# Unit II

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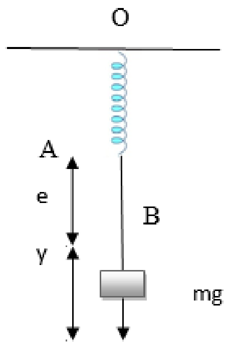
## 2.1 General Solution of Higher Order Ordinary Linear Differential Equations with Constant coefficients

Many engineering problems and applied mathematics problems such as study of small-amplitude mechanical oscillations, analysis of electrical networks, oscillations of mass-spring system, bending of beams, detection of diabetes, etc. leads to the formulation and solution of linear differential equations of second and higher order.

**Example:** By applying Kirchoff's voltage law to a L-C-R circuit, the second order differential equation is $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{c}=E\sin pt$ is generated, where $q$ is the charge on a plate of condenser, R is resistor, L is inductor, $t$ is the time and $E\sin pt$ is electromotive force.

**Example:** (**Free Oscillation**) Consider a spring OA suspended vertically from a fixed support at O. let a body of mass being large in comparison with mass of the spring, that the later may be neglected. Let $e=(AB)$be the elongation produced by the mass hanging in equilibrium, and then B is called the position of static equilibrium.




Let $k$ be the restoring force per unit stretch of the spring due to elasticity.

For the equilibrium at $B, mg = ke$

Let the mass be displaced through a further distance y from equilibrium position.

The acceleration of the mass m at this position is $\frac{d^2y}{dx^2}$ and the forces acting upon it are weight mg down wards and restoring force $k\ (e + y)$ upward.

∴By Newton's second law of motion

$$m\frac{d^2y}{dx^2} = mg - k(e + y) = ke - ke - ky = -ky$$

$$\therefore m\frac{d^2y}{dx^2} + ky = 0$$

Writing $w^2 = k/m$

$$\therefore \frac{d^2y}{dx^2} + w^2y = 0$$

**Note:** Linear differential equations are those differential equations in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

**Introduction:**

The general form of linear differential equation of n$^{th}$ order is

$$\frac{d^ny}{dx^n} + p_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}(x)\frac{dy}{dx} + p_n(x)y = R(x) \ldots \ldots \ldots ..(1)$$

Where $p_1(x), p_2(x), \ldots, p_n(x)\ and\ R(x)$ are functions of *x* only.

Linear differential equation with constant coefficients is of the form

$$\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}\frac{dy}{dx} + p_ny = R(x)$$

Where $p_1, p_2, \ldots, p_n$ are constants and $R(x)$ is function of *x* only.

If $R(x) = 0$ in equation (1) for all *x* then it reduces to

$$\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}\frac{dy}{dx} + p_ny = 0$$

and it is called **homogeneous** Linear differential equation of order n**.**

If $R(x) \neq 0$, then equation (1) is called **nonhomogeneous** Linear differential equation of order n.

**Note:**

1. If $y_1(x)$ and $y_2(x)$ are two solutions of the differential equation $\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}\frac{dy}{dx} + p_ny = 0$, then $C_1y_1(x) + C_2y_2(x)(= u)$, where $C_1, C_2$ are constants, is also its solution.

2. The general solution of a linear differential equation of n$^{th}$ order contains *n* arbitrary constants. If $y_1(x), y_2(x), \ldots, y_n(x)$ are *n* solutions of equation (1) then u(x) = $C_1y_1(x) + C_2y_2(x) + \ldots + C_ny_n(x)(= u)$ is also its solution.

3. If *R(X)* ≠ *0,* then the particular solution *v* of the equation (1), which is depending on *R(X).* $y = u + v$ is the general solution of the equation (1).

The part *u* is the solution of the homogeneous differential equation and the second part *v* is free from any arbitrary constants, called the particular integral (P.I. or $y_p$) of equation (1).

**Differential operator:** The symbol $D = \frac{d}{dx}$ stands for the differential operator

$$Dy = \frac{dy}{dx}, D^2y = \frac{d^2y}{dx^2}, \dots, D^ny = \frac{d^ny}{dx^n}.$$

So equation (1) can be written in the operator form

$$D^ny + p_1(x)D^{n-1}y + \cdots + p_{n-1}(x)Dy + p_n(x)y = R(x)$$

$$\text{i.e.}\, f\,(D)y \;=\; R(x),$$

where $f(D) = D^n + p_1(x)D^{n-1} + \cdots + p_{n-1}(x)D + p_n(x)$.

The general form of the homogeneous linear differential equation with constant coefficients is

$$\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}\frac{dy}{dx} + p_ny = 0, \dots\dots\dots..(2)$$

where $p_1, p_2, \dots, p_n$ are constants

In operator form, it can be written as $D^ny + p_1D^{n-1}y + \cdots + p_{n-1}Dy + p_ny = 0$

or $f\,(D)y \;=\; 0$, where $f(D) = D^n + p_1D^{n-1} + \cdots + p_{n-1}D + p_n$.

