Graded Homework 7 - Section 6.1

(3)

$$A = \begin{bmatrix} 2 & 7 & 2 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \quad \vec{x}_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A\vec{x}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix} \qquad A\vec{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \qquad A\vec{x}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = -1 \qquad \lambda = 1 \qquad \lambda = 2$$

(9)

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & -2 \end{bmatrix} \qquad \lambda = -2$$

$$(A - \lambda I_3)\vec{u} = \vec{0} \rightarrow A - \lambda I_3 = \begin{bmatrix} -\lambda & 2 & 0 \\ 2 & -\lambda & 0 \\ 2 & 2 & -2-\lambda \end{bmatrix}$$

$$\rightarrow \left|\begin{array}{ccc|cc} -\lambda & 2 & 0 & -\lambda & 2 \\ 2 & -\lambda & 0 & 2 & -\lambda \\ 2 & 2 & (-2-\lambda) & 2 & 2 \end{array}\right| \rightarrow (-\lambda)(-\lambda)(-2-\lambda) - (2)(2)(-2-\lambda)$$

$$\rightarrow -2\lambda^2 - \lambda^3 - (-8 - 4\lambda)$$

$$\rightarrow -\lambda^3 - 2\lambda^2 + 4\lambda + 8 \rightarrow \lambda = -2 \rightarrow -(-2)^3 - 2(-2)^2 + 4(-2) + 8 = 8 - 8 + -8 + 8 = 0 \rightarrow \lambda = -2 \text{ is an eigenvalue}$$

(11) $A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$, $\lambda = 4 \rightarrow (A - \lambda I_2)\vec{u} = \vec{0}$

$\rightarrow A - \lambda I_2 = \begin{bmatrix} (1-\lambda) & -3 \\ 1 & (5-\lambda) \end{bmatrix} \rightarrow \lambda = 4 \rightarrow$

$\begin{bmatrix} -3 & -3 \\ 1 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 1 & 1 & 0 \end{array}\right] \underset{\sim}{\tfrac{1}{3}R_1 + R_2 \rightarrow R_2}$

$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$ let $x_2 = s_1 \rightarrow -3x_1 - 3s_1 = 0$

$\rightarrow x_1 = -s_1 \rightarrow \vec{x} = s_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ = eigenspace $\rightarrow$

basis for eigenspace: $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

(15) $A = \begin{bmatrix} 6 & -3 & 7 \\ 4 & 1 & 5 \\ 4 & -3 & 9 \end{bmatrix}$, $\lambda = 4 \rightarrow \begin{bmatrix} (6-4) & -3 & 7 \\ 4 & (1-4) & 5 \\ 4 & -3 & (9-4) \end{bmatrix}$

$\rightarrow \left[\begin{array}{ccc|c} 2 & -3 & 7 & 0 \\ 4 & -3 & 5 & 0 \\ 4 & -3 & 5 & 0 \end{array}\right] \underset{\sim}{\begin{array}{c} -2R_1 + R_2 \rightarrow R_2 \\ -2R_1 + R_3 \rightarrow R_3 \end{array}} \left[\begin{array}{ccc|c} 2 & -3 & 7 & 0 \\ 0 & 3 & -9 & 0 \\ 0 & 3 & -9 & 0 \end{array}\right]$

$\underset{\sim}{-R_2 + R_3 \rightarrow R_3} \left[\begin{array}{ccc|c} 2 & -3 & 7 & 0 \\ 0 & 3 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$ let $x_3 = s_1 \rightarrow$

$3x_2 - 9s_1 = 0 \rightarrow x_2 = 3s_1 \rightarrow 2x_1 - 3(3s_1) + 7s_1 = 0$

$\rightarrow x_1 = s_1 \rightarrow$ eigenspace of $\lambda = 4$: $s_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \rightarrow$

basis for eigenspace of $\lambda = 4$: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$

(17) $A = \begin{bmatrix} 5 & -1 & 2 \\ 2 & 2 & 2 \\ 2 & -1 & 5 \end{bmatrix}$, $\lambda = 6 \rightarrow \begin{bmatrix} (5-6) & -1 & 2 \\ 2 & (2-6) & 2 \\ 2 & -1 & (5-6) \end{bmatrix}$

$\rightarrow \left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & -1 & -1 & 0 \end{array}\right] \underset{\sim}{\overset{2R_1+R_2 \rightarrow R_2}{2R_1+R_3 \rightarrow R_3}} \left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & -3 & 3 & 0 \end{array}\right]$

$\underset{\sim}{-\frac{1}{2}R_2 + R_3 \rightarrow R_3} \left[\begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$ let $x_3 = s_1 \rightarrow$

