Unit I

1.	**First order and First-degree Ordinary Differential Equations:**
1.1	Formation of Ordinary Differential Equation
1.2	Concept of general and particular solutions
1.3	Initial value problems
1.4	Solutions of first order and first degree differential equations: Linear, Bernoulli, Exact and non-exact differential equations

Differential Equation:

An equation containing the independent variables, dependent variable, and derivatives of a dependent variable is called a differential equation.

Examples:

(1) $\frac{dy}{dx} + y = x$

(2) $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

(3) $\left(\frac{dy}{dx}\right)^2 + 2y^2 = 4\left[x + \frac{dy}{dx}\right]$

(4) $\frac{d^2y}{dx^2} = \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}$

(5) $2x^2\left(\frac{d^2y}{dx^2}\right)^3 + 5y\frac{dy}{dx} = 2xy$

(6) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$

(7) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

(8) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = x^2 + y$

are examples of differential equations.

Ordinary differential equation

A differential equation containing ordinary derivatives of a dependent variable is called an ordinary differential equation.

e.g., (1) to (5) are ordinary differential equations.

Partial differential equation

A differential equation containing partial derivatives of a dependent variable is called partial differential equation.

e.g., (6) to (8) are partial differential equations.

Order of a differential equation

The order of a differential equation is the order of highest derivative occurring in the differential equation.

Degree of a differential equation

The degree of a differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalized, that is, made free from radicals and fractions.

Sr. No.	Differential Equation	Order	Degree
(1)	$\frac{dy}{dx}+2y=x$	1	1
(2)	$\left(\frac{dy}{dx}\right)^2+2y=x$	1	2
(3)	$\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=e^x$	3	1
(4)	$\left(\frac{d^2y}{dx^2}\right)^2+\frac{dy}{dx}+3y=0$	2	2
(5)	$y=2x\frac{dy}{dx}+\frac{3}{\left(\frac{dy}{dx}\right)}$	1	2
(6)	$x^2\left(\frac{d^2y}{dx^2}\right)^3+y\left(\frac{dy}{dx}\right)^4+y^4=0$	2	3
(7)	$y=x\frac{dy}{dx}+\sqrt{1+\left(\frac{dy}{dx}\right)^2}$	1	2
(8)	$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=1$	2	1

Tutorial:

1	Find order and degree of differential equation $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{4}}=\frac{d^2y}{dx^2}$. **Ans: order-2, degree-4**
2	Determine the degree of differential equation $\sin\left(\frac{dy}{dx}\right)+\frac{d^2y}{dx^2}=x$. **Ans: order-2, degree- not defined**

3	Find order and degree of $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 1$. **Ans: order-2, degree-1**

1.1 Formation of Ordinary differential equation

Formation of Ordinary differential equation means to eliminate arbitrary constants or arbitrary functions from the given relation between dependent and independent variables by taking order derivative or derivatives.

Tutorial:

1	**Form the differential equation of simple Harmonic motion given by** $x = A\cos(t+B)$ **, where** A **and** B **are arbitrary constants.** **Solution**: Given that $x = A\cos(t+B)$. To eliminate arbitrary constant A take derivate of $x = A\cos(t+B)$ with respect to t as follows $\therefore \frac{dx}{dt} = -A\sin(t+B)$ $\therefore \frac{d^2x}{dt^2} = -A\cos(t+B) = -x$ $\therefore \frac{d^2x}{dt^2} + x = 0$ is the desired differential equation.
2	Form the differential equation satisfied by $y = ax^2 + bx + c$, where *a, b* and *c* are arbitrary constants. **Ans:** $\boldsymbol{\frac{d^3y}{dx^3} = 0}$
3	Form the differential equation satisfied by $y = a\sin x + b\cos x$, where *a* and b are arbitrary constants. **Ans:** $\boldsymbol{\frac{d^2y}{dx^2} + y = 0}$
4	Form the differential equation satisfied by $y = ae^{-2x} + be^{2x}$, where *a* and *b* are arbitrary constants. **Ans:** $\boldsymbol{\frac{d^2y}{dx^2} - 4y = 0}$

1.2 Concept of general and particular solutions

The solution of a differential equation is a relation between the variables (independent and dependent), which is free of derivatives of any order, and which satisfies the differential equation identically. The solution is said to be general solution, if it contains one or more arbitrary constants. The general solution is also known as a complete solution. Number of arbitrary constants in the General solution is equal to the order of differential equation.

A Particular Solution of a differential equation is a solution obtained from the General Solution by assigning specific values to arbitrary constants.

