V	Applications of Partial differentiations

Practice Examples

5.1	Maclaurin's and Taylor's series expansion in two variables

1	Expand $f(x, y) = y^x$ in powers of $(x - 1)$ and $(y - 1)$ up to second order terms. **Answer:** $f(x, y) = 1 + (y - 1) + (x - 1)(y - 1) + \cdots$
2	Obtain terms up to the second degree in the Taylor's series expansion of $f(x, y) = x^2 + xy + y^2$ at the point $(1, 2)$. **Answer:** $f(x, y) = 7 + 4(x - 1) + 5(y - 2) + (x - 1)^2$ $+(x - 1)(y - 2) + (y - 1)^2.$
3	Find the Maclaurin's series expansion of $f(x, y) = \sin 2x + \cos y$ up to second order terms. **Answer:** $f(x, y) = 1 + 2x - \frac{y^2}{2} + \cdots$
4	Show that $x^y = 1 + (x - 1) + (x - 1)(y - 1) + \cdots$
5	Express $f(x, y) = x^2 + 3y^2 - 9x - 9y + 26$ in powers of $(x - 2)$ and $(y - 2)$ using the Taylor's series expansion. **Answer:** $f(x, y) = 6 - 5(x - 2) + 3(y - 2) + (x - 2)^2 + 3(y - 2)^2$.
6	Expand $f(x, y) = e^x \cdot \cos y$ in powers of x and y up to second order terms. **Answer:** $f(x, y) = 1 + x + \frac{1}{2}x^2 - \frac{1}{2}y^2 + \cdots$

5.2	Tangent plane and normal line to a surface

1	Find the equation of tangent plane and normal line to the surface $\frac{x^2}{4} + \frac{y^2}{9} - \frac{z^2}{25} = 1$ at the point $(2, 3, 5)$. **Answer:** $15x + 10y - 6z = 30, \frac{x-2}{15} = \frac{y-3}{10} = \frac{z-5}{-6}$.

2	Find the equation of tangent plane and normal line to the surface $\cos \pi x - x^2 y + e^{xz} + yz = 4$ at the point $(0, 1, 2)$. **Answer:** $2x + 2y + z = 4, \frac{x}{2} = \frac{y-1}{2} = z - 2$.
3	Find the equation of tangent plane and normal line to the surface $x^3 + y^3 + 3xyz = 3$ at the point $(1, 2, -1)$. **Answer:** $x - 3y - 2z + 3 = 0, \frac{x-1}{-1} = \frac{y-2}{3} = \frac{z+1}{2}$.
4	Find the equation of tangent plane and normal line to the surface $z = \sqrt{3 - x^2 - y^2}$ at the point $(1, 1, 1)$. **Answer:** $x + y + z = 26, x - 1, y - 1, z - 1$.
5	Find the equation of tangent plane and normal line to the surface $\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3$ at the point $(-2, 1, -3)$. **Answer:** $-3x + 6y - 2z = 18, \frac{x+2}{-1} = \frac{y-1}{2} = \frac{3(z+3)}{-2}$.
6	Find the equation of tangent plane and normal line to the surface $2xz^2 - 3xy - 4x = 7$ at the point $(1, -1, 2)$. **Answer:** $7x - 3y + 8z = 26, \frac{x-1}{7} = \frac{y+1}{-3} = \frac{z-2}{8}$.

5.3	Maxima and Minima of two variable function

1	Show that the minimum value of the function $f(x, y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ is $3a^2$. **Answer:** $f(x, y)$ is minimum at $(a, a), f(a, a) = 3a^2$.
2	Find the extreme values of the function $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$. **Answer:** $f(x, y)$ is maximum at $(\sqrt{2}, -\sqrt{2}), f(\sqrt{2}, -\sqrt{2}) = -8$ and $f(x, y)$ is maximum at $(-\sqrt{2}, \sqrt{2}), f(-\sqrt{2}, \sqrt{2}) = -8$.
3	Find the values of x and y such that the function $x^2 + y^2 + 6x + 12$ has a minimum value and find this minimum value. **Answer:** $f(x, y)$ is minimum at $(-3, 0), f(-3, 0) = 3$.
4	Find the minimum value of $x^2 + y^2 + z^2$ subject to $ax + by + cz = p$.

	Answer: $\frac{p^2}{(a^2+b^2+c^2)^2}$.
5	Show that the maximum value of $x^2y^3z^4$ subject to $2x+3y+4z=a$ is $\left(\frac{a}{9}\right)^9$.
6	If $\frac{3}{x}+\frac{4}{y}+\frac{5}{z}=6$, then find the values of x,y,z such that $x+y+z$ is minimum. **Answer:** $x=\left(\frac{\sqrt{3}+\sqrt{4}+\sqrt{5}}{6}\right)\sqrt{3}, y=\left(\frac{\sqrt{3}+\sqrt{4}+\sqrt{5}}{6}\right)\sqrt{4}, z=\left(\frac{\sqrt{3}+\sqrt{4}+\sqrt{5}}{6}\right)\sqrt{5}$.