$-6x_2 + 6s_1 = 0 \rightarrow$

$x_2 = s_1 \rightarrow -x_1 - s_1 + 2s_1 = 0 \rightarrow x_1 = s_1 \longrightarrow$

basis for eigenspace of $\lambda = -4$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

(23) $A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \rightarrow (A - \lambda I_2)\vec{v} = \vec{0} \rightarrow$

$\det\left(\begin{bmatrix} (1-\lambda) & -2 \\ 2 & (-3-\lambda) \end{bmatrix} \right) = 0 \rightarrow (1-\lambda)(-3-\lambda) - (-2)(2)$

$\longrightarrow$

$+2\lambda+\lambda^2+1 \rightarrow \lambda^2+2\lambda+1=0 \rightarrow$

characteristic polynomial

$(\lambda+1)^2=0 \rightarrow \lambda=-1$ with multiplicity 2 $\rightarrow$

eigenvalues

$$\left[\begin{array}{cc|c} 2 & -2 & 0 \\ 2 & -2 & 0 \end{array}\right] \rightarrow \underset{\sim}{-R_1+R_2 \rightarrow R_2} \left[\begin{array}{cc|c} 2 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\rightarrow$ let $x_2=s_1 \rightarrow 2x_1-2s_1=0 \rightarrow x_1=s_1$

$\rightarrow$ basis for eigenspace: $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

25

$$A=\begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ -4 & 5 & -1 \end{bmatrix} \rightarrow (A-\lambda I_3)\vec{u}=\vec{0} \longrightarrow$$

$$\det\left(\begin{bmatrix} 3-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -4 & 5 & -1-\lambda \end{bmatrix}\right) \rightarrow (3-\lambda)(2-\lambda)(-1-\lambda)=0$$

characteristic polynomial

$\rightarrow$ eigenvalues: $\{3, 2, -1\}$ all with multiplicity 1

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & 0 \\ -4 & 5 & -4 \end{bmatrix} \underset{\sim}{R_1 \leftrightarrow R_2} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -4 & 5 & -4 \end{bmatrix} \underset{\sim}{4R_1+R_3 \rightarrow R_3}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -4 \end{bmatrix} \overset{R_1+R_3\to R_3}{\sim} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix} \to x_3 = 0$$

$\to$ let $x_2 = s_1 \to x_1 - s_1 = 0 \to x_1 = s_1 \to$

basis of eigenspace for $\lambda = 3$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$$\to \lambda = 2 \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -4 & 5 & -3 & 0 \end{array}\right] \begin{array}{l} -R_1+R_2\to R_2 \\ 4R_1+R_3\to R_3 \\ \sim \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right] \to \text{let } x_3 = s_1 \to$$

$5x_2 - 3s_1 = 0 \to x_2 = \frac{3}{5}s_1 \to$

$x_1 = 0 \to$ basis of eigenspace for $\lambda = 2$: $\left\{ \begin{bmatrix} 0 \\ 3/5 \\ 1 \end{bmatrix} \right\}$

$$\to \lambda = -1 \to \begin{bmatrix} 4 & 0 & 0 \\ 1 & 3 & 0 \\ -4 & 5 & 0 \end{bmatrix} \begin{array}{l} -\frac{1}{4}R_1+R_2\to R_2 \\ R_1+R_3\to R_3 \\ \sim \end{array} \left[\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right]$$

$$-\frac{5}{3}R_2+R_3\to R_3 \sim \left[\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \text{let } x_3 = s_1 \to x_2 = 0$$

$\to x_1 = 0. \to$ eigenbasis for $\lambda = -1$: $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

(27) $A = \begin{bmatrix} 2 & 5 & 1 \\ 0 & -3 & -1 \\ 2 & 14 & 4 \end{bmatrix} \to (A - \lambda I_3)\vec{v} = \vec{0} \to$

$$\left|\begin{matrix} 2-\lambda & 5 & 1 \\ 0 & -3-\lambda & -1 \\ 2 & 14 & 4-\lambda \end{matrix}\right| \begin{matrix} 2-\lambda & 5 \\ 0 & -3-\lambda \\ 2 & 14 \end{matrix} = (2-\lambda)(-3-\lambda)(4-\lambda) - 10 - (-6-2\lambda) -$$

$$(2-\lambda)(-1)(14) = (-6+3\lambda-2\lambda+\lambda^2)(4-\lambda)$$

$$-10+6+2\lambda-(-28+14\lambda) = -24 + 4\lambda + 4\lambda^2$$

$$+6\lambda - \lambda^2 - \lambda^3 - 4 + 2\lambda + 28 - 14\lambda =$$

$$-\lambda^3 + 3\lambda^2 - 2\lambda \to \lambda(-\lambda^2 + 3\lambda - 2) = 0 \to$$

