# Unit V

| 5. | **Improper and Multiple Integrals:** |
|---|---|
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| 5.2 | Definitions and properties of Gamma, Beta and Error functions |
| 5.3 | Evaluation of double integrals |
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## 5.1 Introduction to Improper Integrals

In most of applications of engineering and Science there occurs special function, like gamma function, beta function etc. Which are in the form of integrals which are of special types in which the limits of integration are infinity or the integrand becomes unbounded within the limits. Such types of integrals are known as improper integrals. Beta and gamma functions are very fundamental and hold great importance in various branches of Engineering and physics.

**Improper integrals:**

The integrals of the form $\int_a^b f(x)\,dx$ is said to be an improper integral if

**(i)** one or both limits of integrations are infinite

**(ii)** function $f(x)$ becomes infinite at the end points of the interval or at a point within the interval of integrations.

Improper integrals are classified into following kinds.

**<u>Improper integrals of the first kind (Type-I)</u> (Just for information only)**

If in the definite integral $\int_a^b f(x)dx$, a or b or both a and b are infinite, then the integral is called improper integral of Type-I.

(1) If f(x) is continuous on [a, ∞), then $\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$

(2) If f(x) is continuous on (−∞, b], then $\int_{-\infty}^b f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx$

(3) If f(x) is continuous on (−∞, ∞), then $\int_{-\infty}^\infty f(x)dx = \int_{-\infty}^c f(x)dx + \int_c^\infty f(x)dx$, where c is any real number.

$$i.e. \int_{-\infty}^{\infty} f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx + \lim_{b\to\infty}\int_c^b f(x)dx$$

For example: The integrals $\int_1^\infty \frac{1}{\sqrt{x}}dx$, $\int_1^\infty \frac{1}{x^2}dx$ and $\int_{-\infty}^0 xsinxdx$ are improper integrals of first kind.

**<u>Improper Integrals of second kind (Type-II)</u> (Just for information only)**

If in the definite integral $\int_a^b f(x)dx$, the integrand f(x) becomes infinite at x=a or x=b or at one or more points within the interval (a, b), then the integral is called improper integral of Type-II.

(1) If f(x) is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)dx = \lim_{\delta\to 0+}\int_{a+\delta}^b f(x)dx$$

(2) If f(x) is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)dx = \lim_{\delta\to 0+}\int_a^{b-\delta} f(x)dx$$

(3) If f(x) is discontinuous at $c \in (a,b)$ and continuous on $[a,c)\ U\ (c,b]$, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$i.e. \int_a^b f(x)dx = \lim_{\delta\to 0+}[\int_a^{c-\delta} f(x)dx + \int_{c+\delta}^b f(x)dx]$$

For instance: The integrals $\int_0^3 \frac{1}{\sqrt{3-x}}dx$ and $\int_0^{\frac{\pi}{2}} secxdx$ are improper integrals of second kind.

**5.2 Definitions and properties of Gamma, Beta and Error functions**

- **Gamma function:**

  Let $n$ be any positive real number. Then the Gamma function of $n$ is defined as

$$\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx .$$

- **Beta function:**

  Let $m$ and $n$ be any two positive real numbers. Then the beta function of $m$ and $n$ is defined as $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$.

- **Properties of Gamma and Beta function:**

  **(1)** $\Gamma(1) = 1$

  **(2)** $\Gamma(n+1) = n\Gamma(n) = n!$ for $n \in \mathbb{N}$

  **(3)** $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

  For $m,n$ positive real numbers,

  **(4)** $\beta(m,n) = \beta(n,m)$

  **(5)** $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1} x\cos^{2n-1} x\, dx$

**(6)** $\beta(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

**(7)** $\int_0^{\pi/2}\sin^p x\cos^q x\,dx=\dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$ for positive real numbers $p$ and $q$

## Error function:

- **Error function:**

  The error function of $x$ is defined as $erf(x)=\dfrac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$.

- **Complimentary Error function:**

  The complimentary error function of $x$ is defined as $erf_c(x)=\dfrac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$.

- **Properties of error function:**

  **(1)** $erf(0)=0$

  **(2)** $erf(\infty)=1$

  **(3)** $erf(x)+erf_c(x)=1$

  **(4)** $erf(-x)=-erf(x)$

**Example:** Evaluate $\int_0^{\pi/2}\sqrt{\tan x}\,dx$.

**Sol:** We know that $\int_0^{\pi/2}\sin^p x\cos^q x\,dx=\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)=\dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$.

