III Matrix Algebra- I

3.1 Definition of Matrix, types of matrices and their properties

Introduction:

Definition: A set of mn numbers arranged in the form of a rectangular array having m number of rows and n number of columns called an $m \times n$ matrix.

This arrangement may be enclosed by $(\cdot)$, $[\cdot]$. We denote it by $\left(a_{ij}\right)_{m\times n}$ or $\left[a_{ij}\right]_{m\times n}$.

Types of matrices: Consider a matrix $A = \left(a_{ij}\right)_{m\times n}$.

Row matrix: If $m = 1$, then A is called a row matrix. e.g. $(0 \quad 1)$, $(1 \quad -1 \quad 0)$.

Column matrix: If $n = 1$, then A is called a column matrix.

e.g. $\begin{pmatrix} 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ are column matrix.

Square matrix: If $m = n$, then A is called a square matrix. The entries $a_{11}, a_{22}, \ldots, a_{mm}$ are called the entries of the principal(main) diagonal.

e.g. $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & -2 \end{pmatrix}$ are square matrices.

Null matrix: If $a_{ij} = 0$ for all i and j, then A is called a null matrix.

e.g. $\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ are null matrices.

Triangular matrix:

Upper triangular matrix: If $a_{ij} = 0$ for all i and j such that $i > j$, then A is called an upper triangular matrix.

e.g. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 4 & -3 \\ 0 & 0 & 5 \\ 0 & 0 & 2 \end{pmatrix}$ are upper triangular matrices.

Lower triangular matrix: If $a_{ij} = 0$ for all i and j such that $j > i$, then A is called a lower triangular matrix.

e.g. $\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 1 & -3 & 2 \end{pmatrix}$ lower triangular matrices.

Diagonal matrix: If $a_{ij} = 0$ for all i and j such that $i \neq j$, then A is called a diagonal matrix.

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ are diagonal matrices.

Scalar matrix: If the matrix A is a diagonal matrix and $a_{11} = a_{22} = \cdots = a_{mm}$, then it is called a scalar matrix.

e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ are scalar matrices.

Identity matrix: If the matrix A is a scalar matrix and $a_{11} = a_{22} = \cdots = a_{mm} = 1$, then it is called an identity matrix. We denote it by I_m.

e.g. $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Sub-matrix: A matrix obtained by eliminating some row(s) and/or column(s) from the matrix A is called a sub-matrix of the matrix A.

e.g. $\begin{pmatrix} 2 & 3 \\ 10 & 11 \end{pmatrix}$ is a sub-matrix of the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{pmatrix}$ formed by eliminating 1^{st}, 4^{th} columns and 2^{nd} row.

Equality of two matrices: Two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ are said to be equal if (1) $m = p$, $n = q$ and (2) the elements at the respective positions are equal. i.e. $a_{ij} = b_{ij}$ for all i and j.

Addition: Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ be two matrices. Then the addition is possible only if $m = p$ and $n = q$. If so then $A + B = (a_{ij} + b_{ij})_{m\times n}$.

e.g. Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 5 & 0 \end{pmatrix}$ then $A + B = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 8 & 2 \end{pmatrix}$.

Properties: Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{m\times n}$ and $C = (c_{ij})_{m\times n}$ be three matrices. Then

(1) $A + B = B + A$ (Commutativity). (ii) $(A + B) + C = A + (B + C)$ (Associativity).

Scalar multiplication: Let $A = (a_{ij})_{m\times n}$ be a matrix and k be a number. Then $\cdot A = (k \cdot a_{ij})_{m\times n}$. e.g. Let $A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{pmatrix}$ then $2A = \begin{pmatrix} 2 & -2 \\ 0 & 2 \\ 4 & 6 \end{pmatrix}$.

Properties: Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$ be two matrices, k_1 and k_2 be two numbers. Then

(i) $k_1(A + B) = k_1A + k_1B$.

(ii) $(k_1 + k_2)A = k_1A + k_2A$.

(iii) $(k_1k_2)A = k_1(k_2A) = k_2(k_1A)$.

Matrix multiplication: Let $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$. Then the product is possible only if $n = p$. If so then $A \cdot B = (a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + \cdots + a_{in} \cdot b_{nj})_{m\times q}$.

i.e. $A \cdot B = \left(\sum_{k=1}^{n} a_{ik} \cdot b_{kj}\right)_{m\times q}$.

e.g. Let $A = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 0 \\ 2 & -2 \\ -1 & 3 \end{pmatrix}$ then

$$A \cdot B = \begin{pmatrix} 1\cdot 5 + 0\cdot 2 + (-2)\cdot(-1) & 1\cdot 0 + 0\cdot(-2) + (-2)\cdot 3 \\ 0\cdot 5 + 3\cdot 2 + 2\cdot(-1) & 0\cdot 0 + 3\cdot(-2) + 2\cdot 3 \end{pmatrix} = \begin{pmatrix} 7 & -6 \\ 4 & 0 \end{pmatrix}.$$

Properties:

(i) Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$ and $C = (c_{ij})_{p\times q}$ be three matrices. Then $A(BC) = (AB)C$ (Associativity).

