CSE 202: Design and Analysis of Algorithms

Lecture 18

Instructor: Kamalika Chaudhuri

Announcements

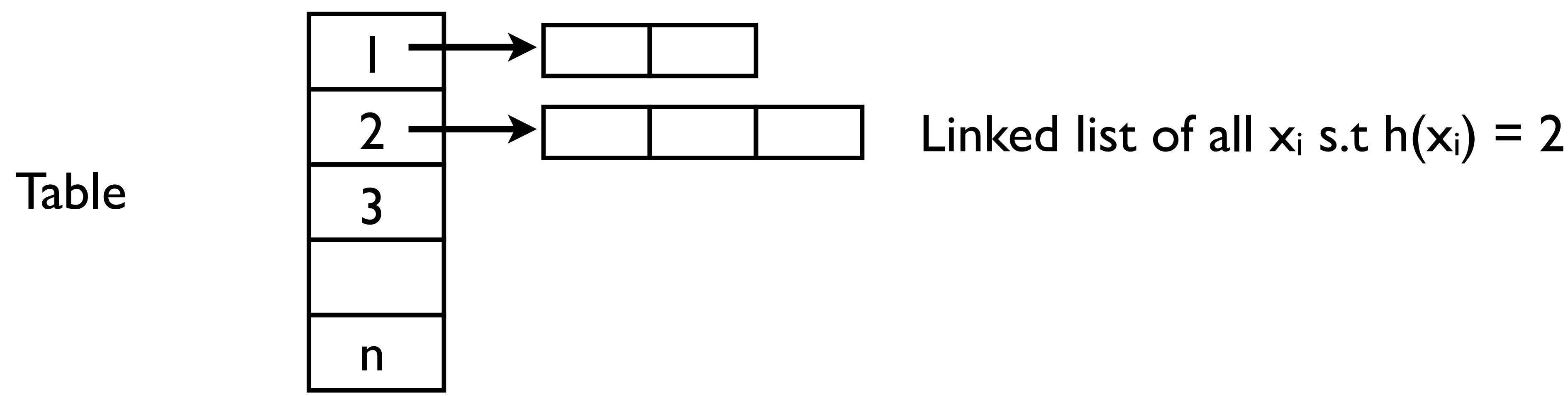
- No TA Office Hours on Thu May 26
- Extra Instructor Office Hours Thu May 26 5-6pm (instead of 2:30-3:30pm), CSE 4110
- No class on Monday May 30 (Memorial Day)
- Extra Instructor Office Hours Tue May 31 5-6pm at CSE 4110
- TA Office Hours on Tue May 31 moved to Wed Jun 1, 11-12, B250A
- Pick up graded Homework 3 after class

Randomized Algorithms

- Contention Resolution
- Some Facts about Random Variables
- Global Minimum Cut Algorithm
- Randomized Selection and Sorting
- Max 3-SAT
- Three Concentration Inequalities
- Hashing and Balls and Bins

Hashing and Balls-n-Bins

Problem: Given a large set S of elements $x_1, .., x_n$, store them using O(n) space s.t it is easy to determine whether a query item q is in S or not



Popular Data Structure: A Hash table

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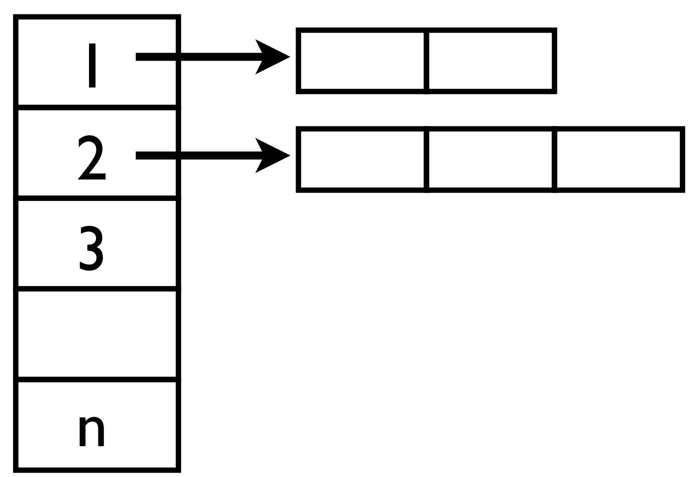

Linked list of all x_i s.t $h(x_i) = 2$

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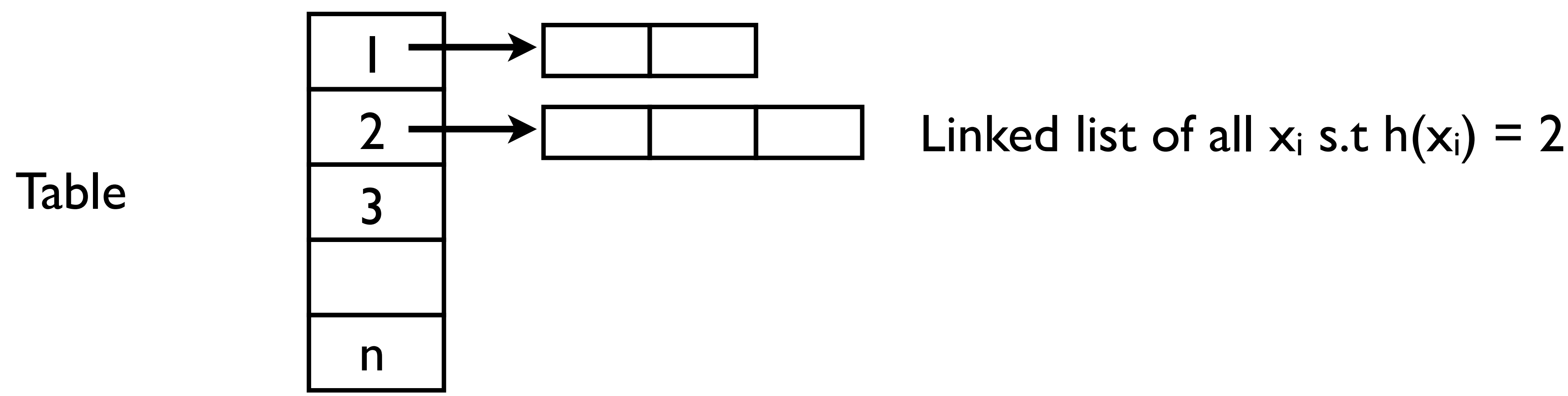

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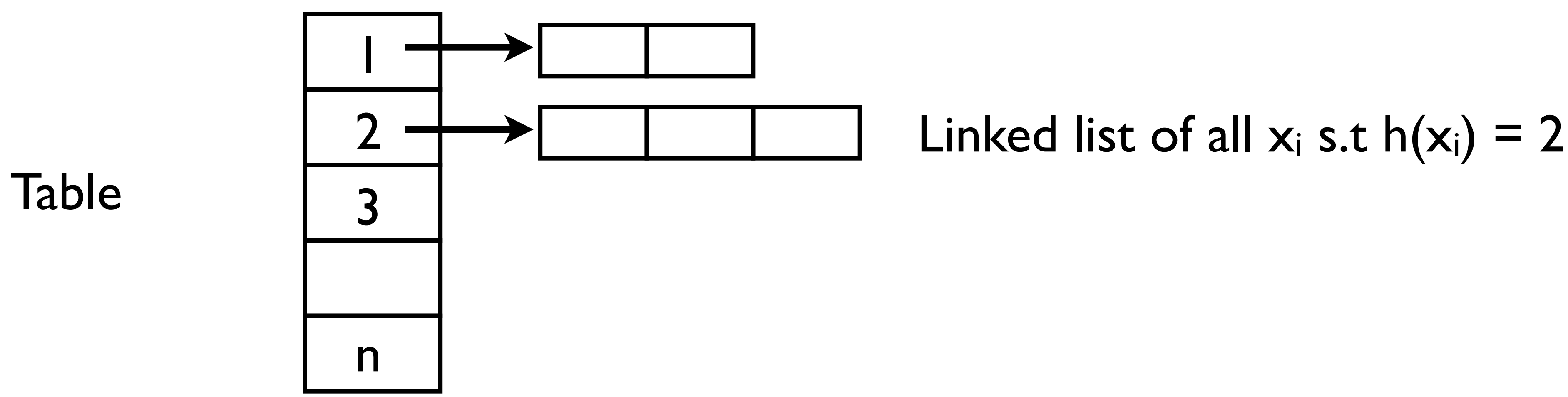

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What is the query time of the algorithm?

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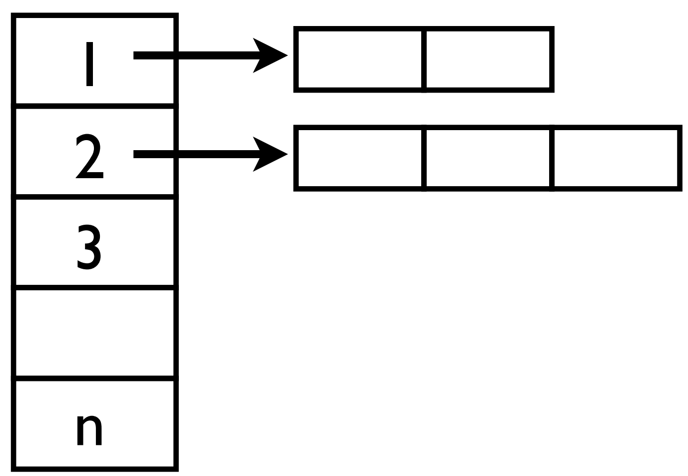

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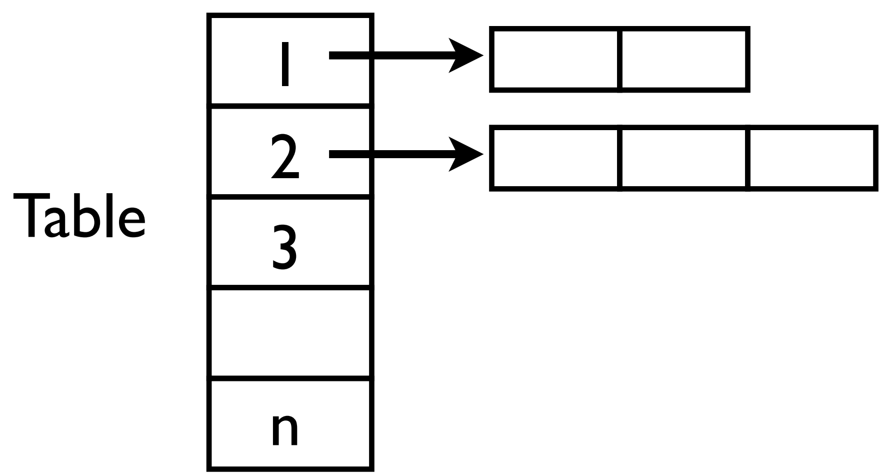

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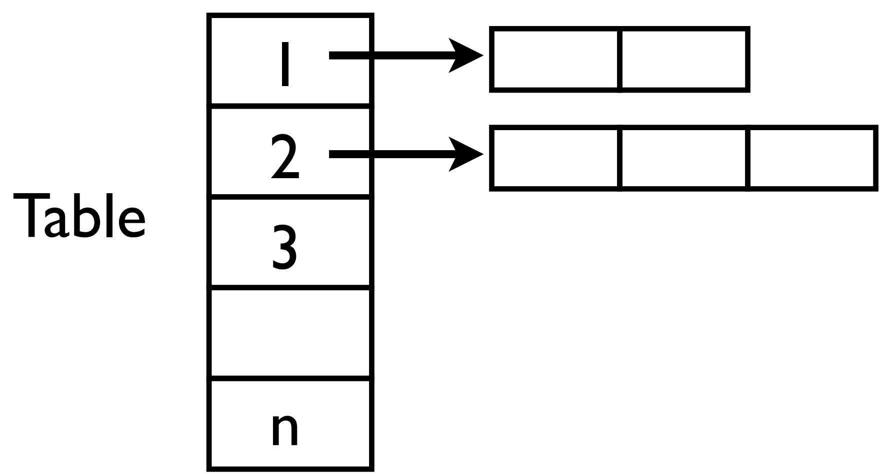

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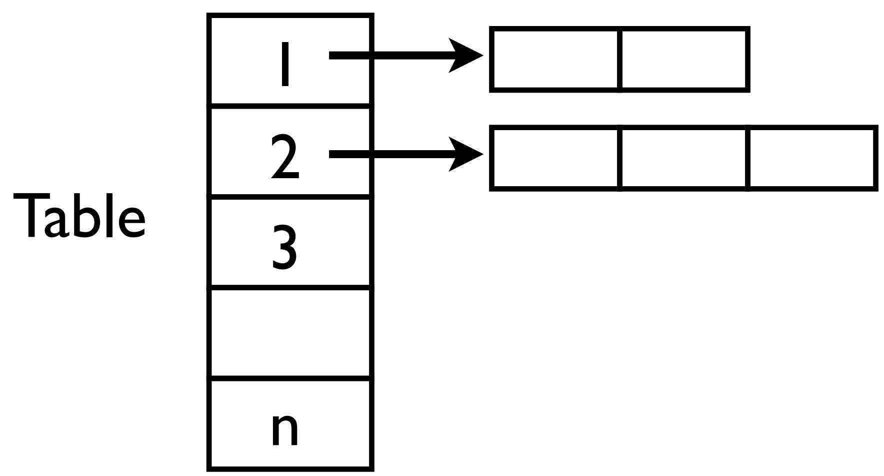

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$$= \frac{1}{n} \sum_{i} (\text{length of list at } T[i]) = \frac{1}{n} \cdot n = 1$$

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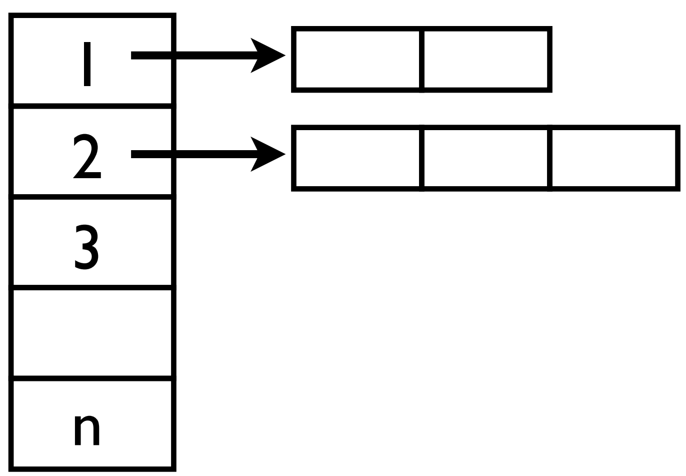

