V	Applications of Partial differentiations

Classwork Examples

5.1	Maclaurin's and Taylor's series expansion in two variables

1	Expand $f(x,y) = xy^2 + xy + 3$ in powers of $(x-1)$ and $(y+2)$ using Taylor's series expansion. **C.W.** **Answer:** $f(x,y) = 5 + 2(x-1) - 3(y+2) - 3(x-1)(y+2) + (y+2)^2 + \cdots$
2	Expand $f(x,y) = e^{xy}$ in powers of $(x-1)$ and $(y-1)$ using Taylor's series expansion. **C.W.** **Answer:** $f(x,y) = e\left[1 + (x-1) + (y-1) + \frac{1}{2}(x-1)^2 + 2(x-1)(y-1) + \frac{1}{2}(y-1)^2 + \cdots\right]$.
3	Expand $f(x,y) = xe^y + 1$ in powers of $(x-1)$ and y using Taylor's series expansion. **H.W.** **Answer:** $f(x,y) = 1 + x + xy + \frac{y^2}{2}$ ...
4	Expand $f(x,y) = e^x \cdot \sin y$ in powers of x and y up to second order terms. **C.W.** **Answer:** $f(x,y) = y + xy + \cdots$
5	Find the expansion of $f(x,y) = \cos x \cdot \cos y$ in powers of x and y up to second order terms. **C.W.** **Answer:** $f(x,y) = 1 - \frac{x^2}{2} - \frac{y^2}{2} + \cdots$
6	Show that $e^y \cdot \log(1+x) = x + xy - \frac{x^2}{2} + \cdots$. **H.W.**

5.2	Tangent plane and normal line to a surface

1	Find the equation of tangent plane and normal line to the surface $x^2 + y^2 + z^2 = 3$ at the point $(1, 1, 1)$. **C.W.** **Answer:** $x + y + z = 3, \frac{x-1}{2} = \frac{y-1}{2} = \frac{z-1}{2}$.
2	Find the equation of tangent plane and normal line to the surface $2x^2 + y^2 + 2z = 3$ at the point $(2, 1, -3)$. **H.W.** **Answer:** $4x + y + z = 6, \frac{x-2}{4} = \frac{y-1}{1} = \frac{z+3}{1}$.
3	Find the equation of tangent plane and normal line to the surface $z = \sqrt{1 - x^2 - y^2}$ at the point $\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)$. **C.W.** **Answer:** $2x + 2y + z = 3, \frac{3x-2}{2} = \frac{3y-2}{2} = \frac{3z-1}{1}$.
4	Find the equation of tangent plane and normal line to the surface $z = \tan^{-1}\left(\frac{y}{x}\right)$ at the point $(1, 1)$. **C.W.** **Answer:** $x - y + 2z = \frac{\pi}{2}, \frac{x-1}{1} = \frac{y-1}{-1} = \frac{z-\frac{\pi}{4}}{2}$.
5	Find the equation of tangent plane and normal line to the surface $xyz = a^2$ at the point (α, β, γ). **C.W.** **Answer:** $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 3, \frac{x-\alpha}{\beta\gamma} = \frac{y-\beta}{\alpha\gamma} = \frac{z-\gamma}{\alpha\beta}$.
6	Find the equation of tangent plane and normal line to the surface $\frac{x^2}{2} - \frac{y^2}{3} = z$ at the point $(2, 3, -1)$. **H.W.** **Answer:** $2x - 2y - z + 1 = 0, \frac{x-2}{2} = \frac{y-3}{-2} = \frac{z+1}{-1}$.

5.3	Maxima and Minima of two variable function

1	Find the maximum and minimum values of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$. **C.W.** **Answer**: $f(x, y)$ is maximum at $(-1, -2), f(-1, -2) = 38$ and minimum at $(1, 2), f(1, 2) = 2$.

2	Find the extreme values of the function $f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$. **C.W.** **Answer:** $f(x,y)$ is maximum at $(0,0)$, $f(0,0) = 4$ and minimum at $(2,0)$, $f(2,0) = 0$.
3	Find the extreme values of the function $f(x,y) = x^2 + y^2 + xy + x - 4y + 5$. **C.W.** **Answer:** $f(x,y)$ is minimum at $(-2,3)$, $f(-2,3) = -2$.
4	Show that the minimum value of the function $f(x,y) = 2x^4 + y^2 - x^2 - 2y$ is $-\frac{9}{8}$. **C.W.** **Answer:** $f(x,y)$ is minimum at $\left(\pm\frac{1}{2}, 1\right)$, $f\left(\pm\frac{1}{2}, 1\right) = -\frac{9}{8}$.
5	Find the extreme values of the function $f(x,y) = x^4 + y^4 - x^2 - y^2 + 1$. **H.W.** **Answer**: $f(x,y)$ is maximum at $(0,0)$, $f(0,0) = 1$ and minimum at $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$, $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$.
6	Show that the maximum value of the function $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ is 4. **H.W.**

5.4	Lagrange's method of undetermined multiplier

1	If the sum of three positive numbers is unity, then find the maximum value of their product. **C.W.** **Answer:** $\frac{1}{27}$.
2	In a triangle, find maximum value of $\sin A \sin B \sin C$, where A, B, and C are three angles of triangle. **C.W.** **Answer:** $\frac{3\sqrt{3}}{8}$.
3	Find the extreme values of $x - 2y + 5z$ on the sphere $x^2 + y^2 + z^2 = 30$. **C.W.** **Answer:** Maximum value is 30 and Minimum value is -30.
4	Find the maximum value of $x^l y^m z^n$ subject to $x + y + z = a$. **H.W.** **Answer:** $\frac{a^{l+m+n}\, l^l m^m n^n}{(l+m+n)^{l+m+n}}$.

