Graded Homework 6 - Section 5.1

(5) $A = \begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 6 & 1 & 2 & 0 & 5 \\ 3 & 2 & 2 & 4 & 4 \\ 5 & 1 & 0 & 0 & 3 \\ 2 & 2 & 4 & 1 & 0 \end{bmatrix}$

$M_{2,3} = \begin{bmatrix} 4 & 3 & 1 & 0 \\ 3 & 2 & 4 & 4 \\ 5 & 1 & 0 & 3 \\ 2 & 2 & 1 & 0 \end{bmatrix}$

$M_{3,1} = \begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 2 & 0 & 5 \\ 1 & 0 & 0 & 3 \\ 2 & 4 & 1 & 0 \end{bmatrix}$

(9) $A = \begin{bmatrix} 6 & 1 & 2 \\ 4 & 3 & 0 \\ 1 & 1 & 1 \end{bmatrix} \rightarrow C_{13} = (-1)^{1+3} \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} \rightarrow$

$(1)[(4)(1) - (3)(1)] = 1 = C_{13} \rightarrow C_{22} = \rightarrow$

$(-1)^{2+2} \begin{vmatrix} 6 & 2 \\ 1 & 1 \end{vmatrix} = (1)[(6)(1) - (2)(1)] = 4 = C_{22}$

(13)

(a) $A = \begin{bmatrix} -1 & 1 & -1 & 2 \\ 0 & 3 & 2 & 0 \\ 1 & 4 & 0 & 1 \\ 0 & -1 & 3 & -1 \end{bmatrix} \rightarrow \det(A) = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23} + a_{24}C_{24}$

$= a_{22} C_{22} + a_{23} C_{23}$ because a_{21} and $a_{24} = 0$ $\rightarrow$ $C_{22} = (-1)^{2+2} \begin{vmatrix} -1 & -1 & 2 \\ 1 & 0 & 1 \\ 0 & 3 & -1 \end{vmatrix}$

$= \cancel{(-1)(0)(-1)} + (-1)\cancel{(1)(0)} + (2)(1)(3) - \cancel{(2)(0)(0)}$
$- (-1)(1)(3) - (-1)(1)(-1) = 6 + 3 - 1 = 4 = C_{22}$

$\rightarrow C_{23} = (-1)^{2+3} \left| \begin{matrix} -1 & 1 & 2 \\ 1 & 4 & 1 \\ 0 & -1 & -1 \end{matrix} \,\middle|\, \begin{matrix} -1 & 1 \\ 1 & 4 \\ 0 & -1 \end{matrix} \right. = -1 \big[(-1)(4)(-1)$
$+ \cancel{(1)(1)(0)} + (2)(1)(-1) - \cancel{(2)(4)(0)} - (-1)(1)(-1)$
$- (1)(1)(-1) \big] = -1 [4 - 2 - 1 + 1] = -2 = C_{23}$

$\rightarrow \det(A) = (3)(4) + (2)(-2) = 8$

$\rightarrow$ $\det(A) = 8$ row method

(b) $\det(A) = a_{11} C_{11} + a_{21} C_{21} + a_{31} C_{31} + a_{41} C_{41}$

$\rightarrow \det(A) = a_{11} C_{11} + a_{31} C_{31}$ because a_{21} and a_{41} are 0. $\rightarrow C_{11} = (-1)^{1+1} \left| \begin{matrix} 3 & 2 & 0 \\ 4 & 0 & 1 \\ -1 & 3 & -1 \end{matrix} \,\middle|\, \begin{matrix} 3 & 2 \\ 4 & 0 \\ -1 & 3 \end{matrix} \right. = \cancel{(3)(0)(-1)}$
$+ (2)(1)(-1) + (0)\cancel{(4)(3)} - (0)\cancel{(0)(-1)} - (3)(1)(3) -$
$(2)(4)(-1) = -2 - 9 + 8 = -3 = C_{11} \longrightarrow$
$C_{31} = (-1)^{3+1} \left| \begin{matrix} 1 & -1 & 2 \\ 3 & 2 & 0 \\ -1 & 3 & -1 \end{matrix} \,\middle|\, \begin{matrix} 1 & -1 \\ 3 & 2 \\ -1 & 3 \end{matrix} \right. = (1)(2)(-1) + \cancel{(-1)(0)(-1)}$

$+(2)(3)(3) - (2)(2)(-1) - (1)\cancel{(0)(3)} - (-1)(3)(-1)$

$= -2 + 18 + 4 - 3 = 17 = C_{31} \rightarrow (-1)(-3) + (1)(17)$

$= 20 = \det(A)$ by column.