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Worst Case Query Time: For any q, what is the query time? (with high probability over the choice of hash functions)

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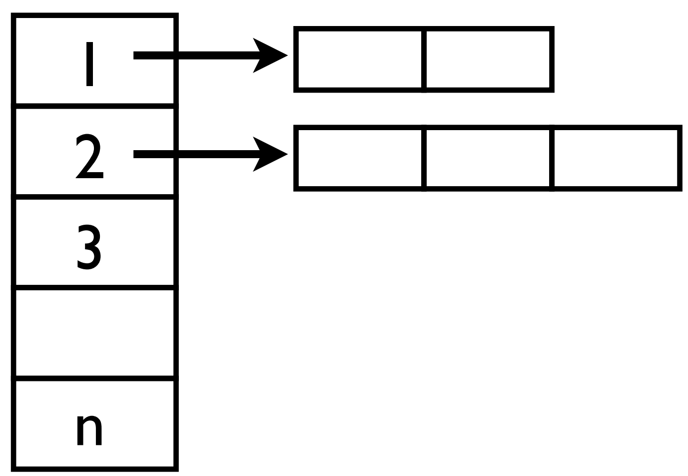

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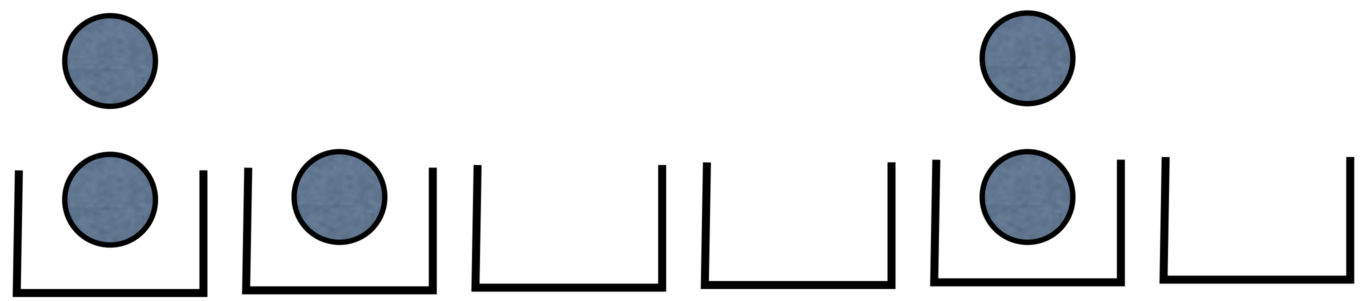
Equivalent to the following Balls and bins Problem:
Suppose we toss n balls u.a.r into n bins. What is the max #balls in a bin with high probability?

With high probability (w.h.p) = With probability 1 - 1/poly(n)

Balls and Bins, again

Suppose we toss n balls u.a.r into n bins. What is the max load of a bin with high probability?



Some Facts:

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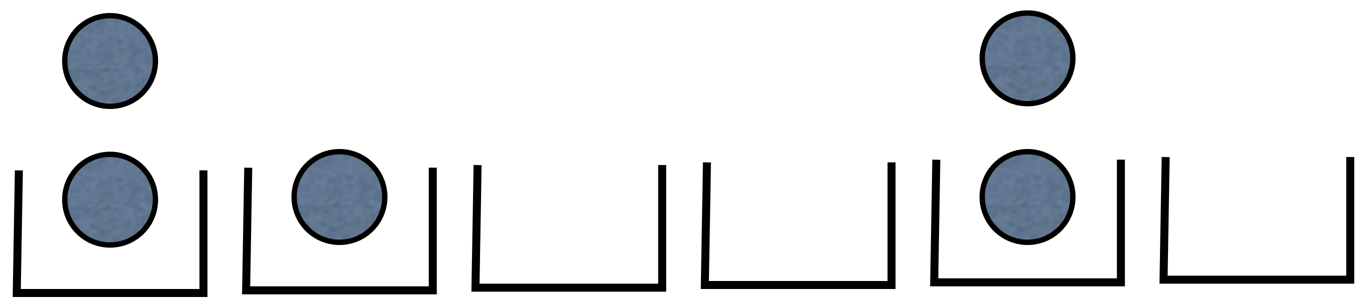

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1. The expected load of each bin is 1

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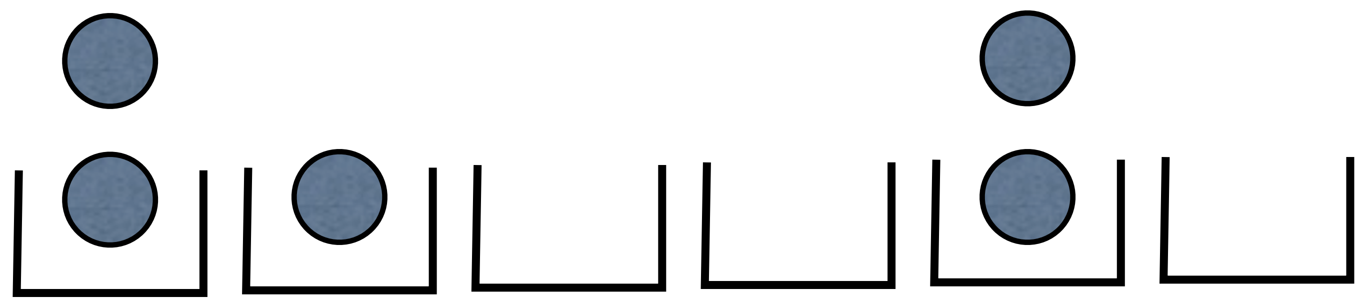

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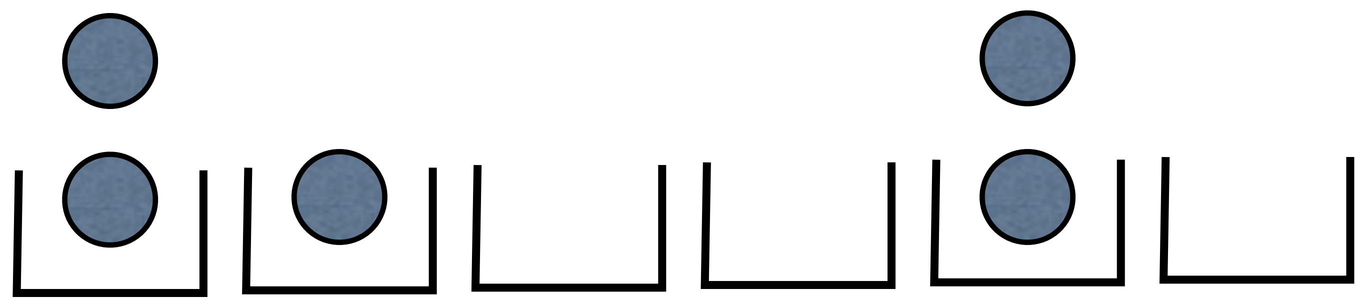

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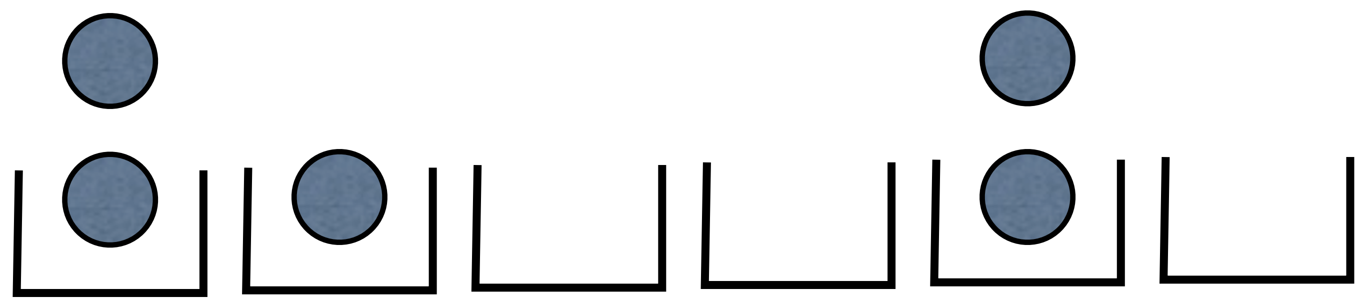

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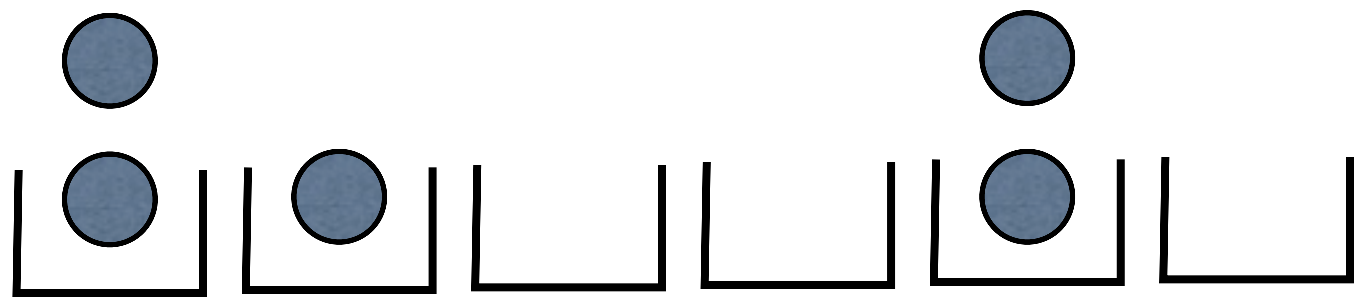

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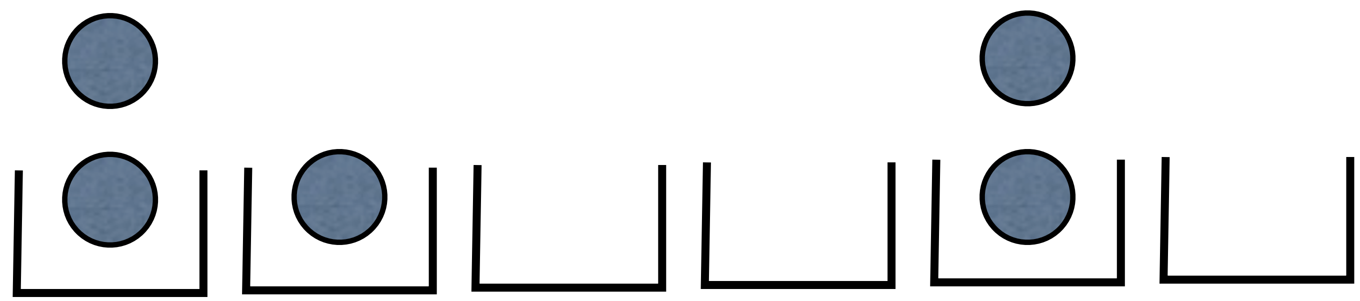

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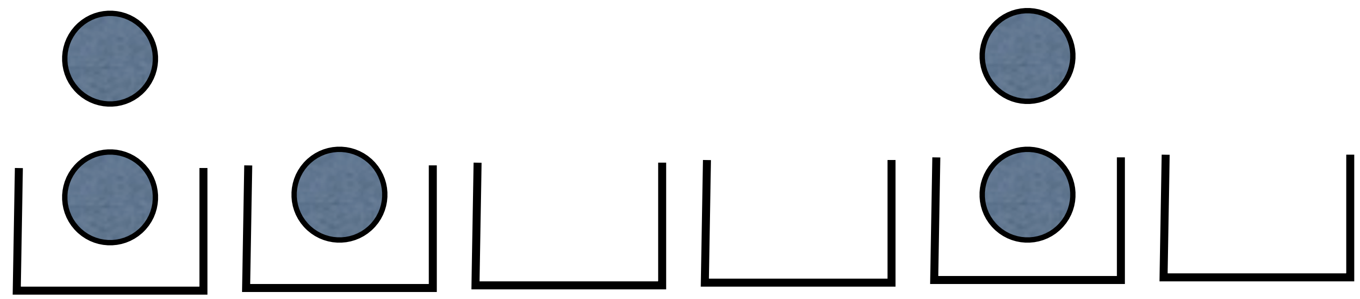

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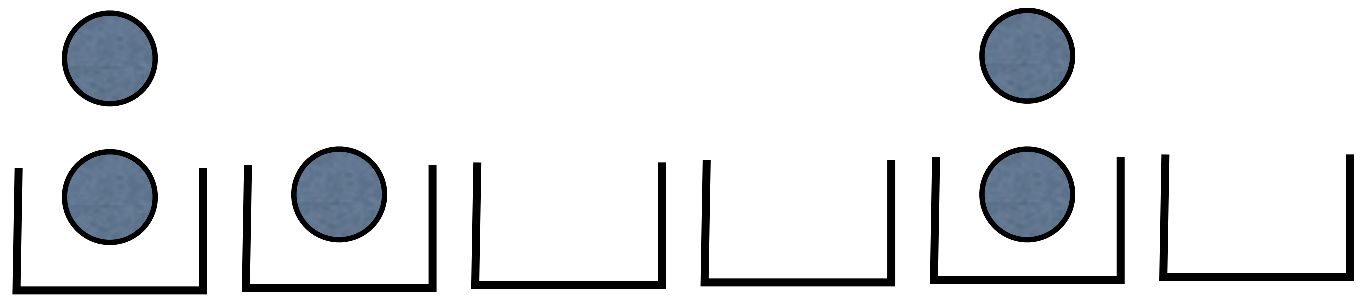

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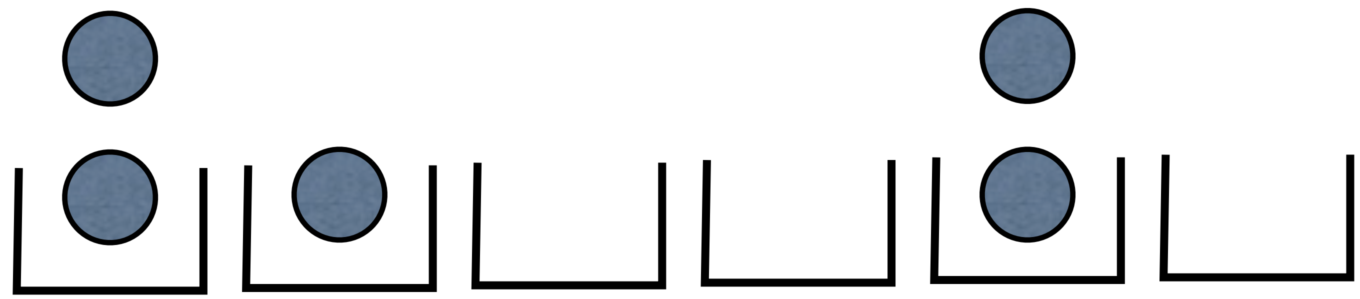

