IV Partial differentiations

4.1 Partial derivative and geometrical interpretation

Partial derivative with respect to x

The partial derivative of $z = f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0+h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists i.e. differentiate with respect to x keeping y constant.

Symbolically we may write it as $f_x(x_0, y_0)$.

Partial derivative with respect to y

The partial derivative of $z = f(x, y)$ with respect to y at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0}\frac{f(x_0, y_0+h) - f(x_0, y_0)}{h},$$

provided the limit exists i.e. differentiate with respect to y keeping x constant.

Symbolically we may write it as $f_y(x_0, y_0)$.

For example, if $f(x, y) = x + y + xy$, then $\frac{\partial f}{\partial x} = 1 + y$ and $\frac{\partial f}{\partial y} = 1 + x$.

Partial derivatives of higher orders

Let $z = f(x, y)$, then $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ being the functions of x and y can further be differentiated partially with respect to x and y.

Symbolically, we write

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}\ or\ \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)\ or\ f_{xx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y\partial x}\ or\ \frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\ or\ f_{xy}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x\partial y}\ or\ \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)\ or\ f_{yx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}\ or\ \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\ or\ f_{yy}$$

Notations: We use the following notation

$$p=\frac{\partial z}{\partial x}, q=\frac{\partial z}{\partial y}, r=\frac{\partial^2 z}{\partial x^2}, s=\frac{\partial^2 z}{\partial x\,\partial y}, t=\frac{\partial^2 z}{\partial y^2}$$

For example, if $f(x,y)=x^2y+\sin(x+y)$, then $\frac{\partial f}{\partial x}=2xy+\cos(x+y)$, $\frac{\partial f}{\partial y}=x^2+\cos(x+y)$, $\frac{\partial^2 f}{\partial x^2}=2y-\sin(x+y)$, $\frac{\partial^2 f}{\partial x\,\partial y}=2x-\sin(x+y)$, $\frac{\partial^2 f}{\partial y\,\partial x}=2x-\sin(x+y)$ and $\frac{\partial^2 f}{\partial y^2}=2x-\sin(x+y)$.

Note: In general, $\frac{\partial^2 f}{\partial x\,\partial y}$ and $\frac{\partial^2 f}{\partial y\,\partial x}$ are not equal.

Tutorial:

Partial derivative and geometrical interpretation

1	Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if $z=x^2y-x\sin(xy)$. **Solution.** Here $z=x^2y-x\sin(xy)---(1)$ Differentiating (1) partially w.r.t $'x'$, we get $$\frac{\partial z}{\partial x}=2xy-\{x\cos(xy)\cdot y+\sin(xy)\}=2xy-xy\cos(xy)-\sin(xy)$$ Differentiating (1) partially w.r.t $'y'$, we get $$\frac{\partial z}{\partial y}=x^2-x\cos(xy)\cdot x=x^2-x^2\cos(xy)=x^2(1-\cos(xy)).$$
2	Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if $z=\log(x^2+y^2)$. **Answer:** $\frac{2x}{x^2+y^2}, \frac{2y}{x^2+y^2}$.
3	If $f(x,y)=x\cos y+ye^x$, then find all second order partial derivatives of f. **Answer:** $f_x=\cos y+ye^x, f_y=-x\sin y+e^x, f_{xx}=ye^x$, $f_{xy}=-\sin y+e^x, f_{yx}=-\sin y+e^x, f_{yy}=-x\cos y$.
4	Verify that $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$, for the function $z=\tan^{-1}\left(\frac{x}{y}\right)$.
5	If $z=e^x(x\cos y-y\sin y)$, then show that $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$.

4.2	Euler's theorem with corollaries and their applications

Homogeneous functions

A function of two variables, say $f(x,y)$, is said to be homogeneous of degree n if for any nonzero constant k, $f(kx,ky) = k^n\, f(x,y)$.

For example, $f(x,y) = x^2 + y^2$ is homogeneous function of degree 2.

Note: In general, a function of k variables, say $f(x_1, x_2, x_3, \ldots, x_k)$, is said to be homogeneous of degree n if for any nonzero constant k, $f(kx_1, kx_2, kx_3, \ldots, kx_k) = k^n\, f(x_1, x_2, x_3, \ldots, x_k)$.

