II	Infinite Series and Complex numbers

2.1	Tests of convergence of series viz., comparison test, ratio test, root test, Leibnitz test

Some concepts to discuss for sequences:

1. An ordered set of real numbers $a_1, a_2, a_3, \ldots \ldots$ is called a sequence and is denoted by $\{a_n\}$. If the number of terms is unlimited, then the sequence is said to be an infinite sequence and a_n is its nth term.
   e.g. $\{3,5,7,9,\ldots..\}$, Here nth term $a_n = 2n+1$.
2. **Limit**: A sequence is said to tend to a limit l, if for every $\varepsilon > 0$, a natural number n_0 can be found such that $|a_n - l| < \varepsilon \; for \; all \; n \geq n_0$.
3. **Convergence:** If a sequence $\{a_n\}$ has a finite limit, it is called a convergent sequence. If the limit of sequence $\{a_n\}$ does not tend to a finite number, it is said to be divergent.
   e.g. $\left\{1, \frac{1}{4}, \frac{1}{9}, \ldots\ldots, \frac{1}{n^2}, \ldots..\right\}$ is a convergent sequence.
   $\{3,5,7,\ldots.,(2n+1),\ldots..\}$ is a divergent sequence.
4. **Bounded sequence:** A sequence $\{a_n\}$ is said to be bounded, if there exists a number $k > 0$ such that $|a_n| < k$ for every n.
5. A sequence $\{a_n\}$ is called **increasing** if $a_n \leq a_{n+1}$ for all n
6. A sequence $\{a_n\}$ is called **decreasing** if $a_n \geq a_{n+1}$ for all n
7. **Monotonic sequence:** A sequence $\{a_n\}$ is called monotonic if it is either increasing or decreasing.
   e.g. $\{1,4,7,10,\ldots..\}$ is a monotonic sequence.
   $\left\{1, \frac{1}{4}, \frac{1}{9}, \ldots\ldots, \frac{1}{n^2}, \ldots..\right\}$ is also a monotonic sequence.
   $\{1,-1,1,-1,\ldots\ldots\}$ is not a monotonic sequence.
   A monotonic sequence always tends to a limit, finite or infinite.
   A sequence which is monotonic and bounded is convergent.

Series:

Definition: If $u_1, u_2, u_3, \dots, u_n, \dots$ is an infinite sequence of real numbers, then $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is called an infinite series. An infinite series is denoted by $\sum_{n=1}^{\infty} u_n$, the sum of its first *n* terms is denoted by $s_n = \sum_{j=1}^{n} u_j$ and it is known as partial sum of *n* terms.

e.g. $1 + 3 + 5 + 7 + \cdots$ is an infinite series.

Convergent and divergent series:

Consider the infinite series $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ and let the sum of the first *n* terms be $s_n = u_1 + u_2 + u_3 + \cdots \dots .. + u_n$.

1. If s_n tends to a finite number as $n \to \infty$, then the series $\sum_{n=1}^{\infty} u_n$ is said to be convergent.

   e.g. Test the convergence of the series $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \cdots \frac{1}{(2n-1).(2n+1)}$

   Solution: Let

$$s_n = \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \cdots \frac{1}{(2n-1).(2n+1)}$$

$$= \frac{1}{2}\left(1 - \frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3} - \frac{1}{5}\right) + \frac{1}{2}\left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \frac{1}{2}\left(\frac{1}{(2n-1)} - \frac{1}{(2n+1)}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{(2n+1)}\right)$$

   $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{2}\left(1 - \frac{1}{(2n+1)}\right) = \frac{1}{2}$, which is a finite quantity.

   the series is convergent.

2. If s_n tends to infinity as $n \to \infty$, then the series $\sum_{n=1}^{\infty} u_n$ is said to be divergent.

   e.g. $(\boldsymbol{i})$ Test the convergence of the series $1 + 2 + 3 + \cdots$.

   Solution: Let $s_n = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$.

   $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{n(n+1)}{2} = \infty$, the series is divergent.

   e.g. $(\boldsymbol{ii})$ Test the convergence of the series $1 + 4 + 9 + 16 + \cdots$.

   Solution. Here $S_n = 1^2 + 2^2 + \cdots + n^2 = \frac{n}{6}(n+1)(2n+1)$.

   $\therefore \lim_{n\to\infty} S_n = \infty$.

   Hence, the given series is divergent.

Some Standard Limits

1. $\lim_{n\to\infty} \frac{logn}{n} = 0$.
2. $\lim_{n\to\infty} \frac{1}{logn} = 0$.