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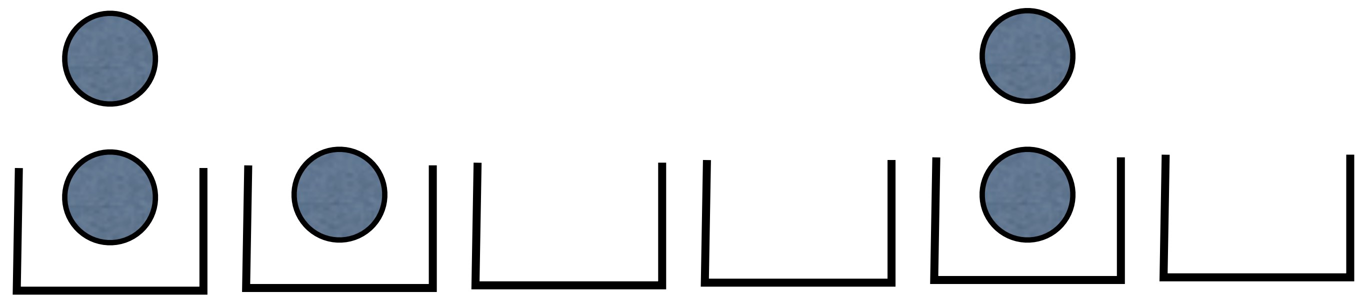

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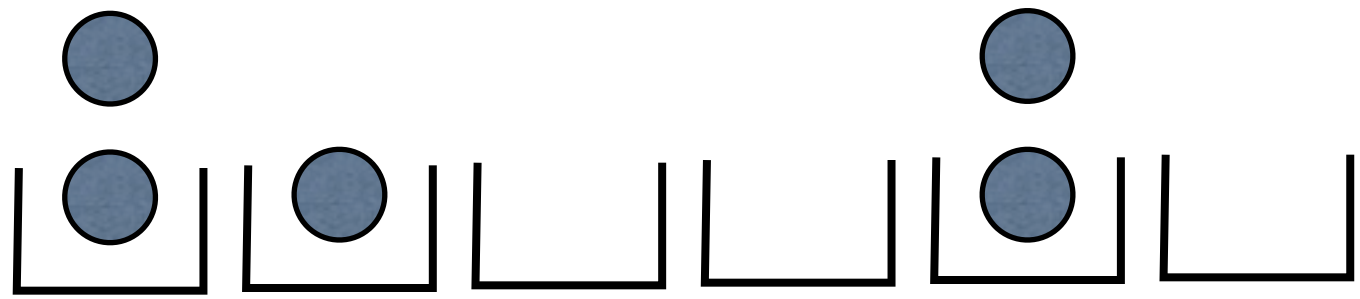

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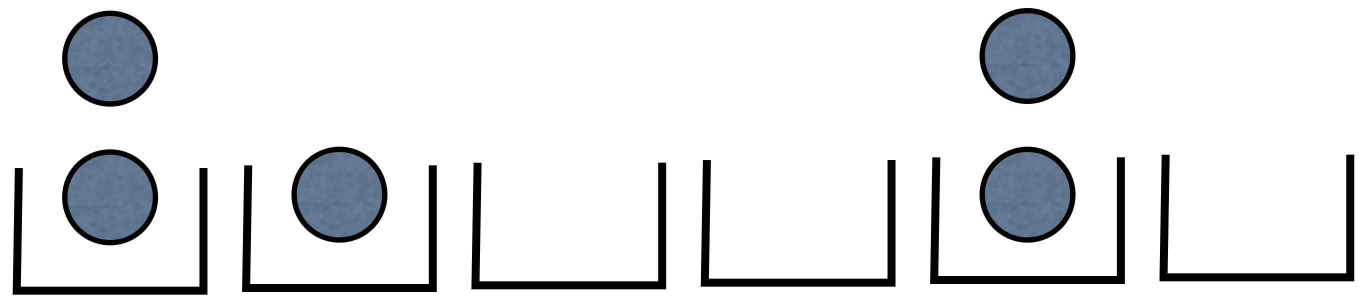

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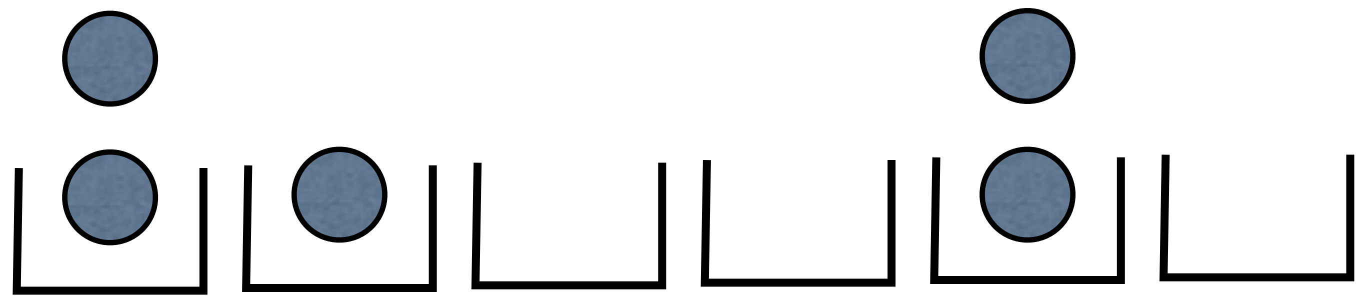

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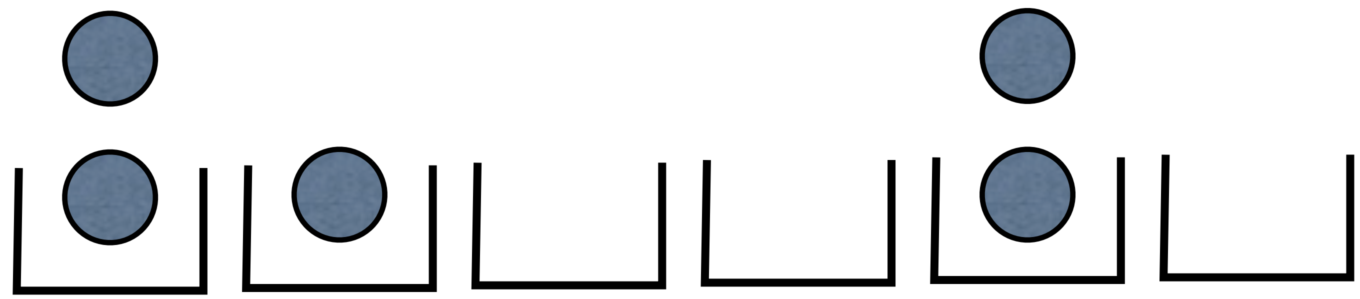

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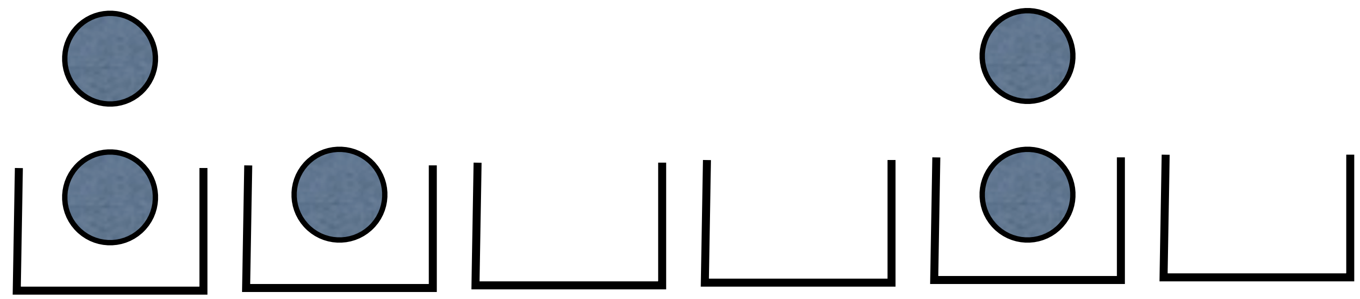

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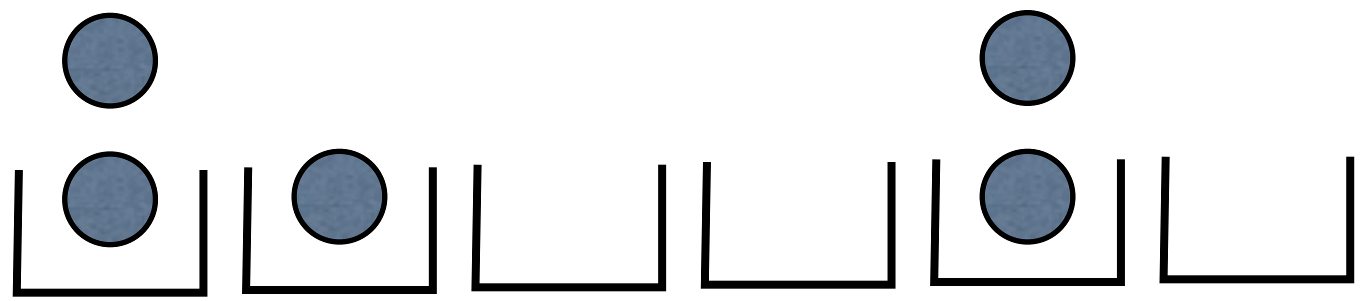

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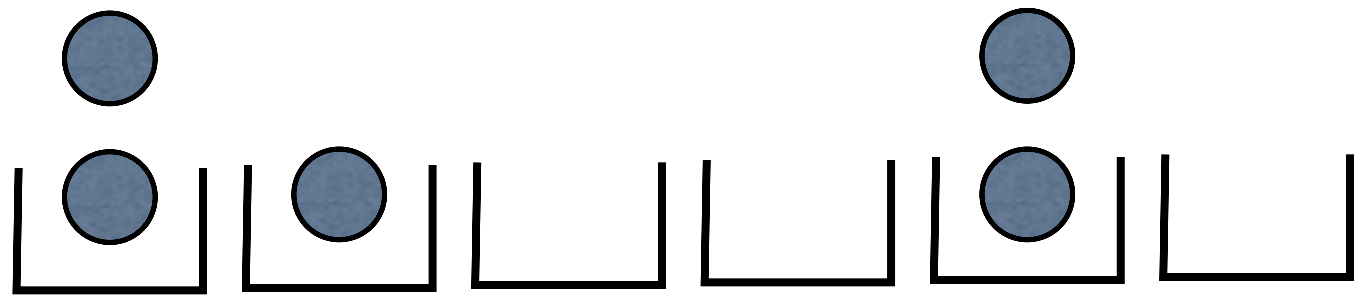

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$$\geq \frac{c}{2}\log n \geq 2\log n, \text{for } c \geq 4$$

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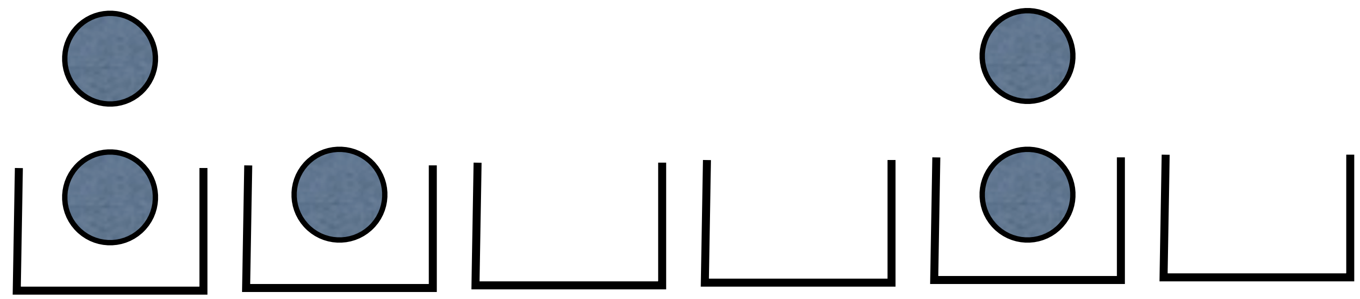

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From **Fact**

Would like this for whp condition

Fact: If n >= k

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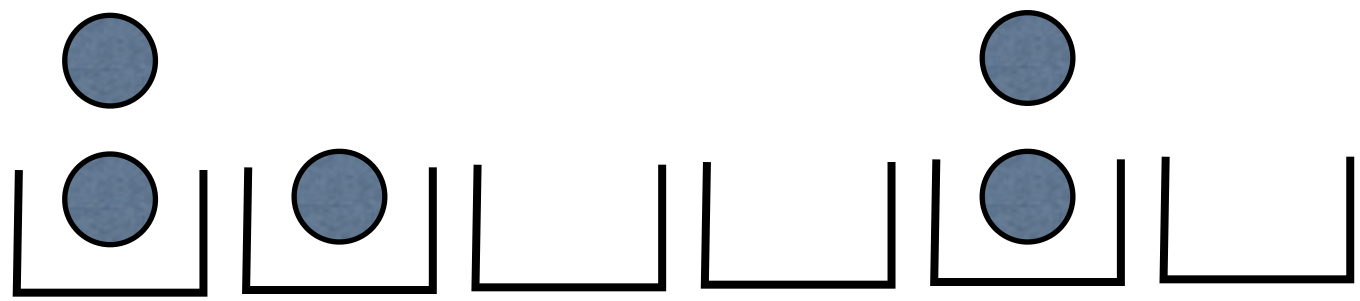

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Applying Union Bound, Pr(All bins have <=t balls) >= 1 - 1/n

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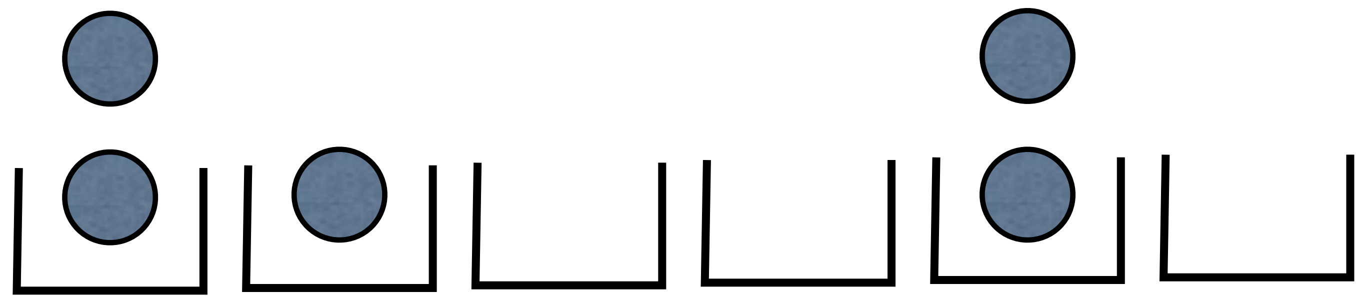