Euler's theorem on homogeneous function

If z is a homogeneous function of x, y of order n, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = n\, z.$$

Note: In general, if u is a homogeneous function of $x_1, x_2, x_3, \ldots, x_k$ of degree n, then

$$x_1\frac{\partial u}{\partial x_1} + x_2\frac{\partial u}{\partial x_2} + x_3\frac{\partial u}{\partial x_3} + \cdots + x_k\frac{\partial u}{\partial x_k} = n\, u\,.$$

Deduction from Euler's theorem

1. If z is a homogeneous function of x, y of order n, then
$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\,\frac{\partial^2 z}{\partial x\,\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n\,(n-1)\,z.$$
2. If z is a homogeneous function of x, y of order n and $z = f(u)$, then
$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}.$$
3. If z is a homogeneous function of x, y of order n and $z = f(u)$, then
$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\,\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = g(u)\,[g'(u) - 1]$, where $g(u) = n\frac{f(u)}{f'(u)}$.

Tutorial:

Euler's theorem with corollaries and their applications

1	If $z = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log_e x - \log_e y}{x^2+y^2}$, then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -2z$. **Solution.** Here $z(x,y) = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log_e x - \log_e y}{x^2+y^2} = \frac{1}{x^2} + \frac{1}{xy} + \frac{\log_e\left(\frac{x}{y}\right)}{x^2+y^2}$. Replacing x by kx and y by ky,

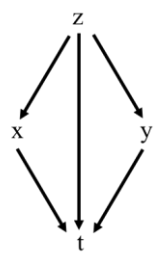
	$= k^{-2}\left(\frac{1}{x^2}+\frac{1}{xy}+\frac{\log_e\left(\frac{x}{y}\right)}{x^2+y^2}\right)$ $= k^{-2}\, z(x,y).$ Thus $z(x,y)$ is homogeneous function of degree -2. By Euler's theorem $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz \Rightarrow x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-2z.$
2	If $u=\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}$, then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=0.$
3	If $z=f\left(\frac{y}{x}\right)+\sqrt{x^2+y^2}$, then find $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}$. **Answer:** 0.
4	If $u=sec^{-1}\left(\frac{x^3+y^3}{x+y}\right)$, then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\cot u$. **Solution.** Here $z(x,y)=f(u)=\sec u=\frac{x^3+y^3}{x+y}$. Replacing x by kx and y by ky, $z(kx,ky)=\frac{(kx)^3+(ky)^3}{kx+ky}$ $= k^2\left(\frac{x^3+y^3}{x+y}\right).$ Thus $z(x,y)=f(u)=\sec u$ is homogeneous function of degree 2. By modified Euler's theorem $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f'(u)} \Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2\frac{\sec u}{\sec u\tan u}=2\cot u.$
5	If $u=\log_e\left(\frac{x^4+y^4}{x+y}\right)$, then prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3.$
6	If $u=\tan^{-1}\left(\frac{y^2}{x}\right)$, then find $(i)\ x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$, $(ii)\ x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}$. **Answer:** $(i)\frac{1}{2}\ sin\ 2u$, $(ii)-sin^2u\ sin\ 2u.$

4.3	Chain rule

Chain rule for function of two independent variables

If $z=f(x,y)$ is differentiable and $x=x(t)$, $y=y(t)$ are differentiable functions of t, then z is a differentiable function of t and

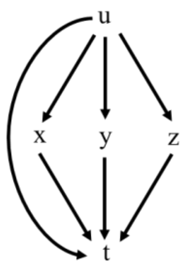
$$\frac{df}{dt}\ or\ \frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}.$$



Chain rule for function of three independent variables

If $u = f(x, y, z)$ is differentiable and $x = x(t)$, $y = y(t), z = z(t)$ are differentiable functions of t, then u is a differentiable function of t and

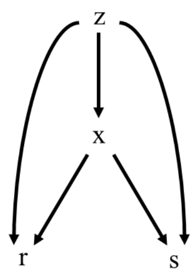
$$\frac{df}{dt} \text{ or } \frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}.$$



