IV Partial differentiations

Classwork Examples

4.1	Partial derivative and geometrical interpretation

Partial derivative and geometrical interpretation

1	Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(4,-5)$, where $f(x,y) = x^2 + 3xy + y - 1$. **C.W.** **Answer:** $\left.\frac{\partial f}{\partial x}\right\|_{(4,-5)} = -7$, $\left.\frac{\partial f}{\partial y}\right\|_{(4,-5)} = 13$.
2	The plane $x = 1$ intersects the paraboloid $z = x^2 + y^2$. Find the slope of the tangent to the parabola at $(1, 2, 5)$. **C.W.** **Answer:** $\left.\frac{\partial z}{\partial y}\right\|_{(1,2)} = 4$.
3	Find the first and second order partial derivatives of $z = x^3 + y^3 - 3axy$, where a is a constant. **C.W.** **Answer:** $\frac{\partial z}{\partial x} = 3x^2 - 3ay, \frac{\partial z}{\partial y} = 3y^2 - 3ax,$ $\frac{\partial^2 z}{\partial x^2} = 6x, \frac{\partial^2 z}{\partial x \partial y} = -3a, \frac{\partial^2 z}{\partial y \partial x} = -3a, \frac{\partial^2 z}{\partial y^2} = 6y.$
4	If $z = x^y + y^x$, then verify that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$. **C.W.**
5	Find the first order partial derivatives for the following functions at the specified point. (i) $f(x,y) = \ln\left(\frac{x}{y}\right)$, at $(2, 3)$. **C.W.** (ii) $f(x,y) = \frac{x}{\sqrt{x^2+y^2}}$, at $(3, 4)$. **H.W.** (iii) $f(x,y,z) = (x^2 + y^2 + z^2)^{-\frac{1}{2}}$, at $(2, 1, 2)$. **C.W.** (iv) $f(x,y,z) = \ln\left(x + \sqrt{y^2 + z^2}\right)$, at $(2, 3, 4)$. **H.W.** **Answer:** (i) $\left.\frac{\partial f}{\partial x}\right\|_{(2,3)} = \frac{1}{2}$, $\left.\frac{\partial f}{\partial y}\right\|_{(2,3)} = -\frac{1}{3}$. (ii) $\left.\frac{\partial f}{\partial x}\right\|_{(3,4)} = \frac{16}{125}$, $\left.\frac{\partial f}{\partial y}\right\|_{(3,4)} = -\frac{12}{125}$.

	$(iii)\ \left.\frac{\partial f}{\partial x}\right\|_{(2,1,2)} = -\frac{2}{27}$, $\left.\frac{\partial f}{\partial y}\right\|_{(2,1,2)} = -\frac{1}{27}$, $\left.\frac{\partial f}{\partial z}\right\|_{(2,1,2)} = -\frac{2}{27}$. $(iv)\ \left.\frac{\partial f}{\partial x}\right\|_{(2,3,4)} = \frac{1}{7}$, $\left.\frac{\partial f}{\partial y}\right\|_{(2,3,4)} = \frac{3}{35}$, $\left.\frac{\partial f}{\partial z}\right\|_{(2,3,4)} = \frac{4}{35}$.
6	Verify that $f_{xy} = f_{yx}$, when f is equal to $\sin^{-1}\left(\frac{y}{x}\right)$. **H.W.**

4.2	Euler's theorem with corollaries and their applications

Euler's theorem with corollaries and their applications

1	If $z = (8x^2 + y^2)(\log x - \log y)$, then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$. **C.W.**
2	If $u = \cos\left(\frac{xy+yz+zx}{x^2+y^2+z^2}\right)$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$. **C.W.**
3	If $z = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$, then find $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}$. **C.W.** **Answer:** 0.
4	If $u = \sin^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$. **C.W.**
5	If $u = e^{x^2+y^2}$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\,u\log u$. **C.W.**
6	If $u = \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{xy}\right)$, then find $x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$. **C.W.** **Answer:** $\tan u\,(1 + \sec^2 u)$.
7	If $u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right)$, then find $(i)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$, $(ii)\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}$. **H.W.** **Answer:** $(i)\ 2u, (ii)\ 2u$.
8	If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, then prove that $(i)\ x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \sin 2u$, $(ii)\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2\cos 3u \sin u\ \ or \sin 4u - \sin 2u$. **H.W.**

4.3	Chain rule

Chain rule

1	Find $\frac{df}{dt}$ at $t = 0$ for a function $f(x, y) = x\cos y + e^x \sin y$, where $x = t^2 + 1, y = t^3 + t$. **C.W.**

