# Unit IV

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### 4.1 Revision of matrices and determinant

**Definition of a Matrix**: A rectangular array of numbers is called a matrix. The horizontal arrays of a matrix are called its rows and the vertical arrays are called its columns. A matrix is said to have the order m× n if it has m rows and n columns. An m× n matrix A can be represented in either of the following forms:

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \text{ or } A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

where $a_{ij}$ is the entry at the intersection of the $i^{th}$ row and $j^{th}$ column, in a more concise manner, we may write $A_{m\times n} = [a_{ij}]$ $or$ $A = [a_{ij}]_{m\times n}$ $or$ $A = [a_{ij}]$. We shall mostly concerned with matrices having real entries.

For example, if $A = \begin{bmatrix} 1 & 3 & 7 \\ 4 & 5 & 6 \end{bmatrix}$ $then$ $a_{11} = 1, a_{12} = 3, a_{13} = 7, a_{21} = 4, a_{22} = 5, a_{23} = 6.$

A matrix having only one column is called a column vector; and a matrix with only one row is called a row vector. Whenever a vector is used, it should be understood from the context whether it is a row vector or a column vector.

Equality of two Matrices: Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$having the same order $m \times n$ and if $a_{ij} = b_{ij}$for each $i = 1,2, \dots, m$ and $j = 1,2, \dots, n$.

In other words, two matrices are said to be equal if they have the same order and their corresponding entries are equal.

**Special Matrices**:

1) Row Matrix: A matrix having only one row and any number of columns is called a row matrix or a row vector.

   For example:$A = [1 \quad 3 \quad -8]$

2) Column matrix: A matrix having only one column and any number of row is called a column matrix or a column vector.

For example: $\begin{bmatrix} 2 \\ 3 \\ -8 \\ 5 \end{bmatrix}$

3) Zero-matrix: A matrix in which each entry is zero is called a zero-matrix, denoted by 0. For example: $0_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
4) Square matrix: A matrix that has the same number of rows as the number of columns, is called a square matrix. A square matrix is said to have order n if it is an $n \times n$ matrix.
5) Diagonal entries: The entries $a_{11}, a_{22}, \dots, a_{nn}$, of an $n \times n$ square matrix $A = [a_{ij}]$ are called the diagonal entries (the principal diagonal) of $A$.
6) Diagonal matrix: A square matrix $A = [a_{ij}]$ is said to be a diagonal matrix if $a_{ij}$= 0 for $i \neq j$. In other words, the non-zero entries appear only on the principal diagonal. For example, the zero matrix $0_n$ and $A = \begin{bmatrix} 3 & 0 \\ 0 & 6 \end{bmatrix}$.
7) Identity matrix: A diagonal Matrix, all of whose diagonal elements are unity is called a identity matrix and is denoted by $I$.
   For example: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
8) Scalar matrix: A diagonal Matrix, all of whose diagonal elements are equal is called a scalar matrix. For example: $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.
9) Upper triangular matrix: A square matrix $A = [a_{ij}]$ is said to be an upper triangular matrix if $a_{ij}$= 0 for $i > j$.
10) Lower triangular matrix: A square matrix $A = [a_{ij}]$ is said to be lower triangular matrix if $a_{ij}$= 0 for $i < j$.
11) Trace of matrix: The sum of all the diagonal elements of a square matrix is called the trace of a matrix.
   For example: If $A = \begin{bmatrix} 3 & 1 & 7 \\ 2 & 9 & 6 \\ 4 & 3 & -5 \end{bmatrix}$ then trace (A) = 3+9-5 = 7
12) Transpose of a matrix: A matrix obtained by interchanging rows to columns (equivalently columns to rows) of a matrix is called transpose of a matrix and is denoted by $A'$ or $A^T$.
   For example: If $A = \begin{bmatrix} 2 & -7 & 3 \\ 4 & 5 & 2 \end{bmatrix}$ then $A'$ or $A^T = \begin{bmatrix} 2 & 4 \\ -7 & 5 \\ 3 & 2 \end{bmatrix}$.
13) Symmetric matrix: A square matrix $A = [a_{ij}]_{n\times n}$ is called symmetric if $a_{ij} = a_{ji}$ For all $i$ and $j$, i.e. $A = A^T$
   For example: $\begin{bmatrix} 3 & 7 \\ 7 & 2 \end{bmatrix}$ and $\begin{bmatrix} 3 & i & 7i \\ i & -1 & 6 \\ 7i & 6 & -5 \end{bmatrix}$
14) Skew symmetric matrix: A square matrix $A = [a_{ij}]_{n\times n}$ is called Skew-symmetric if $a_{ij} = -a_{ji}$, for all $i$ and $j$, i.e. $A = -A^T$,
   Thus the diagonal elements of a skew symmetric matrix are all zero,