**Definition:** The general solution of the homogeneous differential equation $f\,(D)y \;=\; 0$ is known as the complementary function (in short we may write C.F.).

Let $y = e^{mx}$ be the solution of equation (2), where *m* is a constant.

Substituting this value of *y* and its derivatives in equation (2), we get

$$e^{mx}(m^n + p_1m^{n-1} + \cdots + p_{n-1}m + p_n)y = 0.$$

Since $e^{mx} \neq 0$, we get $f(m) = m^n + p_1m^{n-1} + \cdots + p_{n-1}m + p_n = 0$ ……. (3)

The equation (3) , is called the **Auxiliary equation (A.E.)** or **Characteristic equation** of the given equation (2).

Depending of the nature of the roots of the Auxiliary equation (A.E.) or characteristic equation $f(m) = 0$, the following cases arise.

**Case I: All roots are real and different**

If all the roots are real and different say $m_1, m_2, \dots, m_n$ are all distinct real roots, then C.F. is $y = C_1e^{m_1x} + C_2e^{m_2x} + \ldots + C_ne^{m_nx}$, where $C_1, C_2, \dots, C_n$ are arbitrary constants, which is a solution of $f\,(D)y \;=\; 0$, i.e. it is a C.F. of the differential equation.

**Example:** Solve $\frac{d^2y}{dx^2} - 5\,\frac{dy}{dx} + \;6y = 0$.

**Solution:** Rewriting the given differential equation in operator form, we get

$$D^2y - 5\,Dy + \;6y = 0$$

Therefore, auxiliary equation is

$m^2 - 5m + 6 = 0$

$\Rightarrow (m-3)(m-2) = 0$

$\therefore m = 3\,, \quad m = 2$, which are real and distinct roots of the characteristic equation. Thus, general solution of the given D.E. is $y = C_1 e^{3x} + C_2 e^{2x}$, where $C_1$ and $C_2$ are arbitrary constants.

**Case II: All roots are real and equal**

If all roots are real and equal say $m_1 = m_2 = m_3 = \ldots = m_n = m$, then C.F. is $y = (C_1 + C_2 x + \ldots + C_n x^{n-1}) e^{mx}$; where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

**Example:** Solve $y'' - 2y' + y = 0$.

**Solution:** Rewriting the given differential equation in operator form, we get

$$D^2 y - 4\,Dy + 4y = 0$$

Therefore, auxiliary equation is
$m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\therefore m = 2\,, \quad m = 2$

which are real and equal. Thus, general solution (C.F.) of the given D.E. is $y = (C_1 + C_2 x)e^{2x}$, where $C_1$ and $C_2$ are arbitrary constants.

**Case III: Roots are imaginary and distinct**

If the auxiliary equation has only one pair of imaginary roots i.e. $m_1 = \alpha + i\beta, m_2 = \alpha - i\beta$, then C.F. is $e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$, where $C_1$ and $C_2$ are arbitrary constants.

**Example:** Solve $16y'' - 8y' + 5y = 0$.

**Solution:** Rewriting the given differential equation in operator form, we get

$$16D^2 y - 8\,Dy + 5\,y = 0$$

Therefore, Auxiliary equation is $16m^2 - 8m + 5 = 0$

$m = \frac{-8 \pm \sqrt{64 - 4(16)(5)}}{32} = \frac{-8 \pm \sqrt{-256}}{32} = \frac{-8 \pm 16i}{32} == \frac{1}{4} \pm \frac{i}{2}$

which are imaginary and distinct. Thus, general solution of the given D.E. is

$y = e^{\frac{x}{4}} \left( C_1 cos\, \frac{x}{2} + C_2 sin\, \frac{x}{2} \right)$, where $C_1$ and $C_2$ are arbitrary constants.

**Case IV: Roots are imaginary and equal**

If auxiliary equation has two pairs of equal imaginary roots are equal, i.e. $m_1 = m_2 = \alpha + i\beta; m_3 = m_4 = \alpha - i\beta, m_5, m_6, \ldots \ldots, m_n$, then C.F. is $y = e^{\alpha x}\big((c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x) sin\, \beta x\big) + c_5 e^{m_5 x} + \ldots\ldots\ldots + c_n e^{m_n x}$, where $C_1, C_2, \ldots, C_n$ are arbitrary constants.

Similarly if auxiliary equation has more pairs of equal imaginary roots then C.F. is analogous to Case II.