Here $\int_0^{\pi/2}\sqrt{\tan x}\,dx=\int_0^{\pi/2}\sin^{1/2}x\cos^{-1/2}x\,dx=\beta\left(\dfrac{1/2+1}{2},\dfrac{-1/2+1}{2}\right)=\beta\left(\dfrac{3/2}{2},\dfrac{1/2}{2}\right)=\frac{1}{2}\beta\left(\frac{3}{4},\frac{1}{4}\right)$

**Tutorial:**

| | |
|---|---|
| **1** | Prove that $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$. |
| **2** | Evaluate $\int_0^\infty x^7 7^{-x}\,dx$. **(Ans:** $\dfrac{\Gamma(8)}{(\log 7)^8}$ **)** |

| | |
|---|---|
| **3** | Evaluate $\int_0^1 \sqrt{x}(1-x^2)^{1/3}\,dx$. **(Ans:** $\frac{1}{2}\beta\left(\frac{3}{4},\frac{4}{3}\right)$**)** |
| **4** | Show that $\int_{-a}^{a} e^{-(x+a)^2}\,dx = \frac{\sqrt{\pi}}{2}\,erf(2a)$. |
| **5** | Show that $\int_{a}^{\infty} e^{-(2x-a)^2}\,dx = \frac{\sqrt{\pi}}{4}(1-erf(a))$. |
| **6** | Prove that $erf(x) = \alpha(x\sqrt{2})$, where $\alpha(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{-t^2/2}\,dt$. |

## 5.3 Evaluation of double integrals

**Double Integrals:**

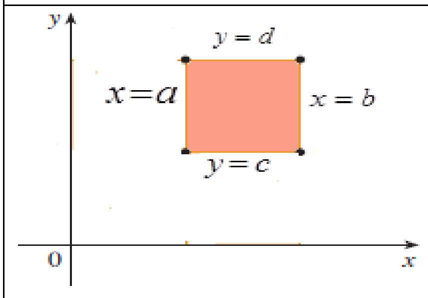
$$\iint_R f(x,y)\,dA = \iint f(x,y)\,dxdy = \iint f(x,y)\,dydx$$

**Fubini's theorem:**

Let $f(x,y)$ be a continuous function defined on a rectangular region $R: a \le x \le b; c \le y \le d$.

Then $\iint_R f(x,y)\,dA = \int_{y=c}^{d}\int_{x=a}^{b} f(x,y)\,dxdy = \int_{x=a}^{b}\int_{y=c}^{d} f(x,y)\,dydx$.

**Evaluation of the double integration:**

<table>
<tr><td>Limits of Integral over Rectangle R:</td><td></td></tr>
<tr><td>
</td><td>Rectangle R whose sides are $x=a$, $x=b$ $y=c$ and $y=d$<br>$\int_a^b\int_c^d f(x,y)\,dydx$<br>OR<br>$\int_c^d\int_a^b f(x,y)\,dx\,dy$</td></tr>
<tr><td colspan="2">Double Integral over general region:</td></tr>
</table>

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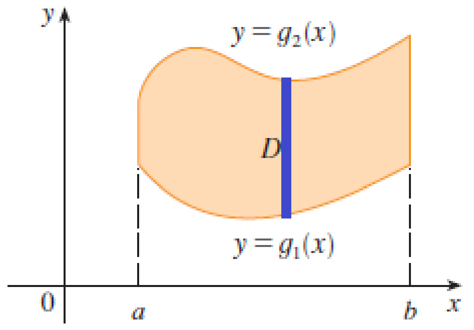
<tr>
<td>
</td>
<td>For $\int \int f(x,y)\,dy\,dx$<br>Draw vertical strip in region,<br>lower end of strip touches curve $y = g_1(x)$ and<br>upper end of strip touches curve $y = g_2(x)$,<br>hence limit of y: From $y = g_1(x)$ to $y = g_2(x)$.<br>This strip moves from $x = a$ to $x = b$.<br>Hence, limit of x: From $x = a$ to $x = b$.<br>$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$</td>
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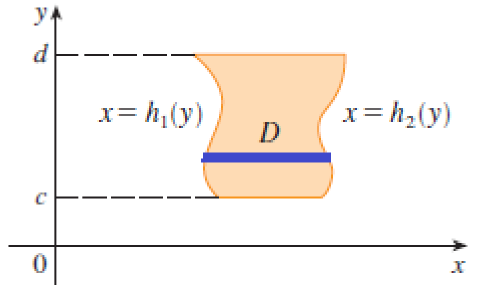
<td>
</td>
<td>For $\int \int f(x,y)\,dx\,dy$<br>Draw horizontal strip in region,<br>left end of strip touches curve $x = h_1(y)$ and right end of the strip touches curve $x = h_2(y)$,<br>hence limit of y: From $x = h_1(y)$ and $x = h_2(y)$.<br>This strip moves from $y = c$ to $y = d$.<br>Hence, limit of x: From $y = c$ to $y = d$.<br>$\int_{y=c}^{y=d} \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy$</td>
</tr>
</table>

