# Lecture 1 - 08/22/2023

- $x_1 + 2x_2 - 3x_3 = 11$
  $3x_1 + 6x_2 - 8x_3 = 32$
  $\underbrace{-2}_{\text{coefficients}}x_1 - \underbrace{x_2}_{\text{variables (unknowns)}} = \underbrace{-7}_{\text{constant terms}}$

  "system of linear equations"

The only solution for this system of linear equations:

$x_1 = 2 \quad x_2 = 3 \quad x_3 = -1$

- Example: find all solutions to

$$\begin{aligned} x_1 + 5x_2 - 4x_3 &= -12 \Rightarrow x_1 + 5(-1) - 4(3) \Rightarrow x_1 = 5 \\ -2x_2 + 3x_3 &= 11 \Rightarrow -2x_2 + 3(3) = 11 \Rightarrow x_2 = -1 \\ 2x_3 &= 6 \Rightarrow x_3 = 3 \end{aligned}$$

"Triangular System"

back-substitution

$\boxed{\text{Unique Solution: } x_1 = 5, \; x_2 = -1, \; x_3 = 3}$

"always (almost)" a unique solution, can backsubstitute

- Example: find all solutions to

$$\begin{aligned} 4x_1 - x_2 + 6x_3 + 7x_4 &= 4 \\ 3x_2 \qquad\qquad - x_4 &= 10 \\ 5x_4 &= -5 \end{aligned}$$

"free variable"

"leading variables"

- For "free variable", let's set it as an unknown

solution: let $x_3 = s_1$ "free parameter"

$$\begin{aligned} 4x_1 - x_2 + 6x_3 + 7x_4 &= 4 \\ 3x_2 \qquad\qquad -x_4 &= 10 \Rightarrow 3x_2 - (-1) = 10 \quad x_2 = 3 \\ 5x_4 &= -5 \Rightarrow x_4 = -1 \end{aligned}$$

$\Rightarrow$

$4 \cdot x_1 - (3) + 6s_1 + 7(-1) = 4 \Rightarrow$

$x_1 = \dfrac{14 - 6s_1}{4} = \dfrac{7 - 3s_1}{2}$

CONTINUED $\longrightarrow$

- Example:

$$x_1 + x_2 + x_3 = 2$$
$$2x_1 + 3x_2 + x_3 = 3$$
$$x_1 - x_2 - 2x_3 = -6$$

$$\downarrow$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & 1 & 2 & -6 \end{array}\right]$$

"Augmented Matrix" – "Augmented" means that it has the right hand side of equations

- Elementary Row Operations:

(1) Interchange two rows

(2) Multiply a row by a non-zero constant.

(3) Add a mult. of one row to a second row, replacing the second row

These are the counterparts that are the same as the elementary equation operations.

these should become zeros for echelon form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{array}\right] \quad -2R_1 + R_2 \to R_2$$

"row"

$\sim$ ← "equivalent"

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 1 & -1 & -2 & -6 \end{array}\right]$$

$\sim \Downarrow \quad -R_1 + R_3 \to R_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{array}\right]$$

Use the $r_2$ because any multiplication of $r_1$ would make $r_3$ lose the "0" in the $x_1$ position.

General solution:

$x_1 = \frac{7-3s_1}{2}$,

$x_2 = 3$

$x_3 = s_1$

$x_4 = -1$,

where $s_1$ can be any real number

$\boxed{08/24/2023 \text{ - Lecture 2}}$

"Echelon form"

$4x_1 - x_2 + 6x_3 + 7x_4 = 4$

$3x_2 - x_4 = 10$

$5x_4 = -5$

General solution:

$x_1 = \frac{7-3s_1}{2}$

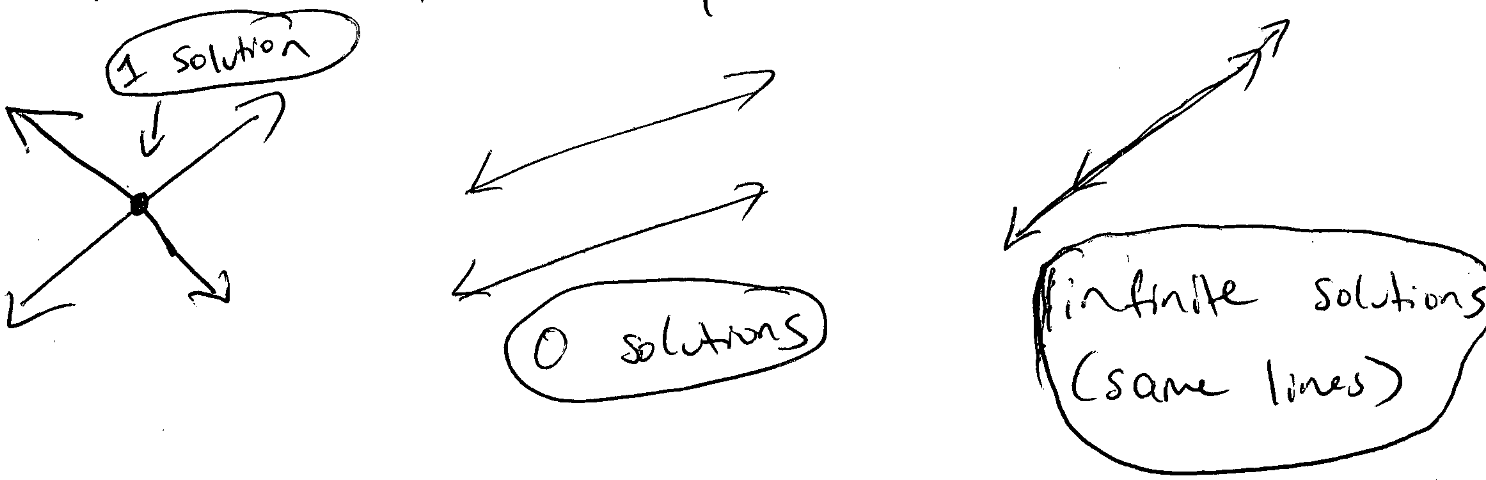
$x_2 = 3$

$x_3 = s_1$

$x_4 = -1$

$s_1$ can be an number; hence infinitely many solutions.

• Remark : Every $\boxed{\text{linear}}$ system has 0, 1, or infinitely many solutions.




• Two special cases :

1) If $0 = 0$ appears in a linear system, it can be ignored!

2) If $0 = a$ $(a \neq 0)$ appears, then the system has no solutions!

END of Section 1.1

# Section 1.2 : Elementary Equation Operations

1) Interchange 2 equations

2) Multiply an equation by a nonzero constant.

3) Add a multiple of one equation to another, replacing the 2nd equation.
   - This is used 90 % of the time

- 1) - 3) above <u>DO NOT</u> change the equations.

- Example:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 11 \\ 3x_1 + 6x_2 - 8x_3 &= 32 \\ -2x_1 - x_2 &= -7 \end{aligned}$$

CONTINUED

$-3\,Eq_1 + Eq_2 \longrightarrow Eq_2$

"Equation 1" "replaces"

$$\begin{cases} x_1 + 2x_2 - 3x_3 = 11 \\ x_3 = -1 \\ -2x_1 - x_2 = -7 \end{cases}$$

$\Downarrow$ $Eq_2 \longleftrightarrow Eq_3$

"interchange" equations

$$\begin{cases} (x_1 + 2x_2 - 3x_3 = 11)2 \\ -2x_1 - x_2 = -7 \\ x_3 = -1 \end{cases}$$

$2Eq_1 + Eq_2 \rightarrow Eq_2$ $\Downarrow$ multiply by 2

$$\begin{cases} x_1 + 2x_2 - x_3 = 11 \\ 3x_2 - 6x_3 = 15 \\ x_3 = -1 \end{cases}$$

After backsubstitution:

$x_1 = 2$

$x_2 = 3$

$x_3 = -1$

$$\sim \quad \Downarrow \quad 2R_2 + R_3 \rightarrow R_3$$

"echelon" form

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -1 \end{array}\right]$$

$$\Downarrow$$

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ x_2 - x_3 &= -1 \\ -5x_3 &= -1 \end{aligned}$$

Solution:
$x_1 = -1$
$x_2 = 1$
$x_3 = 2$

- This process above is known as "Gaussian Elimination"

- $$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \leftarrow \frac{1}{-5} R_3 \rightarrow R_3$$

"this is to make" $R_3$ easy to work with

$\sim$

$R_3 + R_2 \to R_2$

"gets a 0"

$-R_3 + R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

now focus on these two

$\Downarrow$ $-R_2 + R_1 \to R_1$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \Rightarrow \begin{array}{l} x_1 = -1 \\ x_2 = 1 \\ x_3 = 2 \end{array}$$

- Process of ~~[illegible]~~ avoiding backsubstitution and getting matrix of only non-zeros on diagonal (as we did above) is called Gauss-Jordan Elimination

- Example: Find all solutions to

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 1 \\
3x_1 - 6x_2 + 2x_3 &= 1 \\
-x_2 + 3x_3 &= 6
\end{aligned}
$$

$\Downarrow$ Augmented matrix

"start here"

$$\left[\begin{array}{ccc|c} \boxed{1} & -2 & 1 & 1 \\ 3 & -6 & 2 & 1 \\ 0 & -1 & 3 & 6 \end{array}\right]$$

$\Downarrow$ $-3R_1 + R_2 \to R_2$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & -1 & 3 & 6 \end{array}\right]$$

"now focus on here"

$\Downarrow$ $R_2 \longleftrightarrow R_3$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 1 \\ 0 & -1 & 3 & 6 \\ 0 & 0 & -1 & -2 \end{array}\right]$$

swapped

Solution: $x_1 = -1$, $x_2 = 0$, $x_3 = 2$

- Explaining how we are able to add a multiple of one equation to another to replace the second equation.

$$cE_1 + E_2$$

$$-3x_1 - x_2 = 7$$

$$x_1 + x_2 = 4$$

$$2E_1 + E_2 \Rightarrow$$

$$2(x_1 + x_2) + (3x_1 - x_2) = 2(4) + 7$$

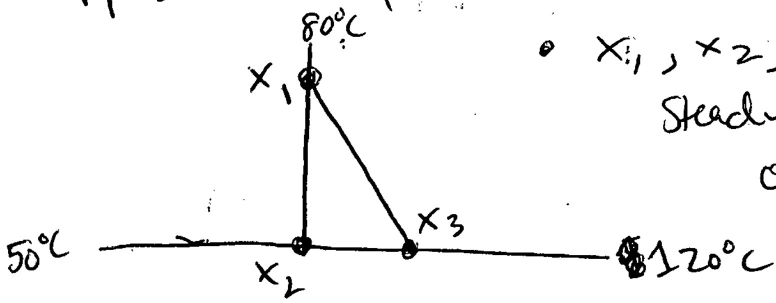
$$\Downarrow$$

$$-x_1 + x_2 = 15$$

## Lecture 3 - 08/29/2023

- Section 1.3 : Applications

- Suppose heavy wires are welded together as shown




- $x_1, x_2, x_3$ are points that are steady state and are average of the endpoints near it.

CONTINUED →

$x_1 = \frac{1}{3}(80 + x_2 + x_3)$

$x_2 = \frac{1}{3}(50 + \cancel{x_4} + x_3)$

$x_3 = \frac{1}{3}(120 + x_1 + x_2)$

multiply by 3
and Isolate
Constant

$3x_1 - x_2 - x_3 = 80$

$-x_1 + 3x_2 - x_3 = 50$

$-x_1 - x_2 + 3x_3 = 120$

Solution:

$x_1 = 82.5$

$x_2 = 75.0$

$x_3 = 92.5$

$$\int \frac{3x-1}{(x-1)(x+1)}\,dx \Longrightarrow \int \frac{3x-1}{(x-1)(x+1)} = \int \frac{A}{x-1} + \int \frac{B}{x+1}$$

add integrals

Find A and B so that...

$$\frac{3x-1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$$

$$(x-1)(x+1)\left(\frac{3x-1}{(x-1)(x+1)}\right) = \left[\frac{A}{\cancel{x-1}} + \frac{B}{x+1}\right]\begin{matrix}(x-1)\\(x+1)\end{matrix}$$

$$3x-1 = A(x+1) + B(x-1)$$

CONTINUED →

$$3x - 1 = (A+B)(x) + (A-B)$$

$$A + B = 3$$
$$A - B = -1$$
$$\Longrightarrow$$

Solution:
$$A = 1$$
$$B = 2$$

this means that

$$\int \frac{3x-1}{(x-1)(x+1)}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{2}{x+1}\,dx$$

- Example: Chemical Reactions

$$Na + H_2O \longrightarrow NaOH + H_2$$

$$\Downarrow$$

$$x_1\,Na + x_2\,H_2O \longrightarrow x_3\,NaOH + x_4\,H_2$$

Na: $x_1 = x_3$

H: $2x_2 = x_3 + 2x_4$

O: $x_2 = x_3$

$$\Rightarrow \quad \begin{aligned} x_1 \quad\quad - x_3 \quad\quad &= 0 \\ 2x_2 - x_3 - 2x_4 &= 0 \\ x_2 - x_3 \quad\quad &= 0 \end{aligned}$$

$$\swarrow$$

General Solution:

$x_1 = 2s_1$ $\quad$ $x_3 = 2s_1$

$x_2 = 2s_1$ $\quad$ $x_4 = s_1$

CONTINUED $\longrightarrow$

$$2s_1\,Na + 2s_1\,H_2O \rightarrow 2s_1\,NaOH + s_1\,H_2$$

For $s_1 = 1 \Rightarrow$ $\boxed{2Na + 2H_2O \rightarrow 2NaOH + H_2}$

## §1.4 Numerical Solutions

- 1st method: <u>Jacobi Iteration</u>

- Example: Approximate the solution to

$$\begin{cases} 5x_1 - 2x_2 = 13 \\ x_1 - 10x_2 = -7 \end{cases} \Rightarrow \text{Solution: } x_1 = 3,\ x_2 = 1$$

using this to validate the Jacobi Iteration.

Reminder that this process is meant for 1000s of variables and 1000s of equations.

$$x_1 = 2.6 + 0.4\,x_2$$
$$x_2 = 0.7 + 0.1\,x_1$$

Step 0 "guess the solution" $\Longrightarrow$ $x_1 = 0,\ x_2 = 0$

we've chosen zero as a placeholder

Step 1

$$x_1 = 2.6 + 0.4(0) = 2.6$$
$$x_2 = 0.7 + 0.1(0) = 0.7$$

CONTINUED →

Step 2 - "plug in the numbers"

$$x_1 = 2.6 + 0.4(0.7) = 2.88$$

$$x_2 = 0.7 + 0.1(2.6) = 0.96$$

$\Downarrow$ Step 3

$$x_1 = 2.6 + 0.4(0.96) = 2.984$$

$$x_2 = 0.7 + 0.1(2.88) = 0.988$$

$\Downarrow$ Step 4

$$x_1 = 2.6 + 0.4(.988) = 2.9952$$

$$x_2 = 0.7 + 0.1(2.984) \quad 0.9984$$

- Jacobi iteration requires the same number of equations as variables. It also will not always converge depending on the order in which you work with it.

- "Diagonally dominant" guarantees convergence and hence, you may need to re-order the equations to get them in this form.