**Example:** $y^{(4)} + 2y^{(2)} + y = 0$

**Solution:** Auxiliary equation is $m^4 + 2m^2 + 1 = 0$, i.e. $(m^2 + 1)^2 = 0$.

$\therefore m = \pm i\,,\quad m = \pm i$ are repeated roots of the A. E.

So the general solution of the given D.E. is $y = \big((C_1 + C_2x)cos\,x + \;(C_3 + C_4x)sin\,x\big)$, where $C_1, C_2, C_3$ and $C_4$ are arbitrary constants.

**Tutorial:**

| | |
|---|---|
| 1 | Solve $y''' - 2y'' - 5y' + 6y = 0, y(0) = 1, y'(0) = 0\;and\;y''(0) = 0$<br>**Ans: y(x)=** $\boldsymbol{e^x + \frac{1}{5}e^{-2x} - \frac{1}{5}e^{3x}}$ |
| 2 | Solve $y'' + 2y' + 4y = 0, y(0) = 2$ and $y'(0) = 1$.<br>**Ans: y(x)=** $\boldsymbol{e^{-x}\,(\sqrt{3}\sin\big(x\sqrt{3}\big) + 2\cos\big(x\sqrt{3}\big))}$ |
| 3 | Solve $y^{(iv)} + y'' - 2y = 0$.<br>**Ans: y(x)=** $\boldsymbol{c_1e^x + c_2e^{-x} + c_3cos\big(x\sqrt{2}\big) + c_4sin\big(x\sqrt{2}\big)}$ |
| 4 | Solve $(D^2 + 1)^3y = 0$ where $D = \frac{d}{dx}$<br>Ans: **y(x)=** $\boldsymbol{((c_1 + c_2x + c_3x^2)cosx + (c_4 + c_5x + c_6x^2)sinx)}$ |

## 2.2 Methods for finding particular integrals viz. variation of parameters and undetermined coefficients

The general form of the nonhomogeneous linear differential equation with constant coefficients is

$$\frac{d^ny}{dx^n} + p_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + p_{n-1}\frac{dy}{dx} + p_ny = R(x);\; R(x) \neq 0, \ldots (1)$$

where $p_1, p_2, \ldots, p_n$ are constants and $R(x)$ is function of *x* only. We may write it as $f\,(D)y = R(x)$, where $f(D) = D^n + p_1D^{n-1} + \cdots + p_{n-1}D + p_n$.

Such equations are important in the study of electro-mechanical vibrations and other engineering problems.

Recall that the solution of homogeneous linear differential equation with constant coefficients $f\,(D)y = 0$ contains *n* arbitrary constants is the called complementary function and the particular integral is a part of the general solution which satisfies the equation $f\,(D)y = R(x)$ but it will not contain arbitrary constants.
The complete solution (general solution) of equation (1) is *y* = C.F.+P.I. (or $y = y_c + y_p$)

**Working rule to solve the equation (1)**, first find the C.F. i.e. the solution of homogeneous differential equation $f\,(D)y = 0$ and then find the particular integral of equation (1). The general solution is the sum of these solutions.

- **Linearly Dependent and Independent set of functions:**

The Wronskian of the functions $y_1(x)$and $y_2(x)$ is denoted and defined as the determinant

$$w(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

Functions $y_1(x)$ *and* $y_2(x)$ defined on the interval (a, b) are said to be linearly dependent if their Wronskian is zero for each x ∈ (a, b).

Functions $y_1(x)$ and $y_2(x)$ are said to be linearly independent if their Wronskian is not zero at some point $x_0 \in (a, b)$.

**Example:** Check whether the following functions are Linearly independent or linearly dependent.

**1.** $\boldsymbol{e^x, e^{-x}}$

**Solution:** Consider $y_1 = e^x, y_2 = e^{-x}$.

Therefore, $w(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix}$ = -2 ≠ 0

∴ Given functions are linearly independent.

**Tutorial:**

| | |
|---|---|
| 1 | Check whether the following functions are Linearly independent or linearly dependent.<br>$\log_e x, \quad \log_e x^n$<br>**Ans: Linearly dependent** |
| 2 | Check whether the following functions are Linearly independent or linearly dependent.<br>$e^{ax} \sin bx, \qquad e^{ax} cosbx$<br>**Ans: Linearly independent** |

- There are many methods to find the particular integral, here we will discuss following methods.

1) Method of variation of parameters
2) Method of undetermined coefficients

**1. Method of variation of parameters**

Method of variation of parameters enables to find the particular solution of any linear nonhomogeneous differential equation with constant coefficients.

Here, the particular integral of the nonhomogeneous linear differential equation is obtained by varying the parameters; that is, by replacing the arbitrary constants in the C.F. by variable functions.

