Graded Homework 10 - Section 7.2

① $v = 3x^2 + 11x - 8$ in span $\{3x^2 + x - 1,\ x^2 - 3x + 2\}$?

$c_1(3x^2 + x - 1) + c_2(x^2 - 3x + 2) = 3x^2 + 11x - 8 \rightarrow$

$(3c_1 + c_2)x^2 + (c_1 - 3c_2)x + (c_1 + 2c_2) = 0 \rightarrow$

$$\begin{aligned} 3c_1 + c_2 &= 3 \\ c_1 - 3c_2 &= 11 \\ -c_1 + 2c_2 &= -8 \end{aligned} \rightarrow \left[\begin{array}{cc|c} 3 & 1 & 3 \\ 1 & -3 & 11 \\ -1 & 2 & -8 \end{array}\right] \begin{array}{l} -\frac{1}{3}R_1 + R_2 \rightarrow R_2 \\ \frac{1}{3}R_1 + R_3 \rightarrow R_3 \end{array} \sim$$

$$\left[\begin{array}{cc|c} 3 & 1 & 3 \\ 0 & -10/3 & 10 \\ 0 & 7/3 & -7 \end{array}\right] \underset{\sim}{\frac{7}{10}R_2 + R_3 \rightarrow R_3} \left[\begin{array}{cc|c} 3 & 1 & 3 \\ 0 & -10/3 & 10 \\ 0 & 0 & 0 \end{array}\right] \rightarrow -\frac{10}{3}x_2 = 10 \rightarrow$$

$x_2 = -3 \rightarrow 3x_1 + 1(-3) = 3 \rightarrow x_1 = 2$

$c_1 = 2,\ c_2 = -3$

⑦ $v = x^2 + 4x + 4$ in span $\{x^3 + x - 2,\ x^2 + 2x + 1,\ x^3 - x^2 + x\}$

$c_1(x^3 + x - 2) + c_2(x^2 + 2x + 1) + c_3(x^3 - x^2 + x) = x^2 + 4x + 4$

$$\begin{aligned} c_1 + c_3 &= 0 \\ c_2 - c_3 &= 1 \\ c_1 + 2c_2 + c_3 &= 4 \\ -2c_1 + c_2 &= 4 \end{aligned} \quad \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 1 & 2 & 1 & 4 \\ -2 & 1 & 0 & 4 \end{array}\right] \begin{array}{l} -R_1 + R_3 \rightarrow R_3 \\ 2R_1 + R_4 \rightarrow R_4 \end{array} \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & 0 & 4 \\ 0 & 1 & 2 & 4 \end{array}\right] \begin{array}{l} -2R_2 + R_3 \rightarrow R_3 \\ -R_2 + R_4 \rightarrow R_4 \end{array} \sim$$

(21) $\left\{ \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} -4 & 2 \\ -2 & -6 \end{bmatrix} \right\} \in \mathbb{R}^{2\times 2} \rightarrow$

$$c_1 \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} + c_2 \begin{bmatrix} -4 & 2 \\ -2 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow$$

$$\left[\begin{array}{cc|c} 2 & -4 & 0 \\ -1 & 2 & 0 \\ 1 & -2 & 0 \\ 3 & -6 & 0 \end{array}\right] \begin{matrix} \frac{1}{2}R_1 + R_2 \to R_2 \\ -\frac{1}{2}R_1 + R_3 \to R_3 \\ -\frac{3}{2}R_1 + R_4 \to R_4 \\ \sim \end{matrix} \left[\begin{array}{cc|c} 2 & -4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow$$

let $x_2 = s_1 \rightarrow x_1 = 2s_1 \rightarrow \vec{x} = s_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow$

not linearly independent.

(33) (a) False, vectors need not be columns of numbers.

(b) False, one can have an infinite set that is linearly independent.

(35) (a) False, vectors can both span a vector space V and be linearly independent.

