II Infinite Series and Complex numbers

Classwork Examples

2.1	Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test.

Test the convergence of the following series

1	$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\frac{1}{7\cdot 9}+\cdots+\frac{1}{(2n-1)\cdot(2n+1)}+\cdots$ **C.W.** **Answer: convergent.**
2	$1+2+3+\cdots+n+\cdots$ **C.W.** **Answer: divergent.**
3	$1+4+9+16+\cdots$ **C.W.** **Answer: divergent.**
4	$1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\cdots$ **C.W.** **Answer: convergent.**

1	Prove that the series $\sum_{n=1}^{\infty} n\cdot\sin\frac{1}{n}$ is divergent. **C.W.**

Test the convergence of the following series

1	$\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$. **C.W.** **Answer: convergent.**
2	$\sum_{n=0}^{\infty}\left(\frac{9}{8}\right)^n$. **C.W.** **Answer: divergent.**
3	$\sum_{n=1}^{\infty}\frac{5^n-1}{6^n}$. **C.W.**

	Answer: convergent.
4	The water treatment plant removes one mth of the impurity in the first stage. In each succeeding stage, the amount of impurity removed is one-mth of the removed in the preceding stage. Show that if $m = 2$, the water can be made as pure as you like, but if $m = 3$, at least one-half of the impurity will remain no matter how many stages are used. **C.W.**

Test the convergence of the following series:

1	$\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^3$. **C.W.** **Answer: convergent.**
2	$\sum_{n=1}^{\infty}\sqrt{n}$. **C.W.** **Answer: divergent.**

Test the convergence of the following series:

1	$\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+\cdots$. **C.W.** **Answer: convergent.**
2	$\sum_{n=1}^{\infty}\left(\sqrt{n^3+1}-\sqrt{n^3}\right)$. **C.W.** **Answer: convergent.**
3	$\frac{2}{1^p}+\frac{3}{2^p}+\frac{4}{3^p}+\cdots$ **C.W.** **Answer:** convergent if $p > 2$ and divergent if $p \leq 2$.
4	$\frac{1}{2.4}+\frac{1}{4.6}+\frac{1}{6.8}+\cdots$. **H.W.** **Answer: convergent.**
5	$\frac{2}{1}+\frac{3}{4}+\frac{4}{9}+\cdots+\frac{n+1}{n^2}+\ldots$. **H.W.** **Answer: divergent.**

Test the convergence of the following series:

1	$\sum_{n=1}^{\infty}\frac{n^2+1}{5^n}$. **C.W.** **Answer: convergent.**
2	$\sum_{n=1}^{\infty}\frac{3^n n!}{n^n}$. **C.W.** **Answer: divergent.**
3	$\frac{1}{10}+\frac{2!}{10^2}+\frac{3!}{10^3}+\cdots$. **C.W.** **Answer: divergent.**
4	$\sum_{n=1}^{\infty}\frac{n\cdot 2^n(n+1)!}{3^n n!}$. **H.W.** **Answer: convergent.**
5	$\sum_{n=1}^{\infty}\frac{n}{e^{-n}}$. **H.W.** **Answer: divergent.**

Test the convergence of the following series:

1	$\sum_{n=1}^{\infty}\frac{1}{\left(1+\frac{1}{n}\right)^{n^2}}$. **C.W.** **Answer: convergent.**
2	$\sum_{n=1}^{\infty}\left(\frac{\log n}{1000}\right)^n$. **C.W.** **Answer: divergent.**
3	$\sum_{n=1}^{\infty}\frac{n^{10}}{10^n}$. **H.W.** **Answer: convergent.**
4	$\sum\left(\frac{n}{n+1}\right)^{n^2}$. **H.W.** **Answer: convergent.**
5	$\left(\frac{2^2}{1^2}-\frac{2}{1}\right)^{-1}+\left(\frac{3^3}{2^3}-\frac{3}{2}\right)^{-2}+\left(\frac{4^4}{3^4}-\frac{4}{3}\right)^{-3}+\cdots$. **H.W.** **Answer: convergent.**

Test the convergence of the following series:

1	Show that the series $\frac{1}{2^3}-(1+2)\frac{1}{3^3}+(1+2+3)\frac{1}{4^3}-\cdots$ is convergent. **C.W.**
2	$\sum_{n=1}^{\infty}(-1)^{n+1}\left(\sqrt{n+1}-\sqrt{n}\right)$. **C.W.** **Answer: convergent.**
3	$1-\frac{1}{2!}+\frac{1}{3!}-\frac{1}{4!}+\ \dots$. **H.W.** **Answer: convergent.**

