Some Studies on Big Bang Model and Early Dark Energy

A Dissertation thesis
submitted to the University of Delhi

by

Yogita Kumari(21026762068)

M.Sc Physics (F)

Under the supervision of

Prof. Patrick Das Gupta

Department of Physics and Astrophysics
University of Delhi-110007

Certificate

This is to certify that the dissertation thesis report entitled **"Some Studies on Big Bang Model and Early Dark Energy"** submitted to Department of Physics and Astrophysics, University of Delhi-110007 by **Yogita Kumari (Roll Number: 21026762068)** for partial fulfilment of the requirements for the degree of **Master of Science** Physics 2021-23 contains work done by her under my supervision and guidance. This work has not been submitted in part or in full to any other Institute or University for any degree..

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Prof. Patrick Das Gupta
Department of Physics and Astrophysics,
University of Delhi-110007

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Date:

Signature of Student
Yogita Kumari
Roll Number : 21026762068
M.Sc.Physics(F)

Contents

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Abstract

The question that has kept human beings busy from early ages till now is that how the universe we are a part of came into existence. Many attempts are available to explain it in science, philosophical views, religious texts, etc. Universe was considered static till the time Hubble found around 1929 that galaxies are receding from us which could be the consequence of expanding universe. Perlmutter , Riess and Schmidt were given a Nobel prize in 2011 to discover that the universe is accelerating via making measurements using Type 1A Supernovae.So far, the most successful explanation of the accelerating universe is Standard Cosmological Model (ΛCDM Model).The discrepancy arises in value of Hubble constant calculated from the Cosmic Microwave background via fitting ΛCDM Model and from local measurements using standard candles like Cepheid Variables,Type 1A Supernovae etc.With $H_0 = (74.03 \pm 1.42) km/s/Mpc$ from Riess et al.(2019) and $H_0 = (67.36 \pm 0.54) km/s/Mpc$ by Planck this tension stand at 4.4σ significance level. This dissertation thesis explores about Hubble tension and the model which can resolve this tension.

Units and Notations

Greek Letters($\mu, \nu, \alpha, ...$) are used to denote space time coordinates while Latin letters ($i, j, k, ...$) are being reserved for spatial coordinates.

Throughout this dissertation thesis , Natural units are being used unless otherwise specified.In natural units, $c = \hbar = \epsilon_0 = k_B = 1$
where,
c=speed of light=2.9979 X 10^8 m s^-1
$\hbar$=reduced Planck constant=1.0546 X 10^{-34} J s
ϵ_0=electric permittivity of free space=8.8542 X 10^-12 C^2 N^{-1} m^{-2}
k_B=Boltzmann Constant=1.3806 X 10^{-23} J K^{-1}
All quantities are being expressed in units of energy(usually GeV) in natural units.SI units ($kg^\alpha m^\beta s^\gamma$) can be converted into natural units $E^{\alpha-\beta-\gamma}$. To revert back SI units we need to multiply by factor $\hbar^{\beta+\gamma} c^{\beta-2\alpha}$. Natural and SI units of certain quantities is begin shown in Table:1.

Quantity	SI Unit	Natural Unit
Mass	kg	E
Length	m	E^{-1}
Time	s	E^{-1}
Energy Density	$kg\ m^{-1}\ s^{-2}$	E^{-4}
Pressure	$kg\ m^{-1}\ s^{-2}$	E^{-4}
Ricci scalar	m^{-2}	E^2
Action	dimensionless	dimensionless
Lagrangian density	m^{-4}	E^4
Gravitational Constant	$kg^{-1} m^3 s^{-2}$	E^{-2}

Table 1: SI and Natural Units

Planck mass M_p is defined as :

$$M_p = \sqrt{\frac{1}{8\pi G}} \tag{1}$$

Symbol	Description
t	Proper time
τ	Conformal time
$g_{\mu\nu}$	metric tensor
$\tilde{g}_{\mu\nu}$	$A^2 g_{\mu\nu}$
R	Ricci Scalar
;	Covariant Derivative
,	Partial Derivative
L_ϵ	Lie Derivative
S	Action
$\cdot$	$\frac{d}{d\tau}$
$'$	$\frac{d}{dt}$
$\tilde{Q}$	Quantity Q in frame with metric $\tilde{g}_{\mu\nu}$
$\bar{Q}$	unperturbed quantity Q in perturbation theory part
Ω	Density Parameter
H_0	Hubble Constant
Γ	Chrisofel Symbol
$\mathfrak{L}$	Lagrangian density
a	Scale factor
$T_{\mu\nu}$	Energy Momentum tensor
ϕ	Scalar field
w	Equation of state parameter
D $h_{\mu\nu}$	Dimension of spacetime Pertubation in metric
ρ	density
p	Pressure
β	Coupling Constant
c_s	Speed of sound
D_A	Angular diameter distance
ρ_c	Critical density
d_L	Lumonsity Distance
r_s	Sound Horizon

Table 2: Symbols with description

Chapter 1

Introduction to Cosmology

Cosmology can be considered the study of the structure of the universe and its evolution. Modelling of our universe is being done based on the cosmological principle. This principle states that the universe is homogeneous and isotropic in space at large scales $\sim$ 200 Mpc (parsec is a unit of distance 1pc=distance at which 1 AU subtends an angle of 1 arcsec = 3.26ly) which implies that the universe is maximally symmetric in space (i.e. the universe is homogeneous and isotropic at hypersurface where t = constant). Cosmic microwave background (CMB) is a clear manifestation of it with anisotropy only at the level of 10^{-5}. The tool that we use to study cosmology is General theory of relativity.

1.1 General Theory of Relativity

On the scale at which cosmology is studied, the interaction in the picture is mainly due to the gravitational field. General theory of relativity is study of this gravitational field, propagated by Einstein in 1915. It is based on the principle of equivalence principle. According to this principle, for a small enough region we can't distinguish between a frame under uniform acceleration and a frame under gravitational field so for small enough region we can make gravity vanish which is not case with any other physical interaction since laws of nature should remain same in all frames so gravity is not a force. According to Einstein, gravity is curvature of space-time (background under which all fields and matter interact). Mathematical representation of curvature need understanding of following concepts-

1.1.1 Metric:

Metric have a notion of distance between two points in space-time which is independent of coordinate system. Line element for space time is given as:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \tag{1.1}$$

here $g_{\mu\nu}$ is metric tensor which is 2-rank symmetric tensor.If two events in space-time are connected in such is way that $ds^2 < 0 (> 0)$ then we get time-like (space-like) separation.In case of $ds^2 = 0$ we get null separation.

1.1.2 Christoffel Symbol:

when a quantity(vector ,tensor) is parallel transported in space time due to curvature of space-time it no longer remain parallel so take into account this change covariant derivative is used which is defined as:

$$A^\mu{}_{\nu;\alpha} = A^\mu{}_{\nu,\alpha} + \Gamma^\mu_{\beta\alpha} A^\beta{}_\nu - \Gamma^\beta_{\nu\alpha} A^\mu{}_\beta \tag{1.2}$$

where $\Gamma^{\mu}_{\alpha\beta}$ is defined as:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}(g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu} + g_{\beta\nu,\alpha}) \tag{1.3}$$

It is not a tensor.

