and ^{16}O .
  - Electromagnetic repulsion between p and ^{16}O.
- **Agent of Interaction :** Nuclear attraction between p and ^{16}O which reflects the next higher energy level region and Coulomb repulsion between p and ^{16}O , that dies down with distance due to screening. These two will result in dynamical equilibrium.
- **Interaction Variables :** The Nuclear interaction will have a depth (V_0) at equilibrium distance (r_m) with the nature of interaction captured using a shape parameter (a_m).
  The screened coulomb interaction will have a strength parameter (V_a) and a screening parameter (a).

2.1.3 Process Description

- **Reference system :** The two body problem in lab frame is replaced by a one-body system in centre of mass frame with the transformations $\bar{r} = \bar{r_1} - \bar{r_2}$ and $\bar{R} = \frac{m_p\bar{r_1}+m_O\bar{r_2}}{m_p+m_O}$. So the reduced mass of the system can be given as

$$\mu = \frac{m_p m_O}{m_p + m_O} \tag{1}$$

where m_p and m_O are the masses of proton and oxygen in MeV/c^2.

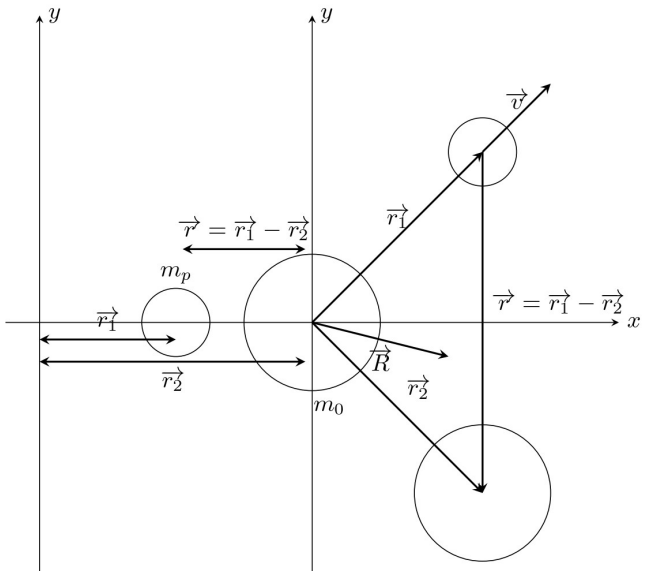

Figure 1: Reduction of two-body problem to one-body problem

- **State Variable :** At the microscopic level, the dynamics are usually controlled by the time-dependent Schrodinger equation, and the wave function of the system determines its current state. However, in this work, we use the phase function method, where the Ricatti equation is used in place of the Schrodinger equation in terms of phase shifts. As a result, the system's state can be described by the scattering phase shifts $\delta_l(k, r)$.

2.2 Formulation Stage

This step involves first going over the **Interaction laws** in their mathematical form, followed by a description of the **Dynamical laws** that control the nucleon's mobility. Next, in order to finish formulating the mathematical model that represents the physical system, we define the initial and boundary conditions, if any.

2.2.1 Interaction law

The total interaction $V_T(r)$ between projectile and target nucleus is modeled by combination of nuclear $V_n(r)$ and Coulomb parts $V_C(r)$ as

$$V_T(r) = V_n(r) + V_C(r) \tag{2}$$

where nuclear interaction is modeled by Morse function:

$$V_n(r) = V_0(e^{-2(r-r_m)/a_m} - 2e^{-(r-r_m)/a_m}) \tag{3}$$

where V_0 represents strength of potential, r_m represents the distance at which interaction is maximum and a_m represents shape parameter.
The Coulomb interaction is modeled by atomic Hulthén potential [20],

$$V_C(r) = V_a \frac{e^{-r/a}}{1 - e^{-r/a}} \tag{4}$$

where V_a is the strength and a is the screening radius of the screened Coulomb potential. [21]. These two parameters V_a and a are related by [25]

$$V_a a = 2K\eta$$

where K is momentum energy in lab frame and η is Sommerfeld parameter defined as

$$\eta = \frac{\alpha}{\hbar v}$$

Here, v is relative velocity of the reactants at large separation and $\alpha = Z_1 Z_2 e^2$. So,

$$V_a a = \frac{Z_1 Z_2 e^2 \mu}{\hbar^2}$$

For $p -^{16} O$, $Z_1 = 1, Z_2 = 8$, $\mu = \frac{m_p m_O}{m_p + m_O} = 882.698 \ \frac{MeV}{c^2}$, $e^2 = 1.44 MeV fm$ and therefore $V_a a = 0.5255 fm^{-1}$. As V_h is in fm^{-2}, so we have to convert it into MeV by

multiplying by a factor of $\frac{\hbar^2}{2\mu}$, which is equal to 22.056 $MeV\ fm^2$.
So, the total interaction potential is given by:

$$V_T(r) = V_0[(e^{-2(r-r_m)/a_m} - 2e^{-(r-r_m)/a_m})] + V_a\frac{e^{-r/a}}{1-e^{-r/a}} \tag{5}$$

Utilising this modeled interaction, we have simulated the phase shifts of $p-^{16}$O by employing phase function method.

2.2.2 Dynamical law

The time independent Schrödinger equation (TISE) can be written as

$$\frac{d^2u_\ell(r)}{dr^2} + \left[k^2 - \frac{\ell(\ell+1)}{r^2} - U(r)\right]u_\ell(r) = 0 \tag{6}$$

where $U(r) = V(r)/(\hbar^2/2\mu)$ & $k_{c.m} = \sqrt{E_{c.m}/(\hbar^2/2\mu)}$ and $E_{c.m} = 0.940E_{\ell ab}$.

For $p-^{16}O$ system, the value of $\hbar^2/2\mu$ = 22.056 MeVfm^2.