For example: $\begin{bmatrix} 0 & -7 \\ 7 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -i & 7i \\ i & 0 & -6 \\ -7i & 6 & 0 \end{bmatrix}$

15) Conjugate of a matrix: A matrix obtained from a given matrix A by replacing its entries by their complex conjugates is called the conjugate of A and is denoted by $\bar{A}$.

For example: If $A = \begin{bmatrix} 4+5i & 3+2i & 3 \\ -2i & 5 & 3-2i \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} 4-5i & 3-2i & 3 \\ 2i & 5 & 3+2i \end{bmatrix}$

16) Transposed conjugate of a Matrix: The conjugate of the transpose of a matrix A is called the transposed conjugate or conjugate transpose of A and is denoted by $A^{\theta}$.

Note that, $A^{\theta} = (\bar{A})^T = \overline{(A^T)}$. For example, If $A = \begin{bmatrix} 3+2i & 2-i & 8 \\ 5+4i & 3-2i & 1-i \\ 5 & 5-9i & 2+4i \end{bmatrix}$, then $A^T = \begin{bmatrix} 3+2i & 5+4i & 5 \\ 2-i & 3-2i & 5-9i \\ 8 & 1-i & 2+4i \end{bmatrix}$ then $A^{\theta} = \begin{bmatrix} 3-2i & 5-4i & 5 \\ 2+i & 3+2i & 5+9i \\ 8 & 1+i & 2-4i \end{bmatrix}$.

17) Hermitian Matrix: A square matrix $A = [a_{ij}]$ is called Hermitian matrix if $a_{ij} = \overline{a_{ji}}$, for all $i$ and $j$, i.e. $A = A^{\theta}$. For example $\begin{bmatrix} 2 & 3+i & 1-i \\ 3-i & 0 & 3+6i \\ 1+i & 3-6i & 8 \end{bmatrix}$.

18) Skew Hermitian matrix: A square matrix $A = [a_{ij}]$ is called Skew Hermitian matrix if $a_{ij} = -\overline{a_{ji}}$, for all $i$ and $j$, i.e. $A = -A^{\theta}$.

Hence diagonal elements of a skew Hermitian matrix must be either purely imaginary or zero, For example : $\begin{bmatrix} i & 2-7i \\ 2+7i & 0 \end{bmatrix}$

19) Unitary Matrix: A square matrix A is called unitary if $AA^{\theta} = A^{\theta}A = I$.

20) Orthogonal matrix: A square matrix A is called orthogonal if $AA^T = A^T A = I$.

21) Let $A = [a_{ij}]_{n\times n}$ be a square matrix. Then the range of the matrix A = $\{Ax' : x = (x_1, x_2, \ldots\ldots, x_n) \in R^n\}$

22) Let $A = [a_{ij}]_{n\times n}$ be a square matrix. Then the kernel of the matrix A = $\{Ax' = 0 : x = (x_1, x_2, \ldots\ldots, x_n) \in R^n\}$

## Matrix Operations:

**Equal matrices:** Two matrices are defined to be equal if they have same size and their corresponding entries are equal.

**Addition and subtraction of matrices:** If A and B are matrices of same size, then the sum A+B is the matrix obtained by adding the entries of B to the corresponding entries of A, and the difference A-B is the matrix obtained by subtracting the entries of B from the corresponding entries of A.