As $\det(A) \neq 0 \Rightarrow T$ is invertible.

(17) $A = \begin{bmatrix} 4 & 2 & 1 & 0 & 1 \\ 0 & 3 & -1 & 1 & 2 \\ 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$

(a) row method:

$|A| = a_{51}C_{51} + a_{52}C_{52} + a_{53}C_{53} + a_{54}C_{54} + a_{55}C_{55}$

$\rightarrow |A| = a_{53}C_{53} + a_{55}C_{55}$ because a_{51}, a_{52} and a_{54} are 0. $\rightarrow C_{53} = (-1)^{5+3} \begin{vmatrix} 4 & 2 & 0 & 1 \\ 0 & 3 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 3 \end{vmatrix}$

let this be $Z \rightarrow |Z| = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41} \rightarrow |Z| = a_{11}C_{11}$ because $a_{21}, a_{31}, a_{41} = 0 \rightarrow |Z| = a_{11}C_{11} \rightarrow C_{11} = (-1)^{1+1} \left|\begin{array}{ccc|cc} 3 & 1 & 2 & 3 & 1 \\ -1 & 0 & 1 & -1 & 0 \\ 1 & 0 & 3 & 1 & 0 \end{array}\right|$

$= (3)\cancel{(0)(3)} + (1)(1)(1) + (2)\cancel{(-1)(0)} - (2)\cancel{(0)(1)} - (3)\cancel{(1)(0)} - (1)(-1)(3) = 4 \rightarrow |Z| = (4)(4) = 16$

$\rightarrow C_{53} = (+1)(16) = +16 \rightarrow$

$$C_{5,5} = (-1)^{5+5} \begin{vmatrix} 4 & 2 & 1 & 0 \\ 0 & 3 & -1 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 \end{vmatrix} \rightarrow \text{let this be } Y$$

$\rightarrow |Y| = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41} \rightarrow$

$|Y| = a_{11}C_{11}$ as $a_{21}, a_{31}, a_{41} = 0. \rightarrow C_{11} = (-1)^{1+1}$

$$\begin{vmatrix} 3 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 2 & 0 \end{vmatrix} \begin{matrix} 3 & -1 \\ -1 & 0 \\ 1 & 2 \end{matrix} = (3)\cancel{(0)(0)} + (-1)\cancel{(0)(1)} + (1)(-1)(2) -$$

$(1)\cancel{(0)}(1) - (3)\cancel{(0)(2)} - (-1)\cancel{(-1)(0)} = -2 \rightarrow$

$|Y| = (4)(-2) = -8 \rightarrow C_{55} = (1)(-8) = -8$

$\rightarrow \det(A) = (1)(+16) + (1)(-8) = 8$ by now

(b) Column of A:

$|A| = a_{11}C_{11} + a_{21}C_{21} + a_{31}C_{31} + a_{41}C_{41} + a_{51}C_{51} \rightarrow$

$|A| = a_{11}C_{11}$ as $a_{21}, a_{31}, a_{41}, a_{51} = 0 \rightarrow C_{11} =$

$$(-1)^{1+1} \begin{vmatrix} 3 & -1 & 1 & 2 \\ -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \end{vmatrix} \rightarrow \text{let this be } Z$$

$$|Z| = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43} \rightarrow$$

$|Z| = a_{13}C_{13}$ because $a_{23} = a_{33} = a_{43} = 0.$

$$\rightarrow C_{13} = (-1)^{1+3} \left|\begin{array}{ccc|cc} -1 & 0 & 1 & -1 & 0 \\ 1 & 2 & 3 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right| \rightarrow$$

$$(-1)(2)(1) + \cancel{(0)(3)(0)} + (1)(1)(1) - \cancel{(1)(2)(0)} -$$

$$(-1)(3)(1) - \cancel{(0)(1)(1)} = -2 + 1 + 3 = 2$$

$\rightarrow |Z| = (1)(2) = 2 \rightarrow$ $C_{11} =$

$(1)(2) = 2 \rightarrow$ $|A| = a_{11}C_{11} = (4)(2) = 8$ by column method

$T(x)$ invertible as $|A| \neq 0$

(23)