To find particular integral of second order differential equation

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = R(x).$$

Let the complementary function be

$$y_c(x) = Ay_1(x) + By_2(x)$$

such that $y_1(x)$ and $y_2(x)$ satisfy

$$\frac{d^2y}{dx^2} + p_1\frac{dy}{dx} + p_2 y = R(x).$$

Let us assume particular integral $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$, where $u$ and $v$ are unknown functions of $x$.

It can be obtained by

$$u(x) = -\int \frac{R(x)y_2(x)}{W}\,dx;\ v(x) = \int \frac{R(x)y_1(x)}{W}\,dx,$$

where $W$ is the Wronskian of $y_1(x)$ and $y_2(x)$ which is

$$w(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_2(x)y_1'(x) \neq 0\ .$$

Also, one can write

$$y_p(x) = -y_1(x)\int \frac{R(x)y_2(x)}{W}dx + y_2(x)\int \frac{R(x)y_1(x)}{W}dx$$

**Working Rule to solve a linear differential equation of order two.**

1. Find out the C.F. $y_c(x) = A_1y_1(x) + A_2y_2(x)$
2. Find out Wronskian of $y_1(x)$ and $y_2(x)$.
3. Find $u(x)$ and $v(x)$ by formulae $u(x) = -\int \frac{R(x)y_2(x)}{W(x)}\,dx$ and $v(x) = \int \frac{R(x)y_1(x)}{W(x)}\,dx$.
4. Find out particular integral $y_p(x) = u(x)y_1(x) + v(x)y_2(x)$ by substituting *u(x)* and *v(x).*
5. The general solution is $y(x) = y_c(x) + y_p(x)$.

**Example.** Solve $\left(D^2 - 2D + 2\right)y = e^x \tan x$ using the method of variation of parameter.

**Solution:** Auxiliary equation is $m^2 - 2m + 2 = 0 \Rightarrow m_1 = 1 + i, m_2 = 1 - i$

So C.F. $y_c = e^x\left(c_1 \cos x + c_2 \sin x\right)$

Let $y_1(x) = e^x \cos x, y_2(x) = e^x \sin x, R(x) = e^x \tan x.$

Then wronskian is

$$W(x)=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}=\begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x(\cos x-\sin x) & e^x(\sin x+\cos x) \end{vmatrix}=e^{2x}.$$

$$u(x)=\int\frac{-y_2(x)R(x)}{W(x)}\,dx=-\int\frac{e^x\sin x.e^x\tan x}{e^{2x}}\,dx=-\int\frac{\sin^2 x}{\cos x}\,dx$$

$$u(x)=-\log(\sec x+\tan x)+\sin x,$$

$$v(x)=\int\frac{y_1(x)R(x)}{W(x)}\,dx=\int\frac{e^x\cos x.e^x\tan x}{e^{2x}}\,dx=\int\sin x\,dx=-\cos x.$$

Substitute these values in $y_p=u(x)y_1(x)+v(x)y_2(x)$.

Therefore

$$y_p=[-\log(\sec x+\tan x)+\sin x]e^x\cos x+[-\cos x]e^x\sin x=-e^x\cos x\log(\sec x+\tan x).$$

Hence general solution is $y=y_c+y_p=e^x(c_1\cos x+c_2\sin x)-e^x\cos x.\log(\sec x+\tan x)$,

where $c_1$ and $c_2$ are arbitrary constants.

**Tutorial:**

| | |
|---|---|
| **1** | Apply the method of variation of parameters to solve $(D^2+a^2)y=sec\ ax$.<br>**Ans:** $\boldsymbol{y(x)=c_1cosx+c_2sinx+\frac{1}{a}xsinx+\frac{1}{a^2}cosxlogcosx}$ |
| **2** | Apply the method of variation of parameters to solve $y''-y=2e^{2x}$.<br>**Ans:** $\boldsymbol{y(x)=c_1e^{-x}+c_2e^x+\frac{2e^{2x}}{3}}$. |
| **3** | Apply the method of variation of parameters to solve $y''-2y'+y=\frac{e^x}{x^3}$.<br>**Ans:** $\boldsymbol{y(x)=(c_1+c_2x)e^x+\frac{1}{2}\frac{e^x}{x}}$ |

- **Method of undetermined coefficients**

To find the particular integral of

$$\frac{d^ny}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+\cdots+p_{n-1}\frac{dy}{dx}+p_ny=R(x);\ R(x)\neq 0,\ldots(1)$$

where $p_1,p_2,\ldots,p_n$ are constants and $R(x)$ is function of *x* only.

we assume a trial solution containing unknown constants which are determined by substituting in the given equation. The trial solution to be assumed in each case, depends on the term of *R(x)*.

Here, *R(x)* can only contain terms such as $b,x^p,e^{kx},sin\ kx,cos\ kx$ and finite number of combinations of such terms.