1.1.3 Curvature Tensor

Curvature of space-time is defined as:

$$R^{\mu}_{\alpha\beta\gamma} = \Gamma^{\mu}_{\alpha\gamma,\beta} - \Gamma^{\mu}_{\alpha\beta,\gamma} + \Gamma^{\nu}_{\alpha\gamma}\Gamma^{\mu}_{\nu\beta} - \Gamma^{\nu}_{\alpha\beta}\Gamma^{\mu}_{\nu\gamma} \tag{1.4}$$

Ricci tensor

$R_{\mu\nu}$ is Ricci Tensor which we get via contracting Curvature tensor in first and third indices .

$$R_{\alpha\gamma} = R^{\mu}_{\alpha\mu\gamma} = \Gamma^{\mu}_{\alpha\gamma,\mu} - \Gamma^{\mu}_{\alpha\nu,\gamma} + \Gamma^{\nu}_{\alpha\gamma}\Gamma^{\mu}_{\nu\mu} - \Gamma^{\nu}_{\alpha\mu}\Gamma^{\mu}_{\nu\gamma} \tag{1.5}$$

Ricci Scalar

R is Ricci Scalar which we find via contracting Ricci tensor

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu} R_{\mu\nu} \tag{1.6}$$

1.1.4 Lie Derivative

when a quantity is moved from one point to another point in spacetime along a vector ξ^{μ}then change in quantity is given as

$$L_{\xi} A^{\alpha}{}_{\beta} = \xi^{\mu} A^{\alpha}{}_{\beta;\mu} - \xi^{\alpha}{}_{;\mu} A^{\mu}{}_{\beta} + \xi^{\mu}{}_{;\beta} A^{\alpha}{}_{\mu} \tag{1.7}$$

1.1.5 Energy Momentum Tensor

In case we have a system of particles in which we do not require to know the motion of individual particle, we define energy-momentum tensor $T_{\mu\nu}$ which describes macroscopic quantities like pressure, and density. $T_{\mu\nu}$ is symmetric tensor.

1.1.6 Einstein Equation

To understand any phenomena that are occurring in the universe, the First thing that needs to understand is the structure of space-time in which all interactions propagate except gravity which is a kind of manifestation of space-time itself. Gravity can be understood as the background on which all other forces are going to propagate. The relationship between gravity and space-time can be better explained via Einstein's equation-

$$M_P^2 \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = T_{\mu\nu} \tag{1.8}$$

here Left hand side can be considered as information about the space time. $T_{\mu\nu}$ is energy momentum tensor which is associated with matter. So this relation clearly make a relationship between gravity(Since it is associated with matter) and Spacetime.

1.1.7 Classical Field Theory

for a field in spacetime, action is defined as -

$$S = \int d^4x\sqrt{-g}\mathfrak{L} \tag{1.9}$$

where $\mathfrak{L}$ is known as Lagrangian Density.

Lagrangian density for gravity is given by

$$\mathfrak{L}_{\mathfrak{G}} = \frac{M_P^2}{2}R \tag{1.10}$$

Energy momentum tensor for matter or field can be expressed as:

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta L}{\delta g^{\mu\nu}}$$

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\left\{\frac{\partial L}{\partial g^{\mu\nu}} - \partial_\alpha\frac{\partial L}{\partial g^{\mu\nu}_{,\alpha}}\right\} \tag{1.11}$$

According to principle of least action, variation of total action need to be zero

1.2 Applying GTR to large scale structure

With a mathematical background of general theory of relativity let's explore large scale structure in mathematical terms.

1.2.1 Metric for universe

The cosmological Principle requires a spatially maximally symmetric universe which implies that the spatial part of the metric need to be maximally symmetric. This requirement is being fulfilled by the following metric.

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}dx^i dx^j \tag{1.12}$$

where γ_{ij} is function of spatial part only

$$ds^2 = -dt^2 + R^2(t)\left(d\chi^2 + S_k^2(\chi)d\Omega^2\right) \tag{1.13}$$

where

$$S_k(\chi) = \begin{cases} sin(\chi), & k = +1 \\ \chi, & k = 0 \\ sinh(\chi), & k = -1 \end{cases}$$

for $k = 0$ spatial part of metric is simply flat Euclidean space with no curvature. For $k = +1$ spatial part represent the metric for the 3-sphere(closed surface or positive curvature)and $k = -1$ spatial part represents 3-D space of constant negative curvature (hyperbolic space). If one tries to measure in different spaces then one will get different results. or a more popular way to write it down is as follows-

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right) \tag{1.14}$$

where t is the cosmological time which is the proper time It is known as Friedmann Lemaitre Robertson Walker(FLRW) metric. where $a(t)$ is scale factor which is evolving only in time.

$$\kappa = \begin{cases} \kappa = 0, & \text{for zero curvature} \\ \kappa < 0, & \text{for negative curvature} \\ \kappa > 0, & \text{for positive curvature} \end{cases} \tag{1.15}$$

1.2.2 Energy Momentum Tensor

The dynamics of a system is explained by Einstein equations. Metric for the universe is given by FLRW metric from where we can Ricci tensor and Ricci scalar using eq(1.5) and eq(1.5) respectively.So the left part of Einstein equation is given. The right part of equation is energy momentum tensor.Universe is usually taken as made of components like perfect fluid and/or scalar field.

For perfect fluid,energy momentum tensor is given as:

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu} \tag{1.16}$$

where u_μ is four-velocity

For a scalar field,energy momentum tensor can be evaluated as follows:

$$\begin{aligned}
\mathfrak{L}_\phi =& -\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi) && (1.17)\\
T_{\lambda\nu} =& -\frac{2}{\sqrt{-g}}\frac{\delta(\sqrt{-g}\mathfrak{L}_\phi)}{\delta g^{\lambda\nu}} && (1.18)\\
=& -\frac{2}{\sqrt{-g}}\left\{\frac{\partial(\sqrt{-g}\mathfrak{L}_\phi)}{\partial g^{\lambda\nu}} - \partial_\gamma \frac{\partial(\sqrt{-g}\mathfrak{L}_\phi)}{\partial g^{\lambda\nu}{}_{,\gamma}}\right\}\\
=& -\frac{2}{\sqrt{-g}}\left\{\sqrt{-g}\frac{\partial\mathfrak{L}_\phi}{\partial g^{\lambda\nu}} + \mathfrak{L}_\phi\frac{\partial(\sqrt{-g})}{\partial g^{\lambda\nu}}\right\}\\
=& -\frac{2}{\sqrt{-g}}\left[\sqrt{-g}\frac{\partial\{-\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi)\}}{\partial g^{\lambda\nu}} + \{-\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi)\}\frac{\partial(\sqrt{-g})}{\partial g^{\lambda\nu}}\right]\\
=& -\frac{2}{\sqrt{-g}}\left[\sqrt{-g}\left(-\frac{1}{2}\phi_{,\lambda}\phi_{,\nu}\right) + \{-\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi)\}\left(-\frac{1}{2\sqrt{-g}}\right)\frac{\partial g}{\partial g^{\lambda\nu}}\right]\\
=& \phi_{,\lambda}\phi_{,\nu} - \frac{1}{2}g_{\lambda\nu}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - g_{\lambda\nu}V(\phi) && (1.19)\\
T^\mu{}_\nu =& g^{\mu\lambda}T_{\lambda\nu} = g^{\mu\lambda}\phi_{,\lambda}\phi_{,\nu} - \frac{1}{2}\delta^\mu_\nu g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \delta^\mu_\nu V(\phi) && (1.20)
\end{aligned}$$

1.3 A Discussion on flat space

Since theoretical explanation and observational evidences are in favour of a flat space so this section is devoted to discuss flat space.