CONTINUED $\longrightarrow$

$$5x_1 - 2x_2 = 13$$
$$x_1 - 10x_2 = -7$$

→ "Diagonally dominant"

Absolute value of each variable in the equation order ($x_1$ for 1st equation, $x_2$ for 2nd equation) is greater than sum of all other ~~coef~~ coefficients ABSOLUTE VALUE

$$7x_1 - 3x_2 + x_3 = 15$$
$$2x_1 + 6x_2 + 3x_3 = 5$$
$$x_1 - x_2 + 9x_3 = 0$$

- Gauss-Seidel : faster convergence

New innovation: Use most recent variable values.

$$x_1 = 2.6 + 0.4x_2$$
$$x_2 = 0.7 + 0.1x_1$$

Step 0 ⇒ $x_1 = 0$, $x_2 = 0$

⇓ Step 1

$x_1 = 2.6 + 0.4(0) = 2.6$

$x_2 = 0.7 + 0.1(2.6) = 0.96$

Step 2

$x_1 = 2.6 + 0.4(0.96) = 2.984$

$x_2 = 0.7 + 0.1(2.984) = 0.9984$

## Section 2.1 - Vectors

- Definition: A vector is a list of numbers

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

← The set of all such vectors is $\mathbb{R}^n$, where there are $n$ components

- Example:

$$\vec{u} = \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix}, \quad \vec{v} \begin{bmatrix} 7 \\ 0 \\ 5 \\ 4 \end{bmatrix} \quad \left( \text{These are in } \mathbb{R}^4 \right)$$

$$\vec{U} + \vec{v} = \begin{bmatrix} 1+7 \\ -2+0 \\ 3+5 \\ 0+4 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \\ 8 \\ 4 \end{bmatrix}$$

$$2\vec{u} = (2) \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 6 \\ 0 \end{bmatrix}$$

"scalar"

- A linear combination:

$$-2\vec{u} + 3\vec{v} = (-2) \begin{bmatrix} 1 \\ -2 \\ 3 \\ 0 \end{bmatrix} + (3) \begin{bmatrix} 7 \\ 0 \\ 5 \\ -4 \end{bmatrix}$$

$$= \begin{bmatrix} 19 \\ 4 \\ 9 \\ -12 \end{bmatrix}$$

# Lecture 4 - 08/31/2023

- <u>Definition:</u> Let $\vec{v}_1, \vec{v}_2, \dots \vec{v}_m$ be vectors in $\mathbb{R}^n$, and let $c_1, c_2, \dots, c_m$ be real numbers ("scalars"). Then:

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots c_m \vec{v}_m$$

is called a <u>linear combination</u> of $\vec{v}_1, \dots, \vec{u}_m$

- Example:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 11 \\ 3x_1 + 6x_2 - 8x_3 &= 32 \\ -2x_1 - x_2 \phantom{- 8x_3} &= -7 \end{aligned}$$

$$\underbrace{x_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ -8 \\ 0 \end{bmatrix}}_{\text{linear combination}} = \begin{bmatrix} 11 \\ 32 \\ -7 \end{bmatrix}$$

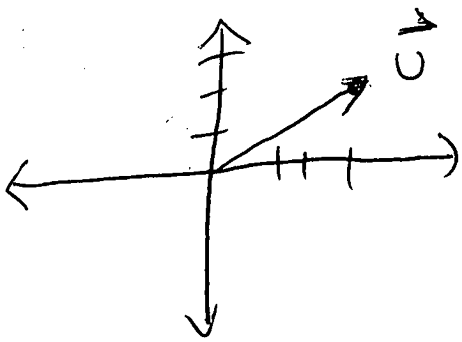
- Example: general solutions

$$\begin{cases} x_1 = 11 - 2s_1 + s_2 \\ x_2 = s_1 \\ x_3 = 5 + 3s_2 \\ x_4 = s_2 \end{cases}$$

$$\vec{x} = \begin{bmatrix} 11 \\ 0 \\ 5 \\ 0 \end{bmatrix} + s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} 1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

- Visualizing vectors:

$\vec{u} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$




## $\boxed{\text{Section 2.2 – Span}}$

- <u>Definition:</u> let $\{\vec{u}_1, \ldots \vec{u}_m\}$ be a set of vectors in $\mathbb{R}^n$. Then, the set of all possible linear combinations:

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 \ldots c_m \vec{u}_m$$

CONTINUED →

is called the <u>span</u> of $\{\vec{u}_1, \ldots, \vec{u}_n\}$

- Example: Span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$ $\qquad \vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

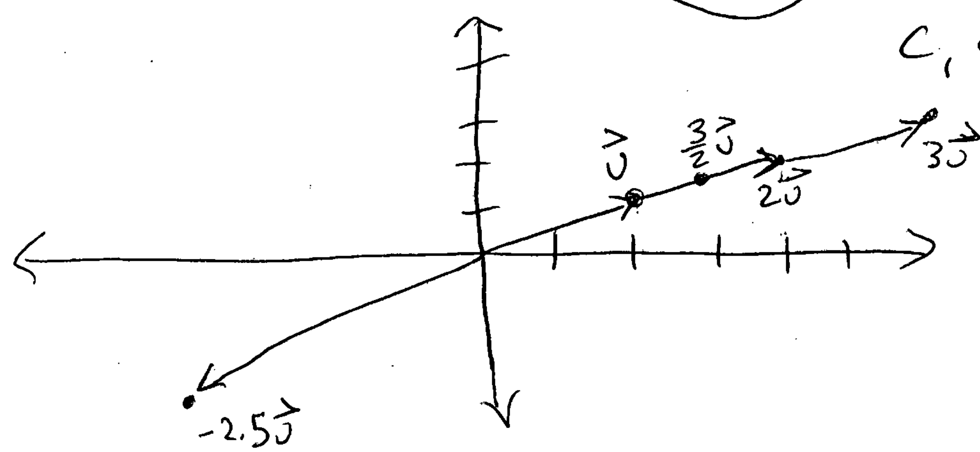
$$c_1 \vec{u}_1 = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\Downarrow$$

$$c_1 = 2$$

$$\Downarrow$$

$$\begin{bmatrix} 4 \\ 2 \end{bmatrix}$$




The line is the span $\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$

- Example $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$

You also get a line when looking at span $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right\}$

- Example:

Span $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix} \right\}$ gives you a plane

$$c_1 \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 5 \\ 4 \\ 0 \end{bmatrix}$$

- Determine if $\vec{v}$ is in span $\{\vec{u}_1, \vec{u}_2\}$ for

$$\vec{v} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}, \quad \vec{u}_1 = \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

Solution:

$$x_1 \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -7 \end{bmatrix}$$

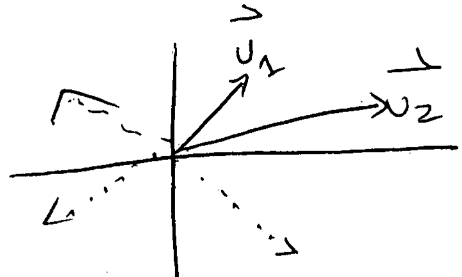
We need to find $x_1$ and $x_2$

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 5x_2 &= 4 \\ -2x_1 + x_2 &= -7 \end{aligned}$$

Augmented Matrix Conversion

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 3 & 5 & 4 \\ -2 & 1 & -7 \end{array}\right] \rightarrow$$

This has a solution: $x_1 = 3$, $x_2 = -1$

- 


Span of $\vec{u}_1$ and $\vec{u}_2$ is all of $\mathbb{R}^2$

- Example / General Tool:

Does the set $\{\vec{u}_1, \ldots, \vec{u}_m\}$ span $\mathbb{R}^n$?

<u>Solution:</u>

Step 1: form $A = [\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n]$

Step 2: Transform $A \underset{\substack{\uparrow \\ \text{row operations}}}{\sim} B$ (echelon form)

$$\begin{array}{cccc} \begin{bmatrix} 2 & 1 & -1 & 1 \\ 2 & 0 & 2 & 3 \\ -4 & 0 & -4 & -6 \end{bmatrix} \\ \begin{array}{cccc} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 & \vec{u}_4 \end{array} \\ A \end{array} \sim \begin{bmatrix} 2 & 1 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

pivots

NOTE: pivots are used when this is not a "system" but a table of numbers

- Does every row have a pivot?

If yes, set spans $\mathbb{R}^n$

If no, set does not span.

In the above example, the row of 0's means that the set does not span.

# Lecture 5 - 09/05/2023

## Section 2.3 - Linear Independence

- Let $\{\vec{a}_1, \dots, \vec{a}_m\}$ where all vectors have the same dimension.

  Informal: This set of vectors is linearly independent if each vector is not in the span of the other vectors.

- Example: $S = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

  For $S$ to be linearly independent we need:

  (a) $\vec{a}_1$ is not in span $\{\vec{a}_2, \vec{a}_3\}$

  (b) $\vec{a}_2$ is not in span $\{\vec{a}_1, \vec{a}_3\}$

  (c) $\vec{a}_3$ is not in span $\{\vec{a}_1, \vec{a}_2\}$

- Formal definition of linear independence:

  $S = \{\vec{a}_1, \dots, \vec{a}_m\}$ is linearly independent if the system

CONTINUED →

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \cdots + x_m \vec{a}_m = \vec{0}$$

has only the trivial solution $x_1 = 0, \ldots x_m = 0$ )

If there are other solutions, then $S$ is linearly dependent.

- Example:

Is $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -2 \\ 5 \end{bmatrix} \right\}$ linearly independent?

Solution: the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 1 & -1 & -2 & 0 \\ -1 & 3 & 5 & 0 \end{array}\right] \xrightarrow{\text{after putting into echelon form}}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow$$ backsubstituting gives you only the trivial solution $x_1 = x_2 = x_3 = 0$ which means <u>linear independence</u>

- Recall from Section 2.2:

$S = \{\vec{a}_1 \ldots \vec{a}_m\}$ in $\mathbb{R}^n$

CONTINUED $\longrightarrow$

$A = [\vec{a_1} \ldots \vec{a_m}]$

$A \sim B \leftarrow$ echelon form

<u>Question 1</u>: Does Span $(S) = \mathbb{R}^n$?

Yes if every row of B has a pivot and no otherwise.

<u>Question 2</u>: Is S linearly independent?

Yes if every <u>column</u> of B has a pivot and no otherwise.

- Is S linearly independent?

$$S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 10 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ -5 \end{bmatrix} \right\}$$

echelon form

$$A = \begin{bmatrix} 1 & 4 & -3 \\ 3 & -2 & -1 \\ -1 & 10 & -5 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 4 & -3 \\ 0 & \boxed{-14} & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

No pivot
So S does not span $\mathbb{R}^n$

No pivot means S is linearly dependent

CONTINUED $\longrightarrow$

- NOTE: you generally need $n$ vectors to span $\mathbb{R}^n$

- Example:

$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$\vec{a}_1$ $\vec{a}_2$

To be linearly independent

(1) $\vec{a}_1$ is not in the Span $\{\vec{a}_2\}$ (i.e. $\vec{a}_1 \neq c\vec{a}_2$ for any $c$ )

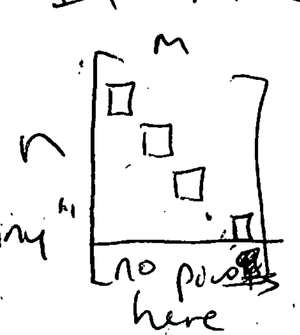
(2) $\vec{a}_2$ is not in the Span $\{\vec{a}_1\}$

- 2 Special cases:

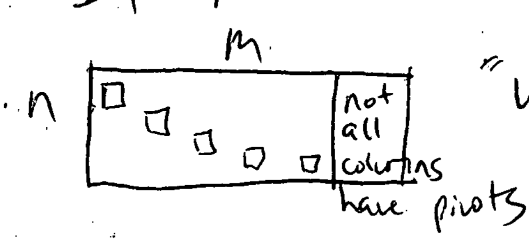
$S = \{\vec{a}_1, \ldots \vec{a}_m\}$ in $\mathbb{R}^n$

① If $m < n \Rightarrow$ $S$ does not span $\mathbb{R}^n$

"tall and skinny"




② If $m > n \Rightarrow S$ is not linearly independent




"wide and long"

- The set $\{\vec{0}, \vec{a}_1, \dots, \vec{a}_m\}$ is <u>always</u> linearly dependent.

$$A = [\vec{0}, \vec{a}_1 \dots \vec{a}_m]$$

- <u>The Unifying Theorem</u>: (version 1)

$$S = \{\vec{a}_1 \dots \vec{a}_n\} \text{ in } \mathbb{R}^n$$

$$A = [\vec{a}_1 \dots \vec{a}_n]$$

Then, the following are equivalent:

"This means that all conditions are either all <u>true</u> or <u>all false</u>"

(a) $S$ spans $\mathbb{R}^n$

$$\begin{array}{cc} & n \\ n & \begin{bmatrix} \square & & & & \\ & \square & & & \\ & & \square & & \\ & & & \square & \\ & & & & \square \end{bmatrix} \end{array}$$

(b) $S$ is linearly independent

(c) $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$ in $\mathbb{R}^n$

# Section 3.1 - Linear Transformations

- Let $A = \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & -2 \\ -7 & 5 \end{bmatrix}}_{m=2}$, $\vec{x} = \left.\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right\} n=2$

Then, $A\vec{x} = x_1 \begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$

Now define:

$T(\vec{x}) = A\vec{x}$, where $T$ is a linear transformation

For instance:

$$\underset{\text{Input}}{T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right)} = (2)\begin{bmatrix} 1 \\ 3 \\ -7 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} = \underset{\text{output}}{\begin{bmatrix} 2 \\ 8 \\ -19 \end{bmatrix}}$$

In general:

$$T(\vec{x}) = \begin{bmatrix} x_1 \\ 3x_1 - 2x_2 \\ -7x_1 + 5x_2 \end{bmatrix}$$

Notation:

$$T: \underset{\text{(domain)}}{\mathbb{R}^2} \longrightarrow \underset{\text{(codomain)}}{\mathbb{R}^3}$$

- General Definition: A function $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation if

(a) $T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^m$

(b) $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ in $\mathbb{R}^m$ and any scalar $c$.

☆ $\boxed{\text{Lecture 6 - 09/07/2023}}$ ☆

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- Example: Suppose that

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + 3x_2 - x_3 \\ 5x_1 - 4x_2 + 7x_3 \end{bmatrix}$$

$$T: \underset{\text{(domain)}}{\mathbb{R}^3} \to \underset{\text{(codomain)}}{\mathbb{R}^2}$$

$T$ is a linear transformation that can be written $T(\vec{x}) = A\vec{x}$

$$x_1 \begin{bmatrix} -2 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -4 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 7 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 & 3 & -1 \\ 5 & -4 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\vec{x}}$$

- Definition:

(a) If $T(\vec{v}) = \vec{w}$, then $\vec{w}$ is called the <u>image of $\vec{v}$ under $T$</u>.