**Note:** This method can be used to find P.I. only if linearly independent derivatives of *R(x)* are finite in number. For $R(x) = \tan x, \sec x$ or $1/x$, this method fails.

The following table is useful for trial solution

| **Term of $R(x)$** | | **Assumption of particular integral ($y_p$)** |
|---|---|---|
| $Ae^{ax}$ (Exponential function) | | $Ae^{ax}$ |
| $Asin\ (ax)$ or $Acos\ (ax)$ (Trigonometry function) | | $A\ sin\ ax + B\ cos\ ax$ |
| $Ax^n$ (Polynomial function) | | $c_1x^n + c_2x^{n-1} + \cdots + c_n$ |
| $Ae^{ax}\sin ax$ or $Ae^{ax}\cos ax$ (combination of Exponential and trigonometry function) | | $e^{ax}(A\sin ax + B\cos ax)$ |

Before assuming the particular integral, it is necessary to compare the terms of $R(x)$ with the Complimentary function. While comparing the terms following different cases arise.

**Case I:** If no terms of $R(x)$ occurs in the complementary function, then P.I. is assumed from the table depending on the nature of $R(x)$.

**Example:** Find general solution of the differential equation $y'' + 2y' + 10y = 25x^2 + 3$ by the method of undetermined coefficients.

**Solution:** Rewriting the given differential equation in operator form

$$D^2y + 2Dy + 10y = 25x^2 + 3$$

Therefore, auxiliary equation is $m^2 + 2m + 10 = 0$. So $m = -1 \pm 3i$.

Here the roots of the characteristic equation are complex and distinct.

Thus, the complementary function of the given D.E. is

$y_c = e^{-x}(c_1\ cos\ 3x + c_2 sin\ 3x)$, where $c_1$ and $c_2$ are arbitrary constants.

Here we are given that $R(x) = 25x^2 + 3$, which is not appeared in C.F.

So we may take a trial solution of D. E. as $y_p = Ax^2 + Bx + C$.

Therefore $y_p' = 2Ax + B$ and $y_p'' = 2A$.

Substituting values of $y_p,\ y_p'\ \ and\ y_p''$ in given differential equation, we get

$2A + 4Ax + 2B + 10Ax^2 + 10Bx + 10C = 25x^2 + 3$

i.e. $10Ax^2 + (4A + 10B)x + (2A + 2B + 10C) = 25x^2 + 3$

On equating the coefficients, we get

$$10A = 25\ \ i.e.\ A = \frac{5}{2}.$$

As $4A + 10B = 0, B = -1$.

As $2A + 2B + 10C = 3,\ C = 0$.

$\therefore y_p = \frac{5}{2}x^2 - x$

Therefore, the general solution $y = y_c + y_p = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{5}{2}x^2 - x$, where $c_1$ and $c_2$ are arbitrary constants.

**Case II (Modification rule):** If a term $u$ of $R(x)$ is also a term of C.F. corresponding to an $r$-fold root, then assumed P.I. corresponding to u should be multiplied by $x^r$.

**Example:** Find general solution of the differential equation $y'' - y' - 2y = 3e^{2x}$ by the method of undetermined coefficients.

**Solution:** Rewriting the given differential equation in operator form, we get

$$(D^2 - D - 2)y = 3e^{2x}$$

Therefore, auxiliary equation is $m^2 - m - 2 = 0$

$$\therefore m = -1, m = 2$$

Here the roots of the characteristic equation are real and distinct.

Thus, complementary function of the given D.E. is

$y_c = c_1 e^{-x} + c_2 e^{2x}$, where $c_1$ and $c_2$ are arbitrary constants.

Here we are given that $R(x) = 3e^{2x}$.

Since a solution of the homogeneous equation (C. F.) contains $e^{2x}$, by modification rule we consider a trial solution of D.E. as $y_p = Axe^{2x}$.

$\therefore y_p' = Axe^{2x} + 2x\, Axe^{2x}$ and $\quad y_p'' = A(4e^{2x} + 4xe^{2x})$

Substituting the values of $y_p, y_p', and\ y_p''$ in given differential equation, we get

$3Ae^{2x} = 3e^{2x}$

$\therefore A = 1$

Therefore $y_p = xe^{2x}$.

Thus, the general solution is $y = y_c + y_p = c_1 e^{-x} + c_2 e^{2x} + xe^{2x}$, where $c_1$ and $c_2$ are arbitrary constants.

**Example:** Find general solution of the differential equation $y'' + 6y' + 9y = 50e^{-x}\cos x$ by the method of undetermined coefficients.

**Solution:** The auxiliary equation is $m^2 + 6m + 9 = 0$ . Its solution is $m = -3,\ m = -3$.
Here the roots of the characteristic equation are real and same.