(ii) Let $A = (a_{ij})_{m\times n}$, $B = (b_{ij})_{n\times p}$ and $C = (c_{ij})_{n\times p}$ be three matrices. Then $A(B + C) = AB + AC$ (Distributive).

(iii) In general $AB \neq BA$.

e.g. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then $AB = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and

$BA = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Here, $AB \neq BA$.

Transpose: Transpose of a matrix $A = (a_{ij})_{m\times n}$ is $A' = (b_{ij})_{n\times m}$ where $b_{ij} = a_{ji}$.

Sometimes it is denoted by A^T. e.g. Let $A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & -2 \end{pmatrix}$. Then $A' = \begin{pmatrix} 2 & 1 \\ 1 & 3 \\ 3 & -2 \end{pmatrix}$.

Trace: Trace of a matrix $A = (a_{ij})_{m\times m}$ is $a_{11} + a_{22} + \cdots + a_{mm}$. We denote it by $tr(A)$.

e.g. Let $A = \begin{pmatrix} 1 & 7 & -2 \\ -2 & 4 & 2 \\ 3 & 1 & 3 \end{pmatrix}$. Then $tr(A) = 8$.

3.2	Determinant and their properties

Determinant:

2×2 matrix: Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{22} & a_{22} \end{pmatrix}$. Then the determinant of A is $a_{11}a_{22} - a_{12}a_{21}$.

3×3 matrix: Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$. Then the determinant of A is

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Minor: Let $\left(a_{ij}\right)_{m\times m}$ be a square matrix. Then the minor of an element a_{st} is the determinant of sub-matrix formed by eliminating s^{th} raw and t^{th} column. We denote it by M_{st}.

Co-factor: Co-factor of an element a_{st} is $(-1)^{s+t}M_{st}$. We denote it by C_{st}.

$\boldsymbol{m\times m}$ **matrix:** Let $A=\left(a_{ij}\right)_{m\times m}$ be a square matrix. Then the determinant of A is

$a_{11}C_{11}+a_{12}C_{12}+\cdots+a_{1m}C_{1m}$. We denote it by $det(A)$ or $|A|$.

Properties:

Let $A=\left(a_{ij}\right)_{m\times m}$ and $B=\left(b_{ij}\right)_{m\times m}$ be two matrices. Then

(i) $det(AB)=det(A)\,det(B)$

(ii) $det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ 0 & a_{22} & a_{23} & \cdots & a_{2m} \\ 0 & 0 & a_{33} & \cdots & a_{3m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{mm} \end{pmatrix}=a_{11}a_{22}a_{33}\cdots a_{mm}.$

(iii) $det\begin{pmatrix} a_{11} & 0 & 0 & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix}=a_{11}a_{22}a_{33}\cdots a_{mm}.$

(i.e. The determinant of a triangular matrix is the product of all its principal diagonal entries.)

(iv) $det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pm} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & a_{q2} & \cdots & a_{qm} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix}=$

$-det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & a_{q3} & \cdots & a_{qm} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & a_{p2} & \cdots & a_{pm} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix}\ \left(R_p\leftrightarrow R_q\right)$

(Here $p\neq q$ and p^{th} row and q^{th} row are interchanged. i.e. Interchanging of two rows changes the sign of the determinant.)

(v) $det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ka_{p1} & ka_{p2} & ka_{p3} & \cdots & ka_{pm} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix}=$

$$k \cdot det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pm} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix}$$

(Here p^{th} row is multiplied by a non-zero number k. i.e. If any one row is multiplied by a number, then the determinant is also multiplied by that number.)

(vi) $$det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p1} & a_{p2} & a_{p3} & \cdots & a_{pm} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & a_{q2} & \cdots & a_{qm} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix} =$$

$$det\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{p1}+ka_{q1} & a_{p2}+ka_{q2} & a_{p3}+ka_{q3} & \cdots & a_{pm}+ka_{qm} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & a_{q3} & \cdots & a_{qm} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mm} \end{pmatrix}$$

$\left(R_p \to R_p + kR_q\right)$

(Here p^{th} row is replaced by multiplying q^{th} row by k and added to it. i.e. If a row is replaced by multiplying another row by a number and add it to that row, then the determinant remains unchanged)

(vii) In a matrix if two rows are equal, then its determinant is zero.

(viii) In a matrix if one row is zero, then its determinant is zero.

(ix) $det(kA) = k^m det(A)$.

(x) $det(A) = det(A^T)$.

Remark: Properties (iv), (v), (vi), (vii) and (viii) are also true if the term "row" is replaced by "column".