(b) False as you could have g be the 0 function and f be a non-zero function such that $f(1) = 0$ which equals $-3g(1) = 0$ but $f(x \neq 1) = \mathbb{R} \neq 0$.

37 (a) A proper subset of a subset that spans a space
False does not need to span the space.

(b) False as it's possible that 0 is introduced to the set if a specific choice of v is included.

Graded Homework 10 - Section 7.3

① No as $\dim(P^2) = 3$ but there are only 2 vectors.

③ Possible as $\dim(\mathbb{R}^{2\times 2}) = 4$ and there are 4 vectors.

⑤ No as $\dim(P^4) = 5$ and there are only 4 vectors.

⑦ $V = \{2x^2 + x - 3,\ x+1,\ -5\}$, basis for P^2?

$c_1(2x^2+x-3) + c_2(x+1) + c_3(-5) = 0$

$$\begin{aligned} 2c_1 &= 0 \\ c_1 + c_2 &= 0 \\ -3c_1 + c_2 - 5c_3 &= 0 \end{aligned} \rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -3 & 1 & -5 & 0 \end{array}\right] \begin{array}{l} -\tfrac{1}{2}R_1 + R_2 \rightarrow R_2 \\ \tfrac{3}{2}R_1 + R_3 \rightarrow R_3 \\ \sim \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -5 & 0 \end{array}\right] \begin{array}{l} -R_2 + R_3 \rightarrow R_3 \\ \sim \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -5 & 0 \end{array}\right] \rightarrow c_1 = c_2 = c_3 = 0 \Rightarrow$$ V is a basis for P^2

⑨ $V = \left\{ \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} \right\}$ - Basis for $\mathbb{R}^{2\times 2}$?

$$c_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} + c_4 \begin{bmatrix} 3 & 3 \\ 3 & 4 \end{bmatrix} = 0$$

$$1c_1 + 3c_2 + 2c_3 + 3c_4 = 0$$
$$2c_1 + 1c_2 + 2c_3 + 3c_4 = 0$$
$$2c_1 + 0c_2 + 1c_3 + 3c_4 = 0$$
$$1c_1 + 3c_2 + 1c_3 + 4c_4 = 0$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 3 & 0 \\ 2 & 1 & 2 & 3 & 0 \\ 2 & 0 & 1 & 3 & 0 \\ 1 & 3 & 1 & 4 & 0 \end{array}\right] \begin{array}{l} -2R_1 + R_2 \to R_2 \\ -2R_1 + R_3 \to R_3 \\ -R_1 + R_4 \to R_4 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 3 & 0 \\ 0 & -5 & -2 & -3 & 0 \\ 0 & -6 & -3 & -3 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] -\frac{6}{5} R_2 + R_3 \to R_3 \sim$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 3 & 0 \\ 0 & -5 & -2 & -3 & 0 \\ 0 & 0 & -3/5 & 3/5 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \begin{array}{l} -5/3\ R_3 + R_4 \to R_4 \\ \sim \end{array} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 3 & 0 \\ 0 & -5 & -2 & -3 & 0 \\ 0 & 0 & -3/5 & 3/5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \to$$

let $x_4 = s_1 \to -3/5\, x_3 + \frac{3}{5} s_1 = 0 \to x_3 = s_1 \to$

$-5x_2 - 2s_1 - 3s_1 = 0 \to x_2 = -s_1 \to x_1 \overset{-3s_1}{} + 2s_1 + 3s_1 = 0$

$\to x_1 = -2 \to \vec{x} = s_1 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow$ V is not a basis for $\mathbb{R}^{2\times 2}$

(11) Not a basis for $\mathbb{R}^\infty$ as you cannot get the infinite 1 sequence $(1, 1, 1, \ldots)$ from any finite subset of vectors as they will contain at least 1 zero.

(13) $S =$ subspace $\in \mathbb{R}^{3\times 3}$ s.t. $tr(A) = 0, \forall A \in S$

For example, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ where $a_{11} + a_{22} + a_{33} = 0$

$\Rightarrow a_{11} = -a_{22} - a_{33}$

$\rightarrow$ Non-diagonal terms account for free parameters.