Tutorial:

1	Show that $y = e^{-5x}$ is the particular solution of differential equation $y'' + 6y' + 5y = 0$.
2	Show that $y = Ae^{\frac{x^2}{2}}$ is the general solution of the differential equation $\frac{dy}{dx} = xy$, where A is an arbitrary constant.
3	Show that $y = A\cos x + B\sin x$ is the general solution of the differential equation $y'' + y = 0$, where A and B are arbitrary constants.

1.3 Initial value problems

In the field of differential equations, an **initial value problem** (also called a **Cauchy problem**) is an ordinary differential equation together with a specified value of the solution of the differential equation. The initial condition means value of the unknown solution at a given point in the domain of the solutions.

Initial condition is used to derive particular solution from the general solution by finding value of arbitrary constants.

For example, consider the differential equation $\frac{dy}{dx} + 2y = x,\ y(0) = 1$. The general solution of the differential equation is given by $ye^{2x} = \frac{e^{2x}}{4}(2x-1) + c$. From, the initial condition $y(0) = 1$,

$Take\ x = 0\ y = 1$

$$\therefore 1 = 0 - \frac{1}{4} + c$$

$$\therefore c = 1 + \frac{1}{4} = \frac{5}{4}$$

$ye^{2x} = \frac{e^{2x}}{4}(2x-1) + \frac{5}{4}$ is a particular solution.

Tutorial:

1.	Solve the differential equation $\frac{dy}{dx}=2y+1,\ \textit{given that}\ y(0)=1$. **Ans:** $\mathbf{2y=3e^{2x}-1}$
2.	Solve the differential equation $\frac{dy}{dx}=y\cot x,\ \ \textit{given that}\ y\left(\frac{\pi}{2}\right)=1$. **Ans:** $\mathbf{y=\sin x}$

1.4 Solutions of first order and first degree differential equations

Following are the methods to find solution of 1st ordinary differential equation.

1.4.1 Linear differential equation or Leibnitz's differential equation
1.4.2 Bernoulli's differential equation
1.4.3 Exact and non-exact differential equation

1.4.1 Linear differential equation

The general form of First Order Linear Differential equation in the dependent variable y and independent variable x is $\frac{dy}{dx}+P(x)y=Q(x)$, where P and Q are functions of x or constants. Integrating factor (I.F.) for a linear differential equation is given by $e^{\int P(x)dx}$. The solution of this equation is $y(IF)=\int Q(x)\times(I.F.)\,dx+c$, where c is an arbitrary constant.

Tutorial:

1	Solve the differential equation $(x+1)\frac{dy}{dx}-y=e^{3x}(x+1)^2$. **Solution**: On dividing the differential equation by (*x+1*) on both sides yields $\frac{dy}{dx}-\frac{y}{(x+1)}=e^{3x}(x+1)$. This is a linear differential equation. Here $P(x)=\frac{1}{(x+1)}$ & $Q(x)=e^{3x}(x+1)$. $I.F.=e^{\int P(x)dx}=e^{-\int\frac{1}{(x+1)}dx}=e^{-\log(x+1)}=(x+1)^{-1}$. Thus the solution is

	$y(I.F.) = \int Q(x) \times (I.F.)\, dx + c$ $\therefore\ y(x+1)^{-1}. = \int e^{3x}(x+1) \times (x+1)^{-1}.dx + c$ $= \int e^{3x}.dx + c$ $= \frac{e^{3x}}{3} + c.$ i.e. $y = (1+x)\left(\frac{e^{3x}}{3} + c\right)$ is a solution of the differential equation.
2	Solve the differential equation $\frac{dx}{dy} + \frac{x}{y\log y} = \frac{2}{y}$ **Ans:** $\boldsymbol{x\log y = (\log y)^2 + c}$
3	Solve the differential equation $(x^3 - x)\frac{dy}{dx} - (3x^2 - 1)y = x^5 - 2x^3 + x$. **Ans:** $\boldsymbol{y = (x^3 - x)logx + (x^3 - x)C}$

1.4.2 Bernoulli's differential equation

The general form of the Bernoulli's differential equation in the dependent variable y and independent variable x is $\frac{dy}{dx} + P(x)y = Q(x)y^n$ where P and Q are functions of x or constants; n is neither zero nor 1.

On dividing by y^n on both sides of the equation produces

$$y^{-n}\frac{dy}{dx} + P(x)y^{1-n} = Q(x).$$

Putting $y^{1-n} = v,$ we have $(1-n)y^{-n}\frac{dy}{dx} = \frac{dv}{dx}$. So the last equation yields

$$\frac{1}{1-n}\frac{dv}{dx} + P(x)v = Q(x)\,.$$

This is a linear differential equation of the form $\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x)$.