Thus, complementary function of the given D.E. is $y_c = (c_1 + c_2x)e^{-3x}$, where $c_1$ and $c_2$ are arbitrary constants.

Here we are given that $R(x) = 50e^{-x}cos\, x$.
So we may take a trial solution of D.E. is $y_p = e^{-x}\,(Acos\, x + B\, sin\, x)$

$$y_p{}' = e^{-x}\,(-Acos\, x - B\, sin\, x - Asinx + B\, cos\, x)$$

$$= e^{-x}(-A + B)cos\, x - e^{-x}(A + B)sin\, x$$

$$y_p'' = 2e^{-x}(A\, sinx - B\, cos\, x\,).$$

Substituting the values of $y_p, y_p' and\ y_p''$ in given differential equation, we get

$$2e^{-x}(A\, sinx - B\, cos\, x\,) + 6e^{-x}\,(-Acos\, x - B\, sin\, x - Asinx + B\, cos\, x) + 9e^{-x}\,(Acos\, x + B\, sin\, x) = 50e^{-x}cos\, x$$

$$(2A - 6B - 6A + 9B)e^{-x}\, sin\, x + (-2B - 6A + 6B + 9A)e^{-x}cos\, x = 50e^{-x}cos\, x$$

$$(-4A + 3B)e^{-x}\, sin\, x + (3A + 4B)e^{-x}cos\, x = 50e^{-x}cos\, x$$

Comparing the co efficient, we get

$$-4A + 3B = 0 \quad ,3A + 4B = 50$$

$$\therefore A = 6\,, B = 8$$

$$\therefore\ y_p = e^{-x}\,(6cos\, x + 8\, sin\, x)$$

Therefore the general solution is

$$y = y_c + y_p = (c_1 + c_2x)e^{-3x} + e^{-x}\,(6cos\, x + 8\, sin\, x),$$

where $c_1$ and $c_2$ are arbitrary constants.

**Tutorial:**

| | |
|---|---|
| **1** | Apply the method of undetermined coefficients to solve $y''' + 3y'' + 2y' + y = 7e^{-x}$.<br>**Ans:** $(\boldsymbol{c_1 + c_2e^{-x} + c_3e^{-2x} - 7xe^{-x}}$ |
| **2** | Solve $(D^2 + 2D + 1)y = 2\cos 2x + 3x + 2 + 3e^x$ using method of undetermined coefficients.<br>**Ans:** $(\boldsymbol{c_1 + c_2x)e^{-x} + \frac{8}{25}sin\ 2x - \frac{6}{25}cos\ 2x + \frac{3}{4}e^x + 3x - 4}$ |
| **3** | Solve $y'' + 2y' + 5y = e^{0.5\,x} + 40\cos 10\,x - 190\sin 10\,x$ using method of undetermined coefficients.<br>**Ans:** $\boldsymbol{e^{-x}(A\ cos\ 2x + B\ sin\ 2x + 0.16\ e^{0.5x} + 2\ sin10x)}$ |

## 2.3 Linear Differential Equation of higher order with variable coefficients: Legendre's Equations (Special case: Cauchy-Euler equation)

**Legendre's linear differential equation**

A differential equation of the form $(ax+b)^n \dfrac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \dfrac{d^{n-1}y}{dx^{n-1}} + ...... + k_n\, y = R(x)$,

……. (4) where *k*'s are constants and *R(x)* is a function of *x*, is called Legendre's linear differential equation.

Such equations can be reduced to linear equations with constant coefficients by the substitution $ax+b=e^t$, *i.e.* $t=\log(ax+b)$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx} = \dfrac{a}{ax+b} \cdot \dfrac{dy}{dt}$ i.e. $(ax+b)\dfrac{dy}{dx} = a\,Dy$ , where *D=d/dt*.

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\frac{dy}{dt}\right)$$

$$= \frac{-a^2}{(ax+b)^2}\frac{dy}{dt} + \frac{a}{ax+b}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dy}{dx}$$

$$= \frac{a^2}{(ax+b)^2}\left(\frac{d^2y}{dt^2} - \frac{dy}{dt}\right).$$

$$\therefore (ax+b)^2 \frac{d^2y}{dx^2} = a^2\, D(D-1)\,y\,.$$

Similarly $(ax+b)^3 \dfrac{d^3y}{dx^3} = a^3\, D(D-1)(D-2)\,y$ and so on.

After making these replacements in (4), we get

$$\left\{a^n\left[D(D-1)...(D-n+1)\right] + k_1 a^{n-1}\left[D(D-1)...(D-n+2)\right] + ...... + k_{n-1}aDy + k_n\right\} y$$

$$= F(t); \text{ where } D = \frac{d}{dt}$$

which is a linear differential equation with constant coefficients. It can be solved by the method of variation of parameters or method of undetermined coefficients.