	Answer: $\left.\frac{df}{dt}\right\|_{t=0} = e$.
2	Find $\frac{df}{dt}$ at $t = 0$ for a function $f(x, y, z) = x^3 + xz^2 + y^3 + xyz$, where $x = e^t, y = \cos t, z = t^3$. **C.W.** **Answer:** $\left.\frac{df}{dt}\right\|_{t=0} = 3$.
3	If $z = f(ax + by)$, then prove that $b\frac{\partial z}{\partial x} - a\frac{\partial z}{\partial y} = 0$, where a and b are constants. **C.W.**
4	If $z = f(x, y)$, $x = e^{2u} + e^{-2v}, y = e^{-2u} + e^{2v}$, then show that $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = 2\left[x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}\right]$. **C.W.**
5	If $w = f(u, v)$ and $u = \sqrt{x^2 + y^2}, v = \cot^{-1}\left(\frac{y}{x}\right)$, then prove that $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = \frac{1}{x^2+y^2}\left[(x^2 + y^2)\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right]$. **C.W.**
6	If $u = f(x - y, y - z, z - x)$, then prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$. **C.W.**
7	If $z = f(x, y)$ and $x = u\cos\alpha - v\sin\alpha, y = u\sin\alpha + v\cos\alpha$, where α is a constant, then prove that $\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$. **H.W.**
8	If $z = \ln(u^2 + v), u = e^{x+y^2}, v = x + y^2$, then prove that $2y\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$. **H.W.**

4.4	Implicit functions

Implicit functions

1	Find $\frac{dy}{dx}$ at $(1, 1)$ if $y^3 + y^2 + 5y - x^2 + 4 = 0$. **C.W.** **Answer:** $\left.\frac{dy}{dx}\right\|_{(1,1)} = \frac{1}{5}$.
2	Find $\frac{dy}{dx}$ at $(-1, 1)$ if $xy + y^2 - 3x - 3 = 0$. **C.W.** **Answer:** $\left.\frac{dy}{dx}\right\|_{(-1,1)} = 2$.
3	Assume that the equation $y^2 + xz + z^2 - e^z - c = 0$ defines z as a function of x and y say $z = f(x, y)$. Find a value of the constant c such that $f(0, e) = 2$ and compute the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(x, y) = (0, e)$. **C.W.** **Answer:** $\left.\frac{\partial z}{\partial x}\right\|_{(0,e,2)} = \frac{2}{e^2-4}$, $\left.\frac{\partial z}{\partial y}\right\|_{(0,e,2)} = \frac{2e}{e^2-4}$.

4	Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1,1,1)$, if $z^3 - xy + yz + y^3 - 2 = 0$. **C.W.** **Answer:** $\left.\frac{\partial z}{\partial x}\right\|_{(1,1,1)} = \frac{1}{4}$, $\left.\frac{\partial z}{\partial y}\right\|_{(1,1,1)} = -\frac{3}{4}$.
5	Find $\frac{dy}{dx}$ at $(0, \ln 2)$, if $xe^y + \sin xy + y - \ln 2 = 0$. **H.W.** **Answer:** $\left.\frac{dy}{dx}\right\|_{(0,\ln 2)} = -(2 + ln\ 2)$.
6	Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at (π, π, π), if $\sin(x+y) + \sin(y+z) + \sin(x+z) = 0$. **H.W.** **Answer:** $\left.\frac{\partial z}{\partial x}\right\|_{(\pi,\pi,\pi)} = -1$, $\left.\frac{\partial z}{\partial y}\right\|_{(\pi,\pi,\pi)} = -1$.

4.5	Total differentials

Total differentials

1	Find total derivative of the function $z = 4x^3 - xy^2 + 3y - 7$. **C.W.** **Answer:** $dz = (12x^2 - y^2)\ dx + (3 - 2xy)dy$.
2	If $u = xyz$, then obtain du. **C.W.** **Answer:** $du = yz\ dx + zx\ dy + xy\ dz$.
3	Find total derivative of the function $z = x\cos y - y\sin x$. **H.W.** **Answer:** $dz = (\cos y - y\cos x)\ dx + (-x\sin y - \sin x)dy$.
4	If $u = \ln(x^2 + y^2 + z^2)$, then obtain du. **C.W.** **Answer:** $du = \left(\frac{2x}{x^2+y^2+z^2}\right) dx + \left(\frac{2y}{x^2+y^2+z^2}\right) dy + \left(\frac{2z}{x^2+y^2+z^2}\right) dz$.