I Higher order derivatives and applications

1.1 Lagrange's Mean Value Theorem, Local Maxima and Minima of function of one variable

Continuity of Function: Consider a function $f: \mathbb{R} \to \mathbb{R}$ and $a \in \mathbb{R}$.

1. If $\lim_{x\to a^-} f(x) = f(a)$, then we say that f is left continuous at a.
2. If $\lim_{x\to a^+} f(x) = f(a)$, then we say that f is right continuous at a.
3. If $\lim_{x\to a} f(x) = f(a)$, then we say that f is continuous at a.

Note that if f is continuous at a, then it is left as well as right continuous at a.

If f is not continuous at a, then f is said to be **discontinuous at $x = a$,** and a is called a point of discontinuity.

Differentiability of Functions: Consider function $f(x)$ defined on a closed interval $[a, b]$. Let $c \in (a, b)$. Then the function $f(x)$ is said to be differentiable at $x = c$, if following is $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists. The limit is called the derivative of $f(x)$ with respect to x at c and it is denoted by $\frac{df}{dx}$ or $f'(x)$ at $x = c$ or $\left(\frac{df}{dx}\right)_{x=c}$ or $f'(c)$.

Remark: $f'(a)$ is the slope of the tangent line of the curve $y = f(x)$ at $x = a$.

The derivative of $\frac{df}{dx}$ w.r.t. x is called the second order derivative of f w.r.t x and is denoted by $\frac{d^2f}{dx^2}$.

Note:

1. If $f: (a, b) \to R$ is differentiable at $x \in (a, b)$, then it is continuous at x.
2. If $f: (a, b) \to R$ is continuous at $x \in (a, b)$, then it may not be differentiable at x.

   For example: $f(x) = |x|$ is continuous on $\mathbb{R}$ but it is not differentiable at $x = 0$.

$$\left(\begin{array}{l} \because \lim_{x\to 0^+} \dfrac{f(x)-f(0)}{x} = 1 \\ \quad while \\ \quad \lim_{x\to 0^-} \dfrac{f(x)-f(0)}{x} = -1 \end{array} \right)$$

Formulas for differentiation:

1. $\frac{d(c)}{dx} = 0$, where c is any constant.
2. $\frac{d(x^n)}{dx} = nx^{n-1}$, where n is any rational number.
3. $\frac{d(e^x)}{dx} = e^x$
4. $\frac{d(a^x)}{dx} = a^x \log_e a$, where $a > 0$.
5. $\frac{d(\log_e x)}{dx} = \frac{1}{x}$.
6. $\frac{d(sinx)}{dx} = cosx$
7. $\frac{d(cosx)}{dx} = -sinx$
8. $\frac{d(tanx)}{dx} = \sec^2 x$
9. $\frac{d(cotx)}{dx} = -cosec^2x$
10. $\frac{d(secx)}{dx} = secx\ tanx$
11. $\frac{d(cosecx)}{dx} = -cosecx\ cotx$
12. $\frac{d(sinhx)}{dx} = coshx$
13. $\frac{d(coshx)}{dx} = sinhx$
14. $\frac{d(\sin^{-1} x)}{dx} = \frac{1}{\sqrt{1-x^2}}$
15. $\frac{d(cos^{-1} x)}{dx} = -\frac{1}{\sqrt{1-x^2}}$
16. $\frac{d(\tan^{-1} x)}{dx} = \frac{1}{1+x^2}$
17. $\frac{d(cot^{-1} x)}{dx} = -\frac{1}{1+x^2}$
18. $\frac{d(\sec^{-1} x)}{dx} = \frac{1}{x\sqrt{x^2-1}}$
19. $\frac{d(cosec^{-1} x)}{dx} = \frac{-1}{x\sqrt{x^2-1}}$

Rules of differentiation: Suppose u and v are functions of x.

1. $\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$
2. $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
3. $\frac{d\left(\frac{u}{v}\right)}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}, v \neq 0$

Derivative of the function of a function (Derivative of composition of functions):

If y is function of u and u is a function of x, then $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$. This rule is known as the chain rule.

Note: If $y = f(t)$ and $x = g(t)$, where t is a parameter, then $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{f'(t)}{g'(t)}$, where $g'(t) \neq 0$.

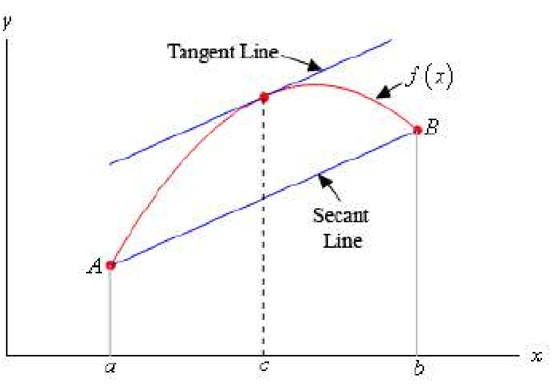
Tutorial:

1	Evaluate: $\lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{x^2}$. **Answer**: $\frac{1}{4\sqrt{2}}$.
2	Evaluate: $\lim_{x\to a^+} \frac{\tan^{-1} x-\tan^{-1} a}{tan\, x-\tan a}$. **Answer**: $\frac{\cos^2 a}{1+a^2}$.
3	Discuss continuity of the function $f(x) = \begin{cases} \frac{\lvert x\rvert}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$ at $x = 0$. **Answer**: f is not continuous.
4	Let $f(x) = \begin{cases} ax + b, & x > 1 \\ 6, & x = 1 \\ 5ax - b, & x < 1 \end{cases}$. If f is continuous at $x = 1$, find a and b. **Answer**: $a = 2, b = 4$
5	Find $\frac{dy}{dx}$ for $y = x^x + (\sin x)^x$. **Answer**: $x^x(1 + \log x) + (\sin x)^x(x \cot x + logsin\, x)$.
6	If $x^y = e^{x-y}$, prove that $\frac{dy}{dx} = \frac{logx}{(logex)^2}$.
7	Find $\frac{d^2y}{dx^2}$ for $x = acos^3 t, y = bsin^3 t$. **Answer**: $\frac{b}{3a^2} \cdot \frac{1}{\cos^4 t \sin t}$.

Mean Value Theorem:

Suppose that f is defined and continuous on a closed interval $[a, b]$, and suppose that derivative of f exists on the open interval (a, b). Then there exists a point c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

The Mean value theorem says that there is a tangent line to the curve whose slope is same as the slop of the line joing the points $A(a, f(a))$ and $B(b, f(b))$ in the figure.



Tutorial:

1	Check whether the Mean Value Theorem can be applicable to the function $f(x)=3x^{\frac{2}{3}}-2x$ on the closed interval $[0,1]$. If so, find a value of c which satisfies the Mean value theorem in $(0,1)$. **Solution:** Here $f(x)=3x^{\frac{2}{3}}-2x$ is defined for all values of x in the closed interval $[0,1]$. Clearly $f(x)=3x^{\frac{2}{3}}-2x$ is continuous at all values of x in the interval $[0,1]$. We compute the derivative of the function f, which is given by $f'(x)=3\left(\frac{2}{3}\right)x^{-\frac{1}{3}}-2=\frac{2}{\sqrt[3]{x}}-2$ $f'(x)=\frac{2}{\sqrt[3]{x}}-2$ is defined for all values in $(0,1)$. Thus, f is differentiable at each point of x in the interval $(0,1)$. Thus, $f(x)$ satisfies all condition of Mean Value Theorem. From Mean Value theorem $f'(c)=\frac{f(b)-f(a)}{b-a}=\frac{f(1)-f(0)}{1-0}$ $\left(\begin{array}{l}\because\ f(a)=f(0)=3(0)^{2/3}-2(0)=0-0=0\\ \quad f(b)=f(1)=3(1)^{2/3}-2(1)=3-2=1\end{array}\right)$ and for the derivative $f'(x)=\frac{2}{\sqrt[3]{x}}-2$, $\therefore\ f'(c)=\frac{2}{\sqrt[3]{c}}-2$. $\therefore 1=\frac{2}{\sqrt[3]{c}}-2$ Therefore $c=\frac{8}{27}\in(0,1)$.
2	Check whether the Mean Value Theorem can be applied to the function $\tan^{-1}x$ on the closed interval $[-1,1]$. If so, find a value of c which satisfies the Mean value theorem in $(-1,1)$. **Answer:** $\pm\sqrt{\frac{4}{\pi}-1}$.
3	Check whether the Mean Value Theorem can be applied to the function x^3+12x^2+7x on the closed interval $[-4,4]$. If so, find a value of c which satisfies the Mean value theorem in $(-4,4)$. **Answer:** 0.62.

Local Maxima and Minima:

Maximum value of a function: If the value of a function $f(x)$ at $x = a$ is maximum in the small interval $(a - h, a + h)$ then we say that $f(x)$ is maximum at $x = a$.

The following two conditions must be satisfied for a function $f(x)$ to be maximum at $x = a$.

1. $f'(a) = 0$ (Necessary condition)
2. $f''(a) < 0$ (Sufficient condition).

Minimum value of a function: If the value of a function $f(x)$ at $x = a$ is minimum in the small interval $(a - h, a + h)$ then we say that $f(x)$ is minimum at $x = a$.

