

~~A~~ Graded Homework 7 - Section 6.1 ~~A~~

(3)

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix}, \vec{x}_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A\vec{x}_1 = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

$$A\vec{x}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$A\vec{x}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$\lambda = -1$$

$$\lambda = 1$$

$$\lambda = 2$$

(4)

$$A = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 2 & 2 & -2 \end{bmatrix} \quad \lambda = -2$$

$$(A - \lambda I_3) \vec{u} = \vec{0} \rightarrow A - \lambda I_3 = \begin{bmatrix} -\lambda & 2 & 0 \\ 2 & -\lambda & 0 \\ 2 & 2 & -2-\lambda \end{bmatrix}$$

$$\rightarrow \begin{array}{|ccc|ccc|} \hline & -\lambda & 2 & 0 & -\lambda & 2 & \\ \hline 2 & -\lambda & 0 & 2 & -\lambda & 2 & \\ 2 & 2 & -2-\lambda & 2 & 2 & -2-\lambda & \\ \hline \end{array} \rightarrow (-\lambda)(-\lambda)(-2-\lambda) - (2)(2)(-2-\lambda)$$

$$\rightarrow -2\lambda^2 - \lambda^3 - (-8 - 4\lambda)$$

$$\rightarrow -\lambda^3 - 2\lambda^2 + 4\lambda + 8 \rightarrow \lambda = -2 \rightarrow -(-2)^3 - 2(-2)^2 + 4(-2) + 8 = 8 - 8 + -8 + 8 = 0 \rightarrow \lambda = -2 \text{ is an eigenvalue}$$

$$⑪ A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}, \lambda = 4 \rightarrow (A - \lambda I_2) \vec{u} = \vec{0}$$

$$\rightarrow A - \lambda I_2 = \begin{bmatrix} (1-\lambda) & -3 \\ 1 & (5-\lambda) \end{bmatrix} \rightarrow \lambda = 4 \rightarrow$$

$$\begin{bmatrix} -3 & -3 \\ 1 & 1 \end{bmatrix} \rightarrow \left[\begin{array}{cc|c} -3 & -3 & 0 \\ 1 & 1 & 0 \end{array} \right] \xrightarrow{\frac{1}{3}R_1 + R_2 \rightarrow R_2}$$

$$\left[\begin{array}{cc|c} -3 & -3 & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow \text{let } x_2 = s_1 \rightarrow -3x_1 - 3s_1 = 0$$

$$\rightarrow x_1 = -s_1 \rightarrow \vec{x} = s_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \text{eigenspace} \rightarrow$$

basis for eigenspace: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$$⑯ A = \begin{bmatrix} 6 & -3 & 7 \\ 4 & 1 & 5 \\ 4 & -3 & 9 \end{bmatrix}, \lambda = 4 \rightarrow \begin{bmatrix} (6-4) & -3 & 7 \\ 4 & (1-4) & 5 \\ 4 & -3 & (9-4) \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & -3 & 7 & 0 \\ 4 & -3 & 5 & 0 \\ 4 & -3 & 5 & 0 \end{array} \right] \xrightarrow{-2R_1 + R_2 \rightarrow R_2} \left[\begin{array}{ccc|c} 2 & -3 & 7 & 0 \\ 0 & 3 & -9 & 0 \\ 0 & 3 & -9 & 0 \end{array} \right]$$

$$\xrightarrow{-R_2 + R_3 \rightarrow R_3} \left[\begin{array}{ccc|c} 2 & -3 & 7 & 0 \\ 0 & 3 & -9 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \text{let } x_3 = s_1 \rightarrow$$

$$3x_2 - 9s_1 = 0 \rightarrow x_2 = 3s_1 \rightarrow 2x_1 - 3(3s_1) + 7s_1 = 0$$

$\rightarrow x_1 = s_1$ → eigenspace of $\lambda = 4$: $s_1 \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \rightarrow$

basis for eigenspace of $\lambda = 4$: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} \right\}$

$$\textcircled{17} \quad A = \begin{bmatrix} 5 & -1 & 2 \\ 2 & 2 & 2 \\ 2 & -1 & 5 \end{bmatrix}, \lambda = 6 \rightarrow \begin{bmatrix} (5-6) & -1 & 2 \\ 2 & (2-6) & 2 \\ 2 & -1 & (5-6) \end{bmatrix}$$

