Que 1

mean $= \theta_1$

variance $= \theta_2$ Standard devi $= (\theta_2)^{1/2}$

The likelyhood function of sample $x_1 x_2$
x_n can be written as

$$L = \sum_{i=1}^{n} p(x_i, \mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta_2}} e^{-\frac{1}{2}\frac{(x_i - \theta_1)^2}{\theta_2}}$$

$$\Rightarrow \frac{1}{\sqrt{2\pi\theta_2}} e^{-(x_1-\theta_1)^2/2\theta_2} * \frac{1}{\sqrt{2\pi\theta_2}} e^{-(x_2-\theta_1)^2/2\theta_2}$$

$$\cdots \frac{1}{\sqrt{2\pi\theta_2}} e^{-(x_n-\theta_1)^2/2\theta_2}.$$

$$= \left(\frac{1}{\sqrt{2\pi\theta_2}}\right)^n e^{-\frac{1}{2}\sum_{i=1}^{n}\frac{(x_i-\theta_1)^2}{\theta_2}}$$

$$= \left(\frac{1}{\sqrt{2\pi\theta_2}}\right)^n e^{-\frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\theta_1)^2}.$$

taking log on both side

$$\log L = n \log \frac{1}{\sqrt{2\pi\theta_2}} + \log e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$$

$$= n(\cancel{\log}^{0} 1 - \log\sqrt{2\pi\theta_2}) - \frac{1}{2\theta_2}\sum_{i=1}^{n}(x_i-\mu)^2$$

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$$= -n \log (2\pi \theta_2)^{1/2} - \frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

$$= -\frac{n}{2} \log (2\pi\theta_2) - \frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

$$= -\frac{n}{2} (\log \theta_2 + \log 2\pi) - \frac{1}{2\theta_2} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

differentiate partially wrt to u and σ^2, we get

$$\frac{d}{du} \log L = -\frac{n}{2}(0+0) - \frac{1}{2\sigma^2} \sum_{i=1}^{n} 2(x_i - \theta_1)(-1)$$

$$= \frac{1}{\theta_2} \sum_{i=1}^{n} (x_i - u)$$

and

$$\frac{d}{d\sigma^2} (\log L) = -\frac{n}{2} \left(\frac{1}{\theta_2} + 0 \right) + \frac{1}{2\theta_2^2} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

Equating ① to zero and solving for u, we get

$$\frac{d}{du} \log L = 0 \Rightarrow \frac{1}{\theta_2} \sum_{i=1}^{n} (x_i - \theta_1) \Rightarrow \sum_{i=1}^{n} x_i - n\theta_1$$

$$\Rightarrow \theta_1 = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \Rightarrow \hat{\theta}_1 = \bar{x}$$

Equating ② to zero and solving for σ^2 and after substting $\mu = \bar{x}$ we get

$$\frac{d}{d\sigma^2} \log L = 0 \Rightarrow \frac{-n}{2\theta_2} + \frac{1}{2\theta_2^2} \sum_{i=1}^{n} (x_i - \theta_1)^2 = 0$$

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 = n\,\theta_2$$

$$\hat{\theta}_2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2, \text{ the sample varian } s$$

$\underline{V}$ Again partial derivative wrt to μ

$$\frac{d^2}{d\theta_1^2} - \frac{1}{\theta_2} \sum_{i=1}^{n} (1)(x-\theta_1)^{1-1}(-1)$$

$$= \frac{-n}{\theta_2} < 0.$$

$\hat{\theta}_1 = \bar{x}$ is mle for θ_1

Similarly differentiate wrt to σ^2, we get

$$\frac{d^2}{d(\theta_2)^2} = \frac{n}{2}(\theta_2)^{-2} + (-1)(\theta_2)^{-3} \sum_{i=1}^{n} (x_i - \theta_1)^2$$

$$= \frac{n}{2\theta_2^2} - \frac{1}{\theta_2^3} (x_i - \theta_1)^2.$$

$$\frac{d^2}{d(\theta_2)^2}(\log L) \text{ at } \hat{\theta}_2 = s^2$$

$$= -\left(-\frac{n^2}{2s^4} + \frac{1}{s^6}\sum_{i=1}^{n}(x_i-\bar{x})^2\right) < 0$$

$\hat{\theta}_2 = s^2$ is MLE of σ^2.

