Homework 5 – Section 4.2

(3) No as 3 vectors in $\mathbb{R}^2$ are not a linearly independent set of vectors.

(5) $\begin{bmatrix} 1 & -4 \\ -5 & 20 \end{bmatrix} \underset{\sim}{5R_1+R_2} \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix} \rightarrow$ β for $S = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix} \right\}$

dimension 1

(7) $\begin{bmatrix} 1 & 3 & -2 \\ 2 & 4 & 1 \\ -1 & 1 & -8 \end{bmatrix} \underset{\sim}{\begin{matrix} -2R_1+R_2 \to R_2 \\ R_1+R_3 \to R_3 \end{matrix}} \begin{bmatrix} 1 & 3 & -2 \\ 0 & -2 & 5 \\ 0 & 4 & -10 \end{bmatrix} \underset{\sim}{2R_2+R_3 \to R_3}$

$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -2 & 5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow$ β for $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix} \right\}$, dimension = 2

(11) $\begin{bmatrix} 1 & 4 \\ 3 & -12 \end{bmatrix} \underset{\sim}{-3R_1+R_2 \to R_2} \begin{bmatrix} 1 & 4 \\ 0 & -24 \end{bmatrix} \rightarrow$ β for $S = \left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -12 \end{bmatrix} \right\}$

dimension = 2

(15) $\begin{bmatrix} 1 & 2 & 0 \\ -1 & -5 & 1 \\ 0 & 9 & -3 \\ 2 & 7 & -1 \end{bmatrix} \underset{\sim}{\begin{matrix} R_1+R_2 \to R_2 \\ -2R_1+R_4 \to R_4 \end{matrix}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & 9 & -3 \\ 0 & 3 & -1 \end{bmatrix} \underset{\sim}{\begin{matrix} 3R_2+R_3 \to R_3 \\ R_2+R_4 \to R_4 \end{matrix}}$

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \beta \text{ for } S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 9 \\ 7 \end{bmatrix} \right\}$$

dimension = 2

(19) β for $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

dimension = 1

(31) $A = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & -3 \end{bmatrix} \rightarrow$ let $x_4 = s_1 \rightarrow x_3 - 3s_1 = 0$

$\rightarrow x_3 = 3s_1 \rightarrow$ let $x_2 = s_2 \rightarrow x_1 + s_2 + 2(3s_1) + s_1 = 0$

$\rightarrow x_1 = -s_2 - 7s_1 \rightarrow \text{null}(A) = \begin{bmatrix} -7s_1 - s_2 \\ s_2 \\ 3s_1 \\ s_1 \end{bmatrix} \rightarrow$

$\text{null}(A) = s_1 \begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \text{null}(A) = \text{span} \left\{ \begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$\rightarrow$ As all vectors are linearly independent in null(A) $\Rightarrow$

β of $\text{null}(A) = \left\{ \begin{bmatrix} -7 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \rightarrow$ nullity(A) = 2

(33) $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$

(37) $S_1 = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$S_2 = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$S_1 \cap S_2 = \{\vec{0}\}$

(43) (a) False as $s_1 + s_2$ would ~~be~~ make the basis set linearly dependent.

(b) True as a proper subspace of ~~a~~ a 1-D subspace is a 0-dimensional subspace, which is $\{\vec{0}\}$.

(45) (a) False, for a set of vectors to span the subspace means that either ~~[illegible]~~ ~~[illegible]~~ the vectors are linearly independent and no more can be added or that it is linearly dependent and hence, adding more vectors would not create a basis.

(b) True as it could be that there are vectors that could be added to span the subspace.

(47)

(a) False as they do not span the space of $\mathbb{R}^3$.

(b) True as there is nothing exactly in between 3 and 4 dimensions

Homework 5 - Section 4.3

(3) β for $\text{col}(A) = \left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 4 \\ -4 \end{bmatrix} \right\}$

β for $\text{row}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

null(A): $x_4 = 0 \rightarrow$ let $x_3 = s_1 \rightarrow x_2 - (s_1) = 0 \rightarrow$

