Graded Homework 3 - Section 2.3

(3)

$$\left[\begin{array}{cc|c} 7 & 5 & 0 \\ 1 & -3 & 0 \\ -13 & 2 & 0 \end{array}\right] \quad -\frac{1}{7}R_1 + R_2 \to R_2 \quad \sim$$

$$\left[\begin{array}{cc|c} 7 & 5 & 0 \\ 0 & -\frac{26}{7} & 0 \\ -13 & 2 & 0 \end{array}\right] \quad \frac{13}{7}R_1 + R_3 \to R_3 \quad \sim \quad \left[\begin{array}{cc|c} 7 & 5 & 0 \\ 0 & -\frac{26}{7} & 0 \\ 0 & \frac{79}{7} & 0 \end{array}\right]$$

$$-\frac{26}{7}x = -\frac{79}{7} \quad x = \frac{79}{26} \longrightarrow \frac{79}{26}R_2 + R_3 \to R_3 \quad \sim$$

$$\left[\begin{array}{cc|c} 7 & 5 & 0 \\ 0 & -\frac{26}{7} & 0 \\ 0 & 0 & 0 \end{array}\right] = -\frac{26}{7}x_2 = 0 \quad x_2 = 0 \to 7x_1 + 5(0) = 0 \to x_1 = 0$$

As $x_1 = 0$ and $x_2 = 0$, these are linearly independent.

(5)

$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 0 \\ -1 & 4 & 4 & 0 \\ 2 & 1 & 7 & 0 \end{array}\right] \quad \begin{array}{c} R_1 \longleftrightarrow R_3 \\ R_2 \longleftrightarrow R_1 \end{array} \quad \left[\begin{array}{ccc|c} -1 & 4 & 4 & 0 \\ 2 & 1 & 7 & 0 \\ 3 & 0 & 2 & 0 \end{array}\right] \quad \begin{array}{c} 2R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \end{array} \quad \sim$$

$$\left[\begin{array}{ccc|c} -1 & 4 & 4 & 0 \\ 0 & 9 & 15 & 0 \\ 0 & 12 & 14 & 0 \end{array}\right] \underset{\sim}{-\frac{12}{9}R_2 + R_3 \to R_3} \left[\begin{array}{ccc|c} -1 & 4 & 4 & 0 \\ 0 & 9 & 15 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right] \to$$

$-6x_3 = 0 \to x_3 = 0 \to 9x_2 + 15(0) = 0 \to x_2 = 0$

$\to -1x_1 + 4(0) + 4(0) = 0 \to x_1 = 0$

Linearly independent as ~~[illegible]~~ only the trivial solution.

(11) $$\left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 5 & -2 & -1 & 0 \\ 4 & -4 & -3 & 0 \end{array}\right] \underset{\sim}{\begin{array}{c} -\frac{5}{3}R_1 + R_2 \to R_2 \\ -\frac{4}{3}R_1 + R_3 \to R_3 \end{array}} \left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 0 & -\frac{11}{3} & -1 & 0 \\ 0 & -\frac{16}{3} & -3 & 0 \end{array}\right]$$

$-\frac{11}{3}a = \frac{16}{3} \to a = -\frac{16}{11} \to -\frac{16}{11}R_2 + R_3 \to R_3$ $\sim$

$$\left[\begin{array}{ccc|c} 3 & 1 & 0 & 0 \\ 0 & -\frac{11}{3} & -1 & 0 \\ 0 & 0 & -\frac{17}{11} & 0 \end{array}\right]$$

x_1, x_2, and $x_3 = 0$ so linearly independent as trivial solution

(15) $$\left[\begin{array}{cc|c} 8 & 1 & 0 \\ 0 & -1 & 0 \\ -3 & 2 & 0 \end{array}\right] \underset{\sim}{\frac{3}{8}R_1 + R_3 \to R_3} \left[\begin{array}{cc|c} 8 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 0 \end{array}\right]$$

$$\underset{\sim}{2R_2 + R_3 \to R_3} \left[\begin{array}{cc|c} 8 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \to -x_2 = 0 \to x_2 = 0$$