$$\rightarrow \begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & -1 & -1 & 0 \end{array} \xrightarrow{2R_1+R_2 \rightarrow R_2} \begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & -6 & 6 & 0 \\ 2 & -3 & 3 & 0 \end{array} \xrightarrow{2R_1+R_3 \rightarrow R_3} \begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & -3 & 3 & 0 \end{array}$$

$$-\frac{1}{2}R_2 + R_3 \rightarrow R_3 \sim \begin{array}{ccc|c} -1 & -1 & 2 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array} \rightarrow \text{let } x_3 = s_1 \rightarrow \text{row } 2: -6x_2 + 6s_1 = 0 \rightarrow$$

$$x_2 = s_1 \rightarrow -x_1 - s_1 + 2s_1 = 0 \rightarrow x_1 = s_1 \rightarrow$$

basis for eigenspace of $\lambda = -4$: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$$\textcircled{23} \quad A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \rightarrow (A - \lambda I_2) \vec{v} = \vec{0} \rightarrow$$

$$\det \begin{pmatrix} (1-\lambda) & -2 \\ 2 & (-3-\lambda) \end{pmatrix} = 0 \rightarrow (1-\lambda)(-3-\lambda) - (-2)(2) \rightarrow$$

$$8) \quad + 2\lambda + \lambda^2 + 1 \rightarrow \lambda^2 + 2\lambda + 1 = 0 \rightarrow$$

characteristic polynomial

$$(\lambda + 1)^2 = 0 \rightarrow \lambda = -1 \text{ with multiplicity 2} \rightarrow$$

$$\begin{bmatrix} 2 & -2 & 0 \\ 2 & -2 & 0 \end{bmatrix} \xrightarrow{-R_1 + R_2 \sim R_2} \begin{bmatrix} 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

eigenvalues

$$\rightarrow \text{let } x_2 = s_1 \rightarrow 2x_1 - 2s_1 = 0 \rightarrow x_1 = s_1$$

$$\rightarrow \text{basis for eigenspace: } \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$65) \quad A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 2 & 0 \\ -4 & 5 & -1 \end{bmatrix} \rightarrow (A - \lambda I_3) \vec{u} = \vec{0} \rightarrow$$

$$\det \left(\begin{bmatrix} 3-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -4 & 5 & -1-\lambda \end{bmatrix} \right) \rightarrow (3-\lambda)(2-\lambda)(-1-\lambda) = 0$$

characteristic polynomial

$$\rightarrow \text{eigenvalues: } \{ 3, 2, -1 \} \text{ all with multiplicity 1}$$

$$\rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ -4 & 5 & -1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ -4 & 5 & -4 \end{bmatrix} \xrightarrow[=]{4R_1 + R_3 \rightarrow R_3}$$

$$\left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -4 \end{array} \right] \xrightarrow{\sim} \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -4 \end{array} \right] \rightarrow x_3 = 0$$

$$\rightarrow \text{let } x_2 = s_1 \rightarrow x_1 - s_1 = 0 \rightarrow x_1 = s_1 \rightarrow$$

basis of eigenspace for $\lambda = 3$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$\rightarrow \lambda = 2 \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ -4 & 5 & -3 & 0 \end{array} \right] \xrightarrow{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 5 & -3 & 0 \end{array} \right] \rightarrow \text{let } x_3 = s_1 \rightarrow$$

$$5x_2 - 3s_1 = 0 \rightarrow x_2 = \frac{3}{5}s_1 \rightarrow$$

$x_1 = 0 \rightarrow$ basis of eigenspace for $\lambda = 2$: $\left\{ \begin{bmatrix} 0 \\ 3/5 \\ 1 \end{bmatrix} \right\}$

$$\rightarrow \lambda = -1 \rightarrow \left[\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ -4 & 5 & 0 & 0 \end{array} \right] \xrightarrow{\sim} \left[\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\xrightarrow{\sim} \left[\begin{array}{ccc|c} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \text{let } x_3 = s_1 \rightarrow x_2 = 0$$