Characteristic polynomial

$$\lambda(-\lambda+1)(\lambda-2) = 0 \to \text{eigenvalues: } \{0, 1, 2\}$$

$$\left[\begin{matrix} 2 & 5 & 1 \\ 0 & -3 & -1 \\ 2 & 14 & 4 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \underset{\sim}{-R_1+R_3 \to R_3} \left[\begin{matrix} 2 & 5 & 1 \\ 0 & -3 & -1 \\ 0 & 9 & 3 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \underset{\sim}{3R_2+R_3 \to R_3}$$

$$\left[\begin{matrix} 2 & 5 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \end{matrix}\middle|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \to \text{let } x_3 = s_1 \to -3x_2 - s_1 = 0 \to$$

$$x_2 = -\frac{1}{3}s_1 \to 2x_1 - \frac{5}{3}s_1 + \frac{3}{3}s_1 = 0$$

$\to 2x_1 = \frac{2}{3}s_1 \to x_1 = \frac{1}{3}s_1 \to$ eigenbasis for $\lambda = 0$: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\}$

$\to \lambda = 1 \to \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 2 & 14 & 3 & 0 \end{array}\right]$

$\underset{\sim}{-2R_1 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right] \underset{\sim}{R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\to$ let $x_3 = s_1 \to -4x_2 - s_1 = 0 \to x_2 = -\frac{1}{4}s_1$

$\to x_1 - \frac{5}{4}s_1 + s_1 = 0 \to x_1 = \frac{1}{4}s_1 \to$

eigenbasis for $\lambda = 1$: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\}$ $\to \lambda = 2$

$\to \left[\begin{array}{ccc|c} 0 & 5 & 1 & 0 \\ 0 & -5 & -1 & 0 \\ 2 & 14 & 2 & 0 \end{array}\right] \underset{\sim}{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 2 & 14 & 2 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & 5 & 1 & 0 \end{array}\right] \underset{\sim}{R_2 + R_3 \to R_3}$

$\left[\begin{array}{ccc|c} 2 & 14 & 2 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to$ let $x_3 = s_1 \to -5x_2 - s_1 = 0 \to$

$x_2 = -\frac{1}{5}s_1 \to 2x_1 - \frac{14}{5}s_1 + \frac{10}{5}s_1$

$= 0 \to 2x_1 = \frac{\frac{4}{5}s_1}{2} \to x_1 = \frac{2}{5}s_1 \to$

eigenbasis for $\lambda = 2$: $\left\{ \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \right\}$

(37) (a) False as λ can be zero but eigenvector cannot be zero.

(b) True by definition.

(39) (a) True as the diagonal - λ means that the eigenvalue is a root of the characteristic polynomial

(b) False as the columns of A would not span $\mathbb{R}^n$ by the unifying theorem.

(41) (a) False, any non-zero matrix with zero diagonal and upper or lower triangular can have only the 0 eigenvalue.

(b) True or else a constant would mean that all λ terms are cancelled out but a constant would be equal to zero which is impossible for all constants not 0.

(47) (a) 6x6 matrix

(b) eigenvalues: $\{3, 2, -1\}$

(c) A is invertible

(d) 3 is largest possible dimension for eigenspace of A

Graded Homework 7 - Section 6.2

(1) $A = PDP^{-1}$, A^5, $P = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$

$$\rightarrow P^{-1} = \frac{1}{1}\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \rightarrow$$

$$D^5 = \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 128 & -3 \\ 32 & -1 \end{bmatrix}\begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 131 & -396 \\ 33 & -100 \end{bmatrix}$$

(3) $P = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, A^5

$$P^{-1} = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{\begin{matrix} 2R_3 + R_2 \rightarrow R_2 \\ R_3 + R_1 \rightarrow R_1 \end{matrix}} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right]$$

$$\underset{\sim}{3R_2 + R_1 \rightarrow R_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 7 \\ 0 & -1 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array}\right] \underset{\sim}{\begin{matrix} -R_2 \rightarrow R_2 \\ -R_3 \rightarrow R_3 \end{matrix}}$$

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 7 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array}\right] \rightarrow P^{-1} = \begin{bmatrix} 1 & 3 & 7 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow$$

$$D^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 96 & -1 \\ 0 & -32 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 7 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -93 & -184 \\ 0 & 32 & 66 \\ 0 & 0 & -1 \end{bmatrix}$$