(b) The <u>range of $T$</u> is the set of all images under $T$

$$T(\vec{x}) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$T: \mathbb{R}^2 \to \mathbb{R}^2 \qquad \text{range}(T) = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

- <u>Theorem:</u>

let $A = [\vec{a}_1, \vec{a}_2 \ldots \vec{a}_m]$ and $T(\vec{x}) = A\vec{x}$

<u>Then:</u>

<u>(a)</u> $\vec{w}$ is in range $(T)$ exactly when $A\vec{x} = \vec{w}$ is consistent (has a solution)

<u>(b)</u> $\text{range}(T) = \text{span}\{\vec{a}_1, \vec{a}_2 \ldots \vec{a}_m\}$

- <u>Definition</u>: let $T: \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a linear transformation.

<u>(a)</u> $T$ is <u>one-to-one</u> if for all output vectors $(\vec{w})$ in $\mathbb{R}^n$ there is at most one $\vec{v}$ in $\mathbb{R}^m$ with $T(\vec{v}) = \vec{w}$

(b) T is onto / surjective if for all $\vec{w}$ in $\mathbb{R}^n$ there is at ☆ least ☆ one $\vec{v}$ in $\mathbb{R}^m$ with $T(\vec{v}) = \vec{w}$ $\longrightarrow$ range$(T) = \mathbb{R}^n$

- Suppose $T(\vec{x}) = A\vec{x}$, with $A$ having $n$ rows and $m$ columns ($n \times m$) $\leftarrow$ Theorem

Then:

(a) T is one-to-one exactly when the columns of A are linearly independent.

(b) T is onto exactly when the columns of A span $\mathbb{R}^n$

(c) Suppose $A \sim B$ ← echelon form

(1) Do all rows of B have a pivot?

Yes → T is onto

No → T is not onto

CONTINUED →

(2) Do all columns of $B$ have a pivot?

yes → $T$ is one-to-one

No → $T$ is not one-to-one

- Example:

$T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\vec{x}) = A\vec{x}$

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 1 \\ 2 & -1 \end{bmatrix}$$

Is $T$ one-to-one? Onto?

Solution: $A \sim \begin{bmatrix} 1 & 3 \\ 0 & 7 \\ 0 & 0 \end{bmatrix}$ ← no pivot → $T$ is not onto

Both columns have a pivot ⇒ $T$ is one-to-one

- The Unifying Theorem - Version 2

Let $S = \{\vec{a}_1, \dots \vec{a}_n\}$ in $\mathbb{R}^n$

$A = [\vec{a}_1 \dots \vec{a}_n]$, $T(x) = A\vec{x}$

Then these are equivalent:

CONTINUED →

(a) $S$ spans $\mathbb{R}^n$

(b) $S$ is linearly independent

(c) $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b}$ in $\mathbb{R}^n$

(d) $T$ is onto

(e) $T$ is one-to-one

- Example:

$T: \mathbb{R}^3 \to \mathbb{R}^3$ with $T(\vec{x}) = A\vec{x}$,

$$A = \begin{bmatrix} 1 & -1 & -1 \\ -3 & 2 & 1 \\ 5 & -4 & -3 \end{bmatrix}$$

Is $T$ one-to-one? Onto?

Solution:

$$\begin{bmatrix} \boxed{1} & -1 & -1 \\ 0 & \boxed{-1} & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

no pivot $\longrightarrow$ not onto

No pivot $\longrightarrow$ not one-to-one

# Section 3.2 - Matrix Algebra

- Example:

$$A = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 4 & -7 \\ 1 & 3 & 2 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 9 & 3 & -5 \\ 4 & 3 & 3 \end{bmatrix}$$

$$-2A = \begin{bmatrix} -8 & 2 & -4 \\ -6 & 0 & -2 \end{bmatrix}$$

let $b_1 = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$ $\quad A\vec{b}_1 = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} (4)(1) + (-1)(3) + (2)(2) \\ (3)(1) + (0)(3) + (1)(2) \end{bmatrix}$$

$$\begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

"linear combination version"

$$(1)\begin{bmatrix} 4 \\ 3 \end{bmatrix} + (3)\begin{bmatrix} -1 \\ 0 \end{bmatrix} + (2)\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

let $B = [\vec{b}_1 \;\; \vec{b}_2] \Rightarrow AB \; [A\vec{b}_1 \;\; A\vec{b}_2]$

# Lecture 7 - 09/12/2023

## Section 3.3 - Inverses

- Inverses only involve square matrices.
- $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$, $B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ $\Rightarrow$

$$AB = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underbrace{I_2}_{\text{"2x2"}}$$

$$BA = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

- $xy = 1 \Rightarrow y = \frac{1}{x} = x^{-1}$ (if $x \neq 0$)
- Definition: An $n \times n$ matrix $A$ is invertible (or nonsingular) if there exists another matrix ($n \times n$) $B$ such that $AB = I_n$.

(If there is no $B$, then $A$ is not invertible and hence, singular)

- Remarks:

(a) If $AB = I_n$, then $BA = I_n$

(b) The inverse of $A$ is unique ($A^{-1}$)

(c) Not all square matrices have an inverse.

Example:

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$ ← no inverse as you cannot create an identity matrix from all 0's.

(d) $T: \mathbb{R}^n \to \mathbb{R}^n$, $T(\vec{x}) = A\vec{x}$,

$\uparrow$ $n \times n$

then $T^{-1}(\vec{x}) = A^{-1}\vec{x}$, if $A^{-1}$ exists

- Theorem:

Let $A, B$ be $n \times n$, invertible matrices

(a) $(A^{-1})^{-1} = A$

(b) $(AB)^{-1} = B^{-1}A^{-1}$

(c) If $AC = AD$, then $C = D$

CONTINUED →

(d) If $AC = 0$, then $C = 0$

"0" matrices

- Example:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \quad \text{Find } A^{-1}$$

We want to find $B$ s.t. $AB = I_2$

$$A \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{cases} A\vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ A\vec{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{cases}$$

$$\left[ A \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix} \right] \Rightarrow \left[ \begin{matrix} 1 & 3 \\ 2 & 5 \end{matrix} \,\middle|\, \begin{matrix} 1 \\ 0 \end{matrix} \right]$$

$$\left[ A \,\middle|\, \begin{matrix} 0 \\ 1 \end{matrix} \right] \Rightarrow \left[ \begin{matrix} 1 & 3 \\ 2 & 5 \end{matrix} \,\middle|\, \begin{matrix} 0 \\ 1 \end{matrix} \right]$$

$$\left[ \underbrace{\begin{matrix} 1 & 3 \\ 2 & 5 \end{matrix}}_{A} \,\middle|\, \underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} \right]$$

$$[A \mid I_2] \sim [I_2 \mid A^{-1}]$$

you have to find exactly the identity matrix.

CONTINUED →

Solution:

$$\left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 5 & 0 & 1 \end{array}\right] \underset{\sim}{-2R_1+R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right] \underset{\sim}{-R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{array}\right] \underset{\sim}{-3R_2+R_1 \to R_1}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -5 & 3 \\ 0 & 1 & 2 & -1 \end{array}\right]$$

$$\underbrace{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}}_{I_2} \quad \underbrace{\begin{matrix} -5 & 3 \\ 2 & -1 \end{matrix}}_{A^{-1}}$$

- Example: Find $A^{-1}$ for $A = \begin{bmatrix} 1 & -3 \\ -4 & 12 \end{bmatrix}$

Solution:

$$\left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -4 & 12 & 0 & 1 \end{array}\right] \underset{\sim}{4R_1+R_2 \to R_2} \left[\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{array}\right]$$

$$\underbrace{0 \quad 0}_{\text{No } A^{-1}}$$

• Example: Find $A^{-1}$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 5 \\ 1 & -2 & 2 \end{bmatrix} \to \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 2 & -3 & 5 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$-2R_1 + R_2 \to R_2$
$-R_1 + R_3 \to R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$-3R_3 + R_2 \to R_2$
$-R_3 + R_1 \to R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

$2R_2 + R_1 \to R_1$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 2 & -7 \\ 0 & 1 & 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$

• Unifying Theorem (version 3)

• Same preamble = $(S, A, T)$

(a) $S$ spans $\mathbb{R}^n$

(b) $S$ is linearly independent

CONTINUED →

⋮

(e) $T$ is one-to-one

(f) $A$ is invertible (non-singular)

$\boxed{\text{Lecture 8 - 09/14/2023}}$

- Find the solution to:

$$x_1 - 2x_2 + x_3 = 2$$
$$2x_1 - 3x_2 + 5x_3 = -1 \longrightarrow A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 5 \\ 1 & -2 & 2 \end{bmatrix}$$
$$x_1 - 2x_2 + 2x_3 = 4$$

$$\vec{b} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \longrightarrow A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$\longrightarrow \vec{x} = A^{-1}\vec{b} \rightarrow$ Previously, we found

$$A^{-1} = \begin{bmatrix} 4 & 2 & -7 \\ 1 & 1 & -3 \\ -1 & 0 & 1 \end{bmatrix} \longrightarrow A^{-1}\vec{b} = \begin{bmatrix} -22 \\ -11 \\ 2 \end{bmatrix}$$

# Section 3.4 - LU-factorization

• Example: Solve

$$\begin{aligned} 2x_1 + \phantom{x_2} \quad x_3 &= 2 \\ 4x_1 + x_2 - x_3 &= 9 \\ -2x_1 + 3x_2 - 12x_3 &= 17 \end{aligned}$$

Solution:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 4 & 1 & -1 \\ -2 & 3 & -12 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & -2 \end{bmatrix}}_{U}$$

"lower triangular" "upper triangular"

$$A\vec{x} = \vec{b} = \begin{bmatrix} 2 \\ 9 \\ 17 \end{bmatrix}$$

$$A = LU \rightarrow LU\vec{x} = \vec{b}$$

$$L(\overbrace{U\vec{x}}^{\vec{y}}) = \vec{b} \Rightarrow L\underbrace{\vec{y}}_{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}} = \vec{b} \rightarrow \begin{aligned} y_1 \quad\quad\quad &= 2 \\ 2y_1 + y_2 \quad &= 9 \\ -y_1 + 3y_2 + y_3 &= 17 \end{aligned}$$

$\rightarrow \quad \vec{y} = \begin{bmatrix} 2 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = U\vec{x} \quad \rightarrow$ After back substitution, we get $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$

- NOTE: LU Factorization can have more than 1 solution.

- Find an LU Factorization for

$$A = \begin{bmatrix} 3 & 1 & -2 \\ 6 & 4 & -7 \\ -3 & 5 & -8 \end{bmatrix} \xrightarrow[R_1 + R_3 \to R_3]{-2R_1 + R_2 \to R_2} \begin{bmatrix} 3 & 1 & -2 \\ 0 & 2 & -3 \\ 0 & 6 & -10 \end{bmatrix}$$

$$\xrightarrow{-3R_2 + R_3 \to R_3} \begin{bmatrix} 3 & 1 & -2 \\ 0 & 2 & -3 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution:

divide by pivot

divide by pivot

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 3 & 1 \end{bmatrix}$$

divide by pivot

$U$

- If $A$ can be transformed to echelon form <u>without</u> interchanging rows, then $A$ has an LU-factorization

- Example:

$$A = \begin{bmatrix} \boxed{1} & 3 & 1 \\ -1 & -3 & 2 \\ 1 & 1 & 4 \end{bmatrix} \underset{-R_1+R_3 \to R_3}{\overset{R_1+R_2 \to R_2}{\sim}} \begin{bmatrix} 1 & 3 & 1 \\ 0 & \boxed{0} & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

We can stop here as we know there is no LU Factorization

Solution:

$$L = \begin{bmatrix} 1 & & \\ -1 & & \\ 1 & & \end{bmatrix}$$

# Section 3.5 – Markov Chains

- Example: For some town, it is known that if it rained yesterday, then there is a 60% chance of rain today. If it didn't rain yesterday, then there is an 80% chance of no rain today.

<u>Question:</u> If it rained yesterday, what is the chance of rain tomorrow?

<u>Solution:</u> start with

$$\begin{array}{cccl} & \multicolumn{2}{c}{\text{yesterday}} & \\ & \text{rain} & \text{no rain} & \\ A = & \left[\begin{matrix} .60 \\ .40 \end{matrix}\right. & \left.\begin{matrix} .20 \\ .80 \end{matrix}\right] & \begin{matrix} \text{rain} \\ \text{no rain} \end{matrix} \quad \text{today} \end{array}$$

"transition matrix"

(generally, "$\underbrace{\text{stochastic matrix}}$")

↳ Each column adds up to 1 and there are no negative values

## Initial State Vector

$$\vec{x}_0 = \begin{array}{c} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \text{rain} \\ \text{no rain} \end{array} \quad \text{(100\% chance of rain yesterday)}$$

$$\vec{x}_1 = A\vec{x}_0 = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .6 \\ .4 \end{bmatrix}$$

$$\vec{x}_2 = A\vec{x}_1 = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} \begin{bmatrix} .6 \\ .4 \end{bmatrix} = \begin{bmatrix} .44 \\ .56 \end{bmatrix}$$

44% chance of rain tomorrow

- **Notes:**

(a) The sequence $\underbrace{\vec{x}_0, \vec{x}_1, \vec{x}_2, \ldots}_{\text{"state vectors"}}$ is a Markov chain.

(b) The "state vectors" are stochastic (their probabilities add to 1 and have non-negative elements).

(c) Frequently, the sequence will converge to a <u>steady-state</u> vector.

To <u>find this:</u>

(a) $\vec{x}_{n+1} = A\vec{x}_n$

(b) $x_{n+1} \approx x_n$ for large $n$

$\to \vec{x}_n \approx A\vec{x}_n$

We need: $A\vec{x} = \vec{x} \to A\vec{x} - \vec{x} = \vec{0}$

$\to A\vec{x} - I\vec{x} = \vec{0} \to (A - I)\vec{x} = \vec{0}$

$\to A - I = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -.4 & .2 \\ .4 & -.2 \end{bmatrix}$

$\to \left[\begin{array}{cc|c} -.4 & .2 & 0 \\ .4 & -.2 & 0 \end{array}\right] \underset{\sim}{R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} -.4 & .2 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\to -.4x_1 + .2x_2 = 0 \to$ let $x_2 = s_1 \to$

$-.4x_1 + .2s_1 = 0 \to x_1 = \frac{1}{2}s_1 \to x_1 = s_1 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$

$\to$ we need entries to add to 1, so set

$$s_1 = \frac{1}{\frac{1}{2}+1} = \frac{2}{3} \rightarrow \vec{x} = \begin{bmatrix} (\frac{1}{2})(\frac{2}{3}) \\ (1)(\frac{2}{3}) \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$

## Lecture 9 - 09/19/2023

- A is <u>stochastic</u> if
  - a) A is square
  - b) Columns add to 1
  - c) No negative numbers

  Ex: $A = \begin{bmatrix} .6 & .2 \\ .4 & .8 \end{bmatrix}$ $\vec{x}_0,\ \vec{x}_1 = A\vec{x}_0,\ \vec{x}_2 A \vec{x}_1$

  $\Downarrow$

  Steady-state $= \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix}$

- <u>Definition</u>: A stochastic matrix A is <u>regular</u> if $A^K$ has all positive entries for some $K \geq 1$.
  - For example, the example before this ("A") is regular

- Example:

$$A = \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix} \begin{bmatrix} .5 & 1 \\ .5 & 0 \end{bmatrix} = \begin{bmatrix} .75 & .5 \\ .25 & .5 \end{bmatrix}$$

All positive

A is regular

This is important so we know it will converge to a steady state vector.