2.2	Complex numbers and their geometric representation

1	Find the real and imaginary part of the complex numbers (a) $2+i$ (b) $i-1$ (c) -3. **C.W.** **Answer:** (a) $Re(z)=2$ and $Im(z)=1$ (b) $Re(z)=-1$ and $Im(z)=1$ (c) $Re(z)=-3$ and $Im(z)=0$.
2	The total impedance Z of a circuit containing an inductor with inductance L and a resistor with resistance R in series is given by $Z=R+i\ 2\pi fL$, where $f=$ frequency. If $Z=(50+200i)\Omega$ and $f=50$ Hz, find R and L. **C.W.** **Answer:** $R=50\ \Omega, \mathrm{L}=\frac{2}{\pi}\ H\ (henry)$.
3	Sketch the following complex numbers. **C.W.** (a) $z_1=3+2i$ (b) $z_2=3-2i$ (c) $z_3=-3-2i$ (d) $z_4=-3+2i$
4	If $w_1=-2+2i$, $w_2=1-\frac{\sqrt{3}}{2}i$ and $w_3=4-6i$. Find (a) $\mathrm{w}_1-\mathrm{w}_2$ (b) w_1+w_3 **C.W.** **Answer:** (a) $\mathrm{w}_1-\mathrm{w}_2=-3+\left(\frac{4+\sqrt{3}}{2}\right)i$ (b) $w_1+w_3=2-4i$.
5	Solve for x and y if $3x+4i=(2y+x)+ix$. **C.W.** **Answer:** $x=y=4$.
6	Given $w_1=-2+2i$ and $w_2=4-6i$. Find (a) $w_1\cdot w_2$ (b) $-3w_1$. **C.W.** **Answer:** (a) $w_1\cdot w_2=4+20i$ (b) $-3w_1=6-6i$.

7	Given that $z_1 = 1 - 2i,\ z_2 = -3 + 4i$. Find $\frac{z_1}{z_2}$, and express it in the form $a + ib$. **C.W.** **Answer:** $\frac{z_1}{z_2} = \frac{-11+2i}{25}$.

2.3	Complex numbers in polar and exponential forms

1	Represent following complex numbers in polar form, with the principal argument. (a) $-4 + 4i$ (b) $-25i$ **C.W.** **Answer:** (a) $4\sqrt{2}\left[cos\left(\frac{3\pi}{4}\right) + isin\left(\frac{3\pi}{4}\right)\right]$ (b) $25\left[cos\left(\frac{\pi}{2}\right) - isin\left(\frac{\pi}{2}\right)\right]$.
2	Find the modulus and argument of the following complex numbers. (a) $\frac{1+i}{2i-1}$ (b) $-5i$ **C.W.** **Answer:** (a) $\sqrt{\frac{2}{5}}\ , -tan^{-1} 3$ (b) $5\ , -\frac{\pi}{2}$.
3	Express $z = -2 - \sqrt{3}i$ in polar form. **C.W.** **Answer:** $\sqrt{7}[cos\,(-\pi + \alpha) + isin(-\pi + \alpha)]$, where $\alpha = tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$.
4	Find the $Arg\ (z), arg\ (z)$ and convert into polar form, if (a) $z = -1 - i$ (b) $z = 1 + i$ (c) $z = -2 - 2\sqrt{3}\, i$. **C.W.** **Answer:** (a) $Arg(z) = -\frac{3\pi}{4}, arg(z) = -\frac{3\pi}{4} \pm 2k\pi, k = 0,1,2, \ldots, \sqrt{2}\left[cos\left(\frac{3\pi}{4}\right) - isin\left(\frac{3\pi}{4}\right)\right]$ (b) $Arg(z) = \frac{\pi}{4}, arg(z) = \frac{\pi}{4} \pm 2k\pi, k = 0,1,2, \ldots, \sqrt{2}\left[cos\left(\frac{\pi}{4}\right) + isin\left(\frac{\pi}{4}\right)\right]$ (c) $rg(z) = \frac{-2\pi}{3}, arg(z) = \frac{-2\pi}{3} \pm 2k\pi, k = 0,1,2, \ldots, 4\left[cos\left(\frac{2\pi}{3}\right) - isin\left(\frac{2\pi}{3}\right)\right]$.
5	Find $mod(z), Arg(z)$ and $arg(z)$ for (a) $z = -\frac{2}{1+\sqrt{3}i}$ (b) $z = -8i$. **C.W.** **Answer:** (a) $mod(z) = 1, Arg(z) = \frac{2\pi}{3}, arg(z) = \frac{2\pi}{3} \pm 2k\pi, k = 0,1,2, \ldots$ (b) $mod(z) = 8, Arg(z) = -\frac{\pi}{2}, arg(z) = -\frac{\pi}{2} \pm 2k\pi, k = 0,1,2, \ldots$.
6	Evaluate $(1 + i)^{100} + (1 - i)^{100}$ **Answer:**$\mathbf{-2^{51}}$