1.3.1 Metric Tensor for flat space

In flat space($\kappa = 0$) metric is given as:

$$ds^2 = -dt^2 + a^2(t)\left(dr^2 + r^2 d\Omega^2\right) \tag{1.21}$$

or in Cartesian coordinate system-

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{1.22}$$

Now taking the transformation $a(t) \to a(\tau)$(here function form of a also changes just writing same notation for convenience) such that $dt = a(\tau)d\tau$ metric becomes:

$$ds^2 = a^2(\tau)\left(-d\tau^2 + dx^2 + dy^2 + dz^2\right) \tag{1.23}$$

here τ is conformal time.

$$x^0 = \tau \quad x^1 = x \quad x^2 = y \quad x^3 = z \tag{1.24}$$

Component of metric and inverse metric $((g_{\mu\alpha}g^{\alpha\nu} = \delta^{\nu}{}_{\mu})$ are given as follows:

$$\begin{aligned} g_{00} = -a^2 \quad g_{11} = a^2 \quad g_{22} = a^2 \quad g_{33} = a^2 \\ g^{00} = -\tfrac{1}{a^2} \quad g^{11} = \tfrac{1}{a^2} \quad g^{22} = \tfrac{1}{a^2} \quad g^{33} = \tfrac{1}{a^2} \end{aligned} \tag{1.25}$$

1.3.2 Christofel Symbols for flat space

Calculating christofel symbols:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu} + g_{\beta\nu,\alpha})$$

$$\begin{aligned} \Gamma^0_{00} = \tfrac{\dot{a}}{a} \qquad \Gamma^0_{11} = \tfrac{\dot{a}}{a} \qquad \Gamma^0_{22} = \tfrac{\dot{a}}{a} \qquad \Gamma^0_{33} = \tfrac{\dot{a}}{a} \\ \Gamma^1_{10} = \tfrac{\dot{a}}{a} = \Gamma^1_{01}, \quad \Gamma^2_{20} = \tfrac{\dot{a}}{a} = \Gamma^2_{02}, \quad \Gamma^3_{30} = \tfrac{\dot{a}}{a} = \Gamma^3_{03} \end{aligned} \tag{1.26}$$

all other christofel symbols turn out to be zero.

1.3.3 Ricci Tensor and Ricci Scalar for flat space

Curvature tensor is given by:

$$R^{\mu}_{\alpha\beta\gamma} = \Gamma^{\mu}_{\alpha\gamma,\beta} - \Gamma^{\mu}_{\alpha\beta,\gamma} + \Gamma^{\nu}_{\alpha\gamma}\Gamma^{\mu}_{\nu\beta} - \Gamma^{\nu}_{\alpha\beta}\Gamma^{\mu}_{\nu\gamma}$$

Calculating Ricci tensor and Ricci Scalar:

$$R_{\alpha\gamma} = \Gamma^{\mu}_{\alpha\gamma,\mu} - \Gamma^{\mu}_{\alpha\nu,\gamma} + \Gamma^{\nu}_{\alpha\gamma}\Gamma^{\mu}_{\nu\mu} - \Gamma^{\nu}_{\alpha\mu}\Gamma^{\mu}_{\nu\gamma}$$

$$\begin{aligned} R_{00} &= 3\left(-\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \\ R_{ij} &= \delta_{ij}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \end{aligned} \tag{1.27}$$

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu} = \frac{6\ddot{a}}{a^3} \tag{1.28}$$

1.3.4 Energy Momentum Tensor in flat space

for Perfect Fluid

$$T_{00} = a^2\rho \tag{1.29}$$
$$T_{ij} = pa^2\delta_{ij} \tag{1.30}$$

for Scalar Field

$$\begin{aligned} T^0{}_0 =& g^{0\lambda}\phi_{,\lambda}\phi_{,0} - \frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - V(\phi) \\ =& -\frac{1}{2}\frac{\dot{\phi}^2}{a^2} - V(\phi) \end{aligned} \tag{1.31}$$

$$T^0{}_k = 0 \tag{1.32}$$

$$\begin{aligned} T^i{}_j =& g^{i\lambda}\phi_{,\lambda}\phi_{,j} - \frac{1}{2}\delta^i_j g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \delta^i_j V(\phi) \\ =& \frac{1}{2}\delta^i_j\frac{\dot{\phi}^2}{a^2} - \delta^i_j V(\phi) \end{aligned} \tag{1.33}$$

1.3.5 Equation of state

For the total description of universe, we need to know the relation between ρ and p.

For barotropic fluid,

$$p = w\rho \tag{1.34}$$

where w is equation of state parameter . Values for w for some specific cases are as shown in Table:1.1

1.3.6 Einstein Equation for flat space

$$M_P^2 G_{\mu\nu} = M_P^2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) = T_{\mu\nu}$$

Calculating different components of left hand side:

$$G_{00} = 3\frac{\dot{a}^2}{a^2} \tag{1.35}$$
$$G_{ij} = \delta_{ij}\left(\frac{-2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \tag{1.36}$$

Now from eq(1.35),eq(1.29) ,eq (1.36) and eq(1.30)

$$3M_P^2\frac{\dot{a}^2}{a^2} = a^2\rho \tag{1.37}$$
$$M_P^2\left(\frac{-2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = pa^2 \tag{1.38}$$

Adding eq(1.37) and 3 X eq(1.38)

$$3M_P^2\frac{d}{d\tau}\left(\frac{\dot{a}}{a}\right) = -\frac{(\rho + 3p)a^2}{2} \tag{1.39}$$

Eq (1.37) and (1.39) are known as Friedmann Equations which give the evolution of scale factor a .

1.3.7 Continuity Equation for flat space

$$T^{\mu}{}_{\nu;\mu} = 0 = T^{\mu}{}_{\nu,\mu} + \Gamma^{\mu}{}_{\alpha\mu}T^{\alpha}{}_{\nu} - \Gamma^{\alpha}{}_{\nu\mu}T^{\mu}{}_{\alpha} \tag{1.40}$$

taking case $\nu = 0$ and $\nu = j$

$$\begin{aligned} T^{\mu}{}_{0;\mu} &= 0 \\ T^{\mu}{}_{0,\mu} + \Gamma^{\mu}{}_{\alpha\mu}T^{\alpha}{}_{0} - \Gamma^{\alpha}{}_{0\mu}T^{\mu}{}_{\alpha} &= 0 \\ T^{\mu}{}_{j;\mu} &= 0 \\ T^{\mu}{}_{j,\mu} + \Gamma^{\mu}{}_{\alpha\mu}T^{\alpha}{}_{j} - \Gamma^{\alpha}{}_{j\mu}T^{\mu}{}_{\alpha} &= 0 \end{aligned}$$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0 \tag{1.41}$$

$$p_{,j} = 0 \tag{1.42}$$

Now, using eq(1.41) and eq(1.34)

$$\frac{d\rho}{\rho} = -3(1+w)\frac{da}{a}$$

integration and taking limits a at some time t with density ρ to a_0 at present time $t_0$with density ρ_0

$$\frac{\rho}{\rho_0} = \left(\frac{a}{a_0}\right)^{-3(1+w)} \tag{1.43}$$

It shows how density evolves with scale factor. Now from F1

$$3M_P^2\frac{a'^2}{a^2} = \rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)}$$

where $'$ implies $\frac{d}{dt}$,here t is proper time . Now taking $\frac{a}{a_0} = x$ and reshuffling the equation we are going to get-

$$x^{\frac{3(1+w)}{2}}dx = \sqrt{\frac{\rho_0}{3M_P^2}}dt$$

now integrating ,taking limit as $t \to 0$ $x(t) \to 0$ and putting back value of $x(t)$

$$a = a_0\left(\frac{2}{3(1+w)}\sqrt{\frac{\rho_0}{3M_P^2}}t\right)^{\frac{2}{3(1+w)}} \tag{1.44}$$

1.4 Content of Universe:

From Friedmann equations it is clear evolution depends on the content of the universe. In this section, we are going to discuss how the universe with particular content evolves over time.

1.4.1 Single Component Universe

Table:1.1 has shown evolution equations for some of the single component universe.