3. $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e.$
4. $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a.$
5. $\lim_{n\to\infty}(1+n)^{1/n} = e.$
6. $\lim_{n\to\infty}(n)^{\frac{1}{n}} = 1.$
7. $\lim_{n\to\infty}(n!)^{\frac{1}{n}} = \infty.$
8. $\lim_{n\to 0} e^n = 1.$
9. $\lim_{n\to\infty} x^n = 0$ if $|x| < 1.$
10. $\lim_{n\to\infty} x^n = \infty$ if $x > 1.$
11. $\lim_{n\to\infty} n.x^n = 0$ if $|x| < 1.$
12. $\lim_{n\to\infty}\frac{x^n}{n!} = 0$ for all values of x.
13. $\lim_{n\to\infty}\left[\frac{(n!)}{n}\right]^{1/n} = \infty.$

General properties of infinite series:

1. The nature (convergence or divergence) of an infinite series does not change:
   - **(i)** by multiplication of all terms by a non-zero number k.
   - **(ii)** by addition or deletion of a finite number of terms.
2. If two series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are convergent, then $\sum_{n=1}^{\infty}(u_n+v_n)$ is also convergent.

Necessary condition for convergence

If a series $\sum_{n=1}^{\infty} u_n$ is convergent, then $\lim_{n\to\infty} u_n = 0$.

Note: The converse of this result need not be true.

Consider, for instance, the series $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+..+\frac{1}{\sqrt{n}}+\cdots$.

Since the term go on descending,

$s_n = 1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+..+\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n}}+..+\frac{1}{\sqrt{n}} = \frac{n}{\sqrt{n}}$

So, $\lim_{n\to\infty} s_n = \lim_{n\to\infty}\sqrt{n} \longrightarrow \infty.$

Thus the series is divergent even though $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0.$

$\lim_{n\to\infty} u_n = 0$ is a necessary but not sufficient condition for the convergence of $\sum_{n=1}^{\infty} u_n$.

We have simple test for divergence from above:

If $\lim_{n\to\infty} u_n \neq 0$, then the series $\sum_{n=1}^{\infty} u_n$ does not converge.

Geometric series:

Geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ is

- **i.** convergent if $|r| < 1$.
- **ii.** divergent if $r \geq 1$.

  If $|r| < 1$, then the sum of the Geometric series is $S = \frac{a}{1-r}$.

e.g. Test the convergence of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$

Solution: Let $s_n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{n}$.

Here series is a Geometric series with common ratio $r = \frac{1}{2}$.

$$\lim_{n\to\infty} s_n = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2.$$

Hence, the series is convergent.

Tutorial Work:

Test the convergence of the following series

1	$log2 + log\frac{3}{2} + log\frac{4}{3} + \cdots$ **Answer:** divergent.
2	$\frac{1}{1.5} + \frac{1}{5.9} + \frac{1}{9.13} + \frac{1}{13.17} + \cdots$ **Answer:** convergent.

P- Test: The *p* series $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$ is

- **i.** convergent if $p > 1$.
- **ii.** divergent if $p \leq 1$.
- **iii.** It is not always possible to find the partial sum S_n for every series easily. Thus it becomes necessary to use other tests for series with all terms positive. Using these tests, we can discuss the convergence/divergence of series.

Limit Comparison Test:

If two positive term series $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ are such that $\lim_{n\to\infty} \frac{u_n}{v_n}$ = finite and is nonzero, then $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ converge or diverge together.

e.g. Test the convergence of the series $\frac{2}{1}+\frac{3}{4}+\frac{4}{9}+\cdots+\frac{n+1}{n^2}+\cdots$.

Solution: Let $u_n=\frac{n+1}{n^2}$.

(In general take $v_n=\frac{1}{n^p}$ can be obtained from u_n , where p = (the highest power n in denominator of u_n) - (the highest power n in numerator of u_n)).

So here in this case $p=1$.

i.e. $v_n=\frac{n}{n^2}=\frac{1}{n}$.

$\therefore \lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\left(\frac{n+1}{n^2}\right)}{\left(\frac{1}{n}\right)}=\lim_{n\to\infty}\frac{n+1}{n}=1$, finite & non-zero.

$\therefore$ By limit comparison test, both the series $\sum_{n=1}^{\infty}u_n$ and $\sum_{n=1}^{\infty}v_n$ converges or diverges together.

Now, $\sum_{n=1}^{\infty}v_n=\sum_{n=1}^{\infty}\frac{1}{n}$ is a p-series with p=1.

$\therefore \sum_{n=1}^{\infty}v_n$ is divergent.

So, $\sum_{n=1}^{\infty}u_n$ is also divergent.

e.g. Test the convergence of the series $\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\cdots$

Solution: Let $u_n=\frac{1}{(2n-1)(2n)}$.

Take $v_n=\frac{1}{n^2}$.

$\therefore \quad \frac{u_n}{v_n}=\frac{n^2}{(2n-1)(2n)}=\frac{1}{2\left(2-\frac{1}{n}\right)}$.