- $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^2 = A^3 \dots$ hence, not regular as the zeros will never go away.

- Theorem: Suppose A is a regular matrix, (which means that it has to be stochastic),

  (a) For any initial state $\vec{x}_0$, the chain $\vec{x}_0, \vec{x}_1, \vec{x}_2 \longrightarrow \vec{x}$ (the same steady-state vector)

CONTINUED →

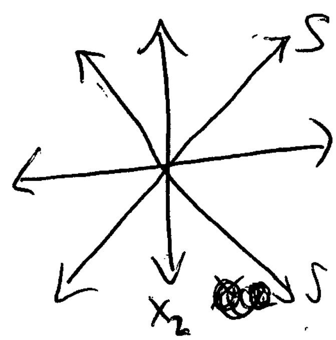
(b) The sequence $A, A^2, A^3, \ldots$ converges to $\underbrace{[\vec{x}, \vec{x}, \ldots, \vec{x}]}_{n \text{ copies of } \vec{x}}$

# Section 4.1 – Subspaces

- Example: $S$ is a subset of $\mathbb{R}^2$:

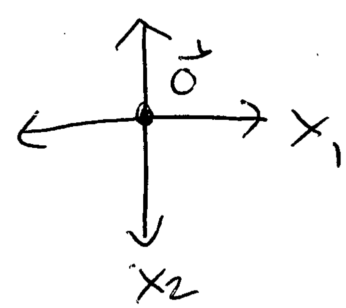
(a) $S$ is a line through the origin

Possibilities




(b) $S$ is equal to $\mathbb{R}^2$ ($S = \mathbb{R}^2$)

(c) $S = \{\vec{0}\}$




- <u>Definition</u>: A subset of $\mathbb{R}^n$ is a <u>subspace</u> if:

(a) $S$ contains the zero vector $\vec{0}$

(b) if $\vec{u}, \vec{v}$ are in $S$, so is $\vec{u} + \vec{v}$.
("closed under addition")

(c) if $\vec{u}$ is in $S$, then so is $c\vec{u}$ for all $c$.

("closed under scalar multiplication")

Example: $S$ is a subset of $\mathbb{R}^3$

(a) $S$ is a line through the origin

(b) $S = \mathbb{R}^3$

(c) $S = \{\vec{0}\}$

(d) $S$ is a plane through the origin

Example: Show that $S = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}_{\vec{u}_1}, \underbrace{\begin{bmatrix} 3 \\ 7 \\ 1 \end{bmatrix}}_{\vec{u}_2} \right\}$ is a subspace of $\mathbb{R}^3$

Solution:

(a) All vectors in $S$ are $c_1\vec{u}_1 + c_2\vec{u}_2$

$0\vec{u}_1 + 0\vec{u}_2 = \vec{0}$ is in $S$

(b) $\underbrace{\vec{v}}_{\text{some vector in } S} = c_1\vec{u}_1 + c_2\vec{u}_2$, $\underbrace{\vec{w}}_{\text{some vector in } S} = d_1\vec{u}_1 + d_2\vec{u}_2$

CONTINUED →

$$\vec{v} + \vec{w} = (c_1\vec{u}_1 + c_2\vec{u}_2) + (d_1\vec{u}_1 + d_2\vec{u}_2)$$
$$= (c_1 + d_1)\vec{u}_1 + (c_2 + d_2)\vec{u}_2$$

(c) $\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 \longrightarrow c\vec{v} = c(c_1\vec{u}_1 + c_2\vec{u}_2)$

Some vector in S

$$= (cc_1)\vec{u}_1 + (cc_2)\vec{u}_2$$

- **Theorem:** If $S = \text{span}\{\vec{u}_1, \ldots \vec{u}_m\}$ in $\mathbb{R}^n$, then $S$ is a subspace of $\mathbb{R}^n$

- **Theorem:** If $S$ is the set of solutions to $A\vec{x} = \vec{b}$, then $S$ is a subspace exactly when $b = \vec{0}$

$A\vec{x} = \vec{0} \rightarrow$ subspace

$A\vec{x} = \vec{b} \neq \vec{0} \rightarrow$ not a subspace.

$\boxed{\text{Lecture 11 - 09/28/2023}}$

CONTINUED $\longrightarrow$

- Is $S$ a subspace? Check in this order
  (a) is $\vec{0}$ in $S$?
  (b) is $S$ the span of vectors?
  (c) $\vec{u}, \vec{v}$ in $S$ means $\vec{u}+\vec{v}$ in $S$
  (d) $\vec{u}$ in $S$ means $c\vec{u}$ in $S$

- Example: $S$ is the set of the form

$$\vec{s} = \begin{bmatrix} a \\ a^2 \end{bmatrix} \text{ for any } a$$

possible values of $\vec{s} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \end{bmatrix}, \begin{bmatrix} -\pi \\ \pi^2 \end{bmatrix}$

- Is this a subspace?

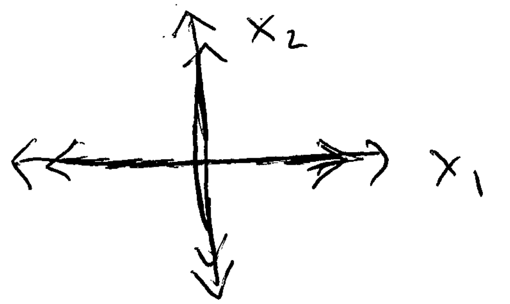
(a) $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0^2 \end{bmatrix}$ in $S$

(b) Inconclusive

(c) Try examples:

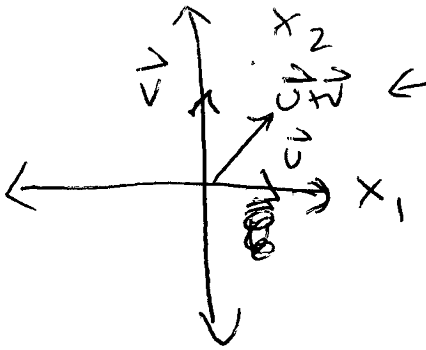
$$\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \rightarrow \vec{u}+\vec{v} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \neq \begin{bmatrix} 3 \\ 9 \end{bmatrix}$$

hence, $S$ is not a subspace.

- 


where the vectors in $S$ are just the horizontal and vertical lines.

In this example, it is not closed under addition so it is not a subspace.




← this is not part of the set of vectors.

- The solutions of $A\vec{x} = \vec{0}$ are called the **null space** of $A$, written $\text{null}(A)$

- **Example:** Find $\text{null}(A)$ for $A = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 6 & -3 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ -3 & 6 & -3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} \boxed{1} & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

After backsubstitution:

$$x_1 = 2s_1 - s_2$$
$$x_2 = s_1$$
$$x_3 = s_2$$

$$\Rightarrow \text{null}(A) = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

- <u>Definition</u>: $T: \mathbb{R}^m \to \mathbb{R}^n$

<u>(1)</u> range$(T)$ = "set of outputs of $T$"
(subspace of $\mathbb{R}^n$) = col$(A)$

<u>(2)</u> $\underbrace{\ker(T)}_{\text{"Kernel of T"}}$ = "set of inputs with $T(\vec{x}) = \vec{0}$
(subspace of $\mathbb{R}^m$) = null$(A)$

- <u>Unifying Theorem - version 4</u>:

$S = \{\vec{a}_1, \ldots, \vec{a}_n\}$ ; $A = [\vec{a}_1 \ldots \vec{a}_n]$ , $T(\vec{x}) = A\vec{x}$.

The following are equivalent:

<u>(a)</u> $S$ spans $\mathbb{R}^n$

<u>(b)</u> $S$ is linearly independent

⋮

<u>(f)</u> $A$ is invertible

<u>(g)</u> $\ker(T) = \{\vec{0}\}$

# Lecture 10 - 09/26/2023

## Section 4.2 - Basis and Dimension

- Definition: If $\mathcal{B} = \{\vec{u}_1 \dots \vec{u}_m\}$ is in a subspace $S$, then $\mathcal{B}$ is a basis if:

  (a) $\mathcal{B}$ spans $S$

  (b) $\mathcal{B}$ is linearly independent.

- Important: If $\mathcal{B}$ is a basis, then every vector in $S$ is a unique linear combination of the vectors in $\mathcal{B}$.

- Find a basis for

$$S = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 11 \\ -2 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -4 \\ -6 \end{bmatrix} \right\}$$

- Theorem: Suppose $A \sim B$, then the rows of $A$ span the same subspace as the rows of $B$.

• Solution 1 for problem that is previously described:

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 3 & 11 & -2 & 11 \\ 2 & 2 & -4 & -6 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & -1 & 2 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = B$$

$$S = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

these are the rows of B.

we can throw this out.

these 2 are a basis for S

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 5 \end{bmatrix} \right\}$$

• Solution 2

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 11 & 2 \\ -1 & -2 & -4 \\ 2 & 11 & -6 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 3 & 2 \\ 0 & \boxed{1} & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

pivots

$$\text{Basis} = \left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 11 \\ -2 \\ 11 \end{bmatrix} \right\}$$

The pivot columns determine these as a basis.

- Theorem: Every basis for a subspace $S$ has the same number of vectors. This is the <u>dimension</u> of $S$.
- You do not need to find multiple bases to find the dimension. You can use any given basis.
- <u>Example:</u> In $\mathbb{R}^n$ these form the <u>standard basis</u>:

$$\underbrace{\left\{ \underset{\vec{e}_1}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \underset{\vec{e}_2}{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}, \ldots, \underset{\vec{e}_n}{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}} \right\}}_{n \text{ vectors}} \Rightarrow \dim(\mathbb{R}^n) = n$$

- Exception: $S = \{\vec{0}\}$ is a subspace, the only one without a basis. (There are no linearly dependent subsets). $\dim(\{\vec{0}\}) = 0$

• Example: Find the dimension of the subspace of solutions to

$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

Solution: Free variables – $x_2 = s_1$, $x_3 = s_2$, $x_4 = s_3$

$$x_1 = -2s_1 - 3s_2 - 4s_3$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} (-2s_1 - 3s_2 - 4s_3) \\ (s_1) \\ (s_2) \\ (s_3) \end{bmatrix} \longrightarrow$$

$$s_1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \text{Span} \underbrace{\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}}_{\text{also linearly independent}}$$

dim = 3

basis $\longrightarrow$ also linearly independent

• Theorem: Let $\mathcal{U} = \{\vec{u}_1, \ldots, \vec{u}_m\}$ be vectors in a subspace $S \neq \{\vec{0}\}$ of $\mathbb{R}^n$

(a) If $\dim(S) = m$, then if $\mathcal{U}$ spans $S$ or is linearly independent, then $\mathcal{U}$ is a basis.

(b) Suppose $\dim(S) = k$

(i) If $k > m$, then $\mathcal{U}$ cannot span $S$.

(ii) if $m > k$, then $\mathcal{U}$ cannot be linearly independent.

(c) If $\mathcal{U}$ is linearly independent, then vectors can be added to $\mathcal{U}$ to form a basis.

(d) If $\mathcal{U}$ spans $S$, then vectors can be removed from $\mathcal{U}$ to form a basis.

- **Theorem:** If $S_1 \subseteq S_2$, then $\dim(S_1) \leq \dim(S_2)$

- **The Unifying Theorem – Version 5:**

$S = \{\vec{a}_1, \ldots, \vec{a}_n\}$, $A = [\vec{a}_1 \ldots \vec{a}_n]$, $T(\vec{x}) = A\vec{x}$

The following are equivalent:

(a) $S$ spans $\mathbb{R}^n$

⋮

(g) $\ker(T) = \{\vec{0}\}$

(h) $S$ is a basis for $\mathbb{R}^n$.

# Section 4.3 : Row and Column Spaces

## Lecture 11 - 09/28/2023

- Example:

Suppose $A = \begin{bmatrix} 3 & 2 \\ -1 & 4 \\ 0 & 5 \end{bmatrix}$

"row space of A" = $\text{row}(A) = \text{Span}\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \end{bmatrix} \right\}$

"column space of A" = $\text{col}(A) = \text{Span}\left\{ \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix} \right\}$

$\text{row}(A)$ = subspace of $\mathbb{R}^2$

$\text{col}(A)$ = subspace of $\mathbb{R}^3$

NOTE: $\text{row}(A) \neq \text{col}(A)$

Generally, row(A) will not equal to col(A) and in rare cases, a square matrix could be the only possibility.

- Theorem: $\underbrace{\dim(\text{row}(A)) = \dim(\text{col}(A))}_{= \text{rank}(A)}$

**Example:** We have

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 3 & 11 & -2 & 11 \\ 2 & 2 & -4 & -6 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 3 & -1 & 2 \\ 0 & \boxed{2} & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \text{row}(A) = \text{span}\underbrace{\left\{ \begin{bmatrix} 1 \\ 3 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 5 \end{bmatrix} \right\}}_{\text{A basis}} \Rightarrow \text{rank}(A) = 2$$

$$\to \text{col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 11 \\ 2 \end{bmatrix} \right\}$$

These 2 are used as they are the columns with pivots

**Next:** The set of solutions to $A\vec{x} = \vec{0}$ is called the null space of A

Notation: $\text{null}(A)$

For our A, $\text{null}(A) = \text{span}\left\{ \begin{bmatrix} 5 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 11 \\ -5 \\ 0 \\ 2 \end{bmatrix} \right\}$

$\text{nullity}(A) = \dim(\text{null}(A)) = 2$

- **Rank - Nullity Theorem:**

  If $A$ is $n \times m$, then $\text{rank}(A) + \text{nullity}(A) = m$ for all $A$.