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 2 & 0 & 4 \\ 1 & 6 & 1 \end{bmatrix} \rightarrow |A| = \left|\begin{array}{ccc|cc} 3 & 1 & -1 & 3 & 1 \\ 2 & 0 & 4 & 2 & 0 \\ 1 & 6 & 1 & 1 & 6 \end{array}\right| \longrightarrow$$

$$(3)\cancel{(0)(1)} + (1)(4)(1) + (-1)(2)(6) - \cancel{(-1)(0)(1)}$$

$$- (3)(4)(6) - (1)(2)(1) = 4 - 12 - 72 - 2$$

$$= -82 = |A|$$

(25)

$$A = \begin{bmatrix} 6 & 1 & 2 & 1 \\ 3 & 1 & 0 & 0 \\ 0 & 2 & 2 & 1 \\ 1 & 2 & 3 & -1 \end{bmatrix} \rightarrow$$

Shortcut (Rule of Sarrus) not possible as $A \in \mathbb{R}^{n \times n}$, $n > 3$

(31)

$$A = \begin{bmatrix} 1 & -1 & 3 \\ 0 & a & -2 \\ 2 & 4 & 3 \end{bmatrix}, \quad \{a \mid \nexists A^{-1}\}$$

$$|A| = (1)(a)(3) + (-1)(-2)(2) + \cancel{(3)(0)(4)} - (3)(a)(2) - (1)(-2)(4) - \cancel{(3)(0)(-1)}$$

$$= 3a + 4 - 6a + 8 \rightarrow 3a + 4 - 6a + 8 = 0 \rightarrow$$

$$-3a + 12 = 0 \rightarrow a = 4 \rightarrow a = \{4\}$$

(37) No determinant of A as there is a column of 0's which means the set of the columns of A are linearly dependent.

(47) $\det(A - \lambda I_2) = 0$, find all possible λ.

$$A = \begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} \rightarrow \det\left(\begin{bmatrix} 1 & 0 \\ -5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 1-\lambda & 0 \\ -5 & 1-\lambda \end{bmatrix}\right) = 0 \rightarrow (1-\lambda)(1-\lambda) - (0)(-5) = 0$$

$$\rightarrow (1-\lambda)^2 = 0 \rightarrow \boxed{\lambda = 1}$$

(55)

$$A = \begin{bmatrix} 3 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 2 & 1 \end{bmatrix} \rightarrow |A| = \left|\begin{array}{ccc|cc} 3 & 1 & 0 & 3 & 1 \\ 1 & 2 & 3 & 1 & 2 \\ 0 & 2 & 1 & 0 & 2 \end{array}\right| \rightarrow$$

$$(3)(2)(1) + (1)\cancel{(3)(0)} + (0)\cancel{(1)(2)} - (\cancel{0)(2})(0) -$$

$$(3)(3)(2) - (1)(1)(1) = 6 - 18 - 1 = -13$$

$$A \underset{\sim}{R_2 \longleftrightarrow R_1} B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \rightarrow |B| =$$

$$\left|\begin{array}{ccc|cc} 1 & 2 & 3 & 1 & 2 \\ 3 & 1 & 0 & 3 & 1 \\ 0 & 2 & 1 & 0 & 2 \end{array}\right| \rightarrow |B| = (1)(1)(1) + (2)\cancel{(0)(0)} + (3)(3)(2$$

$$- (3)\cancel{(1)(0)} - (1)\cancel{(0)(2)} - (2)(3)(1)$$

$$= 1 + 18 - 6 = \boxed{13}$$

Conjecture: interchanging a row from A to get B means $|A| = -|B|$ and equivalently $|B| = -|A|$

(61) $A \in \mathbb{R}^{2\times 2}$ s.t. $|A| = 12$

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

(65) $A \in \mathbb{R}^{3\times 3}$ s.t. $M_{11} = \begin{bmatrix} 0 & 4 \\ 6 & -3 \end{bmatrix} \wedge$

$$M_{23} = \begin{bmatrix} 5 & -1 \\ 2 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 5 & -1 & 10 \\ 2 & 0 & 4 \\ 2 & 6 & -3 \end{bmatrix}$$

(69)

(a) False as determinants are only for square matrices.

