Graded Homework 8 - Section 8.1

① ⓐ $\vec{u}_1 \cdot \vec{u}_5 = (-3)(2) + (1)(1) + (2)(1) = -3$

ⓑ $\vec{u}_3 \cdot (-3\vec{u}_2) = -3[(2)(1) + (1)(0) + (-1)(1)] = -3$

ⓒ $\vec{u}_4 \cdot \vec{u}_7 = (1)(3) + (-4)(-3) + (-2)(2) = 11$

ⓓ $2\vec{u}_4 \cdot \vec{u}_7 = 2 \cdot 11 = 22$

③ ⓐ $\|\vec{u}_7\| = \sqrt{(3)^2 + (-4)^2 + (-2)^2} = \sqrt{29}$

ⓑ $\|-\vec{u}_7\| = |-1| \, \|\vec{u}_7\| = \|\vec{u}_7\| = \sqrt{29}$

ⓒ $\|2\vec{u}_5\| = 2\|\vec{u}_5\| = 2\left[\sqrt{(2)^2+(1)^2+(1)^2}\right] = 2\sqrt{6}$

ⓓ $\|-3\vec{u}_5\| = |-3| \, \|\vec{u}_5\| = 3\|\vec{u}_5\| = 3\sqrt{6}$

⑤ ⓐ $\|\vec{u}_1 - \vec{u}_2\| = \left\| \begin{matrix} -4 \\ 0 \\ 1 \end{matrix} \right\| = \sqrt{(-4)^2+(0)^2+(1)^2} = \sqrt{17}$

ⓑ $\|\vec{u}_3 - \vec{u}_8\| = \left\| \begin{matrix} 3 \\ 1 \\ -4 \end{matrix} \right\| = \sqrt{(3)^2 + (1)^2 + (-4)^2}$

$= \sqrt{26}$

ⓒ $\|2\vec{u}_6 - (-\vec{u}_3)\| = \left\| \begin{matrix} 2 \\ 6 \\ -3 \end{matrix} \right\| = \sqrt{(2)^2+(6)^2+(-3)^2}$

$= 7$

(d) $\|-3\vec{u}_2 - 2\vec{u}_5\| \rightarrow -3\vec{u}_2 - 2\vec{u}_5 =$

$$-3\begin{bmatrix}1\\1\\1\end{bmatrix} - 2\begin{bmatrix}2\\1\\1\end{bmatrix} = \begin{bmatrix}-7\\-5\\-5\end{bmatrix} \rightarrow \left\|\begin{matrix}-7\\-5\\-5\end{matrix}\right\| =$$

$$\sqrt{(-7)^2+(-5)^2+(-5)^2} = \sqrt{99}$$

(7) (a) $\vec{u}_1 \cdot \vec{u}_3 = (-3)(2) + (1)(0) + (2)(-1)$

$= -8 \rightarrow$ hence, not orthogonal

(b) $\vec{u}_3 \cdot \vec{u}_4 = (2)(1) + (0)(-3) + (-1)(2) = 0 \Rightarrow$ orthogonal

(c) $\vec{u}_2 \cdot \vec{u}_5 = (1)(2) + (1)(1) + (1)(1) = 4 \rightarrow$ not orthogonal

(d) $\vec{u}_1 \cdot \vec{u}_8 = (-3)(-1) + (1)(-1) + (2)(3) = 8 \rightarrow$ not orthogonal

(11) $u = \begin{bmatrix}2\\a\\-3\\-1\end{bmatrix}, v = \begin{bmatrix}-5\\4\\6\\a\end{bmatrix} \rightarrow$ find a s.t. $u \cdot v = 0 \wedge u \cdot v$ orthogonal

$u \cdot v = (2)(-5) + (4)(a) + (6)(-3) + (-1)(a) = 3a - 28 \rightarrow$

$3a - 28 = 0 \rightarrow a = \frac{28}{3}$

(15) $\vec{u}_1 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ $\vec{u}_2 = \begin{bmatrix} -5 \\ 13 \\ 16 \end{bmatrix}$ $\vec{u}_3 = \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}$ $\rightarrow$

does $\{ \vec{u}_1, \vec{u}_2, \vec{u}_3 \}$ form an orthogonal basis?