2.4	De Moivre's theorem and its applications

1	Prove that $(1+cos\theta+isin\theta)^n+(1+cos\theta-isin\theta)^n=2^{n+1}cos^n\left(\frac{\theta}{2}\right)cos\left(\frac{n\theta}{2}\right)$, where n is an integer. **C.W.**
2	Evaluate $\left(\frac{1+sin\alpha+icos\alpha}{1+sin\alpha-icos\alpha}\right)^n$, where n is an integer. **C.W.** **Answer:** $\cos\ n\left(\frac{\pi}{2}-\alpha\right)+isin\,n\left(\frac{\pi}{2}-\alpha\right)$.
3	Let n be a positive integer. Then prove that $(1+i)^n+(1-i)^n=2^{\frac{n+2}{2}}cos\left(\frac{n\pi}{4}\right)$. **C.W.**
4	Simplify $\frac{(cos3\theta+isin3\theta)^4(cos4\theta-isin4\theta)^5}{(cos4\theta+isin4\theta)^3(cos5\theta+isin5\theta)^{-4}}$. **C.W.** **Answer:** 1.
5	Find all the values of $\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{\frac{3}{4}}$. **C.W.** **Answer:** $\cos\left(\frac{(2n+1)\pi}{4}\right)+i\sin\left(\frac{(2n+1)\pi}{4}\right), n=0,1,2,3.$
7	Solve $x^5=\ 1+i$ and find the continued product of the roots. **C.W.** **Answer:** $2^{\frac{1}{10}}\left[cos\frac{1}{5}\left(2n\pi+\frac{\pi}{4}\right)+isin\frac{1}{5}\left(2n\pi+\frac{\pi}{4}\right)\right]$, where $n=0,1,2,3,4$, and 1+i.
6	Find and graph all sixth roots of unity in the complex plane. **C.W.** **Answer:** $1,-1,w,w^2,w^4,w^5$, where $w=e^{\frac{i\pi}{3}}$
7	If ω is a cube root of unity, then prove that $(1-\omega)^6=-27$. **H.W.**
8	Solve $x^5-1=0$. **H.W.** **Answer:** $cos\left(\frac{2n\pi}{5}\right)+isin\left(\frac{2n\pi}{5}\right), n=0,1,2,3,4.$
9	Find all the values of $(-1)^{\frac{1}{6}}$. **H.W.** **Answer:** $\pm i,\frac{1}{2}(\sqrt{3}\pm i),\frac{1}{2}(-\sqrt{3}\pm i)$.

2.5	Exponential, Logarithmic, Trigonometric and Hyperbolic functions.

1	Find the value of $Log(1+i)+Log(1-i)$. **C.W.** **Answer:** $og2+4n\pi i$.
2	Find the values of $\log(6+8i)$. **C.W.** **Answer:** $\log(6+8i)=\log(10)+i\left[tan^{-1}\left(\frac{4}{3}\right)+2n\pi\right], n\in\mathbb{Z}$.
3	Show that $i^i=e^{-\frac{(4n+1)\pi}{2}}$. **C.W.**
4	Evaluate $(1+i)^{(1+i)}$. **C.W.** **Answer:** $(1+i)^{(1+i)}=e^{A+iB}=e^{A}e^{iB}=e^{A}(cosB+isinB)$, where $A=\frac{1}{2}\log 2-\left(2n\pi+\frac{\pi}{4}\right)$ and $B=\frac{1}{2}\log 2+\left(2n\pi+\frac{\pi}{4}\right)$.
5	Prove that $\lvert cos\,z\rvert^2=cos^2x+sinh^2\,y$ and $\lvert sin\,z\rvert^2=sin^2x+sinh^2y$, where $z=x+iy$. **C.W.**
6	If $cosh(u+iv)=x+iy$, then prove that $\frac{x^2}{cosh^2u}+\frac{y^2}{sinh^2u}=1$ and $\frac{x^2}{cos^2v}-\frac{y^2}{sin^2v}=1$. **C.W.**
7	Prove that $(coshx-sinhx)^n=cosh\,nx-sinh\,nx$. **C.W.**
8	Separate the real and imaginary parts of (a) $sinh(x+iy)$ **C.W.** **Answer:** (a) $sinh(x+iy)=\sinh x\,cosy+i\,cosh\,xsiny$ (b) $\tan(x+iy)=\frac{sin2x}{cos2x+cosh2x}+i\frac{sinh2y}{cos2x+cosh2y}$.