The following two conditions must be satisfied for a function f(x) to be minimum at x=a.

1. $f'(a) = 0$ (Necessary condition)
2. $f''(a) > 0$ (Sufficient condition).

Stationary points:

The point at which a function obtains its maximum or minimum values is called a stationary points. $f'(x) = 0$ is a necessary condition for obtaining stationary points.

Working rules for finding Maxima and Minima:

1. For a given function $y = f(x)$, obtain $\dfrac{dy}{dx}$ or $f'(x)$.
2. Take $f'(x) = 0$ and solve this equation to find the roots. Let the roots be $a, b, c, ...$
3. Obtain the second derivative $f''(x)$ of f.
4. Substitute the roots $a, b, c, ...$ in $f''(x)$ one by one. Suppose $x = a$ is substitute in $f''(x)$.
   i. If $f''(a) < 0$, then f is maximum at $x = a$ and the maximum value of $f(x)$ is $f(a)$.
   ii. If $f''(a) > 0$, then f is minimum at $x = a$ and the minimum value of $f(x)$ is $f(a)$.
   iii. If $f''(a) = 0$, then we cannot draw any conclusion about the maximum and minimum value of f at $x = a$.

eg. Find the extreme values of the function $f(x) = x^5 - 5x^4 + 5x^3 - 1$.

Solution:

$f(x) = x^5 - 5x^4 + 5x^3 - 1,\ \ f'(x) = 5x^2(x-1)(x-3),\ \ f''(x) = 10x(2x^2 - 6x + 3)$

For maxima or minima, $f'(x) = 0 \ \therefore 5x^2(x-1)(x-3) = 0 \Rightarrow x = 0, 1, 3.$

When $x = 0$, $f''(0) = 0$. Hence we cannot draw any conclusion about maxima or minima of $f(x)$.

When $x = 1$, $f''(1) = -10 < 0$. Therefore $f(x)$ is maximum at $x = 1$ and its maximum value is 0.

When $x = 3$, $f''(3) = 90 > 0$. Therefore $f(x)$ is minimum at $x = 3$ and its minimum value is -28.

Note: These are local maxima/local minima.

Tutorial:

1	Find the extreme value of the function $y = \left(\frac{1}{x}\right)^x, x > 0$. **Solution** : Here $y = \left(\frac{1}{x}\right)^x, x > 0$ $\therefore \log y = x \log \frac{1}{x} = -x \log x$ Differentiating w.r.t x, we get $\therefore \frac{1}{y}\frac{dy}{dx} = -x\frac{1}{x} - \log x = -(1 + \log x)$ $\therefore \frac{dy}{dx} = -y(1 + \log x) = -\left(\frac{1}{x}\right)^x (1 + \log x)$. Now taking $\frac{dy}{dx} = 0$, we have $\left(\frac{1}{x}\right)^x (1 + \log x) = 0.$ $\therefore (1 + \log x) = 0$ i.e $\log x = -1$. $\therefore x = \frac{1}{e}$. Now $\therefore \frac{d^2y}{dx^2} = -\left\{\frac{dy}{dx}(1 + \log x) + y \cdot \frac{1}{x}\right\}$ At $x = \frac{1}{e}, \frac{d^2y}{dx^2} = -\left\{0 + (e)^{\frac{1}{e}} \cdot e\right\} = -(e)^{\frac{1}{e}+1} < 0$.

	Therefore y is maximum at $x=\frac{1}{e}$ and the maximum value is $(e)^{\frac{1}{e}}$.
2	Find the extreme values of the function $f(x)=-2x^2+4x+1$. **Answer:** 3.
3	Find the extreme values of the function $y=\sin(cos2x)$. **Answer:** $\sin(1)$ and $-\sin(1)$.

1.2	Successive differentiation: n[th] derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic

Successive Differentiation:

Successive Differentiation is the process of differentiating a given function successively times and the results of such differentiation are called successive derivatives. The n[th] order derivative of function $y=f(x)$ is denoted by y_n or $y^{(n)}$ or $f_n(x)$ or $f^{(n)}(x)$.

Some standard results on n[th] order derivative:

1	If $y=e^{ax+b}$, then $y_n=a^n e^{ax+b}$. **Proof:** Here, we have $y=e^{ax+b}$. Next, taking derivative of y with respect to x gives $y_1=ae^{ax+b}$. Taking derivative of y_1 with respect to x gives $y_2=a^2e^{ax+b}$. Similarly taking n[th] order derivative of y with respect to x gives $y_n=a^n e^{ax+b}$.
2	If $y=a^{bx}$, then $y_n=b^n(loga)^n a^{bx}$. **Proof:** Here, we have $y=a^{bx}$. Next, taking derivative of y with respect to x gives $y_1=b\,(loga)\,a^{bx}$. Taking derivative of y_1 with respect to x gives $y_2=b^2(loga)^2a^{bx}$. Similarly taking n[th] order derivative of y with respect to x gives $y_n=b^n(loga)^n a^{bx}$.
3	If $y=\sin(ax+b)$, then $y_n=a^n sin\left(ax+b+\frac{n\pi}{2}\right)$. **Proof:** Here, we have $\mathrm{y}=\sin(ax+b)$. Next, taking derivative of y with respect to x gives $y_1=a\,cos(ax+b)=a\,sin\left(ax+b+\frac{\pi}{2}\right)$. Taking derivative of y_1 with respect to x gives

	$y_2 = a^2 cos\left(ax + b + \frac{\pi}{2}\right) = a^2 sin\left(ax + b + \frac{2\pi}{2}\right)$. Similarly taking nth order derivative of y with respect to x gives $y_n = a^n sin\left(ax + b + \frac{n\pi}{2}\right)$.
4	If $y = cos(ax + b)$, then $y_n = a^n cos\left(ax + b + \frac{n\pi}{2}\right)$. **Proof:** Here, we have $y = \cos(ax + b)$. Next, taking derivative of y with respect to x gives $y_1 = -a\, sin(ax + b) = a\, cos\left(ax + b + \frac{\pi}{2}\right)$. Taking derivative of y_1 with respect to x gives $y_2 = -a^2 sin\left(ax + b + \frac{\pi}{2}\right) = a^2 cos\left(ax + b + \frac{2\pi}{2}\right)$. Similarly taking nth order derivative of y with respect to x gives $y_n = a^n cos\left(ax + b + \frac{n\pi}{2}\right)$.
5	If $y = e^{ax}\sin(bx + c)$, then $y_n = r^n e^{ax} sin(bx + c + n\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$. **Proof:** Here, we have $y = e^{ax}\sin(bx + c)$. Next, taking derivative of y with respect to x gives $y_1 = ae^{ax}\sin(bx + c) + be^{ax}\cos(bx + c)$ $\quad = e^{ax}[a\sin(bx + c) + b\cos(bx + c)]$. Put $a = rcos\theta$ and $b = rsin\theta$, so that $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$ $\therefore y_1 = e^{ax}[rcos\theta\sin(bx + c) + rsin\theta\cos(bx + c)]$ $\quad = re^{ax}[cos\theta\sin(bx + c) + sin\theta\cos(bx + c)]$ $\quad = re^{ax}\sin(bx + c + \theta)$. $\quad (\because \sin(A + B) = sinAcosB + cosAsinB)$ Taking derivative of y_1 with respect to x gives $y_2 = rae^{ax}\sin(bx + c + \theta) + rbe^{ax}\cos(bx + c + \theta)$ $\quad = re^{ax}[a\sin(bx + c + \theta) + b\cos(bx + c + \theta)]$ $(\because a = rcos\theta$ and $b = rsin\theta)$ $\quad = re^{ax}[rcos\theta\sin(bx + c + \theta) + rsin\theta\cos(bx + c + \theta)]$ $\quad = r^2e^{ax}[cos\theta\sin(bx + c + \theta) + sin\theta\cos(bx + c + \theta)]$ $\quad = r^2e^{ax}\sin(bx + c + 2\theta)$. $(\because \sin(A + B) = sinA\, cosB + cosA\, sinB)$ Similarly, we have $y_3 = r^3e^{ax}sin(bx + c + 3\theta)$,

