I	**Higher order derivatives and applications**

Practice Examples

1.1	Lagrange's Mean Value Theorem, Local Maxima and Minima of function of one variable

1	Check whether the Mean Value Theorem can be applicable to the function $f(x) = x(x-1)(x-2)$ on the closed interval $[0,3]$. If so, find a value of c which satisfies the Mean value theorem in $(0,3)$. **Answer:** 2.
2	Check whether the Mean Value Theorem can be applicable to the function $f(x) = \frac{6}{x} - 3$ on the closed interval $[1,2]$. If so, find a value of c which satisfies Mean value theorem in $(1,2)$. **Answer:** $\sqrt{2}$.
3	Check whether the Mean Value Theorem can be applicable to the function $f(x) = 3x^2 + 5x - 2$ on the closed interval $[-1,1]$. If so, find a value of c which satisfies Mean value theorem in $(-1,1)$. **Answer:** 0.
4	Check whether the Mean Value Theorem can be applied to the function $x^3 + 24x - 16$ on the closed interval $[0,4]$. If so, find a value of c which satisfies the Mean value theorem in $(0,4)$. **Answer:** $\frac{4\sqrt{3}}{3}$.
5	Check whether the Mean Value Theorem can be applied to the function $f(x) = 8x + e^{-3x}$ on the closed interval [-2,3]. If so, find a value of c which satisfies the Mean value theorem in (-2, 3). **Answer:** -1.0973.

6	Suppose we know that $f(x)$ is continuous and differentiable on the interval $[-7, 0]$ that $f(-7) = -3$ and that $f'(x) \leq 2$. What is the largest possible value for $f(0)$? **Answer:** 11.

1	Find the extreme values of the function $f(x) = x^3 + x^2 - x + 1$. **Answer:** 2 and $\frac{22}{27}$.
2	Find the extreme values of the function $f(x) = 2x^3 - 21x^2 + 36x - 20$. **Answer:** -3 and -128.
3	Find the extreme values of the function $f(x) = 4x^3 - 21x^2 + 36x + 30$. **Answer:** 50 and 201/4.
4	Find the extreme value of the function$f(x) = a^{x+1} - a^x - x \ (a > 1)$. **Answer:** $\frac{\log[(ae-e)loga]}{loga}$.
5	Suppose an open cylinder of given surface area having maximum volume. Prove that its height is equal to the radius of its base.
6	Show that semi vertical angle of a cone of given slant height and maximum volume is $\tan^{-1}\sqrt{2}$.

1.2	Successive differentiation: n^{th} derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic

1	Find the n^{th} order derivative of the function $y = cos^2x\ sinx$. **Answer:** $y_n = \frac{1}{4}\left[3^n sin\left(3x + \frac{n\pi}{2}\right) + sin\left(x + \frac{n\pi}{2}\right)\right]$.
2	Find the n^{th} order derivative of the function $y = \frac{1}{6x^2-5x+1}$. **Answer:** $y_n = (-1)^{n+1}n!\left[\frac{3^{n+1}}{(1-3x)^{n+1}} - \frac{2^{n+1}}{(1-2x)^{n+1}}\right]$.
3	Show that the n^{th} order derivative of the function $y = \log(4x^2 - 9)$ is

	$y_n = (-1)^{n-1}(n-1)!\,2^n\left[\frac{1}{(2x-3)^n} + \frac{1}{(2x+3)^n}\right]$.
4	Find the n[th] order derivative of the function $y = \sinh(3x)$. **Answer:** $y_n = \frac{1}{2}[3^n e^{3x} - (-3)^n e^{-3x}]$.
5	Find the n[th] order derivative of the function $y = sin^4 x$. **Answer:** $y_n = -2^{n-1}cos\left(2x + \frac{n\pi}{2}\right) + \frac{4^{n-1}}{2}cos\left(4x + \frac{n\pi}{2}\right)$.
6	If $y = xlog\left(\frac{x-1}{x+1}\right)$, show that $y_n = (-1)^{n-2}(n-2)!\left[\frac{x-n}{(x-1)^n} - \frac{x+n}{(x+1)^n}\right]$ for $n \geq 2$.