- Find nullity of $A^T$ given:

$$A = \begin{bmatrix} 1 & 3 & -1 & 2 \\ 3 & 11 & -2 & 11 \\ 2 & 2 & -4 & -6 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 11 & 2 \\ -1 & -2 & -4 \\ 2 & 11 & -6 \end{bmatrix}$$

$\text{row}(A) = \text{Col}(A^T)$

$\text{col}(A) = \text{row}(A^T)$

$\rightarrow$ hence, $\text{rank}(A^T) = \text{rank}(A)$

$\downarrow$

$\text{rank}(A) = 2$ and $\text{rank}(A^T) = 2$

$\downarrow$

$$\underset{\substack{\uparrow \\ 2}}{\text{rank}(A^T)} + \underset{\substack{\uparrow \\ 1}}{\text{nullity}(A^T)} = 3 \leftarrow 3 \text{ columns of } A^T$$

$\downarrow$

$\text{nullity}(A^T) = 1$

# The Unifying Theorem - Version 6

$S = \{\vec{a}_1, \dots, \vec{a}_n\}$ $A = [\vec{a}_1 \dots \vec{a}_n]$, $T(\vec{x}) = A\vec{x}$

The following are equivalent:

(a) $S$ spans $\mathbb{R}^n$

⋮

(h) $S$ is a basis for $\mathbb{R}^n$

(i) $\operatorname{col}(A) = \mathbb{R}^n$

(j) $\operatorname{row}(A) = \mathbb{R}^n$

(k) $\operatorname{rank}(A) = n$

# Lecture 12 - 10/12/2023

## Section 5.1 - Determinants

- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

<u>Definition</u>: For A above, we have

$\det(A) = ad - bc$

<u>Alternate notation</u>: $|A| = \det(A)$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

(determinant)

- <u>$3 \times 3$ matrices</u>:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \Rightarrow |A| =$$

$$(a_{11})\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + (a_{12})(-1)\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + \cdots$$

$(a_{13}) \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

- Example:

$$A = \begin{bmatrix} 3 & 2 & -4 \\ 1 & 0 & 2 \\ -1 & 3 & -2 \end{bmatrix} \Rightarrow |A| = (3) \begin{vmatrix} 0 & 2 \\ 3 & -2 \end{vmatrix}$$

$$-(2) \begin{vmatrix} 1 & 2 \\ -1 & -2 \end{vmatrix} + (-4) \begin{vmatrix} 1 & 0 \\ -1 & 3 \end{vmatrix} \rightarrow 3(\overset{0}{\cancel{\phantom{00}}} - \overset{6}{\cancel{\phantom{00}}})$$

$$-(2)(-2-(-2)) - 4(3-0) = -18 + 0 - 12 = -30$$

- <u>Shortcut Method:</u> (<u>only for 3x3</u>)

$$\left|\begin{array}{ccc|cc} 3 & 2 & -4 & 3 & 2 \\ 1 & 0 & 2 & 1 & 0 \\ -1 & 3 & -2 & -1 & 3 \end{array}\right. \rightarrow |A| = (0 + (-4) + (-12)) - (0 + 18 + (-4))$$

$$= -16 - 14 = -30$$

CONTINUED $\longrightarrow$

- $A = \begin{bmatrix} 4 & 7 & -1 & 0 \\ 3 & -1 & -1 & 2 \\ 0 & 5 & 4 & 1 \\ 8 & -9 & 2 & 0 \end{bmatrix}$

Then:

$$M_{32} = \begin{bmatrix} 4 & -1 & 0 \\ 3 & -1 & 2 \\ 8 & 2 & 0 \end{bmatrix}, \quad M_{13} = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 5 & 1 \\ 8 & -9 & 0 \end{bmatrix}$$

row column

eliminate these columns and rows

<u>Definition</u>: (1) $\det(M_{ij})$ = "minor of $a_{ij}$"

(2) $(-1)^{i+j} \det(M_{ij}) = C_{ij}$ = "cofactor of $a_{ij}$"

(3) The determinant of $A$ is, where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & & \end{bmatrix}, \quad |A| = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

- **Theorem:** (Cofactor Expansions)
  (Laplace Expansion)

(a) $|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in} + C_{in}$

(expansion on row $i$)

(b) $|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$

(expansion on column $j$)

- Example:

$$A = \begin{bmatrix} 3 & -1 & 2 & 4 \\ 2 & 1 & 0 & 0 \\ 1 & 3 & 0 & 3 \\ -1 & 5 & 0 & 1 \end{bmatrix} \longrightarrow |A| = 2C_{13} + \cancel{0C_{23}} + \cancel{0C_{33}} + \cancel{0C_{43}}$$

3rd column has 0's

$$= 2(-1)^{1+3} \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 3 \\ -1 & 5 & 1 \end{vmatrix}$$

$$= (2)(1)(-28) = \underline{-56}$$

The lesson here is to always choose a row or column with the most number of 0's.

- Alternate Solution:

$$|A| = 2C_{21} + 1C_{22} = (2)(-1)^{2+1} \begin{vmatrix} -1 & 2 & 4 \\ 3 & 0 & 3 \\ 5 & 0 & 1 \end{vmatrix} + 1(-1)^{2+2} \begin{vmatrix} 3 & 2 & \\ 1 & 0 & 3 \\ -1 & 0 & 1 \end{vmatrix}$$

$\rightarrow (-2)(24) + (-8) = \underline{-56}$

- **Theorem** (Properties):

**(a)** $\det(I_n) = 1$

**(b)** If $A$ is upper <u>or</u> lower triangular, then $|A|$ is the product of the diagonal entries.

- Ex: $A = \begin{bmatrix} 3 & 7 & -4 \\ 0 & 2 & 5 \\ 0 & 0 & -2 \end{bmatrix} \Rightarrow |A| = (3)(2)(-2)$
$= -12$

**(c)** $\det(A) = \det(A^T)$

**(d)** If $A$ has a row or column of 0's, then $|A| = 0$

**(e)** If $A$ has two identical rows or columns, then $|A| = 0$

CONTINUED $\longrightarrow$

(f) If $A$ and $B$ are $n \times n$, then $\det(AB) = \det(A) \cdot \det(B)$. Hence, $\det(AB) = \det(BA)$ as $\det(A)$ and $\det(B)$ are constants.

- Example:

$$A = \begin{bmatrix} 3 & 5 \\ 4 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ 4 & 3 \end{bmatrix}$$

$|A| = (6-20) = -14 \qquad |B| = (-3-0) = -3$

$|A|\,|B| = (-14)(-3) = 42$

$AB = \begin{bmatrix} 17 & 15 \\ 4 & 6 \end{bmatrix} \Rightarrow |AB| = 102 - 60 = 42$

- Unifying Theorem - v7

All of version 6 and:

(l) $\det(A) \neq 0$

# Lecture 13 - 10/16/2023

## Section 5.2 - Properties of Determinants

- Suppose $A$ is $n \times n$, and we get $B$ from $A$ by:

(a) Interchanging two rows of $A \Rightarrow |A| = -|B|$

Example:

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \Rightarrow |A| = 2$$

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |B| = -2$$

interchanging rows of A

(b) Multiplying by a constant for a row of $A$ with constant $c$.

$$|A| = \frac{1}{c}|B|$$

Example:

$$B = \begin{bmatrix} 9 & 12 \\ 1 & 2 \end{bmatrix} \Rightarrow |B| = 6 \Rightarrow \frac{1}{3}|B| = 2$$

CONTINUED $\longrightarrow$

(c) Add a multiple of one row to another

$|A| = |B|$

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix} \underset{\sim}{\xrightarrow{2R_1+R_2 \to R_2}} \begin{bmatrix} 3 & 4 \\ 7 & 10 \end{bmatrix} = B$$

$|B| = 30 - 28 = 2$

- Example:

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ 2 & 4 & 0 & -1 \\ -1 & 1 & 7 & -3 \\ 1 & 4 & 2 & -1 \end{bmatrix} \begin{matrix} -2R_1+R_2 \to R_2 \\ R_1+R_3 \to R_3 \\ -R_1+R_4 \to R_4 \\ \sim \end{matrix} \longrightarrow$$

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 3 & 6 & -3 \\ 0 & 2 & 3 & -1 \end{bmatrix} \begin{matrix} ☆ \\ R_2 \leftrightarrow R_3 \\ \frac{1}{3}R_2 \to R_2 \\ \sim \; ☆ \end{matrix} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 2 & 3 & -1 \end{bmatrix}$$

$$\underset{\sim}{-2R_2+R_4 \to R_4} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

CONTINUED

$\frac{1}{2}R_3 + R_4 \to R_4$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} = B$$

$$|B| = (1)(1)(2)(\tfrac{1}{2}) = 1$$

$$|B| = 1 \Rightarrow (-1)\left(\frac{1}{\frac{1}{3}}\right)|B| = (-1)(3)(1)$$

$$= -3$$

## ☆ Section 5.3 - Applications of Determinants ☆

- Notation:

$$A = [\vec{a}_1, \vec{a}_2 \dots \vec{a}_n] \text{ (} n \times n \text{)}, \quad \vec{b} \text{ is in } \mathbb{R}^n$$

Define:

$$A_i = [\vec{a}_1 \dots \vec{a}_{i-1} \ \vec{b} \ \vec{a}_{i+1} \dots \vec{a}_n]$$

replace $\vec{a}_i$ with $\vec{b}$

Can be done for any $i$

Example:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 5 & 2 \\ 6 & 4 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 5 \\ 3 & 6 \end{bmatrix}$$

<u>Theorem:</u> (Cramer's Rule)

Suppose $A$ is invertible. Then $A\vec{x} = \vec{b}$ has a unique solution $\vec{x}$ with components:

$$x_i = \frac{|A_i|}{|A|}$$

<u>Example:</u>

$$\begin{aligned} x_1 + 2x_2 &= 5 \\ 3x_1 + 4x_2 &= 6 \end{aligned} \Rightarrow x_1 = \frac{|A_1|}{|A|} = \frac{20-12}{4-6} = -4$$

$$x_2 = \frac{|A_2|}{|A|} = \frac{6-15}{-2} = \frac{9}{2}$$

Cofactor matrices: $C_{ij} = (-1)^{i+j} |M_{ij}|$

CONTINUED $\longrightarrow$

- **Definition:** cofactor matrix

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & \ddots & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

- The adjoint of A:

$$\operatorname{adj}(A) = C^T = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & \ddots & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}$$

- **Theorem:**

If $A$ is invertible, then:

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}(A)$$

- **Example:**

$$A = \begin{bmatrix} 3 & 7 & -6 \\ 1 & -2 & 1 \\ 2 & -3 & 0 \end{bmatrix}$$

CONTINUED →

To find $A^{-1}$:

$$C_{11} = (-1)^{1+1}\begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} = (-2)(0)-(1)(-3) = 3$$

$$C_{12} = (-1)^{1+2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = (-1)(0-2) = 2$$

The others:

$C_{13} = 1$, $C_{14} = 12$, $C_{23} = 5$, $C_{31} = -5$,

$C_{32} = -3$, $C_{33} = -1$

$$\Rightarrow C = \begin{bmatrix} 3 & 2 & 1 \\ 18 & 12 & 5 \\ -5 & -3 & -1 \end{bmatrix} \Rightarrow \operatorname{adj}(A) =$$

$$C^T = \begin{bmatrix} 3 & 18 & -5 \\ 2 & 12 & -3 \\ 1 & 5 & -1 \end{bmatrix} \rightarrow$$ It turns out that $|A| = -1$

$$\Rightarrow A^{-1} = \frac{1}{(-1)}\operatorname{adj}(A) \Rightarrow \begin{bmatrix} -3 & -18 & 5 \\ -2 & -12 & 3 \\ -1 & -5 & 1 \end{bmatrix}$$

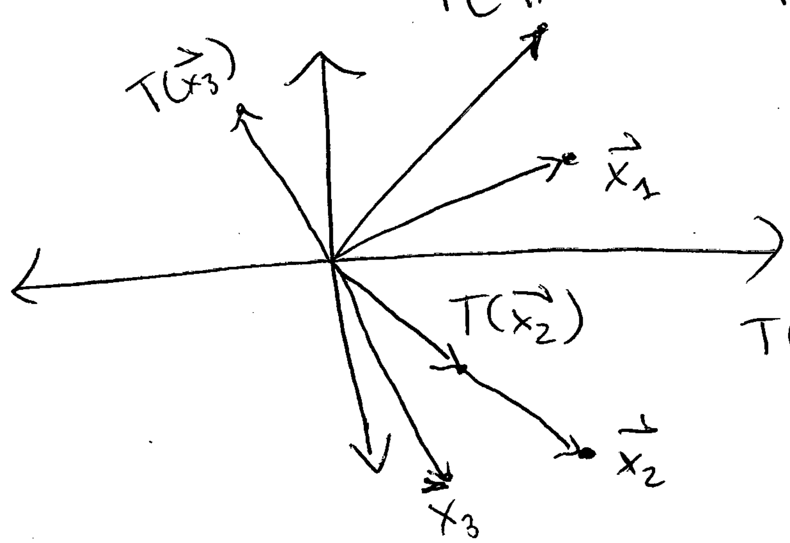
# Lecture 14 - 10/17/2023

## Section 6.1 - Eigenvalues and Eigenvectors

- Eigenvalues and eigenvectors are only for square matrices.

- Suppose $T: \mathbb{R}^2 \to \mathbb{R}^2$ is given by $T(\vec{x}) = A\vec{x}$ for some $2\times 2$ matrix $A$.

$T(\vec{x}_1) \neq c\,\vec{x}_1$




$T(\vec{x}_2) = c_2\vec{x}_2$ for some $c_2$

- <u>Definition</u>: Let $A$ be $n\times n$.

Then $\vec{u} \neq 0$ is an <u>eigenvector of $A$</u> if:

$A\vec{u} = \lambda\vec{u}$ is true for some scalar $\lambda$ ("eigenvalue")

Example:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

(a) $\vec{x}_1 = \begin{bmatrix} -3 \\ 2 \end{bmatrix} \Rightarrow A\vec{x}_1 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$,

$= \underset{\text{eigenvalue}}{-1} \underset{\text{eigenvector}}{\begin{bmatrix} -3 \\ 2 \end{bmatrix}}$

(b) $\vec{x}_2 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$, $A\vec{x}_2 = \begin{bmatrix} 7 \\ 10 \end{bmatrix} \neq \lambda \vec{x}_2$ for any $\lambda$.

Not an eigenvector of A

- Find all eigenvectors associated with $\lambda = -1$

We need the solutions to:

$$A\vec{u} = (-1)\vec{u} \rightarrow A\vec{u} - (-1)\vec{u} = \vec{0}$$

$$A\vec{u} - (-1) I_2 \vec{u} = \vec{0} \rightarrow (A - (-1) I_2) \vec{u} = \vec{0}$$

$$\rightarrow A - (-1) I_2 = A + I_2 = \begin{bmatrix} 1+1 & 3+0 \\ 2+0 & 2+1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

$$\rightarrow \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 2 & 3 & 0 \end{array}\right] \underset{\sim}{-R_1 + R_2 \rightarrow R_2} \left[\begin{array}{cc|c} 2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow 2u_1 + 3u_2 = 0 \rightarrow \text{let } u_2 = s_1 \rightarrow u_1 = -\frac{3}{2} s_1$$

General Solution:

$$\vec{v} = s_1 \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$$

- The set $S$ of eigenvectors of matrix $A$ associated with some eigenvalue $\lambda$ and the $\vec{0}$ vector forms the <u>eigenspace of $\lambda$</u> (a subspace).

- <u>Finding eigenvalues</u>: For eigenvectors, we solve $(A - \lambda I)\vec{v} = \vec{0}$. We need to find $\lambda$ so that this has nontrivial solutions. This happens when $|A - \lambda I| = 0$

<u>Example</u>:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \Rightarrow A - \lambda I_2 = \begin{bmatrix} (1-\lambda) & 3 \\ 2 & 2-\lambda \end{bmatrix}$$

$$\longrightarrow |A - \lambda I_2| = (1-\lambda)(2-\lambda) - 6 = 0 \longrightarrow$$

NOTE: the polynomial degree matches the dimension of $A$

$2-\lambda-2\lambda+\lambda^2-6=0 \rightarrow \lambda^2-3\lambda-4=0$

$\rightarrow (\lambda-4)(\lambda+1)=0 \rightarrow \lambda=4$ or $\lambda=-1$

Eigenvectors associated with $\lambda=4$.

$$A-4I_2 \rightarrow \left[\begin{array}{cc|c} -3 & 3 & 0 \\ 2 & -2 & 0 \end{array}\right] \underset{2R_1+R_2\rightarrow R_2}{\overset{\frac{1}{3}R_1\rightarrow R_1}{\sim}} \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\rightarrow -v_1+v_2=0 \rightarrow$ let $v_2=s_1 \rightarrow v_1=v_2=s_1$

$\rightarrow \vec{v}=s_1\begin{bmatrix}1\\1\end{bmatrix} \rightarrow$ Basis for eigenspace $\left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$

Test: $A\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1&3\\2&2\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}4\\4\end{bmatrix}$

- Definitions: $A$ is $n\times n$.