Table 1.1: Content of our universe

	w	Equation of state		
matter(dust)	0	$p=0$	$\frac{\rho}{\rho_0}=\left(\frac{a}{a_0}\right)^{-3}$	$a \; \alpha \; t^{\frac{2}{3}}$
Radiation	$\frac{1}{3}$	$p=\frac{1}{3}\rho$	$\frac{\rho}{\rho_0}=\left(\frac{a}{a_0}\right)^{-4}$	$a \; \alpha \; t^{\frac{1}{2}}$
Cosmological Constant(Λ)	-1	$p=-\rho=-\Lambda M_P^2$	$\frac{\rho}{\rho_0}=1$	$a \; \alpha \; exp\left(\frac{\sqrt{\Lambda}}{3}t\right)$

Multi-component Universe

If the universe consists of more than one component universe then Friedmann equations are described as follows:

$$3M_P^2\frac{\dot{a}^2}{a^2}=a^2\sum_i \rho_i \tag{1.45}$$

$$3M_P^2\frac{d}{d\tau}\left(\frac{\dot{a}}{a}\right)=-\frac{\sum_i(\rho_i+3p_i)a^2}{2} \tag{1.46}$$

here i refers to i^{th} component of universe

Continuity equation is given as:

$$\sum_i\left\{\dot{\rho}_i+3\frac{\dot{a}}{a}\left(\rho_i+p_i\right)\right\}=0 \tag{1.47}$$

$$\tag{1.48}$$

from eq(1.47) it can be seen that total energy density needs to be conserved but one component need not to be conserved. It depends on whether components are interacting with each other or not. In the case of a non-interacting scenario, all components are individually conserved.

1.5 Deriving some useful expressions

Two metrics are related to each other via the following relation

$$\tilde{g}_{\mu\nu}=A^2(\phi(x))g_{\mu\nu}=A^2(x)g_{\mu\nu} \tag{1.49}$$

where $A^2(x)$ is a smooth, non-vanishing function of space-time dependent function. This transformation is different then coordinate transformation which does not change geometry. Let's see how is it going to change geometry-

1. Inverse Metric:

$$\begin{aligned}
\tilde{g}_{\mu\nu}\tilde{g}^{\mu\sigma}&=\delta^\sigma_\nu\\
\text{using eq(1.49)}&\\
A^2g_{\mu\nu}\tilde{g}^{\mu\sigma}&=\delta^\sigma_\nu\\
\text{multiplying } g^{\lambda\nu}&\\
A^2g^{\lambda\nu}g_{\mu\nu}\tilde{g}^{\mu\sigma}&=g^{\lambda\nu}\delta^\sigma_\nu\\
A^2\delta^\lambda_\mu\tilde{g}^{\mu\sigma}&=g^{\lambda\sigma}\\
A^2\tilde{g}^{\lambda\sigma}&=g^{\lambda\sigma}\\
\tilde{g}^{\lambda\sigma}&=\frac{1}{A^2}g^{\lambda\sigma}
\end{aligned} \tag{1.50}$$

2. Determinant of metric:

   By definition

$$\frac{\delta g}{g} = g^{\mu\nu}\delta g_{\mu\nu} \tag{1.51}$$

$$\begin{aligned}
\frac{\delta \tilde{g}}{\tilde{g}} &= \tilde{g}^{\mu\nu}\delta \tilde{g}_{\mu\nu} \\
&= \frac{1}{A^2} g^{\mu\nu}\delta(A^2 g_{\mu\nu}) \\
&= \frac{1}{A^2} g^{\mu\nu}(A^2\delta g_{\mu\nu} + 2A\delta A g_{\mu\nu}) \\
&= \frac{\delta g}{g} + 2D\frac{\delta A}{A}
\end{aligned}$$

$$\tilde{g} = gA^{2D} \tag{1.52}$$

3. Christofel Symbol:

$$\begin{aligned}
\tilde{\Gamma}^{\mu}_{\alpha\beta} &= \frac{1}{2}\tilde{g}^{\mu\nu}(g_{\nu\alpha\tilde{,}\beta} - \tilde{g}_{\alpha\beta\tilde{,}\nu} + \tilde{g}_{\beta\nu\tilde{,}\alpha}) \\
&= \frac{A^{-2}}{2} g^{\mu\nu}((A^2 g_{\nu\alpha})_{,\beta} - (A^2 g_{\alpha\beta})_{,\nu} + (A^2 g_{\beta\nu})_{,\alpha}) \\
&= \frac{1}{2} g^{\mu\nu}(g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu} + g_{\beta\nu,\alpha}) + \frac{2AA^{-2}}{2} g^{\mu\nu}(g_{\nu\alpha}A_{,\beta} - g_{\alpha\beta}A_{,\nu} + g_{\beta\nu}A_{,\alpha}) \\
&= \Gamma^{\mu}_{\alpha\beta} + \frac{1}{A}(\delta^{\mu}_{\alpha}A_{,\beta} + \delta^{\mu}_{\beta}A_{,\alpha} - g_{\alpha\beta}g^{\mu\nu}A_{,\nu})
\end{aligned} \tag{1.53}$$

   A useful form can be given as-

$$\begin{aligned}
\tilde{\Gamma}^{\mu}_{\alpha\mu} &= \Gamma^{\mu}_{\alpha\mu} + \frac{1}{A}(\delta^{\mu}_{\alpha}A_{,\mu} + \delta^{\mu}_{\mu}A_{,\alpha} - g_{\alpha\mu}g^{\mu\nu}A_{,\nu}) \\
&= \Gamma^{\mu}_{\alpha\mu} + D\frac{A_{,\alpha}}{A}
\end{aligned} \tag{1.54}$$

4. Action

$$\begin{aligned}
\tilde{S} &= S \\
\int d^D x\sqrt{-\tilde{g}}\tilde{\mathcal{L}} &= \int d^D x\sqrt{-g}\mathcal{L} \\
\int d^D x A^D\sqrt{-g}\tilde{\mathcal{L}} &= \int d^D x\sqrt{-g}\mathcal{L} \\
A^D\tilde{\mathcal{L}} &= \mathcal{L} \\
\tilde{\mathcal{L}} &= A^{-D}\mathcal{L}
\end{aligned} \tag{1.55}$$

5. Energy Momentum Tensor

$$\begin{aligned}\tilde{T}^{\alpha\beta} &= \frac{-2}{\sqrt{-\tilde{g}}}\frac{\delta(\sqrt{-\tilde{g}}\tilde{\mathcal{L}})}{\delta\tilde{g}_{\alpha\beta}} \\ &= \frac{-2}{A^D\sqrt{-g}}\frac{\delta g_{\gamma\sigma}}{\delta\tilde{g}_{\alpha\beta}}\frac{\delta(A^D\sqrt{-g}A^{-D}\mathcal{L})}{\delta g_{\gamma\sigma}} \\ &= \frac{-2}{A^D\sqrt{-g}}\frac{\delta(A^{-2}\tilde{g}_{\gamma\sigma})}{\delta\tilde{g}_{\alpha\beta}}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\gamma\sigma}} \\ &= \frac{-2}{\sqrt{-g}}A^{-D-2}\frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g_{\alpha\beta}} \\ &= A^{-D-2}T^{\alpha\beta}\end{aligned} \tag{1.56}$$

with 1 index up

$$\begin{aligned}\tilde{g}_{\beta\mu}\tilde{T}^{\alpha\beta} &= A^2 g_{\beta\mu}A^{-D-2}T^{\alpha\beta} \\ \tilde{T}^{\alpha}{}_{\mu} &= A^{-D}T^{\alpha}{}_{\mu}\end{aligned} \tag{1.57}$$

taking trace

$$\begin{aligned}\tilde{T}^{\alpha}{}_{\alpha} &= A^{-D}T^{\alpha}{}_{\alpha} \\ \tilde{T} &= A^{-D}T\end{aligned} \tag{1.58}$$