5	Find the volume of the largest rectangular parallelepiped that can be inscribed in the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. **C.W.** **Answer:** $\frac{8abc}{3\sqrt{3}}$.
6	Find the dimensions of the rectangular box of maximum capacity whose surface area is given when box is closed. **H.W.** **Answer:** $x=\sqrt{\frac{S}{6}}, y=\sqrt{\frac{S}{6}}, z=\sqrt{\frac{S}{6}},\ V=\left(\frac{S}{6}\right)^{\frac{3}{2}}$.

5.5	Jacobian

1	If $u=e^x\cos y$ and $v=e^x\sin y$, then find $\frac{\partial(u,v)}{\partial(x,y)}$. **C.W.** **Answer:** e^{2x} .
2	If $x=r\cos\theta$ and $y=\mathrm{r}\sin\theta$, then find $\frac{\partial(x,y)}{\partial(r,\theta)}$ and $\frac{\partial(r,\theta)}{\partial(x,y)}$. **C.W.** **Answer:** r and $\frac{1}{r}$.
3	If $u=xyz,\ v=xy+yz+zx$, and $w=x+y+z$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. **C.W.** **Answer:** $(x-y)(y-z)(z-x)$.
4	If $u=\frac{x}{y-z},\ v=\frac{y}{z-x}$, and $w=\frac{z}{x-y}$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. **H.W.** **Answer:** 0.
5	If $u=\frac{yz}{x},\ v=\frac{zx}{y}$, and $w=\frac{xy}{z}$, then find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. **C.W.** **Answer:** 4.
6	If $x=u(1-v), y=uv$, then prove that $\frac{\partial(x,y)}{\partial(u,v)}\cdot\frac{\partial(u,v)}{\partial(x,y)}=1$. **H.W.**

5.6	Errors and approximations

1	If the sides and angles of a plane triangle vary in such a way that its circumradius remains constant, then prove that $\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=0$. **C.W.**

2	The power dissipated in a resistor is given by $P = \frac{E^2}{R}$. Find the approximate percentage change in P when E is increased by 3 percentage and R is decreased by 2 percentage. **C.W.** **Answer:** 8 percentage.
3	The diameter and altitude of a can in the shape of a right circular cylinder are measured as 40 cm and 64 cm respectively. The possible error in each measurement is ± 5 percentage. Find approximately the maximum possible error in the computed value for the volume and the lateral surface. **C.W.** **Answer:** $3840\pi\ cm^3$, $256\pi\ cm^2$.
4	Find the possible percentage error in computing the parallel resistance R of the three resistances R_1, R_2 and R_3 from the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ if R_1, R_2, R_3 are each in error by 1.2 percentage. **H.W.** **Answer:** 1.2 percentage.
5	As a result of deformation of the radius of a cone changes from 30 cm to 30.1 cm and its height changes from 60 cm to 59.5 cm. Find the approximate change in volume of a cone. **C.W.** **Answer:** $30\pi\ cm^3$.
6	If kinetic energy $K = \frac{wv^2}{2g}$, where g is constant, then find approximately the change in kinetic energy as w changes from 49 to 49.5 and v changes from 1600 to1590. **H.W.** **Answer:** $-\frac{144000}{g}$.
7	Find an approximate value of $(4.1)^2 + (2.9)^2$ using the theory of approximations. **C.W.** **Answer:** 25.02.
8	Find an approximate value of $\sqrt[5]{(3.8)^2 + 2(2.1)^3}$ using the theory of approximations. **H.W.** **Answer:** 2.01.

9	Find an approximate value of $\sqrt{(0.98)^2+(2.01)^2+(1.94)^2}$ using the theory of approximations. **C.W.** **Answer:** 2.96.
10	Find an approximate value of $\sin 58^0 \cdot \cos 46^0$ using the theory of approximations. **C.W.** **Answer:** $\frac{\sqrt{3}}{2\sqrt{2}} - \frac{\pi}{180}\left(\frac{2+\sqrt{3}}{2\sqrt{2}}\right)$.
11	Find an approximate value of $\sin 29^0 \cdot \cos 58^0$ using the theory of approximations. **C.W.** **Answer:** $\frac{1}{4} + \frac{\sqrt{3}}{4} \cdot \frac{\pi}{180}$.
12	Find an approximate value of $(27.1)^{\frac{2}{3}} + \sqrt{26}$ using the theory of approximations. **H.W.** **Answer:** 14.1222.