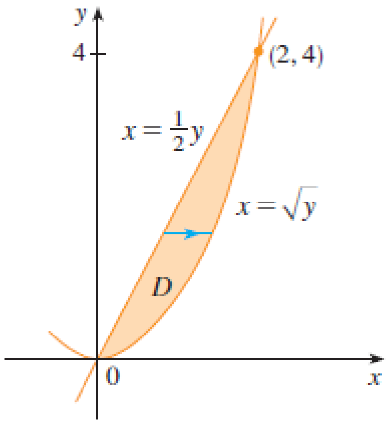
**Example:** Evaluate $\int_0^a \int_0^{\sqrt{ay}} xy\,dxdy$.

**Solution:** Here $\int_0^a \int_0^{\sqrt{ay}} xy\,dxdy = \int_0^a \left[ \int_0^{\sqrt{ay}} xydx \right] dy = \int_0^a y \left[ \frac{x^2}{2} \right]_0^{\sqrt{ay}} dy$

$$= \int_0^a y \left[ \frac{x^2}{2} \right]_0^{\sqrt{ay}} dy = \frac{1}{2} \int_0^a ay^2\,dy = \frac{a}{2} \left[ \frac{y^3}{3} \right]_0^a = \frac{a^4}{6}.$$

**Example:** Evaluate the $\iint_R \left(x^2 + y^2\right) dA$, where R is the region bounded by the line $y = 2x$ and the parabola $y = x^2$.

**Solution:** (As $2x = y = x^2, x = 0$ or $x = 2$. So both curves intersect at $(0,0)$ and $(2,4)$.)




From figure with horizontal strip,
Limits of $x$: $x=\frac{y}{2}$ $to$ $x=\sqrt{y}$
Limits of $y$: $y=0$ $to$ $y=4$

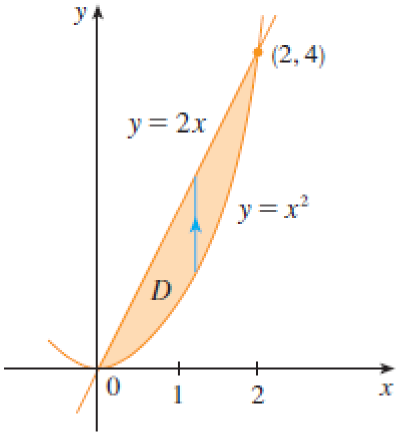
$$I=\iint_R \left(x^2+y^2\right)dA$$

$$=\int_0^4\int_{\frac{y}{2}}^{\sqrt{y}}\left(x^2+y^2\right)dx\,dy$$

$$=\int_0^4\left(\frac{x^3}{3}+xy^2\right)_{\frac{y}{2}}^{\sqrt{y}}dy$$

$$=\int_0^4\left(\frac{y^{\frac{3}{2}}}{3}+y^{\frac{5}{2}}-\frac{13y^3}{24}\right)dy$$

$$=\left(\frac{2y^{\frac{5}{2}}}{15}+\frac{2y^{\frac{7}{2}}}{7}-\frac{13y^4}{96}\right)_0^4=\frac{216}{35}$$