Tutorial:

Determinant:

1	Show that the $det\begin{pmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{pmatrix} = 0.$ **Solution:** $det\begin{pmatrix} b-c & c-a & a-b \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{pmatrix} = det\begin{pmatrix} b-a & c-b & a-c \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{pmatrix}$

	(Property: (vi) i.e. $R_1 \to R_1 + R_2$) $= det\begin{pmatrix} 0 & 0 & 0 \\ c-a & a-b & b-c \\ a-b & b-c & c-a \end{pmatrix}$ (Property: (vi) i.e. $R_1 \to R_1 + R_3$) $= 0$
2	$det\begin{pmatrix} 1 & 3 & 7 \\ 4 & 9 & 1 \\ 2 & 7 & 6 \end{pmatrix} =$ _________. **Solution:** $det\begin{pmatrix} 1 & 3 & 7 \\ 4 & 9 & 1 \\ 2 & 7 & 6 \end{pmatrix} = 1(54-7) - 3(24-2) + 7(28-18) = 47 - 66 + 70 = 51.$
3	Find the determinant of $\begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$. **Answer:** 1.
4	Find the determinant of $\begin{pmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{pmatrix}$. **Answer:** $(a+b+c)^3$.

3.3	Rank and nullity of a matrix

Rank: A number r is said to be the rank of a matrix A if

(i) there is at least one non-zero minor of order r.

(ii) all the minors of order greater than r are zero.

We denote it by $r(A)$ or $\rho(A)$

Properties:

(i) A matrix A is a null matrix if and only if $r(A) = 0$.

(ii) Let A be an $m \times n$ matrix. Then $r(A) \leq min\{m, n\}$.

(iii) $r(A) = r(A^T)$.

Nullity: Nullity of an $m \times n$ matrix A is, $n - r(A)$. We denote it by $n(A)$.

e.g. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Here $det(A) = 0$ and $det\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = 3 \neq 0$.

Therefore $r(A) = 2$ and $n(A) = 3 - 2 = 1$.

Elementary row operations:

(i) Interchanging two rows.

$R_i \leftrightarrow R_r$ or R_{ir}.

($i \neq r$ and i^{th} row is interchanged with r^{th} row.)

(ii) Multiplying a row by a non-zero number.

$R_i \to kR_i$ or $R_i(k)$.

(i^{th} row is multiplied by $k \neq 0$.)

(iii) Addition of a row to another row by multiplying a non-zero number.

$R_i \to R_i + kR_r$ or $R_{ir}(k)$.

(r^{th} row is multiplied by k and add to i^{th} row.)

Elementary column operations:

(i) Interchanging two columns.

$C_j \leftrightarrow C_s$ or C_{js}.

($j \neq s$ and j^{th} column is interchanged with s^{th} column.)

(ii) Multiplying a column by a non-zero number.

$C_j \to kC_j$ or $C_j(k)$.

(j^{th} column is multiplied by $k \neq 0$.)

(iii) Addition of a column to another column by multiplying a non-zero number.

$C_j \to C_j + kC_s$ or $C_{js}(k)$.

(s^{th} column is multiplied by k and add to j^{th} column.)

3.4 Determination of rank

Row-echelon form:

Leading entry: The first nonzero entry of i^{th} row is known as the leading entry of i^{th} row. And the column containing the leading entry of i^{th} row is denoted by $l(i)$.

A matrix A is said to be in row-echelon form if

(i) all zero rows are at the bottom of the matrix.

(ii) all leading entries must be 1.

(iii) leading entries moves from left to right if we go down. i.e. $l(i) < l(i+1)$.

e.g. $\begin{pmatrix} 0 & 1 & 2 & 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 2 & -5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

Reduced row-echelon form: A matrix A is said to be in reduced row-echelon form if

(i) the matrix is in row-echelon form.

(ii) all the entries in a column containing the leading entry must be zero other than the leading entry.

e.g. $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

Remark: The rank of a matrix in row-echelon form is equal to the number of non-zero rows of the matrix (i.e. total number of leading entries).

Tutorial:

Rank using minors:

1	Determine the rank of $A = \begin{pmatrix} 1 & 1 & 1 \\ a & 1 & 1 \\ 1 & b & 1 \end{pmatrix}$ using minors. **Solution:** $det\begin{pmatrix} 1 & 1 & 1 \\ a & 1 & 1 \\ 1 & b & 1 \end{pmatrix} = (1-b)(1-a)$ **(i)** If $a \neq 1$ and $b \neq 1$, then rank is 3. **(ii)** If $a = 1$ and $b \neq 1$, then det(A) = 0 and $det\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} = ab - 1 \neq 0$. Therefore the rank is 2. **(iii)** If $a \neq 1$ and $b = 1$, then det(A) = 0 and $det\begin{pmatrix} a & 1 \\ 1 & b \end{pmatrix} = ab - 1 \neq 0$. Therefore the rank is 2. **(iv)** If $a = 1$ and $b = 1$, then det(A)= 0 and all 2-rowed minors are $det\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 0$. Since 1-rowed minor is $det(1) = 1 \neq 0$. Therefore the rank is 1.
2	Determine the rank of $\begin{pmatrix} 1 & 2 \\ -2 & 1 \\ -3 & 3 \end{pmatrix}$ using minors.

