V Applications of Partial differentiations

5.1 Maclaurin's and Taylor's series expansion in two variables

Taylor's series expansion in two variables

If $f(x+h, y+k)$ is a given function which can be expanded into a series of positive ascending powers of h and k, then

$$f(x+h,y+k) = f(x,y) + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(x,y) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(x,y) + \cdots$$

$$= f(x,y) + \left[h\,\frac{\partial f(x,y)}{\partial x} + k\frac{\partial f(x,y)}{\partial y}\right]$$

$$+\frac{1}{2!}\left[h^2\frac{\partial^2 f(x,y)}{\partial x^2} + 2hk\frac{\partial^2 f(x,y)}{\partial x \partial y} + k^2\frac{\partial^2 f(x,y)}{\partial y^2}\right] + \cdots$$

Putting $x = a$ and $y = b$ in the above series, we have

$$f(a+h,b+k) = f(a,b) + \left[h\frac{\partial f(a,b)}{\partial x} + k\frac{\partial f(a,b)}{\partial y}\right]$$
$$+\frac{1}{2!}\left[h^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2hk\frac{\partial^2 f(a,b)}{\partial x \partial y} + k^2\frac{\partial^2 f(a,b)}{\partial y^2}\right] + \cdots \tag{1}$$

Putting $a + h = x$ and $b + k = y$ in the above series, we have

$$f(x,y) = f(a,b) + \left[(x-a)\frac{\partial f(a,b)}{\partial x} + (y-b)\frac{\partial f(a,b)}{\partial y}\right]$$

$$+\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f(a,b)}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f(a,b)}{\partial y^2}\right] + \cdots$$

Maclaurin's series expansion in two variables

By replacing a, b by zeros and h, k by x, y respectively in (1), we get

$$f(x,y) = f(0,0) + \left[x\frac{\partial f(0,0)}{\partial x} + y\frac{\partial f(0,0)}{\partial y}\right]$$

$$+\frac{1}{2!}\left[x^2\frac{\partial^2 f(0,0)}{\partial x^2} + 2xy\frac{\partial^2 f(0,0)}{\partial x \partial y} + y^2\frac{\partial^2 f(0,0)}{\partial y^2}\right] + \cdots$$

Tutorial:

Maclaurin's and Taylor's series expansion in two variables

1 Expand $f(x,y) = x^2y + 3y - 2$ in powers of $(x-1)$ and $(y+2)$ up to second degree terms.

Solution. Here $f(x,y) = x^2y + 3y - 2, a = 1, b = -2$

By Taylor's series expansion,

$$\begin{aligned}
f(x,y) &= f(a,b) + \left[(x-a)\frac{\partial f(a,b)}{\partial x} + (y-b)\frac{\partial f(a,b)}{\partial y}\right] \\
&\quad + \frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f(a,b)}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f(a,b)}{\partial x \partial y}\right. \\
&\quad \left. + (y-b)^2\frac{\partial^2 f(a,b)}{\partial y^2}\right] + \cdots \\
&= f(1,-2) + \left[(x-1)\frac{\partial f(1,-2)}{\partial x} + (y+2)\frac{\partial f(1,-2)}{\partial y}\right] \\
&\quad + \frac{1}{2!}\left[(x-1)^2\frac{\partial^2 f(1,-2)}{\partial x^2} + 2(x-1)(y+2)\frac{\partial^2 f(1,-2)}{\partial x \partial y}\right. \\
&\quad \left. + (y+2)^2\frac{\partial^2 f(1,-2)}{\partial y^2}\right] + \cdots \qquad ---(1)
\end{aligned}$$

$f(x,y) = x^2y + 3y - 2$

$f(1,-2) = (1)^2(-2) + 3(-2) - 2$
$= -2 - 6 - 2 = -10$

$\frac{\partial f(x,y)}{\partial x} = 2xy$ $\qquad$ $\frac{\partial f(1,-2)}{\partial x} = 2(1)(-2) = -4$