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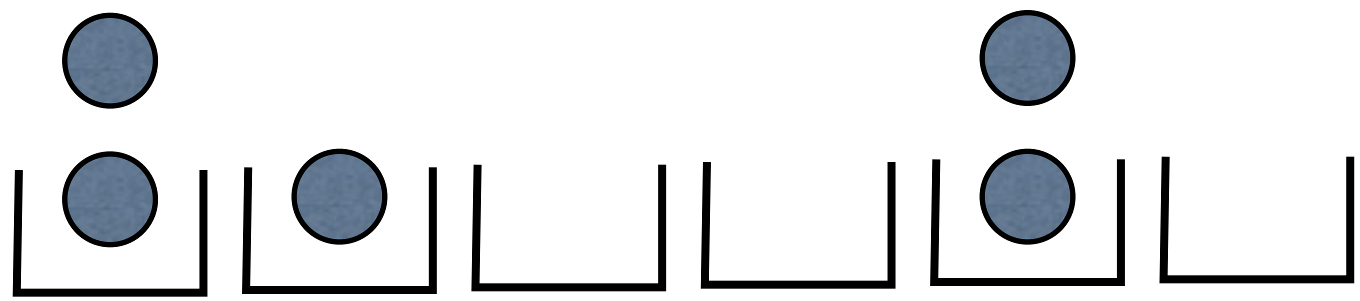

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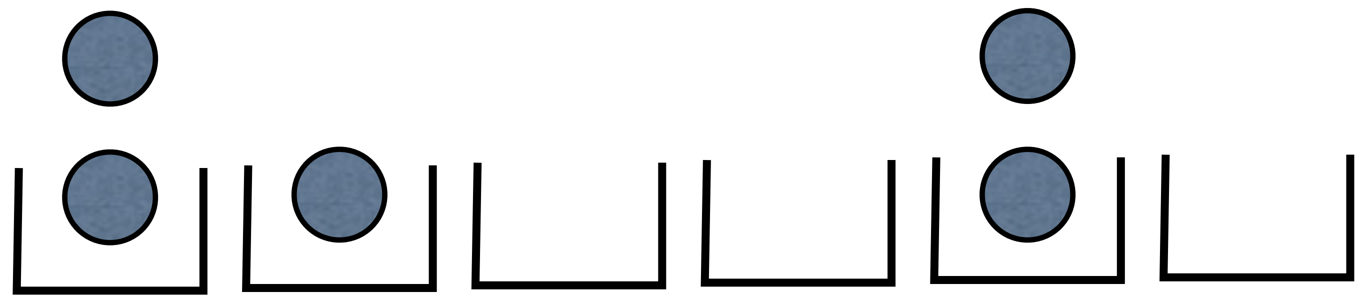

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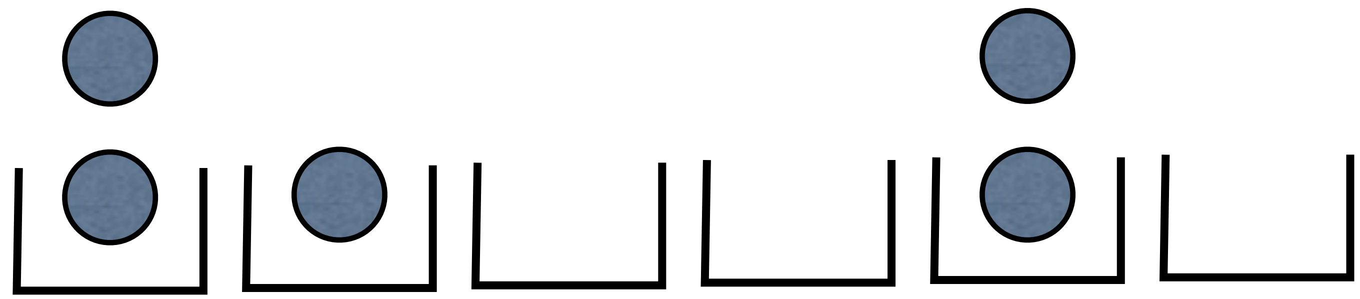

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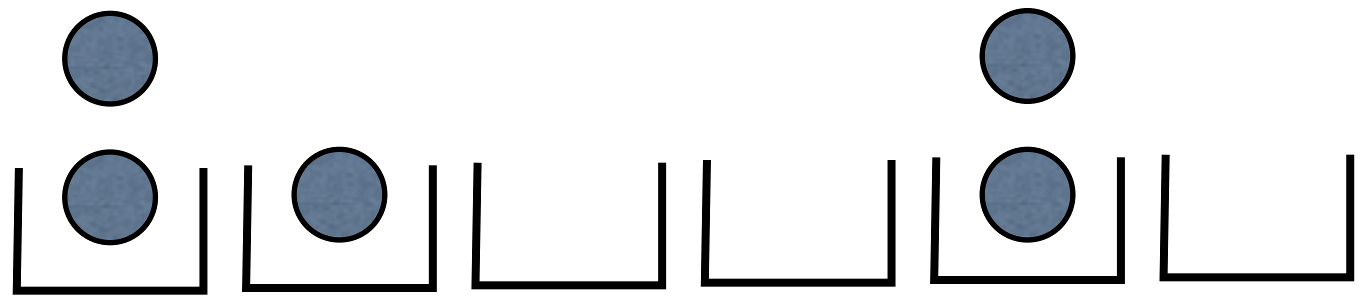

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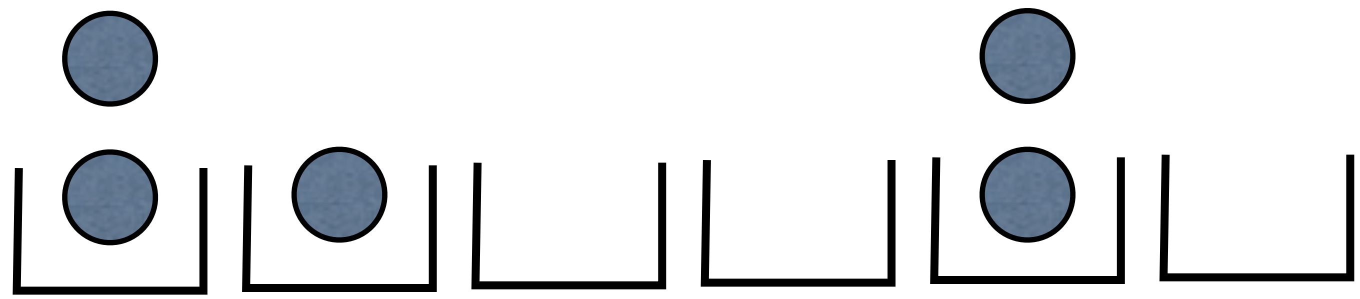

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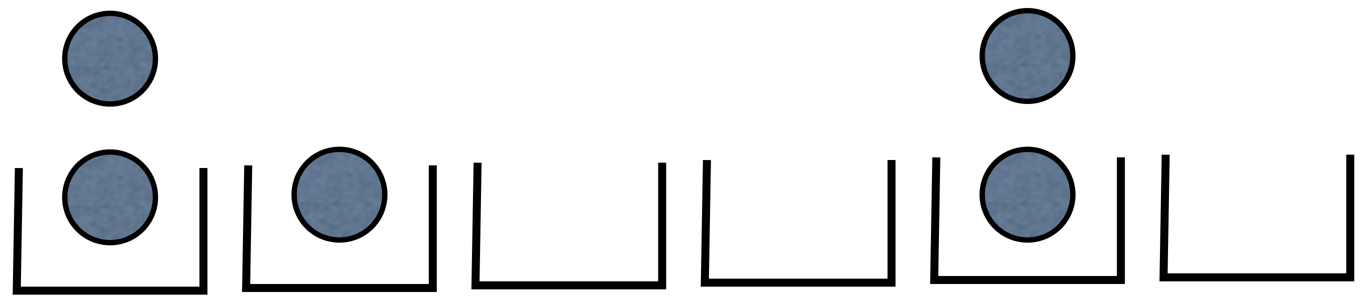

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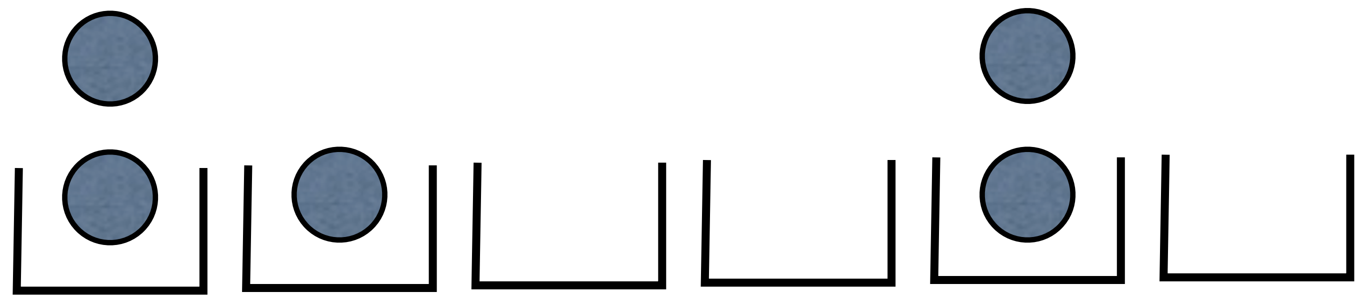

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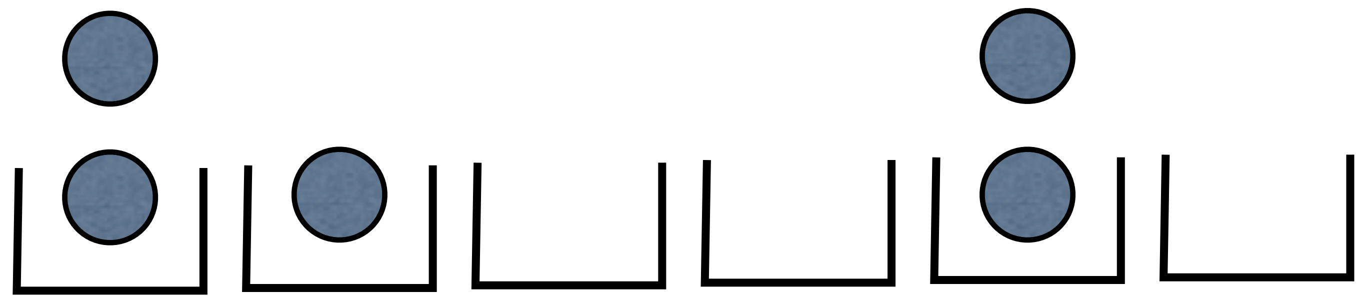

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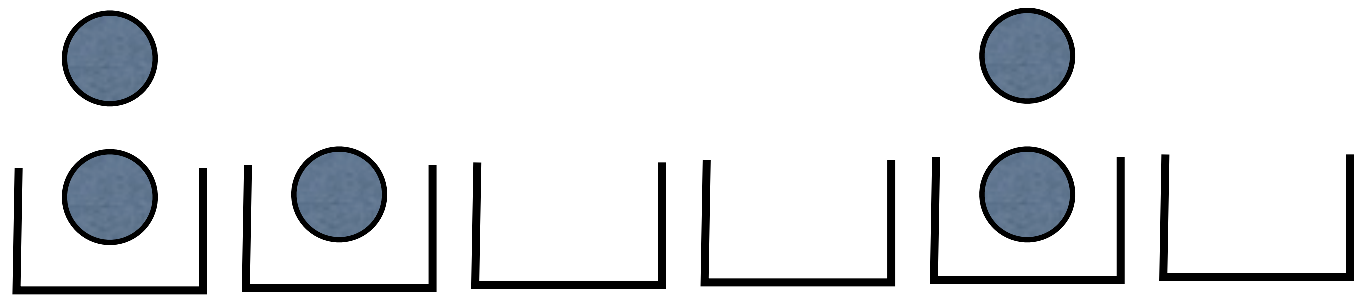

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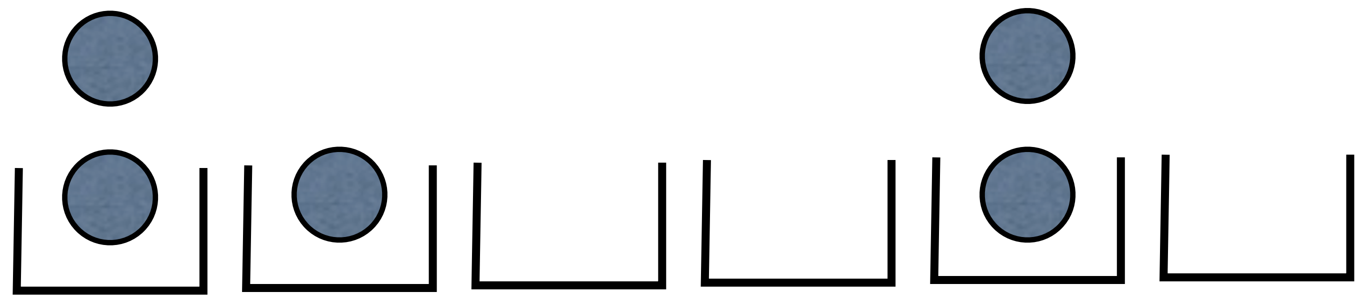

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Using Chebyshev, Pr(|Y - E[Y]| >= E[Y]) <= $\mathrm{Var}(Y)/E(Y)^2$

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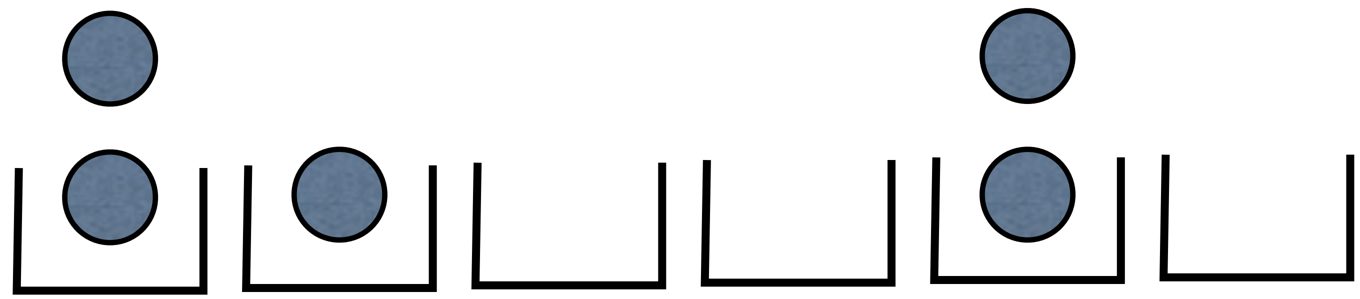