(b) False as the cofactor is also multiplied by 1 or -1 to an $(n-1) \times (n-1)$ matrix

(97)

(a) False as one of the diagonal entries may be 0 and hence $\det(A) = 0$.

Graded Homework 6 - Section 5.2

(3) $A = \begin{bmatrix} 1 & -1 & -3 \\ -2 & 2 & 6 \\ -3 & -3 & 10 \end{bmatrix} \underset{\sim}{\overset{2R_1+R_2 \to R_2}{3R_1+R_3 \to R_3}} \begin{bmatrix} 1 & -1 & -3 \\ 0 & 0 & 0 \\ 0 & -6 & 1 \end{bmatrix}$

$\to$ $|A| = 0$ because there is a 0 in the diagonal as we convert A to R.E.F.

(5) $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \underset{\sim}{\overset{R_1 \longleftrightarrow R_2}{R_3 \longleftrightarrow R_4}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to$

$|A| = (-1)(-1)(1)(1)(1)(1) = 1 = |A|$

(7) $|A| = (-1)[(1)(-4) - (-5)(0)] = 4 \Rightarrow A$ is invertible.

(11)

$$B = \begin{bmatrix} -6 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix} \rightarrow |B| = 0 \rightarrow |A| = \left(-\tfrac{1}{7}\right)(\det(B))$$

$\rightarrow$ $|A| = 0$. A is not invertible.

(13)

$$B = \begin{bmatrix} 1 & 0 & 5 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{bmatrix} \rightarrow |B| = (1)(1)(2)(4) = 8 \rightarrow$$

$$|A| = (-1)(-1)(|B|) = 8 = |A|$$

A is invertible.

(15)

$$\det(A) = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 3$$

$$\begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix} = -3$$

(19)

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ 3 & 7 \end{bmatrix} \rightarrow |A| = -11 \rightarrow$$

$$|B| = -3 \rightarrow AB = \begin{bmatrix} 9 & 19 \\ -12 & -29 \end{bmatrix} \rightarrow |AB| = -33$$

$$\rightarrow (-11)(-3) = -33 \rightarrow A+B = \begin{bmatrix} 2 & 2 \\ 4 & 3 \end{bmatrix}$$

$|A+B| = -2 \neq -33 = |AB|$

(25) $A, B \in \mathbb{R}^{n \times n} \wedge |A| = 3 \wedge |B| = -2$

(a) $|A^2 B^3| = |A| \cdot |A| \cdot |B| \cdot |B| \cdot |B| = -72 = |A^2 B^3|$

(b) $|AB^{-1}| = |A| \cdot \frac{1}{|B|} = -\frac{3}{2} = |AB^{-1}|$

(c) $|B^3 A^T| = |B| \cdot |B| \cdot |B| \cdot |A^T| =$
$|B| \cdot |B| \cdot |B| \cdot |A| = -24 = |B^3 A^T|$

(d) $|A^2 B^3 B^T| = |A| \cdot |A| \cdot |B| \cdot |B| \cdot |B| \cdot |B^T|$
$= |A| \cdot |A| \cdot |B| \cdot |B| \cdot |B| \cdot |B| = 144$
$= |A^2 B^3 B^T|$

(39)
$A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & -2 & 2 \\ 6 & -7 & -1 \end{bmatrix}, \vec{b} = \begin{bmatrix} -3 \\ 9 \\ 4 \end{bmatrix}$, $A\vec{x} = \vec{b}$ has a unique solution?

$|A| = (1)(-2)(-1) + (1)(2)(6) + (-2)(3)(-7) - (-2)(-2)(6)$
$- (1)(2)(-7) - (-1)(3)(1) = 2 + 12 + 42 - 24$
$+ 14 + 3 = 49 = |A| \Rightarrow$ unique solution exists

(41) $A \in \mathbb{R}^{2\times 2}$ s.t. $A \neq 0^{2\times 2} \wedge 3|A| = |3A|$

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \rightarrow$$

$|A| = (1)(4)-(2)(2) = 0 \rightarrow 3|A| = (3)(0) = 0$

$\rightarrow 3A = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix} \rightarrow |3A| = (3)(12)-(6)(6)$

$= 0$

(47) (a) False, interchanging the rows of a matrix means that the computed determinant must be multiplied by -1.

(b) True as $\det(A) \neq 0 \Rightarrow A^{-1}$ exists which requires linear independence in the columns of A.