From figure with vertical strip,
Limits of $y$: $y=x^2$ $to$ $y=2x$
Limits of $x$: $x=0$ $to$ $x=2$

$$I=\iint_R \left(x^2+y^2\right)dA$$

$$=\int_0^2\int_{x^2}^{2x}\left(x^2+y^2\right)dy\,dx$$

$$=\int_0^2\left(x^2y+\frac{y^3}{3}\right)_{x^2}^{2x}dx$$

$$=\int_0^2-\left(x^4+\frac{x^6}{3}-\frac{14x^3}{3}\right)dx$$

$$=-\left(\frac{x^5}{5}+\frac{x^7}{21}-\frac{7x^4}{6}\right)_0^2\quad=\frac{216}{35}$$

**Tutorial:**

| | |
|---|---|
| **1** | Evaluate $\int_0^1\int_0^1\frac{1}{\sqrt{(1-x^2)(1-y^2)}}dxdy$. **(Ans:** $\frac{\pi^2}{4}$**)** |
| **2** | Evaluate $\iint_R xdxdy$; where $R$ is the region bounded by the curves $x=0; y=0; x=\frac{y^2}{4}; y=4$. **(Ans:** $\frac{32}{5}$**)** |

| | |
|---|---|
| **3** | Evaluate $\iint\limits_R xy(x+y)\,dA$, where R is the region bounded by $y = x$ & $y = x^2$. **(Ans: 1/24)** |

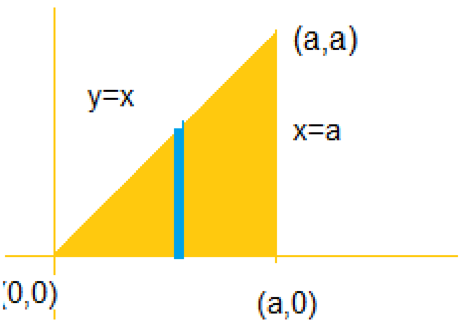
**5.4 Change of order of double integration, Transformation to polar coordinates, Applications of double integrals: Area**

In case of double integral with variable limits, the limits of the integration changes with the change of order of integration. The new limits are obtained by sketching the region of integration. Sometime in changing the order of integration, it is required to split up the region of integration, and the given integral is expressed as the sum of number of double integrals with the changed limits.

Change the order of integration from either $dxdy$ to $dydx$ or $dydx$ to $dxdy$ by changing the corresponding limits of $x$ and $y$.

**Example:** Change the order of integration in $\int_0^2\int_y^2 \frac{xdxdy}{x^2+y^2}$, and evaluate the same.

**Solution:**




From the limits of integration, it is clear that the region of integration is bounded by y=x, x=2 and y= 2. Thus, the region of integration as shown in figure.

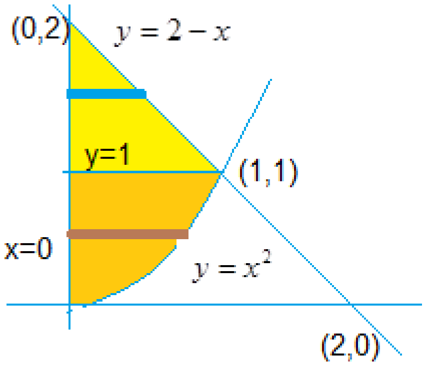
Draw vertical strip. So new limits of integration are
Limits of x: from y=0 to y=x
Limits of y: from x=0 to x=2

$$\int_0^2\int_0^x \frac{xdydx}{x^2+y^2} = \int_0^2 x\left(\frac{1}{x}\tan^{-1}\left(\frac{y}{x}\right)\right)_0^x dx$$

$$= \int_0^2 \frac{\pi}{4}dx = \frac{\pi}{4}(x)_0^2 = \frac{\pi}{2}$$

**Example:** Change the order of integration $\int_0^1\int_{x^2}^{2-x} xydy\,dx$.

**Solution:**




Here given that the region is bounded by $x^2 = y$, $y = 2 - x$ , $x = 0$ and $x = 1$

To change the order of integration, we divide region into following two regions:

Region 1 :

Limits of x: $x = 0$ to $x = \sqrt{y}$

Limits of y: y=0 to y= 1

Region 2:

Limits of x: $x = 0$ to $x = 2 - y$

Limits of y: y=1 to y=2

$$I_1 = \int_0^1 \int_0^{\sqrt{y}} xy\,dx\,dy$$

$$= \int_0^1 \left( \frac{x^2}{2} \right)_0^{\sqrt{y}} y\,dy = \int_0^1 \frac{y^2}{2} dy = \left( \frac{y^3}{6} \right)_0^1 = \frac{1}{6}$$

$$I_2 = \int_1^2 \int_0^{2-y} xy\,dx\,dy = \int_1^2 \left( \frac{x^2}{2} \right)_0^{2-y} y\,dy$$

$$= \int_0^1 \frac{(2-y)^2 y}{2} dy = \int_1^2 \frac{\left(4y - 4y^2 + y^3\right)}{2} dy$$

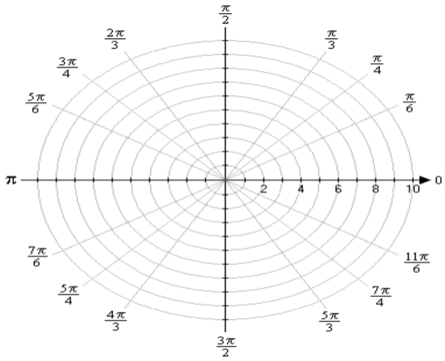
$$= \left( y^2 - \frac{2}{3} y^3 + \frac{y^4}{8} \right)_1^2 = \frac{5}{24}$$

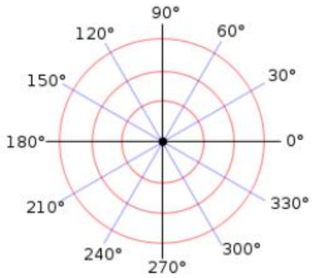
$$\therefore I = I_1 + I_2 = \frac{1}{6} + \frac{5}{24} = \frac{3}{8}$$

| **Tutorial:** | |
|---|---|
| **1** | Evaluate $\int_0^1 \int_{4y}^4 dxdy$ by changing the order of integration. **(Ans: 2)** |
| **2** | Evaluate $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2}{\sqrt{y^4 - a^2 x^2}} dxdy$ by changing the order of integration.<br><br>**(Ans:** $\frac{\pi a^2}{6}$ **)** |
| **3** | Evaluate $\int_0^2 \int_{2-x}^{\sqrt{4-x^2}} x\,dA$ by changing the order of integration. **(Ans:** $\frac{4}{3}$**)** |

**Change of Cartesian coordinates to Polar coordinates:**

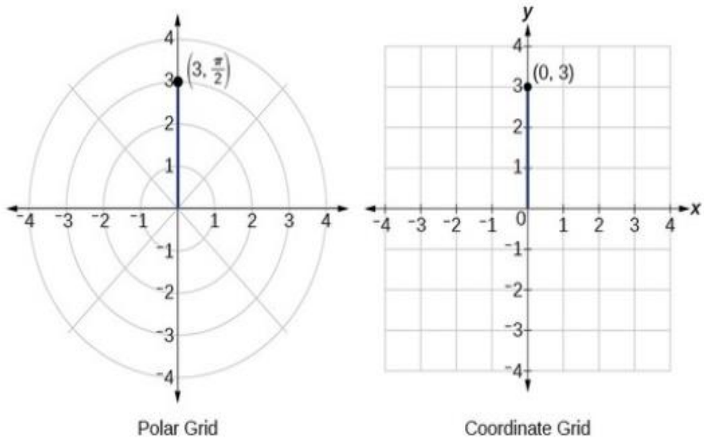
In a polar coordinate grid, as shown below, there will be a series of circles extending out from the pole (or origin in a rectangular coordinate grid) and five different lines passing through the pole to represent the angles at which the exact values are known for the trigonometric functions.






The distance from the pole is called the radial coordinate or radius, and the angle is called the angular coordinate or polar angle. The radial coordinate is often denoted by r, and the angular coordinate by θ.

Representation of (0,3) in cartesian coordinate system and polar coordinate system:




**Polar and Coordinate Grid of Equivalent Points:** The rectangular coordinate $(0, 3)$ is the same as the polar coordinate $(3, \frac{\pi}{2})$ as plotted on the two grids above.