Phase Function Method is one of the important tools in scattering studies for both local and non-local interactions [24] . The TISE in Eq.6 can been transformed to a non-linear Riccati equation of first order, which directly deals with SPS information, given by:

$$\delta_\ell\prime(k,r) = -\frac{U(r)}{k}\left[\cos(\delta_\ell(k,r))\hat{j}_\ell(kr) - \sin(\delta_\ell(k,r))\hat{\eta}_\ell(kr)\right]^2 \tag{7}$$

The Riccati Hankel function of first kind is given by $\hat{h}_\ell(r) = -\hat{\eta}_\ell(r) + i\,\hat{j}_\ell(r)$, where $\hat{j}_\ell(kr)$ is Ricatti-Bessel and $\hat{\eta}_\ell(kr)$ Riccati-Neumann function. By substituting the expressions for different ℓ-values of these two later functions, we obtain the respective phase equations as:

$$\delta_0\prime(k,r) = -\frac{U(r)}{k}\sin^2[\delta_0 + \kappa] \tag{8}$$

This equation is called the phase equation for $\ell = 0$, which is a non-linear first order differential equation of Ricatti type. The value of phase shift (δ) at $r = 0$ is chosen to be zero, as interaction with potential is not present, since $u_0(r = 0) = 0$. This Eq. 8 can not be solved using analytical techniques and hence we resort to numerical approach . The state of system , for $\ell = 0$ is described by its wavefunction $\psi(r,t)$ and is obtained by solving radial time independent schrodinger equation, given by

$$\frac{d^2u_0(r)}{dr^2} + \frac{2\mu}{\hbar^2}[E - V(r)]u_o(r) = 0 \tag{9}$$

The wave function must satisfy $u_a(0) = 0$ at r=0. at a distance r_0 beyond which V(r) is zero , wavefunction and its derivative both need to be continuous. That is, choosing $u_a(0)$ to be a asymptotic solution of eq. 9, we must have

$$u_0(r)|_{r=r_0} = u_a(r)|_{r=r_a} \tag{10}$$

and

$$\frac{du_0(r)}{dr}|_{r=r_0} = \frac{du_a(r)}{dr}|_{r=r_a} \tag{11}$$

These two conditions are combined together into a single equation by considering logarithmic derivative to satisfy boundary condition ,obtained as

$$\frac{1}{u_0(r)}\frac{du_0(r)}{dr}\ |_{r=r_0} = \frac{1}{u_a(r)}\frac{du_a(r)}{dr}|_{r=r_a} \tag{12}$$

2.2.3 Derivation of phase equation:

To understand the concept of phase shift, let's assume that the interaction is just a simple spherical square well potential of width $r_0 = r_D$. For the case of $E > 0$ singlet state, the solution within the potential well is given by

$$u_0(r) = Asin(k_1 r) + Bcos(k_1 r) \tag{13}$$

here $k_1 = \sqrt{\frac{2\mu(E-V_0)}{\hbar^2}}$.

Using boundary condition $u_0(r = 0) = 0$

$$0 = Asin(k_1 0) + Bcos(k_1 0) \qquad B = 0 \tag{14}$$

Outside the well, the asymptotic solution obtained using the Schrodinger equation is given by

$$u_a(r) = A_1 sin(kr + \delta) \tag{15}$$

here, $k = \sqrt{\frac{2\mu_D E}{\hbar^2}}$

At $r = r_D$, Both wavefunctions and their derivatives need to be the same.

$$k_1 cot(k_1 r_D) = kcot(kr_D + \delta) \tag{16}$$

from here we can determine δ A particle of energy E comes under the influence of interaction potential.

for $r > r_0$, $V(r) = 0$ so wavefunction is given by

$$u_a(r) = \alpha sin(kr + \delta) \tag{17}$$

here α and δ are constants,and $k = \sqrt{\frac{2\mu E}{\hbar^2}}$.

Logarithmic Derivative, $A(r)$ is

$$A(r) = \frac{1}{u_0(r)}\frac{du_0(r)}{dr} = \frac{1}{u_a(r)}\frac{du_a(r)}{dr} \tag{18}$$

$$A(r) = k\cot(kr+\delta) \tag{19}$$

let's take the derivative of eq. 18

$$\frac{A(r)}{dr} = \frac{d}{dr}\left(\frac{1}{u_0(r)}\frac{du_0(r)}{dr}\right) = \frac{1}{u_0(r)}\frac{d^2u_0(r)}{dr^2} - \frac{1}{u_0^2(r)}\left(\frac{du_0(r)}{dr}\right)$$

$$\frac{A(r)}{dr} + \frac{1}{u_0^2(r)}\left(\frac{du_0(r)}{dr}\right) = \frac{1}{u_0(r)}\frac{d^2u_0(r)}{dr^2}$$

$$\frac{1}{u_0(r)}\frac{d^2u_0(r)}{dr^2} = \frac{dA}{dr} + [A(r)]^2 \tag{20}$$

Schrodinger equation is given by

$$\frac{1}{u_0(r)}\frac{d^2u_0(r)}{dr^2} + \frac{2\mu}{\hbar^2}(E - V(r)) = 0 \tag{21}$$

using eq. 20

$$\frac{dA}{dr} + [A(r)]^2 = \frac{2\mu}{\hbar^2}(V(r) - E) \tag{22}$$

Using eq. 19 to find $\frac{dA}{dr}$,

$$\frac{dA}{dr} = -\frac{(k^2 + k\frac{d\delta}{dr})}{\sin^2(kr+\delta)} \tag{23}$$