$\rightarrow x_1 = 0 \rightarrow$ eigenbasis for $\lambda = -1$: $\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$②7) A = \begin{bmatrix} 2 & 5 & 1 \\ 0 & -3 & -1 \\ 2 & 14 & 4 \end{bmatrix} \rightarrow (A - \lambda I_3) \vec{z} = \vec{0} \rightarrow$$

$$\begin{vmatrix} 2-\lambda & 5 & 1 & | & 2-\lambda & 5 \\ 0 & -3-\lambda & -1 & | & 0 & -3-\lambda \\ 2 & 14 & 4-\lambda & | & 2 & 14 \end{vmatrix} = (2-\lambda)(-3-\lambda)(4-\lambda)$$

~~$$(2-\lambda)(-1)(14) = (-6 + 3\lambda - 2\lambda + \lambda^2)(4-\lambda)$$~~

$$-10 + 6 + 2\lambda - (-28 + 14\lambda) = -24 + 4\lambda + 4\lambda^2$$
~~$$+ 6\lambda - \lambda^2 - \lambda^3 - 4 + 2\lambda + 28 - 14\lambda =$$~~

$$-\lambda^3 + 3\lambda^2 - 2\lambda \rightarrow \lambda(-\lambda^2 + 3\lambda - 2) = 0 \rightarrow$$

characteristic polynomial

$$\lambda(-\lambda + 1)(\lambda - 2) = 0 \rightarrow \text{eigenvalues: } \{0, 1, 2\}$$

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 2 & 14 & 4 & 0 \end{array} \right] \xrightarrow{R_1 + R_3 \rightarrow R_3} \sim \left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 9 & 3 & 0 \end{array} \right] \xrightarrow{3R_2 + R_3 \rightarrow R_3} \sim$$

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \text{let } x_3 = s_1 \rightarrow -3x_2 - s_1 = 0 \rightarrow$$

$$x_2 = -\frac{1}{3}s_1 \rightarrow 2x_1 + \frac{-5}{3}s_1 + \frac{3}{3}s_1 = 0$$

$$\rightarrow 2x_1 = \frac{2}{3}s_1 \rightarrow x_1 = \frac{1}{3}s_1 \rightarrow \text{eigenbasis for}$$

$$\lambda=0: \left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\}$$

$$\rightarrow \lambda=1 \rightarrow \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 2 & 14 & 3 & 0 \end{array} \right]$$

$$-2R_1 + R_3 \rightarrow R_3 \sim \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 4 & 1 & 0 \end{array} \right] R_2 + R_3 \rightarrow R_3 \sim \left[\begin{array}{ccc|c} 1 & 5 & 1 & 0 \\ 0 & -4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\rightarrow \text{let } x_3 = s_1 \rightarrow -4x_2 - s_1 = 0 \rightarrow x_2 = -\frac{1}{4}s_1$$

$$\rightarrow x_1 + -\frac{5}{4}s_1 + s_1 = 0 \rightarrow x_1 = \frac{1}{4}s_1 \rightarrow$$

$$\text{eigenbasis for } \lambda=1: \left\{ \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix} \right\} \rightarrow \lambda=1$$

$$\rightarrow \left[\begin{array}{ccc|c} 0 & 5 & 1 & 0 \\ 0 & -5 & -1 & 0 \\ 2 & 14 & 2 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 2 & 14 & 2 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & 5 & 1 & 0 \end{array} \right] R_1 \leftrightarrow R_3 \sim \left[\begin{array}{ccc|c} 2 & 14 & 2 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] R_2 + R_3 \rightarrow R_3 \sim$$

$$\left[\begin{array}{ccc|c} 2 & 14 & 2 & 0 \\ 0 & -5 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \text{let } x_3 = s_1 \rightarrow -5x_2 - s_1 = 0 \rightarrow x_2 = -\frac{1}{5}s_1 \rightarrow 2x_1 - \frac{14}{5}s_1 + \frac{10}{5}s_1 = 0 \rightarrow$$

$$2x_1 = \frac{4}{5}s_1 \rightarrow x_1 = \frac{2}{5}s_1 \rightarrow$$

$$\text{eigenbasis for } \lambda=2: \left\{ \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \right\}$$

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a)

False as λ can be zero but eigenvector cannot be zero.

b)

True by definition.

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a)

True as the diagonal - ~~and~~ means that the eigenvalue is a root of the characteristic polynomial

b)

False as the columns of A would not span \mathbb{R}^n by the unifying theorem.