**Product of matrices:** If A is $m \times r$ matrix and B is $s \times n$ matrix, then matrix product is possible only if $r = s$.

Consider $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mr} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ b_{r1} & b_{r2} & \cdots & b_{rn} \end{bmatrix}$. Then the matrix

product is AB = $[c_{ij}] = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + \dots + a_{1r}b_{r1} & \dots. & a_{11}b_{1n} + a_{12}b_{2n} + \dots + a_{1r}b_{rn} \\ \vdots & \ddots & \vdots \\ a_{m1}b_{11} + a_{m2}b_{21} + \dots + a_{mr}b_{rn} & \dots. & a_{m1}b_{1n} + a_{m2}b_{2n} + \dots + a_{mr}b_{rn} \end{bmatrix}$,
where $c_{ij} = \sum_{k=1}^{r} a_{ik}b_{kj}$, for all i= 1, 2,..., m and j= 1, 2,..., n.

**<u>Determinant</u>**: Every square matrix can be associated to an expression or a number which is known as its determinant OR

Consider a square matrix A=$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$. The determinant of A is denoted by

det(A) or $\Delta$ or $|A|$ or $\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

Note: Determinant of a non-square matrix is not defined.

We shall deal with a determinant of matrices of order 1 or 2 or 3 only.

- Determinant of a matrix of order 1:

  If A =$[a_{11}]$ is a square matrix of order 1, then $|A|=a_{11}$

- Determinant of a matrix of order 2:

  If A= $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2, then $|A|$ = $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$ = $a_{11}a_{22} - a_{21}a_{12}$

- Determinant of a matrix of order 3:

  If A= $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3, then $|A|$ = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ =

  $= a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

  = $a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$.

**Inverse of a matrix:** Suppose A is a square matrix. If there is a square matrix B (of same size that of A) such that AB = BA= I. Then A is called invertible or regular or nonsingular matrix and B is called the inverse of A and it is denoted by $A^{-1}$.

**Minors and Co-factors:**

Suppose $A = [a_{ij}]$ is a square matrix of order n. For i, j=1, 2,..., n, minor of $a_{ij}$ is the determinant of the matrix obtain from A by removing $i^{th}$ row and $j^{th}$ column.

Suppose $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. To find the minor of $a_{23}$ , if the row and column containing the element $a_{23}$( second row and third column) are removed, we get the determinant $\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}$ which is the minor of the element $a_{23}$.

The $ij^{th}$ cofactor of A is $A_{ij} = (-1)^{i+j}$ (minor of $a_{ij}$)

The co-factor of $a_{23}$ is

$$A_{23} = (-1)^{2+3}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = -(a_{11}a_{32} - a_{12}a_{31})$$

The co-factor of the elements of A are:

$$A_{11} = Co-factor\ of\ a_{11} = (-1)^{1+1}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$A_{13} = Co-factor\ of\ a_{13} = (-1)^{1+3}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$A_{21} = Co-factor\ of\ a_{21} = (-1)^{2+1}\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

**Adjoint of a square matrix:**

Consider $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. The co-factor matrix of A is $\begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$.

The transpose of the Co-factor matrix of A $\begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$ is called the adjoint of the matrix $A$. It is denoted by $adj(A)$.

**Inverse of a matrix by determinant method (Adjoint method):** If $det(A) = 0$ then inverse does not exist and if $det(A) \neq 0$ then $A^{-1} = \frac{1}{|A|} adjA$.