**Double integral in polar coordinate system:**

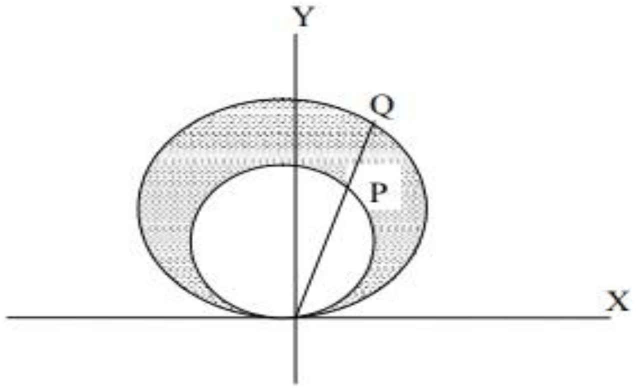
$\iint_R f(r,\theta)drd\theta$; where $r$ is distance from the pole (radial coordinate), $\theta$ is the angular coordinate.

Here $r_1 = f_1(\theta)$ to $r_2 = f_2(\theta)$ and $\theta_1 = \alpha$ to $\theta_2 = \beta$ .

$\iint_R f(r,\theta)drd\theta = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta)drd\theta$

**Example -1:** Evaluate $\iint_R r^3 dr d\theta$ over the area included between the $r = 2\,sin\theta$ and $r = 4\,sin\theta$.

**Solution:** Here given that the region bounded between $r = 2\,sin\theta$ and $r = 4\,sin\theta$.




Therefore the limits are

$r = 2\,sin\theta$ and $r = 4\,sin\theta$ and $\theta = 0$ *to* $\theta = \pi$

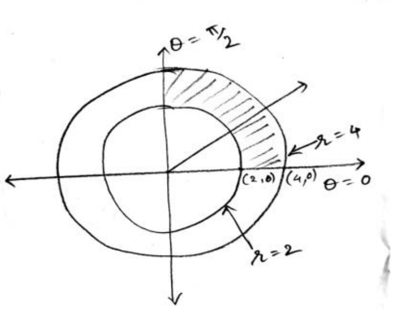
Therefore $\iint_R r^3 dr d\theta = \int_{\theta=0}^{\pi} \int_{r=2\sin\theta}^{4\sin\theta} r^3 dr d\theta = \int_{\theta=0}^{\pi} \left[ \frac{r^4}{4} \right]_{2\sin\theta}^{4\sin\theta} d\theta = \int_{\theta=0}^{\pi} 60 \sin^4 \theta d\theta$

$= 60 \int_{\theta=0}^{\pi} sin^4\,\theta d\theta = 120 \int_{\theta=0}^{\frac{\pi}{2}} sin^4\,\theta d\theta = 120\,.\frac{3}{4}.\frac{1}{2}.\frac{\pi}{2} = \frac{45\pi}{2}$ (By the property (7) of the Gamma and Beta functions).

$$60 \int_{\theta=0}^{\pi} \sin^4 \theta d\theta = 120 \int_{\theta=0}^{\pi/2} \sin^4 \theta d\theta = 120 \frac{3}{2} \frac{1}{2} \frac{\pi}{2} = \frac{45\pi}{2}.$$

**Example-2** Evaluate: $\iint_R r^3 \sin 2\theta dr d\theta$, where R is the region bounded in the first quadrant between $r = 2$ & $r = 4$.

**Sol:** Here given that the region bounded in the first quadrant between $r = 2$ & $r = 4$.



Therefore the limits are

$$r = 2 \; to \; r = 4 \text{ and } \theta = 0 \; to \; \theta = \frac{\pi}{2}$$

Therefore $$\iint_R r^3 \sin 2\theta dr d\theta = \int_{\theta=0}^{\pi/2} \int_{r=2}^{4} r^3 \sin 2\theta dr d\theta$$

$$= \int_{\theta=0}^{\pi/2} \sin 2\theta \left[ \frac{r^4}{4} \right]_{r=2}^{4} d\theta$$

$$= 60 \int_{\theta=0}^{\pi/2} \sin 2\theta d\theta$$

$$= 60 \left[ \frac{-\cos 2\theta}{2} \right]_{\theta=0}^{\pi/2} = -30[\cos \pi - \cos 0] = -30[-1-1] = 60.$$

**Change of Variables:**

To convert Cartesian system into Polar system, take $x = r\cos\theta, y = r\sin\theta$ and Jacobian $J = \frac{\partial(x,y)}{\partial(r,\theta)}$. Then for the polar system the order of integration becomes