Chain rule for function of two independent variables and one intermediate variable

If $z = f(x)$ and $x = g(r, s)$, then z has partial derivatives with respect to r and s, given by the formulas

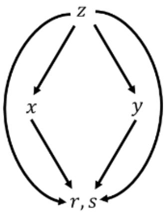
$$\frac{\partial z}{\partial r} = \frac{dz}{dx} \cdot \frac{\partial x}{\partial r} \text{ and } \frac{\partial z}{\partial s} = \frac{dz}{dx} \cdot \frac{\partial x}{\partial s}.$$



Chain rule for function of two independent variables and two intermediate variables

If $z = f(x, y), x = g(r, s)$, and $y = h(r, s)$, then z has partial derivatives with respect to r and s, given by the formulas

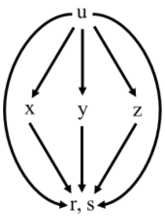
$$\frac{\partial z}{\partial r}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} \text{ and } \frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}.$$



Chain rule for function of two independent variables and three intermediate variables

Suppose that $u=f(x,y,z), x=g(r,s), y=h(r,s)$, and $z=k(r,s)$. If all four functions are differentiable, then u has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial u}{\partial r}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial r} \text{ and } \frac{\partial u}{\partial s}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial s}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial s}.$$



Tutorial:

Chain rule

1	Given $z=\sin\left(\frac{x}{y}\right), x=e^t$ and $y=t^2$. Find $\frac{dz}{dt}$ as a function of t. **Solution.** Here $z(x,y)=\sin\left(\frac{x}{y}\right), x=e^t, y=t^2$. So $$\frac{\partial z}{\partial x}=\frac{1}{y}\cos\left(\frac{x}{y}\right), \frac{\partial z}{\partial y}=-\frac{x}{y^2}\cos\left(\frac{x}{y}\right), \frac{dx}{dt}=e^t, \frac{dy}{dt}=2t.$$ Now $$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$ $$=\frac{1}{y}\cos\left(\frac{x}{y}\right)\ e^t-\frac{x}{y^2}\cos\left(\frac{x}{y}\right)\ 2t$$ Substituting x and y, we get $$\frac{dz}{dt}=\frac{1}{t^2}\cos\left(\frac{e^t}{t^2}\right)\ e^t-\frac{e^t}{t^4}\cos\left(\frac{e^t}{t^2}\right)\ 2t$$ $$=\frac{1}{t^2}\ e^t\cos\left(\frac{e^t}{t^2}\right)\left(1-\frac{2}{t}\right).$$
2	If $z=\tan^{-1}\left(\frac{y}{x}\right)$ where $x=e^t-e^{-t}$ and $y=e^t+e^{-t}$, then find $\frac{dz}{dt}$.

	Answer: $\frac{dz}{dt} = -\frac{2}{e^{2t}+e^{-2t}}$.
3	If $u = x^2 + y^2 + z^2$ where $x = e^t, y = e^t \sin t\,, z = e^t \cos t$, then find $\frac{du}{dt}$. **Answer:** $\frac{du}{dt} = 4e^{2t}$.
4	Express $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$ in terms of r and s if $$z = x^2 + y^2, \qquad x = r - s, \qquad y = r + s\,.$$ **Solution.** Here $z(x, y) = x^2 + y^2, x = r - s,\ y = r + s$. So $$\frac{\partial z}{\partial x} = 2x, \frac{\partial z}{\partial y} = 2y, \frac{\partial x}{\partial r} = 1, \frac{\partial y}{\partial r} = 1, \frac{\partial x}{\partial s} = -1, \frac{\partial y}{\partial s} = 1.$$ Now $$\begin{aligned}\frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r}\\ &= (2x)(1) + (2y)(1)\\ &= 2(r-s) + 2(r+s)\\ &= 4r\end{aligned}$$ and $$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}\\ &= (2x)(-1) + (2y)(1)\\ &= -2(r-s) + 2(r+s)\\ &= 4s.\end{aligned}$$
5	If $u = f(r, s),\ r = x + at, s = y + bt$ and x, y, t are independent variables, then show that $\frac{\partial u}{\partial t} = a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y}$.
6	If $u = f(2x - 3y, 3y - 4z, 4z - 2x)$, then prove that $\frac{1}{2}\frac{\partial u}{\partial x} + \frac{1}{3}\frac{\partial u}{\partial y} + \frac{1}{4}\frac{\partial u}{\partial z} = 0$.