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a)

False, any non-zero matrix with zero diagonal and upper or lower triangular ~~is~~ can have only the 0 eigenvalue.

b)

True or else a constant would mean that all λ terms are cancelled out but a constant would be equal to zero which is impossible for all constants not 0.

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a)

6x6 matrix

b)

eigenvalues: $\{3, 2, -1\}$

c)

A is invertible

d)

3 is largest possible dimension for eigenspace of A

~~★~~ Graded Homework 7 - Section 6.2 ~~★~~

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$$\textcircled{1} \quad A = PDP^{-1}, \quad A^5, \quad P = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\rightarrow P^{-1} = \frac{1}{1} \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \rightarrow$$

$$D^5 = \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 32 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} 128 & -3 \\ 32 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 131 & -396 \\ 33 & -100 \end{bmatrix}}$$

$$\textcircled{3} \quad P = \begin{bmatrix} 1 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad A^5$$

$$P^{-1} = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} 2R_3 + R_2 \rightarrow R_2 \\ R_3 + R_1 \rightarrow R_1 \end{array} \left[\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 1 \\ 0 & \cancel{-1} & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array} \right]$$

$$3R_2 + R_1 \rightarrow R_1 \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 7 \\ 0 & -1 & 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} -R_2 \rightarrow R_2 \\ -R_3 \rightarrow R_3 \end{array} \sim$$

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 7 \\ 0 & 1 & 0 & 0 & -1 & -2 \\ 0 & 0 & 1 & 0 & 0 & -1 \end{array} \right] \rightarrow P^{-1} = \left[\begin{array}{ccc} 1 & 3 & 7 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{array} \right] \rightarrow$$

$$P^5 = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 7 \\ 0 & 32 & 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{array} \right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 32 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{array} \right] \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 3 & 7 \\ 0 & 32 & 0 & 0 & -1 & -2 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{array} \right] \left[\begin{array}{ccc|ccc} 1 & 3 & 7 & 1 & -93 & -184 \\ 0 & -1 & -2 & 0 & 32 & 66 \\ 0 & 0 & -1 & 0 & 0 & -1 \end{array} \right]$$

⑨ $A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \rightarrow A = PDP^{-1} \rightarrow$

Eigenvalues of A : $(A - \lambda I_2) \vec{v} = \vec{0} \rightarrow \begin{bmatrix} 1-\lambda & -2 \\ 0 & 1-\lambda \end{bmatrix}$

$\rightarrow \det \left(\begin{bmatrix} 1-\lambda & -2 \\ 0 & 1-\lambda \end{bmatrix} \right) = (1-\lambda)^2 \rightarrow \lambda = 1$ with
multiplicity 2. \rightarrow eigenspace
associated with $\lambda = 1$: $\begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \rightarrow$ let $x_1 =$

$\begin{cases} \rightarrow -2x_2 = 0 \rightarrow x_2 = 0 \rightarrow \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \rightarrow \end{cases}$

Cannot diagonalize as eigenvectors cannot form basis
for \mathbb{R}^2

$$(15) \quad A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 2 \end{bmatrix} \rightarrow A = PDP^{-1} \rightarrow \text{eigenvalues}$$

$$\text{of } A: (A - \lambda I_3) \vec{u} = \vec{0} \rightarrow \begin{vmatrix} -\lambda & 1 & -1 & -1 \\ 1 & -\lambda & 1 & 1 \\ 1 & 1 & 2-\lambda & -1 \end{vmatrix}$$

$$= (-\lambda)(-\lambda)(2-\lambda) + 1 + -\lambda - \lambda \cancel{(2-\lambda)} =$$

$$2\lambda^2 - \lambda^3 - \lambda = 0 \rightarrow \lambda(-\lambda^2 + 2\lambda - 1) = 0 \rightarrow$$

~~$$\lambda(\lambda^2 - 2\lambda + 1) = 0 \rightarrow \lambda(\lambda - 1)^2 = 0$$~~