$\vec{u}_1 \cdot \vec{u}_2 = \vec{u}_2 \cdot \vec{u}_3 = \vec{u}_1 \cdot \vec{u}_3 = 0$ must hold.

$\vec{u}_1 \cdot \vec{u}_2 = (2)(-5) + (2)(13) + (-1)(16) = 0$

$\vec{u}_2 \cdot \vec{u}_3 = (-5)(5) + (13)(-4) + (16)(2) = -45$

Hence, they do not form an orthogonal basis.

(17) $\vec{u}_1 = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 6 \\ a \\ 3 \end{bmatrix}$

$\vec{u}_1 \cdot \vec{u}_3 = 0 \wedge \vec{u}_2 \cdot \vec{u}_3 = 0 \rightarrow$

$(-1)(6) + 0a + (2)(3) = 0 \rightarrow a = \mathbb{R} \rightarrow$

$(4)(6) + 3a + (2)(3) = 0 \rightarrow a = -10 \rightarrow$

$\vec{u}_3 = \begin{bmatrix} 6 \\ -10 \\ 3 \end{bmatrix}$

(23) $\|u+v\|^2 = \|u\|^2 + \|v\|^2$, $\vec{u}_1 = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$

$(\vec{u_1+u_2}) = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix} \to \|(\vec{u_1+u_2})\|^2 = (6)(6) + (0)(0) + (2)(2)$

$= 40 \to \|\vec{u}_1\|^2 = (2)(2) + (-3)(-3) + (1)(1) = 14 \to$

$\|\vec{u}_2\|^2 = (4)(4) + (3)(3) + (1)(1) = 26 \to$

$\|\vec{u+v}\|^2 = \|\vec{u}\|^2 + \|v\|^2 \to$ $40 = 26+14$ ✓

(25) $\vec{u}_1 \cdot \vec{u}_2 = 0$, $\|\vec{u}_1\| = 2$, $\|\vec{u}_2\| = 5$, find

$\|\vec{3u_1+4u_2}\| \to \|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$

$\to \|\vec{3u_1+4u_2}\|^2 = \|3\vec{u}_1\|^2 + \|4\vec{u}_2\|^2$

$\to \|\vec{3u_1+4u_2}\| = \sqrt{\|3\vec{u}_1\|^2 + \|4\vec{u}_2\|^2}$

$= \sqrt{[(3)(2)]^2 + [(4)(5)]^2} = \sqrt{436} = \|\vec{3u_1+4u_2}\|$

(31) $\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, $S = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \right\}$

Is $\vec{u}$ in $S^\perp$?

$\vec{u} \cdot \vec{s}_1 = (2)(1) + (-3)(2) + (1)(-1) = -3$

$\vec{u}$ is not orthogonal to S.

(35) $S = \text{span}\left\{\begin{bmatrix}1\\1\\-2\end{bmatrix}\right\}$, β for $S^{\perp}$

$\rightarrow$ need to satisfy $(1)(x_1) + (1)(x_2) + (-2)(x_3) = 0$

$\rightarrow \left[\begin{array}{ccc|c}1 & 1 & -2 & 0\end{array}\right] \rightarrow$ let $x_3 = s_1 \rightarrow$ let $x_2 = s_2$

$\rightarrow x_1 + s_2 - 2s_1 = 0 \rightarrow x_1 = 2s_1 - s_2 \rightarrow$

$$\vec{x} = \begin{bmatrix}2s_1 - s_2\\ s_2\\ s_1\end{bmatrix} = s_1\begin{bmatrix}2\\0\\1\end{bmatrix} + s_2\begin{bmatrix}-1\\1\\0\end{bmatrix} \rightarrow$$

β for $S^{\perp}$: $\left\{\begin{bmatrix}2\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$