$$\begin{aligned}\tilde{T}^{\mu\nu} &= (\tilde{\rho}+\tilde{p})\tilde{u}^{\mu}\tilde{u}^{\nu} + \tilde{p}\tilde{g}^{\mu\nu} \\ \text{here } \tilde{u}^{\mu} = \frac{dx^{\mu}}{d\tilde{s}} = \frac{1}{A}\frac{dx^{\mu}}{ds} &= A^{-1}u^{\mu} \\ \tilde{T}^{\mu\nu} &= (\tilde{\rho}+\tilde{p})A^{-2}u^{\mu}u^{\nu} + \tilde{p}A^{-2}g^{\mu\nu} \\ A^{-D-2}T^{\mu\nu} &= (\tilde{\rho}+\tilde{p})A^{-2}u^{\mu}u^{\nu} + \tilde{p}A^{-2}g^{\mu\nu} \\ T^{\mu\nu} &= A^{D}(\tilde{\rho}+\tilde{p})u^{\mu}u^{\nu} + A^{D}\tilde{p}g^{\mu\nu}\end{aligned}$$

comparing with $T^{\mu\nu} = (\rho+p)u^{\mu}u^{\nu} + pg^{\mu\nu}$

$$\tilde{p} = A^{-D}p \tag{1.59}$$

$$\tilde{\rho} = A^{-D}\rho \tag{1.60}$$

6. Continuity Equation:

$$\tilde{T}^{\mu}{}_{\nu;\mu} = 0 \tag{1.61}$$

$$\begin{aligned}\tilde{T}^{\mu}{}_{\nu,\mu} + \tilde{\Gamma}^{\mu}{}_{\alpha\mu}\tilde{T}^{\alpha}{}_{\nu} - \tilde{\Gamma}^{\alpha}{}_{\nu\mu}\tilde{T}^{\mu}{}_{\alpha} &= 0 \\ (A^{-D}T^{\mu}{}_{\nu})_{,\mu} + A^{-D}T^{\alpha}{}_{\nu}(\Gamma^{\mu}_{\alpha\mu} + D\frac{A_{,\alpha}}{A}) & \\ -(\Gamma^{\alpha}{}_{\nu\mu} + \frac{1}{A}(\delta^{\alpha}_{\mu}A_{,\nu} + \delta^{\alpha}_{\nu}A_{,\mu} - g_{\mu\nu}g^{\alpha\beta}A_{,\beta}))A^{-D}T^{\mu}{}_{\alpha} &= 0 \\ T^{\mu}{}_{\nu,\mu} + \Gamma^{\mu}{}_{\alpha\mu}T^{\alpha}{}_{\nu} - \Gamma^{\alpha}{}_{\nu\mu}T^{\mu}{}_{\alpha} - D\frac{A_{,\mu}}{A}T^{\mu}{}_{\nu} + D\frac{A_{,\alpha}}{A}T^{\alpha}{}_{\nu} & \\ -\frac{1}{A}(T^{\mu}{}_{\mu}A_{,\nu} + T^{\mu}{}_{\nu}A_{,\mu} - T^{\beta}{}_{\nu}A_{,\beta}) &= 0\end{aligned}$$

$$T^{\mu}{}_{\nu;\mu} = \frac{A_{,\nu}}{A}T \tag{1.62}$$

1.6 Perturbation Theory

This part is intended to understand how small fluctuations in the isotropic universe evolve with time. Maximally symmetric universe in space can be expressed as 1.12

If the universe is not maximally symmetric in space but can be expressed as a small perturbation to FLRW universe then metric can be written as:

$$ds^2 = (\bar{g}_{\mu\nu} + \underline{h}_{\mu\nu})dx^\mu dx^\nu \tag{1.63}$$

$$ds^2 = a^2(\tau)\left(-d\tau^2 + \gamma_{ij}dx^i dx^j + h_{\mu\nu}dx^\mu dx^\nu\right) \tag{1.64}$$

here $\underline{h}_{\mu\nu} = a^2 h_{\mu\nu}$

$\underline{h}_{\mu\nu}$ is perturbation to metric. Inverse metric to $g_{\mu\sigma}$ is $g^{\mu\rho}$ such that :

$$\begin{aligned} g^{\mu\rho}g_{\mu\sigma} &= \delta^\rho_\sigma \\ \left(\bar{g}^{\mu\rho} + \underline{h}^{\mu\rho}\right)\left(\bar{g}_{\mu\sigma} + \underline{h}_{\mu\sigma}\right) &= \delta^\rho_\sigma \\ \text{where } \underline{h}^{\mu\nu} &= g^{\mu\nu} - \bar{g}^{\mu\nu} \\ \bar{g}^{\mu\rho}\bar{g}_{\mu\sigma} + \underline{h}^{\mu\rho}\bar{g}_{\mu\sigma} + \bar{g}^{\mu\rho}\underline{h}_{\mu\sigma} &= \delta^\rho_\sigma \\ \text{multiplying both side with } \bar{g}^{\nu\sigma} \\ \bar{g}^{\nu\sigma}\bar{g}^{\mu\rho}\bar{g}_{\mu\sigma} + \bar{g}^{\nu\sigma}\underline{h}^{\mu\rho}\bar{g}_{\mu\sigma} + \bar{g}^{\nu\sigma}\bar{g}^{\mu\rho}\underline{h}_{\mu\sigma} &= \bar{g}^{\nu\sigma}\delta^\rho_\sigma \\ \bar{g}^{\nu\rho} + \underline{h}^{\mu\rho}\delta^\nu_\mu + \bar{g}^{\nu\sigma}\bar{g}^{\mu\rho}\underline{h}_{\mu\sigma} &= \bar{g}^{\nu\rho} \\ \underline{h}^{\nu\rho} &= -\bar{g}^{\nu\sigma}\bar{g}^{\rho\mu}\underline{h}_{\mu\sigma} \end{aligned} \tag{1.65}$$

so

$$\underline{h}^{\mu\nu} = -\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}\underline{h}_{\rho\sigma} \tag{1.66}$$

$h_{\mu\nu}$ has 10 degree of freedom (for symmetric tensor of rank-2 and dimension N,degree of freedom is given as $\frac{N(N+1)}{2}$). In these 10 degrees of freedom only 6 physical degrees of freedom and 4-gauge dependent degree of freedom(Changing $(\tau, \mathbf{x})$ will lead to change in $h_{\mu\nu}$ since universe is no longer maximally symmetric on hypersurface at constant τ so they are more than one way of choosing these hypersurfaces).

1. **Christofel Symbols** :

$$\begin{aligned} \Gamma^\mu_{\alpha\beta} &= \frac{1}{2}g^{\mu\nu}(g_{\nu\alpha,\beta} - g_{\alpha\beta,\nu} + g_{\beta\nu,\alpha}) \\ &= \frac{1}{2}(\bar{g}^{\mu\nu} + \underline{h}^{\mu\nu})(\bar{g}_{\nu\alpha,\beta} + \underline{h}_{\nu\alpha,\beta} - \bar{g}_{\alpha\beta,\nu} - \underline{h}_{\alpha\beta,\nu} + \bar{g}_{\beta\nu,\alpha} + \underline{h}_{\beta\nu,\alpha}) \\ &= \bar{\Gamma}^\mu_{\alpha\beta} + \frac{1}{2}\bar{g}^{\mu\nu}(\underline{h}_{\nu\alpha,\beta} - \underline{h}_{\alpha\beta,\nu} + \underline{h}_{\beta\nu,\alpha}) + \frac{1}{2}\underline{h}^{\mu\nu}(\bar{g}_{\nu\alpha,\beta} - \bar{g}_{\alpha\beta,\nu} + \bar{g}_{\beta\nu,\alpha}) \end{aligned}$$

using eq(1.66)

$$\begin{aligned} \Gamma^\mu_{\alpha\beta} - \bar{\Gamma}^\mu_{\alpha\beta} &= \frac{1}{2}\bar{g}^{\mu\nu}(\underline{h}_{\nu\alpha,\beta} - \underline{h}_{\alpha\beta,\nu} + \underline{h}_{\beta\nu,\alpha}) - \frac{1}{2}\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}\underline{h}_{\rho\sigma}(2\bar{\Gamma}_{\nu\alpha\beta}) \\ \delta\Gamma^\mu_{\alpha\beta} &= \frac{1}{2}\bar{g}^{\mu\nu}(\underline{h}_{\nu\alpha,\beta} - \underline{h}_{\alpha\beta,\nu} + \underline{h}_{\beta\nu,\alpha} - 2\underline{h}_{\nu\sigma}\bar{\Gamma}^\sigma_{\alpha\beta}) \end{aligned} \tag{1.67}$$