$\therefore \quad \lim_{n\to\infty}\frac{u_n}{v_n}=\frac{1}{4}$ (finite).

So, both series are convergent or divergent together.

But we know that $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is convergent. ($\because$ p = 2 in p series $\sum\frac{1}{n^p}$)

Therefore the given series is convergent by limit comparison test.

Tutorial:

Test the convergence of the following series:

1	$\sum_{n=1}^{\infty}\frac{1}{n^2+1}$. **Answer:** convergent.
2	$\sum_{n=1}^{\infty}\sqrt{n+1}-\sqrt{n}$. **Answer:** divergent.

3	$\sum_{n=1}^{\infty} \frac{n^p}{\sqrt{n+1}+\sqrt{n}}$. **Answer:** convergent if $p < -1/2$ and divergent if $p \geq -1/2$.

D'alembert's Ratio Test:

If $\sum_{n=1}^{\infty} u_n$ is a positive term series and $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = l$, then

- **i.** $\sum_{n=1}^{\infty} u_n$ is convergent if $l < 1$.
- **ii.** $\sum_{n=1}^{\infty} u_n$ is divergent if $l > 1$.
- **iii.** If $l = 1$, the ratio test fails. i.e., no conclusion can be drawn about the convergence or divergence of the series.

Example: Test the convergence of the series $\frac{1}{3} + \frac{1.2}{3.5} + \frac{1.2.3}{3.5.7} + \cdots$.

Solution: Here $u_n = \frac{1.2.3\ldots n}{3.5.7\ldots(2n+1)}$.

$\therefore \quad u_{n+1} = \frac{1.2.3\ldots n(n+1)}{3.5.7\ldots(2n+1)(2n+3)}$,

$\therefore \quad \frac{u_{n+1}}{u_n} = \frac{n+1}{2n+3} = \frac{1+\frac{1}{n}}{2+\frac{3}{n}}$,

Here $\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2+\frac{3}{n}} = \frac{1}{2} < 1$.

So, the given series is convergent using ratio test.

Tutorial:

Test the convergence of the following series:

1	$\sum_{n=1}^{\infty} \frac{n!}{n^2}$. **Answer:** divergent.
2	$\sum_{n=1}^{\infty} \frac{n^3+2}{2^n+2}$. **Answer:** convergent.
3	$1 + \frac{2^p}{2!} + \frac{3^p}{3!} + \frac{4^p}{4!} + \cdots$. **Answer:** convergent.
4	$\sum_{n=1}^{\infty} \frac{n!}{n^n}$. **Answer:** convergent.

Cauchy's Root Test:

If $\sum u_n$ is a positive term series and $\lim_{n\to\infty} (u_n)^{\frac{1}{n}} = l$, then

- **i.** $\sum u_n$ is convergent if $l < 1$.
- **ii.** $\sum u_n$ is divergent if $l > 1$.
- **iii.** If $l = 1$, the root test fails. i.e., no conclusion can be drawn about the convergence or divergence of the series.

Test the convergence of the series $\sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$.

Solution: Here $u_n = \frac{1}{(log\ n)^n}$.

$\therefore \quad (u_n)^{\frac{1}{n}} = \left(\frac{1}{(log\ n)^n}\right)^{\frac{1}{n}} = \frac{1}{log\ n}$,

$\therefore \quad \lim_{n\to\infty} (u_n)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{log\ n} = 0 < 1$.

So, the given series is convergent using Root test.

Tutorial:

Test the convergence of the following series:

1	$\sum_{n=1}^{\infty} \frac{(n-logn)^n}{2^n.n^n}$. **Answer:** convergent.
2	$1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$. **Answer:** convergent .

Alternating series:

An infinite series with alternate positive and negative terms is called an alternating series.

Leibnitz's test for alternating series:

An alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$, where $u_n > 0$ is convergent if

- **i.** $\{u_n\}$ is strictly decreasing, i.e., $u_{n+1} < u_n$ for all n.
- **ii.** $\lim_{n\to\infty} u_n = 0$.

Example: Show that the series $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots$ is convergent.

Solution: Here $u_n = \frac{1}{\sqrt{n}}$

- **i.** The given series is $\sum_{n=1}^{\infty}(-1)^{n+1} u_n = \sum_{n=1}^{\infty}(-1)^{n+1} \frac{1}{\sqrt{n}}$ an alternating series.

ii. Clearly $\sqrt{n+1} > \sqrt{n} \Rightarrow \frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \Rightarrow u_{n+1} < u_n$.

So the sequence $\{u_n\}$ is strictly decreasing.

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. By Leibnitz's test, the series is convergent.

Tutorial:

Test the convergence of the following series:

1	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{log\,(n+1)}$. **Answer:** convergent.
2	$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ \dots$. **Answer:** convergent.