**Elementary row operations**:

The following are elementary row operations on a matrix

1. Interchanging of two different rows. Symbolically we may write
$$R_{ik} : interchanging\ of\ the\ i^{th}\ and\ k^{th}\ row, (i \neq k).$$
2. The multiplication of the entries of a row by any non-zero number. Symbolically we may write $\alpha R_i : Multiplication\ of\ i^{th}\ row\ by\ a\ non\ zero\ number\ of\ \alpha.$
3. A multiplication of a row by any number $\alpha$ and add it in another row. Symbolically we may write
$$R_i + \alpha . R_k : Addition\ of\ \alpha\ times\ the\ k^{th}\ row\ to\ the\ i^{th}\ row.$$

**Elementary matrices**: A matrix obtained by applying any one elementary row operation to the identity matrix.

**Equivalence of matrices**: If a matrix B obtained from a matrix $A$ by applying finite number of elementary row operations is called equivalent to A. Symbolically, we may write $A \sim B$.

**Row-Echelon form of a matrix**:

Consider a matrix A. The leading entry of $r^{th}$ row is the first non zero entry of $r^{th}$ row. A matrix A is called in row echelon form if

1. All zero rows of A are at the bottom of the matrix.
2. All leading entry must be 1.
3. The leading entries of rows must move from left to right if we go from top to the bottom.
4. All entries of the column below the leading entry must be zero.

The following matrices are in row echelon form.

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 7 \\ 0 & 1 & 8 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 5 & -1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 & 9 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

**Reduced row-echelon form of a matrix**:
A matrix is called in reduced row echelon form if it satisfies conditions (1), (2), (3) above and all entries of the column containing leading entry except the leading entry must be zero.

The following matrices are in reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

**4.2 and 4.3 Eigenvalues and Eigenvectors of matrices.**

**Eigenvalues and Eigenvector:**

Suppose A is a square matrix of order n. If there is a nonzero vector $x$ ($x$ is an $n \times 1$ matrix) and a number $\lambda$ such that , $Ax = \lambda x$, then $\lambda$ is called eigenvalue of A and $x$ is called the eigenvector of A corresponding to eigenvalue $\lambda$.
We may call $\lambda$ as eigenvalue or characteristic root or characteristic value of the matrix A.
**Remark:** If $x$ is an eigenvector corresponding to eigenvalue $\lambda$, then for all real number $\alpha \neq 0$, $\alpha x$ is also an eigenvector corresponding to $\lambda$.
Note that there exists $x \neq 0$ such that $Ax = \lambda x = \lambda I x$ if and only if $(A - \lambda I)x = 0$ if and only if $A - \lambda I$ is not invertible if and only if $\det(A - \lambda I) = 0$.
Let A be a square matrix of order $n$. The matrix $A - \lambda I$ is called the characteristic matrix of A, where $I$ is the identity matrix of order n.
The determinant

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix}$$

which is a polynomial in $\lambda$ of degree n, is called the characteristic polynomial of A.
The equation $\det(A - \lambda I)$=0 is called the characteristic equation of A and the roots of this equation are the eigenvalues of the matrix A. The space spanned by all eigenvectors corresponding to $\lambda$ is called the eigenspace of A corresponding to $\lambda$. In fact the eigenspace

of A corresponding to $\lambda$ is $ker(A - \lambda I)$. The set of all eigenvalues of A is called the spectrum of A, and it is defined by $\sigma(A)$.

Note : 1. The characteristic equation of the matrix A of order 2 can be obtained from

$$\lambda^2 - S_1\lambda + S_2 = 0$$

Where $S_1 = trace(A)$ and $S_2 = \det(A)$.

2. The characteristic equation of the matrix A of order 3 can be obtained from

$$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$$

Where $S_1 = trace(A)$,

$S_2 = Sum\ of\ Minors\ of\ Principal\ diagonal\ elements$ and

$S_3 = \det(A)$.

**Properties of eigenvalues and eigenvectors:**

1) The trace of a matrix $A$ is the sum of all eigenvalues of $A$. i.e. $trace(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$, where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are eigen values of A.
2) The determinants of $A$ is the product of all eigenvalues of $A$.
   i.e. $det(A) = \lambda_1.\lambda_2 \ldots \lambda_n$, where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are eigenvalues of A.
3) A square matrix $A$ and its transpose $A^T$ have the same eigenvalues but in general there is no relation between eigenvectors of $A$ and eigenvectors of $A^T$.
4) The eigenvalues of a diagonal matrix and triangular matrix are precisely its diagonal elements.
5) A matrix $A$ is an invertible matrix if and only if $'0'$ is not an eigenvalue of A.
6) If a matrix A has an eigenvalues $\lambda$, then for any positive integer $k$, $A^k$ has eigenvalues $\lambda^k$ with same eigenvector.
7) If $A$ is an invertible matrix with eigenvalue $\lambda$, then $A^{-1}$ has eigenvalue $\lambda^{-1}$ with same eigenvector.