$\dim(S) = 8 \rightarrow \beta$ for S: $\left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \right\}$

(15) $S =$ subspace $\in \mathbb{R}^{2\times 2}$ s.t. components of A equal zero.

For example, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $a + b + c + d = 0$

$\longrightarrow$

$a = -b - c - d \rightarrow \dim(S) = 3$ and

β for S: $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

37

(a) False as S could be a trivial subspace of V.

(b) True as $\dim(\mathbb{R}^{4\times3}) = \dim(P^{11}) = 12$

39

(a) False as you can have different families of vectors with the same dimension.

(b) If $S \subseteq V$ and $\dim(S) = \dim(V) = 6$, then $S = V$

True.

41

(a) False as either the set is linearly dependent or vectors can be removed to make it so.

(b) True as subspaces of finite dimensional vector spaces are finite in dimension.

Graded Homework 10 - Section 10.1

① $\langle \vec{u}, \vec{v} \rangle$, $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$, $\langle \vec{u}, \vec{v} \rangle = 2u_1v_1 + 3u_2v_2 + 1u_3v_3$

$\langle \vec{u}, \vec{v} \rangle = 6 + 24 + 2 = 32$

③ $\langle \vec{p}, \vec{q} \rangle$, $p(x) = 3x+2$, $q(x) = -x+1$, $\langle \vec{p}, \vec{q} \rangle = p(x_0)q(x_0) + \dots + p(x_n)q(x_n)$, $x_0 = -1$, $x_1 = 0$, $x_2 = 2$

$(3(-1)+2)(-(-1)+1) + (3(0)+2)(-(0)+1) + (3(2)+2)(-(2)+1)$

$= -2 + 2 - 8 = -8$

⑤ $\langle \vec{f}, \vec{g} \rangle$, $f(x) = x+3$, $g(x) = x^2$, $\langle \vec{f}, \vec{g} \rangle = \int_{-1}^{1} f(x)g(x)\,dx$

$\rightarrow \int_{-1}^{1} (x+3)(x^2)\,dx = \int_{-1}^{1} x^3 + 3x^2\,dx = \left[\frac{x^4}{4} + x^3\right]_{-1}^{1}$

$\rightarrow \frac{5}{4} - \left(-\frac{3}{4}\right) = 2$

⑦ $\langle A, B \rangle = tr(A^T B)$, $A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}$

$\rightarrow tr\left(\begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 5 & 2 \\ -3 & -2 \end{bmatrix}\right) \rightarrow tr\left(\begin{bmatrix} 1 & -2 \\ -17 & -10 \end{bmatrix}\right) = -9$

⑨ $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $\langle \vec{u}, \vec{v} \rangle = 0$, $\langle \vec{u}, \vec{v} \rangle = t_1u_1v_1 + \dots + t_3u_3v_3$

$t_1 = 3$, $t_2 = 1$, $t_3 = a$

$\to (3)(1)(2) + (1)(0)(1) + (-1)(2)(a) = 0 \to$

$6 - 2a = 0 \to a = 3$

(11) $p(x) = x+2,\ q(x) = -3x+1,\ \langle p, q\rangle = 0,\ x_0 = -1,$

$x_1 = a,\ x_2 = 2,\ \langle p, q\rangle = p(x_0)q(x_0) + \dots + p(x_n)q(x_n)$

$(-1+2)(-3(-1)+1) + (a+2)(-3a+1) + (2+2)(-3(2)+1) = 0$

$\to 4 - 3a^2 - 5a + 2 \quad -20 = 0 \to -3a^2 - 5a - 14 = 0 \to$

no real solutions for a so not possible for orthogonality.

(13) $f(x) = 2x,\ g(x) = x+b,\ b$ s.t. $\langle f, g\rangle = 0,$

$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

$\int_{-1}^{1} (2x)(x+b)\,dx = \int_{-1}^{1} 2x^2 + 2xb\,dx = \frac{4}{3} \neq 0$

So they are not orthogonal for any value.