$\rightarrow 8x_1 + 1(0) = 0 \rightarrow x_1 = 0$

x_1 and x_2 are zero so no non-trivial solutions

(17)

$$\left[\begin{array}{ccc|c} -1 & 3 & 1 & 6 \\ 4 & -3 & -1 & 0 \\ 3 & 0 & 5 & 0 \end{array}\right] \underset{\sim}{\begin{array}{c} 4R_1 + R_2 \rightarrow R_2 \\ 3R_1 + R_3 \rightarrow R_3 \end{array}} \left[\begin{array}{ccc|c} -1 & 3 & 1 & 0 \\ 0 & 9 & 2 & 0 \\ 0 & 9 & 8 & 0 \end{array}\right]$$

$$\underset{\sim}{-R_2 + R_3 \rightarrow R_3} \left[\begin{array}{ccc|c} -1 & 3 & 1 & 0 \\ 0 & 9 & 2 & 0 \\ 0 & 0 & 6 & 0 \end{array}\right] \rightarrow 6x_3 = 0 \rightarrow x_3 = 0$$

$\rightarrow 9x_2 + 2(0) = 0 \rightarrow x_2 = 0 \rightarrow -1x_1 + 3(0) + 1(0) = 0$

$\rightarrow x_1 = 0$

$x_1, x_2, x_3 = 0$ so only trivial solution

(19)

$V = \frac{1}{2}u$ so linearly dependent

(23) Linearly dependent as $\vec{0}$ is included

(27)

$$\left[\begin{array}{ccc|c} 4 & 3 & -5 & 0 \\ -1 & 5 & 7 & 0 \\ 3 & -2 & -7 & 0 \end{array}\right] \underset{\sim}{R_2 \leftrightarrow R_1}$$

$$\left[\begin{array}{ccc|c} -1 & 5 & 7 & 0 \\ 4 & 3 & -5 & 0 \\ 3 & -2 & -7 & 0 \end{array}\right] \quad \begin{matrix} 4R_1 + R_2 \to R_2 \\ 3R_1 + R_3 \to R_3 \\ \sim \end{matrix} \quad \left[\begin{array}{ccc|c} 1 & 5 & 7 & 0 \\ 0 & 23 & 23 & 0 \\ 0 & 13 & 14 & 0 \end{array}\right]$$

$$-\frac{13}{23} R_2 + R_3 \to R_3 \sim \left[\begin{array}{ccc|c} 1 & 5 & 7 & 0 \\ 0 & 23 & 23 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \to x_3 = 0 \to$$

$$23x_2 + 23(0) = 0 \to x_2 = 0 \to x_1 + 5(0) + 7(0) = 0$$

$$\to x_1 = 0$$

These vectors are linearly independent and so it is not possible that any of the vectors are in the span of the others.

(31) $\left[\begin{array}{cc|c} 6 & -9 & 0 \\ -4 & 6 & 0 \end{array}\right] \quad \frac{4}{6} R_1 + R_2 \to R_2 \sim \left[\begin{array}{cc|c} 6 & -9 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\to$ let $x_2 = s_1 \to 6x_1 - 9s_2 = 0 \to x_1 = -\frac{9}{6} s_2$

Infinite solutions

so linearly dependent and cannot find solutions for all b

(33)

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ -3 & 4 & 5 & 0 \end{array}\right] \quad \begin{array}{l} -\frac{1}{2} R_1 + R_2 \to R_2 \\ \frac{3}{2} R_1 + R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 5/2 & 5 & 0 \end{array}\right] \quad -5 R_2 + R_3 \to R_3 \quad \sim$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & \frac{1}{2} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Not a unique solution as the set of equations are not linearly independent and do not span $\mathbb{R}^3$.

(39)

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

(43)

(a) False, a set of vectors that are multiples of each other (an example of linear dependence) would not span $\mathbb{R}^n$

(b) False, you can have a set of vectors that span a space and add a linearly dependent set to the aforementioned and you would get ~~[illegible]~~ a linearly ~~[illegible]~~ dependent set that still spans $\mathbb{R}^n$

(45) (a) False, we just know that this ~~[illegible]~~ set spans the space.

(b) True by definition

(47) (a) False as they could be multiples of each other

(b) False for the same reason as above

(49) (a) False as $\vec{u}_4$ could be a linear combination

(b) True no matter what as any ~~[illegible]~~ vector being a combination of others is a linearly dependent set.

51 (a) False by definition of linear independence

(b) True by definition

53 a, b, c are the only possible vectors that can be linearly independent.