$\rightarrow \lambda = 0$ and $\lambda = 1$ are eigenvalues with multiplicity 1 and 2 respectively. \rightarrow eigenspace

for $\lambda = 0 \rightarrow \left[\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 \end{array} \right] \xrightarrow{\substack{R_1 \leftrightarrow R_3 \\ R_2 \leftrightarrow R_3}} \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right]$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{\substack{-R_1 + R_3 \rightarrow R_3 \\ }} \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array} \right] \xrightarrow{\substack{-R_2 + R_3 \rightarrow R_3 \\ }} \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{aligned} &\text{let } x_3 = s_1 \rightarrow x_2 - s_1 = 0 \rightarrow \\ &x_2 = s_1 \rightarrow x_1 - s_1 + 2s_1 = 0 \end{aligned}$$

$\rightarrow x_1 = -s_1 \rightarrow \text{Span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\} \rightarrow \text{eigenspace}$

for $\lambda = 1$

$$\rightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 1 & -1 & 1 & 0 \\ 1 & -1 & 1 & 0 \end{array} \right] \xrightarrow{\substack{R_1 + R_2 \rightarrow R_2 \\ R_2 + R_3 \rightarrow R_3}} \sim$$

$$\left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \text{let } x_3 = s_1 \rightarrow \text{let } x_2 = s_2 \rightarrow$$

$$-x_1 + s_2 - s_1 = 0 \rightarrow x_1 = s_2 - s_1$$

$$S_2 - S_1 \rightarrow \cancel{x_1 = s_1} \quad \vec{x} = s_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow$$

$$\text{Span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \rightarrow P = \begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\rightarrow D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow P^{-1} = \begin{bmatrix} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \sim \left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{array} \right] \xrightarrow{\substack{R_1 + R_2 \rightarrow R_2 \\ R_1 + R_3 \rightarrow R_3}} \sim$$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 2 & 1 & 1 & 0 \end{array} \right] \xrightarrow{\substack{R_2 \leftrightarrow R_3 \\ -R_3 + R_1 \rightarrow R_1}} \sim \left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right] \xrightarrow{\substack{-2R_3 + R_2 \rightarrow R_2 \\ }} \sim$$

$$\left[\begin{array}{ccc|ccc} -1 & -1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 0 & -1 & 1 & -2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right] \xrightarrow{-R_1 \rightarrow R_1} \sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right]$$

$$\xrightarrow{-R_2 + R_1 \rightarrow R_1} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array} \right] \rightarrow A = PDP^{-1} \rightarrow$$

$$A = \left[\begin{array}{ccc} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{array} \right] \left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array} \right] \left[\begin{array}{ccc} -1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 0 & 1 \end{array} \right]$$

② $A = \begin{bmatrix} 7 & -8 \\ 4 & -5 \end{bmatrix}$, find A^{1000}

Eigenvalues of A : $(A - \lambda I_2) \vec{x} = \vec{0} \rightarrow \begin{vmatrix} 7-\lambda & -8 \\ 4 & -5-\lambda \end{vmatrix}$

$$= -35 - 2\lambda + \lambda^2 + 32 \rightarrow \lambda^2 - 2\lambda - 3 = 0 \rightarrow$$

$$(\lambda - 3)(\lambda + 1) = 0 \rightarrow \lambda_1 = 3 \text{ and } \lambda_2 = -1 \rightarrow$$

eigenspace associated with $\lambda = 3 \rightarrow \left[\begin{array}{cc|c} 4 & -8 & 0 \\ 4 & -8 & 0 \end{array} \right]$

$$\xrightarrow{-R_1 + R_2 \rightarrow R_2} \left[\begin{array}{cc|c} 4 & -8 & 0 \\ 0 & 0 & 0 \end{array} \right] \rightarrow \text{let } x_2 = s_1 \rightarrow 4x_1 - 8s_1 = 0$$

$$\rightarrow x_1 = 2s_1 \rightarrow \text{Span} \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} \rightarrow$$

eigenspace associated with $\lambda = -1 \rightarrow \begin{bmatrix} 8 & -8 & | & 0 \\ 4 & -4 & | & 0 \end{bmatrix}$

$$-\frac{1}{2}R_1 + R_2 \rightarrow R_2 \quad \begin{bmatrix} 8 & -8 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \rightarrow \text{let } x_2 = s_1 \rightarrow$$

$$8x_1 - 8s_1 = 0 \rightarrow x_1 = s_1 \rightarrow \text{span } \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$$

$$\rightarrow P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \rightarrow P^{-1} = \begin{bmatrix} 2 & 1 & | & 0 \\ 1 & 1 & | & 0 \end{bmatrix}$$