$dA = |J| dr d\theta = r dr d\theta.$

$$\iint_R f(x,y) dA = \iint_R f(r,\theta)\, r\, dr d\theta = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f(r,\theta)\, r\, dr d\theta$$

**Tutorial:**

| | |
|---|---|
| **1** | Evaluate the integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} y^2\sqrt{x^2+y^2}dydx$ by changing into polar coordinates. **(Ans: $\boldsymbol{\frac{a^5\pi}{20}}$)** |
| **2** | Evaluate $\iint \frac{4xy}{x^2+y^2} e^{-(x^2+y^2)} dxdy$ over the region bounded by the circle $x^2+y^2-x=0$ in the first quadrant. **(Ans: $\boldsymbol{\frac{1}{e}}$)** |
| **3** | Find the area for the region bounded by the curves $y = 2 - x$ and $x^2+y^2=4$ **(Ans: $\boldsymbol{\pi - 2}$)** |

**5.5 Evaluation of triple integrals, Transformation cylindrical coordinates, Applications of triple integrals: Volume**

**Triple Integrals:** The triple integration is defined as

$$\iiint_V f(x,y,z)\,dV = \iiint f(x,y,z)\,dxdydz = \iiint f(x,y,z)\,dydzdx = \iiint f(x,y,z)\,dzdxdy$$

Note that other possibilities of order $dV$ are also there, but mainly we will use above mentioned possibilities of order.

**Example:** Evaluate $\int_1^2 \int_x^{2x} \int_0^{y-x} dz\,dy\,dx$.

**Solution:** $\int_1^2 \int_x^{2x} \int_0^{y-x} dz\,dy\,dx = \int_1^2 \int_x^{2x} z\Big|_0^{y-x} dy\,dx = \int_1^2 \int_x^{2x} (y-x)\,dy\,dx = \int_1^2 \left[\frac{y^2}{2} - xy\right]\Big|_x^{2x} dx$

$$= \int_1^2 \left[\left(\frac{(2x)^2}{2} - x\cdot 2x\right) - \left(\frac{x^2}{2} - x\cdot x\right)\right] dx = \int_1^2 \frac{x^2}{2}\,dx = \frac{x^3}{6}\Big|_1^2 = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}$$

**Example:** Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

**Solution:**

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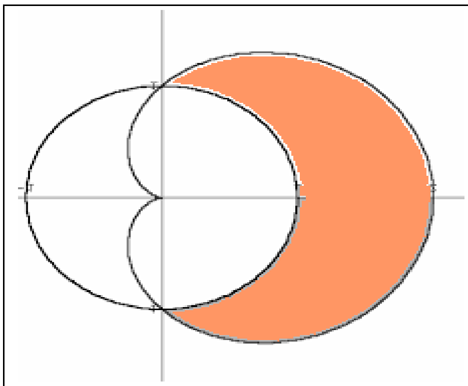
<tr>
<td></td>
<td>From the figure<br>
$\iint_R r\,dr\,d\theta$<br>
$= 2\int_0^{\frac{\pi}{2}} \int_1^{1+\cos\theta} r\,dr\,d\theta$<br>
$= 2\int_0^{\frac{\pi}{2}} \left(\frac{r^2}{2}\right)_1^{1+\cos\theta} d\theta$<br>
$= \int_0^{\frac{\pi}{2}} \left[(1+\cos\theta)^2 - 1\right] d\theta$<br>
$= \int_0^{\frac{\pi}{2}} \left(1 + 2\cos\theta + \cos^2\theta - 1\right) d\theta$<br>
$= \int_0^{\frac{\pi}{2}} \left(2\cos\theta + \cos^2\theta\right) d\theta$</td>
</tr>
</table>

| | |
|---|---|
| | $=\int_0^{\frac{\pi}{2}}\left(2\cos\theta+\frac{1+\cos 2\theta}{2}\right)d\theta$<br>$=\left(2\sin\theta+\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right)_0^{\frac{\pi}{2}}$<br>$=\left(2+\frac{\pi}{4}\right)$ |