1.3	Leibnitz rule for the n[th] order derivatives of product of two functions

1	If $y = acos(logx) + bsin(logx)$, then show that $x^2 y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$.
2	If $y = sin^{-1}x$, then show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$.
3	If $y = (tan^{-1}x)^2$, then show that $(1+x^2)^2 y_2 + 2x(x^2+1)y_1 - 2 = 0$.
4	Find the n[th] order derivative of the function $y = x^3\log(3x)$.
5	If $y = e^{acos^{-1}x}$, then show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+a^2)y_n = 0$.
6	If $y = cos^{-1}x$, then show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$.

1.4	Power series expansion of a function: Maclaurin's and Taylor's series expansion.

1	Find the Taylor's series expansion of $f(x) = x^3 - 2x + 4$ about $a = 2$. **Answer:** $f(x) = 8 + 10(x-2) + 6(x-2)^2 + (x-2)^3$.
2	Expand $sin\left(\frac{\pi}{4} + x\right)$ in powers of x. Find the value of $\sin 44^0$. **Answer:** $\sin 44^0 = 0.69467$.

3	Expand $\log\sqrt{\frac{1+x}{1-x}}$ into Maclaurin's series. **Answer:** $\log\sqrt{\frac{1+x}{1-x}} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$.
4	Using Taylor's series, find $\sqrt[3]{27.12}$ correct to four decimal places. **Answer:** $\sqrt[3]{27.12} = 3.00443$.
5	Expand $e^{a\cos bx}$ using Maclaurin's series. **Answer:** $e^{a\cos bx} = e^a - e^a ab^2\frac{x^2}{2} + e^a ab^4(1+3a)\frac{x^4}{24} - \cdots$.
6	Expand $\frac{e^x}{cosx}$ into Maclaurin's series. **Answer:** $\frac{e^x}{cosx} = 1 + x + x^2 + \frac{2x^3}{3} + \cdots$.

1.5	L'Hospital's rule and related applications, Indeterminate forms

1	Evaluate $\lim_{x\to 1}\frac{x^x - x}{x-1-\log x}$. **Answer:** 2.
2	Evaluate $\lim_{x\to y}\frac{x^y - y^x}{x^x - y^y}$. **Answer:** $\frac{1-log\ y}{1+log\ y}$.
3	Evaluate $\lim_{x\to\infty}\frac{x\log x}{x+\log x}$. **Answer:** ∞.
4	Evaluate $\lim_{x\to 0}\frac{tan^2x - x^2}{x^2tan^2x}$. **Answer:** $\frac{2}{3}$.
5	Find the value of a and b if $\lim_{x\to 0}\frac{x(1+a\cos x) - b\sin x}{x^3} = 1$. **Answer:** $a = -\frac{5}{2}, b = -\frac{3}{2}$.
6	Evaluate $\lim_{x\to\infty}\frac{\log(1+e^{3x})}{x}$.

	Answer:3.
7	Evaluate $\lim_{x\to 1}(1-x)tan\left(\frac{\pi x}{2}\right)$. **Answer:**$\frac{2}{\pi}$.
8	Evaluate $\lim_{x\to 0}\left[\frac{1}{x^2}-\frac{1}{sin^2x}\right]$. **Answer:** $-\frac{1}{3}$.
9	Evaluate $\lim_{x\to 1}(x-1)^{x-1}$. **Answer:** 1 .
10	Prove that $\lim_{x\to 0} x\ logx = 0$.
11	Evaluate $\lim_{x\to\frac{\pi}{2}}(secx - tanx) = 0$.
12	Evaluate $\lim_{x\to 0}(\cot x)^{1/\log x}$. **Answer:**$e^{-1}$.