I Higher order derivatives and applications

Classwork Examples

1.1	Lagrange's Mean Value Theorem

1	Check whether the Mean Value Theorem can be applied to the function $2x^2+7x+10$ on the closed interval $[2,5]$. If so, find a value of c which satisfies the Mean value theorem in (2,5). **C.W.** **Answer:** $\mathbf{\frac{7}{2}}$**.**
2	Check whether the Mean Value Theorem can be applied to the function $4t^3-8t^2+7t-2$ on the closed interval $[0,5]$. If so, find a value of c which satisfies the Mean value theorem in $(0,5)$. **H.W.** **Answer: 3.**
3	Check whether the Mean Value Theorem can be applied to the function $f(x)=\log x$ on the closed interval $[1,e]$. If so, find a value of c which satisfies the Mean value theorem in $(1,e)$. **H.W.** **Answer:** $\boldsymbol{e-1}$**.**
4	Using Mean Value Theorem for the function $f(x)=\tan^{-1}x, x\in R$, prove that $\frac{x}{1+x^2}<\tan^{-1}x<x$. **C.W.**
5	Check whether the Mean Value Theorem can be applied to the function $f(x)=\sin x$ on the closed interval $\left[0,\frac{\pi}{2}\right]$. If so, find a value of c which satisfies the Mean value theorem in $\left(0,\frac{\pi}{2}\right)$. **H.W.** **Answer:** $\boldsymbol{cos^{-1}\left(\frac{2}{\pi}\right)}$**.**
6	Check whether the Mean Value Theorem can be applied to the function

	$f(x) = \frac{6}{x} - 3$ on the closed interval $[-1,2]$. If so, find a value of c which satisfies the Mean value theorem in $(-1,2)$. **C.W.** **Answer: Mean Value Theorem does not apply.**
7	A truck travels on a toll road with a speed limit of $80\ km/hr$. The truck completes a $164\ km$ journey in $2\ hours$. At the end of the toll road the trucker is issued with a speed violation notice. Justify this using the Mean Value Theorem. **C.W.**
8	A thermometer was taken from a freezer and placed in a boiling water. It took $22\ seconds$ for the thermometer to raise from -10℃ to 100℃. Show that the rate of change of temperature at some time t is $5°C$ per second. **H.W.**

1.1	Local Maxima and Minima of function of one variable

1	Show that the function $y = x^x$ is minimum at $x = e^{-1}$. **C.W.**
2	Show that the function $y = sinx\ (1 + cosx)$ is maximum when $x = \frac{\pi}{3}$. **C.W.**
3	Investigate for maxima and minima of the function given by $f(x) = \frac{\log x}{x}$ in $(0,\infty)$. **H.W.** **Answer:** $\boldsymbol{\frac{1}{e}}$**.**
4	Find the extreme values of the function $f(x) = 2x^3 - 9x^2 + 12x + 1$. **H.W.** **Answer: 6 and 5.**
5	The reaction of the body to a doze of medicine can sometimes be represented by an equation of the form $R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$ where C is a positive constant and M is the amount of medicine absorbed in the blood and R is the amount of reaction (either blood pressure or temperature in their respective units). Determine the amount of the medicine to which the body is most sensitive (rate of change in R with respect to M). **C.W.** **Answer:** $\boldsymbol{M = \frac{C}{2}}$

6	P is the perimeter of a rectangle, show that its area is maximum when it is a square. **C.W.**
7	A steel plant is capable of producing x tonnes per day of a low-grade steel and y tonnes per day of a high-grade steel, where $y = \frac{40-5x}{10-x}$. If the fixed market price of low-grade steel is half that of high-grade steel, then what should be optimal productions in low-grade steel and high-grade steel in order to have maximum receipts. **H.W.** **Answer:** $\boldsymbol{x = 10 - 2\sqrt{5}, y = 5 - \sqrt{5}}$**.**

1.2	Successive differentiation: nth derivative of elementary functions: rational, logarithmic, trigonometric, exponential and hyperbolic