	$y_4 = r^4 e^{ax} sin(bx + c + 4\theta)$. In general, the n[th] order derivative of y is given by $y_n = r^n e^{ax} sin(bx + c + n\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$.
6	If $y = e^{ax}\cos(bx + c)$, then $y_n = r^n e^{ax} cos(bx + c + n\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$. **Proof:** Here, we have y $= e^{ax}\cos(bx + c)$. Next, taking derivative of y with respect to x gives $y_1 = ae^{ax}\cos(bx + c) - be^{ax}\sin(bx + c)$ $= e^{ax}[a\cos(bx + c) - b\sin(bx + c)]$. Put $a = rcos\theta$ and $b = rsin\theta$, so that $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$ $\therefore y_1 = e^{ax}[rcos\theta\cos(bx + c) - rsin\theta\sin(bx + c)]$ $= re^{ax}[cos\theta\cos(bx + c) - sin\theta\sin(bx + c)]$ $= re^{ax}\cos(bx + c + \theta)$. $(\because \cos(A + B) = cosAcosB - sinAsinB)$ Taking derivative of y_1 with respect to x gives $y_2 = rae^{ax}\cos(bx + c + \theta) - rbe^{ax}\sin(bx + c + \theta)$ $= re^{ax}[a\cos(bx + c + \theta) - b\sin(bx + c + \theta)]$ $= re^{ax}[rcos\theta\cos(bx + c + \theta) - rsin\theta\sin(bx + c + \theta)]$ ($\because a = rcos\theta$ and $b = rsin\theta$) $= r^2 e^{ax}\cos(bx + c + 2\theta)$. Similarly, we have $y_3 = r^3 e^{ax} cos(bx + c + 3\theta)$, $y_4 = r^4 e^{ax} cos(bx + c + 4\theta)$. In general, the n[th] order derivative of y is given by $y_n = r^n e^{ax} cos(bx + c + n\theta)$, where $r = \sqrt{a^2 + b^2}$ and $\theta = tan^{-1}\left(\frac{b}{a}\right)$.
7	If $y = log(ax + b)$, then $y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$. **Proof:** Here, we have y $= log(ax + b)$. Next, taking derivative of y with respect to x gives $y_1 = \frac{a}{ax+b}$. Taking derivative of y_1 with respect to x gives

	$y_2 = \frac{a^2(-1)}{(ax+b)^2}$. Taking derivative of y_2 with respect to x gives $y_3 = \frac{a^3(-1)(-2)}{(ax+b)^3} = \frac{a^3(-1)^2(1\times2)}{(ax+b)^3} = \frac{a^3(-1)^2(2!)}{(ax+b)^3}$. Taking derivative of y_2 with respect to x gives $y_3 = \frac{a^3(-1)(-2)}{(ax+b)^3} = \frac{a^3(-1)^2(1\times2)}{(ax+b)^3} = \frac{a^3(-1)^2(2!)}{(ax+b)^3}$. Taking derivative of y_3 with respect to x gives $y_4 = \frac{a^4(-1)(-2)(-3)}{(ax+b)^4} = \frac{a^4(-1)^3(1\times2\times3)}{(ax+b)^4} = \frac{a^4(-1)^3(3!)}{(ax+b)^4}$. Similarly, we have $y_5 = \frac{a^5(-1)^4(4!)}{(ax+b)^5}$. In general, the n[th] order derivative of y is given by $y_n = \frac{(-1)^{n-1}(n-1)!a^n}{(ax+b)^n}$.
8	If $y = (ax+b)^m$, then $y_n = \begin{cases} \frac{m!}{(m-n)!} a^n (ax+b)^{m-n}, & If\ \ m > n > 0 \\ 0, & If\ n > m > 0 \\ n!a^n, & If\ \ m = n \\ \frac{(-1)^n n!a^n}{(ax+b)^{n+1}}, & If\ m = -1. \end{cases}$ **Proof:** Here, we have y $= (ax+b)^m$. **Case 1:** $m > n > 0$ Next, taking derivative of y with respect to x gives $y_1 = am(ax+b)^{m-1}$. Taking derivative of y_1 with respect to x gives $y_2 = a^2m(m-1)(ax+b)^{m-2}$. Similarly, we have $y_3 = a^3m(m-1)(m-2)(ax+b)^{m-3}$, $y_4 = a^4m(m-1)(m-2)(m-3)(ax+b)^{m-4}$. In general, n[th] order derivative of y is given by $y_n = a^nm(m-1)\ldots\big(m-(n-1)\big)(ax+b)^{m-n}$ $= a^nm(m-1)\ldots(m-n+1)\times\frac{(m-n)!}{(m-n)!}(ax+b)^{m-n}$ $= \frac{m!}{(m-n)!}a^n(ax+b)^{m-n}$.

	Case 2: If $n > m > 0$, then one of the terms of $m(m-1)\dots(m-n+1)$ in y_n will be zero. Therefore, $y_n = 0$. For example, $n = 5$ and $m = 3$ gives $m(m-1)\dots(m-n+1)$ $= 3 \times 2 \times 1 \times 0 \times -1$ $=0$. **Case 3:** $m = n$ Here, we have y $= (ax+b)^n$. Next, taking derivative of y with respect to x gives $y_1 = an(ax+b)^{n-1}$. Taking derivative of y_1 with respect to x gives $y_2 = a^2 n(n-1)(ax+b)^{n-2}$. Similarly, we have $y_3 = a^3 n(n-1)(n-2)(ax+b)^{n-3}$, $y_4 = a^4 n(n-1)(n-2)(n-3)(ax+b)^{m-n}$. In general, n^th order derivative of y is given by $y_n = a^n n(n-1)\dots\big(n-(n-1)\big)(ax+b)^{n-n}$ $= a^n n(n-1)\dots(n-(n-2))\big(n-(n-1)\big)$ $= a^n n(n-1)(n-2)\dots 3 \times 2 \times 1$ $= a^n n!$. **Case 4:** $m = -1$ Here, we have y $= (ax+b)^{-1}$. Next, taking derivative of y with respect to x gives $y_1 = a(-1)(ax+b)^{-2}$. Taking derivative of y_1 with respect to x gives $y_2 = a^2(-1)(-2)(ax+b)^{-3} = a^2(-1)^2(2!)(ax+b)^{-3}$. Similarly, we have $y_3 = a^3(-1)(-2)(-3)(ax+b)^{-4} = a^3(-1)^3(3!)(ax+b)^{-4}$, $y_4 = a^4(-1)(-2)(-3)(-4)(ax+b)^{-5} = a^4(-1)^4(4!)(ax+b)^{-5}$. In general, n^th order derivative of y is given by $y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$.
9	If $y = sinhax$, then $y_n = \begin{cases} a^n sinhax, & if\ n\ is\ even \\ a^n coshax, & if\ n\ is\ odd \end{cases}$.

	Proof: Here, we have y $= sinhax$. We know that $sinhax = \frac{e^{ax}-e^{-ax}}{2}$ and $\cosh ax = \frac{e^{ax}+e^{-ax}}{2}$. Taking derivative of y with respect to x gives $y_1 = \frac{d}{dx}(sinhax) = \frac{d}{dx}\left(\frac{e^{ax}-e^{-ax}}{2}\right) = \frac{ae^{ax}-(-a)e^{-ax}}{2} = a\left(\frac{e^{ax}+e^{-ax}}{2}\right) = acoshax.$ Taking derivative of $y_1$with respect to x gives $y_2 = \frac{d}{dx}(acoshax) = a\frac{d}{dx}\left(\frac{e^{ax}+e^{-ax}}{2}\right) = a\left[\frac{ae^{ax}+(-a)e^{-ax}}{2}\right] = a^2\left(\frac{e^{ax}-e^{-ax}}{2}\right) =$ $a^2sinhax.$ Similarly, $y_3 = a^3coshax,$ $y_4 = a^4sinhax.$ Thus, In general, n[th] order derivative of y is given by $y_n = \begin{cases} a^n sinhax, \ if\ n\ is\ even \\ a^n coshax, \ if\ n\ is\ odd \end{cases}$.
10	If $y = \cosh ax$, then $y_n = \begin{cases} a^n \cosh ax\,,\ if\ n\ is\ even \\ a^n \sinh ax\,,\ if\ n\ is\ odd \end{cases}$. **Proof:** Here, we have $y = \cosh ax$. We know that $\cosh ax = \frac{e^{ax}+e^{-ax}}{2}$ and $\sinh ax = \frac{e^{ax}-e^{-ax}}{2}$. Taking derivative of y with respect to x gives $y_1 = \frac{d}{dx}(\cosh ax) = \frac{d}{dx}\left(\frac{e^{ax}+e^{-ax}}{2}\right) = \frac{ae^{ax}-ae^{-ax}}{2} = a\left(\frac{e^{ax}-e^{-ax}}{2}\right) = a\sinh ax$ Taking derivative of y_1 with respect to x gives $y_2 = \frac{d}{dx}(a\sinh ax) = a\frac{d}{dx}\left(\frac{e^{ax}-e^{-ax}}{2}\right) = a\left[\frac{ae^{ax}-(-a)e^{-ax}}{2}\right] = a^2\left(\frac{e^{ax}+e^{-ax}}{2}\right) =$ $a^2\cosh ax$ Similarly, $y_3 = a^3 \sinh ax,$ $y_4 = a^4 \cosh ax$ Thus, In general, n^{th} order derivative of y is given by $y_n = \begin{cases} a^n \cosh ax\,,\ if\ n\ is\ even \\ a^n \sinh ax\,,\ if\ n\ is\ odd \end{cases}$.

