IV Partial differentiations

Practice Examples

4.1	Partial derivative and geometrical interpretation

1	Evaluate $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, if $z = x^2 \sin\left(\frac{y}{x}\right)$. **Answer:** $\frac{\partial z}{\partial x} = 2x \sin\left(\frac{y}{x}\right) - y\cos\left(\frac{y}{x}\right), \frac{\partial z}{\partial y} = x\cos\left(\frac{y}{x}\right)$.
2	Verify that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$, for the function $z = \log\left(\frac{x^2+y^2}{xy}\right)$.
3	If $z = \log(\tan x + \tan y)$, then prove that $\sin 2x \frac{\partial z}{\partial x} + \sin 2y \frac{\partial z}{\partial y} = 2$.
4	If resistors R_1, R_2 and R_3 ohms are connected in parallel to make R ohm resistor, the value of R can be found from the equation $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$. Find $\frac{\partial R}{\partial R_2}$ when $R_1 = 30, R_2 = 45,$ and $R_3 = 90$ ohms. **Answer:** $\frac{1}{9}$.
5	Find the first order partial derivatives for the following functions at the specified point. $(i)\ f(x,y) = \cot^{-1}(x+y)$, at $(1,2)$. $(ii)\ f(x,y) = \ln\left(\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}\right)$, at $(3,4)$. $(iii)\ f(x,y) = (xy)^{\sin z}$, at $\left(5,3,\frac{\pi}{2}\right)$. **Answer:** $(i) \left.\frac{\partial f}{\partial x}\right\|_{(1,2)} = -\frac{1}{10}, \left.\frac{\partial f}{\partial y}\right\|_{(1,2)} = -\frac{1}{10}$. $(ii) \left.\frac{\partial f}{\partial x}\right\|_{(3,4)} = -\frac{2}{5}, \left.\frac{\partial f}{\partial y}\right\|_{(3,4)} = \frac{3}{10}$. $(iii) \left.\frac{\partial f}{\partial x}\right\|_{\left(5,3,\frac{\pi}{2}\right)} = 3, \left.\frac{\partial f}{\partial y}\right\|_{\left(5,3,\frac{\pi}{2}\right)} = 5, \left.\frac{\partial f}{\partial z}\right\|_{\left(5,3,\frac{\pi}{2}\right)} = 0$.
6	Find the first and second order partial derivatives of $z = (1-2xy+y^2)^{-\frac{1}{2}}$. **Answer:** $f_x = y\,(1-2xy+y^2)^{-\frac{3}{2}}, f_y = (x-y)(1-2xy+y^2)^{-\frac{3}{2}}$,

	$f_{xx} = 3y^2\,(1-2xy+y^2)^{-\frac{3}{2}}$, $f_{xy} = 3y(x-y)(y^2-2xy+1)^{-\frac{5}{2}} + (y^2-2xy+1)^{-\frac{3}{2}} = f_{yx}$, $f_{yy} = 3(x-y)^2(y^2-2xy+1)^{-\frac{5}{2}} - (y^2-2xy+1)^{-\frac{3}{2}}$.

4.2	Euler's theorem with corollaries and their applications

1	If $z = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{x}{y}\right)$, then prove that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.
2	If $z = \frac{xy}{x+y}$, then find the value of $x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2}$. **Answer:** 0.
3	If $u = \sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\sin u}{\cos u}$, $(ii)\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = -\frac{\sin u \cos 2u}{4\cos^3 u}$.
4	If $z = \log\left(\frac{x^2-y^2}{x^2+y^2}\right)$, then show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.
5	If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, then show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u$.
6	If $u = \sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$, then show that (i) $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$, $(ii)\ x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 9\tan^3 u + 6\tan u$.

4.3	Chain rule

1	If $z = y + f(u), u = \frac{x}{y}$, then show that $u\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 1$.
2	If $u = f(x^2+2yz, y^2+2zx)$, then prove that $(y^2-zx)\frac{\partial u}{\partial x} + (x^2-yz)\frac{\partial u}{\partial y} + (z^2-xy)\frac{\partial u}{\partial z} = 0$.
3	Express $\frac{\partial u}{\partial r}$ and $\frac{\partial u}{\partial s}$ in terms of r and s if $u = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r.$

	Answer: $\frac{\partial u}{\partial r} = 12r + \frac{1}{s}, \frac{\partial u}{\partial s} = \frac{2}{s} - \frac{r}{s^2}$.
4	If $u = xy + yz + zx$ where $x = \frac{1}{t}, y = e^t, z = e^{-t}$, then find $\frac{du}{dt}$. **Answer:** $\frac{du}{dt} = -\frac{1}{t^2}(e^t + e^{-t}) + \frac{1}{t}(e^t - e^{-t})$.
5	If $u = f(r)$ where $r^2 = x^2 + y^2$, then prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$.
6	If $u = f(e^{y-z}, e^{z-x}, e^{x-y})$, then show that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

4.4	Implicit functions

1	Find $\frac{dy}{dx}$ if $y^2 - x^2 - \sin xy = 0$. **Answer:** $\frac{dy}{dx} = \frac{2x+y\cos(xy)}{2y-x\cos(xy)}$.
2	If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, then show that $\frac{dy}{dx} = -\frac{ax+hy+g}{hx+by+f}$.
3	Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(2, 3, 6)$, if $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} - 1 = 0$. **Answer:** $\frac{\partial z}{\partial x} = -9, \frac{\partial z}{\partial y} = -4$.
4	Find $\frac{dy}{dx}$, if $x^2y^2 + \sin(xy) + y^3 = 0$. **Answer:** $\frac{dy}{dx} = -\frac{2xy^2+y\cos(xy)}{2x^2y+x\cos(xy)+3y^2}$.
5	Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(1, 1, 1)$, if $x^2 + y^4 - z^3 + 3xy^2 = 8$. **Answer:** $\frac{\partial z}{\partial x} = \frac{5}{3}, \frac{\partial z}{\partial y} = \frac{10}{3}$.
6	Given $xyz = \cos(x + y + z)$, then find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$. **Answer:** $\frac{\partial z}{\partial x} = -\frac{yz+\sin(x+y+z)}{xy+\sin(x+y+z)}, \frac{\partial z}{\partial y} = -\frac{xz+\sin(x+y+z)}{xy+\sin(x+y+z)}$.

4.5	Total differentials

1	Find the total derivative of the function $z = 2x\sin y - 3x^2y^2$. **Answer:** $dz = (2\sin y - 6xy^2)dx + (2x\cos y - 6x^2y)dy$.
2	Find the total differential of $f(x,y) = 5x^3 - 4xy^{-1} + 3y^4$. **Answer:** $df = (15x^2 - 4y^{-1})dx + (4xy^{-2} + 12y^3)dy$.
3	Find the total differential of $f(x,y,z) = xy + yz + zx$. **Answer:** $df = (y+z)dx + (x+z)dy + (x+y)dz$.
4	If $g(x,y) = (x-y)^{-1}$, then obtain dg. **Answer:** $dg = (x-y)^{-2}\,(dy - dx)$.
5	Find the total differential of $f(x,y,z) = e^{xyz}$. **Answer:** $df = e^{xyz}(yz\,dx + zx\,dy + xy\,dz)$.
6	If $u = e^{yz} - \cos(xz)$, then find du. **Answer:** $du = \big(z\,sin\,(xz)\big)\,dx + (z\,e^{yz})\,dy + \big(y\,e^{yz} + x\,sin\,(xz)\big)\,dz$.