	Answer: 2.
3	Determine the rank of $\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 1 \\ 0 & 1 & a \end{pmatrix}$ using minors. **Answer:** **(i)** If $a \neq 1$ and $a \neq -1$ then the rank is 3. **(ii)** If $a = 1$ then the rank is 2. **(iii)** If $a = -1$ then the rank is 2.

Tutorial:

Row-echelon/ Reduced row-echelon form:

1	Reduce the matrix $\begin{pmatrix} 0 & 6 & 7 \\ -5 & 4 & 2 \\ 1 & -2 & 0 \end{pmatrix}$ to row-echelon/ reduced row-echelon form and hence determine its rank. **Solution:** The given matrix is $\begin{pmatrix} 0 & 6 & 7 \\ -5 & 4 & 2 \\ 1 & -2 & 0 \end{pmatrix}$ $R_1 \leftrightarrow R_3$ $\sim \begin{pmatrix} 1 & -2 & 0 \\ -5 & 4 & 2 \\ 0 & 6 & 7 \end{pmatrix}$ $R_2 \rightarrow R_2 + 5R_1$ $\sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & -6 & 2 \\ 0 & 6 & 7 \end{pmatrix}$ $R_3 \rightarrow R_3 + R_2$ $\sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & -6 & 2 \\ 0 & 0 & 9 \end{pmatrix}$ $R_2 \rightarrow -\frac{1}{6} R_2, R_3 \rightarrow \frac{1}{9} R_3$ $\sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 1 \end{pmatrix}$ (Row-echelon form) $R_2 \rightarrow R_2 + (1/3) R_3$ $\sim \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $R_1 \rightarrow R_1 + 2 R_2$ $\sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (Reduced row-echelon form) Number of non-zero rows in row-echelon form/ reduced row-echelon form is 3. Therefore the rank is 3.
2	Reduce the matrix $\begin{pmatrix} 1 & 4 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 4 & 8 & 9 & -1 \end{pmatrix}$ to row-echelon/reduced row-echelon form and hence determine the rank.

	Answer: $\begin{pmatrix} 1 & 4 & 3 & -1 \\ 0 & 1 & 3/8 & -3/8 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 3/2 & 1/2 \\ 0 & 1 & 3/8 & -3/8 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, rank is 2.
3	Reduce the matrix $\begin{pmatrix} 7 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & -4 & 2 \end{pmatrix}$ to row-echelon/reduced row-echelon form and hence determine the rank. **Answer:** $\begin{pmatrix} 1 & -9 & -8 \\ 0 & 1 & 31/32 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, rank is 3.

3.5	Solution of a system of linear equations by Gauss elimination and Gauss Jordan Methods.

System of linear equations:

$$\left.\begin{matrix} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n = b_3 \\ \vdots \quad \vdots \quad \vdots \quad \ddots \quad \vdots \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{matrix}\right\} \qquad (1)$$

Matrix form:

The system (1) can be represented as

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_m \end{pmatrix} \qquad (2)$$

Homogeneous system: If $b_i = 0$ for all $i = 1, 2, 3, ..., m$ in the system (1), then it is known as homogeneous system.

Coefficient matrix: The matrix $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix}$ is called the coefficient matrix of the system (1).

Augmented matrix: The matrix $(A|b) = \left(\begin{array}{ccccc|c} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} & b_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} & b_m \end{array}\right)$ is known as augmented matrix of the system (1).

Solution of homogeneous system:

(i) A homogeneous system always has at least one solution namely $(0,0,\ldots,0)$(n-tuple) which is known as the trivial solution.

(ii) If $r(A)=n$, then the trivial solution is the only solution of the system.

(iii) If $r(A)<n$, then the system has infinitely many solutions.

e.g.

(i) $x+y=0, x-y=0$

The coefficient matrix is

$A=\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$ whose rank is 2.

i.e. $r(A)=2$ (Number of unknowns).

Therefore the system has only the trivial solution.

i.e. the solution is $(0,0)$.

(ii) $x+y=0, 2x+2y=0$

The coefficient matrix is

$A=\begin{pmatrix}1 & 1\\ 2 & 2\end{pmatrix}$ whose rank is 1 which is less than 2.

i.e. $r(A)<2$ (Number of unknowns).

Therefore the system has infinitely many solutions.

The solution set is $\{(k,-k)|k\in\mathbb{R}\}$.

Solution of non-homogeneous system:

(i) If $r(A|b)\neq r(A)$, then the system is said to be inconsistent. (i.e. the system does not have any solution.)

(ii) If $r(A|b)=r(A)=n$, then the system has unique solution.

(iii) If $r(A|b)=r(A)<n$, then the system has infinitely many solutions.

e.g.