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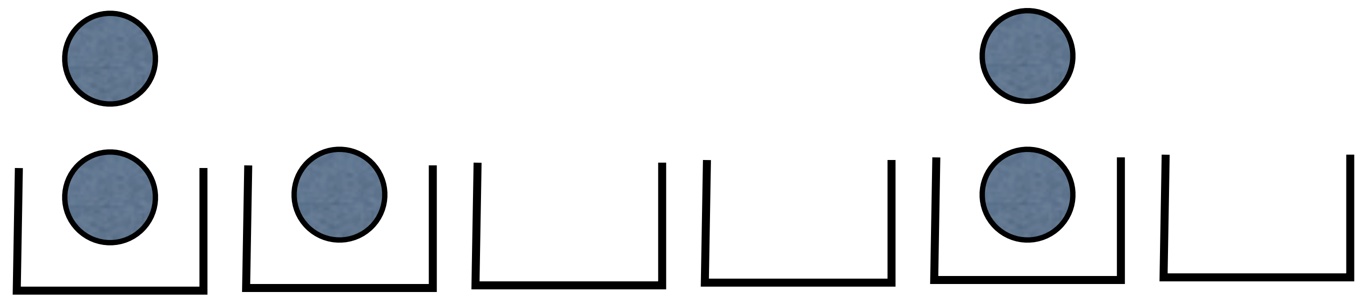

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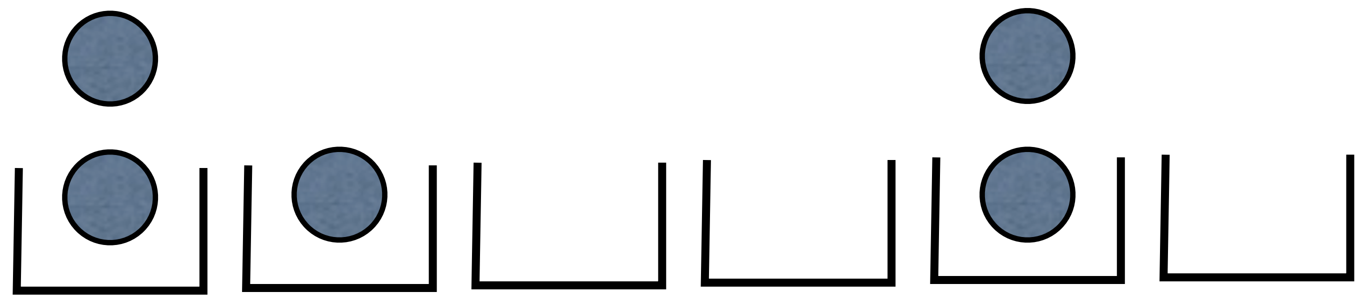

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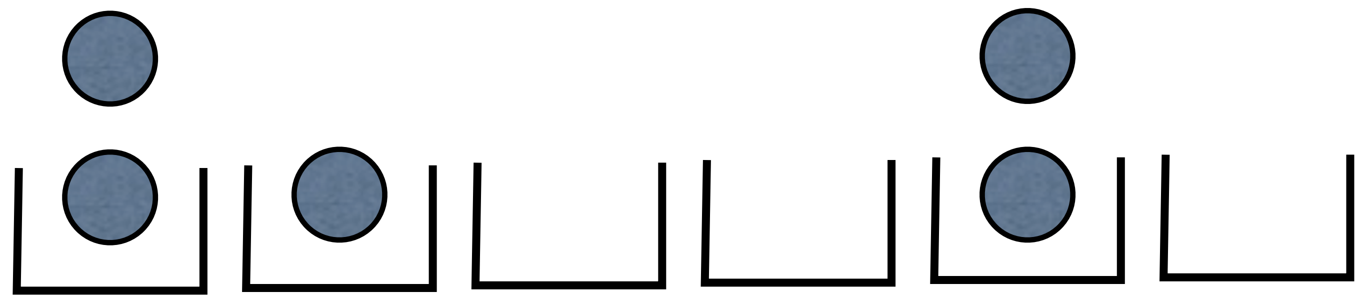

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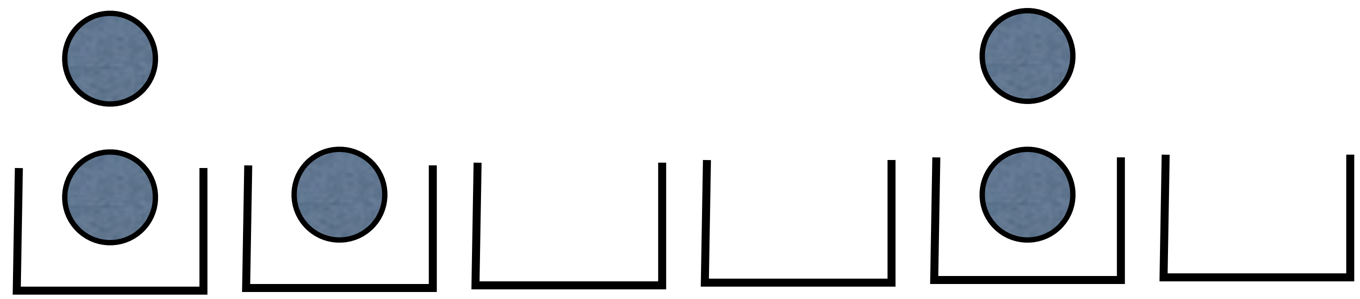

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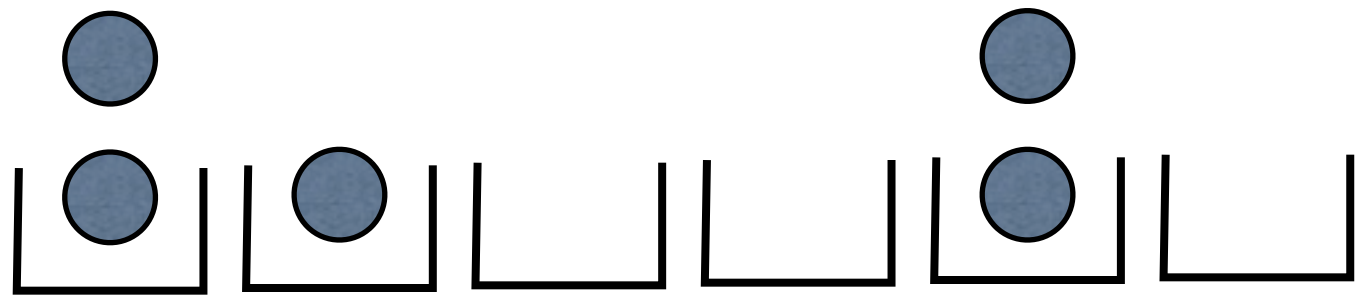

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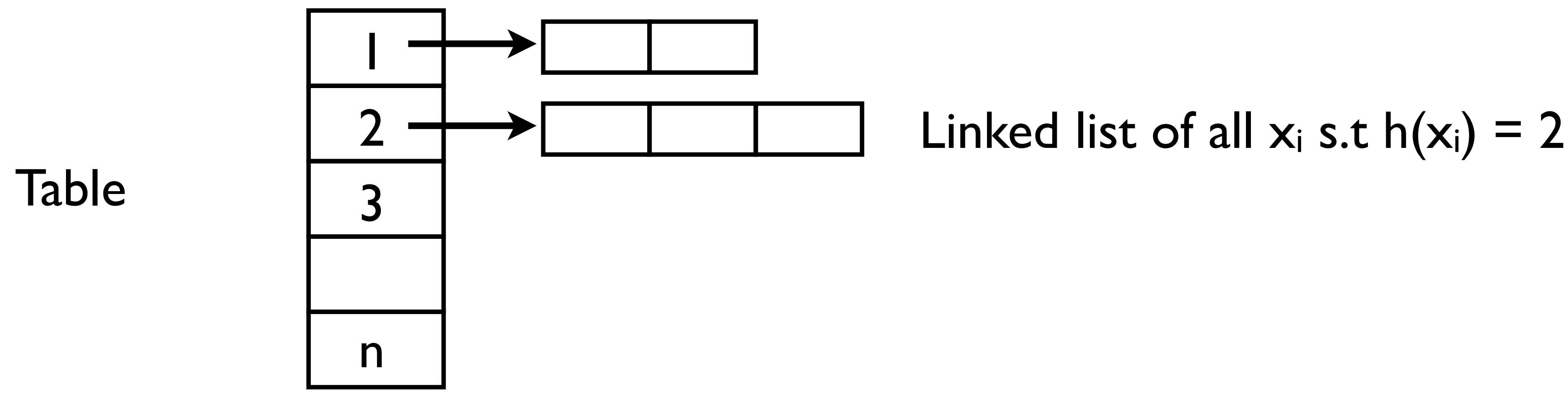
$$\Pr(Y = 0) \leq \frac{Var(Y)}{E(Y)^2} \leq \frac{ne^2}{n^{4/3}} \leq \frac{e^2}{n^{1/3}}$$

Randomized Algorithms

- Contention Resolution
- Some Facts about Random Variables
- Global Minimum Cut Algorithm
- Randomized Selection and Sorting
- Max 3-SAT
- Three Concentration Inequalities
- Hashing and Balls and Bins
  - The Power of Two Choices

The Power of Two Choices

Problem: Given a large set S of elements $x_1, .., x_n$, store them using O(n) space s.t it is easy to determine whether a query item q is in S or not



Algorithm:

1. Pick **two** completely random functions $h_1 : \mathcal{U} \to \{1, \ldots, n\},$ and $h_2 : \mathcal{U} \to \{1, \ldots, n\}$
2. Create a table of size n, initialize it to null
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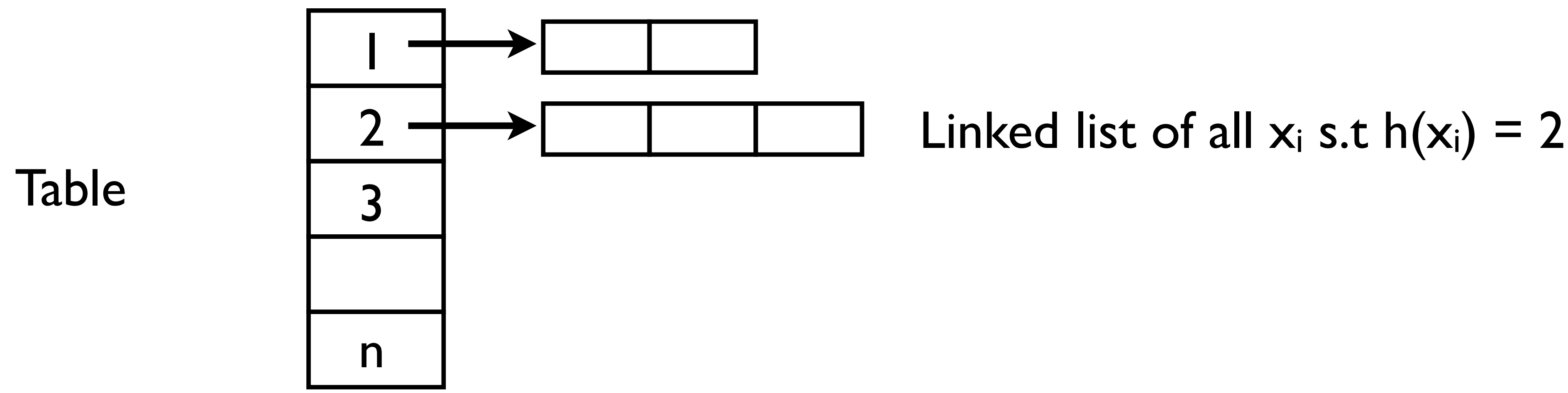

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Equivalent to the following Balls and Bins Problem: Toss n balls into n bins. For each ball, pick two bins u.a.r and put the ball into the lighter of the two bins.

What is the worst case query time?

Power of Two Choices

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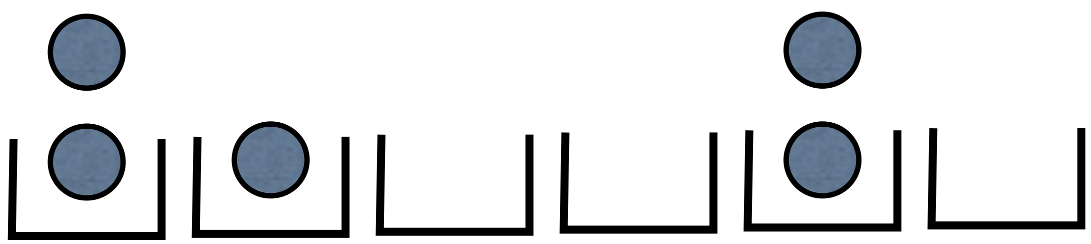

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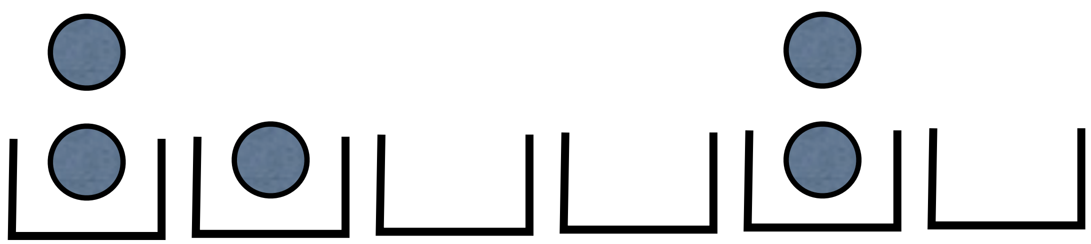

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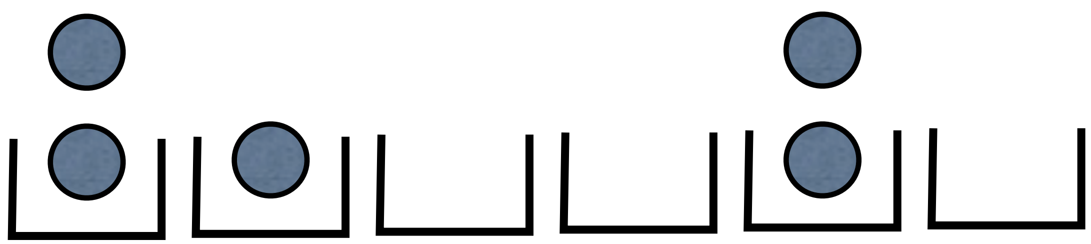

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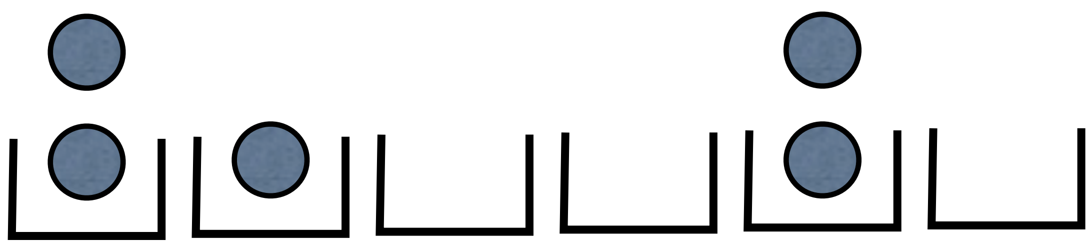

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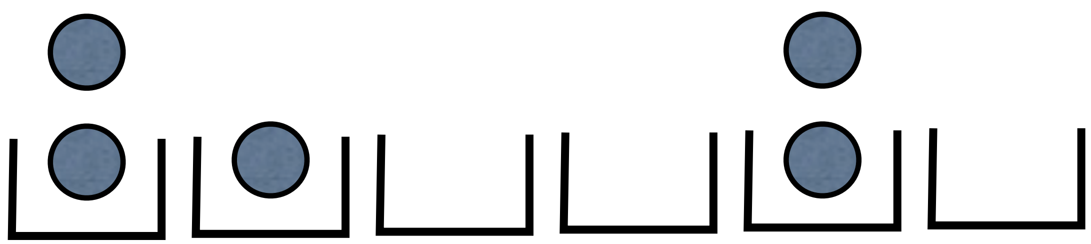

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Now we prove this formally. But what makes the proof hard?