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The pmf of x_i is given by
for easy considering θ as p.

$$P(x_i \mid m, p) = \binom{m}{x_i} p^{x_i} (1-p)^{m-x_i}$$

$, x_i \in \{0, 1, 2 \ldots m\}$

the likelihood function is

$$L(p) = \prod_{i=1}^{n} P(x_i \mid m, p) = \prod_{i=1}^{n} \binom{m}{x_i} p^{x_i} (1-p)^{m-x_i}$$

$$\Rightarrow \prod_{i=1}^{m} \binom{m}{x_i} \prod_{i=1}^{n} p^{x_i} \prod_{i=1}^{n} (1-p)^{m-x_i}$$

$$= \prod_{i=1}^{m} \binom{m}{x_i} p^{\sum_{i=1}^{n} x_i} (1-p)^{\sum_{i=1}^{n} m-x_i}$$

$$\Rightarrow \prod_{i=1}^{n} \binom{m}{x_i} p^{\sum_{i=1}^{n} x_i} (1-p)^{n \cdot m - \sum_{i=1}^{n} x_i}.$$

$\Rightarrow$ The log likelyhood function.

$$\ell(p) = \log L(p)$$

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$$= \log\left(\prod_{i=1}^{n}\binom{m}{x_i} p^{\sum_{i=1}^{n} x_i} (1-p)^{nm-\sum_{i=1}^{n} x_i}\right)$$

$$= \log\left(\prod_{i=1}^{n}\binom{m}{x_i}\right) + \log\left(p^{\sum_{i=1}^{n} x_i}\right)$$

$$+ \log\left((1-p)^{nm-\sum_{i=1}^{n} x_i}\right)$$

$$\Rightarrow \log\left[\prod_{i=1}^{n}\binom{m}{x_i}\right] + \log(p)\left(\sum_{i=1}^{n} x_i\right)$$

$$+ \log(1-p)\left(nm - \sum_{i=1}^{n} x_i\right).$$

differentiating w.r.t to p

$$\frac{d\ell(p)}{dp} = \frac{1}{p}\sum_{i=1}^{n} x_i + \frac{1}{1-p}\left(nm - \sum_{i=1}^{n} x_i\right)(-1)$$

$$\frac{d^2\ell(p)}{dp^2} = -\frac{1}{p^2}\sum_{i=1}^{n} x_i - \frac{1}{(1-p)^2}\left(nm - \sum_{i=1}^{n} x_i\right)$$

to find the MLE of p, $\hat{p}$, we solve the following.

$$\left.\frac{d\ell(p)}{dp}\right|_{p=\hat{p}} = 0 \Rightarrow \frac{1}{\hat{p}}\sum_{i=1}^{n} x_i - \frac{1}{1-p}\left(nm - \sum_{i=1}^{n} x_i\right)$$

$$\Rightarrow \frac{1}{1-\hat{p}}\left(nm - \sum_{i=1}^{n} x_i\right) = \frac{1}{\hat{p}} \sum_{i=1}^{n} x_i$$

$$\Rightarrow \frac{nm - \sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} x_i} = \frac{1-\hat{p}}{\hat{p}}$$

$$\frac{nm}{\sum_{i=1}^{n} x_i} - \cancel{1} = \frac{1}{\hat{p}} - \cancel{1}$$

$$\hat{p} = \frac{1}{nm} \sum_{i=1}^{n} x_i \Rightarrow \frac{\overline{X}_n}{m}$$

to prove mle is maximize $p = \hat{p}$

we must show $\left.\frac{d^2 \ell(p)}{dp^2}\right|_{p=\hat{p}} < 0$

Note:

$$\frac{d^2 \ell(p)}{dp^2} = -\left(\frac{1}{p^2} \sum_{i=1}^{n} x_i + \frac{1}{(1-p)^2}\left(nm - \sum_{i=1}^{n} x_i\right)\right)$$

$$< 0$$

$\forall\, p\,(0,1)$