1	Find the nth order derivative of the function $y = e^{2x+4} + 6^{2x+4}$. **C.W.** **Answer:** $\boldsymbol{y_n = 2^n[e^{2x+4} + (log\ 6)^n 6^{2x+4}]}$**.**
2	Find the nth order derivative of the function $y = \log[(4x+3) \cdot e^{6x+7}]$. **C.W.** **Answer:** $\boldsymbol{y_n = \frac{(-1)^{n-1}(n-1)!4^n}{(4x+3)^n}}$**.**
3	Find the nth order derivative of the function $y = \frac{x}{2x+5}$. **C.W.** **Answer:** $\boldsymbol{y_n = \frac{(-5)(-1)^n n! 2^n}{2(2x+5)^{n+1}}}$.
4	Show that the nth order derivative of the function $y = \frac{x}{(x-1)(2x+3)}$ is $y_n = \frac{(-1)^n n!}{5}\left[\frac{1}{(x-1)^{n+1}} + \frac{3(2)^n}{(2x+3)^{n+1}}\right]$. **C.W.**
5	Find the nth order derivative of the function $y = \frac{x^n-1}{x-1}$. **C.W.** **Answer:** $\boldsymbol{y_n = 0}$.
6	Find the nth order derivative of the function $y = e^x\ sinx\ cos2x$. **C.W.** **Answer:** $\boldsymbol{y_n = \frac{1}{2}\left[10^{\frac{n}{2}} e^x sin(3x + ntan^{-1}3) - 2^{\frac{n}{2}} e^x sin(x + ntan^{-1}1)\right]}$.
7	Find the nth order derivative of the function $y = sin2x\ cos2x$. **H.W.** **Answer:** $\boldsymbol{y_n = \frac{4^n}{2} sin\left(4x + \frac{n\pi}{2}\right)}$.

8	Find the nth order derivative of the function $y = \frac{2}{(x-1)(x-2)}$. **H.W.** **Answer:** $\boldsymbol{y_n = 2(-1)^n n!\left[\frac{1}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]}$.

1.3	Leibnitz rule for the n^{th} order derivatives of product of two functions

1	Find the n^{th} order derivative of $y = x^3 \cdot \log(2x+1)$. **C.W.**
2	Find the n^{th} derivative of the function $y = x^3 e^{3x}$. **H.W.** **Answer:** $\boldsymbol{y_n = x^3 3^n e^{3x} + 3nx^2 3^{n-1} e^{3x} + 3n(n-1)x3^{n-1}e^{3x}}$ $\boldsymbol{+n(n-1)(n-2)3^{n-2}e^{3x}}$.
3	If $y = tan^{-1}x$, prove that $(1+x^2)y_{n+1} + 2nxy_n + n(n-1)y_{n-1} = 0$. **C.W.**
4	If $y = \sin log(x^2+2x+1)$, show that $(x+1)^2 y_{n+2} + (2n+1)(x+1)y_{n+1} + (n^2+4)y_n = 0$. **C.W.**
5	If $y = \sqrt{\frac{1+x}{1-x}}$, show that $(1-x^2)y_n - [2(n-1)x+1]y_{n-1} - (n-1)(n-2)y_{n-2} = 0$. **C.W.**
6	If $y = (sin^{-1}x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$. **C.W.**
7	If $y = (cos^{-1}x)^2$, show that $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$. **H.W.**
8	If $y^{\frac{1}{m}} + y^{-\frac{1}{m}} = 2x$, prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$.

1.4	Power series expansion of a function: Maclaurin's and Taylor's series expansion.

1	Use Taylor's series to find the expansion of $\log_e x$ in powers of $(x-1)$. Find the value of $\log 1.1$. **C.W.** **Answer:** $\boldsymbol{\log_e x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots, \log_e 1.1 \approx 0.09531}$
2	Expand $\tan^{-1} x$ in powers of $x - \frac{\pi}{4}$. **H.W.**