(37) $S = \text{span}\left\{\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-1\\4\end{bmatrix}\right\}$, $\vec{s} = \begin{bmatrix}1\\2\\-2\end{bmatrix}$

$(1)(1) + (1)(-1) + (0)(4) = 0 \rightarrow$

basis for S is orthogonal $\rightarrow \vec{s} = c_1\vec{s}_1 + c_2\vec{s}_2$

$$\rightarrow c_1 = \frac{\vec{s}_1 \cdot \vec{s}}{\|\vec{s}_1\|^2} = \frac{(1)(1) + (1)(2) + (0)(-2)}{(1)^2 + (1)^2 + (0)^2}$$

$$= \frac{3}{2}$$

$$c_2 = \frac{\vec{s}_2 \cdot \vec{s}}{\|\vec{s}_2\|^2} = \frac{(1)(1) + (-1)(2) + (4)(-2)}{(1)^2 + (-1)^2 + (4)^2}$$

$= -\frac{1}{2} \rightarrow \vec{s} = \frac{3}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\-1\\4\end{bmatrix}$

(41) $\vec{u}$ s.t. $\sqrt{u^T \cdot u} = 1 \wedge \vec{u} \cdot \begin{bmatrix} -2 \\ 1 \end{bmatrix} = 0$

$\rightarrow \sqrt{1} = 1$ so $u_1^2 + u_2^2 = 1 \rightarrow \begin{bmatrix} -2 & 1 & | & 0 \end{bmatrix}$

$\rightarrow$ let $x_2 = s_1 \rightarrow -2x_1 + s_1 = 0 \rightarrow x_1 = \frac{1}{2} s_1 \rightarrow$

span $\left\{ \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right\} \rightarrow (\frac{1}{2} s_1)^2 + (s_1)^2 = 1 \rightarrow$

$\frac{5}{4} s_1^2 = 1 \rightarrow s_1^2 = 4/5 \rightarrow s_1 = \pm\sqrt{4/5} \rightarrow$

$$\vec{u} = \begin{bmatrix} \frac{1}{2}\sqrt{4/5} \\ \sqrt{4/5} \end{bmatrix}$$

(49) ⓐ False, $\|\vec{u-v}\| = 3 \rightarrow \|2\vec{u} - 2\vec{v}\| =$

$\sqrt{(2\vec{u}-2\vec{v}) \cdot (2\vec{u}-2\vec{v})} = \sqrt{4\vec{u}^2 - 8\vec{v}\vec{u} + 2\vec{v}^2} \neq 4\sqrt{\vec{u}^2 - 2\vec{u}\vec{v} + \vec{v}^2}$

ⓑ True as the multiplication and addition of non-negative numbers will be equal or greater than zero.

(53) True

ⓐ $\|\vec{u} - \vec{v}\| = \sqrt{\vec{u}^2 - 2\vec{u}\vec{v} + \vec{v}^2} = \sqrt{\vec{u}^2 + \vec{v}^2}$ if $u \cdot v = 0 \rightarrow$

$\sqrt{\|\vec{u}\|^2 + \|\vec{v}\|^2}$

ⓑ True $\|\vec{u} - \vec{v}\| = \sqrt{\vec{u}^2 - 2\vec{u}\vec{v} + \vec{v}^2}$, $\|\vec{u} + \vec{v}\| = \sqrt{\vec{u}^2 + 2\vec{u}\vec{v} + \vec{v}^2} \rightarrow$

$\|\vec{u} - \vec{v}\| = \|u + v\|$ only when $\vec{u} \cdot \vec{v} = \vec{0}$

Graded Homework 8 - Section 8.2

① ⓐ $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\text{proj}_{\vec{u}_3} \vec{u}_2 =$

$$\frac{\vec{u}_3 \cdot \vec{u}_2}{\|\vec{u}_3\|^2} \vec{u}_3 = \frac{1}{5}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 0 \\ -1/5 \end{bmatrix}$$