(i) $x+y=1, x+y=2$

The augmented matrix is

$(A|b)=\left(\begin{array}{cc|c}1 & 1 & 1\\ 1 & 1 & 2\end{array}\right)$ whose rank is 2.

And the rank of co-efficient matrix $A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}$ is 1.

i.e. $r(A|b)\neq r(A)$.

Therefore the system is inconsistent.

(In fact these are two parallel lines, so they do not intersect.)

(ii) $x + y = 1, x - y = 1$

The augmented matrix is

$(A|b) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & -1 & 2 \end{array}\right)$ whose rank is 2.

Also, the rank of co-efficient matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ is 2.

i.e. $r(A|b) = r(A) = 2$ (Number of unknowns).

Therefore the system is consistent and it has unique solution.

The solution is $(1, 0)$.

(iii)$x + y = 1, 2x + 2y = 2$

The augmented matrix is

$(A|b) = \left(\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right)$ whose rank is 1.

Also, the rank of co-efficient matrix $A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$ is 1.

i.e. $r(A|b) = r(A) < 2$ (Number of unknowns).

Therefore the system is consistent and it has infinitely many solutions.

The solution set is $\{(1 - k, k) | k \in \mathbb{R}\}$.

Tutorial:

System of linear equations:

1	Solve the system: $x + 2y + 3z = 0$ $3x + 4y + 4z = 0$ $7x + 20y + 12z = 0$ by Gauss elimination/Gauss-Jordan method. **Solution:** The coefficient matrix is $\sim \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 20 & 12 \end{pmatrix}$ $R_2 \leftrightarrow R_2 - 3R_1, R_3 \to R_3 - 7R_1$ $\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & -5 \\ 0 & 6 & -9 \end{pmatrix}$ $R_2 \to \frac{1}{2} R_2$ $R_3 \to -\frac{1}{24} R_3, R_2 \to (-1) R_2$ $\sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 5/2 \\ 0 & 0 & 1 \end{pmatrix}$ (1) $R_1 \to R_1 - 3R_3, R_2 \to R_2 - \frac{5}{2} R_3$ $\sim \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $R_1 \to R_1 + 2R_2$

	$\sim\begin{pmatrix}1 & 2 & 3\\ 0 & -1 & -5/2\\ 0 & 6 & -9\end{pmatrix}$ $R_3 \leftrightarrow R_3 + 6R_2$ $\sim\begin{pmatrix}1 & 2 & 3\\ 0 & -1 & -5/2\\ 0 & 0 & -24\end{pmatrix}$	$\sim\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ (2)

Gauss elimination method:

By (1) the equivalent system to the given system is:

$x + 2y + 3z = 0$ (3)

$y + \frac{5}{2}z = 0$ (4)

$z = 0$

Substituting $z = 0$ in (4) we get $y = 0$.

And substituting $y = 0$ and $z = 0$ in (3) we get $x = 0$.

Gauss-Jordan method:

By (2) the solution is $(0, 0, 0)$.

2 Solve the system:

$x - y + z = 1$

$2x + y - z = 2$

$5x - 2y + 2z = 5$

by Gauss elimination/Gauss-Jordan method, if it is consistent.

Solution:

The augmented matrix is $(A\|b) = \left(\begin{array}{ccc\|c}1 & -1 & 1 & 1\\ 2 & 1 & -1 & 2\\ 5 & -2 & 2 & 5\end{array}\right)$ $R_2 \leftrightarrow R_2 - 2R_1, R_3 \to R_3 - 5R_1$ $\sim\left(\begin{array}{ccc\|c}1 & -1 & 1 & 1\\ 0 & 3 & -3 & 0\\ 0 & 3 & -3 & 0\end{array}\right)$ $R_3 \to R_3 - R_2$ $\sim\left(\begin{array}{ccc\|c}1 & -1 & 1 & 1\\ 0 & 3 & -3 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$	$R_2 \to \frac{1}{3}R_2$ $\sim\left(\begin{array}{ccc\|c}1 & -1 & 1 & 1\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$ (1) $R_1 \to R_1 + R_2$ $\sim\left(\begin{array}{ccc\|c}1 & 0 & 0 & 1\\ 0 & 1 & -1 & 0\\ 0 & 0 & 0 & 0\end{array}\right)$ (2)

$r(A|b) = r(A) = 2 < 3 =$ Number of unknowns.

The number of independent solutions is 3 – 2 = 1.

	Gauss elimination method: By (1) the equivalent system to the given system is: $x - y + z = 1$ (3) $y - z = 0$ (4) Consider $z = k$, an arbitrary constant. Substituting $z = k$ in (4) we get $y = k$. And substituting $y = k$ and $z = k$ in (3) we get $x = 1$. **Gauss-Jordan method:** By (2) the equivalent system to the given system is: $x = 1$ (5) $y - z = 0$ (6) Consider $z = k$, an arbitrary constant. Substituting $z = k$ in (4) we get $y = k$. Therefore the solution set is $\{(1, k, k) \mid k \in \mathbb{R}\}$.
3	Solve the system: $x - y + z = 3$ $2x - 3y + 5z = 10$ $x + y + 4z = 4$ by Gauss elimination/Gauss-Jordan method, if it is consistent. **Answer:** $(1, -1, 1)$.
4	Find the value of k so that the system: $x + y + 3z = 0$ $4x + 3y + kz = 0$ $2x + y + 2z = 0$ has non-trivial solution. **Answer:** If $k = 8$, then it has non-trivial solution.