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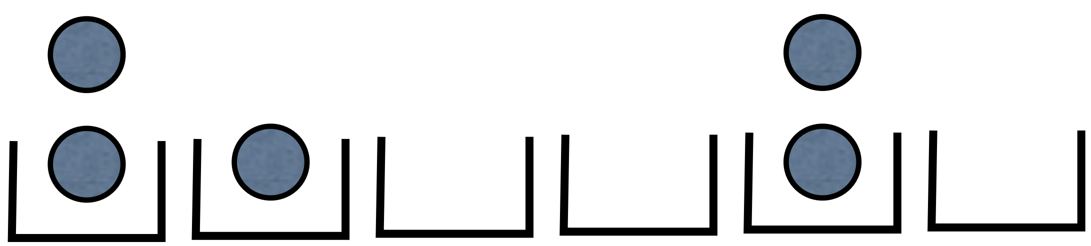

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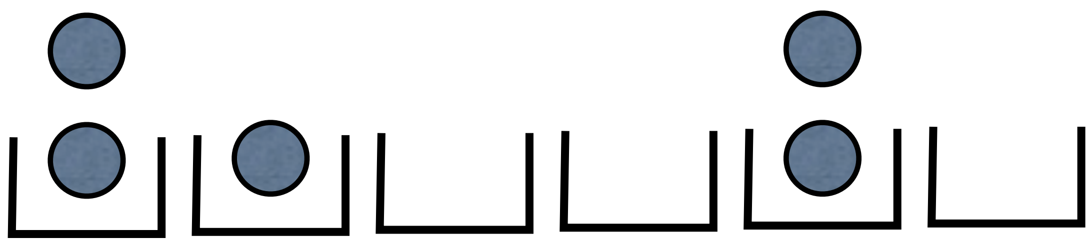

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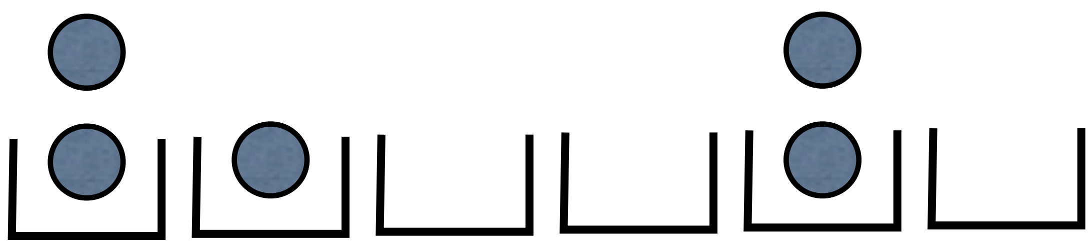

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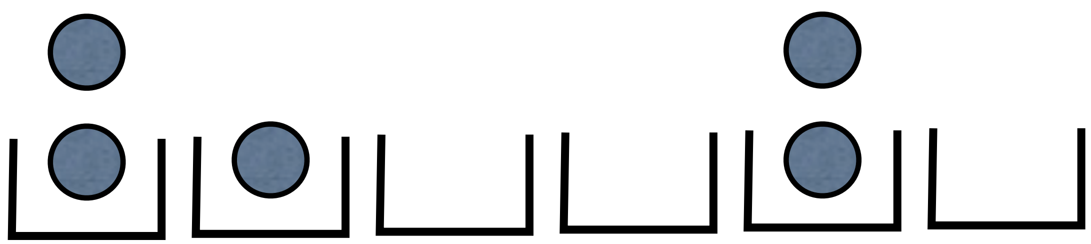

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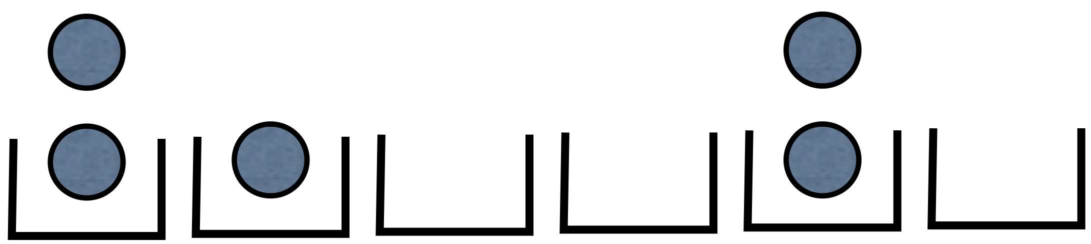

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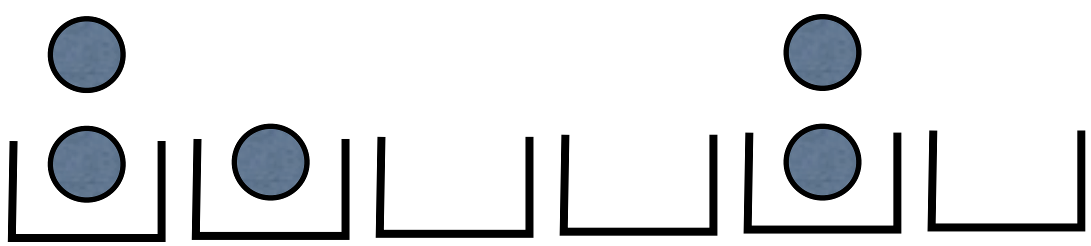

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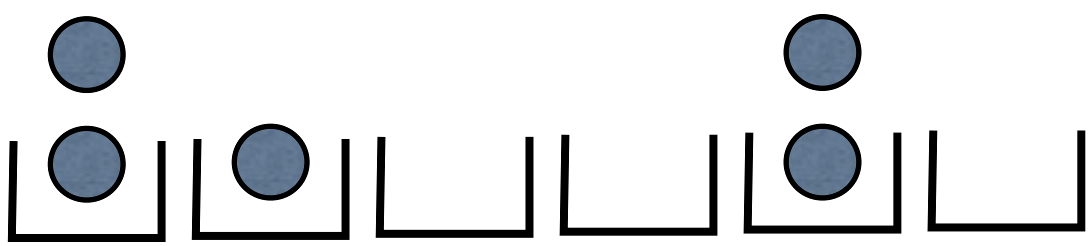

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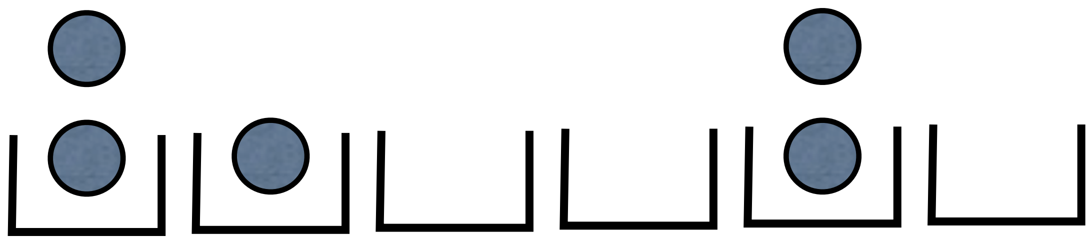

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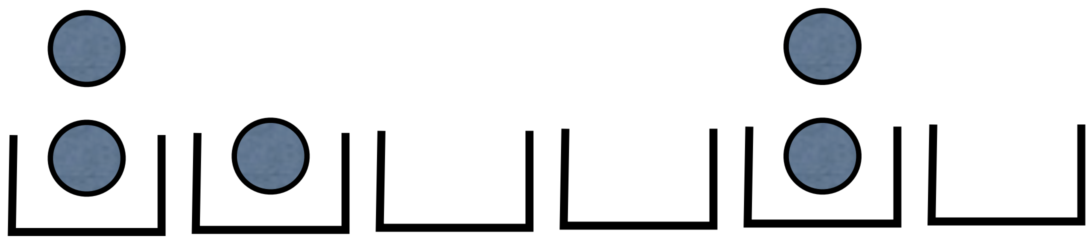

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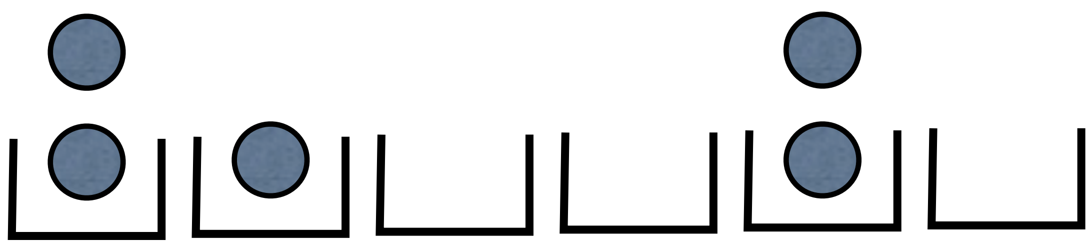

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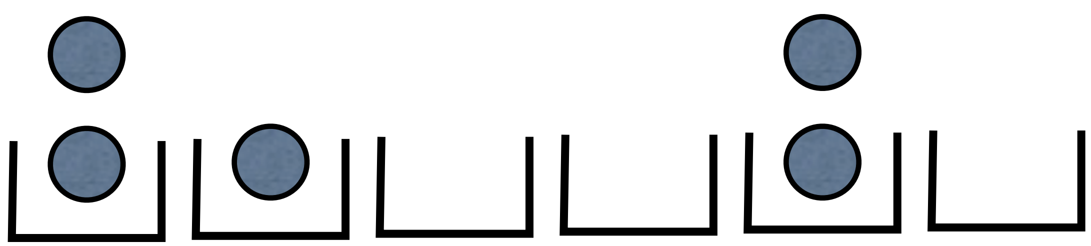

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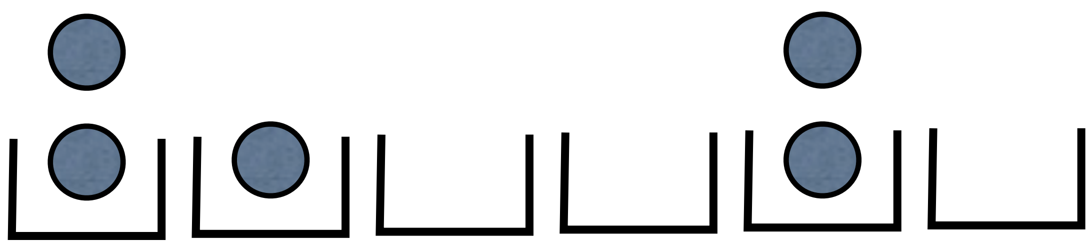

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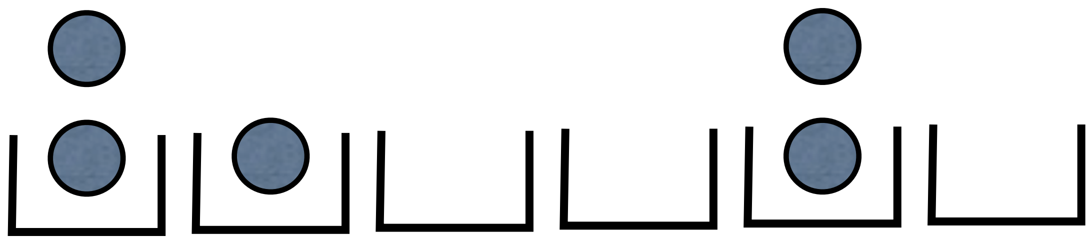

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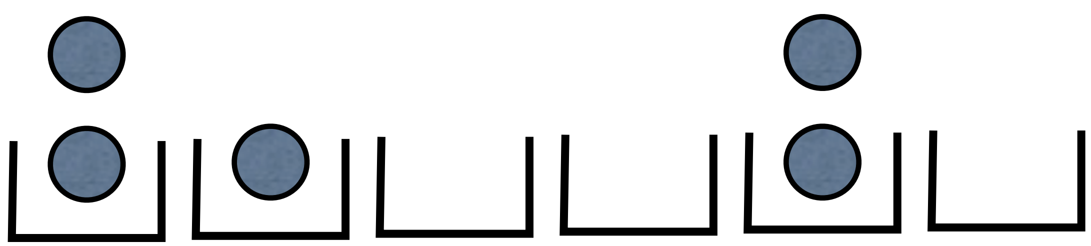

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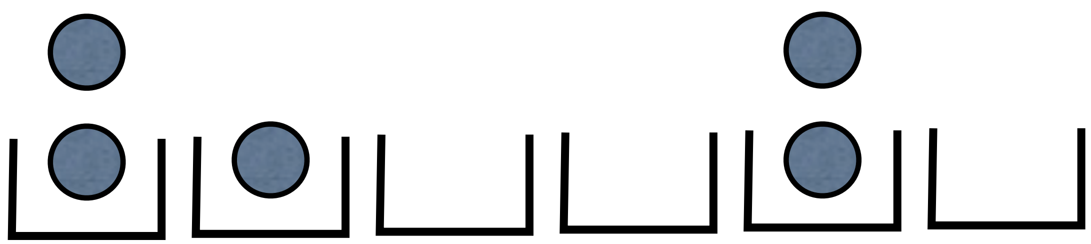