(9) $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \rightarrow A = PDP^{-1} \rightarrow$

Eigenvalues of A: $(A - \lambda I_2)\vec{v} = \vec{0} \rightarrow \begin{bmatrix} 1-\lambda & -2 \\ 0 & 1-\lambda \end{bmatrix}$

$\rightarrow \det\left(\begin{bmatrix} 1-\lambda & -2 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2 \rightarrow \lambda = 1$ with multiplicity 2. $\rightarrow$ eigenspace associated with $\lambda = 1$: $\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow$ let $x_1 = s_1 \rightarrow -2x_2 = 0 \rightarrow x_2 = 0 \rightarrow \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\} \rightarrow$

Cannot diagonalize as eigenvectors cannot form basis for $\mathbb{R}^2$

(15) $A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \rightarrow A = PDP^{-1} \rightarrow$ eigenvalues

of A: $(A - \lambda I_3)\vec{u} = \vec{0} \rightarrow \left| \begin{matrix} -\lambda & 1 & -1 \\ 1 & -\lambda & 1 \\ 1 & -1 & 2-\lambda \end{matrix} \right| \begin{matrix} -\lambda & 1 \\ 1 & -\lambda \\ 1 & -1 \end{matrix}$

$= (-\lambda)(-\lambda)(2-\lambda) \cancel{+1 +1} - \lambda - \lambda \cancel{-(2-\lambda)} =$

$2\lambda^2 - \lambda^3 - \lambda = 0 \rightarrow \lambda(-\lambda^2 + 2\lambda - 1) = 0 \rightarrow$

$\lambda(\lambda^2 - 2\lambda + 1) = 0 \rightarrow \lambda(\lambda - 1)^2 = 0$

$\rightarrow \lambda = 0$ and $\lambda = 1$ are eigenvalues with multiplicity 1 and 2 respectively. $\rightarrow$ eigenspace

for $\lambda = 0 \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \begin{matrix} R_1 \leftrightarrow R_3 \\ R_2 \leftrightarrow R_3 \\ \sim \end{matrix}$

$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \underset{\sim}{-R_1 + R_3 \rightarrow R_3} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \underset{\sim}{-R_2 + R_3 \rightarrow R_3}$

$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$ let $x_3 = s_1 \rightarrow x_2 - s_1 = 0 \rightarrow$

$x_2 = s_1 \rightarrow x_1 - s_1 + 2s_1 = 0$

$\rightarrow x_1 = -s_1 \rightarrow \text{Span}\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \rightarrow$ eigenspace

for $\lambda = 1 \rightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right]$ $R_1 + R_2 \rightarrow R_2$, $R_1 + R_3 \rightarrow R_3$ $\sim$

$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow$ let $x_3 = s_1 \rightarrow$ let $x_2 = s_2 \rightarrow$ $-x_1 + s_2 - s_1 = 0 \rightarrow x_1 =$

$s_2 - s_1 \rightarrow \vec{x} = s_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow$

$\text{Span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \rightarrow P = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

$\rightarrow D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow P^{-1} = \left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$R_2 \leftrightarrow R_3$ $\sim$ $\left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array}\right]$ $R_1 + R_2 \rightarrow R_2$, $R_1 + R_3 \rightarrow R_3$ $\sim$

$\left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 2 & 1 & 1 & 0 \end{array}\right]$ $R_2 \leftrightarrow R_3$ $\sim$ $\left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$ $-2R_3 + R_2 \rightarrow R_2$, $-R_3 + R_1 \rightarrow R_1$ $\sim$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \begin{array}{c} -R_1 \to R_1 \\ -R_2 \to R_2 \\ \sim \end{array} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right]$$

$$\begin{array}{c} -R_2 + R_1 \to R_1 \\ \sim \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right] \to A = PDP^{-1} \to$$

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$$

2) $A = \begin{bmatrix} 7 & -8 \\ 4 & -5 \end{bmatrix}$, find A^{1000}

Eigenvalues of A: $(A - \lambda I_2)\vec{u} = \vec{0} \to \begin{vmatrix} (7-\lambda) & -8 \\ 4 & (-5-\lambda) \end{vmatrix}$

$= -35 - 2\lambda + \lambda^2 + 32 \to \lambda^2 - 2\lambda - 3 = 0 \to$

$(\lambda - 3)(\lambda + 1) = 0 \to \lambda_1 = 3$ and $\lambda_2 = -1 \to$

eigenspace associated with $\lambda = 3 \to \left[\begin{array}{cc|c} 4 & -8 & 0 \\ 4 & -8 & 0 \end{array}\right]$