(a) $|A-\lambda I|$ is the characteristic polynomial of $A$. (degree of polynomial $=n$)

(b) $|A-\lambda I|=0$ is the characteristic equation of $A$.

- Example:

Find the eigenvalues and a basis for each eigenspace for:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Solution:

We start with

$$|A - \lambda I_3| = \left| \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix} \right| = \underbrace{-\lambda^3 + 3\lambda + 2}_{\text{Characteristic polynomial}}$$

## $\boxed{\text{Lecture 15 - 10/19/2023}}$

$$= -(\lambda^3 - 3\lambda - 2) \rightarrow 2 = \lambda \text{ gives us } 0 \rightarrow$$

$$-(\lambda - 2)(\lambda^2 + 2\lambda + 1) \rightarrow -(\lambda - 2)(\lambda + 1)^2$$

CONTINUED

Eigenvalues: $\lambda_1 = 2, \lambda_2 = -1$

Eigenvectors: $(A - \lambda I_3)\vec{u} = \vec{0}$

$\lambda_1 = 2$ :

$A - \lambda_1 I_3 \vec{u} = \vec{0}$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

General solution: $\vec{u} = s_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow$ Basis $= \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\lambda_2 = -1$ :

$A - \lambda_2 I_3 \vec{u} = \vec{0}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

General Solution: $\vec{u} = s_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$\Rightarrow \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$

CONTINUED $\longrightarrow$

- Definition: The multiplicity of an eigenvalue $\lambda$ is equal to the number of times $\lambda$ is repeated as a root in the characteristic equation.

- Example:

$$|A-\lambda I| = (\lambda-3)^4(\lambda+1)^2(\lambda-7)$$

$\rightarrow$ $\lambda=3$ has multiplicity of 4.
$\lambda=-1$ has multiplicity of 2.
$\lambda=7$ has multiplicity of 1.

- Theorem: The multiplicity of $\lambda$ is always $\geq$ (greater than or equal to the dimension of the eigenspace associated with $\lambda$.

- The Unifying Theorem - version 8

All of version 7

(m) $\lambda=0$ is not an eigenvalue of A.

- Don't forget the main point about finding eigenvectors and eigenvalues.

  We want $\lambda$ and $\vec{v}$ (all that satisfy):

$$A\vec{v} = \lambda\vec{v}$$

- IMPORTANT TERMINOLOGY:

  - Algebraic multiplicity is the number of times a root is repeated.
  - Geometric multiplicity is the dimension of the eigenspace associated with that root.

- Example: $A = \begin{bmatrix} 15 & -6 & 2 \\ 35 & -14 & 5 \\ 7 & -3 & 2 \end{bmatrix}$

Characteristic polynomial: $(\lambda - 1)^3$

Eigenvalue, 1, has algebraic multiplicity 3

Basis for eigenspace: $\left\{ \begin{bmatrix} -1 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 0 \end{bmatrix} \right\}$

This has geometric multiplicity of 2 which is less than algebraic multiplicity.

# Section 6.2 - Diagonalization

- Note: If

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \vec{u} = \begin{bmatrix} 5 \\ 4 \\ 6 \end{bmatrix} \rightarrow D\vec{u} = \begin{bmatrix} (3)(5) \\ (-1)(4) \\ (2)(6) \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 15 \\ -4 \\ 12 \end{bmatrix}$$

- <u>Definition</u>: An $n \times n$ matrix $A$ is <u>diagonalizable</u> if there exists an $n \times n$ $D$ (diagonal) and $n \times n$ $P$ (invertible) such that:

$$A = PDP^{-1}$$

- Example:

$$A = \begin{bmatrix} -22 & 10 \\ -50 & 23 \end{bmatrix} \rightarrow D = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}, P = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix}$$

CONTINUED →

$$P^{-1} = \frac{1}{(-1)} \begin{bmatrix} 2 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}$$

remember: this is det(P)

$$PDP^{-1} = \begin{bmatrix} 2 & 1 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -2 \\ 15 & -4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -22 & 10 \\ -50 & 23 \end{bmatrix}$$

- Eigenvalues of A:

$$|A - \lambda I_2| = \begin{vmatrix} (-22-\lambda) & 10 \\ -50 & (23-\lambda) \end{vmatrix}$$

$$= (-22-\lambda)(23-\lambda) + 500 = \lambda^2 - \lambda - 6$$

$$= (\lambda - 3)(\lambda + 2)$$

$$\lambda_1 = 3 \quad \lambda_2 = -2$$

CONTINUED →

Eigenvectors for $\lambda_1 = 3$ : $\vec{v}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$

Eigenvectors for $\lambda_2 = -2$ : $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

- <u>Example:</u> Diagonalize $A = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$

<u>Solution:</u> Eigenvalues

$$|A - \lambda I_2| = \begin{vmatrix} (5-\lambda) & -4 \\ 6 & (-5-\lambda) \end{vmatrix} = (5-\lambda)(-5-\lambda)+24$$

$$= \lambda^2 - 1 = (\lambda+1)(\lambda-1) \longrightarrow \lambda_1 = 1, \ \lambda_2 = -1$$

$$D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

<u>Eigenvectors:</u> $(A - \lambda I_2)\vec{v} = \vec{0}$

$$\lambda_1 = 1 \longrightarrow \left[\begin{array}{cc|c} 4 & -4 & 0 \\ 6 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \longrightarrow$$

$$\longrightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = -1 \longrightarrow \left[\begin{array}{cc|c} 6 & -4 & 0 \\ 6 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow v_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\rightarrow P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$

$$A = PDP^{-1} = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$$

- Theorem:

An $n \times n$ matrix $A$ is diagonalizable exactly when $A$ has eigenvectors that form a basis of $\mathbb{R}^n$. Another way to say it is that you need $n$ eigenvectors.

- Remember: Diagonalization is not unique for a matrix. Similar to basis vectors, there are many possibilities.

- Theorem: Eigenvectors associated with distinct eigenvalues are linearly independent.

- Theorem:

An $n \times n$ matrix is diagonalizable when each eigenspace has dimension equal to the multiplicity of the corresponding eigenvalue.

- Special case: If $A$ has all distinct eigenvalues, then $A$ is diagonalizable.

## Lecture 16 - 10/24/2023

- Eigenvectors associated with distinct eigenvalues are linearly independent. (Theorem)

- Example:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow |A - \lambda I_2| = (1-\lambda)^2$$

$\lambda = 1$, multiplicity $= 2$

Eigenvectors for eigenvalue, $\lambda = 1$:

$$(A - \lambda I_2)\vec{x} = \vec{0} \rightarrow \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \end{array} \rightarrow$$

$x_1 =$ free variable $= s_1$

$x_2 =$ leading $= 0$

General solution: $\vec{x} = \begin{bmatrix} s_1 \\ 0 \end{bmatrix} = s_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

eigenspace has dim $= 1$

- $A = \begin{bmatrix} 3 & 6 & 5 \\ 3 & 2 & 3 \\ -5 & -6 & -7 \end{bmatrix}$ $\rightarrow$ $\lambda = -2$ with multiplicity 2

  $\rightarrow$ eigenspace has dimension 1.

  $\rightarrow$ hence, cannot be diagonalized

- Find $A^{10001}$ for $A = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$

<u>Solution</u>: If $A = PDP^{-1}$ then

$$A^2 = AA = (PDP^{-1})(PDP^{-1})$$

$$= PD(P^{-1}P)DP^{-1}$$

$$= PD^2P^{-1}$$

Similarly, $A^3 = AA^2 = (PDP^{-1})(PD^2P^{-1})$

$$= PD^3P^{-1}$$

and so on....

$$A^{10001} = PD^{10001}P^{-1}$$

Now recall:

$$P = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow D^{10001} = \begin{bmatrix} 1^{10001} & 0 \\ 0 & (-1)^{10001} \end{bmatrix} = D$$

$$\rightarrow A^{10001} = PDP^{-1} = A$$

- IMPORTANT:

A defective matrix is a square matrix that does not have a complete basis of eigenvectors and hence, is not diagonalizable.

Example:

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$$

Characteristic polynomial: $(3-\lambda)^2$

Basis for eigenspace: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

Hence, the algebraic and geometric multiplicities do not match.

# Section 6.5 - Approximation Methods

- Let $A = \begin{bmatrix} 10 & -18 \\ 6 & -11 \end{bmatrix} \rightarrow \lambda_1 = -2, \vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\lambda_2 = 1, \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Start with $\vec{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Then:

$$\vec{x}_1 = A\vec{x}_0 = \begin{bmatrix} 10 & -18 \\ 6 & -11 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ -5 \end{bmatrix}$$

$$\vec{x}_2 = A\vec{x}_1 = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$$

$$\vec{x}_3 = A\vec{x}_2 = \begin{bmatrix} -26 \\ -17 \end{bmatrix}$$

$$\vec{x}_4 = A\vec{x}_3 = \begin{bmatrix} 46 \\ 31 \end{bmatrix}$$

Modify the above with a "scaling factor"

$A\vec{x}_0 = \begin{bmatrix} -8 \\ -5 \end{bmatrix} \Rightarrow$ let $s_0 = -8 \Rightarrow$ set $\vec{x}_1$

$$= \frac{1}{s_0} A\vec{x}_0$$

$$= \begin{bmatrix} 1 \\ 0.625 \end{bmatrix}$$

$$A\vec{x}_1 = \begin{bmatrix} -1.25 \\ 0.875 \end{bmatrix} \Rightarrow \text{let } s_1 = -1.25 \Rightarrow$$

$$\vec{x}_2 = \frac{1}{s_1} A\vec{x}_1$$

$$= \begin{bmatrix} 1 \\ .7 \end{bmatrix}$$

$$A\vec{x}_2 = \begin{bmatrix} -2.6 \\ -1.7 \end{bmatrix} \Rightarrow \text{let } s_2 = -2.6 \Rightarrow \vec{x}_3 = \frac{1}{s_2} A\vec{x}_2$$

$$= \begin{bmatrix} 1 \\ 0.654 \end{bmatrix}$$

$$A\vec{x}_3 = \begin{bmatrix} -1.769 \\ -1.192 \end{bmatrix} \Rightarrow \text{let } s_3 = -1.769$$

$$\Rightarrow x_4 = \begin{bmatrix} 1 \\ 0.674 \end{bmatrix}$$

- The Power Method: For each $k \geq 0$,

(a) Let $s_k$ be the largest component of $A\vec{x}_k$ (the scaling factor)

(b) Set $\vec{x}_{k+1} = \frac{1}{s_k} A\vec{x}_k$

- **Definition:** Suppose $A$ has eigenvalues

$$|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \dots \geq |\lambda_n|$$

Then $\lambda_1$ is the *dominant eigenvalue* of $A$.

- **Theorem:** If $A$ has a dominant eigenvalue $\lambda_1$ and corresponding eigenvector $\vec{u}_1$, then for most $\vec{x}_0$, the power method will converge to $\lambda_1$ and a multiple of $\vec{u}$.

# Lecture 17 - 10/31/2023

## Section 8.1 - Dot Products and Orthogonal Sets

- $\vec{u} = \begin{bmatrix} 2 \\ 4 \\ -1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}$

$$\vec{u} \cdot \vec{v} = (2)(5) + (4)(0) + (-1)(2) = 8$$

- <u>Definition</u>:

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

- <u>Notation</u>:

$$\begin{bmatrix} u_1 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \vec{u} \cdot \vec{v} \end{bmatrix}$$

$$\vec{u}\cdot\vec{v} = \vec{u}^T\vec{v}$$

- Properties of dot products:

(a) $\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

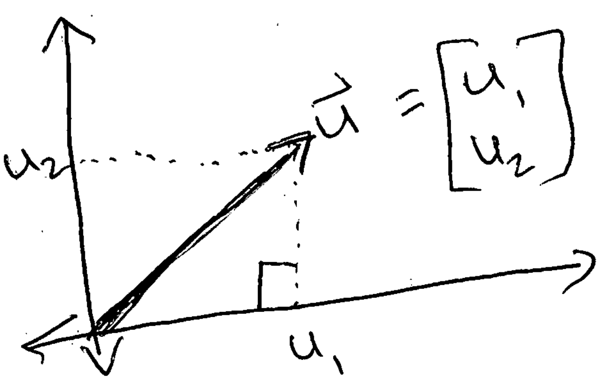
(b) $(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w} + \vec{v}\cdot\vec{w}$

(c) $c\vec{u}\cdot\vec{v} = c\cdot(\vec{u}\cdot\vec{v})$ for scalar $c$

$= \vec{u}\cdot(c\vec{v})$

(d) $\vec{u}\cdot\vec{u} \geq 0$ and $\vec{u}\cdot\vec{u} = 0$ only when $\vec{u} = \vec{0}$

- Example:




length of $\vec{u}$ =

$$\sqrt{u_1^2+u_2^2} = \sqrt{\vec{u}\cdot\vec{u}}$$

- Definition:

The length of a vector ("norm")

is:

$$\|\vec{u}\| = \sqrt{\vec{u}\cdot\vec{u}}$$

- Example:

If $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ 1 \end{bmatrix} \Rightarrow ||\vec{u}|| = \sqrt{1+4+16+1}$

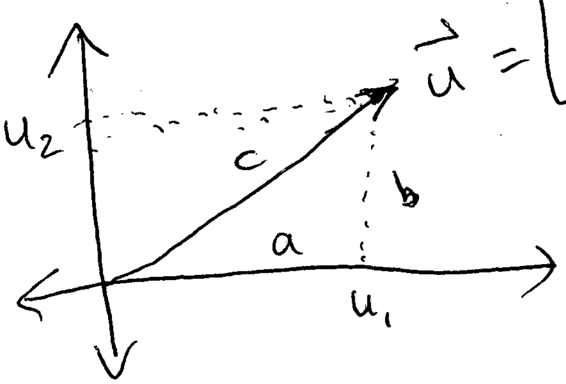
$= \sqrt{22}$

- Definition: $\vec{u}$ and $\vec{v}$ are orthogonal (perpendicular) if $\vec{u} \cdot \vec{v} = \vec{0}$

- Example:

For $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 7 \\ 3 \\ 0 \\ 1 \end{bmatrix}$

$\Rightarrow \vec{u} \cdot \vec{v} = (-1)(7) + (2)(3) + (4)(0) + (1)(1) = -7 + 6 + 1 = 0$

- 


$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Pythagorean theorem:

$a^2 + b^2 = c^2$

**Theorem:** $\vec{u}$ and $\vec{v}$ are orthogonal exactly when

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$$

Proof: $\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v})$

$$= \vec{u}\cdot\vec{u} + \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{u} + \vec{v}\cdot\vec{v} =$$

$$\|\vec{u}\|^2 + \|\vec{v}\|^2 + \underbrace{2(\vec{u}\cdot\vec{v})}$$

0 exactly when $\vec{u}$ and $\vec{v}$ are orthogonal.