Matrix

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(1) $\det\begin{bmatrix} 1 & 3 & 7 \\ 4 & 9 & 1 \\ 2 & 7 & 6 \end{bmatrix}$

$= 54 - 7 - 3(24-2) + 7(28-18)$

$= 47 - 66 + 70 = 51$

(2) $\det\begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix}$

$-3 + 4 = 1$

(3) $\det\begin{bmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{bmatrix}$

$R_1 \to R_1 + R_2 + R_3$

$\begin{vmatrix} a-b-c+2a+2c & 2a+b-c-a+2c & 2a+2b+c-b-a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$C_1 \to C_1 - C_2$

$(a+b+c)\begin{vmatrix} 0 & 1 & 1 \\ a+b+c & b-c-a & 2b \\ 0 & 2c & c-a-b \end{vmatrix}$

$(a+b+c)\,(a+b+c)\,(-1)\,(-a-b-c)$

$(a+b+c)^3$

Find the range using minor of matrix.

$$\begin{bmatrix} 1 & 2 \\ -2 & 1 \\ -3 & 3 \end{bmatrix} \quad |1| = 1 \neq 0$$

$$\begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1+4 = 5$$

$\therefore$ 3x3 and 4x4 is not posible

$\therefore$ Rank is 2.

Find value of α so that rank of a matrix is either 3 or 2.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & 1 \\ 0 & 1 & \alpha \end{bmatrix}$$

For rank to be 3 $|A| \neq 0$.

$1(\alpha^2 - 1) \neq 0$

$(\alpha-1)(\alpha+1) \neq 0$

$\alpha \neq \pm 1$

For rank 2,

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} \quad 1(1-1) = 0 \qquad \begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{vmatrix} \quad 1(1-1) = 0$$

If $\alpha \neq \pm 1$ then rank is 3.

If $\alpha = 1, -1$ then rank is 2

Find value of a and b so that rank of matrix is 3, 2 or 1

$$A = \begin{bmatrix} 1 & 1 & 1 \\ a & 1 & 1 \\ 1 & b & 1 \end{bmatrix}$$

For rank = 3, $|A| \neq 0$

$1 - b - 1(a-1) + 1(ab-1) \neq 0$

$ab + 1 - a - b \neq 0$

$(a-1)(b-1) \neq 0$

Case I: $a \neq 1$ $b \neq 1$ $\longrightarrow$ rank $= 3$

Case II: $a \neq 1$ $b = 1$

Case III: $a = 1$ $b \neq 1$ $\Big\} \longrightarrow$ rank $= 2$

Case IV: $b = 1$ $a = 1$ $\longrightarrow$ rank $= 1$

For case II and III

$a = 1$ $b \neq 1$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & b & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ 1 & b \end{vmatrix} = 1 - b \neq 0, \quad b \neq 1$$

$b = 1$ $a \neq 1$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} a & 1 \\ 1 & 1 \end{vmatrix} = 1 - a \neq 0, \quad a \neq 1$$

$$\begin{bmatrix} 0 & 6 & 7 \\ -5 & 4 & 2 \\ 1 & -2 & 0 \end{bmatrix}$$

$|7| \neq 0$

$$\begin{vmatrix} 6 & 7 \\ 4 & 2 \end{vmatrix} = 12 - 28 \neq 0$$

$-6(-5-2) + 7(10-4)$

$42 + 7(6) \neq 0$

~~$1(12-28)+2(35)$~~

Rank $= 3$

$$\begin{bmatrix} 1 & 4 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 4 & 8 & 9 & -1 \end{bmatrix}$$

$|1| = 1 \neq 0$

$$\begin{vmatrix} 1 & 4 \\ 2 & 0 \end{vmatrix} = 0 - 8 = -8 \neq 0$$

$$\begin{vmatrix} 1 & 4 & 3 \\ 2 & 0 & 3 \\ 4 & 8 & 9 \end{vmatrix} = 3 \begin{vmatrix} 1 & 4 & 1 \\ 2 & 0 & 1 \\ 4 & 8 & 3 \end{vmatrix} = 1(-8) - 4(6-4) + 1(16)$$