(51)

(a) True as $|A| = \frac{1}{2}|B| \equiv |B| = 2|A|$

(b) False if $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, $\nexists M^{-1}_{ij} \ \forall \ i,j \in n$

Graded Homework 6 – Section 5.3

① $A = \begin{bmatrix} 6 & -5 \\ -2 & 7 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} 12 \\ 0 \end{bmatrix} \rightarrow$ find $\vec{x}$ with Cramer's rule. $\rightarrow |A| = (6)(7)-(-5)(-2) = 32$

$\rightarrow x_1 = \dfrac{\det(A_1)}{\det(A)} = \dfrac{\begin{vmatrix} 12 & -5 \\ 0 & 7 \end{vmatrix}}{32} = \dfrac{(12)(7)-5(0)}{32} =$

$\dfrac{84}{32} = x_1 \rightarrow x_2 = \dfrac{\det(A_2)}{\det(A)} = \dfrac{\begin{vmatrix} 6 & 12 \\ -2 & 0 \end{vmatrix}}{32} =$

$\dfrac{(6)(0)-(12)(-2)}{32} = \dfrac{24}{32} = \dfrac{3}{4} = x_2 \rightarrow \vec{x} = \begin{bmatrix} 84/32 \\ 3/4 \end{bmatrix}$

⑤ $A = \begin{bmatrix} 1 & 1 & -2 \\ 3 & -2 & 2 \\ 6 & -7 & -1 \end{bmatrix}$, $\vec{b} = \begin{bmatrix} -3 \\ 9 \\ 4 \end{bmatrix} \rightarrow$ find $\vec{x}$ with Cramer's rule.

$\rightarrow x_1 = \dfrac{\det(A_1)}{\det(A)} \rightarrow \det(A) = \left|\begin{array}{ccc|cc} 1 & 1 & -2 & 1 & 1 \\ 3 & -2 & 2 & 3 & -2 \\ 6 & -7 & -1 & 6 & -7 \end{array}\right| =$

$(1)(-2)(-1) + (1)(2)(6) + (-2)(3)(-7) - (-2)(-2)(6)$

$-(1)(2)(-7) - (1)(3)(-1) = 2 + 12 + 42 - 24$

$+ 14 + 3 = 49 \rightarrow$ ~~det(A)~~ CONTINUED $\rightarrow$

$$|A_1| = \left|\begin{array}{ccc|cc} -3 & 1 & -2 & -3 & 1 \\ 9 & -2 & 2 & 9 & -2 \\ 4 & -7 & 1 & 4 & -7 \end{array}\right| = (-3)(-2)(1) + (1)(2)(4) + (-2)(9)(-7) - (-2)(-2)(4) - (-3)(2)(-7) - (1)(9)(1)$$

$$= 6 + 8 + 126 - 16 - 42 - 9 = 73 \rightarrow$$

$$x_1 = \frac{73}{49} \rightarrow x_2 = \frac{|A_2|}{|A|} \rightarrow |A_2| = \rightarrow$$

$$\left|\begin{array}{ccc|cc} 1 & -3 & -2 & 1 & -3 \\ 3 & 9 & 2 & 3 & 9 \\ 6 & 4 & -1 & 6 & 4 \end{array}\right| = (1)(9)(-1) + (-3)(2)(6) + (-2)(3)(4) - (-2)(9)(6) - (1)(2)(4) - (-3)(3)(-1) = -9 - 36 - 24 + 108 - 8 - 9 = 108 - 86$$

$$= 22 \rightarrow x_2 = \frac{22}{49} \rightarrow |A_3| = \left|\begin{array}{ccc|cc} 1 & 1 & -3 & 1 & 1 \\ 3 & -2 & 9 & 3 & -2 \\ 6 & -7 & 4 & 6 & -7 \end{array}\right|$$

$$= (1)(-2)(4) + (1)(9)(6) + (-3)(3)(-7) - (-3)(-2)(6) - (1)(9)(-7) - (1)(4)(3) = -8 + 54 + 63 - 36 + 63 - 12 = 124 \rightarrow x_3 = \frac{124}{49} \rightarrow$$

$$\vec{x} = \begin{bmatrix} 73/49 \\ 22/49 \\ 124/49 \end{bmatrix}$$

(9) $A = \begin{bmatrix} 3 & 0 & 2 \\ 0 & 3 & 2 \\ 2 & 3 & 1 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} \longrightarrow$ find x_2