$\frac{\partial f(x,y)}{\partial y} = x^2 + 3$ $\qquad$ $\frac{\partial f(1,-2)}{\partial y} = (1)^2 + 3 = 4$

$\frac{\partial^2 f(x,y)}{\partial x^2} = 2y$ $\qquad$ $\frac{\partial^2 f(1,-2)}{\partial x^2} = 2(-2) = 4$

$\frac{\partial^2 f(x,y)}{\partial x \partial y} = 2x$ $\qquad$ $\frac{\partial^2 f(1,-2)}{\partial x \partial y} = 2(1) = 2$

$\frac{\partial^2 f(x,y)}{\partial y^2} = 0$ $\qquad$ $\frac{\partial^2 f(1,-2)}{\partial y^2} = 0$

and so on.

Putting these values in equation (1), we get

	$f(x,y) = -10 + [(x-1)(-4) + (y+2)(4)]$ $+\frac{1}{2!}[(x-1)^2(-4) + 2(x-1)(y+2)(2) + (y+2)^2(0)] + \cdots$ $= -10 - 4(x-1) + 4(y+2) - 2(x-1)^2 + 2(x-1)(y+2) + \cdots$
2	Expand $f(x,y) = x^2y + \sin y + e^x$ in powers of $(x-1)$ and $(y-\pi)$. **Answer:** $f(x,y) = \pi + e + (x-1)(2\pi + e) + \frac{1}{2}(x-1)^2(2\pi + e)$ $+2(x-1)(y-\pi) + \cdots$
3	Expand $f(x,y) = e^x \log(1+y)$ in powers of x and y. **Answer:** $f(x,y) = y + xy - \frac{y^2}{2} + \cdots$

5.2	Tangent plane and normal line to a surface

Tangent plane and normal line to a surface

Let $f(x,y,z) = 0$ be any surface. Then the equation of tangent plane at $P(x_0, y_0, z_0)$ is

$$(x-x_0)\left(\frac{\partial f}{\partial x}\right)_P + (y-y_0)\left(\frac{\partial f}{\partial y}\right)_P + (z-z_0)\left(\frac{\partial f}{\partial z}\right)_P = 0$$

and equation of normal line at $P(x_0, y_0, z_0)$ is

$$\frac{x-x_0}{\left(\frac{\partial f}{\partial x}\right)_P} = \frac{y-y_0}{\left(\frac{\partial f}{\partial y}\right)_P} = \frac{z-z_0}{\left(\frac{\partial f}{\partial z}\right)_P}.$$

Tutorial:

Tangent plane and normal line to a surface

1	Find the equation of tangent plane and normal line to the surface $x^2 + 2y^2 + 3z^2 = 12$ at the point $(1,2,-1)$. **Solution.** Let $f(x,y,z) = x^2 + 2y^2 + 3z^2 - 12$. Then $\frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 4y, \frac{\partial f}{\partial z} = 6z$. At point $(1,2,-1)$, $\frac{\partial f}{\partial x} = 2, \frac{\partial f}{\partial y} = 8, \frac{\partial f}{\partial z} = -6$. $\therefore$ The equation of the tangent plane at $(1,2,-1)$ is $2(x-1) + 8(y-2) - 6(z+1) = 0$ $\Rightarrow 2x + 8y - 6z - 24 = 0$ $\Rightarrow x + 4y - 3z = 12$ and the equation of the normal line at $(1,2,-1)$ is $\frac{x-1}{2} = \frac{y-2}{8} = \frac{z+1}{-6} \Rightarrow \frac{x-1}{1} = \frac{y-2}{4} = \frac{z+1}{-3}$.
2	Find the equation of tangent plane and normal line to the surface $xyz = 6$ at the point $(1,2,3)$.

	Answer: $6x + 3y + 2z = 18, \frac{x-1}{6} = \frac{y-2}{3} = \frac{z-3}{2}$.
3	Find the equation of tangent plane and normal line to the surface $z = \sqrt{21 - x^2 - y^2}$ at the point $(1, 4, -2)$. **Answer:** $x + 4y - 2z = 21, \frac{x-1}{1} = \frac{y-4}{4} = \frac{z+2}{-2}$.