2.2	Complex numbers and their geometric representation

A number of the form $z = x + iy$, where x and y are real numbers and $i = \sqrt{-1}$ i.e. $i^2 = -1$, is called a **Complex number**. This is known as Cartesian form of a Complex number. A complex number $z = x + iy$ may be written as a pair (x, y).

Here, x is known as the real part of z and y is known as the imaginary part of z and are often denoted as, $Re(z) = x$ and $Im(z) = y$ respectively.

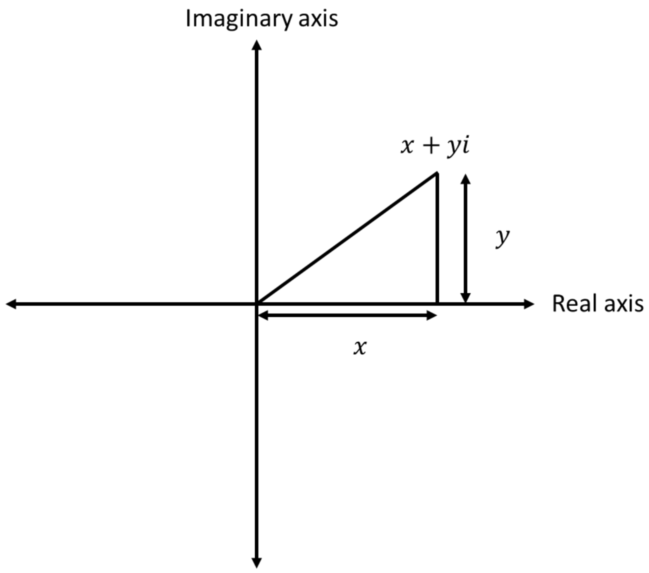
If a complex number has a zero real part i.e. $z = 0 + iy = iy$. We call it a purely imaginary number.

A complex number has a zero imaginary part i.e. $z = x + 0.i = x$. We can see that such complex number represents a real number.

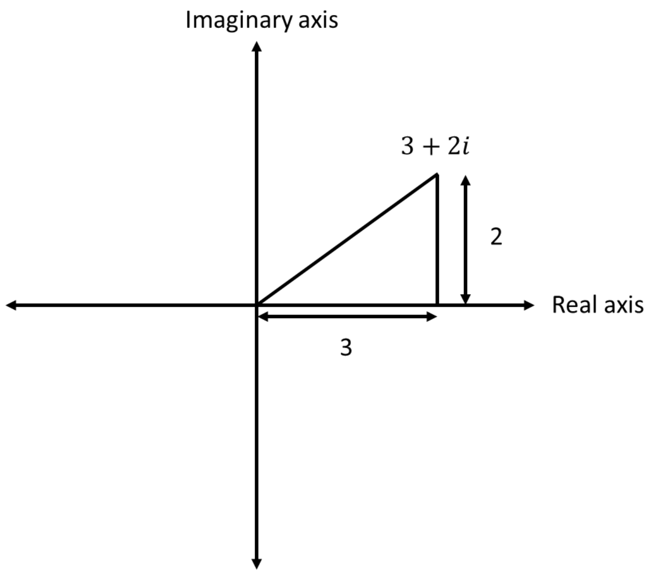
The set of complex numbers is denoted by $\mathbb{C}$.

The roots of quadratic equation $x^2 - 2x + 5 = 0$ are $1 - 2i$ and $1 + 2i$. Which are complex numbers. $z = 1 + 2i, 3i$ are also complex numbers.

We can represent the complex number $z = x + iy$ using Argand Diagram.



Sketch the complex number 3+2i:



Equality of two complex numbers:

Two complex numbers $z_1 = a + ib$ and $z_2 = c + id$ are equal. i.e. $z_1 = z_2$, if $a = c$ and $b = d$. In other words, two complex numbers are equal iff their real parts are equal and their imaginary parts are equal.

Modulus: The modulus or absolute value of $z = x + iy$ is $|z| = \sqrt{x^2 + y^2}$. It can be denoted by $mod(z)$. In fact, $|z|$ is the distance of z to the origin.

Algebraic operations on complex numbers:

Addition and subtraction:

If $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers, then

$z_1 \pm z_2 = (a + c) \pm (b + d)i.$

Multiplication:

If $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers, and k is a real number, then

(i) $z_1.z_2 = (a + ib)(c + id) = (ac - bd) + (ad + bc)i$

(ii) $kz_1 = ka + kbi$

Complex Conjugate:

If $z = a + ib$, then the conjugate of z is defined as $\bar{z} = a - bi$.

Note that $z.\bar{z} = (a + ib)(a - ib) = a^2 + b^2 = |z|^2$.