- Test case from previous example:

$$\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 7 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

$$\|\vec{u}\|^2 = 22 \qquad \|\vec{v}\|^2 = 59$$

$$22 + 59 = 81$$

$$\vec{u}+\vec{v} = \begin{bmatrix} 6 \\ 5 \\ 4 \\ 2 \end{bmatrix} \Rightarrow \|\vec{u}+\vec{v}\|^2 = 36+25+16+4$$

$$= 81$$

- <u>Definition</u>: A set of vectors $V$ in $\mathbb{R}^n$ is an <u>orthogonal set</u> if $v_i \cdot v_j = 0$ for all $i$ and $j$ where $i \neq j$

- <u>Example</u>: This set is orthogonal:

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

"orthogonal" basis

- <u>Theorem</u>: A set of non-zero orthogonal vectors is linearly independent.

- <u>Theorem</u>: Suppose $S$ has a basis $\{\vec{s}_1, \dots, \vec{s}_K\}$ of orthogonal vectors. Then, every vector $\vec{s}$ in $S$ is equal to

$$\vec{s} = c_1 \vec{s}_1 + c_2 \vec{s}_2 + \dots c_K \vec{s}_K$$

where $c_i = \dfrac{\vec{s} \cdot \vec{s}_i}{\vec{s}_i \cdot \vec{s}_i} = \dfrac{\vec{s} \cdot \vec{s}_i}{\|\vec{s}_i\|^2}$

• Example: Express $\vec{s} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}$ as a linear

combination of this basis:

$$\left\{ \underset{\vec{s}_1}{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}, \underset{\vec{s}_2}{\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}}, \underset{\vec{s}_3}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} \right\}$$

we need:

$$\left.\begin{aligned} \vec{s} \cdot \vec{s}_1 &= 10 - 4 + 3 = 9 \\ \vec{s}_1 \cdot \vec{s}_1 &= 4 + 1 + 1 = 6 \end{aligned}\right\} \rightarrow c_1 = \frac{9}{6}$$

$$\left.\begin{aligned} \vec{s} \cdot \vec{s}_2 &= 6 \\ \vec{s}_2 \cdot \vec{s}_2 &= 3 \end{aligned}\right\} c_2 = \frac{6}{3} = 2$$

$$\left.\begin{aligned} \vec{s} \cdot \vec{s}_3 &= 7 \\ \vec{s}_3 \cdot \vec{s}_3 &= 2 \end{aligned}\right\} c_3 = \frac{7}{2}$$

$$\begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix} = \frac{2}{3} \vec{s}_1 + 2 \vec{s}_2 + \frac{7}{2} \vec{s}_3$$

- Definition:

$S$ is a subset of $\mathbb{R}^n$

(a) A vector $\vec{u}$ in $\mathbb{R}^n$ is orthogonal to $S$ if $\vec{u} \cdot \vec{s} = \vec{0}$ for every vector $\vec{s}$ in $S$

(b) The set of all such vectors above, $\vec{u}$, is the orthogonal complement of $S$.

(Notation: $S^{\perp}$)

- Theorem: If $S$ is a subspace of $\mathbb{R}^n$, then so is $S^{\perp}$.

- Example: Find $S^{\perp}$ for

$$S = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}}_{\vec{s}_1}, \underbrace{\begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}}_{\vec{s}_2} \right\}$$

Solution:

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \Rightarrow$ We need:

$\vec{u} \cdot \vec{s}_1 = 0$

$\vec{u} \cdot \vec{s}_2 = 0$

$u_1 + 3u_2 + 2u_3 = 0$

$2u_1 + 5u_2 + u_3 = 0$

$$\left[\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 2 & 5 & 1 & 0 \end{array}\right] \longrightarrow$$

General solution: $\vec{u} = t_1 \begin{bmatrix} -13 \\ 5 \\ 1 \end{bmatrix}$

"free variable"

$S^{\perp} = \text{span}\left\{ \begin{bmatrix} -13 \\ 5 \\ 1 \end{bmatrix} \right\} \leftarrow$ dimension $= 1$

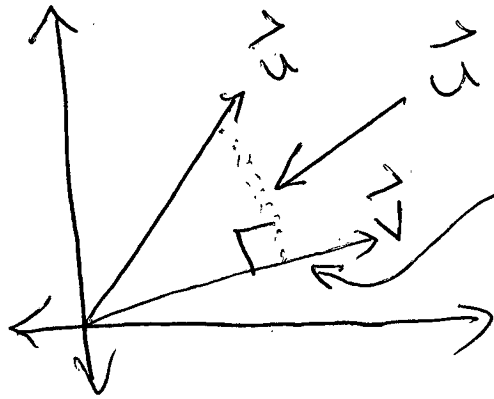
- Remember that $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \right\}$

has dimension 2

In general: $\dim(S) + \dim(S^{\perp}) = n$

from $\mathbb{R}^n$

# Section 8.2 - Projection and Gram-Schmidt Process

- 


Pick a point on the subspace of span $\{\vec{v}\}$ that gets a right angle.

$\text{proj}_{\vec{v}} \vec{u}$ = "The projection of $\vec{u}$ onto $\vec{v}$"

We need: $\text{proj}_{\vec{v}} \vec{u} = c\vec{v}$ for some scalar $c$

Also: $\vec{v} \cdot (\vec{u} - c\vec{v}) = 0$ ← "the right" angle

$$\vec{v} \cdot \vec{u} - c(\vec{v} \cdot \vec{v}) = 0 \rightarrow c = \frac{\vec{v} \cdot \vec{u}}{\vec{v} \cdot \vec{v}} \quad (\vec{v} \neq 0)$$

- $\text{proj}_{\vec{v}} \vec{u} = c\vec{v}$

$$= \left( \frac{\vec{v} \cdot \vec{u}}{\vec{v} \cdot \vec{v}} \right) \vec{v} = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|^2} \vec{v}$$

# Lecture 18 - 11/02/2023

- Example:

$$\vec{u} = \begin{bmatrix} 5 \\ 4 \\ 3 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \operatorname{proj}_{\vec{v}} \vec{u}$$

$$= \frac{(-5 - 4 + 3)}{1+1+1} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

- <u>Definition</u>: Suppose $S$ has orthogonal basis $\{\vec{s}_1, \ldots, \vec{s}_k\}$. Then, the projection of $\vec{u}$ onto $S$ is $\operatorname{proj}_S \vec{u} =$

$$\operatorname{proj}_S \vec{u} = \operatorname{proj}_{\vec{s}_1} \vec{u} + \ldots + \operatorname{proj}_{\vec{s}_k} \vec{u}$$

- Example:

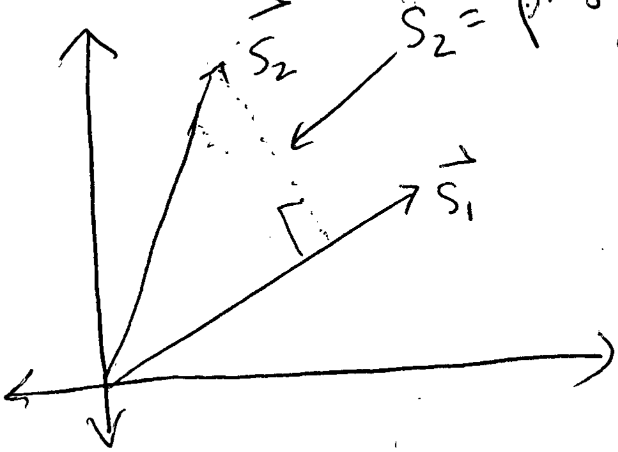
$$\vec{u} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad S = \operatorname{span}\left\{ \underbrace{\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}}_{\vec{s}_1}, \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{\vec{s}_2} \right\}$$

$$\text{proj}_S \vec{u} = \frac{\vec{s}_1 \cdot \vec{u}}{\vec{s}_1 \cdot \vec{s}_1}\vec{s}_1 + \frac{\vec{s}_2 \cdot \vec{u}}{\vec{s}_2 \cdot \vec{s}_2}\vec{s}_2$$

$$= \frac{(9+1+2)}{(9+1+1)}\begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + \frac{(0-1+2)}{(0+1+1)}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 36/11 \\ -13/22 \\ 35/22 \end{bmatrix}$$

- <u>Gram - Schmidt Process</u>




$S = \text{span}\{\vec{s}_1, \vec{s}_2\}$

New basis:

$\vec{v}_1 = \vec{s}_1$

$\vec{v}_2 = \vec{s}_2 - \text{proj}_{\vec{v}_1} \vec{s}_2$

(Then):

(a) $\vec{v}_1 \cdot \vec{v}_2 = 0$

(b) $S = \text{span}\{\vec{v}_1, \vec{v}_2\}$

- Example:

$$S = \text{span}\left\{ \underset{\vec{s}_1}{\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}}, \underset{\vec{s}_2}{\begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix}} \right\}$$

$$\vec{v}_1 = \vec{s}_1 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{v}_2 = \vec{s}_2 - \frac{(\vec{v}_1 - \vec{s}_2)}{(\vec{v}_1 \cdot \vec{v}_1)} \vec{v}_1$$

$$= \begin{bmatrix} 5 \\ -3 \\ -1 \end{bmatrix} - \frac{(14)}{(14)} \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix}$$

$$S = \text{Span}\left\{ \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 1 \end{bmatrix} \right\}$$

- <u>Gram-Schmidt Process</u>:

Suppose $S$ has a basis $\{\vec{s}_1, \ldots, \vec{s}_k\}$

Then we let:

CONTINUED $\longrightarrow$

$$\vec{v}_1 = \vec{s}_1$$

$$\vec{v}_2 = \vec{s}_2 - \text{proj}_{\vec{v}_1} \vec{s}_2$$

$$\vec{v}_3 = \vec{s}_3 - \text{proj}_{\vec{v}_1} \vec{s}_3 - \text{proj}_{\vec{v}_2} \vec{s}_3$$

$$\vdots$$

$$\vec{v}_K = \vec{s}_K - \text{proj}_{\vec{v}_1} \vec{s}_K - \cdots - \text{proj}_{\vec{v}_{K-1}} \vec{s}_K$$

$\Rightarrow \{\vec{v}_1, \ldots, \vec{v}_K\}$ is an orthogonal basis.

- Example:

$$S = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}}_{\vec{s}_1}, \underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}}_{\vec{s}_2}, \underbrace{\begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix}}_{\vec{s}_3} \right\}$$

$\Rightarrow \vec{v}_1 = \vec{s}_1$

$$\Rightarrow \vec{v}_2 = \vec{s}_2 - \frac{(\vec{v}_1 \cdot \vec{s}_2)}{(\vec{v}_1 \cdot \vec{v}_1)} \vec{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \end{bmatrix} - \frac{(2+1+0+0)}{(1+1+0+1)} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{v}_3 = \vec{s}_3 - \frac{(\vec{v}_1 \cdot \vec{s}_3)}{(\vec{v}_1 \cdot \vec{v}_1)} \vec{v}_1 - \frac{(\vec{v}_2 \cdot \vec{s}_3)}{(\vec{v}_2 \cdot \vec{v}_2)} \vec{v}_2$$

$$= \begin{bmatrix} -1 \\ -1 \\ 1 \\ 3 \end{bmatrix} - \left(\frac{3}{3}\right) \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \left(\frac{-3}{3}\right) \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\text{New basis} = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

- Recall: If $\{\vec{v}_1, \ldots, \vec{v}_k\}$ is an orthogonal basis for $S$ and $\vec{s}$ is in $S$ then:

$$\vec{s} = \frac{\vec{v}_1 \cdot \vec{s}_1}{\|\vec{v}_1\|^2} \vec{v}_1 + \ldots + \frac{\vec{v}_k \cdot \vec{s}}{\|\vec{v}_k\|^2} \vec{v}_k$$

New addition: if $\|\vec{v}_i\| = 1$ for all $i$,

$$\vec{s} = (\vec{v}_1 \cdot \vec{s}) \vec{v}_1 + \ldots + (\vec{v}_k \cdot \vec{s}) \vec{v}_k$$

CONTINUED →

A set of vectors $\{\vec{w}_1, \ldots, \vec{w}_k\}$ is <u>orthonormal</u> if they are orthogonal and $\|w_i\| = 1$ for all $i$.

---

<u>Remark</u>: if $\vec{v} \neq \vec{0}$, then

$$\vec{w} = \frac{1}{\|\vec{v}\|}\vec{v} \text{ satisfies } \|\vec{w}\| = 1.$$

- Example:

$$\beta = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

$$\|\vec{v}_1\| = \sqrt{3} \to \vec{w}_1 = \frac{1}{\sqrt{3}}\vec{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \end{bmatrix}$$

$= \vec{v}_1$

$$\vec{w}_2 = \frac{1}{\sqrt{3}}\vec{v}_2 = \begin{bmatrix} 1/\sqrt{3} \\ 0 \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \to \vec{w}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$

$\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an orthonormal basis.

## Section 8.3 - Diagonalizing Symmetric Matrices

- Recall: $A = PDP^{-1}$

  If $A$ is symmetric:

  (a) $A$ is diagonalizable

  (b) It's possible to find an orthogonal set of eigenvectors (and also orthonormal).

- Example:

  $A$ is $3 \times 3$ with $\lambda_1$ (multiplicity 1) which is $\vec{u}_1$ and $\lambda_2$ (multiplicity 2) which is $\vec{u}_2, \vec{u}_3 \rightarrow$ Use Gram-Schmidt to get $\vec{v}_2$ and $\vec{v}_3$ from $\vec{u}_2$ and $\vec{u}_3 \rightarrow \vec{v}_2$ and $\vec{v}_3$ can be converted to orthonormal vectors $\vec{w}_2$ and $\vec{w}_3$.

$\vec{u}_1$ can be $\vec{w}_1$ so we have $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$.

Now let:

$$P = \underbrace{\begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix}}_{\text{"orthogonal Matrix"}}$$

$P^{-1} = P^T \rightarrow$ Diagonalization: $A = PDP^T$

$$P^T P = \underbrace{\begin{bmatrix} \vec{w}_1^{\,T} \\ \vec{w}_2^{\,T} \\ \vec{w}_3^{\,T} \end{bmatrix}}_{P^T} \begin{bmatrix} \vec{w}_1 & \vec{w}_2 & \vec{w}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

## Lecture 19 - 11/09/2023

## Section 7.1 - Vector Spaces and Subspaces

- **Definition**: A vector space consists of a set of vectors $V$ together with operations of addition and scalar multiplication that satisfy:

  (1) if $\vec{V}_1$ and $\vec{V}_2$ are in $V$, then so is $\vec{V}_1 + \vec{V}_2$

  (2) if $\vec{v}_1$ is in $V$, then so is $c\vec{v}_1$ for any real scalars, $c$.

  (3) There is a unique zero vector $\vec{0}$ such that $\vec{v} + \vec{0} = \vec{v}$ for all $\vec{v}$ in $V$.

  (4) For each $\vec{v}$ in $V$, there exists an additive inverse $-\vec{v}$ s.t. $\vec{v} + (-\vec{v}) = \vec{0}$

(5) For all $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $V$, we have

(a) $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$

(b) $(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$

(c) $c(\vec{v}_1 + \vec{v}_2) = c\vec{v}_1 + c\vec{v}_2$

(d) $(c_1 + c_2)\vec{v}_1 = c_1\vec{v}_1 + c_2\vec{v}_2$

(e) $(c_1 c_2)\vec{v}_1 = c_1(c_2\vec{v}_1)$

(f) $1 \cdot \vec{v}_1 = \vec{v}_1$

- Example: $\mathbb{P}^2$ = all real polynomials of degree 2 or less w/ regular addition and scalar multiplication of polynomials.

