II Infinite Series and Complex numbers

Practice Examples

2.1	Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test

Test the convergence of the following series

1	$\sum_{n=1}^{\infty}\left(\sqrt{n+1}-\sqrt{n}\right)$. **Answer:** divergent.
2	$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\cdots$ **Answer:** convergent.
3	$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots$ **Answer:** convergent.
4	$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots$ **Answer:** convergent.
5	$\sum_{n=1}^{\infty}\frac{n^2-1}{n^2+1}$. **Answer:** divergent.
6	$\sqrt{\frac{1}{2}}+\sqrt{\frac{2}{3}}+\sqrt{\frac{3}{4}}+\cdots$ **Answer:** divergent.
7	$\sum_{n=2}^{\infty}\frac{1}{\sqrt{n^2-1}}$. **Answer:** divergent.
8	$\sum_{n=1}^{\infty}\frac{1}{n^2+1}$.

	Answer: convergent.
9	$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}+1}$. **Answer:** convergent.
10	$\frac{1}{1+\sqrt{2}} + \frac{2}{1+2\sqrt{3}} + \frac{3}{1+3\sqrt{4}} + \cdots$ **Answer:** divergent.
11	$\sum_{n=1}^{\infty} \frac{5n^3+3}{3n^5+4}$. **Answer:** convergent.
12	$\sum_{n=1}^{\infty} \frac{n!}{n^n}$. **Answer:** convergent.
13	$\sum_{n=1}^{\infty} e^{-n^3}$. **Answer:** convergent.
14	$\sum_{n=1}^{\infty} \frac{2^n n!}{n^n}$. **Answer:** convergent.
15	$\frac{1}{3} + \left(\frac{2}{5}\right)^2 + \left(\frac{3}{7}\right)^3 + \cdots$. **Answer:** convergent.
16	$\sum_{n=1}^{\infty} \left(\frac{n+1}{3n}\right)^n$. **Answer:** convergent.
17	$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$. **Answer:** divergent.
18	$1 - \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} - \frac{1}{4\sqrt{4}} + \cdots$. **Answer:** convergent.
19	$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{log(n+1)}{(n+1)^2}$. **Answer:** convergent.

20	$\frac{1}{4}-\frac{1}{7}+\frac{1}{10}-\cdots$. **Answer:** convergent.
21	$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{5^n}$. **Answer:** convergent.
22	$\sum_{n=1}^{\infty}\frac{2n^2+3n}{5+n^5}$. **Answer:** convergent.
23	$\sum_{n=1}^{\infty}\frac{n}{(n^2+1)^{2/3}}$. **Answer:** divergent.
24	$\sum_{n=1}^{\infty}\sqrt{n^4+1}-\sqrt{n^4-1}$. **Answer:** convergent.
25	$\sum_{n=1}^{\infty}\frac{n^2}{3^n}$. **Answer:** divergent.
26	$1+\frac{3}{2!}+\frac{5}{3!}+\frac{7}{4!}+\cdots$. **Answer:** convergent.

2.2	Complex numbers and their geometric representation

1	Find the real and imaginary parts of the following complex numbers (a) $z=2+3i$ (b) $z=4i^2+i-2i^3$. **Answer:** (a) $x=2, y=3$ (b) $=-4, y=3$.
2	Solve the equation $xy-2i+x+2xyi-5=\frac{3}{2}-3i$ for x and y. **Answer:** $x=7, y=-\frac{1}{14}$**.**
3	Let $z=\frac{-2+3i}{3-2i}$. Find the complex conjugate, $\bar{z}$. Write your answer in the form $a\ +\ ib$.

	Answer: $-\frac{12}{13}+i\frac{5}{13}$.
4	Given $z_1=-2+2i$, and $z_2=4-6i$. Find $\frac{2}{\overline{z_1}+\overline{z_2}}$. **Answer:** $\frac{1}{5}-\frac{2}{5}i$.
5	If $z_1=i$, $z_2=-3i$ and $z_3=-1-4i$. Find (a) z_2-z_3 (b) z_2+z_3 (c) $\frac{z_2}{z_3}$ (d) $z_1\cdot z_3$ (e) z_1-z_3 (f) z_1+z_3 (g) $\frac{z_2}{z_1}$ (h) $z_2\cdot z_3$. **Answer:** (a) $1+i$ (b) $-1-7i$ (c) $\frac{12}{17}+\frac{3}{17}i$ (d) $4-i$ (e) $1+5i$ (f) $-1-3i$ (g) -3 (h) $-12+3i$.
6	Express the following numbers into $x+iy$ form: (a) $\frac{1}{1+i}$ (b) $\frac{1}{(2+i)^2}-\frac{1}{(2-i)^2}$. **Answer:** (a) $\frac{1}{2}-\frac{1}{2}i$ (b) $-\frac{8}{25}i$.
7	Reduce each of the following quantities to a real number: (a) $\frac{1+2i}{3-4i}+\frac{2-i}{5i}$ (b) $\frac{5i}{(1-i)(2-i)(3-i)}$. **Answer:** (a) $-\frac{2}{5}$ (b) $-\frac{1}{2}$.
8	Simplify $\frac{1+i}{i}+\frac{i}{1-i}$ into $x+iy$ form. **Answer:** $-\frac{1}{2}-\frac{1}{2}i$.
9	Convert $\frac{(1+i)^3}{1-i}$ into $x+iy$ form. **Answer:** $-2+0i$.