5.3	Maxima and Minima

Maxima and Minima

Working Rules to Find Extremum (Maximum and Minimum) Values

1. Find out $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y^2}$.
2. Solve the equations $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$ for x and y. Let (a_i, b_i) be solutions. (a_i, b_i) are known as stationary points.
3. Find the value of $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}, t = \frac{\partial^2 f}{\partial y^2}$ at the point (a_i, b_i) obtained in 2.
4. If $rt - s^2 > 0$ and
   a) $r < 0$, then $f(x, y)$ has a maximum value at (a_i, b_i).
   b) $r > 0$, then $f(x, y)$ has a minimum value at (a_i, b_i).
5. If $rt - s^2 < 0$, then $f(x, y)$ has neither maximum value nor minimum value at these points. Such points are called saddle points.
6. If $rt - s^2 = 0$, then we can't say about maximum value or minimum value.

For examples,

(1) $f(x, y) = x^2 + y^2$ has a minimum at $(0, 0)$.

We are given that $f(x, y) = x^2 + y^2$.

$\therefore \frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = 2y, r = \frac{\partial^2 f}{\partial x^2} = 2, s = \frac{\partial^2 f}{\partial x \partial y} = 0, t = \frac{\partial^2 f}{\partial y^2} = 2.$

For stationary points,

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 2x = 0, 2y = 0$$

$$\Rightarrow x = 0, y = 0$$

The stationary point is $(0, 0)$.

point	r	s	t	$rt - s^2$	conclusion
$(0, 0)$	2	0	2	4	minimum

(2) $f(x,y) = 1 - x^2 - y^2$ has a maximum at $(0,0)$.

We are given that $f(x,y) = x^2 + y^2$.

$\therefore \frac{\partial f}{\partial x} = -2x, \frac{\partial f}{\partial y} = -2y, r = \frac{\partial^2 f}{\partial x^2} = -2, s = \frac{\partial^2 f}{\partial x \partial y} = 0, t = \frac{\partial^2 f}{\partial y^2} = -2.$

For stationary points,

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 2x = 0, 2y = 0$$

$$\Rightarrow x = 0, y = 0$$

The stationary point is $(0,0)$.

point	r	s	t	$rt - s^2$	conclusion
$(0,0)$	-2	0	-2	4	maximum

(3) $f(x,y) = x^2 - y^2$ has a saddle point at $(0,0)$.

We are given that $f(x,y) = x^2 + y^2$.

$\therefore \frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = -2y, r = \frac{\partial^2 f}{\partial x^2} = 2, s = \frac{\partial^2 f}{\partial x \partial y} = 0, t = \frac{\partial^2 f}{\partial y^2} = -2.$

For stationary points,

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0$$

$$\Rightarrow 2x = 0, 2y = 0$$

$$\Rightarrow x = 0, y = 0$$

The stationary point is $(0,0)$.

point	r	s	t	$rt - s^2$	conclusion
$(0,0)$	2	0	-2	-4	saddle

Tutorial:

Maxima and Minima

1	Find the extreme values of the function $f(x,y) = xy(a - x - y), a > 0.$ **Solution.** We are given that $f(x,y) = xy(a - x - y), a > 0.$ $$\therefore \frac{\partial f}{\partial x} = ay - 2xy - y^2, \frac{\partial f}{\partial y} = ax - x^2 - 2xy,$$ $$r = \frac{\partial^2 f}{\partial x^2} = -2y, s = \frac{\partial^2 f}{\partial x \partial y} = a - 2x - 2y \text{ and } t = \frac{\partial^2 f}{\partial y^2} = -2x.$$

	For extreme values, $$\frac{\partial f}{\partial x} = 0$$ $$\Rightarrow ay - 2xy - y^2 = 0$$ $$\Rightarrow y(a - 2x - y) = 0$$ $$\Rightarrow y = 0, a - 2x - y = 0$$ and $$\frac{\partial f}{\partial y} = 0$$ $$\Rightarrow ax - x^2 - 2xy = 0$$ $$\Rightarrow x(a - x - 2y) = 0$$ $$\Rightarrow x = 0, a - x - 2y = 0.$$ $\therefore$ The stationary points are $(0,0), (a,0), (0,a), \left(\frac{a}{3}, \frac{a}{3}\right)$.