5.4	Lagrange's method of undetermined multiplier

1	If $xyz=8$, then find the value of $x,y,$ and z for which $u=\frac{40}{x+2y+4z}$ is maximum. **Answer:** $x=4, y=2, z=1$.
2	The pressure P at a point (x,y,z) in space is $P=400\,xyz^2$. Find the highest pressure at the surface of a unit sphere $x^2+y^2+z^2=1$. **Answer:** 50.
3	Find the area of a greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. **Answer:** $2ab$.
4	Find the minimum value of $x^2+y^2+z^2$ subject to $ax+by+cz=p$. **Answer:** $\frac{p^2}{(a^2+b^2+c^2)^2}$.
5	Show that the maximum value of $x^2y^3z^4$ subject to $2x+3y+4z=a$ is $\left(\frac{a}{9}\right)^9$.
6	If $\frac{3}{x}+\frac{4}{y}+\frac{5}{z}=6$, then find the values of x,y,z such that $x+y+z$ is minimum. **Answer:** $x=\left(\frac{\sqrt{3}+\sqrt{4}+\sqrt{5}}{6}\right)\sqrt{3}, y=\left(\frac{\sqrt{3}+\sqrt{4}+\sqrt{5}}{6}\right)\sqrt{4}, z=\left(\frac{\sqrt{3}+\sqrt{4}+\sqrt{5}}{6}\right)\sqrt{5}$.

5.5	Jacobian

1	Find the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ for $u=e^x\sin y$ and $v=x\log(\sin y)$. **Answer:** $e^x\cos y\,[x-\log(\sin y)]$.

2	If $x = uv$ and $y = \frac{u+v}{u-v}$, then show that $\frac{\partial(x,y)}{\partial(u,v)} = \frac{4uv}{(u-v)^2}$.
3	Evaluate $\frac{\partial(u,v,w)}{\partial(x,y,z)}$ at$(1,-1,0)$ for $u = x + 3y^2 - z^3, v = 4x^2yz$ and $w = 2z^2 - xy$. **Answer:** 20.
4	Find the Jacobian of the transformation $u = ax + by, v = cx + dy$. **Answer:** $ad - bc$.
5	If $x = u^3 + v^2 - 2uv$ and $y = u + v$, then find $\frac{\partial(x,y)}{\partial(u,v)}$. **Answer:** $3u^2 + 2u - 4v$.
6	If $u = x^2 + yz, v = y^2 + xz$ and $w = z^2 + xy$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. **Answer:**$-2x^3 - 2y^3 - 2z^3 + 10xyz$.

5.6	Errors and approximations

1	The focal length of a mirror is found from the formula $\frac{2}{f} = \frac{1}{v} - \frac{1}{u}$. Find the percentage error in f if u and v both have error by 2% each. **Answer:** 2%.
2	The two resistors x and y are connected in parallel so that the total resistance R is given by $\frac{xy}{x+y}$. If x and y are measured to be $200\ \Omega$ (ohms) and $300\ \Omega$ respectively, with the increase of $1.5\ \Omega$ in x and decrease of $4\ \Omega$ in y, then find the change in R. **Answer:** $0.1\ \Omega$.
3	In calculating the volume of a right circular cylinder, errors of 2% and 1% are found in measuring height and base radius respectively. Find the percentage error in the volume of the cylinder. **Answer:** 4%.
4	Find an approximate value of $(1.04)^{3.01}$ using the theory of approximations. **Answer:** 1.12.
5	If $f(x,y,z) = e^{xyz}$, then find an approximate value of f when $x = 0.01, y = 1.01$, $z = 2.01$.

	Answer: 1.02.
6	Find an approximate value of $(1.99)^2(3.01)^3(0.98)^{\frac{1}{10}}$ using the theory of approximations. **Answer:** 107.784.
7	The deflection at the centre of a rod having length l and diameter d supported at its ends and loaded at the centre with a weight w varies as wl^3d^{-4}.What is the percentage increase in the deflection corresponding to the percentage increase in w, l, and d of $3, 2$, and 1 respectively? **Answer:** 5%.
8	Find the percentage error in the area of ellipse, when an error of 0.05% percentage made in measuring semi-major and semi-minor axis. **Answer:** 0.1%.
9	The ideal gas law $PV = nRT$ is used to find pressure P when temperature T and volume V are given but there is an error of 0.3% in measuring T and an error of 0.8% in measuring V. Find the greatest percentage error in P. (R: ideal gas constant, n: amount of substance) **Answer:** 1.1%.
10	Find an approximate value of $\sqrt[3]{(4.1)^2 + 3(3.8)^2}$ using the theory of approximations. **Answer:** 3.9167.
11	Find an approximate value of $\log\left(3\sqrt{1.03} + 4\sqrt{0.98} - 1\right)$ using the theory of approximations. **Answer:** 0.7790.
12	Find an approximate value of $\sin 31^0 \cdot \cos 58^0$ using the theory of approximations. **Answer:** $\frac{1}{4} + \frac{\pi}{180}\left(\frac{3\sqrt{3}}{4}\right)$.