2.3	Complex number in the polar (or exponential) form

1	Express the complex number $\frac{1}{2}-\frac{\sqrt{3}}{2}i$ in the polar form. **Answer:** $\cos\left(\frac{\pi}{3}\right)-i\sin\left(\frac{\pi}{3}\right)$.
2	Express the following complex numbers in their polar form. (a) i (b) $-8i$ **Answer:** (a) $\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)$, (b) $8\left[\cos\left(\frac{\pi}{2}\right)-i\sin\left(\frac{\pi}{2}\right)\right]$.

3	Find Arg(z) , arg(z) and polar form of (a) $z = 1 + i$ (b) $z = 1 - i$. **Answer:** (a) $\sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$ (b) $\sqrt{2}\left[\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right]$.
4	Express the following complex numbers in their polar form. (a) $-2 + i$ (b) -2. **Answer:** (a) $\sqrt{5}\left[\cos\left(\pi - \tan^{-1}\frac{1}{2}\right) + i\sin\left(\pi - \tan^{-1}\frac{1}{2}\right)\right]$ (b) $2[\cos\pi + i\sin\pi]$.
5	Find Arg(z) , arg(z) and polar form of (a) $z = 2i$ (b) $z = -1 + 3i$ (c) $z = 3$. **Answer:** (a) $2\left[\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right]$ (b) $\sqrt{10}[\cos(\pi - \tan^{-1}3) + i\sin(\pi - \tan^{-1}3)]$ (c) $3(\cos 0 + i\sin 0)$.

2.4	De Moivre's theorem and its applications

1	Let n be a positive integer. Show that $(a + ib)^n + (a - ib)^n = 2r^n\cos n\theta$, where $r^2 = a^2 + b^2$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$. Hence deduce that $(1 + i\sqrt{3})^8 + (1 - i\sqrt{3})^8 = -2^8$.
2	Prove that $\frac{(cos5\theta - isin5\theta)^2(cos7\theta + isin7\theta)^{-3}}{(cos4\theta - isin4\theta)^9(cos\theta + isin\theta)^5} = 1$.
3	Find the values of $(1 + i)^{\frac{2}{3}}$. **Answer:** $2^{\frac{1}{3}}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right], 2^{\frac{1}{3}}\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right], 2^{\frac{1}{3}}\left[\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right]$.
4	Solve the equation $x^7 + x^4 + x^3 + 1 = 0$. **Answer:** $\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right), \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right), \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right), \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right), \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right), \cos(\pi) + i\sin(\pi), \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)$.
5	Solve $x^7 - 1 = 0$. **Answer:** $\cos(0) + i\sin(0), \cos\left(\frac{2\pi}{7}\right) + i\sin\left(\frac{2\pi}{7}\right), \cos\left(\frac{4\pi}{7}\right) + i\sin\left(\frac{4\pi}{7}\right), \cos\left(\frac{6\pi}{7}\right) + i\sin\left(\frac{6\pi}{7}\right), \cos\left(\frac{8\pi}{7}\right) + i\sin\left(\frac{8\pi}{7}\right), \cos\left(\frac{10\pi}{7}\right) + i\sin\left(\frac{10\pi}{7}\right), \cos\left(\frac{12\pi}{7}\right) + i\sin\left(\frac{12\pi}{7}\right)$.
6	Find all values of $(-1 + i)^{\frac{2}{5}}$. **Answer:** $2^{\frac{1}{5}}\left[\cos\left(-\frac{\pi}{10}\right) + i\sin\left(-\frac{\pi}{10}\right)\right], 2^{\frac{1}{5}}\left[\cos\left(\frac{3\pi}{10}\right) + i\sin\left(\frac{3\pi}{10}\right)\right], 2^{\frac{1}{5}}\left[\cos\left(\frac{7\pi}{10}\right) +\right.$

	$i\sin\left(\frac{7\pi}{10}\right)\Big], 2^{\frac{1}{5}}\left[\cos\left(\frac{11\pi}{10}\right)+i\sin\left(\frac{11\pi}{10}\right)\right], 2^{\frac{1}{5}}\left[\cos\left(\frac{3\pi}{2}\right)+i\sin\left(\frac{3\pi}{2}\right)\right].$

2.5	Exponential, Logarithmic, Trigonometric and hyperbolic functions

1	Find the value of $Log(-5)$. **Answer:** $\log 5 + i(2n+1)\pi$.
2	Find the value of $Log\ i$. **Answer:** $\frac{\pi}{2}i$.
3	Prove that $Log(-ei) = 1 - \frac{\pi}{2}i$.
4	Find the values of $\log(1+i)$. **Answer:** $\frac{1}{2}\ln 2 + i\left(\frac{\pi}{4}+2n\pi\right), n \in Z$.
5	Separate real and imaginary parts of $(i)\sin(x \pm iy)$ $(ii)\sinh(x \pm iy)$. **Answer:** $(i)\sin(x \pm iy) = \sin x \cosh y \pm i\cos x \sinh y$, $(ii)\sinh(x \pm iy) = \sinh x \cos y \pm i\cosh x \sin y$.
6	Separate real and imaginary parts of $(i)\cos(x \pm iy)$ $(ii)\cosh(x \pm iy)$. **Answer:** $(i)\cos(x \pm iy) = \cos x \cosh y \mp i\sin x \sinh y$, $(ii)\cosh(x \pm iy) = \cosh x \cos y \pm i\sinh x \sin y$.
7	Separate real and imaginary parts of $(i)\tan(x - iy)$ $(ii)\tanh(x \pm iy)$. **Answer:** $(i)\tan(x - iy) = \frac{\sin 2x}{\cosh^2 y - \sin x} - i\,\frac{\sinh 2y}{\cosh^2 y - \sin x}$, $(ii)\tanh(x \pm iy) = \frac{\sinh 2x}{\cosh 2x + \cos 2y} \pm i\frac{\sin 2y}{\cosh 2x + \cos 2y}$.