(15) $\left\| \begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \right\|,\quad \langle \vec{v}, \vec{u}\rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3 \to$

$\sqrt{2(1)^2 + 3(-3)^2 + (2)^2}$

$= \sqrt{33}$

(17) $\|3x-5\| \quad p(x_0)q(x_0) + \dots + p(x_n)q(x_n)$

$= \langle u, v\rangle,\ x_0 = -2,\ x_1 = 1,\ x_2 = 4 \to$

$$\sqrt{(-11)^2 + (-2)^2 + (7)^2} = \sqrt{174}$$

(19) $\|x^3\|$, $\langle u, v\rangle = \int_{-1}^{1} f(x)g(x)\,dx$

$$\sqrt{\int_{-1}^{1} (x^3)(x^3)\,dx} = \sqrt{\int_{-1}^{1} x^6\,dx} = \sqrt{\left[\frac{x^7}{7}\right|_{-1}^{1}}$$

$$= \sqrt{\frac{(1)^7}{7} - \frac{(-1)^7}{7}} = \sqrt{2/7}$$

(21) $\|A\|$, $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$, $\langle A, B\rangle = tr(A^T B)$

$$\sqrt{\left(\begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}\right) tr} = \sqrt{tr\begin{pmatrix} 13 & -3 \\ -3 & 1 \end{pmatrix}}$$

$$= \sqrt{14}$$

(23) $\text{proj}_{\vec{u}} \vec{v}$, $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$, $\langle \vec{u}, \vec{v}\rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$

$$\frac{\vec{u}\cdot\vec{v}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{(2)(1)(3) + (3)(2)(4) + (1)(2)}{2(1)^2 + 3(2)^2 + (1)^2}\vec{v}$$

$$\rightarrow \frac{6 + 24 + 2}{2 + 12 + 1}\vec{v} = \frac{32}{15}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 32/15 \\ 64/15 \\ 32/15 \end{bmatrix}$$

(25) $\text{proj}_p q$, $p(x) = 3x+2$, $q(x) = -x+1$,

$\langle p, q \rangle = p(x_0)q(x_0) + \ldots + p(x_n)q(x_n)$, $x_0 = -1$,

$x_1 = 0$, $x_2 = 2$

$$\text{proj}_p q = \frac{p \cdot q}{p \cdot p} p = \frac{(-1)(2) + (2)(1) + (8)(-1)}{(-1)^2 + (2)^2 + (8)^2} p$$

$$= \frac{-8}{69} p \rightarrow -\frac{8}{69}(3x+2)$$

(27) $\text{proj}_f g$, $f(x) = x$, $g(x) = x^2$, $\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\,dx$

$$\frac{f \cdot g}{f \cdot f} f = \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx} f \rightarrow \frac{\left[\frac{x^4}{4}\right|_{-1}^{1}}{\left[\frac{x^3}{3}\right|_{-1}^{1}} f$$

$$\rightarrow \frac{0}{\frac{2}{3}} f = 0$$

(29) $\text{proj}_A B$, $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}$, $\langle A, B \rangle =$

$\text{tr}(A^T B)$

$$\text{proj}_A B = \frac{A \cdot B}{A \cdot A} A \rightarrow \frac{\text{tr}\left(\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}\right)}{\text{tr}\left(\begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}\right)} A$$

$$\rightarrow \frac{tr\left(\begin{bmatrix} 4 & 4 \\ -2 & -3 \end{bmatrix}\right)}{tr\left(\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}\right)} A \rightarrow \frac{1}{6} A \rightarrow$$

$$\begin{bmatrix} 1/3 & -1/6 \\ 1/6 & 0 \end{bmatrix}$$

(41) (a) True as $\langle 2u, -4v \rangle = (2)(-4) \langle u, v \rangle$

(b) False as $\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2 \|uv\|$