Graded Homework #3 - Section 3.1

(3)

$A = \begin{bmatrix} 0 & -4 & 2 \\ 3 & 1 & -2 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 4 \\ -5 \\ -2 \end{bmatrix}$

$T(\vec{u}_1) = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$ $T(\vec{u}_2) = \begin{bmatrix} 16 \\ 11 \end{bmatrix}$

(5) $\begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 6 \end{bmatrix} \rightarrow \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 3 & 2 & 1 & 6 \end{array}\right]$

$\xrightarrow{-3R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 8 & 1 & 15 \end{array}\right] \rightarrow$ let $x_3 = s_1 \rightarrow$

$x_2 = -\frac{1}{8}s_1 + \frac{15}{8} \rightarrow x_1 - 2(-\frac{1}{8}s_1 + \frac{15}{8}) = -3 \rightarrow$

$x_1 = -\frac{1}{4}s_1 + \frac{3}{4}$, y is in the range of T

(9) $T(u_1) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $T(u_2) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$, find $T(-2\vec{u}_1 + 3\vec{u}_2)$

$T(-2\vec{u}_1 + 3\vec{u}_2) = -2T(\vec{u}_1) + 3T(\vec{u}_2) =$

$-2\begin{bmatrix} 2 \\ 1 \end{bmatrix} + 3\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} -13 \\ 4 \end{bmatrix}$

(13)

$$A = \begin{bmatrix} 3 & 1 \\ -2 & 4 \end{bmatrix} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Is a linear transformation

(15)

Not a linear transformation

(17)

$$A = \begin{bmatrix} -4 & 0 & 1 \\ 6 & 5 & 0 \end{bmatrix}$$ linear transformation

(27)

$$\left[\begin{array}{ccc|c} 2 & 8 & 4 & 0 \\ 3 & 2 & 3 & 0 \\ 1 & 14 & 5 & 0 \end{array}\right] \xrightarrow[-\frac{1}{2}R_1 + R_3 \to R_3]{-\frac{3}{2}R_1 + R_2 \to R_2} \sim$$

$$\left[\begin{array}{ccc|c} 2 & 8 & 4 & 0 \\ 0 & 10 & -3 & 0 \\ 0 & 10 & 3 & 0 \end{array}\right] \xrightarrow{R_2 + R_3 \to R_3} \sim \left[\begin{array}{ccc|c} 2 & 8 & 4 & 0 \\ 0 & 10 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

→ A lacks column and row pivots. It does not span $\mathbb{R}^3$ and its columns are not linearly independent.

T is neither one-to-one nor onto.

(35)

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right) = A\vec{x} = \begin{bmatrix}7\\0\end{bmatrix}$$

$$A = \begin{bmatrix}2 & 1\\0 & 0\end{bmatrix} \text{ or } \begin{bmatrix}2 & 1\\2 & -6\end{bmatrix}$$

(39)

(a) False as codomain will always be equal to or larger than the range.

(b) True by definition

(43)

(a) True by definition

(b) True as the entire space needs to be available for an onto relationship.

(45)

(a) True as each T_1 and T_2 output is unique for $\vec{x}$

(b) True as you would need equivalent pivots in columns and rows and the only way to do that is a square matrix.

Graded Homework #3 - Section 3.2

(5)

(a) $\left(\begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix} + \begin{bmatrix} 1 & -4 & -5 \\ -2 & 1 & -3 \\ 0 & 2 & 6 \end{bmatrix} \right) \begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix}$

Not possible to add due to differing dimensions

(b) $\begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix} \left(\begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix}^T + \begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix} \right) =$

$\begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix} \left(\begin{bmatrix} 5 & -1 & 3 \\ 0 & 4 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix} \right) =$

$\begin{bmatrix} 0 & 4 \\ -2 & 5 \end{bmatrix} \left(\begin{bmatrix} 6 & -1 & 0 \\ -2 & 9 & 2 \end{bmatrix} \right) = \begin{bmatrix} -8 & 36 & 8 \\ -22 & 47 & 10 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 4 & -5 \\ -2 & 1 & -3 \\ 0 & 2 & 6 \end{bmatrix} + \left(\begin{bmatrix} 5 & 0 \\ -1 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -3 \\ -2 & 5 & -1 \end{bmatrix} \right) =$

$$\begin{bmatrix} 1 & 4 & -5 \\ 2 & 1 & -3 \\ 0 & 2 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 0 & -15 \\ -9 & 20 & -1 \\ -3 & 15 & -12 \end{bmatrix} = \begin{bmatrix} 6 & 4 & -20 \\ -11 & 21 & -4 \\ -3 & 17 & -6 \end{bmatrix}$$