Putting eq. 23 and eq. 19 in eq. 22

$$-\frac{(k^2 + k\frac{d\delta}{dr})}{\sin^2(kr+\delta)} + k^2\frac{\cos^2(kr+\delta)}{\sin^2(kr+\delta)} = \frac{2\mu}{\hbar^2}(V(r) - E) \tag{24}$$

$$-\frac{(k^2 + k\frac{d\delta}{dr})}{\sin^2(kr+\delta)} + k^2\frac{(1-\sin^2(kr+\delta))}{\sin^2(kr+\delta)} = \frac{2\mu}{\hbar^2}V(r) - k^2 \tag{25}$$

$$-\frac{k^2}{\sin^2(kr+\delta)} - \frac{k\frac{d\delta}{dr}}{\sin^2(kr+\delta)} + \frac{k^2}{\sin^2(kr+\delta)} - k^2 = \frac{2\mu}{\hbar^2}V(r) - k^2 \tag{26}$$

$$\frac{d\delta(r)}{dr} = -\frac{2\mu}{\hbar^2}\frac{V(r)}{k}\sin^2(kr+\delta) \tag{27}$$

This is known as the non-linear differential equation of Ricatti type. To solve this problem, we need to use numerical methods as it can't be solved using analytical methods.

2.2.4 Boundary conditions

$$\delta_0(k, r=0) = 0 \tag{28}$$

The non-linear first order differential equation of Ricatti type can not be easily solved using analytically, so we resort to numerical approach.

Numerical Simulation for $p-^{16}$O scattering :

2.3 Preparation of System for simulation

2.3.1 Choice of units:

As we are working in a nuclear domain so,

- The units of energy should be in MeV.
- The unit of distance should be in fm.
- Mass is taken in $\frac{MeV}{c^2}$
- e^2= 1.44 $MeVfm$
- hc= 197.326 $Mevfm$
- 1 $fm = (10^{-15}m)^2 = 10^{-30}m^2 = 10^{-2}barn$

2.3.2 Region of interest

- Interaction potential and hence the phase shift is calculated in a region r = 0.01 to 60 fm for $\ell = 0$.
- Phase shift are calculated for energies from 0.3 - 2 MeV for partial wave $\ell = 0$.

2.3.3 Choice of numerical technique:

While choosing numerical technique, one must check three key characteristics of technique: stability, accuracy and efficiency for solving the problem. Methods like Euler method, 2nd or 4th order Runge-Kutta (RK) can be utilized for solving phase equation. While both RK-2 and RK-4 methods are stable, they have accuracies given by $O(h^2)$ and $O(h^4)$ respectively. The expected $p-^{16}$O expected phase shift data is available up to two decimal places, so to get 2 decimal accuracy it is preferable to utilise RK-4 technique. Hence, In this work, we have utilized RK 4^{th} order method to solve the phase equation.

2.4 Implementation Stage:

2.4.1 Implementation of RK-4 method in Python

The steps for implementing the Runge-Kutta method in Python are as follows:

- **Initialize the Parameters:** There are two types of parameters: physical parameters and algorithm parameters.
  1. **Physical Parameters:** We need to specify the object and interaction variables of the physical system. The state variables that we aim to determine are considered as outputs.
  2. **Algorithm Parameters:** The algorithm parameters include the step size, initial integration distance, and final integration distance. For this particular case, the step size is set to 0.01, and the integration distance ranges from $r_i = 0.01$ to $r_f = 60$ fm.
- **Define the Modeled Interaction:** After initializing the parameters, we have to define the Morse and atomic Hulthén potential.
- **Initial Condition of RK-4 Method:** To start the RK-4 method, we have to specify the initial condition to solve the phase equation.
- **Implementation of RK-4 Technique:** Write all the steps of RK-4 method and obtain the phase shift in radians.
  Algorithm for Implementing RK-4 method
  The phase equation can be written as

$$\frac{d\delta(r)}{dr} = F(r, k, V, \delta_0(k, r)) \tag{29}$$

  where

$$F(r, k, V, \delta_0(k, r)) = -\frac{2\mu}{\hbar^2}\frac{V(r)}{k}\sin^2(kr + \delta_0) \tag{30}$$

  The method involves calculating value of $\delta(r_{n+1})$ by utilizing previous value at r_n, for $n = 0, 1, ..., n-1$.

$$\delta_{n+1} = \delta_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{31}$$

  where k_1, k_2, k_3 and k_4 are slopes of function F at different points in interval [0, h], defined as:

$$\begin{aligned}
k_1 &= F(r_n, k, V, \delta_n) \\
k_2 &= F\left(r_n + \frac{h}{2}, k, V, \delta_n + \frac{h}{2}k_1\right) \\
k_3 &= F\left(r_n + \frac{h}{2}, k, V, \delta_n + \frac{h}{2}k_2\right) \\
k_4 &= F(r_n + h, k, V, \delta_n + hk_3)
\end{aligned}$$

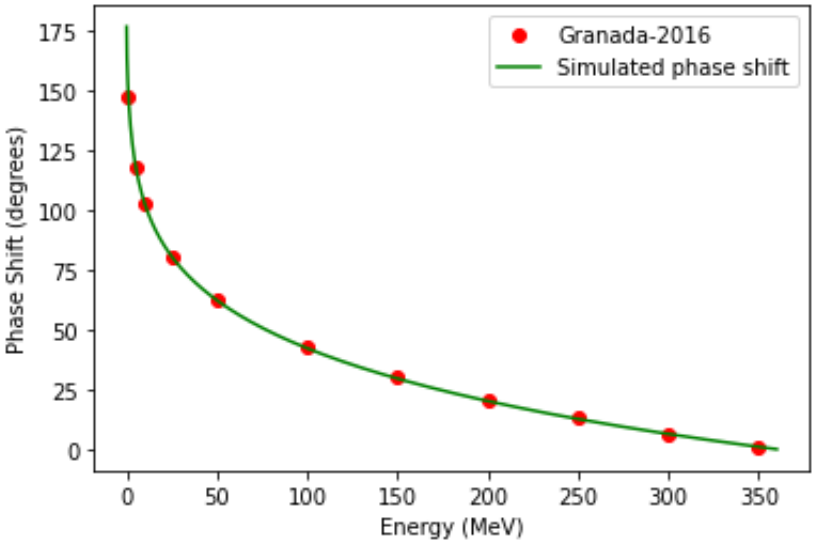

Figure 2: Simulated scattering phase shifts along with the expected phase shifts [18]

where h is the step size. This approach has local truncation of the order of $O(h^4)$, stable and simple to implement in computers.