$$x_2 = \frac{|A_2|}{|A|} \rightarrow |A| = \left|\begin{array}{ccc|cc} 3 & 0 & 2 & 3 & 0 \\ 0 & 3 & 2 & 0 & 3 \\ 2 & 3 & 1 & 2 & 3 \end{array}\right| = \longrightarrow$$

$$(3)(3)(1) + \cancel{(0)(2)(2)} + \cancel{(2)(0)(3)} - (2)(3)(2)$$

$$- (3)(2)(3) - \cancel{(0)(0)(1)} = 9 - 12 - 18 = -21$$

$$\longrightarrow |A_2| = \left|\begin{array}{ccc|cc} 3 & 1 & 2 & 3 & 1 \\ 0 & 3 & 2 & 0 & 3 \\ 2 & -4 & 1 & 2 & -4 \end{array}\right| = (3)(3)(1) +$$

$$(1)(2)(2) + \cancel{(2)(0)(-4)} - (2)(3)(2) - (3)(2)(-4)$$

$$- \cancel{(1)(0)(1)} = 9 + 4 - 12 + 24 = 25$$

$$\longrightarrow x_2 = \frac{|A_2|}{|A|} = -\frac{25}{21}$$

(13) $A = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \longrightarrow$ find adj(A) and $A^{-1} \longrightarrow$

CONTINUED

$\text{adj}(A) = C^T \rightarrow C = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \rightarrow$

$C_{11} = (-1)^{1+1} |7| = 7 \rightarrow C_{12} = (-1)^{1+2} |3|$

$= -3 \rightarrow C_{21} = (-1)^{2+1} |5| = -5 \rightarrow C_{22} = (-1)^{2+2}$

$|2| = 2 \rightarrow C = \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix} \rightarrow C^T = \begin{bmatrix} 7 & -5 \\ -3 & 2 \end{bmatrix}$

$= \text{adj}(A)$

$\rightarrow A^{-1} = \frac{1}{|A|} \text{adj}(A) \rightarrow$

$|A| = (7)(2) - (5)(3) = -1 \rightarrow A^{-1} = \frac{1}{-1} \text{adj}(A)$

$= A^{-1} = \begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$

(15) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow$ find $\text{adj}(A)$ and A^{-1}

$\text{adj}(A) = C^T \rightarrow C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \rightarrow$

$C_{11} = (-1)^{1+1} \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \rightarrow C_{12} = (-1)^{1+2} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$

$= (-1)(-1) = 1 \rightarrow C_{13} = (-1)^{1+3} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$

$\rightarrow C_{21} = (-1)^{2+1} \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0 \rightarrow C_{22} = (-1)^{2+2} \begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix}$

$= 0 \rightarrow C_{23} = (-1)^{2+3} \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = 1 \rightarrow C_{31} =$

$(-1)^{3+1} \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \rightarrow C_{32} = (-1)^{3+2} \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0 \rightarrow$

$C_{33} = (-1)^{3+3} \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0 \rightarrow C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow$

$C^T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = adj(A) \rightarrow |A| = \left|\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}\right| \begin{matrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{matrix}$

$= (0)\cancel{(0)}(0) + (1)(1)(1) + (0)\cancel{(0)}(0) - (0)\cancel{(0)}(1) -$
$(0)\cancel{(1)}(0) - (1)\cancel{(0)}(0) = 1 \rightarrow A^{-1} = \frac{1}{|A|} adj(A)$

$\rightarrow A^{-1} = \frac{1}{1} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

37 $\begin{matrix} x_1 + x_2 = 1 \\ 2x_1 + 2x_2 = 2 \end{matrix} \rightarrow$ cannot be solved by cramer's rule. Proven by the following.

$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \rightarrow x_1 = \frac{|A_1|}{|A|} \rightarrow$

However, $|A| = (1)(2) - (1)(2) = 0 \rightarrow$ Since division by zero is undefined, Cramer's rule cannot be applied.

(43) (a) False as Cramer's rule assumes det(coefficients) $\neq 0$ and hence, the columns of the coefficient matrix would need to be linearly independent which does not need need to be the case.

(b) True because creating adj(A) from A only involves subtraction, addition, and multiplication. Those operations with integers as inputs will always be outputted as integers.