Tutorial:

1	Find the n[th] order derivative of the function $\frac{x}{(x-1)^2(x+1)}$.

	Solution: Let $y = \frac{x}{(x-1)^2(x+1)}$. Using Partial fractions, one can rewrite the function y as follows $\frac{x}{(x-1)^2(x+1)} = \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+1)}$ $\therefore x = A(x-1)(x+1) + B(x+1) + C(x-1)^2$---------(1) In (1), the substitution $x = 1$ gives $1 = 0 + 2B + 0 \Longrightarrow B = \frac{1}{2}$, In (1), the substitution $x = -1$ gives -1 $= 0 + 0 + 4C \Longrightarrow C = -\frac{1}{4}$, In (1), On Comparing coefficient of x^2 on both sides, we get $A + C = 0 \Longrightarrow A = \frac{1}{4}$. Thus, We have $$y = \frac{x}{(x-1)^2(x+1)} = \frac{1}{4(x-1)} + \frac{1}{2(x-1)^2} - \frac{1}{4(x+1)}.$$ Taking n^th order derivative of y, We have $$y_n = (-1)^n n! \left[\frac{1}{4(x-1)^{n+1}} + \frac{n+1}{2(x-1)^{n+2}} - \frac{1}{4(x+1)^{n+1}}\right]$$ using the formula of n^th order derivative of $y = \frac{1}{ax+b}$.
2	If $y = \sin(ax) + \cos(ax)$, then prove that $y_n = a^n\sqrt{1 + (-1)^n \sin(2ax)}$.
3	Find the n^th order derivative of the function $y = \cos x \cos 2x \cos 3x$. **Answer:** $y_n = \frac{1}{4}\left[2^n cos\left(2x + \frac{n\pi}{2}\right) + 4^n cos\left(4x + \frac{n\pi}{2}\right) + 6^n cos\left(6x + \frac{n\pi}{2}\right)\right]$.

1.3	Leibnitz rule for the n^th order derivatives of product of two functions

Leibnitz rule for the n^th order derivatives of product of two functions:

If u and v are functions of x such that their n^th derivatives exist, then the n^th derivative of their product is given by

$$(uv)_n = u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots + u v_n.$$

Tutorial:

1	If $y = \sin(msin^{-1}x)$, show that $(1 - x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0$. **Solution:** Here, we have $y = \sin(msin^{-1}x)$. $\therefore sin^{-1}y = msin^{-1}x$ On taking derivative to this function differentiating with respect to x gives

	$\frac{1}{\sqrt{1-y^2}}y_1 = \frac{m}{\sqrt{1-x^2}}$ $\therefore (1-x^2)y_1^2 = m^2(1-y^2)$. On differentiating again with respect to x, we get $(1-x^2)2y_1y_2 - 2xy_1^2 = -2m^2yy_1$ $\therefore (1-x^2)y_2 - xy_1 + m^2y=0$ (Dividing by $2y_1$) On differentiating n times by Leibnitz's theorem If u and v are functions of x such that their nth derivatives exist, then the nth derivative of their product is given by $(uv)_n = u_nv + \binom{n}{1}u_{n-1}v_1 + \binom{n}{2}u_{n-2}v_2 + \cdots + uv_n$, we have $\left((1-x^2)y_2\right)_n - \left(xy_1\right)_n + (m^2y)_n=0$ $\therefore (1-x^2)y_2 + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - xy_{n+1} - n(1)y_n + m^2y_n = 0$ Thus, we get the required expression in the form $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$.
2	Find the nth derivative of the function $y = x^2\sin(x-7)$. **Solution:** Here, we have $y = x^2\sin(x-7)$. Leibnitz's theorem If u and v are functions of x such that their nth derivatives exist, then the nth derivative of their product is given by $$(uv)_n = u_nv + \binom{n}{1}u_{n-1}v_1 + \binom{n}{2}u_{n-2}v_2 + \cdots + uv_n.$$ Taking $u = \sin(x-7)$ and $v = x^2$, we have $(y)_n = (x^2\sin(x-7))_n = x^2\sin\left(x-7+\frac{n\pi}{2}\right) + 2nx\sin\left(x-7+\frac{(n-1)\pi}{2}\right)$ $$+n(n-1)\sin\left(x-7+\frac{(n-2)\pi}{2}\right)$$ is the required expression.
3	Find the nth derivative of the function $y = x^4\cos(5x-4)$.
4	If $y = e^{a\sin^{-1}x}$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$.

1.4	Power series expansion of a function: Maclaurin’s and Taylor’s series expansion

In this section, we shall discuss infinite series representation of a function, especially in the form of a power series. Such series expansions are useful to approximate the function numerically by polynomials e.g. Functions such as $\sin x$, $\log x$, e^x, etc. For that we shall use Taylor’s series and Maclaurin’s series expansions in the powers of some variable.

Definition: If $c_1, c_2, c_3, \ldots$ and a are constants, then an infinite series expression of the form $\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$ is called a power series in $(x-a)$.

If we put $a = 0$ then $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots$ is called a power series in x.

Taylor’s theorem with remainder: (Only for Information)

Suppose that a function f can be differentiable $n+1$ times at each point in an interval containing the point a. Then for each x in that interval there is at least one point c in the interval such that,

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + R_n,$$

where $R_n = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$, which is called the Taylor’s theorem with Lagrange’s form of the remainder $R_n(x)$.

Note: In the Taylor’s theorem, if $R_n \to 0$ as $n \to \infty$, then it becomes a Taylor’s series.

Taylor’s series:

If $f(x)$ possess derivatives of all orders at the point a, then

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-1)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + \cdots \qquad \_\_\_\_ (i)$$

Result-1: Replacing x by $a+h$ in Taylor’s series (i), we get another form of the series.

$f(a+h) = f(a) + \frac{h^1}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^n}{n!} f^n(a) + \cdots$, which is known as Taylor’s series expansion of the function $f(x)$ in the neighborhood of the point a.

Result-2: If we put $a = 0$ in Taylor’s series (i), then we get

$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$, which known as Maclaurin’s series expansion of the function $f(x)$.

Example: Expand $e^{\sin x}$ in Maclaurin’s series.

Solution: Maclaurin’s series expansion is

$$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^n(0) + \cdots$$

We are given that $f(x) = e^{\sin x} \Rightarrow f(0) = 1$.

$\therefore \quad f'(x) = \cos x \, e^{\sin x} \quad \Rightarrow f'(0) = 1$.

$f''(x) = -\sin x \, e^{\sin x} + cos^2 x e^{\sin x} = f(x)(-\sin x + cos^2 x) \quad \Rightarrow f''(0) = 1$.

$f'''(x) = f'(x)(-\sin x + cos^2 x) + f(x)(-\cos x - \sin 2x) \quad \Rightarrow f'''(0) = 0$

$f^{iv}(x) = f''(x)(-\sin x + cos^2 x) + f'(x)(-\cos\ x + 2\cos x\ (-\sin x)) +$

$\qquad f'(x)(-cosx - \sin 2x) + f(x)(\sin x - 2\cos 2x)$

$\Rightarrow f^{iv}(0) = 1 + (-1) + (-1) + (-2) = -3$.

$f^{v}(0) = -8$ and so on.

Substituting these values in Maclaurin's series, we get

$$f(x) = e^{\sin x} = 1 + x + \frac{x^2}{2!} - 3\frac{x^4}{4!} - 8\frac{x^5}{5!} + \cdots$$

Example: Expand e^x in powers of $(x-1)$ using Taylor's series.

Solution: Taylor's series expansion is

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-1)^2}{2!} f''(a) + \cdots + \frac{(x-a)^n}{n!} f^n(a) + \cdots.$$

Here $a = 1$ and $f(x) = e^x \ \therefore\ f(1) = e$.

$\therefore \quad f'(x) = e^x \quad \Rightarrow f'(1) = e$.

$f''(x) = e^x \quad \Rightarrow f''(1) = e$.

$f'''(x) = e^x \quad \Rightarrow f'''(1) = e$ and so on.

$$f(x) = f(1) + \frac{(x-1)^1}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \cdots + \frac{(x-a)^n}{n!} f^n(1) + \ldots$$

We get, $e^x = f(1) + \frac{(x-1)^1}{1!} f'(1) + \frac{(x-1)^2}{2!} f''(1) + \cdots$

$$e^x = e\left[1 + \frac{(x-1)^1}{1!} + \frac{(x-1)^2}{2!} + \cdots\right]$$

Tutorial:

1	Expand $\sin x$ in powers of $\left(x - \frac{\pi}{2}\right)$. Find the value of $\sin 91^0$. **Answer:** $\sin 91^0 \approx 0.9998$.
2	Find the value of $\sqrt{10}$ correct to four decimal places by Taylor's series. **Answer:** $\sqrt{10} \approx 3.6123$.
3	Obtain the Maclaurin's series of $\frac{e^x}{e^x+1}$.

	Answer: $\frac{e^x}{e^x+1} = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \cdots$.

1.5	L'Hospital's rule and related applications, Indeterminate forms

Indeterminate forms, L'Hospital's rule and related applications:

Some limits can be determined the following rules:

- $q + \infty = \infty$ if $q \neq -\infty$
- $q \times \infty = \infty$ if $q > 0$
- $q \times \infty = -\infty$ if $q < 0$
- $\frac{q}{\infty} = 0$ if $q \neq \infty$ and $q \neq -\infty$
- $\infty^q = 0$ if $q < 0$
- $\infty^q = \infty$ if $q > 0$
- $q^\infty = 0$ if $0 < q < 1$
- $q^\infty = \infty$ if $q > 1$
- $q^{-\infty} = \infty$ if $0 < q < 1$
- $q^{-\infty} = 0$ if $q > 1$

Here we will discuss seven types of indeterminate forms;

(i) $\frac{0}{0}$, (ii) $\frac{\infty}{\infty}$, (iii) $0 \times \infty$, (iv) $\infty - \infty$, (v) 1^∞, (vi) 0^0, (vii) ∞^0.