- **Convert Phase Shift from Radians into Degrees:** As phase shifts are expected in degrees, we have to convert the obtained phase shift into degrees.

utilising this algorithm, we have written a code in python. The code has been given in Appendix and tested for neutron-proton to check its correctness. Using the parameters of Morse potential [18], By employing the RK-4th order approach to solve the phase equation, we have computed the scattering phase shifts. The phase shifts obtained are displayed in Fig. 2.

2.5 Validation Stage:

The RK -method code has been tested for n-p scattering and the obtained results are shown in Fig. 2. Form Fig. 2, we observed that the phase shifts matches very well with the expected one [18] and thus validates our code. Hence, on careful examination we observed that one should determine that model parameters that best describe the scattering phase shifts. For the determination of parameters, various optimisation methods are available in literature, where one can minimize the cost function like mean square error. Typically one can use least square minimisation for this process, but if the relationship between parameters is nonlinear, least squares method may not provide accurate estimates. Also in case of over-determined system, where there are more observations than parameters, least squares minimization may lead to overfitting, meaning it fits the noise in the data rather than the underlying signal. Hence, we have utilized Variational Monte Carlo technique for the optimization of parameters.

2.5.1 Variational Monte-Carlo (VMC) as an optimization procedure:

Variational Monte Carlo (VMC) is a computational technique widely utilized for parameter optimization and system characterization. The essence of VMC lies in its ability to efficiently explore the parameter space of complex models by leveraging Monte Carlo sampling methods while simultaneously minimizing a chosen cost function.

2.5.2 Optimisation Procedure

1. **Initialization of parameters:** Initialize the model parameters $V = [V_0, r_m, a_m]$ by using random number generator. Initially, Random numbers were generated within specific intervals for V_0, r_m and a_m, namely [0.01,500], [0.01, 5] and [0.01, 5], respectively.

2. **Determination of phase shifts using RK-4 method:** Using the randomly generated model parameters, we proceeded to solve the phase equation using RK-4 method and determined the scattering phase shifts.

3. **Cost function:** With the obtained phase shifts, we have calculated the mean square error(MSE) and stored as MSE_{old}. MSE is defined as:

$$MSE = \frac{1}{n}\sum_{i=0}^{N} |\delta_k^{exp} - \delta_k^{sim}|^2 \tag{32}$$

   Here,δ_k^{sim} and δ_k^{exp} the expected and simulated phase-shifts

4. **Monte Carlo Step:**

   - Within the range $[-I, I]$, generate the random integer 'a'.
   - Change a single parameter by a certain amount, like V_0:$p + V_0 = V_0new$
   - Re-calculate the simulated phase shifts using V_0new and determine the Mean Squared error (MSE), denoted as MSE_{new}

5. **Variational Step:**

   - If $MSE_{new} < MSE_{old}$, then the new parameter is accepted.
   - Update V_0 to V_{0new} , otherwise, maintain the current value..

   Terminate the process once the desired level of accuracy has been achieved.

6. **Iterative Minimization:**

   - For every model parameter, iterate through steps 3 and 4 continuously.
   - Continue the cycle of iterations until the MSE change loses significance.
   - Reduce the interval size R to enhance the precision of parameter adjustments.
   - Terminate the process once the desired level of accuracy has been reached.

The python code for optimising the model parameters are given in Appendix.

3 Results and Discussions

We have constructed the interaction potential between $p-^{16}$O by utilising Morse and atomic Hulthén potential for nuclear and Coulomb potential. The modeled nuclear interaction has three parameters which we need to optimised and the atomic Hulthén potential has one parameter 'a' . In the present work, we have chosen a to be 5 fm. To optimise the model parameters, we have taken the scattering phase shift data from Dubovichenko *et.al.* [23] at energy up to about 2.0 MeV. We have utilized the Variational Monte Carlo (VMC) technique to optimised the model parameters V_0, r_m and a_m of the modeled interaction. Initially we have created a random space for these modeled parameters by taking V_0, r_m and a_m to be [0.01,500], [0.01,5] and [0.01,5] respectively and obtained various regions where we get minimas. We have taken 5-10 minimas from the generated sample space. Now, we will take these minimas as initial point for the modeled parameters and solve the phase equation iteratively by minimising Mean Square Error (MSE). Different initial points gives rise to different interaction potentials, so we have taken that potential which best fitted the expected scattering phase shifts and thus giving best possible interaction potential between $p-^{16}$O.

Table 1: Optimised Model parameters using VMC technique for $^2S_{1/2}$ state of $p-^{16}$Oalong with the corresponding MSE

State	$V_0(MeV)$	$r_m(fm)$	$a_m(fm)$	MSE
$^2S_{1/2}$	62.012	3.455	0.843	1.11

With the use of optimised parameters, we have plotted a interaction potential, for the $^2S_{1/2}$ state as shown in Fig. 3. From the obtained interaction, we have observed that there is potential depth of $V_d = -55.997\ MeV$ at an equilibrium distance of $r_d = 3.605\ fm$. Additionally, there is a Coulomb barrier of $V_{CB} = 0.740\ MeV$ occurring at $r_{CB} = 8.937\ fm$.