2. **Ricci Tensor**:

$$\begin{aligned} R_{\alpha\gamma} &= \Gamma^\mu_{\alpha\gamma,\mu} - \Gamma^\mu_{\alpha\mu,\gamma} + \Gamma^\nu_{\alpha\gamma}\Gamma^\mu_{\nu\mu} - \Gamma^\nu_{\alpha\mu}\Gamma^\mu_{\nu\gamma} \\ \delta R_{\alpha\gamma} &= \left(\delta\Gamma^\mu_{\alpha\gamma}\right)_{,\mu} - \left(\delta\Gamma^\mu_{\alpha\mu}\right)_{,\gamma} + \delta\Gamma^\nu_{\alpha\gamma}\bar{\Gamma}^\mu_{\nu\mu} + \bar{\Gamma}^\nu_{\alpha\gamma}\delta\Gamma^\mu_{\nu\mu} - \delta\Gamma^\nu_{\alpha\mu}\bar{\Gamma}^\mu_{\nu\gamma} - \bar{\Gamma}^\nu_{\alpha\mu}\delta\Gamma^\mu_{\nu\gamma} \end{aligned} \tag{1.68}$$

3. **Ricci Scalar**

$$
\begin{aligned}
R &= g^{\mu\nu} R_{\mu\nu} \\
&= (\bar{g}^{\mu\nu} + \underline{h}^{\mu\nu})(\bar{R}_{\mu\nu} + \delta R_{\mu\nu}) \\
R &= \bar{R} + \bar{g}^{\mu\nu} \delta R_{\mu\nu} + \underline{h}^{\mu\nu} \bar{R}_{\mu\nu} \\
\delta R &= \bar{g}^{\mu\nu} \delta R_{\mu\nu} - \bar{g}^{\mu\rho} \bar{g}^{\nu\sigma} \underline{h}_{\rho\sigma} \bar{R}_{\mu\nu}
\end{aligned} \tag{1.69}
$$

4. **Energy Momentum Tensor**:

   - for perfect fluid:

$$
\begin{aligned}
T_{\mu\nu} &= p g_{\mu\nu} + (\rho + p) u_\mu u_\nu \\
&= (\bar{p} + \delta p)(\bar{g}_{\mu\nu} + \underline{h}_{\mu\nu}) + (\bar{\rho} + \delta\rho + \bar{p} + \delta p)(\bar{u}_\mu + \delta u_\mu)(\bar{u}_\nu + \delta u_\nu) \\
&= \bar{p}\bar{g}_{\mu\nu} + \bar{p}\underline{h}_{\mu\nu} + \delta p \bar{g}_{\mu\nu} + (\bar{\rho} + \bar{p})\bar{u}_\mu \bar{u}_\nu + (\delta\rho + \delta p)\bar{u}_\mu \bar{u}_\nu \\
&\quad + (\bar{\rho} + \bar{p})(\bar{u}_\mu \delta u_\nu + \bar{u}_\nu \delta u_\mu) \\
&= \bar{T}_{\mu\nu} + \bar{p}\underline{h}_{\mu\nu} + \delta p \bar{g}_{\mu\nu} + (\delta\rho + \delta p)\bar{u}_\mu \bar{u}_\nu \\
&\quad + (\bar{\rho} + \bar{p})(\bar{u}_\mu \delta u_\nu + \bar{u}_\nu \delta u_\mu) \\
T_{\mu\nu} - \bar{T}_{\mu\nu} = \delta T_{\mu\nu} &= \bar{p}\underline{h}_{\mu\nu} + \delta p \bar{g}_{\mu\nu} + (\delta\rho + \delta p)\bar{u}_\mu \bar{u}_\nu + (\bar{\rho} + \bar{p})(\bar{u}_\mu \delta u_\nu + \bar{u}_\nu \delta u_\mu)
\end{aligned} \tag{1.70}
$$

   The form that is being used is energy momentum tensor is with one index raised for that

$$
\begin{aligned}
g^{\mu\lambda} T_{\lambda\nu} &= (\bar{g}^{\mu\lambda} + \underline{h}^{\mu\lambda})(\bar{T}_{\lambda\nu} + \delta T_{\lambda\nu}) \\
T^\mu{}_\nu &= \bar{T}^\mu{}_\nu + \bar{g}^{\mu\lambda} \delta T_{\lambda\nu} + \underline{h}^{\mu\lambda} \bar{T}_{\lambda\nu} \\
\delta T^\mu{}_\nu &= \bar{g}^{\mu\lambda} \delta T_{\lambda\nu} - \bar{g}^{\mu\rho} \bar{g}^{\lambda\sigma} \underline{h}_{\rho\sigma} \bar{T}_{\lambda\nu} \\
&= \bar{g}^{\mu\lambda} \delta T_{\lambda\nu} - \bar{g}^{\mu\lambda} \underline{h}_{\lambda\sigma} \bar{T}^\sigma{}_\nu \\
\delta T^\mu{}_\nu &= \bar{g}^{\mu\lambda} (\delta T_{\lambda\nu} - \underline{h}_{\lambda\sigma} \bar{T}^\sigma{}_\nu)
\end{aligned} \tag{1.71}
$$

   different components:

$$
\begin{aligned}
\delta T^0{}_0 &= \bar{g}^{0\lambda}(\delta T_{\lambda 0} - \underline{h}_{\lambda\sigma} \bar{T}^\sigma{}_0) = -\frac{1}{a^2} \delta T_{00} \\
&= -\frac{1}{a^2} \{\bar{p}\underline{h}_{00} + \delta p \bar{g}_{00} + (\delta\rho + \delta p)\bar{u}_0 \bar{u}_0 + (\bar{\rho} + \bar{p})(\bar{u}_0 \delta u_0 + \bar{u}_0 \delta u_0)\} \\
&= -\frac{1}{a^2} \{-a^2 \delta p + a^2(\delta\rho + \delta p)\} \\
&= -\delta\rho
\end{aligned} \tag{1.72}
$$

$$
\begin{aligned}
\delta T^0{}_k &= \bar{g}^{0\lambda}(\delta T_{\lambda k} - \underline{h}_{\lambda\sigma} \bar{T}^\sigma{}_k) = -\frac{1}{a^2} \delta T_{0k} \\
&= -\frac{1}{a^2} \{\bar{p}\underline{h}_{0k} + \delta p \bar{g}_{0k} + (\delta\rho + \delta p)\bar{u}_0 \bar{u}_k + (\bar{\rho} + \bar{p})(\bar{u}_0 \delta u_k + \bar{u}_k \delta u_0)\} \\
&= -\frac{1}{a}(\bar{\rho} + \bar{p})\delta u_k
\end{aligned} \tag{1.73}
$$

$$
\begin{aligned}
\delta T^i{}_j &= \bar{g}^{i\lambda}(\delta T_{\lambda j} - \underline{h}_{\lambda\sigma}\bar{T}^\sigma{}_j) = \frac{1}{a^2}\delta^{ik}(\delta T_{kj} - \underline{h}_{kl}\bar{T}^l{}_j) \\
&= \frac{1}{a^2}\delta^{ik}(\bar{p}\underline{h}_{kj} + \delta p\bar{g}_{kj} + (\delta\rho + \delta p)\bar{u}_k\bar{u}_j + (\bar{\rho} + \bar{p})(\bar{u}_j\delta u_k + \bar{u}_k\delta u_j) - \underline{h}_{kl}\delta^l_j\bar{p}) \\
&= \delta p \qquad (1.74)
\end{aligned}
$$