If limit is of any one of the above form, then it can be evaluated by using L'Hospital's Rule.

Some Important Limits

1. $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$
2. $\lim\limits_{x \to 0} (1 + x)^{\frac{1}{x}} = e$
3. $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$
4. $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \log_e a$
5. $\lim\limits_{x \to 0} \frac{e^x - 1}{x} = \log_e e = 1$
6. $\lim\limits_{x \to 0} \frac{(1+x)^n - 1}{x} = n$
7. $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$
8. $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$

9. $\lim_{x\to 0} \cos x = 1$

10. If K is a constant function, then $\lim_{x\to a} K = K$.

(a) L' Hospital's Rule for the indeterminate form $\left(\frac{0}{0}\right)$:

If $\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$; provided the limit exists.

Example: Evaluate $\lim_{x\to 0} \frac{\log(1-x^2)}{\log \cos x}$.

Solution: Given limit is of the form $\left(\frac{0}{0}\right)$. So, using L'Hospital's rule, we get

$$\lim_{x\to 0} \frac{\log(1-x^2)}{\log \cos x} = \lim_{x\to 0} \frac{\frac{1}{1-x^2}\ (-2x)}{\frac{1}{\cos x}\ (-\sin x)}$$

$$= \lim_{x\to 0} \frac{2x}{(1-x^2)\ tan x}$$

$$= \lim_{x\to 0} \frac{2x}{(1-x^2)\ x} \quad \left(\because \lim_{x\to 0} \frac{\tan x}{x} = 1\right)$$

$$= \lim_{x\to 0} \frac{2}{(1-x^2)} = 2\ .$$

(b) L'Hospital's Rule for the indeterminate form $\left(\frac{\infty}{\infty}\right)$ or $\left(\frac{-\infty}{\infty}\right)$ or $\left(\frac{\infty}{-\infty}\right)$:

If $\lim_{x\to a} f(x) = \infty = \lim_{x\to a} g(x)$, then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$; provided the limit on the right exists.

Remark:

Suppose that functions f, g are *n* times differentiable and

$f(a) = f'(a) = f''(a) = \cdots = f^{(n-1)}(a) = 0.$

$g(a) = g'(a) = g''(a) = \cdots = g^{(n-1)}(a) = 0.$

Suppose that $g^{(n)}(a) \neq 0$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f^{(n)}(a)}{g^{(n)}(a)}$.

Example: Evaluate $\lim_{x\to 0} \frac{\log x}{\cot x}$

Solution: Given limit is of the form $\left(\frac{\infty}{\infty}\right)$. So, using L'Hospital's rule, we get

$$\lim_{x\to 0} \frac{\log x}{\cot x} = \lim_{x\to 0} \frac{\frac{1}{x}}{-cosec^2 x} \quad \text{(again it is of the form } \left(\frac{\infty}{\infty}\right)\text{)}$$

$$= \lim_{x\to 0} \frac{-sin^2 x}{x} \quad \text{which is of the form } \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-2 \sin x . \cos x}{1} = 0.$$

Tutorial:

1	Evaluate $\lim_{x\to 0}\frac{\tan x-\sin x}{x^2}$. **Answer: 0**.
2	Evaluate $\lim_{x\to\frac{\pi}{2}}\frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}$. **Answer: 0.**
3	Prove that $\lim_{x\to 0}\log_x \sin x = 1$.

Following are the indeterminate forms which can be reduced to either $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ form by simple transformations.

(a) $\mathbf{0}\cdot\infty$ form

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a}\, g(x) = \infty$, then we can write

$\lim_{x\to a} f(x)\,.\,g(x) = \lim_{x\to a}\frac{f(x)}{\frac{1}{g(x)}}$ or $\lim_{x\to a}\frac{g(x)}{\frac{1}{f(x)}}$ which can be solved using L'Hospital's rule.

Example: Evaluate $\lim_{x\to 0} x^n \log x, n > 0$

Solution: Given limit is of the form $0\cdot\infty$ **.**

$$\lim_{x\to 0} x^n \log x = \lim_{x\to 0}\frac{\log x}{x^{-n}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{x\to 0}\frac{1/x}{-nx^{-n-1}}$$

$$= \lim_{x\to 0}\left(-\frac{x^n}{n}\right) = 0$$

(b) $(\infty-\infty)$ form

To evaluate the limits of the type $\lim_{x\to a}[f(x)-g(x)]$, when $\lim_{x\to a}\, f(x) = \infty$ and $\lim_{x\to a}\, g(x) = \infty$, we reduce the expression in the form of $\left(\frac{0}{0}\right)$ or $\left(\frac{\infty}{\infty}\right)$ by taking LCM or by rearranging the terms and then apply L'Hospital's rule.

Example: Prove that $\lim_{x\to 0}\left[\frac{1}{2x}-\frac{1}{x(e^{\pi x}+1)}\right] = \frac{\pi}{4}$.

Solution: Given limit is of the form $(\infty-\infty)$.

$$\lim_{x\to 0}\left[\frac{1}{2x}-\frac{1}{x(e^{\pi x}+1)}\right] = \lim_{x\to 0}\frac{e^{\pi x}+1-2}{2x(e^{\pi x}+1)} \text{ which is indeterminate form of } \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\pi e^{\pi x}}{2[(e^{\pi x}+1)+x(\pi e^{\pi x})]}$$

$$= \frac{\pi}{2} \frac{e^{0}}{(e^{0}+1)} = \frac{\pi}{4}.$$

(c) $0^{0}, \infty^{0}, 1^{\infty}$ form (Exponential indeterminate forms)

$\lim_{x \to a} f(x)^{g(x)}$ is called an indeterminate of the type 0^{0} if $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ Or type ∞^{0} if $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = 0$

or type 1^{∞} if $\lim_{x \to a} f(x) = 1$ and $\lim_{x \to a} g(x) = \infty$.

To evaluate this kind of limit, let $l = \lim_{x \to a} f(x)^{g(x)}$.

So $\log l = \lim_{x \to a} g(x) . \log f(x)$ which is of the form $(\infty \times 0)$.

Example: Prove that $\lim_{x \to 0} (a^{x} + x)^{\frac{1}{x}} = ae$.

Solution: Given limit is of the form 1^{∞}.

$$\text{Let } l = \lim_{x \to 0} (a^{x} + x)^{\frac{1}{x}}$$

$$log\ l = \lim_{x \to 0} \frac{1}{x} . log\ (a^{x} + x)$$

$$= \lim_{x \to 0} \frac{\log(a^{x}+x)}{x} \text{ which is indeterminate form of the type } \left(\frac{0}{0}\right)$$

$$= \lim_{x \to 0} \frac{\frac{1}{(a^{x}+x)}(a^{x} \log a + 1)}{1}$$

$$= \frac{1}{(a^{0}+0)}(a^{0} \log a + 1) \quad = \frac{(\log a + \log e)}{1}.$$

$\log l = \log ae \Rightarrow l = ae$.

Tutorial:

1	Evaluate $\lim_{x \to \frac{\pi}{4}} (1 - tanx) \cdot sec2x$. $(0 \cdot \infty)$ **Answer:** 1.
2	Evaluate $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{e^{x}-1}\right]$. $(\infty - \infty)$ **Answer:** $\frac{1}{2}$.
3	Evaluate $\lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{\frac{1}{x^{2}}}$. (1^{∞}) **Answer:** $e^{\frac{1}{3}}$.

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Derivative

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Evaluate the following limits.

1) $\lim_{x\to 0} \frac{\sqrt{2}-\sqrt{1+\cos x}}{x^2}$ $\left(\frac{0}{0} \text{ form}\right)$

$\lim_{x\to 0} \frac{-(-\sin x)}{2\sqrt{1+\cos x}} \times \frac{1}{2x}$

$\lim_{x\to 0} \frac{\sin x}{4x\sqrt{1+\cos x}}$ $\left(\frac{0}{0} \text{ form}\right)$

$\lim_{x\to 0} \frac{\cos x}{4\left[\frac{x(-\sin x)}{2\sqrt{1+\cos x}} \times \sqrt{1+\cos x}\right]}$

$\frac{1}{4\sqrt{2}}$

2) $\lim_{x\to a} \frac{\tan^{-1}x - \tan^{-1}a}{\tan x - \tan a}$ $\left(\frac{0}{0} \text{ form}\right)$

$\lim_{x\to a} \frac{\frac{1}{1+x^2}}{\sec^2 x}$

$\lim_{x\to a} \frac{1}{(1+x^2)(\sec^2 x)} = \frac{1}{(1+a^2)(\sec^2 a)}$

Continuity of function $f(x)$ at $x=a$.
A function $f(x)$ is said to be continious at $x=a$ if it satisfy following conditions:

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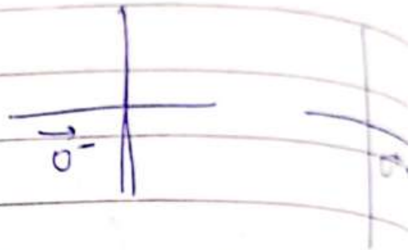
1) $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$ exist and are same.