**<u>Tutorial:</u>**

| | |
|---|---|
| **1.** | Find eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$. |
| | **Solution:** The characteristic equation is $\det(A - \lambda I) = 0$<br>i.e. $\begin{vmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$<br>$\therefore \lambda^2 - S_1\lambda + S_2 = 0$<br>Where $S_1$ = sum of the principal diagonal elements of $A = -5 + (-2) = -7$.<br>$S_2 = \det(A) = \begin{vmatrix} -5 & 2 \\ 2 & -2 \end{vmatrix} = 6$.<br>Hence, the characteristic equation is,<br>$\lambda^2 + 7\lambda + 6 = 0$<br>Eigenvalues of $A$ $are$ $\lambda = -1, -6$ . |

The characteristic vector corresponding to $\lambda = -1$ is given by $(A - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives $-4x_1 + 2x_2 = 0 \therefore 2x_1 - x_2 = 0$ and $2x_1 - x_2 = 0$.

Take $x_2 = K_1$. Then $x_1 = \frac{K_1}{2}$.

Thus $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = K_1 \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ is an eigenvector corresponding to eigen value $\lambda = -1$.

The characteristic vector corresponding to $\lambda = -6$ is given by $(A - \lambda I)x = 0$

$$\Rightarrow \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This gives $x_1 + 2x_2 = 0$ and $2x_1 + 4x_2 = 0 \;\therefore x_1 + 2x_2 = 0$.

Take $x_2 = K_2$. Then $x_2 = \frac{K_2}{2}$.

Thus $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = K_2 \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$ is an eigenvector corresponding to eigen value $\lambda = -6$.

**2.** Find eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

**3.** If $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$, find eigenvalues for the following matrices:

a) $A$ b) $A^T$ c) $A^{-1}$ d) $4A^{-1}$ e) $A^2$

**Ans:** $\boldsymbol{A}$: **2, 3, 6 |** $\boldsymbol{A^T}$: **2, 3, 6 |** $\boldsymbol{A^{-1}}$: $\boldsymbol{\frac{1}{2}, \frac{1}{3}, \frac{1}{6}}$ **|** $\boldsymbol{A^{-1}}$: $\boldsymbol{1, \frac{4}{3}, \frac{2}{3}}$ **|** $\boldsymbol{A^2}$: **4, 9, 36.**

**4.** Find eigenvalues and corresponding eigenvectors of $\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.

**Ans:** $\lambda = 5, -3, -3$ and $X_1 = K_1 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, X_2 = K_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + K_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$, where $K_1, K_2, K_3 \in \mathbb{R}\backslash\{0\}$.

| | |
|---|---|
| **5.** | Find eigenvalues and corresponding eigenvectors of $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.<br><br>**Ans:** $\lambda = 2,3,5$ and $X_1 = K_1 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, X_2 = K_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, X_3 = K_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, where $K_1, K_2, K_3 \in \mathbb{R}\backslash\{0\}$. |
| **6.** | Find eigenvalues and corresponding eigenvectors of $\begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{4} \\ \mathbf{2} & \mathbf{0} & \mathbf{2} \\ \mathbf{4} & \mathbf{2} & \mathbf{3} \end{bmatrix}$.<br><br>**Ans:** $\lambda = -1, -1, 8$ and $X_1 = K_1 \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix} + K_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, X_2 = K_3 \begin{bmatrix} 1 \\ 1/2 \\ 1 \end{bmatrix}$, where $K_1, K_2, K_3 \in \mathbb{R}\backslash\{0\}$. |

### 4.4 Cayley-Hamilton's Theorem and its applications.

**Cayley-Hamilton Theorem:**

Every square matrix satisfies its own characteristic equation i.e. The theorem states that, for a square matrix A of order n.