Stationary points	$r = -2y$	$s = a - 2x - 2y$	$t = -2x$	$rt - s^2$	Conclusion (or Nature of point)
$(0,0)$	0	a	0	$-a^2 < 0$	saddle
$(a,0)$	0	$-a$	$-2a$	$-a^2 < 0$	saddle
$(0,a)$	$-2a$	$-a$	0	$-a^2 < 0$	saddle
$\left(\frac{a}{3}, \frac{a}{3}\right)$	$-\frac{2a}{3} < 0$	$-\frac{a}{3}$	$-\frac{2a}{3}$	$\frac{a^2}{3} > 0$	maximum

Hence, $f(x,y)$ is maximum at $\left(\frac{a}{3}, \frac{a}{3}\right), f\left(\frac{a}{3}, \frac{a}{3}\right) = \frac{a^3}{27}$.

2	Find the maximum and minimum values of the function $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$. **Answer:** $f(x,y)$ is maximum at $(-7,-7), f(-7,-7) = 784$ and minimum at $(3,3), f(3,3) = -216$.
3	Find the extreme values of the function $f(x,y) = x^3 + 3xy^2 - 15x^2 - 15y^2 + 72$. **Answer:** $f(x,y)$ is maximum at $(4,0), f(4,0) = 112$ and minimum at $(6,0), f(6,0) = 108$.

5.4	Lagrange's method of multiplier

Lagrange's method of multiplier

Working Rules for Lagrange's Method of Undetermined Multipliers

1. Write the function to be maximised or minimised $u = f(x, y, z)$ with the condition $g(x, y, z) = 0$.
2. Consider the Lagrange's function $F(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$, where λ is a Lagrange's multiplier.
3. For stationary values,

$$\frac{\partial F}{\partial x} = 0 \; i.e. \frac{\partial f}{\partial x} + \lambda \frac{\partial g}{\partial x} = 0$$

$$\frac{\partial F}{\partial y} = 0 \; i.e. \frac{\partial f}{\partial y} + \lambda \frac{\partial g}{\partial y} = 0$$

$$\frac{\partial F}{\partial z} = 0 \; i.e. \frac{\partial f}{\partial z} + \lambda \frac{\partial g}{\partial z} = 0$$

4. Solving these three equations along with $g(x, y, z) = 0$ and find a solution of x_0, y_0, z_0, and λ.
5. We find the extreme value of $u = f(x, y, z)$ by using x_0, y_0, z_0.

For example, the minimum value of $f(x, y) = x^2 + y^2$ such that $x + y = 1$ is $\frac{1}{2}$.

Here $f(x, y) = x^2 + y^2$ and $g(x, y) = x + y - 1$.

Consider the Lagrange's function

$$F(x, y) = f(x, y) + \lambda\, g(x, y)$$

$$= (x^2 + y^2) + \lambda\, (x + y - 1)$$

For stationary values,

$$\frac{\partial F}{\partial x} = 0 \Rightarrow 2x + \lambda = 0 \Rightarrow \lambda = -2x --- (1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 2y + \lambda = 0 \Rightarrow \lambda = -2y --- (2)$$

From (1) and (2), we have

$$x = y$$

But $x + y = 1 \Rightarrow x + x = 1 \Rightarrow 2x = 1 \Rightarrow x = \frac{1}{2}$ and $y = \frac{1}{2}$.