2) $\lim_{x \to a^-} f(x) = f(a) = \lim_{x \to a^+} f(x)$

If function $f(x)$ failed to satisfy atleast one $cond^n$ then function is said to be discontinous.

Q. Check that the following function is continous at $x=0$ or not

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$



Ans: $\lim_{x \to 0^-} \frac{|x|}{x}$ $\qquad$ $\lim_{x \to 0^+} \frac{|x|}{x}$

$\lim_{x \to 0^-} \frac{-x}{x}$ $\qquad$ $\lim_{x \to 0^+} \frac{x}{x}$ $\qquad$ LHL $\neq$ RHL

$= -1$ $\qquad$ $= 1$

$\therefore$ The function is discontinous at $x=0$

Q. Check wheather the function is contious at $x=1$ or not.

$$f(x) = \begin{cases} x^2+2x+2, & x \neq 1 \\ 4, & x = 1 \end{cases}$$

Ans: $\lim_{x \to 1^-} x^2+2x+2$ $\qquad$ $\lim_{x \to 1^+} x^2+2x+2$

$\lim_{x \to 1^-} 2x+2+x^2$ $\qquad$ $\lim_{x \to 1^+} 2x+2+x^2$

$= 5$ $\qquad$ $= 5$

LHL = RHL

at $x = 1$, x^2+2x+2
$1+2+2 = 5 \neq 4$

$\therefore$ ~~limit~~ function is discontinous at $x = 1$

Check wheather the function is continous at $x = 1$ or not.

$$f(x) = \begin{cases} x^2+2x+2, & x \neq 1 \\ 5, & x = 1 \end{cases}$$

Ans: $\lim_{x \to 1^-} x^2+2x+2 = 5$

$\lim_{x \to 1^+} x^2+2x+2 = 5$.

at $x = 1$, 5

$\therefore$ The function is continous at $x = 1$.

Find value of a and b so that following function is continous at $x = 1$

$$f(x) = \begin{cases} ax+b, & x > 1 \\ 6, & x = 1 \\ 5ax-b, & x < 1 \end{cases}$$

Ans: $\lim_{x \to 1^+} ax+b$ $\qquad$ $\lim_{x \to 1^-} 5ax-b$

$\cancel{a+b=6}$ $\quad a+b = 6$ $\qquad$ $5ax - b = 6$

$\cancel{a=0}$ $\qquad$ $5a = 6+b$

$b = 6 - \frac{6+b}{5}$ $\qquad$ $a = \frac{6+4}{5}$

$b = \frac{30-6-b}{5}$ $\qquad$ $a = \frac{6+4}{5}$

$5b+b = 24$ $\qquad$ $\boxed{a = 2}$

$6b = 24$ $\quad$ $\cancel{4b = 24}$

$\boxed{b = 4}$ $\quad$ $\boxed{\cancel{b=6}}$

OR

Find out a and b for which given function is continuous or
If given function is continous at $x = 1$ then find a and

Find $\frac{dy}{dx}$ for $y = x^x + (\sin x)^x$

Ans: $x \log x + x \log(\sin x)$

$\log x + 1 + \log(\sin x) +$

$W = x^x$

~~W = log~~ $\log W = x \log x$.

$\frac{1}{W} W' = x \times \frac{1}{x} + \log x$.

$W' = W(1 + \log x)$

$W' = x^x(1 + \log x)$ —— (1)

$\log X = x \log \sin x$.

$\frac{1}{X} X' = \log \sin x + x \frac{\cos x}{\sin x}$

$X' = \sin^x x \, (\log \sin x + x \cot x)$ —— (2)

$\frac{dy}{dx} = X' + W'$

$= x^x(1 + \log x) + \sin^x x \, (\log \sin x + x \cot x)$

$x^y = e^{x-y}$ then show that $dy/dx = \frac{\log x}{(\log e x)^2}$

Ans: $y \log x = (x - y) \log e$ $\qquad \log_e = 1$.

$y \log x = x - y$.

$\frac{y}{x} + y' \log x = 1 - \frac{dy}{dx}$

$\frac{dy}{dx} (1 + \log x) = 1 - \frac{y}{x}$

$\frac{dy}{dx} (\log e + \log x) = \frac{x - y}{x}$

$\frac{dy}{dx} = \frac{y \log x}{x \log ex}$.

$\left[1 + \log x = \frac{x}{y} \right.$

$\left. \frac{y}{x} = \frac{1}{1 + \log x} = \frac{1}{\log ex} \right]$

$\frac{dy}{dx} = \frac{\log x}{(\log ex)^2}$.

Find dy/dx and d^2y/dx^2 where
$x = a\cos^3 t$ $y = b\sin^3 t$.

Ans: $\frac{dx}{dt} = 3a\cos^2 t(-\sin t)$ $\frac{dy}{dt} = 3b\sin^2 t(\cos t)$

$\frac{dy}{dx} = \frac{3b\sin^2 t(\cos t)}{3a\cos^2 t(-\sin t)}$ $\frac{dy}{dx} = -\frac{b\sin t}{a\cos t} = -\frac{b}{a}\tan t$

$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$ $\left(\because \frac{dt}{dx} = \frac{1}{dx/dt} \right)$

$= \frac{-b}{a}\sec^2 t \times \frac{1}{3a\cos^2 t(-\sin t)}$

$= \frac{b}{3a^2\cos^4 t(-\sin t)}$

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Mean Value Theorem.

1) Check whether the L.M.V.T can be applicable to the function $f(x) = 3x^{2/3} - 2x$ in $[0, 1]$. If so, find the value c in $(0, 1)$

Ans:

$$f'(x) = 3\left(\frac{2}{3}\right)x^{-1/3} - 2 = \frac{2}{\sqrt[3]{x}} - 2$$

$$f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{1 - 0}{1} = 1$$

$$f'(c) = \frac{2}{\sqrt[3]{c}} - 2$$

$$c = \frac{8}{27}$$

2) $\tan^{-1}x$ in $[-1, 1]$

It is continuous and differentiable in $[-1, 1]$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}(-1) - \tan^{-1}(1)}{-1-1}$$

$$\frac{1}{1+c^2} = \frac{-\pi/4 - \pi/4}{-2}$$

$$\frac{1}{1+c^2} = \frac{2\pi}{4 \times 2} = \frac{\pi}{4}$$

$$\frac{4}{\pi} = 1 + c^2$$

$$c = \pm\sqrt{\frac{4}{\pi} - 1}$$

$$c = 0.52.$$

3) $x^3 + 12x^2 + 7x$ in interval $[-4, 4]$

Ans: It is continuous and differentiable in $[-4, 4]$ and $(-4, 4)$

$$\cancel{f'(x) = \frac{f(b) - f(a)}{b - a}.}$$

$$\cancel{3x^2 + 24x + 7 = 64 + }$$

$$f'(x) = \frac{f(b) - f(a)}{b - a}. \qquad f(a) = -64 + 192 - 28$$

$$f(b) = 64 + 12(16) + 28$$
$$= 64 + 192 + 28$$
$$= 4$$

Here, $f(x)$ is a polynomial function which is continuous and differentiable over R. Therefore, it is continuous in $[-4, 4]$. and differentiable over $(-4, 4)$ Therefore, mean value theorem is applicable.

$$f(b) = 64 + 192 + 28 = 284$$
$$f(a) = 100$$
$$f(b) - f(a) = 284 - 100 = 184$$

$$f'(x) = \frac{184}{4 - (-4)} = \frac{184}{8} = 23.$$

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$3x^2 + 24x + 7 = 23$ $\qquad$ $4 \times 4 \times 3$

$3x^2 + 24x - 16 = 0$ $\qquad$ $2 \times 2 \times 2 \times 2 \times 3$

$c = \frac{-24 \pm \sqrt{576 + 192}}{6}$

$c = \frac{-24 \pm 27.71}{6}$

$c = \frac{3.71}{6} = 0.618$ $\qquad$ $c = -8.6183 \notin (-4, 4)$

Q. Find local maxima or local minima of following functions if they exist.

1) $f(x) = -2x^2 + 4x + 1$

Ans $f'(x) = 0$

$-4x + 4 = 0$

$-4x = -4$

$x = 1$ is a stationary point.

$f''(x) = 0$

$-4 = f''(x)$

$f''(x) < 0$

$x = 1$ is a local maxima.

max. value is $f(1) = -2 + 4 + 1 = 3$.

2) $\sin(\cos 2x)$

Ans: $f'(x) = 0$

$\cos(\cos 2x) + \sin(-\sin 2x)2 = 0$

$\cos(\cos 2x) = \sin(\sin 2x)2$

$\tan(\tan 2x) = \frac{1}{2}$

$\tan 2x = \tan^{-1} \frac{1}{2}$

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22) $f(x) = \sin(\cos 2x)$.