This is a linear differential Equation in dependent variable v. The Integrating factor is $I.F. = e^{\int (1-n)P(x)dx}$ and the solution is $v(I.F.) = \int Q(x)(I.F.)dx + c$

Tutorial:

1	Solve the differential equation $\frac{dy}{dx}+\frac{y}{x}=x^2y^2$. **Solution** : The differential equation $\frac{dy}{dx}+\frac{y}{x}=x^2y^2$ is of the form of the Bernoulli differential equation and divide by y^2 on the both sides, we have $y^{-2}\frac{dy}{dx}-\frac{y^{-1}}{x}=x^2.$ $Take\ y^{-1}=v \Rightarrow -y^{-2}y'=v' \Rightarrow y^{-2}y'=-v',$ $-v'+\frac{v}{x}=x^2 \Rightarrow v'-\frac{v}{x}=-x^2.$ Which is a Linear differential equation in v and x. Compare with $\frac{dv}{dx}+P(x)v=Q(x)$, we have $P(x)=-1/x\ and\ Q(x)=-x^2$ $\therefore I.F.=e^{\int P(x)dx}=e^{-\int\frac{1}{x}dx}=e^{-\log x}=x^{-1}$ $Solution\ is\ \ given\ by$ $v(I.F.)=\int Q(x)dx+c$ $\therefore vx^{-1}=-\int x^{-1}x^2dx+c$ Thus solution is given by $y^{-1}x^{-1}=-\frac{x^2}{2}+c$.
2	Solve the differential equation $2x\frac{dy}{dx}=10x^3y^5+y$. **Ans:** $\boldsymbol{x^2+\left(4x^5+c\right)y^4=0}$
3	Solve the differential equation $\frac{dy}{dx}=2y\tan x+y^2\tan^2 x,\ if\ \ x=0,\ then\ \ y=1.$ **Ans:**$\boldsymbol{\frac{1}{y}\sec^2 x=c-\frac{1}{3}\tan^3 x}$
4	Solve the differential equation $x\frac{dy}{dx}+y=x^3y^6$. **Ans:** $\boldsymbol{\frac{2}{y^5}=5x^3+cx^5}$

1.4.2 Exact and Non-exact differential equation

The differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is called an exact differential equation, if $\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$. The solution of an exact differential equation is given by

$$\int_{y-const.} M(x,y)dx+\int N^{*}(y)dy=c,$$

Where $N^{*}(y)$ is a function derived from $N(x,y)$ by removing terms of involving *x*.

The differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is said to be non-exact differential equation, if $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$. To find solution of a non-exact differential equation, first we find integrating factor by using one of the case from following cases.

Homogeneous function:

A function $f(x,y)$ is said to be homogeneous function of degree *n*, if $f(kx,ky)=k^{n}f(x,y)$.

The Homogeneous differential equation:

The differential equation $M(x,y)dx+N(x,y)dy=0$ is said to be homogeneous differential equation, if $M(x,y)$ & $N(x,y)$ are both homogeneous function of same degree.

Case-I: The differential equation of the form $M(x,y)dx+N(x,y)dy=0$ is a homogeneous differential equation and if $xM(x,y)+yN(x,y)\neq 0$, then $I.F.=\frac{1}{xM(x,y)+yN(x,y)}$.

Case-II: The differential equation of the form $M(x,y)dx+N(x,y)dy=0$ can be rewritten as $f_1(xy)ydx+f_2(xy)xdy=0$ and if $xM(x,y)-yN(x,y)\neq 0$, then $I.F.=\frac{1}{xM(x,y)-yN(x,y)}$.

Case-III: If $\frac{1}{N(x,y)}\left[\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right]=f(x)$ is a function of *x* or a constant, then

$I.F.=e^{\int f(x)dx}$.

Case-IV: If $\frac{1}{M(x,y)}\left[\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right]=g(y)$ is a function of *y* or a constant, then

$I.F.=e^{\int g(y)dy}$.