**Special case: Cauchy-Euler linear differential equation**

Take *a=1* and *b=0* in Legendre's differential equation. We get an equation of the form

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + ...... + k_{n-1}x\frac{dy}{dx} + k_n y = R(x) \qquad \text{...... (5)}$$

where *k*'s are constants and *R(x)* is a function of *x*, is known as Cauchy's linear equation. Such equations can be reduced to linear equations with constant coefficients by the substitution $x=e^t$, *i.e.* $t=\log x$.

**Example:** Solve $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

**Solution:** The given differential equation is Legendre's differential equation.

By substituting $3x+2=e^t, i.e.\ t=\log(3x+2)$ .

$(3x+2)\frac{dy}{dx}=3Dy,\ \ (3x+2)^2\frac{d^2y}{dx^2}=9D(D-1)y$, where $D=d/dt$ .

The given differential equation becomes

$$9D(D-1)y+9Dy-36y=3\left(\frac{e^t-2}{3}\right)^2+4\left(\frac{e^t-2}{3}\right)+1,$$

$\left(D^2-4\right)y=\frac{1}{27}\left(e^{2t}-1\right)$; which is a linear differential equation with constant coefficients. It can be solved by method of variation of parameters or method of undetermined coefficients.

The general solution using any of the method is $y(t)=y_c+y_p=c_1e^{2t}+c_2e^{-2t}+\frac{1}{108}\left(te^{2t}+1\right)$.

The general solution of given equation is obtained by replacing $t=\log(3x+2)$.

$$y(x)=c_1(3x+2)^2+c_2(3x+2)^{-2}+\frac{1}{108}\left((3x+2)^2\log(3x+2)+1\right),$$

where $c_1$ and $c_2$ are arbitrary constants, is the general solution of the given differential equation.

**Tutorial:**

| | |
|---|---|
| 1. | Solve $(1+x)^2\frac{d^2y}{dx^2}+(1+x)\frac{dy}{dx}+y=4\cos\left[\log(1+x)\right]$. **Ans:** $\boldsymbol{y(x)=c_1\cos[\log(1+x)]+c_2 sin[\log(1+x)]+2\log(1+x)\sin[\log(1+x)]}$ |
| 2. | Solve $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=\log x$ . **Ans:** $\boldsymbol{y(x)=c_1\cos(logx)+c_2\sin(logx)+logx}$ |
| 3. | Solve $x^2\frac{d^2y}{dx^2}+4x\frac{dy}{dx}+2y=3x^2$ . **Ans:** $\boldsymbol{y(x)=c_1x^{-1}+c_2x^{-2}+3x^{-2}e^x}$ |
| 4. | Solve $(x-2)^2\frac{d^2y}{dx^2}-3(x-2)\frac{dy}{dx}+4y=2x$ . **Ans:** $\boldsymbol{(x-2)^2[c_1+c_2\log(x-2)]+2x-3}$ |

## 2.4 System of simultaneous first order linear differential equations

Quite often we come across linear differential equations in which there are two or more dependent variables and single independent variable. Such equations are known as simultaneous linear differential equations. The method of solving these equations is based on the process of elimination, as we solve algebraic simultaneous linear equations.

**Example:** Solve the simultaneous differential equations:

$\frac{dx}{dt}+5x-2y=t, \frac{dy}{dt}+2x+y=0,$ given that $x=y=0$ when $t=0.$

**Solution:** Taking $d/dt=D$, the given equations become

$$(D+5)x-2y=t \qquad \text{....... (i)}$$
$$2x+(D+1)y=0 \qquad \text{....... (ii)}$$

Eliminate x as if D were an ordinary algebraic multiplier. Multiplying (i) by 2 and multiplying (ii) by D+5 and then subtracting, we get

$$\left(D^2+6D+9\right)y=-2t.$$

Its auxiliary equation is $D^2+6D+9=0$ *i.e.* $(D+3)^2=0.$

So, C.F. is $y_c=(c_1+c_2t)e^{-3t}$, where $c_1$ and $c_2$ are arbitrary constants.

The particular integral $y_p=-\frac{2t}{9}+\frac{4}{27}$ is obtained by Method of variation of parameters or method of undetermined coefficients.

Hence $y=(c_1+c_2t)e^{-3t}-\frac{2t}{9}+\frac{4}{27}$ is the general solution. .......(iii)

Now to find $x$, either eliminate $y$ from (i) and (ii) and solve the resulting equation OR substitute the value of $y$ in (ii). Here, it is more convenient to adopt the second method.