- for scalar field

$$
\begin{aligned}
T^\mu{}_\nu =& g^{\mu\lambda}\phi_{,\lambda}\phi_{,\nu} - \frac{1}{2}\delta^\mu_\nu g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} - \delta^\mu_\nu V(\phi) \\
=& (\bar{g}^{\mu\lambda} + \underline{h}^{\mu\lambda})(\bar{\phi} + \delta\phi)_{,\lambda}(\bar{\phi} + \delta\phi)_{,\nu} \\
& - \frac{1}{2}\delta^\mu_\nu(\bar{g}^{\alpha\beta} + \underline{h}^{\alpha\beta})(\bar{\phi} + \delta\phi)_{,\alpha}(\bar{\phi} + \delta\phi)_{,\beta} - \delta^\mu_\nu V(\bar{\phi} + \delta\phi) \\
=& \bar{g}^{\mu\lambda}\bar{\phi}_{,\lambda}\bar{\phi}_{,\nu} - \frac{1}{2}\delta^\mu_\nu\bar{g}^{\alpha\beta}\bar{\phi}_{,\alpha}\bar{\phi}_{,\beta} - \delta^\mu_\nu\{V(\bar{\phi}) + V_{,\phi}\delta\phi\} + \bar{g}^{\mu\lambda}(\delta\phi)_{,\lambda}\phi_{,\nu} + \bar{g}^{\mu\lambda}\phi_{,\lambda}(\delta\phi)_{,\nu} \\
& + \underline{h}^{\mu\lambda}\phi_{,\lambda}\phi_{,\nu} - \frac{1}{2}\delta^\mu_\nu\{2\bar{g}^{\alpha\beta}(\delta\phi)_{,\alpha}\phi_{,\beta} + \underline{h}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta}\} \\
=& \bar{T}^\mu{}_\nu - \delta^\mu_\nu V_{,\phi}\delta\phi + \bar{g}^{\mu\lambda}(\delta\phi)_{,\lambda}\phi_{,\nu} + \bar{g}^{\mu\lambda}\phi_{,\lambda}(\delta\phi)_{,\nu} + \underline{h}^{\mu\lambda}\phi_{,\lambda}\phi_{,\nu} \\
& - \delta^\mu_\nu\bar{g}^{\alpha\beta}(\delta\phi)_{,\alpha}\phi_{,\beta} - \frac{1}{2}\delta^\mu_\nu\underline{h}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \\
\delta T^\mu{}_\nu =& - \delta^\mu_\nu V_{,\phi}\delta\phi + \bar{g}^{\mu\lambda}(\delta\phi)_{,\lambda}\phi_{,\nu} + \bar{g}^{\mu\lambda}\phi_{,\lambda}(\delta\phi)_{,\nu} + \underline{h}^{\mu\lambda}\phi_{,\lambda}\phi_{,\nu} \\
& - \delta^\mu_\nu\bar{g}^{\alpha\beta}(\delta\phi)_{,\alpha}\phi_{,\beta} - \frac{1}{2}\delta^\mu_\nu\underline{h}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \qquad (1.75)
\end{aligned}
$$

different components:

$$
\begin{aligned}
\delta T^0{}_0 =& - V_{,\phi}\delta\phi + \bar{g}^{0\lambda}(\delta\phi)_{,\lambda}\phi_{,0} + \bar{g}^{0\lambda}\phi_{,\lambda}(\delta\phi)_{,0} + \underline{h}^{0\lambda}\phi_{,\lambda}\phi_{,0} - \bar{g}^{\alpha\beta}(\delta\phi)_{,\alpha}\phi_{,\beta} - \frac{1}{2}\underline{h}^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} \\
=& - \frac{\dot{\phi}}{a^2}\dot{\delta}\phi - V_{,\phi}\delta\phi \qquad (1.76) \\
\delta T^0{}_k =& \bar{g}^{0\lambda}(\delta\phi)_{,\lambda}\phi_{,k} + \bar{g}^{0\lambda}\phi_{,\lambda}(\delta\phi)_{,k} + \underline{h}^{0\lambda}\phi_{,\lambda}\phi_{,k} \\
=& - \frac{1}{a^2}\dot{\phi}(\delta\phi)_{,k} \qquad (1.77) \\
\delta T^i{}_j =& - \delta^i_j V_{,\phi}\delta\phi + \bar{g}^{i\lambda}(\delta\phi)_{,\lambda}\phi_{,j} + \bar{g}^{i\lambda}\phi_{,\lambda}(\delta\phi)_{,j} + \underline{h}^{i\lambda}\phi_{,\lambda}\phi_{,j} - \delta^i_j\bar{g}^{\alpha\beta}(\delta\phi)_{,\alpha}\phi_{,\beta} \\
=& - \delta^i_j V_{,\phi}\delta\phi + \frac{1}{a^2}\delta^i_j\dot{\phi}\dot{\delta}\phi \qquad (1.78)
\end{aligned}
$$

5. **Einstein Equation**:

$$
M_p^2\delta G^\mu{}_\nu = \delta T^\mu{}_\nu \qquad (1.79)
$$

$$
\begin{aligned}
M_p^2 G_{\mu\nu} = M_p^2\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right) &= T_{\mu\nu} \\
G_{\mu\nu} &= \bar{R}_{\mu\nu} + \delta R_{\mu\nu} - \frac{1}{2}\left(\bar{g}_{\mu\nu} + \underline{h}_{\mu\nu}\right)(\bar{R} + \delta R) \\
G_{\mu\nu} &= \bar{G}_{\mu\nu} + \delta R_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\delta R - \frac{1}{2}\underline{h}_{\mu\nu}\bar{R} \\
G_{\mu\nu} - \bar{G}_{\mu\nu} = \delta G_{\mu\nu} &= \delta R_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\delta R - \frac{1}{2}\underline{h}_{\mu\nu}\bar{R} \qquad (1.80)
\end{aligned}
$$

Another form with 1 index up

$$
\begin{aligned}
g^{\mu\sigma}G_{\mu\nu} &= \left(\bar{g}^{\mu\sigma} + \underline{h}^{\mu\sigma}\right)\left(\bar{G}_{\mu\nu} + \delta G_{\mu\nu}\right) \\
G^{\sigma}{}_{\nu} &= \bar{G}^{\sigma}{}_{\nu} + \bar{g}^{\mu\sigma}\delta G_{\mu\nu} + \underline{h}^{\mu\sigma}\bar{G}_{\mu\nu} \\
G^{\sigma}{}_{\nu} - \bar{G}^{\sigma}{}_{\nu} &= \bar{g}^{\mu\sigma}\delta G_{\mu\nu} - \bar{g}^{\mu\rho}\bar{g}^{\sigma\lambda}\underline{h}_{\rho\lambda}\bar{G}_{\mu\nu} \\
\delta G^{\sigma}{}_{\nu} &= \bar{g}^{\mu\sigma}(\delta G_{\mu\nu} - \underline{h}_{\rho\mu}\bar{G}^{\rho}{}_{\nu})
\end{aligned}
\tag{1.81}
$$