Hence, the minimum value of $f(x, y) = x^2 + y^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

Tutorial:

Lagrange's method of multiplier

1	Find the dimensions of the rectangular box of maximum capacity whose surface area is given when box is open at top.

Solution. Let x, y, z be the dimensions of the rectangular box, where $x > 0, y > 0, z > 0$. Thus surface area $S = 2(xz + yz) + xy$ and the volume $V = xyz$ which is to be maximized.

Here $f(x, y, z) = xyz$ and $g(x, y, z) = 2(xz + yz) + xy - \mathrm{S}$.

Consider the Lagrange's function

$$F(x, y, z) = f(x, y, z) + \lambda g(x, y, z)$$
$$= xyz + \lambda(2xz + 2yz + xy - \mathrm{S})$$

For stationary values,

$$\frac{\partial F}{\partial x} = 0 \Rightarrow yz + \lambda(2z + y) = 0 \Rightarrow \lambda = -\frac{yz}{2z + y} \quad ---(1)$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow xz + \lambda(2z + x) = 0 \Rightarrow \lambda = -\frac{xz}{2z + x} \quad ---(2)$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow xy + \lambda(2x + 2y) = 0 \Rightarrow \lambda = -\frac{xy}{2x + 2y} \quad ---(3)$$

From (1) and (2), we have

$$-\frac{yz}{2z + y} = -\frac{xz}{2z + x} \Rightarrow 2yz + xy = 2xz + xy \Rightarrow 2yz = 2xz \Rightarrow y = x$$

From (1) and (3), we have

$$-\frac{yz}{2z + y} = -\frac{xy}{2x + 2y} \Rightarrow 2xz + 2yz = 2xz + xy \Rightarrow 2yz = xy \Rightarrow 2z = x$$

But

$$2(xz + yz) + xy = S$$
$$\Rightarrow 2xz + 2yz + xy = S$$
$$\Rightarrow 2x\left(\frac{x}{2}\right) + 2x\left(\frac{x}{2}\right) + x \cdot x = S$$
$$\Rightarrow 3x^2 = S$$
$$\Rightarrow x = \sqrt{\frac{S}{3}}.$$

	Now $$y=\sqrt{\frac{S}{3}}$$ and $$z=\frac{1}{2}\cdot\sqrt{\frac{S}{3}}.$$ $\therefore V=xyz=\left(\sqrt{\frac{S}{3}}\right)\left(\sqrt{\frac{S}{3}}\right)\left(\frac{1}{2}\cdot\sqrt{\frac{S}{3}}\right)=\frac{1}{2}\left(\frac{S}{3}\right)^{\frac{3}{2}}.$
2	Find the minimum value of $x^2+y^2+z^2$ subject to $xyz=a^3$. Answer: $3a^2$.
3	In a triangle, find maximum value of $\cos A\cos B\cos C$, where $A, B,$ and C are three angles of triangle. Answer: $\frac{1}{8}$.

5.5	Jacobian

Jacobian

Let u, v be functions of two variables x, y. Then Jacobian of u and v with respect to x and y is defined as

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}.$$

Similarly, if u, v, w are functions of three variables x, y, z then Jacobian of u, v, w with respect to x, y, z is defined as

$$J=\frac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{vmatrix}.$$

For examples,

(1) If $x=r\cos\theta$, $y=r\sin\theta$, then $\frac{\partial(x,y)}{\partial(r,\theta)}=r$.

(2) If $x = r\cos\theta\,, y = r\sin\theta\,, z = z$, then $\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$.

Tutorial:

Jacobian

1	If $x = u^2 + v^2$ and $y = uv$, then find $\frac{\partial(x,y)}{\partial(u,v)}$. **Solution.** We are given that $x = u^2 + v^2$ and $y = uv$. $\therefore \frac{\partial x}{\partial u} = 2u, \frac{\partial x}{\partial v} = 2v, \frac{\partial y}{\partial u} = v$, and $\frac{\partial y}{\partial v} = u$. Now, $\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2u & 2v \\ v & u \end{vmatrix} = 2u^2 - 2v^2$.
2	If $u = x + y + z, v = x^2 + y^2 + z^2$, and $w = x^3 + y^3 + z^3$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. **Answer:** $6(-x^2y + x^2z + xy^2 - y^2z - xz^2 + yz^2)$.
3	If $u = x + y + z, v = xyz$, and $w = 2x + 2y + 2z$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. **Answer:**0.