Ans: ~~$f'(x) = 0$~~

~~$y = \sin(\cos 2x)$~~

$\frac{dy}{dx} =$

$y' = \cos(\cos 2x)(-2)\sin 2x$

$y' = 0$

$\cos(\cos 2x)(-2)\sin 2x = 0$

We know that $\cos\theta = 0$

$\theta = \cos 2x$

$\theta = (2n+1)\frac{\pi}{2}$

$= (2n+1)(1.57)$

$= \pm 1.57, \pm 4.71 \ldots$

Range of $\cos\theta$ is $[-1, 1]$

$\pm 1.57, \pm 4.71 \notin [-1, 1]$

$\cos x \neq 0$

$x \in [-1, 1]$

~~Find the extreme value~~

$\cos(\cos 2x) \neq 0$

$\sin 2x = 0$

$2x = n\pi \qquad n \in \mathbb{Z}$

$x = \frac{n\pi}{2}$

$y'' = -2[\sin(\cos 2x)(-2)\sin^2 2x + \cos(\cos 2x)\, 2\cos 2x]$

$y''\left(\frac{n\pi}{2}\right) = -2\left[0 + \cos\left(\cos \frac{2n\pi}{2}\right) 2 \cos n\pi\right]$

$= -2[2(-1)^n \cos(-1)^n]$

$= -2(2(-1)^n \cos 1)$ → cos is ~~an even~~ function

n is even $y'' < 0$

n is odd $y'' > 0$.

~~Thus, if $x = n\pi$ while n is even no.~~

The local maxima is $x = \frac{n\pi}{2}$, n is even no.

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If maximum value is given by,

$\sin\left(\cos \frac{2n\pi}{2}\right)$, n is even.

$\sin[(-1)^n]$, n is even.

$\sin 1$.

The local minima is $x = \frac{n\pi}{2}$, n is odd.

The minimum value of y is given by,

$\sin\left(\cos \frac{2n\pi}{2}\right)$, n is odd.

$\sin[(-1)^n]$, n is odd.

$\sin(-1)$

$-\sin 1$.

☑ Find n^{th} order derivative of y. If $y = \sin ax + \cos ax$ then show that $y = a^n\sqrt{1+(-1)^n \sin 2ax}$

$y = \sin ax + \cos ax$

$y^n = a^n \sin\left(ax + \frac{n\pi}{2}\right) + a^n \cos\left(ax + \frac{n\pi}{2}\right)$

$y^n = a^n\left[\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right]$

$y^n = a^n\sqrt{\left[\sin\left(ax + \frac{n\pi}{2}\right) + \cos\left(ax + \frac{n\pi}{2}\right)\right]^2}$

$\because \cos^2 x + \sin^2 x = 1$

$y^n = a^n\sqrt{1 + 2\sin\left(ax + \frac{n\pi}{2}\right)\cos\left(ax + \frac{n\pi}{2}\right)}$

$y^n = a^n\sqrt{1 + \sin(2ax + n\pi)}$ $\quad \because 2\sin x \cos x = \sin 2x$

$y^n = a^n\sqrt{1 + \sin 2ax \cos n\pi + \cos 2ax \sin n\pi}$

$\because \sin n\pi = 0$

$y^n = a^n\sqrt{1 + (-1)^n \sin 2ax}$ $\quad \because \cos n\pi = (-1)^n$

most imp.

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Find n^{th} of $y = \cos x \cos 2x \cos 3x$

Ans: $y_n = a^n \cos(ax+b+n\pi/2)$

$y = \frac{1}{2}(2\cos x \cos 2x)\cos 3x$

$y = \frac{1}{2}[\cos(3x) + \cos x]\cos 3x$

$y = \frac{1}{2}[\cos^2 3x + \cos x \cdot \cos 3x]$

$y = \frac{1}{4}[2\cos^2 3x + 2\cos x \cos 3x]$

$\because \; 1 + \cos 2\theta = 2\cos^2\theta$

$\cos x \cos 3x = \cos(x+3x) + \cos(3x-x)$

$y = \frac{1}{4}[1 + \cos 6x + \cos 4x + \cos 2x]$

$y_n = \frac{1}{4}[6^n \cos(6x+n\pi/2) + 4^n\cos(4x+n\pi/2) + 2^n\cos(2x+n\pi/2)$

n^{th} order derivative of $y = x^4\cos(5x-4)$

Ans: Using Leibnitz rule,

$(UV)_n = U_nV + {}^nC_1 U_{n-1}V_1 + {}^nC_2 U_{n-2}V_2 \; ....$

$U = \cos(5x-4)$

$V = x^4$

$V_1 = 4x^3 \quad V_2 = 12x^2 \quad V_3 = 24x \quad V_4 = 24$

~~$U_{n-1} = 5\sin(5x-4)$~~ ~~$U_{n-2} =$~~

$U_m = 5^m \cos(5x-4+m\pi/2)$

$(x^4\cos(5x-4))_n = 5^n\cos(5x-4+n\pi/2)x^4 + n\,5^{n-1}\cos(5x-4+\frac{(n-1)\pi}{2})$
$4x^3 + 12x^2 \frac{(n)(n-1)}{2} 5^{n-2}\cos(5x-4+(n-2)\pi/2) +$

$\frac{24x(n)(n-1)(n-2)}{6} 5^{n-3}\cos((5x-4)+(n-3)\pi/2) + 24\,n(n-1)(n-2)(n-3)$

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$$\frac{5^{n-4}\cos\left((5x-4)+(n-4)\pi/2\right)}{4!}$$

$(x^4\cos(5x-4))_n$

- $= x^4\, 5^n \cancel{(-5x+)} \cos(5x-4+n\pi/2)$
$+ 4x^3 n\, 5^{n-1}\cos(5x-4+(n-1)\pi/2)$
$+ \frac{\cancel{12}^{6} x^2\, n(n-1)}{\cancel{2}}\, 5^{n-2}\cos(5x-4+(n-2)\pi/2)$
$+ \frac{\cancel{24}^{4} x\, n(n-1)(n-2)}{\cancel{6}}\, 5^{n-3}\cos(5x-4+(n-3)\pi/2)$
$+ \frac{\cancel{24}\, n(n-1)(n-2)}{\cancel{24}}\, 5^{n-4}\cos(5x-4+(n-4)\pi/2)$

- $= x^4\, 5^n\cos(5x-4+n\pi/2)$
$+ 4x^3\, 5^{n-1}\cos(5x-4+(n-1)\pi/2)$
$+ 6x^2\, n(n-1)\, 5^{n-2}\cos(5x-4+(n-2)\pi/2)$
$+ 4x\, n(n-1)(n-2)\, 5^{n-3}\cos(5x-4+(n-3)\pi/2)$
$+ n(n-1)(n-2)(n-3)\, 5^{n-4}\cos(5x-4+(n-4)\pi/2)$

☑ If $y = e^{a\sin^{-1}x}$ then show that $(1-x^2)y_{n+2} - x(2n+1)y_{n+1} - (n^2+a^2)y_n = 0$.

Ans: $y_1 = e^{a\sin^{-1}x} \cdot a \cdot \frac{1}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, y_1 = a y$

$(1-x^2) y_1^2 = a^2 y^2$

~~y_2~~ diff. wrt x.

$(1-x^2) 2y_1y_2 - 2xy_1^2 - a^2 2yy_1 = 0$

divide by $2y_1$

$(1-x^2)y_2 - xy_1 - a^2y = 0$

$[(1-x^2)y_2]_n - [xy_1]_n - a^2[y]_n = 0$

$-a^2y_n + (1-x^2)y_{n+2} + n(-2x)y_{n+1} + \frac{n(n-1)}{2}(-2)y_n - xy_{n+1} - ny_n = 0$

$(1-x^2)y_{n+2} + (-2xn - x)y_{n+1} + (n^2 - n + n - a^2)y_n = 0$

$(1-x^2)y_{n+2} - x(2n+1)y_{n+1} - (n^2+a^2)y_n = 0$

☑ Expand $\sin x$ in power of $x - \frac{\pi}{2}$ using this evalute the appoximate value of $\sin 91°$.

→ Here we are going to use Taylor series. It expand the function in power of $x-a$ or around $x=a$ or at $x=a$. The formula is given by

$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''a + \frac{(x-a)^3}{3!}f'''(a) \ldots$

Ans: $a = \pi/2$

$f(x) = \sin x$ $\quad f(\pi/2) = 1$
$f'(x) = \cos x$ $\quad f'(\pi/2) = 0$
$f''(x) = -\sin x$ $\quad f''(\pi/2) = -1$
$f'''(x) = -\cos x$ $\quad f'''(\pi/2) = 0$

$$\sin x = 1 + \left(x - \frac{\pi}{2}\right)0 + \frac{\left(x - \frac{\pi}{2}\right)(-1)}{4} + 0 \ldots$$

$$= 1 - \frac{(x - \pi/2)^2}{2} \ldots$$

~~Take x = 90°;~~

Take $x = 91°$;

$$\sin 91° = 1 - \frac{(1°)^2}{2} + \ldots$$

$$1° = \frac{\pi}{180} = \frac{22}{7} \times \frac{1}{180} = 0.01746$$

$$\sin 91° \approx 1 - \frac{(0.01746)^2}{2}$$

$$\approx 0.999847$$

☒ Find the approximate value of $\sqrt{10}$ correct upto 4 decimal place.

→ Alternate form of Taylor series.