Division:

If $z_1 = a + ib$ and $z_2 = c + id$ are two complex numbers with $z_2 \neq 0$, then

$\frac{z_1}{z_2} = \frac{(a+ib)}{(c+id)} \times \frac{(c-id)}{(c-id)} = \frac{(ac+bd)+i(-ad+bc)}{c^2+d^2}$. In fact, $\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{|z_2|^2}$.

If we are dividing with a complex number, the denominator must be converted to a real number. In order to do this, multiply both the denominator and numerator by the complex conjugate of the denominator.

Laws of complex numbers: If z_1 and z_2 are two complex numbers, then

(i) (Triangle inequality)$|z_1 + z_2| \leq |z_1| + |z_2|$

(ii) $|z_1 - z_2| \geq \big||z_1| - |z_2|\big|$

(iii) (parallelogram equality) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2[|z_1|^2 + |z_2|^2]$

(iv)$|z_1z_2| = |z_1||z_2|$

(v) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$.

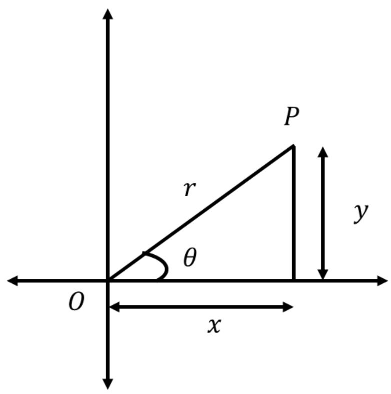
Tutorial:

1	Find the real and imaginary parts of the complex numbers (a) $3 + 4i$ (b) i (c) 10 **Solution:** (a) Here $z = 3 + 4i$ then $Re(z) = 3$ and $Im(z) = 4$. (b) Here $z = i$ then $Re(z) = 0$ and $Im(z) = 1$. You can also say that it is purely imaginary number. (c) Here $z = 10$ then $Re(z) = 10$ and $Im(z) = 0$, here z is a real number.
2	Simplify $\frac{(3+2i)^2}{(1+2i)} + \frac{1+i}{1-i}$. **Solution:** Note that, $(3 + 2i)^2 = (9 - 4) + (6 + 6)i = 5 + 12i,$

	Therefore, $\dfrac{(3+2i)^2}{(1+2i)}=\dfrac{(5+12i)}{(1+2i)}=\dfrac{(5+12i)}{(1+2i)}\times\dfrac{(1-2i)}{(1-2i)}=\dfrac{29+2i}{5}$ and $\dfrac{1+i}{1-i}=\dfrac{1+i}{1-i}\times\dfrac{1+i}{1+i}=\dfrac{(1+i)^2}{(1-i)(1+i)}=\dfrac{1+2i+i^2}{1^2+1^2}=\dfrac{2i}{2}=i$. Thus, $\dfrac{(3+2i)^2}{(1+2i)}+\dfrac{1+i}{1-i}=\dfrac{29+2i}{5}+i=\dfrac{29+7i}{5}=\dfrac{29}{5}+\dfrac{7}{5}i$ Which is the required form.
3	Solve $(3+4i)^2-2(x-iy)=x+iy$ for real numbers *x* and *y*. **Answer:** $x=-\frac{7}{3}, y=-24$.
4	Given $z_1=2+i$ and $z_2=3-4i$, find $\frac{1}{z_1}+\frac{1}{z_2}$ in the form of $a+ib$. **Answer:** $\frac{13}{25}-i\frac{2}{25}$.
5	If $w_1=-1+i$, $w_2=2-3i$ and $w_3=-3-2i$. Find (a) w_2-w_3 (b) w_1+w_3 (c) $\frac{w_2}{w_3}$ (d) $w_2\cdot w_3$. **Answer:** (a) $5-i$ (b) $2+3i$ (c) $0+i$ (d) $-12+5i$.

2.3 Complex numbers in polar and exponential forms

Let P be a point in the complex plane corresponding to the complex number $z=x+iy$.



From the above diagram, we can see that $x=r\cos\theta, y=r\sin\theta$.

As we have seen that, the modulus of $z = x + iy$ is $|z| = \sqrt{x^2 + y^2}$. We take $r = |z|$. In fact, it is the magnitude of the vector OP.

From the above figure, $\tan\theta = \frac{y}{x}, i.e. \theta = tan^{-1}\left(\frac{y}{x}\right)$.

The Argument or amplitude of $z = x + iy$ is defined as $\arg(z) = \theta = tan^{-1}\left(\frac{y}{x}\right)$. In fact it is the angle between x-axis and the line segment OP.

The complex number z can be written as $z = x + iy = re^{i\theta}$, where $e^{i\theta} = \cos\theta + i\sin\theta$

i.e. $z = rcos\,\theta + irsin\,\theta = r(\cos\theta + isin\,\theta)$. In notation, we may write $z = rcis\,\theta$.