If $|A - \lambda I| = (-1)^n \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \cdots + a_n = 0$ is the characteristic equation, then

$$(-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I = 0, \qquad \text{.....(1)}$$

**Application of the Cayley Hamilton Theorem:**

<u>**Cor 1**</u>. Inverse of a non-singular matrix can be found by using Cayley-Hamilton theorem. Suppose A is a non-singular matrix. Then 0 is not an eigenvalue of A and so 0 is not a root of the characteristic polynomial, so $a_n \neq 0$.

Multiplying equation (1) by $A^{-1}$, we get

$$(-1)^n A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \cdots + a_{n-1} I + a_n A^{-1} = 0$$

$$\Rightarrow A^{-1} = -\frac{1}{a_n} [(-1)^n A^{n-1} + a_1 A^{n-2} + a_2 A^{n-3} + \cdots + a_{n-1} I] \ldots\ldots (2)$$

Thus Cayley-Hamilton theorem gives another method for finding the inverse of a matrix.

<u>**Cor 2**</u>. Cayley Hamilton Theorem can be used to find the higher powers of A

$$A^n = -\frac{1}{(-1)^n} [a_1 A^{n-1} + a_2 A^{n-2} + \cdots + a_n I] \ldots\ldots (3)$$

<u>**Tutorial:**</u>

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| **1.** | Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley-Hmilton theorem for this matrix. Find $A^{-1}$ and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. |

**Solution:** Let $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation is, $det(A - \lambda I) = 0$

i.e. $\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$

so, $\lambda^2 - s_1\lambda + s_2 = 0$

Where $s_1$ =the principal diagonal elements of A= 1+3=4,

$s_2 = det(A) = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3 - 8 = -5$

Hence the characteristic equation is, $\lambda^2 - 4\lambda - 5 = 0$.

Thus -1 and 5 are eigenvalues of A.

$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$.

Now, $A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 \ldots\ldots\ldots\ldots\ldots\ldots (1)$

The matrix A satisfies its characteristic equation, hence Cayley-Hamilton theorem is verified.

Pre multiplying equation (1) by $A^{-1}$, we get $A^{-1}(A^2 - 4A - 5I) = 0$

So $A - 4I - 5A^{-1} = 0 \Rightarrow A^{-1} = \frac{1}{5}(A - 4I) = \frac{1}{5}\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}$.

Now, $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I = A^3(A^2 - 4A - 5I) - 2A(A^2 - 4A - 5I) + 3(A^2 - 4A - 5I) + A + 5I = (A^2 - 4A - 5I)(A^3 - 2A + 3) + A + 5I = A + 5I$ (from eq. (1))

Which is a polynomial in A.

**2.** Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 5 & -1 \end{bmatrix}$ and find its inverse.

**Ans:** $\boldsymbol{A^{-1} = \frac{1}{21}\begin{bmatrix} 1 & 4 \\ 5 & -1 \end{bmatrix}}$

**3.** Verify Cayley Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ and hence find $A^{-1}$.

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| **4.** | Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence compute $A^{-1}$. Also express $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$ as a quadratic polynomial.<br><br>**Ans:** $\boldsymbol{A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}, A^2 + A + I}$ |

### 4.5 LU Decomposition:

**Note:**

Recall that a square matrix is said to be triangular if the elements above (or below) of the main diagonal are zero. For example, the matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{11} \end{bmatrix}, B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

are triangular matrices. Here $A$ is an upper triangular matrix and $B$ is a lower triangular matrix.

It can be observed that a triangular matrix is non-singular only when all its diagonal elements are nonzero.