4.4	Implicit functions

A formula for implicit differentiation

Suppose $y = f(x)$ is defined implicitly by $F(x, y) = 0$. Then $\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\cdot\frac{dy}{dx} = 0$.

If $\frac{\partial F}{\partial y} \neq 0$, then $\frac{dy}{dx} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial y}\right)} = -\frac{F_x}{F_y}$.

Suppose $z = f(x, y)$ is defined implicitly by $F(x, y, z) = 0$. Then

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial x} = 0, \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z}\cdot\frac{\partial z}{\partial y} = 0.$$

If $\frac{\partial F}{\partial z} \neq 0$, then $\frac{\partial z}{\partial x} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial z}\right)} = -\frac{F_x}{F_z}$, $\frac{\partial z}{\partial y} = -\frac{\left(\frac{\partial F}{\partial y}\right)}{\left(\frac{\partial F}{\partial z}\right)} = -\frac{F_y}{F_z}$.

Tutorial:

Implicit functions

1	If $x^3 - 3axy + y^3 = 0$, then find $\frac{dy}{dx}$. **Solution.** Here $F(x,y) = x^3 - 3axy + y^3$. So $$\frac{\partial F}{\partial x} = 3x^2 - 3ay, \qquad \frac{\partial F}{\partial y} = 3y^2 - 3ax$$ $$\frac{dy}{dx} = -\frac{\left(\frac{\partial F}{\partial x}\right)}{\left(\frac{\partial F}{\partial y}\right)} = -\frac{3x^2 - 3ay}{3y^2 - 3ax} = \frac{ay - x^2}{y^2 - ax}.$$
2	If $x^y = y^x$, then prove that $\frac{dy}{dx} = \frac{y(y - x\log y)}{x(x - y\log x)}$.
3	Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, \ln 2, \ln 3)$, if $xe^y + ye^z + 2\ln x - 2 - 3\ln 2 = 0$. **Answer:** $\left.\frac{\partial z}{\partial x}\right\|_{(1,\ln 2,\ln 3)} = -\frac{4}{3\ ln\ 2}$, $\left.\frac{\partial z}{\partial y}\right\|_{(1,\ln 2,\ln 3)} = -\frac{5}{3\ ln\ 2}$.

4.5	Total differentials

Total differentials

Suppose $z = f(x,y)$ is a function of two variables x and y. The total differential is defined as

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy.$$

Suppose $u = f(x,y,z)$ is a function of three variables x, y, and z. The total differential is defined as

$$du = \frac{\partial u}{\partial x}\, dx + \frac{\partial u}{\partial y}\, dy + \frac{\partial u}{\partial z}\, dz.$$

Tutorial:

Total differentials

1	Find the total derivative of the function $z = y\tan x^2 - 2xy$. **Solution.** Here $z = y\tan x^2 - 2xy$ then $$\frac{\partial z}{\partial x} = 2xy\sec^2\{x^2\} - 2y, \frac{\partial z}{\partial y} = \tan x^2 - 2x.$$

	The total derivative of a function z is $$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$ $$= (2xy \sec^2\{x^2\} - 2y)\, dx + (\tan x^2 - 2x)\, dy.$$
2	Find the total derivative of the function $z = xe^{2y} + e^{-y}$. **Answer:** $dz = (e^{2y})\, dx + (2xe^{2y} - e^{-y})\, dy.$
3	If $u = x \tan^{-1} z - \frac{y^2}{z}$, then obtain du. **Answer:** $du = (tan^{-1}z)\, dx - \left(\frac{2y}{z}\right)\, dy + \left(\frac{x}{1+z^2} + \frac{y^2}{z^2}\right)\, dz.$