From (iii) $Dy=c_2e^{-3t}+(c_1+c_2t)(-3)e^{-3t}-\frac{2}{9}.$

Substituting for $y$ and $Dy$ in (ii), we get

$$x=-\frac{1}{2}[Dy+y]=\left[\left(c_1-\frac{1}{2}c_2\right)+c_2t\right]e^{-3t}+\frac{t}{9}+\frac{1}{27}. \qquad \text{.......(iv)}$$

Hence (iii) and (iv) constitute the solutions of the given equations.

Since $x=y=0$ when $t=0$ , the equation (iii) and (iv) give

$$c_1=-\frac{4}{27}, c_2=-\frac{2}{9}.$$

Hence desired solutions are

$$x=-\frac{1}{27}(1+6t)e^{-3t}+\frac{1}{27}(1+3t), y=-\frac{2}{27}(2+3t)e^{-3t}+\frac{2}{27}(2-3t).$$

**Tutorial**

| | |
|---|---|
| 1. | Solve the system of simultaneous differential equations $\frac{dx}{dt}+4x+3y=t, \frac{dy}{dt}+2x+5y=e^t$. <br><br> **Ans:** $\boldsymbol{x(t)=c_1e^{-2t}+c_2e^{-7t}-\frac{1}{8}e^t+\frac{5}{14}t-\frac{31}{196}}$ <br><br> $\boldsymbol{y(t)=-\frac{2}{3}c_1e^{-2t}+c_2e^{-7t}+\frac{5}{24}e^t-\frac{1}{7}t+\frac{9}{98}}$ |
| 2. | Solve the system of simultaneous differential equations <br><br> $\frac{dx}{dt}=7x-y$ <br> $\frac{dy}{dt}=2x+5y$ <br><br> **Ans:** $\boldsymbol{x(t)=e^{6t}(c_1cost+c_2sint)}$ <br><br> $\boldsymbol{y(t)=e^{6t}[(c_1-c_2)cost+(c_1+c_2)sint}$ |

**Application:**

| | |
|---|---|
| **1** | Find the steady state oscillation of the mass-spring system governed by the equation $y''+3y'+2y=20cos2t$. <br> **Ans:** $\boldsymbol{c_1e^{-t}+c_2e^{-2t}+3sin2t-cos2t}$ |

**Gate MCQ:**

**1** The general solution of the differential equation $\frac{d^2y}{dx^2}+2\frac{dy}{dx}-5y=0$ in terms of arbitrary constants $K_1$ and $K_2$ is (Gate-2017)

| | | | |
|---|---|---|---|
| (a) | $K_1e^{(-1+\sqrt{6})x}+K_2e^{(-1-\sqrt{6})x}$ | (b) | $K_1e^{(-1+\sqrt{8})x}+K_2e^{(-1-\sqrt{8})x}$ |
| (c) | $K_1e^{(-2+\sqrt{6})x}+K_2e^{(-2-\sqrt{6})x}$ | (d) | $K_1e^{(-2+\sqrt{8})x}+K_2e^{(-2-\sqrt{8})x}$ |

**2** The solution of the differential equation $k^2\frac{d^2y}{dx^2}=y-y_2$ under the boundary conditions

(i) $y=y_1$ at $x=0$

(ii) $y=y_2$ at $x=\infty$

Where $k, y_1$ and $y_2$ are constants. (Gate-2017)

| | | | |
|---|---|---|---|
| (a) | $y=(y_1-y_2)\exp(-x/k^2)+y_2$ | (b) | $y=(y_1-y_2)\exp(-x/k)+y_1$ |
| (c) | $y=(y_1-y_2)\sinh(x/k)+y_1$ | (d) | $y=(y_1-y_2)\exp(-x/k)+y_2$ |

| | | |
|---|---|---|
| **3** | A function $n(x)$ satisfied the differential equation $\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$ where $L$ is a constant. The boundary conditions are: $n(0) = k$ and $n(\infty) = 0$. The solution to this equation is (Gate-2010)<br><br>(a) $n(x) = k\exp\left(\frac{x}{L}\right)$   (b) $n(x) = k\exp\left(-\frac{x}{\sqrt{L}}\right)$<br>(c) $n(x) = k^2\exp\left(-\frac{x}{L}\right)$   (d) $n(x) = k\exp\left(-\frac{x}{L}\right)$ | |
| **4** | With initial values $y(0) = y'(0) = 1$, the solution of the differential equation<br><br>$\frac{d^2y}{dx^2} + \frac{4dy}{dx} + 4y = 0$ at $x = 1$ is ______. (Gate-2014) | |
| **5** | If the characteristic equation of the differential equation $\frac{d^2y}{dx^2} + 2\alpha\frac{dy}{dx} + y = 0$ has two equal roots, then the values of $\alpha$ are (Gate-2014)<br><br>(a) $\pm 1$   (b) 0,0<br>(c) $\pm j$   (d) $\pm\frac{1}{2}$ | |