The following properties hold for triangular matrices:

1) If $A_1$ and $A_2$ are two upper triangular matrices of same order, then $A_1 + A_2$ and $A_1 . A_2$ are also upper triangular matrices of same order. Similar results hold for lower triangular matrices also.
2) The inverse of a non-singular lower triangular matrix is also a lower triangular matrix. Similar results hold good for upper triangular matrix also.

If $A = LU$, where $L$ and $U$ are lower and upper triangular matrices respectively, then $A^{-1} = (LU)^{-1} = (U)^{-1}(L)^{-1}$ .

**LU Decomposition of a Matrix:**

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ be a square matrix such that $A = L \cdot U$,

$$\text{Where } L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l_{21} & 1 & 0 & 0 \\ l_{31} & l_{32} & 1 & 0 \\ . & . & . & . \\ l_{n1} & l_{n2} & \dots & 1 \end{bmatrix} \text{ and } = \begin{bmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ 0 & u_{22} & \dots & u_{2n} \\ 0 & 0 & \dots & u_{3n} \\ . & . & . & . \\ 0 & 0 & \dots & u_{nn} \end{bmatrix}.$$

**Here is the procedure to find $L$ and $U$ for a square matrix of order 3.**

$$\text{Let } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$$

Equating the corresponding entries of both sides, we get

$u_{11} = a_{11};\quad u_{12} = a_{12};\quad u_{13} = a_{13};$

$l_{21}u_{11} = a_{21};\quad l_{21}u_{12} + u_{22} = a_{22};\quad l_{21}u_{13} + u_{23} = a_{23};$

$l_{31}u_{11} = a_{31};\quad l_{31}u_{12} + l_{32}u_{22} = a_{32};\quad l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}.$

From the above equations we obtain

$$u_{11} = a_{11};\qquad u_{12} = a_{12};\qquad u_{13} = a_{13}$$

$$l_{21} = \frac{a_{21}}{a_{11}};\qquad u_{22} = a_{22} - l_{21}u_{12};\qquad u_{23} = a_{23} - l_{21}u_{13}$$

$$l_{31} = \frac{a_{31}}{a_{11}};\qquad l_{32} = \frac{a_{32} - l_{31}u_{12}}{u_{22}};\qquad u_{33} = a_{33} - l_{31}u_{13} - l_{32}u_{23}$$

Substitute all these values, one can generate $L$ and $U$.

**<u>Tutorial:</u>**

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| **1.** | Factorize the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix}$ into the $LU$ form. |
| | **Solution:** Let $A = \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ <br> $= \begin{bmatrix} 1 & 2 & 4 \\ 3 & 8 & 14 \\ 2 & 6 & 13 \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$ |

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| | Now we consider the first row<br>$u_{11} = 1;\ u_{12} = 2;\ u_{13} = 4,$<br>Consider the second row<br>$l_{21}u_{11} = 3, \quad \therefore l_{21} = 3,$<br>$l_{21}u_{12} + u_{22} = 8 \ \therefore\ u_{22} = 8 - (2)(3) = 2,$<br>$l_{21}u_{13} + u_{23} = 14 \ \therefore\ u_{23} = 14 - (3)(4) = 2,$<br>Notice how, at each step, the equation being considered has only one unknown in it, and other quantities that we have already found. This pattern continues on the last row<br>$l_{31}u_{11} = 2 \quad \therefore l_{31} = 2,$<br>$l_{31}u_{12} + l_{32}u_{22} = 6 \quad \therefore\ l_{32} = \frac{6-4}{2} = 1,$<br>$l_{31}u_{13} + l_{32}u_{23} + u_{33} = 13 \quad \therefore\ u_{33} = 13 - 8 - 2 = 3.$<br>The required $L$ and $U$ are,<br>$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$<br>One can verify that $A = LU$. |
| **2.** | Find the $LU$ decomposition of matrix $C=\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$.<br>**Ans:** $\boldsymbol{L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ \frac{1}{2} & -1 & 1 \end{bmatrix}, U = \begin{bmatrix} -2 & 2 & -3 \\ 0 & 3 & -9 \\ 0 & 0 & -\frac{15}{2} \end{bmatrix}}$ |
| **3.** | Find the $LU$ decomposition of matrix<br>$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ . |

## Crout’s method of LU decomposition or factorization:

Consider a system of linear equation in $n$ unknown and $n$ equation $AX = B$, where $A$ is the coefficient matrix.
In this method the matrix $A$ is decomposed into the product LU, where $L$ and $U$ are the lower and upper triangular matrix respectively and the diagonal elements of the matrix U are all $1s$.