Which is known as the polar form of a complex number.

For complex number $z \neq 0$, there corresponds only one value of θ in $-\pi < \theta \leq \pi$. This value of θ is called the value of the principle argument of z and interval is called principal range. It is denoted by $Arg(z)$. Also, $\arg(\mathrm{z}) = \theta \pm 2\mathrm{k}\pi, \mathrm{k} = 0,1,2, \dots$ is called the general value of the argument of z. (Note that $tan\,(\theta \pm 2k\pi) = \tan\theta$)

Note:

(i) Every complex number $x + iy$ can always be express in the form $r(\cos\theta + isin\,\theta)$.

(ii) Consider a complex number $z = a + ib$,

(1) If $b = 0$ and $a \geq 0$, then $\theta = 0$

(2) If $b = 0$ and $a < 0$, then $\theta = \pi$

(3) If $a = 0$ and $b > 0$, then $\theta = \frac{\pi}{2}$

(4) If $a = 0$ and $b < 0$, then $\theta = -\frac{\pi}{2}$.

(iii) Let $z = a + ib$ is a complex number. If $a \neq 0$, $b \neq 0$, $\alpha = \tan^{-1}\left(\left|\frac{b}{a}\right|\right)$ and

(1) $a > 0, b > 0$, then $\theta = \alpha$

(2) $a < 0, b > 0$, then $\theta = \pi - \alpha$

(3) $a < 0, b < 0$, then $\theta = -\pi + \alpha$

(4) $a > 0, b < 0$, then $\theta = -\alpha$

Tutorial:

1	Express the complex number $\left(\frac{1+i}{\sqrt{2}}\right)^4$ in its polar form. **Solution:** Here, $(1 + i)^4 = (1 + i)^2(1 + i)^2 = (2i)(2i) = 4i^2 = -4$ $\therefore z = \left(\frac{1+i}{\sqrt{2}}\right)^4 = \frac{-4}{4} = -1$

	Now, $r = \sqrt{(-1)^2} = 1$ $\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{0}{-1}\right) = \pi$ The required polar form is $z = 1\{\cos(\pi) + isin\,(\pi)\}$.
2	Express the following complex numbers in their polar form. (a) $z = \sqrt{3} + i$ (b) $z = i$ **Answer:** $2\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$.
3	Express $\frac{2+3i}{1-i}$ in polar form after finding $\arg(z)$ and r. **Answer:** $\sqrt{\frac{13}{2}}\,[\cos(\pi - \tan^{-1}5) + i\ \sin(\pi - \tan^{-1}5)\,]$.

2.4	De Moivre's theorem and its applications

De Moivre's Theorem: For any rational number n the value or one of the values of $(cos\theta + isin\theta)^n$ is $cosn\theta + isinn\theta$.

e.g. $(cos\theta + isin\theta)^5 = cos5\theta + isin5\theta$

Tutorial:

1	Express $\frac{(cos\theta+isin\theta)^8}{(sin\theta+icos\theta)^4}$ in the form $(x + iy)$. **Solution:** $\frac{(cos\theta+isin\theta)^8}{(sin\theta+icos\theta)^4} = \frac{(cos\theta+isin\theta)^8}{(i)^4(cos\theta+\frac{1}{i}sin\theta)^4}$ $= \frac{(cos\theta+isin\theta)^8}{(cos\theta-isin\theta)^4} = \frac{(cos\theta+isin\theta)^8}{[cos(-\theta)+isin(-\theta)]^4} = \frac{(cos\theta+isin\theta)^8}{[(cos\theta+isin\theta)^{-1}]^4}$ $= \frac{(cos\theta+isin\theta)^8}{(cos\theta+isin\theta)^{-4}} = (cos\theta + isin\theta)^{12} = cos12\theta + isin12\theta.$
2	Prove that $\frac{(cos\ \alpha+i\ sin\ \alpha)^4}{(sin\ \beta+i\ cos\ \beta)^5} = sin\ (4\alpha + 5\beta) - icos\ (4\alpha + 5\beta)$.
3	Prove that the general value of θ which satisfies the equation $(\cos\theta + i\sin\theta)\,(\cos 2\theta + i\sin 2\theta)\ldots(\cos n\theta + i\sin n\theta) = 1$ is $\frac{4m\pi}{n(n+1)}$, where m is an integer.