$-8 - 8 + 16 = 0$

$$\begin{vmatrix} 1 & 4 & -1 \\ 2 & 0 & 1 \\ 4 & 8 & -1 \end{vmatrix} = 2(-4+8)+1(8-16) = 0$$

$$\boxed{\text{Rank} = 2}$$

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$$3\begin{vmatrix} 4 & 1 & -1 \\ 0 & 1 & 1 \\ 8 & 3 & -1 \end{vmatrix} = \begin{array}{l} 4(-1-3)+8(1+1) \\ -16+16 = 0 \end{array}$$

$$3\begin{vmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 4 & 3 & -1 \end{vmatrix} = 1(-1-3)-1(-2-4)-1(6-4) = -4+6-2 = 0$$

$$\begin{bmatrix} 7 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & -4 & 2 \end{bmatrix}$$

$|7| = 7 \neq 0$

$$\begin{vmatrix} 7 & 1 \\ 3 & 5 \end{vmatrix} = 35-3 = 32 \neq 0$$

$$2\begin{vmatrix} 7 & 1 & 6 \\ 3 & 5 & 7 \\ 2 & -2 & 1 \end{vmatrix} = 2(7-30)+2(49-18)+1(35-3)$$

$$= -46+62+32 \neq 0$$

Rank = 3.

Find row echelon form of following matrix and hence evalute rank and nullity.

1) $A = \begin{bmatrix} \boxed{1} & 4 & 3 & -1 \\ 2 & 0 & 3 & 1 \\ 4 & 8 & 9 & -1 \end{bmatrix}$

$$\underset{R_3-4R_1}{\overset{R_2-2R_1}{\sim}} \begin{bmatrix} \boxed{1} & 4 & 3 & -1 \\ 0 & -8 & -3 & 3 \\ 0 & -8 & -3 & 3 \end{bmatrix}$$

$$\overset{-1/8 R_2}{\sim} \begin{bmatrix} \boxed{1} & 4 & 3 & -1 \\ 0 & \boxed{1} & 3/8 & -3/8 \\ 0 & -8 & -3 & 3 \end{bmatrix}$$

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$$\underset{R_3+8R_2}{\sim}\begin{bmatrix} \boxed{1} & 4 & 3 & -1 \\ 0 & \boxed{1} & 3/8 & -3/8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R(A) = 2$

nullity of A = column $- r(A) = 4-2 = 2$.

Find reduced row echelon form of ~~following matrix and~~

$$A = \begin{bmatrix} 7 & 1 & 6 \\ 3 & 5 & 7 \\ 4 & -4 & 2 \end{bmatrix}$$

$$\underset{R_1\times 1/7}{\sim}\begin{bmatrix} \boxed{1} & 1/7 & 6/7 \\ 3 & 5 & 7 \\ 4 & -4 & 2 \end{bmatrix}$$

$$\underset{\substack{R_2-3R_1\\R_3-4R_1}}{\sim}\begin{bmatrix} \boxed{1} & 1/7 & 6/7 \\ 0 & 32/7 & 31/7 \\ 0 & -32/7 & -10/7 \end{bmatrix}$$

$$\underset{7/32 R_2}{\sim}\begin{bmatrix} \boxed{1} & 1/7 & 6/7 \\ 0 & \boxed{1} & 31/32 \\ 0 & -32/7 & -10/7 \end{bmatrix}$$

$$\underset{\substack{R_1-1/7R_2\\R_3+32/7R_2}}{\sim}\begin{bmatrix} \boxed{1} & 0 & 23/32 \\ 0 & \boxed{1} & 31/32 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\underset{R_3(1/3)}{\sim}\begin{bmatrix} \boxed{1} & 0 & 23/32 \\ 0 & \boxed{1} & 31/32 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

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$R_1 - 23/32\, R_3$
$\sim$
$R_2 - 31/32\, R_3$
$$\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

Rank = 3
nullity = 0

Solve the system

$$\begin{cases} x - y + z = 3 \\ 2x - 3y + 5z = 10 \\ x + y + 4z = 4 \end{cases}$$

Gauss elimination and Gauss Jordan method.

Ans :
$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 3 \\ 2 & -3 & 5 & 10 \\ 1 & 1 & 4 & 4 \end{array}\right]$$

Gauss elimination

$R_2 - 2R_1$
$\sim$
$R_3 - R_1$
$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 3 \\ 0 & -1 & 3 & 4 \\ 0 & 2 & 3 & 1 \end{array}\right]$$

$-R_2$
$\sim$
$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 3 \\ 0 & \boxed{1} & -3 & -4 \\ 0 & 2 & 3 & 1 \end{array}\right]$$

$R_3 - 2R_2$
$\sim$
$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 3 \\ 0 & \boxed{1} & -3 & -4 \\ 0 & 0 & 9 & 9 \end{array}\right]$$

$1/9\, R_3$
$\sim$
$$\left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 3 \\ 0 & \boxed{1} & -3 & -4 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