Consider a system in 3 unknowns as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1,$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2,$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3.$$

The matrix representation of given system is $AX = B$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ be a non-singular square matrix and

if $a_{11} \neq 0, \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$ and so on. i.e., each minors of the matrix A should not be zero, then $A$ can be factorized into the form $LU$,

where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$.

Now, we find $LU$ decomposition of $A$.

After getting $L$ and $U$, we substitute $A = LU$ in the given system $AX = B$.

Then $LUX = B$.

Take $UX = W$ in $LUX = B$ and solve $LW = B$ to obtain the value of $W$.

Put this $W$ in $UX = W$ and solve it to get the value of $X$.

## Tutorial:

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| **1.** | Solve the following system of linear equations using LU decomposition method.<br>$2x + 3y + z = 9; x + 2y + 3z = 6; 3x + y + 2z = 8.$ |
| | **Solution: Solution.** Here, the matrix representation of given system is $AX = B$.<br>Then<br>$\begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$ - - - - - - - - - - - - - (1)<br>Let $A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}$.<br>Then $A$ can be factorized into the form $LU$.<br>Therefore, $A = LU$. |

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| | $\therefore \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \cdot \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$ <br> $= \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix}$ <br> By definition of equality of matrices, we get <br> $u_{11} = 2, u_{12} = 3, u_{13} = 1,$ <br> $l_{21}u_{11} = 1 \Rightarrow l_{21} = \frac{1}{2},$ <br> $l_{21}u_{12} + u_{22} = 2 \Rightarrow u_{22} = \frac{1}{2},$ <br> $l_{21}u_{13} + u_{23} = 3 \Rightarrow u_{23} = \frac{5}{2},$ <br> $l_{31}u_{11} = 3 \Rightarrow l_{31} = \frac{3}{2},$ <br> $l_{31}u_{12} + l_{32}u_{22} = 1 \Rightarrow l_{32} = -7,$ <br> $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2 \Rightarrow u_{33} = 18.$ <br> Hence, <br> $L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix}.$ <br> Now, we take $A = LU$ in (1), we get <br> $\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}$ - - - - - - - - - -(2) <br> We take $UX = W$ in above, we get <br> $W = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ - - - - - - - - - - - -(3) <br> Now, from (2), we have <br> $\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{3}{2} & -7 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \\ 8 \end{bmatrix}.$ <br> Therefore, the system is <br> $w_1 = 9; \frac{w_1}{2} + w_2 = 6 \therefore w_2 = 6 - \frac{9}{2} = \frac{3}{2}.$ |

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| | And $\frac{3}{2}w_1 - 7w_2 + w_3 = 8 \therefore \frac{27}{2} - \frac{21}{2} + w_3 = 8 \therefore w_3 = 5$.<br>Now, from (3), we have<br>$\begin{bmatrix} 9 \\ \frac{3}{2} \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 1 \\ 0 & \frac{1}{2} & \frac{5}{2} \\ 0 & 0 & 18 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$,<br>So, we have<br>$2x + 3y + z = 9; \frac{y}{2} + \frac{5}{2}z = \frac{3}{2} \therefore y + 5z = 3 \; and \; 18z = 5$.<br>Then $z = \frac{5}{18}, y = \frac{29}{18} \; and \; x = \frac{35}{18}$. |
| **2.** | Solve the following system of linear equations using Crout's Method of LU decomposition<br>$x + 2y + z = 3, 2x + 3y + 3z = 10, 3x - y + 2z = 13$.<br>**Ans:** $x = 2, y = -1, z = 3$ |