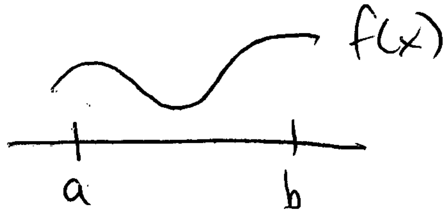
$p(x) = 2x^2 - 7x + 3$

$q(x) = -5x^2 + 2x$

$$p(x) + q(x): \quad \begin{array}{r} 2x^2 - 7x + 3 \\ -5x^2 + 2x + 0 \\ \hline -3x^2 - 5x + 3 \end{array}$$

$$\begin{bmatrix} 2 \\ -7 \\ 3 \end{bmatrix} + \begin{bmatrix} -5 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ -5 \\ 3 \end{bmatrix}$$

• Example: $V = \underbrace{C[a,b]}$, $(a<b)$ → continuous functions on [a, b]




→ Vector

Usual addition and scalar multiplication.

This is a vector space.

• <u>Example:</u> $V = \begin{bmatrix} a \\ b \end{bmatrix}$, a, b, any reals

<u>Scalar multiplication:</u> $c\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} ca \\ cb \end{bmatrix}$

<u>Addition:</u> $\underbrace{\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ b_1 + b_2 \end{bmatrix}}$

↓

No $\vec{0}$ vector.

• <u>Definition:</u> A subset S of a vector space V is a subspace if:

(a) $\vec{0}$ is in S

(b) if $\vec{v}_1, \vec{v}_2$ are in S, then so is $\vec{v}_1 + \vec{v}_2$

(c) if $\vec{v}_1$ is in S, then so is $c\vec{v}_1$ for any real c!

- **Ex:** $V = \mathbb{R}^{3\times 3}$ (3 x 3 matrices with real numbers) with regular (usual) addition and scalar multiplication.

  - $S$ = set of 3x3 diagonal matrices

    Is this a subspace?

    - $\vec{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ← diagonal

    - $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{bmatrix}$ ← closed under addition

    - $4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 12 \end{bmatrix}$ ← closed under multiplication

  - $S$ = 3x3 matrices $A$ with $|A| = 0$

    Is this a subspace?

    - $S$ contains $\vec{0}$

CONTINUED →

- Not closed under Scalar addition

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Downarrow$ det $=0$ $\quad$ $\Downarrow$ det $=0$ $\quad$ $\Downarrow$ det $\neq 0$

- Closed under Scalar multiplication.

- $V = C(\mathbb{R})$ ($\overset{\text{all functions}}{\downarrow}$ continuous on real line)

<u>Example:</u>

- $S =$ differentiable functions on $\mathbb{R}$ (by definition is continuous)

Is this a subspace?

- $\vec{0}$ in $S$
- Closed under addition as adding continuous function still makes you differentiable?
- Closed under multiplication as multiplying continuous function by constant doesn't change differentiability.

- Remark

(1) $\mathbb{R}^3$ is not a subspace of $\mathbb{R}^4$

(2) $\mathbb{P}^2$ is a subspace of $\mathbb{P}^3$.

# Lecture 20 - 11/14/2023

## Section 7.2 - Span and Linear Independence

- Definition: Let $\{\vec{v}_1, \ldots, \vec{v}_m\}$ be vectors in $V$. The span of this set is the collection of all linear combinations.

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots c_m \vec{v}_m$$

for any real scalars.

- Example: Let

$$S = \text{span}\left\{\begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix}\right\}$$

Is $\vec{v}$ in $S$?

$$\vec{v} = \begin{bmatrix} -7 & 0 \\ -2 & 1 \end{bmatrix}$$

Solution: do we have scalars $c_1, c_2$ s.t.

$$c_1\begin{bmatrix} -3 & 1 \\ 0 & 2 \end{bmatrix} + c_2\begin{bmatrix} 1 & 2 \\ -2 & 5 \end{bmatrix} = \begin{bmatrix} -7 & 0 \\ 2 & 1 \end{bmatrix}$$

$-3c_1 + c_2 = -7$ this works

$c_1 + 2c_2 = 0$

$-2c_2 = 2 \longrightarrow c_2 = -1$

$2c_1 + 5c_2 = 1$ this does not work.

Hence, $\vec{v}$ is not in $S$

- <u>Theorem:</u> $S = \operatorname{span}\{\vec{v}_1, \ldots, \vec{v}_n\}$ is always a subspace.

- <u>Example:</u> The set $\{1, x, x^2, x^3\}$ Spans $\mathbb{P}^3$ (degree 3 polynomiar or lower degree) ;

In general: $p(x) = c_0 \cdot 1 + c_1 x + c_2 x^2 + c_3 x^3$

- <u>Definition:</u> The set $\{\vec{v}_1, \ldots, \vec{v}_m\}$ is <u>linearly independent</u> if

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m = \vec{0} \text{ only}$$

when $c_1 = c_2 = \ldots = c_m = 0$. otherwise, the set is linearly dependent.

- **Example:** Is $\{x^2-4,\ 3x+2,\ x^2+2x\}$ linearly independent?

**Solution:** We start with

$$c_1(x^2-4)+c_2(3x+2)+c_3(x^2+2x)=0$$

$$\to \underbrace{(c_1+c_3)}x^2+\underbrace{(3c_2+2c_3)}x+\underbrace{(-4c_1+2c_2)}=0$$

These all need to be zero

$$\left.\begin{aligned} c_1 + \phantom{3c_2 +} c_3 &= 0 \\ 3c_2+2c_3 &= 0 \\ -4c_1+2c_2 \phantom{+2c_3} &= 0 \end{aligned}\right\}$$ only solution is the trivial one $(c_1=c_2=c_3=0)$

- **Example:** Is this set linearly independent?

$$\{x,\ \sin(x),\ \cos(x)\}$$

**Solution:** Start with

$$c_1x+c_2\sin(x)+c_3\cos(x)=0$$

Now let's try specific choices for $x$:

$$\left.\begin{array}{ll} \text{For } x = 0: & c_3 = 0 \\ \text{For } x = \frac{\pi}{2}: & \frac{\pi}{2} c_1 + c_2 = 0 \\ \text{For } x = \pi: & \pi c_1 - c_3 = 0 \end{array}\right\} \quad c_1 = c_2 = c_3 = 0$$

$\Rightarrow$ Linearly independent.

- Theorem: The set $\{\vec{v}_1, \dots, \vec{v}_m\}$ is linearly dependent exactly when some vector in the set is in the span of the others.

## Section 7.3 – Basis and Dimension

- Definition: $\{\vec{v}_1, \dots, \vec{w}_m\}$ is a basis for $V$ if the set spans $V$ and is linearly independent.

- Example: $\{1, x, x^2, x^3\}$ is a basis of $\mathbb{P}^3$.

$$p(x) = c_0 1 + c_1 x + c_2 x^2 + c_3 x^3$$

Linear independence:

$$c_0(1) + c_1 x + c_2 x^2 + c_3 x^3 = 0$$

This only happens when $c_1 = c_2 = c_3 = 0$.

NOTE: $\{1, x, x^2, x^3\}$ is a <u>standard basis.</u>

- Example: This is the standard basis for $\mathbb{R}^{2\times 2}$:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

<u>Remark:</u>

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

- <u>Definition</u>: Every basis for $V$ has the same number of vectors, called the <u>dimension</u>. ($\dim(V)$)

NOTE: If $S = \{\vec{0}\}$, then $S$ does not have a basis and $\dim(S) = 0$.

- <u>Example</u>: Let $\mathbb{P}$ = set of all polynomials with real coefficients of any degree.

$\Rightarrow \{1, x, x^2, x^3, x^4, x^5, \ldots\}$ is a basis for $\mathbb{P}$. $\Rightarrow \dim(P) = \infty$

<u>NOTE</u>: Subspaces also have bases and dimension.

- <u>Example</u>: $\mathbb{P}^n$: $\{1, x, x^2, x^3, \ldots, x^n\}$

$\Rightarrow \dim(\mathbb{P}^n) = n+1$

- <u>Example</u>: $\mathbb{R}^{n\times m}$:

$$\left\{ \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & & & \\ \vdots & & & \ddots & & \\ 0 & \cdots & & & & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & & \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 0 \end{bmatrix} \cdots \right\} \Rightarrow$$

$\dim(\mathbb{R}^{n\times m}) = nm$

- Theorem: $V_1, V_2$ are vector spaces w/
$$V_1 \subseteq V_2 \Rightarrow \dim(V_1) \leq \dim(V_2)$$

- Example: $V = C(\mathbb{R})$ (continuous functions on $\mathbb{R}$ line)

A basis for $V$ is hard to find, but:

$$\mathbb{P} \subseteq V \Rightarrow \underbrace{\dim(\mathbb{P})}_{\infty} \leq \underbrace{\dim(V)}_{\infty}$$

- Theorem: $\{\vec{v}_1, \ldots, \vec{v}_m\}$ in $V$, $\dim(V) = n$

(a) If $m < n$, then the set does not span $V$.

(b) If $m > n$, then the set is not linearly independent.

(c) If $m = n$, then the set spans and is linearly independent or neither is true.

# Lecture 21 - 11/16/2023

## Section 10.1 - Inner Products

"Inner product space"

- <u>Definition</u>: Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $V$.

An <u>inner product</u> $\langle \ , \ \rangle \longrightarrow \mathbb{R}$ is a function that satisfies

(a) $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$

(b) $\langle \vec{u}+\vec{v}, \vec{w} \rangle = \langle \vec{u}+\vec{w} \rangle + \langle \vec{v}+\vec{w} \rangle$

(c) $\langle c\vec{u}, \vec{v} \rangle = \langle \vec{u}, c\vec{v} \rangle = c \langle \vec{u}, \vec{v} \rangle$

(d) $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0$ only if $\vec{u} = \vec{0}$

- <u>Example</u>: the <u>dot product</u> on $\mathbb{R}^n$ is an inner product.

- <u>Example</u>: $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \vec{v} = \begin{bmatrix} d \\ e \\ f \end{bmatrix} \Rightarrow \langle \vec{u}, \vec{v} \rangle =$

$3ad + 2be + 13cf$

• Example:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad B = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \Rightarrow \langle A, B \rangle$$

$$= ae + 2bf + 5cg + 3dh$$

• Example: $A, B$ are in $\mathbb{R}^{n \times n}$

$$\langle A, B \rangle = \underbrace{tr(A^T B)}_{\text{an inner product}}$$

trace = "sum of diagonal" terms

• Definition: $\vec{u}$ and $\vec{v}$ are orthogonal if

$$\langle \vec{u}, \vec{v} \rangle = 0$$

Also $\|\vec{u}\| = \sqrt{\langle \vec{u}, \vec{u} \rangle}$

• Example: $A = \begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}$

$$\langle A, B \rangle = tr(A^T B) \rightarrow \|A\| = \sqrt{\langle A, A \rangle}$$

$$\cancel{= \sqrt{(A^T A)\,\text{trace}} = \sqrt{\begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}}\,\text{trace}}$$

$$= \sqrt{(A^T A)\,\text{trace}} = \sqrt{\text{trace}\left(\begin{bmatrix} 2 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -2 & 0 \end{bmatrix}\right)} =$$

$$\sqrt{\text{trace}\left(\begin{bmatrix} 8 & 2 \\ 2 & 1 \end{bmatrix}\right)} = \sqrt{9} = 3 = \|A\|$$

- <u>Example</u>: Let $p(x)$, $g(x)$ be in $\mathbb{P}$ (family of all polynomials). Then,

$$\langle p, q \rangle = \int_0^1 p(x)\, g(x)\, dx$$

is an inner product on $\mathbb{P}$.

- <u>Example</u>: $p(x) = x$, $q(x) = 3x - 2$

$$\Rightarrow \langle p, q \rangle = \int_0^1 x(3x-2)\, dx$$

$$= \int_0^1 3x^2 - 2x\, dx = \left[ x^3 - x^2 \right|_0^1 = (1-1) - (0-0) = 0$$

$\Rightarrow p(x)$, $g(x)$ are orthogonal

- <u>Theorem</u>: (Pythagorous) if $\vec{u}, \vec{v}$ are orthogonal,

then: $$\|\vec{u}\|^2 + \|v\|^2 = \|\vec{u}+\vec{v}\|^2$$

- (From previous example):

$$\|p\|^2 = \langle p, p \rangle = \int_0^1 x^2 dx = \left[ \tfrac{1}{3}x^3 \right|_0^1 = \frac{1}{3}$$

$$\|q\|^2 = \langle q, q \rangle = \int_0^1 (3x-2)^2 dx =$$

$$\frac{1}{3}(3x-2)^3 \cdot \frac{1}{3} \Big|_0^1 = \frac{1}{9}\left[ (3x-2)^3 \right|_0^1 =$$

$$\frac{1}{9}(1^3 - (-2)^3) = 1$$

$$\|p+q\|^2 = \int_0^1 (4x-2)^2 dx = \left[ \tfrac{1}{12}(4x-2)^3 \right|_0^1 =$$

$$\frac{1}{12}(2^3 - (-2)^3) = \frac{16}{12} = \frac{4}{3}$$

$$\|p+q\|^2 = \|p\| + \|q\| \rightarrow \frac{4}{3} = 1 + \frac{1}{3}$$

- Remember from chapter 8:

$$\text{Proj}_{\vec{v}}\vec{u} = \left( \frac{\vec{v}\cdot\vec{u}}{\vec{v}\cdot\vec{v}} \right) \vec{v}$$

- Now based on this information, more generally the projection is:

$$\text{proj}_{\vec{v}} \vec{u} = \frac{\langle \vec{v}, \vec{u} \rangle}{\langle \vec{v}, \vec{v} \rangle} \vec{v}$$

- <u>Example:</u> $p(x) = e^x$, $q(x) = x$ with

$$\langle p, q \rangle = \int_0^1 p(x)\, q(x)\, dx$$

<u>Find $\text{proj}_p q$</u>

<u>Solution:</u> $\langle p, q \rangle = \int_0^1 e^x x\, dx \rightarrow$

$\int u\,dv = uv - \int v\,du \rightarrow u = x,\ dv = e^x dx$

$\rightarrow du = dx,\ v = e^x \rightarrow \int x e^x dx =$

$\left[x e^x\right|_0^1 - \int_0^1 e^x dx = \left[x e^x\right|_0^1 - \left[e^x\right|_0^1$

$= (e - 0) - (e^1 - 1) = 1$

$\langle p, p \rangle = \int_0^1 e^{2x} dx = \left[\frac{1}{2} e^{2x}\right|_0^1 = \frac{1}{2} e^2 - \frac{1}{2}$

$$\mathrm{proj}_p\, q = \frac{1}{\frac{1}{2}e^2 - \frac{1}{2}} e^x$$

- Cauchy-Schwarz Inequality:

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \, \|\vec{v}\|$$

- Triangle Inequality:

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$