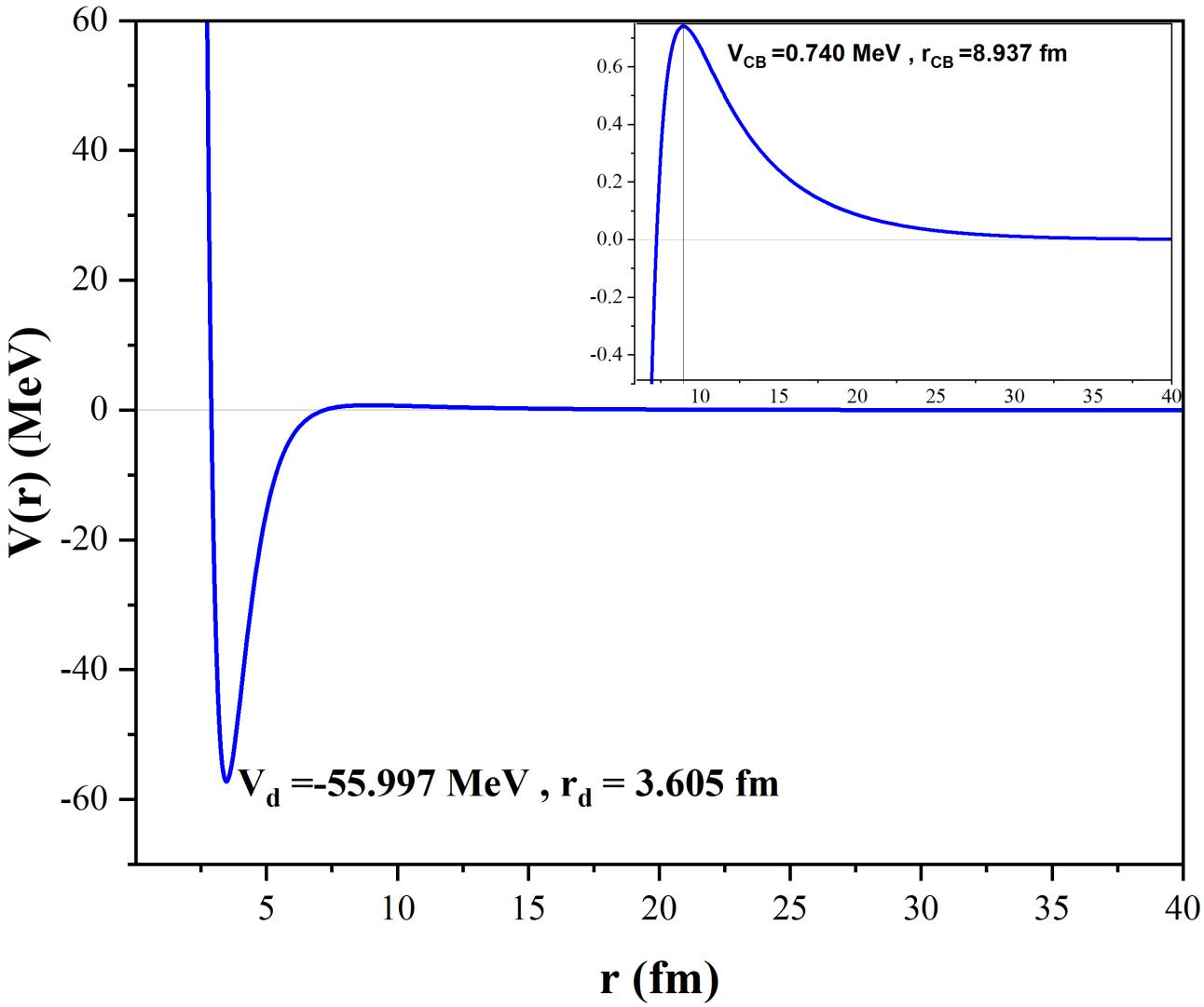

Figure 3: Interaction Potential for S- state of $p - {}^{16}O$ scattering with an inset showing coulomb barrier.

From the obtained interaction potential, the simulated phase shifts are determined using phase function method as given in Table 2 and are contrasted with the results of Dobovichenko *et al.* [23]. Now, we plot a graph between expected phase shift [23] and the simulated phase shift data as shown in Fig. 4. The simulated phase shifts with the expected ones are given in Table 2. From the Table 2, we have observed that the expected phase shifts [23] and our simulated ones matches fairly well. The $^2s_{1/2}$ scattering phase shift, as determined by the VMC method, is nearly equal to 180 degree at very low energies, i.e., up to 0.5 MeV, and is in close agreement with expected ones [23], which is important information for understanding thermonuclear processes of radiative capture at astrophysical energies.

Table 2: Simulated and experimental phase shifts [23] for $p-^{16}$O scattering

$E_{lab}(MeV)$	Exp.phase shift$\delta^+_{1/2}$ [23]	sim.phase shift$\delta^+_{1/2}$
0.3855	179.59	181.92
0.4871	179.74	180.51
0.6162	179.65	178.36
0.6631	178.11	177.52
0.7162	177.96	176.55
0.759	176.46	175.75
0.8108	174.55	174.76
0.8612	173.37	173.79
0.9058	174.19	172.93
0.979	172.17	171.51
1.1063	169.82	169.03
1.2508	166.43	166.22
1.3704	163.61	163.93
1.5898	159.42	159.81
1.7903	155.78	156.14
1.9909	151.09	152.59