	Answer: $\boldsymbol{\tan^{-1} x = \tan^{-1}\frac{\pi}{4} + \frac{x-\frac{\pi}{4}}{1+\frac{\pi^2}{16}} - \frac{\pi}{4}.\frac{\left(x-\frac{\pi}{4}\right)^2}{\left(1+\frac{\pi^2}{16}\right)^2} + \cdots}$
3	Expand $\log\tan\left(\frac{\pi}{4}+x\right)$ in powers of x using the Taylor's series. **C.W.** **Answer:** $\boldsymbol{\log\tan\left(\frac{\pi}{4}+x\right) = 2x + \frac{4}{3}x^3 + \cdots}$
4	Find Maclaurin's series expansion of $(i)\ e^x$ **C.W.**, $(ii)\ \sin x$ **C.W.**, $(iii)\ \cos x$ **H.W.**, $(iv)\ \log(1+x)$ **H.W.**, $(v)\ \sinh x$ **C.W.**, $(vi)\ \cosh x$ **H.W.**, $(vii)\ \frac{1}{1-x}$ **C.W.**.
5	Obtain the Maclaurin's series of $e^{a\sin^{-1}x}$. **C.W.** **Answer:** $\boldsymbol{e^{a\sin^{-1}x} = 1 + ax + \frac{a^2}{2!}x^2 + \frac{a(1^2+a^2)}{3!}x^3 + \frac{a^2(2^2+a^2)}{4!}x^4 + \cdots}$
6	Obtain the Maclaurin's series of $\frac{\sin^{-1}x}{\sqrt{1-x^2}}$. **C.W.** **Answer:** $\boldsymbol{\frac{\sin^{-1}x}{\sqrt{1-x^2}} = x + \frac{2}{3}x^3 + \frac{8}{15}x^5 + ..}$.
7	Obtain the Maclaurin's series of $sinx$ and using it show that $sinx$ is an odd function. **C.W.**
8	Obtain the Maclaurin's series of $\log\sec x$. **H.W.** **Answer:** $\boldsymbol{\log\sec x = \frac{x^2}{2} + \frac{x^4}{12} + \cdots}$

1.5	L'Hospital's rule and related applications, Indeterminate forms

1	Evaluate $\lim\limits_{x\to 0}\frac{e^x+e^{-x}-x^2-2}{sin^2x-x^2}$. **C.W.** **Answer:** $\boldsymbol{-\frac{1}{4}}$.
2	If $\lim\limits_{x\to 0}\frac{ae^x-b\cos x+ce^{-x}}{x\ \sin x} = 2$, find the values of a, b, c. **C.W.** **Answer:** $\boldsymbol{a = 1, b = 2, c = 1}$.
3	Evaluate $\lim\limits_{x\to\infty}\frac{x(\log x)^3}{1+x+x^2}$. **C.W.** **Answer:** $\boldsymbol{0}$.

4	Evaluate $\lim_{x\to 0} \frac{\log(\sin x)}{\log(\tan x)}$. **H.W.** **Answer: 1**.
5	Evaluate $\lim_{x\to\frac{\pi}{2}} \frac{\log \sin x}{(\pi-2x)^2}$. **H.W.** **Answer:** $-\frac{1}{8}$
6	Evaluate $\lim_{x\to\infty} \frac{x^n}{e^x}$. **H.W.** **Answer: 0**.
7	Evaluate $\lim_{x\to\frac{\pi}{2}^{-1}} cosx \cdot \log tanx$. $(0\cdot\infty)$ **C.W.** **Answer: 0**.
8	Evaluate $\lim_{x\to 1} \left[\frac{1}{logx} - \frac{x}{x-1}\right]$. $(\infty-\infty)$ **C.W.** **Answer:** $-\frac{1}{2}$**.**
9	Evaluate $\lim_{x\to\frac{\pi}{2}^{-1}} (\cos x)^{\frac{\pi}{2}-x}$. (0^0) **C.W.** **Answer: 1**.
10	Evaluate $\lim_{x\to 0} \left(\frac{a^x+b^x+c^x}{3}\right)^{\frac{1}{x}}$. (1^∞) **C.W.** **Answer:** $(abc)^{\frac{1}{3}}$ **.**
11	Prove that $\lim_{x\to 1}(1+sec\pi x)tan\frac{\pi x}{2} = 0$. **H.W.**
12	Evaluate $\lim_{x\to 0^+} \left(\frac{1}{x}\right)^{1-\cos x}$**. C.W.** **Answer: 1**.