**Example:** Find the volume of the Sphere $x^2+y^2+z^2=4$.

**Solution:** Required volume is

$$V=\iiint_v dv=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dzdydx=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}[z]_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}}dydx$$

$$=\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}2\sqrt{4-x^2-y^2}dydx=2\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\sqrt{4-x^2-y^2}dydx$$

Taking $4-x^2=b^2$

So, $V=2\int_{-2}^{2}\int_{-b}^{b}\sqrt{b^2-y^2}dydx=4\int_{-2}^{2}\int_{0}^{b}\sqrt{b^2-y^2}dydx=4\int_{-2}^{2}\int_{0}^{b}\left[\frac{y}{2}\sqrt{b^2-y^2}+\frac{b^2}{2}sin^{-1}\left(\frac{y}{b}\right)\right]_0^b dx$

$$=2\int_{-2}^{2}b^2\frac{\pi}{2}dx=\pi\int_{-2}^{2}(4-x^2)dx=2\pi\left[4x-\frac{x^3}{3}\right]_0^2=\frac{32\pi}{3}$$

Therefore, the volume of the sphere is $\frac{32\pi}{3}$.

**Tutorial:**

| | |
|---|---|
| **1** | Evaluate $\int_0^1\int_0^2\int_0^e dydxdz$. **(Ans: $2e$)** |
| **2** | Evaluate $\iiint z^2dx\,dy\,dz$ over the region common to the sphere $x^2+y^2+z^2=4$ and the cylinder $x^2+y^2=2x$. **(Ans: $\frac{64\pi}{15}$)** |
| **3** | Find the volume of the cylinder $x^2+y^2=1$ intercepted by the planes $z=1$ and $z=2$. **(Ans: $\pi$)** |
| **4** | Find the volume of the region bounded by the surfaces $x=0, y=0, z=0$ and $2x+3y+z=6$. **(Ans: 6)** |

**GATE MCQ**

| | |
|---|---|
| **1** | $f(x,y)$ is a continuous function defined over $(x,y) \in [0,1] \times [0,1]$. Given the two constraints $x > y^2$ and $y > x^2$, the volume under $f(x,y)$ is... **(Gate-2009)** |
| | (a) $\int_{y=0}^{y=1} \int_{x=y^2}^{x=\sqrt{y}} f(x,y)\, dxdy$    (b) $\int_{y=x^2}^{y=1} \int_{x=y^2}^{x=1} f(x,y)\, dxdy$    (c) $\int_{y=0}^{y=1} \int_{x=0}^{x=1} f(x,y)\, dxdy$    (d) $\int_{y=0}^{y=\sqrt{x}} \int_{x=0}^{x=\sqrt{y}} f(x,y)\, dxdy$ |
| **2** | The area enclosed between the straight line $y = x$ and the parbola $y = x^2$ in the $x - y$ plane is ... **(Gate-2012)** |
| | (a) 1/6    (b) 1/3    (c) 1/4    (d) 1/2 |
| **3** | The value of the integral of the function $g(x,y) = 4x^3 + 10y^4$ along the straight line segment from the point (0,0) to the point (1,2) in the $x - y$ plane is .... **(Gate-2008)** |
| | (a) 33    (b) 35    (c) 40    (d) 56 |
| **4** | To evaluate the double integral $\int_0^8 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\, dxdy$, we make the substitution $u = \frac{2x-y}{2}$ and $v = \frac{y}{2}$. The integral will reduce to ... **(Gate-2014)** |
| | (a) $\int_0^4 \int_0^2 2u\, dudv$    (b) $\int_0^4 \int_0^1 2u\, dvdu$    (c) $\int_0^4 \int_0^1 u\, dudv$    (d) $\int_0^4 \int_0^2 u\, dudv$ |
| **5** | The area enclosed between the curves $y^2 = 4x$ and $x^2 = 4y$ is.... **(Gate-2009)** |
| | (a) $\frac{16}{3}$    (b) $\frac{32}{3}$    (c) 8    (d) 16 |

| | |
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| | |
| **6** | Changing the order of integration in the double integral<br><br>$I = \int_0^8 \int_{x/4}^2 f(x,y)dydx$ leads to $I = \int_r^s \int_p^q f(x,y)dxdy$. What is q?<br><br>**(Gate-2005)** |

| | | | |
|---|---|---|---|
| (a) | $4y$ | (b) | $x$ |
| (c) | $16y^2$ | (d) | $8$ |