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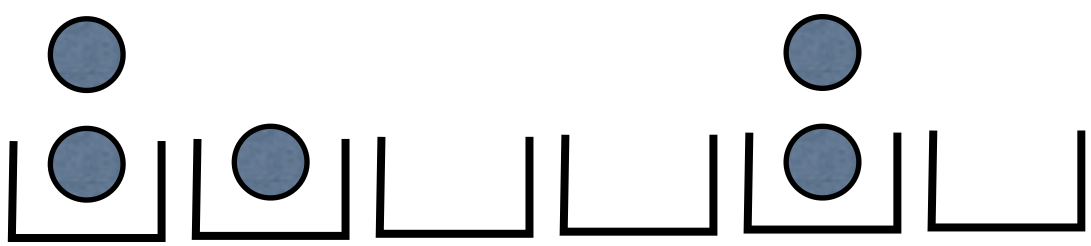

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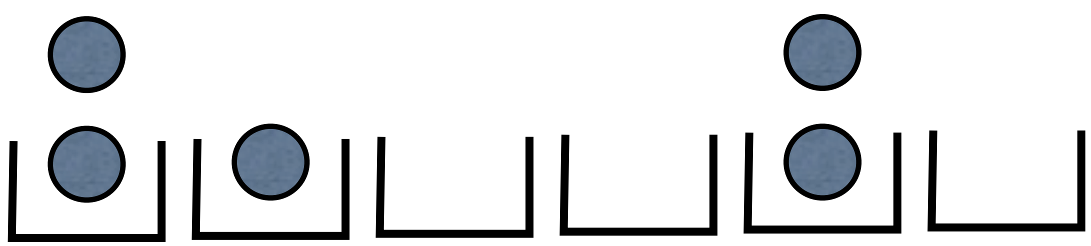

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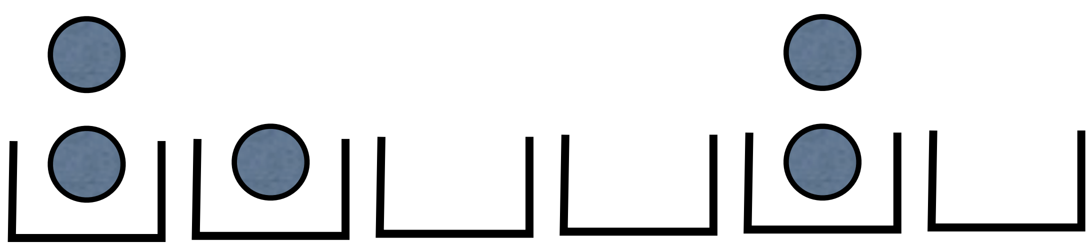

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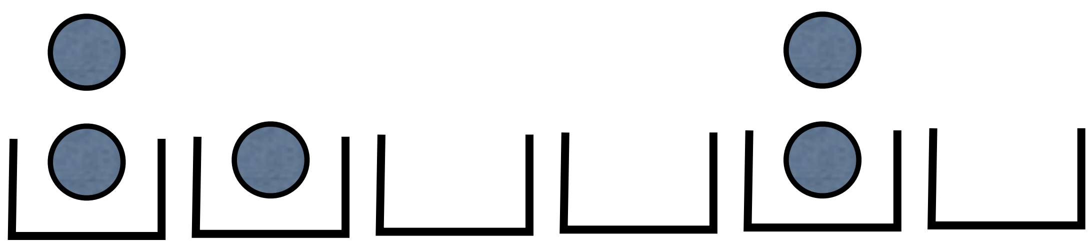

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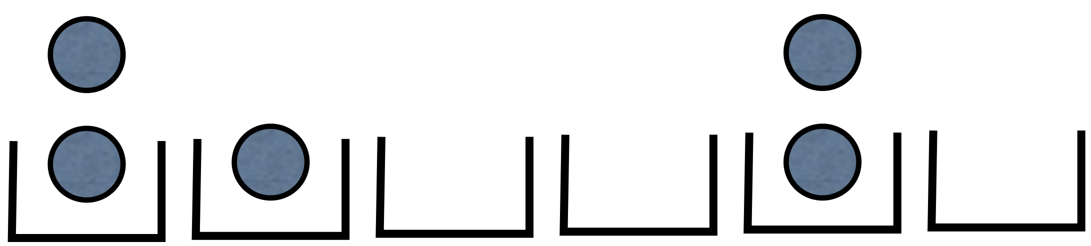

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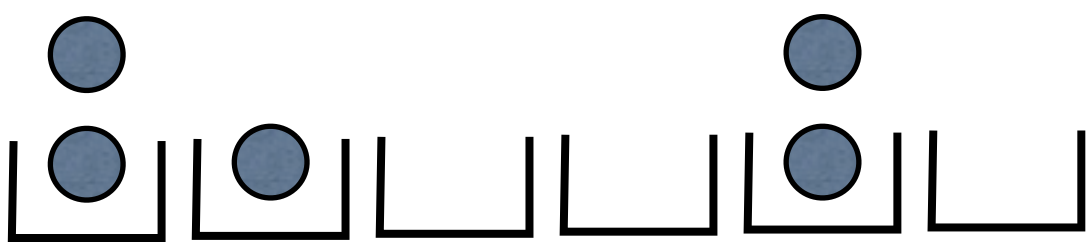

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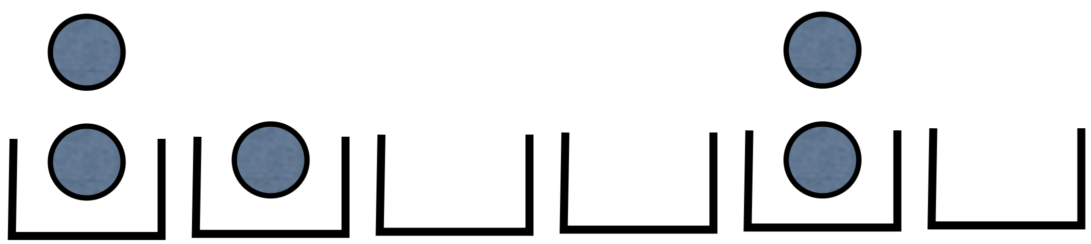

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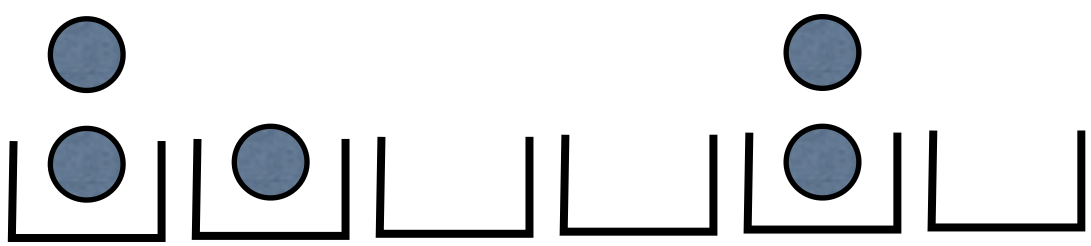

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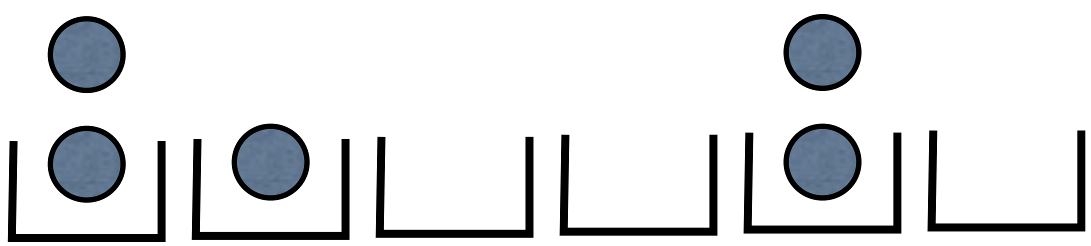

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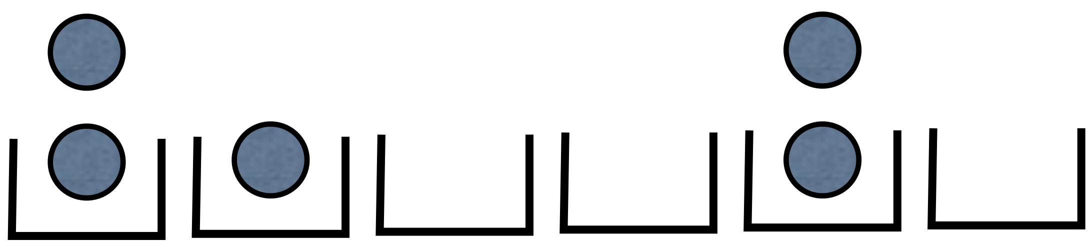

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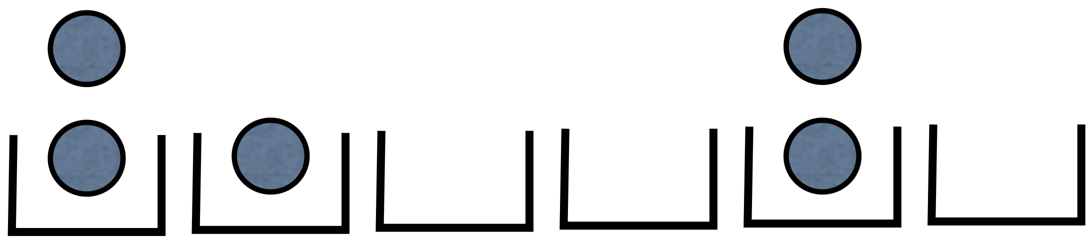

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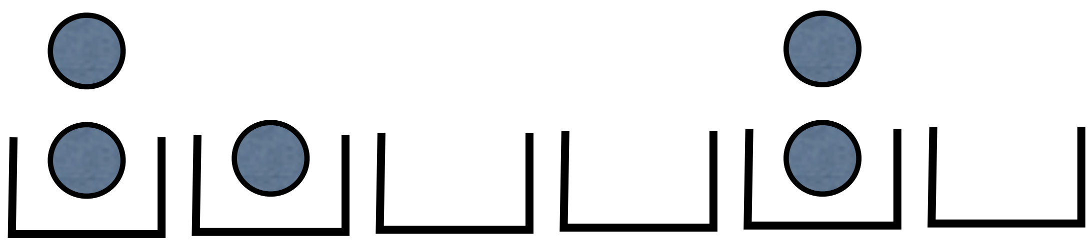

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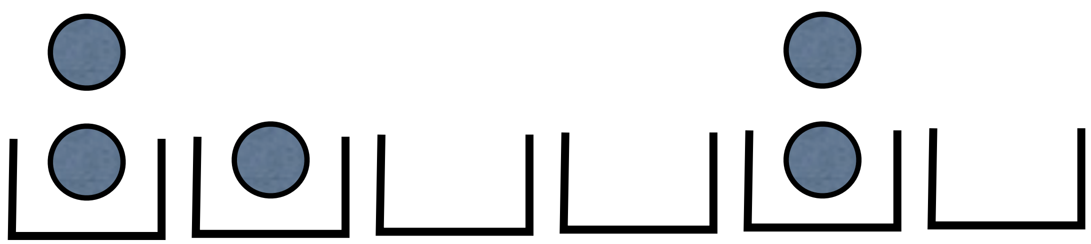

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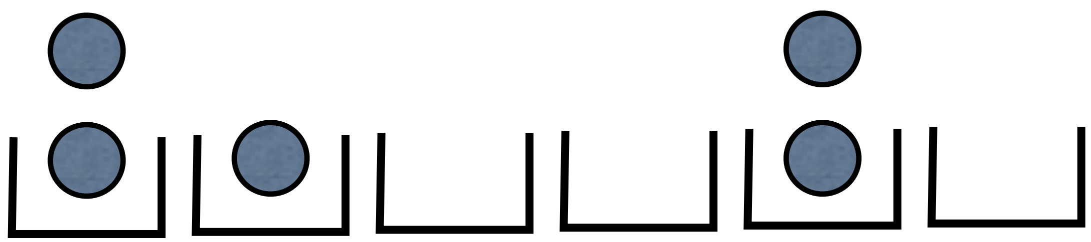

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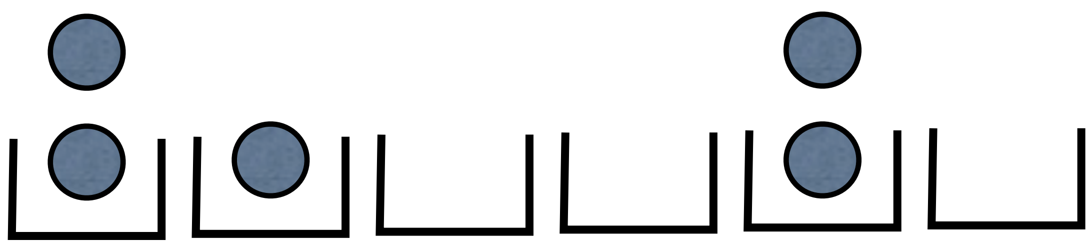

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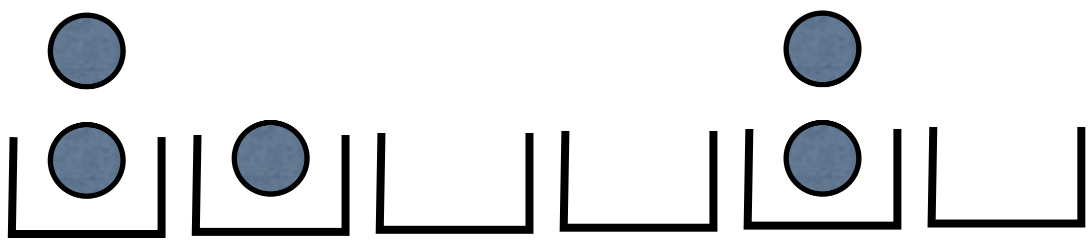

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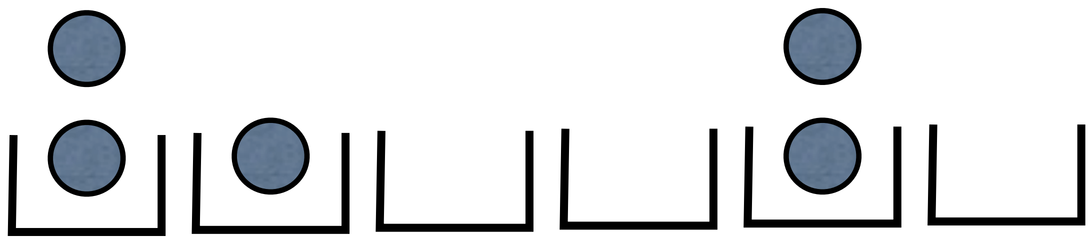