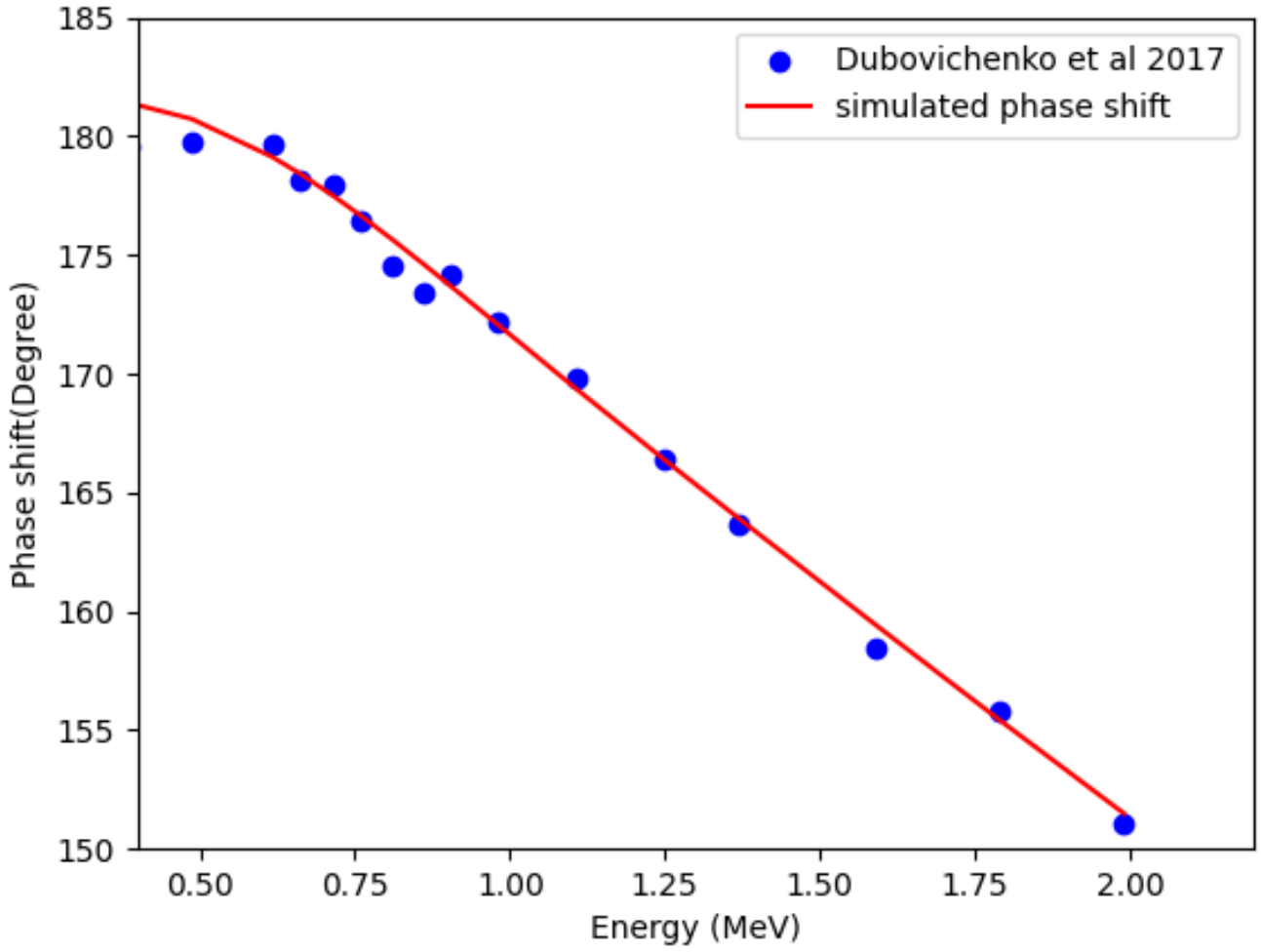

Figure 4: Simulated and expected SPS [23] for $p - {}^{16}O$ scattering

4 Conclusions

Utilising atomic Hulthén potential for describing coulomb interaction and Morse function for nuclear interaction. The present study reports the protons phase characteristics on oxygen in the energy range of 0.4 to 2.0 MeV. On careful investigation, one can conclude that, no resonances are observed in this energy range. The Morse potential and atomic Hulthen potential has been chosen to describe the nuclear and electromagnetic interaction respectively. By minimizing the mean squared error (MSE) between the computed scattering phase shifts using PFM and experimental data, the model parameters are optimized. The obtained phase shifts exhibit good agreement with the anticipated ones [23].

- The model parameters for for the s state of $p-^{16}$O are 62.012 MeV , 3.455 fm and 0.843 fm for V_o,r_m and a_m respectively.
- The MSE for $^2S_{1/2}$ state of system $p-^{16}$O are found to be 1.11.

Our proposed model is anticipated to offer a highly effective explanation for theoretical investigations on nucleon-nucleus elastic scattering data inside the low energy range. These potentials can then be utilized to compute a number of astrophysical issues, including radiative particle capture on light nuclei.

Publications and Participation

- **Poonia, Y.**, Awasthi, A., Sharma, A., Kant, I., & Sastri, O. S. K. S. (2024). Phase shift analysis of $p-^{16}O$ elastic scattering at Astrophysical Energies using phase function method. In Proceedings of the International Conference on Physics for Sustainable Development 2024 (p. 90, P-24).
- Awasthi, A., **Poonia, Y.**, Sharma, A., Kant, I., Sastri, O. S. K. S. (2024). Study of $p-^{16}O$ elastic scattering using phase function method. In Conference Proceedings in Springer Nature (Submitted).

References

[1] Taylor, John R. "A new rigorous approach to Coulomb scattering." Il Nuovo Cimento B Series 11 23.2 (1974): 313-334.

[2] Sahoo, P., B. Swain, and U. Laha. "Elastic Scattering of Proton from Oxygen at Astrophysical Energies." International Journal of Theoretical Physics 62.6 (2023): 129.