$\begin{array}{c} -R_1 + R_2 \to R_2 \\ \sim \end{array} \left[\begin{array}{cc|c} 4 & -8 & 0 \\ 0 & 0 & 0 \end{array}\right] \to$ let $x_2 = s_1 \to 4x_1 - 8s_1 = 0$

$\to x_1 = 2s_1 \to \text{Span}\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right\} \to$

eigenspace associated with $\lambda = -1$ $\rightarrow \left[\begin{array}{cc|c} 8 & -8 & 0 \\ 4 & -4 & 0 \end{array}\right]$

$-\frac{1}{2}R_1 + R_2 \rightarrow R_2 \sim \left[\begin{array}{cc|c} 8 & -8 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$ let $x_2 = s_1 \rightarrow$

$8x_1 - 8s_1 = 0 \rightarrow x_1 = s_1 \rightarrow$ span $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$

$\rightarrow P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow P^{-1} = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right]$

$-\frac{1}{2}R_1 + R_2 \rightarrow R_2 \sim \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} & 1 \end{array}\right] \overset{2R_2 \rightarrow R_2}{\sim} \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right]$

$-R_2 + R_1 \rightarrow R_1 \sim \left[\begin{array}{cc|cc} 2 & 0 & 2 & -2 \\ 0 & 1 & -1 & 2 \end{array}\right] \overset{\frac{1}{2}R_1 \rightarrow R_1}{\sim}$

$\left[\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 2 \end{array}\right] \rightarrow A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

$\rightarrow A^{1000} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\left(\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}\right)^{1000}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \rightarrow$

$\begin{bmatrix} (2)(3)^{1000} & 1 \\ 3^{1000} & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} (2)(3)^{1000} - 1 & -(2)(3)^{1000} + 2 \\ 3^{1000} - 1 & -(3)^{1000} + 2 \end{bmatrix}}$

(23) The dimension of the other eigenspace must be 2.

(31)

(a) True by definition.

(b) False, distinct eigenvectors mean nothing in terms of the diagonalizability of A as there are infinitely many ~~eigenvectors~~

(33) (a) False Matrices without full rank can still be diagonalizable

(b) False, matrix multiplication does not necessarily preserve diagonalizability.

Homework 7 - Section 6.5

(1) $A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}$, $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow A\vec{x}_0 = \vec{x}_1 \rightarrow$

$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \vec{x}_2 = A\vec{x}_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \rightarrow \vec{x}_3 = A\vec{x}_2$

$\rightarrow \vec{x}_3 = \begin{bmatrix} -20 \\ 28 \end{bmatrix}$

(5) $A = \begin{bmatrix} 5 & -1 & 2 \\ 2 & 2 & 2 \\ 2 & -1 & 5 \end{bmatrix}$, $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rightarrow \vec{x}_1 = A\vec{x}_0$

$\rightarrow \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \vec{x}_1 \rightarrow \vec{x}_2 = \begin{bmatrix} 9 \\ 0 \\ -9 \end{bmatrix} \rightarrow \vec{x}_3 = \begin{bmatrix} 27 \\ 0 \\ -27 \end{bmatrix}$

(7) $A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}$, $\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \vec{x}_1 = \frac{1}{s_0} A\vec{x}_0$

$= \frac{1}{1}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \vec{x}_2 = \frac{1}{s_1} A\vec{x}_1 =$

$\frac{1}{-3}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix} = \vec{x}_2$

(11) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -1 & 1 \end{bmatrix}$, $\vec{x}_0 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \rightarrow \vec{x}_1 = \frac{1}{s_0} A \vec{x}_0$

$= \frac{1}{3} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2/3 \end{bmatrix} \rightarrow \vec{x}_2 = \frac{1}{3} \begin{bmatrix} 0 \\ 3 \\ -5/3 \end{bmatrix}$

$= \begin{bmatrix} 0 \\ 1 \\ -5/9 \end{bmatrix} = \vec{x}_2$

(13) Power method converges to $\lambda_3 = 7$

(17) Power method converges to $\lambda_3 = 6$

(29) $A \in \mathbb{R}^{2\times 2} \wedge \vec{x}_0$ s.t. $\vec{x}_0 = \vec{x}_1 = x_k$ as $k \to \infty$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(35a) False, a dominant eigenvalue is sufficient but not necessary for convergence by the Power Method.

(37b) False, it is possible by chance to choose an initial vector, $\vec{x}_0$ s.t. the Power Method does not converge to the same eigenvector as a different choice of $\vec{x}_0$.