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \ldots$$

$x \to (x+h)$ $\qquad a \to (h)$

Ans: $f(x) = \sqrt{x}$ $\qquad f''(x) = -\frac{1}{4} x^{-3/2}$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\qquad f'''(x) = \frac{3}{8} x^{-5/2}$

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$$\sqrt{x+h} = \sqrt{h} + \frac{x}{2\sqrt{h}} - \frac{x^2}{8h^{3/2}} + \frac{3x^3}{48h^{5/2}} \cdots$$

$x = 1 \qquad h = 9$

Take x as small as possible as it reduces error.

$$\sqrt{10} = 3 + \frac{1}{6} - \frac{1}{8\times27} + \frac{3}{48\times243} + \cdots$$

$$= 3 + \frac{1}{6} - \frac{1}{216} + \frac{3}{11664} + \cdots$$

$$= 3.162294$$

Q Obtain power series of $\sin x$ in power of $\left(x-\frac{\pi}{2}\right)$
Find the value of $\sin 91°$.

Solⁿ. Here, $f(x) = \sin x$.
We are going to use taylor series.

$$f(x+a) = f(x) + \frac{f'(a)(x-a)}{1!} + \frac{f''(a)(x-a)^2}{2!} + \cdots$$

$$f(a) = \sin\frac{\pi}{2} = 1.$$

$$f'(x) = \cos x, \quad f'(a) = \cos\frac{\pi}{2} = 0$$

$$f''(x) = -\sin x, \quad f''(a) = -\sin\frac{\pi}{2} = -1$$

$$f'''(x) = -\cos x, \quad f'''(a) = -\cos\pi/2 = 0$$

$$\sin(91^\circ) = 1 - \frac{(x-a)^2}{2} + \cdots \qquad \boxed{x = 91^\circ}$$

$$= 1 - \frac{(91^\circ - 90^\circ)^2}{2} + \cdots$$

$$= 1 - \frac{(1^\circ)^2}{2} + \cdots$$

$$1^\circ = \frac{\pi}{180} = \frac{22}{7 \times 180} = 0.01746$$

$$\sin 91^\circ \approx 1 - \frac{(0.01746)^2}{2}$$

$$= 1 - \frac{0.00030485}{2}$$

$$= \frac{2 - 0.00030485}{2} = \frac{1.99969515}{2}$$

$$= \underline{0.99984}$$

L'Hospital Rule :

Q1. Evaluate $\lim_{x \to 0} \frac{\tan x - \sin x}{x^2}$.

Solⁿ Let $f(x) = \lim_{x \to 0} \frac{\tan x - \sin x}{x^2}$. ——— (1)

The above given function is of the form $\frac{0}{0}$.

Therefore, applying L'Hospital rule, we have

$$\lim_{x \to 0} \frac{\sec^2 x - \cos x}{-2x} \quad \text{———} \quad (2)$$

The above obtained equation is also of the form $\frac{0}{0}$. So again applying L'Hospital rule, we have,

$$\lim_{x \to 0} \frac{2\sec^2 x \tan x + \sin x}{-2}$$

Now applying limit, we have,

$$\lim_{x \to 0} \frac{\tan x - \sin x}{2} = 0.$$

Q2. Evaluate $\lim_{x \to \frac{\pi}{2}} \frac{\log\left(x - \frac{\pi}{2}\right)}{\tan x}$.

Solⁿ Let $f(x) = \lim_{x \to \pi/2} \frac{\log(x - \pi/2)}{\tan x}$

The above eqn is of the form $\frac{\infty}{\infty}$, therefore applying L'Hospital rule, we get,

$$\lim_{x \to \pi/2} \frac{\frac{1}{\left(x-\frac{\pi}{2}\right)}}{\sec^2 x}$$

$$\Rightarrow \lim_{x \to \pi/2} \frac{1}{\left(x-\frac{\pi}{2}\right)\sec^2 x}$$

$$\lim_{x \to \pi/2} \frac{\cos^2 x}{\left(x-\frac{\pi}{2}\right)} \quad \text{——— (2)}$$

The above obtained equation is of the form $\frac{0}{0}$. Therefore again applying L'Hospital rule, we get,

$$\lim_{x \to \pi/2} \frac{-2\sin x\cos x}{1}$$

Now applying limit, we have

$$\rightarrow \lim_{x \to \pi/2} \frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x} = 0.$$

Q3. Prove that $\lim_{x \to 0} \log_x \sin x = 1$.

Soln Let $f(x) = \lim_{x \to 0} \log_x \sin x = \lim_{x \to 0} \frac{\log \sin x}{\log x}$

$$f(x) = \lim_{x \to 0} \frac{\log(\sin x)}{\log x} \quad \text{———} \quad (1)$$

The above equation is of the form $\frac{\infty}{\infty}$. Therefore, we apply L'Hospital rule and we get,

$$\lim_{x \to 0} \frac{\cos x}{\sin x \left(\frac{1}{x}\right)}.$$

$$\lim_{x \to 0} \frac{x \cos x}{\sin x} \quad \text{———} \quad (2)$$

The above obtained equation is of the form $\frac{0}{0}$. Therefore, again applying L'Hospital rule, we have

$$\lim_{x \to 0} \frac{\cos x + x(-\sin x)}{\cos x}.$$

$$= \lim_{x \to 0} \frac{\cos x - x\sin x}{\cos x}.$$

Now applying limit, we get,

$$\lim_{x \to 0} \log_x(\sin x) = 1.$$

$\therefore$ LHS = RHS.

Hence, proved.

Q1. Evaluate $\lim_{x\to \pi/4} (1-\tan x)\cdot \sec 2x$.

Solution: Let $f(x) = \lim_{x\to\pi/4} (1-\tan x)\sec 2x$ —— (1)

$$\Rightarrow \lim_{x\to\pi/4} \frac{(1-\tan x)}{\frac{1}{\sec 2x}}$$

$$\Rightarrow \lim_{x\to\pi/4} \frac{(1-\tan x)}{\cos 2x} \quad \text{—— (2)}$$

The above eq (2) is of the form $\frac{0}{0}$. Therefore by applying L'Hospital's rule, we have

$$\Rightarrow \lim_{x\to\pi/4} \frac{-\sec^2 x}{-\sin 2x(2)}$$

$$\Rightarrow \lim_{x\to\pi/4} \frac{\sec^2 x}{2\sin 2x}$$

Now applying limit to the given function, we have

$$\lim_{x\to\pi/4} (1-\tan x)\sec 2x = \underline{\underline{1}}.$$

Q2 Evaluate $\lim_{x\to 0}\left[\frac{1}{x} - \frac{1}{e^x-1}\right]$. $(\infty-\infty)$

Sol. Let $f(x) = \lim_{x\to 0}\left[\frac{1}{x} - \frac{1}{e^x-1}\right]$

$$\Rightarrow \lim_{x\to 0}\left[\frac{e^x-1-x}{x(e^x-1)}\right]$$ [By taking LCM]

$$\Rightarrow \lim_{x\to 0}\left[\frac{e^x-x-1}{x(e^x-1)}\right] \quad \text{——— (2)}$$

The above eq (2) is of the form $\frac{0}{0}$, therefore by applying L'Hospital's rule, we have,

$$\Rightarrow \lim_{x\to 0}\left[\frac{e^x-1}{e^x-1+x(e^x)}\right]$$

$$\Rightarrow \lim_{x\to 0}\left[\frac{e^x-1}{xe^x+e^x-1}\right] \quad \text{——— (3)}$$

The above eq (3) obtained is of the form $\frac{0}{0}$. So, again applying L'Hospital's rule, we have,

$$\Rightarrow \lim_{x\to 0}\frac{e^x}{(e^x+xe^x+e^x)}$$

Now applying limit we have,

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{e^x-1}\right] = \underline{\underline{\frac{1}{2}}}$$

Q3. Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$ (1^∞)

Soln. Let $l = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$ ——— (1)

Taking log both sides,

$\Rightarrow \log l = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$

$\Rightarrow \log l = \lim_{x\to 0}\frac{\log\left(\frac{\tan x}{x}\right)}{x^2}$ ——— (2)

The above eqn is of the form $\frac{0}{0}$. Therefore applying L'Hospital's rule, we have

$$\Rightarrow \log l = \lim_{x\to 0}\frac{\frac{1}{\left(\frac{\tan x}{x}\right)}\left[\frac{x\sec^2 x - \tan x}{x^2}\right]}{2x}$$

$$\log l = \lim_{x\to 0}\frac{x\sec^2 x - \tan x}{2x^2\tan x}$$

Again applying L'Hospital rule, we have,

$$\lim_{x\to 0}\frac{\sec^2 x + 2x\sec^2 x\tan x - \sec^2 x}{2(x^2\sec^2 x + 2x\tan x)}$$

$$\lim_{x\to 0}\frac{\cancel{x}\sec^2 x\tan x}{x\sec^2 x + 2\tan x}$$

Again applying L'Hospital rule, we have

$$\log l = \lim_{x \to 0} \frac{\sec^2 x \sec^2 x + \tan x \, 2\sec^2 x \tan x}{\sec^2 x + x(2\sec^2 x \tan x) + 2\sec^2 x}$$

Now applying limit, we get,

$$\log l = \frac{1}{3}$$

$$l = \underline{\underline{e^{1/3}}}$$