[3] Yukawa, Hideki. "On the interaction of elementary particles. I." Proceedings of the Physico-Mathematical Society of Japan. 3rd Series 17 (1935): 48-57.

[4] Stoks, V. G. J., et al. "Construction of high-quality NN potential models." Physical Review C 49.6 (1994): 2950.

[5] da Rocha, C. A., and Manoel Roberto Robilotta. "Two pion exchange nucleon-nucleon potential: The Minimal chiral model." Physical Review C 49.4 (1994): 1818.

[6] Obinata, Toshio, and Masanobu Wada. "Nonstatic one-boson-exchange potential with retardation." Progress of Theoretical Physics 53.3 (1975): 732-749.

[7] Naghdi, Mohammad. "Nucleon-nucleon interaction: A typical/concise review." Physics of Particles and Nuclei 45 (2014): 924-971.

[8] Sastri, O. S. K. S., Anil Khachi, and Lalit Kumar. "An innovative approach to construct inverse potentials using variational monte-carlo and phase function method: Application to np and pp scattering." Brazilian Journal of Physics 52.2 (2022): 58.

[9] Awasthi, Shikha, and O. S. K. S. Sastri. "Real and Imaginary Phase Shifts for Nucleon-Deuteron Scattering using Phase Function Method." arXiv preprint arXiv:2304.10478 (2023).

[10] Awasthi, Shikha, O. S. K. S. Sastri, and Anil Khachi. "Low Energy S-Wave Proton-Deuteron Scattering Phase-Shifts using Morse Potential." Journal of Nuclear Physics, Material Sciences, Radiation and Applications 9.2 (2022): 223-228.

[11] Kumar, Lalit, et al. "Phase Shift Analysis of Light Nucleon-Nucleus Elastic Scattering using Reference Potential Approach." arXiv preprint arXiv:2209.00951 (2022).

[12] Khachi, Anil, Lalit Kumar, and O. S. K. S. Sastri. "Alpha–Alpha Scattering Potentials for Various-Channels Using Phase Function Method." Physics of Atomic Nuclei 85.4 (2022): 382-391.

[13] Khachi, Anil, et al. "Phase Shift Analysis for Alpha-alpha Elastic Scattering using Phase Function Method for Gaussian Local Potential." Journal of Nuclear Physics, Material Sciences, Radiation and Applications 9.1 (2021): 1-5.

[14] Khachi, Anil, and O. S. K. S. Sastri. "3He-α Elastic Scattering Phase Shifts in Various Channels Using Phase Function Method with Morse Potential." Journal of Nuclear Physics, Material Sciences, Radiation and Applications 9.2 (2022): 161-167.

[15] Khachi, Anil, et al. "Triton scattering phase-shifts for S-wave using Morse potential." Journal of Nuclear Physics, Material Sciences, Radiation and Applications 9.1 (2021): 81-85.

[16] Kumar, Lalit, et al. "Phase Shift Analysis of $\alpha-^{12}$C Elastic Scattering Using Phase Function Method." Proceedings of the DAE Symp. on Nucl. Phys. Vol. 66. 2022.

[17] Awasthi, Ayushi, and O. S. K. S. Sastri. "Comparative Study of alpha-alpha interaction potentials constructed using various phenomenological models." arXiv preprint arXiv:2307.13207 (2023).

[18] Khachi, Anil, et al. "Inverse potentials for all ℓ channels of neutron-proton scattering using reference potential approach." Physica Scripta 98.9 (2023): 095301.

[19] Hestenes, David. "Toward a modeling theory of physics instruction." American journal of physics 55.5 (1987): 440-454.

[20] J Bhoi, J., and U. Laha. "Elastic scattering of light nuclei through a simple potential model." Physics of Atomic Nuclei 79 (2016): 370-374.

[21] Bhoi, J., and U. Laha. "Hulthén potential models for $\alpha - \alpha$ and $\alpha -^3 He$ elastic scattering." Pramana 88 (2017): 1-6.

[22] Naghdi, Mohammad. "Nucleon-nucleon interaction: A typical/concise review." Physics of Particles and Nuclei 45 (2014): 924-971.

[23] Dubovichenko, Sergey, et al. "New measurements and phase shift analysis of p16O elastic scattering at astrophysical energies." Chinese Physics C 41.1 (2017): 014001.

[24] Sastri, O. S. K. S., Arushi Sharma, and Ayushi Awasthi. "Constructing Inverse Scattering Potentials for $\alpha-\alpha$ System using Reference Potential Approach." arXiv preprint arXiv:2308.11266 (2023).

[25] Yakovlev, D. G., et al. "Simple analytic model for astrophysical S factors." Physical Review C 82.4 (2010): 044609.