ⓑ $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{u}_1 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}$, $\text{proj}_{\vec{u}_1} \vec{u}_2 = \frac{\vec{u}_1 \cdot \vec{u}_2}{\|\vec{u}_1\|^2} \vec{u}_1$

$$= \frac{0}{3}\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} = \vec{0}$$

③ $\text{proj}_S \vec{u}_2$, $S = \text{span}\left\{ \underset{\vec{u}_3}{\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}}, \underset{\vec{u}_4}{\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}} \right\}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$\text{proj}_S \vec{u}_2 = \frac{\vec{u}_3 \cdot \vec{u}_2}{\|\vec{u}_3\|^2} \cdot \vec{u}_3 + \frac{\vec{u}_4 \cdot \vec{u}_2}{\|\vec{u}_4\|^2} \vec{u}_4 =$$

$$\frac{1}{5}\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} + \frac{0}{14}\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \rightarrow \begin{bmatrix} 2/5 \\ 0 \\ -1/5 \end{bmatrix}$$

⑤ ⓐ Normalize $\vec{u}_1$, $\vec{u}_1 = \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} \rightarrow \frac{1}{\|\vec{u}_1\|} \vec{u}_1 \rightarrow$

$$\frac{1}{\sqrt{14}} \vec{u}_1 = \boxed{\begin{bmatrix} -3/\sqrt{14} \\ 1/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix}}$$

(b) Normalize $\vec{u}_4 = \frac{1}{\|\vec{u}_4\|}\vec{u}_4 = \frac{1}{\sqrt{14}}\vec{u}_4 =$

$$\boxed{\begin{bmatrix} 1/\sqrt{14} \\ -3/\sqrt{14} \\ 2/\sqrt{14} \end{bmatrix}}$$

(9) Find orthogonal basis for $S = \text{span}\left\{ \underset{\vec{s}_1}{\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}}, \underset{\vec{s}_2}{\begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}} \right\}$

$\rightarrow \vec{v}_1 = \vec{s}_1 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix} \rightarrow \vec{v}_2 = \vec{s}_2 - \text{proj}_{\vec{v}_1} \vec{s}_2 =$

$\begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} - \frac{0}{9}\begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} \rightarrow$ orthogonal basis $=$ $\text{span}\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} \right\}$

(13) orthogonal basis for $S = \text{span}\left\{ \underset{\vec{s}_1}{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}, \underset{\vec{s}_2}{\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}}, \underset{\vec{s}_3}{\begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix}} \right\}$

$\vec{v}_1 = \vec{s}_1 \rightarrow \vec{v}_2 = \vec{s}_2 - \text{proj}_{\vec{v}_1} \vec{s}_2 = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} - \frac{-2}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$= \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} \rightarrow \vec{v}_3 = \vec{s}_3 - \text{proj}_{\vec{v}_1} \vec{s}_3 - \text{proj}_{\vec{v}_2} \vec{s}_3 = \begin{bmatrix} 4 \\ 1 \\ 6 \end{bmatrix} -$

$\frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \; \frac{24}{24}\begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}$

$\rightarrow$ orthogonal basis for $S =$

$$\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}2\\4\\2\end{bmatrix}, \begin{bmatrix}3\\-3\\3\end{bmatrix}\right\}$$

(17) $\text{proj}_S \vec{u}$, $\vec{u} = \begin{bmatrix}1\\0\\2\end{bmatrix}$, $S = \text{span}\left\{\underset{\vec{s}_1}{\begin{bmatrix}-2\\2\\1\end{bmatrix}}, \underset{\vec{s}_2}{\begin{bmatrix}3\\4\\-2\end{bmatrix}}\right\}$

$$\rightarrow \text{proj}_S \vec{u} = \text{proj}_{\vec{s}_1} \vec{u} + \text{proj}_{\vec{s}_2} \vec{u} = \frac{\vec{s}_1 \cdot \vec{u}}{\|\vec{s}_1\|^2} \vec{s}_1$$

$$+ \frac{\vec{s}_2 \cdot \vec{u}}{\|\vec{s}_2\|^2} \vec{s}_2 = 0 \begin{bmatrix}-2\\2\\1\end{bmatrix} + \frac{-1}{29}\begin{bmatrix}3\\4\\-2\end{bmatrix} = \begin{bmatrix}-3/29\\-4/29\\2/29\end{bmatrix}$$