Roots of a Complex Number:

We know that $cos\theta + isin\theta = \cos(2n\pi + \theta) + i\sin(2n\pi + \theta), n \in Z,$

By De Moivre's theorem one of the value of

$(cos\theta + isin\theta)^{\frac{1}{q}} = cos\left(\frac{2n\pi+\theta}{q}\right) + isin\left(\frac{2n\pi+\theta}{q}\right)$, where q is an integer………… (1)

Giving n the values 0, 1, 2, 3, …, $(q-1)$ successively, we get the following q values of $(cos\theta + isin\theta)^{\frac{1}{q}}$,

$cos\left(\frac{\theta}{q}\right) + isin\left(\frac{\theta}{q}\right)$ (for n=0),

$\cos\left(\frac{2\pi+\theta}{q}\right) + isin\left(\frac{2\pi+\theta}{q}\right)$ (for n=1),

$\cos\left(\frac{4\pi+\theta}{q}\right) + isin\left(\frac{4\pi+\theta}{q}\right)$ (for n=2),

………………………….

$\cos\left(\frac{2(q-1)\pi+\theta}{q}\right) + isin\left(\frac{2(q-1)\pi+\theta}{q}\right)$ (for n=q-1).

Putting n=q in (1), we get a value of

$$(cos\theta + isin\theta)^{\frac{1}{q}} = \cos\left(\frac{2\pi+\theta}{q}\right) + isin\left(\frac{2\pi+\theta}{q}\right) = cos\left(\frac{\theta}{q}\right) + isin\left(\frac{\theta}{q}\right),$$

Which is same as the value of $n = 0$.

Thus, the values of $(cos\theta + isin\theta)^{\frac{1}{q}}$ for n=q, q+1, q+2,... etc. are the repetition of the values obtained as above.

Hence $(cos\theta + isin\theta)^{\frac{1}{q}}$ has q and only q distinct values given as above.

Tutorial:

1	Use De Moivre's theorem to solve the equation $x^4 - x^3 + x^2 - x + 1 = 0$. **Solution:** We have $x^4 - x^3 + x^2 - x + 1 = 0$. From which, $(x+1)\,(x^4 - x^3 + x^2 - x + 1) = 0$. We get $x^5 + 1 = 0$, which can be written as $x^5 = -1 = (cos\pi + isin\pi) = cos(2n\pi + \pi) + isin(2n\pi + \pi)$. Using De Moivre's theorem, we obtain $x = [cos(2n+1)\pi + isin(2n+1)\pi]^{\frac{1}{5}} = cos\left(\frac{(2n+1)\pi}{5}\right) + isin\left(\frac{(2n+1)\pi}{5}\right)$ when n=0,1,2,3,4, the values are $cos\frac{\pi}{5} + isin\frac{\pi}{5},$

	$$cos\frac{3\pi}{5}+isin\frac{3\pi}{5},$$ $$cos\pi+isin\pi,$$ $$cos\frac{7\pi}{5}+isin\frac{7\pi}{5},$$ $$cos\frac{9\pi}{5}+isin\frac{9\pi}{5}.$$ $cos\pi+isin\pi=-1$, which is rejected as it is corresponding to $x+1=0$. Hence, the required roots of the equation $x^4-x^3+x^2-x+1=0$ are $cos\frac{\pi}{5}+isin\frac{\pi}{5}$, $cos\frac{3\pi}{5}+isin\frac{3\pi}{5}$, $cos\frac{7\pi}{5}+isin\frac{7\pi}{5}$, $cos\frac{9\pi}{5}+isin\frac{9\pi}{5}$.
2	Find the roots of the equation $x^3+8=0$. **Answer:** $2\left[\cos\left(\frac{2n\pi+\pi}{3}\right)+isin\left(\frac{2n\pi+\pi}{3}\right)\right]$, where $n=0,1,2$.
3	Find the values of $(-i)^{\frac{1}{6}}$. **Answer:** $cos(4n+1)\frac{\pi}{12}-i\,sin(4n+1)\frac{\pi}{12}$, where $n=0,1,2,3,4,5$.

2.5	Exponential, Logarithmic, Trigonometric and hyperbolic functions.

Exponential Function:

If $z=x+iy$ then the exponential function with complex variable is defined as $e^z=e^{(x+iy)}=e^xe^{iy}=e^x(\cos y+i\sin y)$, where $e^{iy}=\cos y+i\sin y$ is known as the Euler's formula.

Logarithmic Function:

If $z=x+iy$ and $w=u+iv$ such that $e^w=z$, then w is said to be a logarithm of z to the base e and written as $w=\log_e z$.

If $z=r(\cos\theta+i\sin\theta)$, then $r(\cos\theta+i\sin\theta)=z=e^w=e^u(\cos v+i\sin v)$ implies $r=e^u$ or $u=\log r$ and $v=\theta+2n\pi$, $\theta\in(-\pi,\pi]$

Hence, $\log z=\log r+i(\theta+2n\pi)$, $n\in\mathbb{Z}$ i.e. the logarithm of a complex number has an infinite number of values, therefore, it is a multi-valued function. If $n=0$ then it is called the principal value and it is denoted by $Log\ z$. i.e. $Log\ z=\log|z|+i\theta$, $\theta+2n\pi$ is known as the argument of z and denoted by $arg(z)$. θ is known as the principal argument of z and denoted by $Arg(z)$.