(7)

$$2b - a = 3$$
$$3b + 2 = 5 \longrightarrow -6 + 2a = -8 \rightarrow a = -1$$
$$-6 + 2a = -8$$
$$-13 = c$$

$\longrightarrow 3b+2=5 \longrightarrow b = 1 \longrightarrow$

$2(1) - (-1) = 3 \rightarrow 3 \overset{\checkmark}{=} 3$

Solution:

$a = -1, \; b = 1, \; c = -13$

(9)

$$2a + 0 - 2c = 4 \longrightarrow c = -3 \longrightarrow 2a - 2(-3) = 4$$
$$b + 0 + 4c = -6$$
$$-a + 3b - 2 = d$$
$$-3 - 2b + 4 = -5$$

$\longrightarrow a = -1 \longrightarrow b = 3$

$\longrightarrow$ ~~[illegible]~~

$-(-1) + 3(3) - 2 = d \rightarrow d = 8$

Solution:

$a = -1, \; b = 3, \; c = -3, \; d = 8$

(11) $\begin{bmatrix}5 & -10\\ a & -4\end{bmatrix}\begin{bmatrix}5 & -10\\ a & -4\end{bmatrix} = \begin{bmatrix}5 & -10\\ a & -4\end{bmatrix} \longrightarrow$

$25 - 10a = 5$ $\quad a = 2$

$5a - 4a = a \rightarrow$ $\quad a \overset{\checkmark}{=} a$

$-50 + 40 = -10$ $\quad -10 \overset{\checkmark}{=} -10$

$-10a + 16 = -4$ $\quad a = 2$

$a = 2$

(17) $(A + B^2)(BA - A) \longrightarrow ABA - A^2 + B^3A - B^2A$

(21) $(A - B)(A + B) \longrightarrow A^2 - BA + AB - B^2$

$\neq A^2 - B^2 \iff -BA + AB \neq I$

$A = \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}$ $\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix} = B \longrightarrow \begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} - \begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix} \longrightarrow$

$\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix} - \begin{bmatrix}2 & 3\\ 6 & 11\end{bmatrix} = \begin{bmatrix}5 & 7\\ 9 & 11\end{bmatrix} = A^2 - B^2 \longrightarrow$

$\underbrace{\begin{bmatrix}7 & 10\\ 15 & 22\end{bmatrix}}_{A^2} - \underbrace{\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}}_{BA} + \underbrace{\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}\begin{bmatrix}0 & 1\\ 2 & 3\end{bmatrix}}_{AB} - \underbrace{\begin{bmatrix}2 & 3\\ 6 & 11\end{bmatrix}}_{B^2}$

$$\underset{A^2}{\begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}} - \underset{BA}{\begin{bmatrix} 3 & 4 \\ 11 & 16 \end{bmatrix}} + \underset{AB}{\begin{bmatrix} 4 & 7 \\ 8 & 15 \end{bmatrix}} - \underset{B^2}{\begin{bmatrix} 2 & 3 \\ 6 & 11 \end{bmatrix}} =$$

$$A^2 - BA + AB + B^2 = \begin{bmatrix} 6 & 10 \\ 6 & 10 \end{bmatrix} \neq \underset{A^2 - B^2}{\begin{bmatrix} 5 & 7 \\ 9 & 11 \end{bmatrix}} =$$

47

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \underset{B}{\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(51)

(a) $AB \neq O_{nn} \rightarrow \begin{bmatrix} (a_{11}b_{11} + \cdots + a_{1n}b_{n1}) & & \\ & \ddots & \\ & & \ddots \end{bmatrix}$

does not necessarily mean that $A+B \neq O_{nn}$.

Example:

let $A = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -1 \\ -3 & -4 \end{bmatrix}$

$A+B = O_{22}$ but $AB = \begin{bmatrix} -7 & -6 \\ -18 & -19 \end{bmatrix}$

(b) True as $(A+B^T)^T = A^T + B^{T^T} = A^T + B$

(53) (a) True by definition

(b) False, it's possible to get a zero matrix

(57) (a) True by definition.

(b) False

$A = \begin{bmatrix} a_{11} & & & & \\ 0 & a_{22} & & & \\ 0 & 0 & a_{33} & & \\ 0 & 0 & 0 & \ddots & \\ 0 & 0 & 0 & 0\ 0 & a_{nn} \end{bmatrix} \quad A^T = \begin{bmatrix} a_{11} & 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 & 0 \\ a_{31} & & a_{33} & 0 & 0 \\ & & & \ddots & \end{bmatrix}$