$x_2 = s_1 \rightarrow x_1 + 2s_1 = 0 \rightarrow x_1 = -2s_1$

$\vec{x} = s_1 \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow$ β for $\text{null}(A)$: $\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$

rank(A) + nullity(A) = # columns of A

$3 + 1 = 4$

(7) $A = \begin{bmatrix} 1 & 3 & 2 & 0 \\ 3 & 11 & 7 & 1 \\ 1 & 1 & 4 & 0 \end{bmatrix} \begin{matrix} -3R_1 + R_2 \rightarrow R_2 \\ -R_1 + R_3 \rightarrow R_3 \\ \sim \end{matrix} \begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & -2 & 2 & 0 \end{bmatrix} \begin{matrix} R_2 + R_3 \rightarrow R_3 \\ \sim \end{matrix}$

$$\begin{bmatrix} 1 & 3 & 2 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 3 & 1 \end{bmatrix} \rightarrow \beta \text{ for } col(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 11 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 4 \end{bmatrix} \right\}$$

$$\rightarrow \beta \text{ for } row(A) = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\} \rightarrow$$

$$null(A) = \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 3 & 1 & 0 \end{array}\right] \rightarrow \text{let } x_4 = s_1 \rightarrow 3x_3 + s_1 = 0$$

$$\rightarrow x_3 = -\tfrac{1}{3}s_1 \rightarrow 2x_2 + (-\tfrac{1}{3}s_1) + s_1 = 0$$

$$\rightarrow x_2 = \frac{-\frac{2}{3}s_1}{2} \rightarrow x_2 = -\frac{1}{3}s_1 \rightarrow$$

$$x_1 + 3(-\tfrac{1}{3}s_1) + 2(-\tfrac{1}{3}s_1) = 0 \rightarrow x_1 = \tfrac{5}{3}s_1 \rightarrow$$

$$\vec{x} = s_1 \begin{bmatrix} 5/3 \\ -1/3 \\ -1/3 \\ 1 \end{bmatrix} \rightarrow \beta \text{ for } null(A) = \begin{bmatrix} 5/3 \\ -1/3 \\ -1/3 \\ 1 \end{bmatrix}$$

rank $(A) = 3$, nullity $(A) = 1$

$3+1=4$ = # columns of A

(9) $x \neq 8, \ \forall x \in \mathbb{R}$

(13) $A \in \mathbb{R}^{6\times 8} \wedge \dim(\text{row}(A)) = 5$

$\dim(\text{col}(A)) = 5$

(15) $A \in \mathbb{R}^{4\times 7} \wedge \beta$ (echelon form) has one row of zeros.

$\text{row}(A) = \text{col}(A) = 3$ ~~$\rightarrow$~~ $\text{nullity}(A) + 3 = 7 \rightarrow$

~~$\rightarrow$~~ $\text{nullity}(A) = 4$

(19) $A \in \mathbb{R}^{7\times 11} \wedge \text{rank}(A) = 4$

$\text{nullity}(A) + 4 = 11 \rightarrow \text{nullity}(A) = 7$

(23) $A \in \mathbb{R}^{13\times 5} \wedge T(\vec{x}) = A\vec{x} \wedge T$ is one-to-one

$\dim(A) = 5 \rightarrow$ ~~$\rightarrow$~~ $5 + \text{nullity}(A) = 5 \rightarrow$

$\text{nullity}(A) = 0$

(27) $A \in \mathbb{R}^{9\times 5} \wedge \beta = \text{ref}(A) \wedge \beta$ has 3 non-zero rows $\rightarrow \text{rank}(A) = 3$

(31) B has 3 non-zero rows.

(35) A s.t. $A \in \mathbb{R}^{2\times 3} \wedge \text{nullity}(A) = 1$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(43) $A \in \mathbb{R}^{3\times 3}$ s.t. $\text{null}(A)$ is a plane

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$$

$$A \underset{\sim}{\overset{-R_1+R_2 \to R_2}{\overset{-R_1+R_3 \to R_3}{}}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \text{let } x_3 = s_1$$

$\to$ let $x_2 = s_2 \to x_1 + 2s_2 + 3s_1 = 0 \to$

$$x_1 = -2s_2 - 3s_1 \to \vec{x} = s_1 \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

(47) (a) True as the number of non-zero rows in the echelon form of A equals the number of pivots.

(b) False as the order of the components also matters.

(5) (a) $A \in \mathbb{R}^{9 \times 5} \wedge T(\vec{x}) = A\vec{x}$

Not possible as ref(A) does not span $\mathbb{R}^9$

(b) $A \in \mathbb{R}^{9 \times 5} \wedge T(\vec{x}) = A\vec{x}$

T could be one-to-one if the 5 columns of ref(A) contain pivots.