5.6	Errors and approximations

Errors and approximations

If δf is the error in f, then

1. $\frac{\delta f}{f}$ is known as relative error in f.
2. $\frac{\delta f}{f} \times 100$ is known as percentage error in f.

Here, $\delta f = \frac{\partial f}{\partial x} \cdot \delta x + \frac{\partial f}{\partial y} \cdot \delta y$.

Tutorial:

Errors and approximations

1	Find the greatest percentage error in calculating the area of a rectangle when an error of 3% made in measuring each of its sides. **Solution.** Let x and y be the sides of the rectangle and A be its area. Here $A = xy$, then $\frac{\partial A}{\partial x} = y$, and $\frac{\partial A}{\partial y} = x$.

	Now, $$\delta A = \frac{\partial A}{\partial x} \cdot \delta x + \frac{\partial A}{\partial y} \cdot \delta y$$ $$\Rightarrow \delta A = y \cdot \delta x + x \cdot \delta y$$ $$\Rightarrow \frac{1}{A}\,\delta A = \frac{1}{x}\,\delta x + \frac{1}{y}\,\delta y$$ $$\Rightarrow \frac{\delta A}{A} \times 100 = \frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100.$$ Putting $\frac{\delta x}{x} \times 100 = \frac{\delta y}{y} \times 100 = 3$, we have $$\frac{\delta A}{A} \times 100 = 3 + 3 = 6.$$ Hence, percentage error in calculating area is 6%.
2	The period T of a simple pendulum is $T = 2\pi\sqrt{\frac{l}{g}}$. Find the maximum percentage error in T due to possible errors up to 1% in l and 2.5% in g. **Answer:** 1.75%.
3	If the fiber glass sheet costs Rs. 45 per square feet. Find approximate the greatest cost of fiber glass sheet 3.012 feet wide and 5.982 feet long. **Answer:** 810.81 Rs.
4	Find an approximate value of $\sqrt{(299)^2 + (399)^2}$ using the theory of approximations. **Solution.** Let $z = \sqrt{x^2 + y^2} = (x^2 + y^2)^{\frac{1}{2}}$, then $\frac{\partial z}{\partial x} = \frac{x}{(x^2+y^2)^{\frac{1}{2}}}, \frac{\partial z}{\partial y} = \frac{y}{(x^2+y^2)^{\frac{1}{2}}}$ Now, $$\delta z = \frac{\partial z}{\partial x} \cdot \delta x + \frac{\partial z}{\partial y} \cdot \delta y$$ $$\Rightarrow \delta z = \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} \cdot \delta x + \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} \cdot \delta y$$ Putting $x = 300, y = 400, \delta x = (299 - 300) = -1, \delta y = (399 - 400) = -1,$ $x^2 + y^2 = 250000$, we have $$\delta z = \left(\frac{300}{500}\right)(-1) + \left(\frac{400}{500}\right)(-1) = -\frac{3}{5} - \frac{4}{5} = -\frac{7}{5} = -1.4$$ Approximate value $= z + \delta z = (250000)^{\frac{1}{2}} - 1.4 = 500 - 1.4 = 498.6.$

5	Find an approximate value of $\sqrt[4]{(5.1)^2(2.9)+(2.9)^2}$ using the theory of approximations. **Answer:** 3.0027.
6	Find an approximate value of $\sin 44^0 \cdot \cos 62^0$ using the theory of approximations. **Answer:** $\frac{1}{2\sqrt{2}} - \frac{\pi}{180}\left(\frac{1+2\sqrt{3}}{2\sqrt{2}}\right)$.