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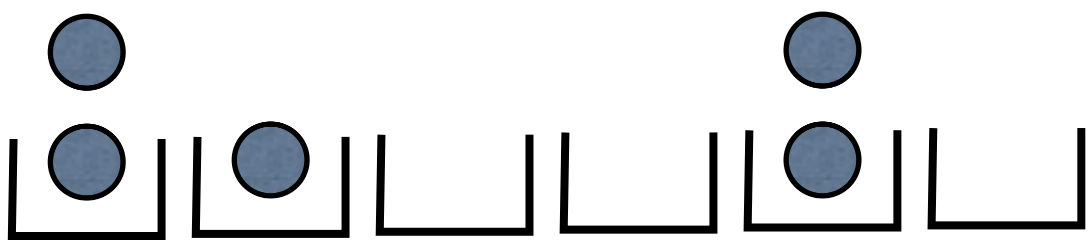

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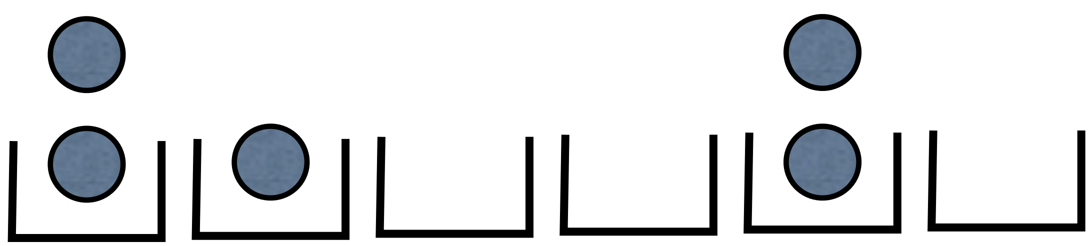

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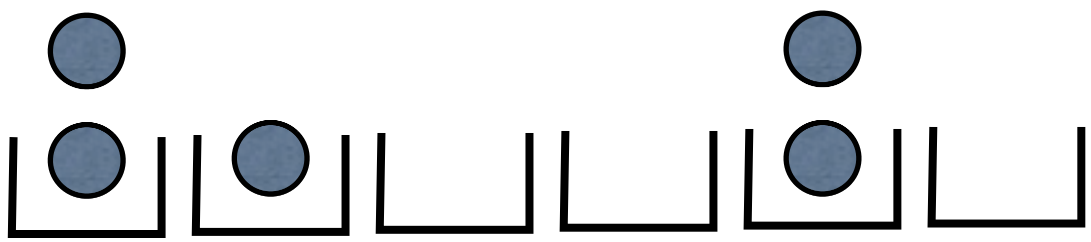

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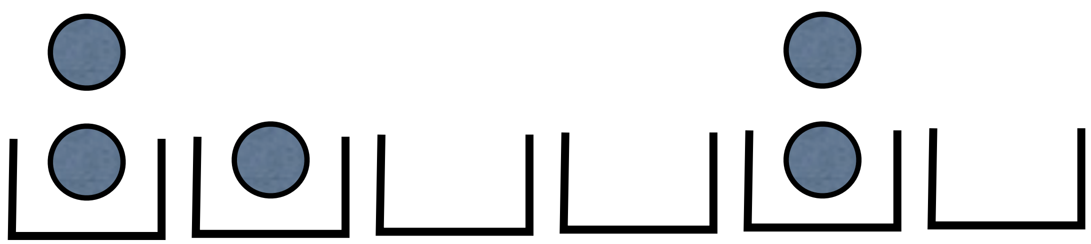

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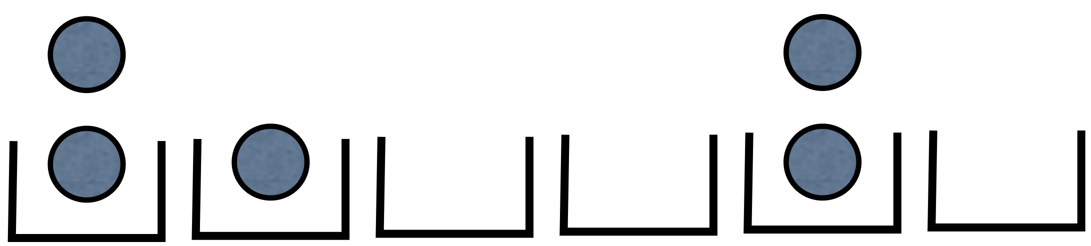

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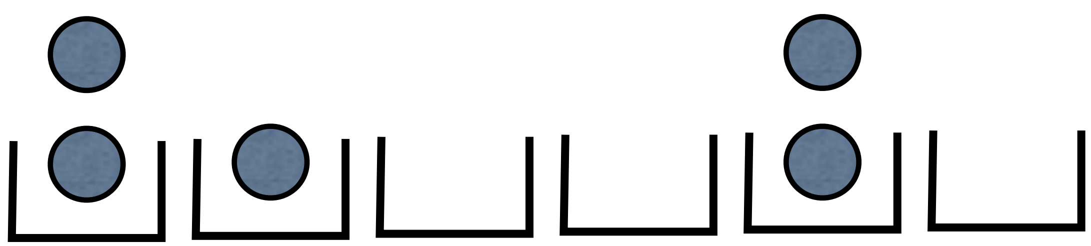

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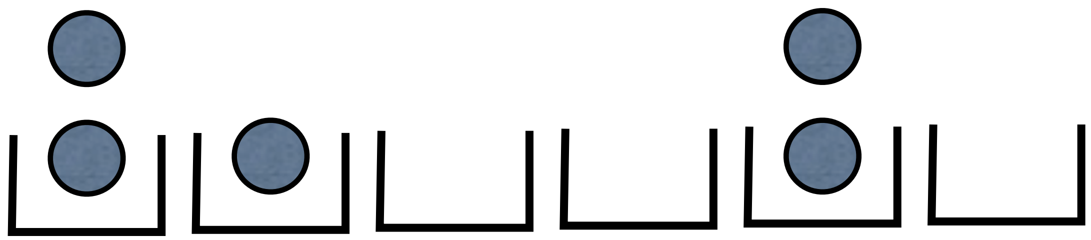

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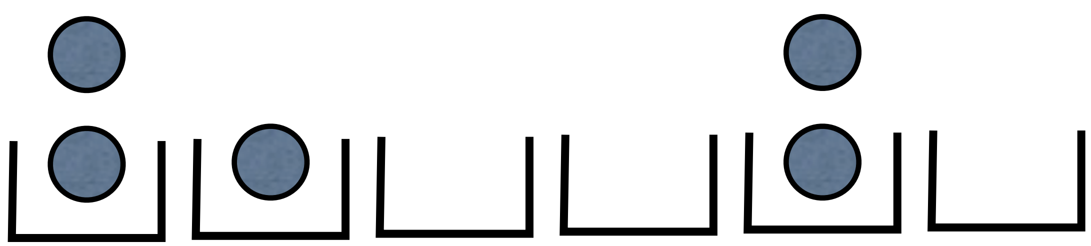

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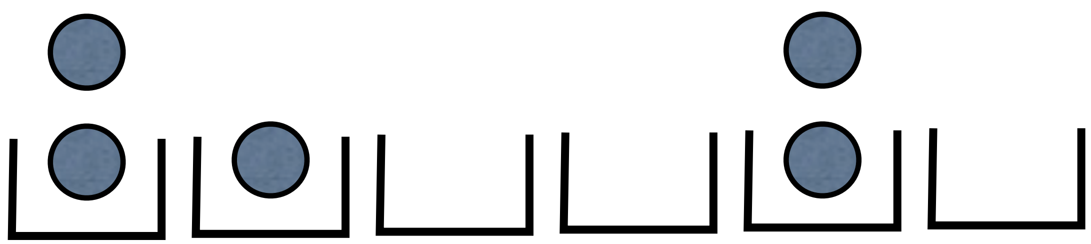

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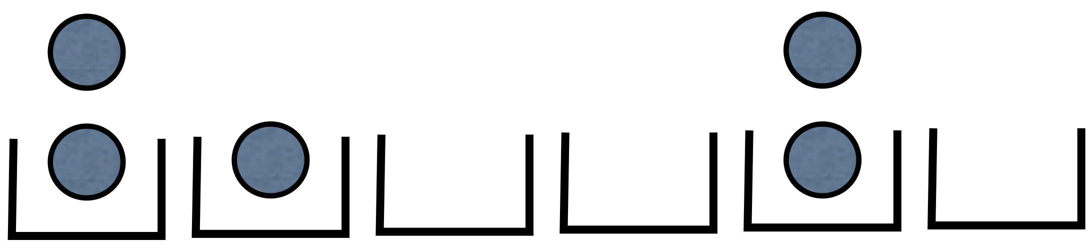

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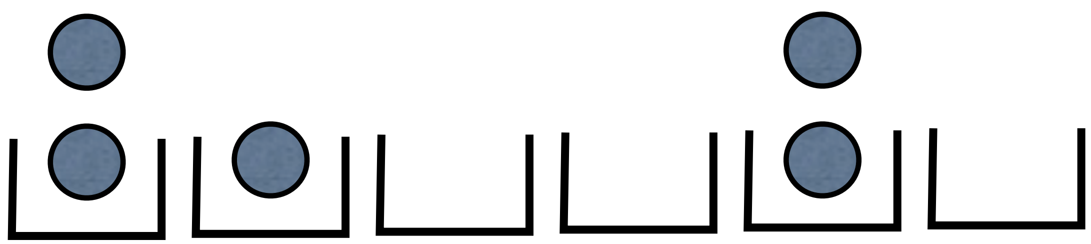

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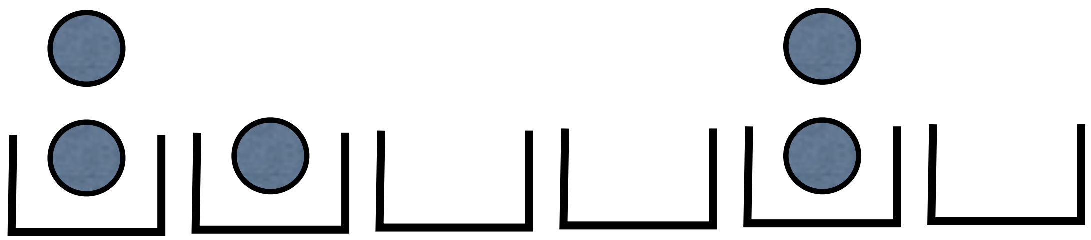

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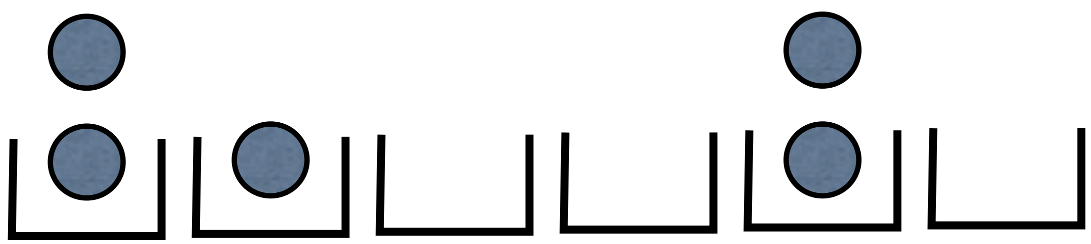

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Lemma 3: If $a_i^2 \geq 6 \ln n / n$ then $\Pr(not(E_{i+1})) \leq (i+1)/n^2$

Let $I^* = \min_i \{a_i^2 \leq 6 \ln n / n\}$ Then, $\Pr[not(E_{I^*})] \leq I^*/n^2 \leq 1/n$

$\Pr(N_{I^*+1} > 16 \ln n) \leq 2/n$

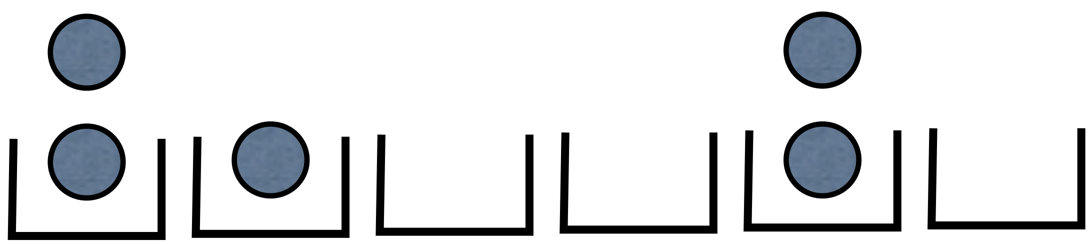
$$\Pr(N_{I^*+2} \geq 1) \leq \Pr(Bin(n, \frac{(16 \ln n)^2}{n}) \geq 1) + \Pr(N_{I^*+1} > 16 \ln n)$$

$$\leq E(Bin(n, \frac{(16 \ln n)^2}{n})) + 2/n \quad \leq O(\log^2 n / n)$$

Lemma 4: Wp (1 - $\log^2 n/n$), there are no bins with load I* + 2 or more

Power of Two Choices

Toss n balls into n bins. For each ball, we pick two bins u.a.r and put it in the lighter of the two. What is the maximum load of a bin with high probability?



Fact: Max load of a bin = O(log log n), w.p. 1-1/n

Define: N_i = #bins with load i or more

Lemma 1: For any a, $\Pr(N_{i+1} > t | N_i \leq an) \leq \frac{\Pr(Bin(n, a^2) > t)}{\Pr(N_i \leq an)}$

Lemma 2: If $a^2 \geq 6 \ln n / n$ then $\Pr(Bin(n, a^2) > 2na^2) \leq 1/n^2$

If $a^2 < 6 \ln n / n$ then $\Pr(Bin(n, a^2) > 16 \ln n) \leq 1/n^2$

Define: Sequence of a_i values: a_4 = 1/4 $a_{i+1} = 2a_i^2$ E_i = good event ($N_i <= a_i$ n)

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Let $I^* = \min_i \{a_i^2 \leq 6 \ln n / n\}$ Then, $\Pr[not(E_{I^*})] \leq I^*/n^2 \leq 1/n$

Lemma 4: Wp (1 - $\log^2 n/n$), no bins with load I^* + 2 or more

Can be shown that $a_i \approx \frac{1}{2^{2^i}}$ from which $I^* = O(\log \log n)$