Gauss Jordan

$$\overset{R_2-2R_1}{\underset{R_3-R_1}{\sim}} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 3 \\ 0 & -1 & 3 & 4 \\ 0 & 2 & 3 & 1 \end{array}\right]$$

$$\overset{-R_2}{\sim} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 1 & 3 \\ 0 & 1 & -3 & -4 \\ 0 & 2 & 3 & 1 \end{array}\right]$$

$$\overset{R_1+R_2}{\underset{R_3-2R_2}{\sim}} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -2 & -1 \\ 0 & \boxed{1} & -3 & -4 \\ 0 & 0 & 9 & 9 \end{array}\right]$$

$$\overset{R_3/9}{\sim} \left[\begin{array}{ccc|c} \boxed{1} & 0 & -2 & -1 \\ 0 & \boxed{1} & -3 & -4 \\ 0 & 0 & \boxed{1} & \boxed{1} \end{array}\right]$$

$r(A) = r(A|B) = 3 =$ no. of unknown.

$\therefore$ System is consistent

$x = 1$; $y = -1$; $z = 1$

$x + y + 3z = 0$
$4x + 3y + kz = 0$
$2x + y + 2z = 0$
has non-trivial sol^n

Ans:
$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 4 & 3 & k & 0 \\ 2 & 1 & 2 & 0 \end{array}\right]$$

$$\overset{R_2R_3}{\sim} \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 1 & 2 & 0 \\ 4 & 3 & k & 0 \end{array}\right]$$

$$R_2-2R_1 \sim R_3-4R_1 \left[\begin{array}{ccc|c} \boxed{1} & 1 & 3 & 0 \\ 0 & -1 & -4 & 0 \\ 0 & -1 & K-12 & 0 \end{array}\right]$$

$$-R_2 \sim \left[\begin{array}{ccc|c} \boxed{1} & 1 & 3 & 0 \\ 0 & \boxed{1} & 4 & 0 \\ 0 & -1 & K-12 & 0 \end{array}\right]$$

$$R_3+R_2 \sim \left[\begin{array}{ccc|c} \boxed{1} & 1 & 3 & 0 \\ 0 & \boxed{1} & 4 & 0 \\ 0 & 0 & K-8 & 0 \end{array}\right]$$

If system has unknown solⁿ

$\rho(A) = \rho(A|B) =$ no. of unknown.

$K \neq 8$

$z = 0$

$y + 4z = 0 \Rightarrow y = 0$

$x + y + 3z = 0 \Rightarrow z = 0$

We get trival solⁿ.

If $k = 8$.

$\rho(A) = \rho(A|B) <$ no. of unknown $= 3$

$y + 4z = 0$

$x + y + 3z = 0$

Arb. constant $= 3 - 2 = 1$

~~$y = k$.~~ Take $y = k'$

~~$k = 4z$~~ $k' = -4z$

~~$z = 2$~~ $x - 4z + 3z = 0$

$x = z$

∴ We get non-trival solⁿ.

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$2x+y-z=0$
$x-y+2z=0$
$-x+2y-kz=0$

Find value of k so that trivial solⁿ.

Ans $\left[\begin{array}{ccc|c} 2 & 1 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ -1 & 2 & -k & 0 \end{array}\right]$

$\frac{1}{2}R_1 \sim \left[\begin{array}{ccc|c} 1 & 1/2 & -1/2 & 0 \\ 1 & -1 & 2 & 0 \\ -1 & 2 & -k & 0 \end{array}\right]$

$\begin{array}{c} R_2-R_1 \\ \sim \\ R_3+R_1 \end{array} \left[\begin{array}{ccc|c} \boxed{1} & 1/2 & -1/2 & 0 \\ 0 & -3/2 & 5/2 & 0 \\ 0 & 5/2 & -k-1/2 & 0 \end{array}\right]$

$-\frac{2}{3}R_2 \sim \left[\begin{array}{ccc|c} \boxed{1} & 1/2 & -1/2 & 0 \\ 0 & \boxed{1} & -5/3 & 0 \\ 0 & 5/2 & -k-1/2 & 0 \end{array}\right]$

$-\frac{2}{5}R_2 \sim \left[\begin{array}{ccc|c} \boxed{1} & 1/2 & -1/2 & 0 \\ 0 & \boxed{1} & -5/3 & 0 \\ 0 & 0 & -k+1/6 & 0 \end{array}\right]$

$\rho(A) = \rho(A|B) =$ no. of unkown.

$k = 1/6.$

if $k = 1/6$

$x + \frac{1}{2}y - \frac{1}{2}z = 0.$

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Here we have homogeneous system so, it is always consistent.

Our given ~~has~~ is get trivial sol^n i.e, $x=y=z=0$ which is an unique sol^n.

$\therefore R(A) = R(A|B) = 3$

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 2 \\ -1 & 2 & -k \end{bmatrix}$$

$R(A) = 3 \quad \therefore |A| \neq 0$

$2(k-4) - 1(-k+2) - 1(2-1) \neq 0$

$2k - 8 + k - 2 - 1 \neq 0$

$3k - 11 \neq 0$

$k \neq \frac{11}{3}$.

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