A Appendix

Python code for Implementing RK method

```
import warnings
import numpy as np
import numba
import matplotlib.pyplot as plt

# Constants
M_p = 938.272046  # MeV / c ^ 2
M_n = 939.565379  # MeV / c ^ 2
hbarc = 197.3269718  # MeV fm
mu = (M_p * M_n) / (M_p + M_n)
den = (hbarc * hbarc) / (mu)
den2 = (hbarc * hbarc) / (2 * mu)
com = M_p / (M_p + M_n)

# Data
E = np.array([1, 5, 10, 25, 50, 100, 150, 200, 250,
300, 350]).astype(np.float64)
pS = np.array([147.7480, 118.1690, 102.5870, 80.559, 62.6450, 43.0880, 30.6440,
21.2440, 13.5510, 6.9660, 1.1760]).astype(np.float64)

# Pre-calculated values
abs_pS_inv = np.abs(pS ** -1)
k = np.sqrt(E / den)
k_1 = k ** -1
den_1 = -den2 ** -1

# Step size
h = 0.01
h_2 = h / 2
h_6 = h / 6

# Convert radians to degrees
rad2deg = 180. / np.pi

# Morse potential function
def morse():
    x1 = np.arange(0.01, 5.0125, 0.0025)
    a1 = 114.153
    a2 = 0.841
    a3 = 0.350
    v_new = (a1 * (np.exp(-2 * (x1 - a2) / a3) - 2 * np.exp(-(x1 - a2) / a3)))
```

```
    plt.figure()
    plt.plot(x1, v_new)
    plt.legend(['$^{3}S_{1}$', 'Morse Potential'])
    plt.xlabel('r (fm)')
    plt.ylabel('V(r) (MeV)')
    plt.axis([0, 5, -160, 150])
    plt.grid()
    plt.title('Potential Plot')
    return v_new

# Differential equation function
@numba.jit(nopython=True)
def f(x_in, y_in, v_new):
    E1 = np.arange(0.01, 360, 0.01)
    k11 = np.sqrt(E1 / den)
    k_2 = k11 ** -1
    kx = k11 * x_in

    # For l=0 state non-linear equation
    res = den_1 * k_2 * v_new * (np.cos(y_in) * np.sin(kx)
 + np.sin(y_in) * np.cos(kx)) ** 2

    return res

# Runge-Kutta 4th Order method
def rk4_method():
    idx = 0
    x = 0.01
    E1 = np.arange(0.01, 360, 0.01)
    k11 = np.sqrt(E1 / den)
    k_2 = k11 ** -1
    y = np.zeros_like(k11)
    v_new = morse()
    for _ in range(500):
        k1 = f(x, y, v_new[idx])
        k2 = f(x + h_2, y + h_2 * k1, v_new[idx + 1])
        k3 = f(x + h_2, y + h_2 * k2, v_new[idx + 1])
        k4 = f(x + h, y + h * k3, v_new[idx + 2])
        y = y + h_6 * (k1 + 2 * k2 + 2 * k3 + k4)
        idx += 4
        x += h
    return y

# Main function
if __name__ == "__main__":
```

```
    warnings.filterwarnings('ignore')

    # Calculate phase shift using RK4 method
    delta_best = rk4_method() * rad2deg

    # Plotting
    E1 = np.arange(0.01, 360, 0.01)
    plt.figure()
    plt.plot(E, pS, 'ro')
    plt.grid()
    plt.plot(E1, delta_best, 'g-')
    plt.xlabel('Energy (MeV)')
    plt.ylabel('Phase Shift (degrees)')
    plt.legend(['Granada-2016', 'Simulated phase shift'])
    plt.title('Phase Shift Plot')
    plt.show()
```

Python code for optimizing the model parameters:

```
import random
import numpy as np
import pandas as pd

mp =938.272046
mo = 14902.973
mu = mp * mo / (mp + mo)
hc = 197.3269718
Elab = np.array([0.3855,0.4871,0.6162,0.6631,0.7162,0.759,0.8108,0.8612,0.9058,
0.979,1.1063,1.2508,1.3704,1.5898,1.7903,1.9909])
X=np.array([179.59,179.74,179.65,178.11,177.96,176.46,174.55,
173.37,174.19,172.17,169.82,166.43,163.61,158.42,155.78,151.09])
Ecom =(mo)/(mp+mo)*Elab
n = 1000
a = 5.0
e2 = 1.44
def phaseshift(V):
    def f(x, y):
        s = (1 * 16 * e2 * mu / (hc) ** 2)
        V0 = 2 * s / a
        Vhp = V0 * (np.exp(-x / a) * (1 - np.exp(-x / a)) ** (-1))
        Vmp = V[0] * (np.exp((-2) * (x - V[1]) / V[2]) - (2 * np.exp((-1)
        * (x - V[1]) / V[2])))
        Vf = Vhp + Vmp
        k = (2 * (Ecom) * mu) ** (0.5) / hc
        t = (-2 * mu * Vf / ((hc) ** 2 * k)) * (np.sin(k * x + y)) ** 2
```

```
        return t
    y0 = 0.0
    x0 = 0.01
    h = 0.01
    z = 4000.0
    for i in np.arange(0, z):
        k1 = h * f(x0, y0)
        k2 = h * f(x0 + 0.5 * h, y0 + 0.5 * k1)
        k3 = h * f(x0 + 0.5 * h, y0 + 0.5 * k2)
        k4 = h * f(x0 + h, y0 + k3)
        y1 = y0 + (1 / 6) * (k1 + 2 * (k2 + k3) + k4)
        y0 = y1
        x0 += h
    y1 = y1 * 180 / np.pi
    return y1
for i in range(n):
    Vo = 62.011763
    ro = 3.4551374
    ao = 0.8430056791

    V = [Vo, ro, ao]
    V_new = [Vo, ro, ao]
    X_t = phaseshift(V)
    print('trial phase shift is:', X_t)
    c2 = ((X_t) - (X))**2
    c2_mean = np.mean(abs(c2))
    print("Trial c2:", c2_mean)
    for i in range(n):
        x = random.randint(0, 2)
        if x == 0:
            y = random.uniform(-10, 10)
        else:
            y = random.uniform(-0.1, 0.1)
        V_new[x] = V[x] + y

        X_t_new = phaseshift(V_new)
        c2_new =((X_t_new) - (X))**2
        c2_new_mean = np.mean(abs(c2_new))
        print(f"phase shifts: {X_t_new} | v0:{V_new[0]} | r0:{V_new[1]} |
        a0:{V_new[2]}  | c2_new:{c2_new_mean}")

        if c2_new_mean < c2_mean:
            c2_mean = c2_new_mean
            V[x] = V_new[x]
```

```
print(f"Final phase shifts: {phaseshift(V)} | Final v0:{V[0]} |
    Final r0:{V[1]} | Final a0:{V[2]}  | Final c2:{c2_mean}")
```