~~$$-\frac{1}{2}R_1 + R_2 \rightarrow R_2 \quad \begin{bmatrix} 2 & 1 & | & 1 & 0 \\ 0 & \frac{1}{2} & | & -\frac{1}{2} & 1 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & | & 1 & 0 \\ 0 & 1 & | & -1 & 2 \end{bmatrix}$$~~

~~$$-R_2 + R_1 \rightarrow R_1 \quad \begin{bmatrix} 2 & 0 & | & 2 & -2 \\ 0 & 1 & | & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 0 & | & 2 & -2 \\ 0 & 1 & | & -1 & 2 \end{bmatrix}$$~~

$$\begin{bmatrix} 1 & 0 & | & 1 & -1 \\ 0 & 1 & | & -1 & 2 \end{bmatrix} \rightarrow A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\rightarrow A^{1000} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \left(\begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \right)^{1000} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} (2)(3)^{1000} & 1 \\ 3^{1000} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \rightarrow \boxed{\begin{bmatrix} (2)(3)^{1000} - 1 & -(2)(3)^{1000} + 2 \\ 3^{1000} - 1 & -(3)^{1000} + 2 \end{bmatrix}}$$

(23) The dimension of the other eigenspace must be ~~be~~ 2.

(31) (a) True by definition.

(b) False, distinct eigenvectors mean nothing in terms of the diagonalizability of A as there are infinity many eigenvectors

(33) (a) False
Matrices without full rank can still be diagonalizable

(b) ~~False~~ False, matrix multiplication does not necessarily preserve diagonalizability.

~~Homework~~ 7 - Section 6.5

$$\textcircled{1} \quad A = \begin{bmatrix} 1 & -3 \\ 1 & 5 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow A\vec{x}_0 = \vec{x}_1 \rightarrow$$

$$\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \vec{x}_2 = A\vec{x}_1 = \begin{bmatrix} -2 \\ 6 \end{bmatrix} \rightarrow \vec{x}_3 = A\vec{x}_2$$

$$\rightarrow \vec{x}_3 = \boxed{\begin{bmatrix} -20 \\ 28 \end{bmatrix}}$$

$$\textcircled{5} \quad A = \begin{bmatrix} 5 & -1 & 2 \\ 2 & 2 & 2 \\ 2 & -1 & 5 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \rightarrow \vec{x}_1 = A\vec{x}_0$$

$$\rightarrow \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = \vec{x}_1 \rightarrow \vec{x}_2 = \begin{bmatrix} 9 \\ 0 \\ -9 \end{bmatrix} \rightarrow \boxed{\vec{x}_3 = \begin{bmatrix} 27 \\ 0 \\ -27 \end{bmatrix}}$$

$$\textcircled{7} \quad A = \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \vec{x}_1 = \frac{1}{S_0} A\vec{x}_0$$

$$= \frac{1}{1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \vec{x}_2 = \frac{1}{S_1} A\vec{x}_1 =$$

$$-\frac{1}{3} \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ -\frac{1}{3} \end{bmatrix} = \vec{x}_2}$$

(11) $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -1 & 1 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \rightarrow \vec{x}_1 = \frac{1}{s_0} A \vec{x}_0$

 $= \frac{1}{3} \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2/3 \end{bmatrix} \rightarrow \vec{x}_2 = \frac{1}{3} \begin{bmatrix} 0 \\ 3 \\ -5/3 \end{bmatrix}$

$\vec{x}_2 = \boxed{\begin{bmatrix} 0 \\ 1 \\ -5/9 \end{bmatrix}} = \vec{x}_2$

(13) Power method converges to $\lambda_3 = 7$

(17) Power method converges to $\lambda_3 = 6$

(29) $A \in \mathbb{R}^{2 \times 2} \wedge \vec{x}_0 \text{ s.t. } \vec{x}_0 = \vec{x}_1 = \vec{x}_k \text{ as } k \rightarrow \infty$

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \vec{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(35a) False, a dominant eigenvalue is sufficient but not necessary for convergence by the Power method.

(37b) False, it is possible by chance to choose an initial vector, \vec{x}_0 s.t. the Power method does not converge to the same eigenvector as a different choice of \vec{x}_0 .