(21) $\text{proj}_S \vec{u}$ where $\vec{u} = \begin{bmatrix}1\\0\\2\end{bmatrix}$, $S = \text{span}\left\{\underset{\vec{s}_1}{\begin{bmatrix}-1\\0\\1\end{bmatrix}}, \underset{\vec{s}_2}{\begin{bmatrix}2\\4\\2\end{bmatrix}},\right.$

$$\left.\underset{\vec{s}_3}{\begin{bmatrix}3\\-3\\3\end{bmatrix}}\right\} \rightarrow \text{proj}_S \vec{u} = \frac{\vec{s}_1 \cdot \vec{u}}{\|\vec{s}_1\|^2} \vec{s}_1 + \frac{\vec{s}_2 \cdot \vec{u}}{\|\vec{s}_2\|^2} \vec{s}_2 +$$

$$\frac{\vec{s}_3 \cdot \vec{u}}{\|\vec{s}_3\|^2} \vec{s}_3 = \frac{1}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix} + \frac{6}{24}\begin{bmatrix}2\\4\\2\end{bmatrix} + \frac{9}{27}\begin{bmatrix}3\\-3\\3\end{bmatrix}$$

$$= \begin{bmatrix}-1/2\\0\\1/2\end{bmatrix} + \begin{bmatrix}1/2\\1\\1/2\end{bmatrix} + \begin{bmatrix}1\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\2\end{bmatrix}$$

(25) orthonormal basis for $S = \text{span}\left\{\underset{\vec{s}_1}{\begin{bmatrix}-2\\2\\1\end{bmatrix}}, \underset{\vec{s}_2}{\begin{bmatrix}3\\4\\-2\end{bmatrix}}\right\}$

$$= \left\{\frac{1}{\|\vec{s}_1\|}\vec{s}_1, \frac{1}{\|\vec{s}_2\|}\vec{s}_2\right\} \rightarrow \left\{\frac{1}{3}\begin{bmatrix}-2\\2\\1\end{bmatrix}, \frac{1}{\sqrt{29}}\begin{bmatrix}3\\4\\-2\end{bmatrix}\right\}$$

$$\left\{ \begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}, \begin{bmatrix} 3/\sqrt{29} \\ 4/\sqrt{29} \\ -2/\sqrt{29} \end{bmatrix} \right\}$$

(29) orthonormal basis for $S = \text{span}\left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \right\} \rightarrow \left\{ \frac{1}{\|\vec{s}_1\|} \vec{s}_1, \frac{1}{\|\vec{s}_2\|} \vec{s}_2, \frac{1}{\|\vec{s}_3\|} \vec{s}_3 \right\}$

$$\rightarrow \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2\sqrt{6}} \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix}, \frac{1}{3\sqrt{3}} \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} \right\} \rightarrow$$

$$\left\{ \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \right\}$$

(33) $\vec{u}, \vec{v}$ s.t. $\vec{u}, \vec{v} \in \mathbb{R}^2 \wedge \text{proj}_{\vec{v}} \vec{u} = 0$

$$\vec{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \vec{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(37)

(a) False as orthogonal vectors can be non-zero and have $\text{proj}_{\vec{u}} \vec{v} = 0$

(b) False if $\vec{u} \cdot \vec{v} = 0$

(39) (a) True as projection onto subspace means that the projection is in the subspace.

(b) False as $\text{proj}_{\vec{v}} \vec{u}$ is a multiple of $\vec{v}$.

(41) (a) True as projecting onto a perpendicular space will give the zero vector.

(b) True as the projection onto a space for a vector already in the space is that vector itself.