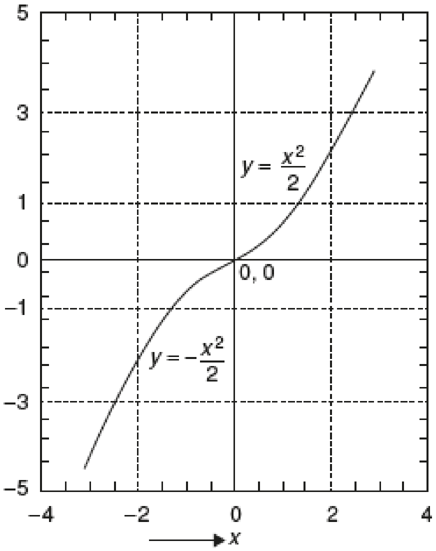
Tutorial:

1	**Solve the differential equation** $(y^2e^{xy^2}+4x^3)dx+(2xye^{xy^2}-3y^2)dy=0$. **Solution**: Compare given differential equation with $M(x,y)dx+N(x,y)dy=0$, we get $M(x,y)=y^2e^{xy^2}+4x^3 \Rightarrow M_y=2ye^{xy^2}+2xy^3e^{xy^2}$, $N(x,y)=2xye^{xy^2}-3y^2 \Rightarrow N_x=2ye^{xy^2}+2xy^3e^{xy^2}$. Thus given differential equation is a exact differential equation. Solution is given by $\int\limits_{y-cons.} M(x,y)dx+\int N^*(y)dy=c,\ N^*(y)=-3y^2$ $\therefore \int\limits_{y-cons.} (y^2e^{xy^2}+4x^3)dx+\int(-3y^2)dy=c,$ $\therefore \dfrac{y^2e^{xy^2}}{y^2}+x^4-y^3=c.$ *The solution is given by* $e^{xy^2}+x^4-y^3=c.$
2	**Solve the differential equation** $(xy^3+y)dx+(2x^2y^2+2x+2y^4)dy=0$. **Solution**: Compare given differential equation with $M(x,y)dx+N(x,y)dy=0$ $M(x,y)=xy^3+y \Rightarrow M_y=3xy^2+1,$ $N(x,y)=2(x^2y^2+x+y^4) \Rightarrow N_x=2(2xy^2+1)=4xy^2+2 \Rightarrow M_y \neq N_x.$ Therefore, the differential equation is non-exact. Next we find integrating factor $\dfrac{1}{M(x,y)}(N_x-M_y)=\dfrac{(4xy^2+2-3xy^2-1)}{xy^3+y}=\dfrac{xy^2+1}{y(xy^2+1)}=\dfrac{1}{y}=f(y),$ $\therefore IF=e^{\int f(y)dy}=e^{\int\frac{1}{y}dy}=e^{\log y}=y.$ Multiplying given differential equation by *y,* we get $(xy^4+y^2)dx+2(x^2y^3+xy+y^5)dy=0$ -----(*) Compare (*) with $A(x,y)dx+B(x,y)dy=0$ gives $A(x,y)=xy^4+y^2 \Rightarrow A_y=4xy^3+2y$ $B(x,y)=2(x^2y^3+xy+y^5) \Rightarrow B_x=4xy^3+2y \Rightarrow A_y=B_x$ Thus, differential equation (*) is exact differential equation. Solution is given by $\int\limits_{y-cons.} A(x,y)dx+\int B^*(y)dy=c,\ B^*(y)=2y^5$ $\therefore \dfrac{x^2y^4}{2}+xy^2+\dfrac{y^6}{3}=c.$

3	Solve the differential equation $(2xycosx^2 - 2xy + 1)dx + (sinx^2 - x^2 + 3)dy = 0.$ **Ans:** $\boldsymbol{ysinx^2 - yx^2 + x + 3y = C}$
4	Solve the differential equation $y(x^2 y + e^x)dx - e^x dy = 0$. **Ans:** $\boldsymbol{\frac{x^3}{3} + \frac{e^x}{y} = c}$
5	Solve the differential equation $x(x-y)\frac{dy}{dx} = y(x+y)$. **Ans:** $\boldsymbol{xy^2 = c}$
6	Solve the differential equation $(2x\log x - xy)dy + 2ydx = 0$. **Ans:** $\boldsymbol{2y\log(x) - \frac{y^2}{2} = c}$
7	Solve $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$ **Ans:** $\boldsymbol{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c}$
8	Solve $y(1 + xy)dx + x(1 - xy)dy = 0.$ **Ans:** $\boldsymbol{xylog\left(\frac{y}{x}\right) = cxy - 1}$
9	Solve $(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0.$ **Ans:** $\boldsymbol{\frac{x}{y} - 2logx + 3logy = c}$
10	Solve $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \left(\frac{x+xy^2}{4}\right)dy = 0.$ **Ans:** $\boldsymbol{3x^4y + x^4y^3 + x^6 = c}$

Gate MCQ:

1	Which one of the following is a linear non-homogeneous differential equation, where *x* and *y* are the independent and dependent variables respectively? **(GATE-2014)** **(a)** $\frac{dy}{dx} + xy = e^{-x}$ **(b)** $\frac{dy}{dx} + xy = 0$ **(c)** $\frac{dy}{dx} + xy = e^{-y}$ **(d)** $\frac{dy}{dx} + e^{-y} = 0$	
Sol.	Solution: (a) Option (b) is homogeneous linear differential equation. Option (c) is non-linear because of e^{-y}. Option (d) is again non-linear because of e^{-y}. Hence, the correct option is (a).	
2	The general solution of the differential equation $\frac{dy}{dx} = \frac{1+\cos 2y}{1-\cos 2x}$ is _____. **(GATE-2015)** **(a)** $\tan y - \cot x = c$ **(b)** $\tan x - \cot y = c$ **(c)** $\tan y + \cot x = c$ **(d)** $\tan x + \cot y = c$	
Sol.	The differential equation $\frac{dy}{dx} = \frac{1+\cos 2y}{1-\cos 2x}$ $\therefore \frac{dy}{1+cos2y} = \frac{dx}{1-cos2x}$ $\therefore \frac{dy}{2\cos^2 y} = \frac{dx}{2\sin^2 x}$ $\sec^2 y dy = -\text{cosec}^2 x\, dx$ Integrate both the sides $tany = -cotx + c$ $tany + cotx = c$ Solution: (c)	

3	The figure shows the plot of *y* as a function of *x*. The function shown in the solution of the differential equation (assuming all initial conditions to be zero) is  **(GATE-2014)**	

(a)	$\frac{\mathrm{dy}}{\mathrm{dx}} = \mathrm{x}$	**(b)**	$\frac{d^2y}{dx^2} = 0$
(c)	$\frac{\mathrm{dy}}{\mathrm{dx}} = -\mathrm{x}$	**(d)**	$\frac{dy}{dx} = \lvert x\rvert$

Sol. Here given function is

$$y(x) = \frac{x^2}{2}, x \geq 0$$

$$= -\frac{x^2}{2}, x \leq 0$$

Therefore

$$\frac{dy}{dx} = x \ \ , x \geq 0$$

$$= -x, x \leq 0$$

Solution: (d)

4	The families of curves represented by the solution of the equation $$\frac{dy}{dx} = -\left(\frac{x}{y}\right)^n$$ For n=1 and n=-1 respectively are________. **(GATE-2015)** **(a)** Hyperbolas and Circles; **(b)** Parabolas and Circles; **(c)** Circles and Hyperbolas; **(d)** Circles and Parabolas	
Sol.	Here the equation is $\frac{dy}{dx} = -\left(\frac{x}{y}\right)^n$ Put n=-1 $\therefore \frac{dy}{dx} = -\left(\frac{x}{y}\right)^{-1}$ $\therefore \frac{dy}{dx} = -\left(\frac{y}{x}\right)$ $\therefore \frac{dy}{y} = -\left(\frac{dx}{x}\right)$ ln y = -ln x+ln c $xy = c$ The graph of xy = c is **a hyperbola**. Here the equation is $\frac{dy}{dx} = -\left(\frac{x}{y}\right)^n$ Put n=1 $\therefore \frac{dy}{dx} = -\left(\frac{x}{y}\right)^1$ $\therefore \frac{dy}{dx} = -\left(\frac{x}{y}\right)$ $\therefore y\,dy = -x\,dx$ $y^2 + x^2 = c$ The graph of $y^2 + x^2 = c$ is a circles.	

5	Which one of the following differential equations has a solution given by the function $y = 5\sin(3x + \frac{\pi}{3})$ is________. **(GATE-2019)** **(a)** $\frac{dy}{dx} - \frac{5}{3}cos(3x) = 0$ **(b)** $\frac{dy}{dx} + \frac{5}{3}cos(3x) = 0$ **(c)** $\frac{d^2y}{dx^2} - 9y = 0$ **(d)** $\frac{d^2y}{dx^2} + 9y = 0$	
Sol.	Here $y = 5\sin(3x + \frac{\pi}{3})$ Differentiate it twice w.r.t. x, we get $\therefore \frac{dy}{dx} = 5 * 3\cos(3x + \frac{\pi}{3})$ $\frac{d^2y}{dx^2} = -5 * 3^2 \sin\left(3x + \frac{\pi}{3}\right) = -9 * 5\sin(3x + \frac{\pi}{3})$ Thus $\frac{d^2y}{dx^2} + 9y = 0$ which is the required differential equation.	