Circular Functions:

Since, $e^{iy} = cosy + isiny$ and $e^{-iy} = cosy - isiny$. Then

$siny = \frac{e^{iy}-e^{-iy}}{2i}$ and $cosy = \frac{e^{iy}+e^{-iy}}{2}$.

Similarly one can define circular function of a complex variable z by the equations:

$sinz = \frac{e^{iz}-e^{-iz}}{2i}, cosz = \frac{e^{iz}-e^{-iz}}{2}$, $tanz = \frac{sinz}{cosz}$, $cosecz = \frac{2i}{e^{iz}-e^{-iz}}$, $secz = \frac{2}{e^{iz}-e^{-iz}}$ and $cotz = \frac{cosz}{sinz}$.

Hyperbolic Functions:

If x is real or complex, then $\frac{e^x-e^{-x}}{2}$ is defined as hyperbolic sine of x and is written as $sinh\ x$.

$\frac{e^x+e^{-x}}{2}$ is defined as hyperbolic cosine of x and is written as $cosh\ x$.

Also we define,

$\tan hx = \frac{\sin hx}{\cos hx} = \frac{e^x-e^{-x}}{e^x+e^{-x}}$; $cothx = \frac{1}{tanhx} = \frac{e^x+e^{-x}}{e^x-e^{-x}}$; $sechx = \frac{1}{coshx} = \frac{2}{e^x+e^{-x}}$ and $cosechx = \frac{1}{\sin hx} = \frac{2}{e^x-e^{-x}}$

Relations between hyperbolic and circular functions:

$\sin ix = \mathrm{i} \sin\ \mathrm{hx}$

$\cos ix = \cos \mathrm{hx}$

$\tan ix = itanhx$

$\sinh ix = isinx$

$\cosh ix = \cos x$

$\tanh ix = \mathrm{i} \tan x$

Formulae of Hyperbolic Functions:

$cosh^2x - sinh^2x = 1$

$sech^2x = 1 - tanh^2x$

$cosech^2x = coth^2x - 1$

$\sinh(x \pm y) = sinhxcoshy \pm coshxsinhy$

$\cosh(x \pm y) = cosxcoshy \pm sinhxsinhy$

$$\tanh(x \pm y) = \frac{tanhx \pm \tanh y}{1 \pm tanhxtanhy}$$

$sin\ h2x = 2sinhxcoshx$

$\cosh 2x = cosh^2x + sinh^2x$

$cosh\ 2x = 2cosh^2x - 1$

$cosh\,2x = 1 + 2sinh^2x$

$\tanh 2x = \dfrac{2tanhx}{1 + tanh^2x}$

$\sinh x + \sinh y = 2\sinh\left(\dfrac{x+y}{2}\right) cosh\left(\dfrac{x-y}{2}\right)$

$\sinh x - \sinh y = 2\cosh\left(\dfrac{x+y}{2}\right) sinh\left(\dfrac{x-y}{2}\right)$

$\cosh x + \cosh y = 2\cosh\left(\dfrac{x+y}{2}\right) cosh\left(\dfrac{x-y}{2}\right)$

$\cosh x - \cosh y = 2\sinh\left(\dfrac{x+y}{2}\right) sin\left(\dfrac{x-y}{2}\right)$

Tutorial:

1	Find the value of $log(-i)$. **Solution:** We know that $\log(z) = \log r + i\,(2n\pi + Arg(z)), n \in \mathbb{Z}$. Here $r = 1$ and $Arg(z) = -\frac{\pi}{2}$. $log(-i) = \log 1 + \left(2n\pi - \frac{\pi}{2}\right) i = (4n-1)\frac{\pi}{2} i, n \in \mathbb{Z}$.
2	Separate the real and imaginary parts of $(-i)^{-(1-i)}$. **Answer:** 0 and $e^{\frac{\pi}{2}}$
3	Show that $\sin ix = isinhx, x \in R$. **Solution:** We know that $sinx = \frac{e^{ix}-e^{-ix}}{2i}$. Then $sinix = \frac{e^{-x}-e^{x}}{2i} = i\,\frac{e^{x}-e^{-x}}{2} = isinhx$.
4	If $\tan(x+iy) = \sin(u+iv)$, then show that $\frac{\sin 2x}{\sinh 2y} = \frac{\tan u}{\tanh v}$.
5	If $\sin(A+iB) = x+iy$, then prove that $\frac{x^2}{\cosh^2 B} + \frac{y^2}{\sinh^2 B} = 1$.