6. **Continuity Equation**:

$$
\begin{aligned}
T^{\mu}{}_{\nu;\mu} &= 0 \\
T^{\mu}{}_{\nu,\mu} + \Gamma^{\mu}{}_{\alpha\mu}T^{\alpha}{}_{\nu} - \Gamma^{\alpha}{}_{\nu\mu}T^{\mu}{}_{\alpha} &= 0 \\
(\bar{T}^{\mu}{}_{\nu} + \delta T^{\mu}{}_{\nu})_{,\mu} + (\bar{\Gamma}^{\mu}{}_{\alpha\mu} + \delta\Gamma^{\mu}{}_{\alpha\mu})(\bar{T}^{\alpha}{}_{\nu} + \delta T^{\alpha}{}_{\nu}) - (\bar{\Gamma}^{\alpha}{}_{\nu\mu} + \delta\Gamma^{\alpha}{}_{\nu\mu})(\bar{T}^{\mu}{}_{\alpha} + \delta T^{\mu}{}_{\alpha}) &= 0 \\
\delta T^{\mu}{}_{\nu,\mu} + \bar{\Gamma}^{\mu}{}_{\alpha\mu}\delta T^{\alpha}{}_{\nu} + \delta\Gamma^{\mu}{}_{\alpha\mu}\bar{T}^{\alpha}{}_{\nu} - \bar{\Gamma}^{\alpha}{}_{\nu\mu}\delta T^{\mu}{}_{\alpha} - \delta\Gamma^{\alpha}{}_{\nu\mu}\bar{T}^{\mu}{}_{\alpha} &= 0
\end{aligned}
\tag{1.82}
$$

1.6.1 Scalar Vector Decomposition

To deal with these small perturbations Lifshitz gave a concept of scalar vector tensor decomposition based on 3+1 split of components of perturbation

Scalar Vector Tensor Decomposition for metric

for metric, we can decompose perturbation in the following way:

$$
h_{00} = \psi \qquad h_{0i} = w_i \qquad h_{ij} = \frac{h}{3}\gamma_{ij} + S_{ij} \tag{1.83}
$$

here,h_{00}, $h_{0i}$and$h_{ij}$are scalar vector and tensor respectively in spatial coordinates therefore indices lowering raising can be done using γ_{ij}. h_{ij} can be written as trace part(h) and traceless part(S_{ij}).Perturbation with 10 degrees of freedom have been split into 1+3+(1+5). Now further splitting can be performed as follows:

- $$
w_i = w_i^{||} + w_i^{\perp} \tag{1.84}
$$

  where $w_i^{||}$and$w_i^{\perp}$ are longitudinal and transverse part of vector w_i such that $\nabla \times \mathbf{w}^{||} = \nabla.\mathbf{w}^{\perp} = \mathbf{0}$ where ∇ is spatial covariant derivative. so $w_i^{||}$ can be written as $\nabla_i\phi_w$ where ϕ_w is scalar field. so vector w_i can be decomposed into scalar and part that can't be obtained from a scalar. So 3 degrees of freedom associated with w_i can be split into 1+2 degree of freedom.

- $$
S_{ij} = S_{ij}^{||} + S_{ij}^{\perp} + S_{ij}^{T} \tag{1.85}
$$

  where $S_{ij}^{||}$, $S_{ij}^{\perp}$ and S_{ij}^{T} are for both indices longitudinal, one is transverse and both transverse respectively such that $\gamma^{jk}\nabla_k S_{ij} = \gamma^{jk}\nabla_k S_{ij}^{||} + \gamma^{jk}\nabla_k S_{ij}^{\perp}$. Divergence of double transverse part is zero so S_{ij}^{T} have degree of freedom 5-3(3 constraints)=2) .So 5 degree of freedom associated with S_{ij} can split into 1+2+2.Now $S_{ij}^{||}$ can be further decompose as $\left(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2\right)\mu$ so we can obtain it from scalar . Single transverse part can be written in term of a vector $S_{ij}^{\perp} = \nabla_i A_j^{\perp} + \nabla_j A_i^{\perp}$

Hence, perturbation $h_{\mu\nu}$ can be split into 4 scalars(ψ, h, μ, ϕ_w) with dof 4 in which 2 are gauge dependent, known as **scalar mode**,2 vectors ($A_i, w_i^{\perp}$) with dof freedom 4 in which 2 are gauge dependent, known as **vector mode** and 1 tensor (S_{ij}^T) with 2 degree of freedom, known as **tensor mode**.

Since from Einstein equation , it is known that geometry and energy -momentum tensor are related to each other small perturbation in geometry will lead to perturbation in energy -momentum tensor($\delta T_{\mu\nu}$) and vice versa.

Scalar Vector Tensor Decomposition for Energy Momentum Tensor

Similar to metric tensor,$\delta T_{\mu\nu}$ can also be split as follows (using eq(1.71)):

$$\delta T^0{}_0 = -\delta\rho \tag{1.86}$$

$$\delta T^0{}_i = (\bar{\rho}+\bar{p})v_i = (\bar{\rho}+\bar{p})(\nabla_i\Theta + v_i^{\perp}) \tag{1.87}$$

$$\delta T^i{}_0 = -(\bar{\rho}+\bar{p})v^i = -(\bar{\rho}+\bar{p})(\nabla^i\Theta + v^{i\perp}) \tag{1.88}$$

$$\begin{aligned}\delta T^i{}_j &= \delta p\delta^i_j + \Sigma^i{}_j = \delta p\delta^i_j + \Sigma^i{}_j^{||} + \Sigma^i{}_j^{\perp} + \Sigma^i{}_j^{T} \\ &= \delta p\delta^i_j + \left(\nabla^i\nabla_j - \frac{1}{3}\delta^i{}_j\nabla^2\right)\phi_\Sigma + (\nabla_i\Sigma_j^{\perp} + \nabla_j\Sigma_i^{\perp}) + \Sigma_{ij}^T\end{aligned} \tag{1.89}$$

where Σ is anisotropic term in case of non perfect fluid. In case of linear perturbations, evolution of scalar, vector and tensor mode are going to be mutually exclusive. For structure formation we just need scalar modes so further discussion will be only for scalar modes only.

for scalar modes- metric perturbations are given as follows-

$$h_{00} == \psi \qquad h_{0i} = \nabla_i\phi_w \qquad h_{ij} = \frac{h}{3}\gamma_{ij} + \left(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2\right)\mu \tag{1.90}$$

Energy momentum perturbations are given as follows-

$$\delta T^0{}_0 = -\delta\rho \qquad \delta T^0{}_i = (\bar{\rho}+\bar{p})\nabla_i\Theta \qquad \delta T^i{}_j = \delta p\delta_{ij} + \left(\nabla_i\nabla_j - \frac{1}{3}\gamma_{ij}\nabla^2\right)\phi_\Sigma \tag{1.91}$$

1.6.2 Perturbation in Flat space

$\gamma_{ij} = \delta_{ij}$

Gauge Choice

Since for scalar field we have 2 gauge dependent dof so we can choose a coordinate system which is convenient to work in. One choice is **Synchronous gauge** which we will use.In synchronous gauge, $\psi = 0$ and $\phi_w = 0$, i.e. g_{00} and g_{0j} part remain unperturbed.So scalar field h and μ are going to characterize the perturbation. so line element is given by-

$$ds^2 = a^2(\tau)(-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j) \tag{1.92}$$

Perturbed metric tensor

$$g_{00} = -a^2 \quad g_{0i} = 0 \quad g_{ij} = \delta_{ij} + h_{ij} \tag{1.93}$$

so pertrubation in metric elements :

$$\delta g_{00} = h_{00} = 0 \quad \delta g_{0i} = h_{0i} = 0 \quad \delta g_{ij} = h_{ij} \tag{1.94}$$

Perturbed Christofel Symbols

Using eq(1.67)

Case:1 $\mu = 0$

$$\begin{aligned}\delta\Gamma^0_{\alpha\beta} &= \frac{1}{2}\bar{g}^{0\nu}(\underline{h}_{\nu\alpha,\beta} - \underline{h}_{\alpha\beta,\nu} + \underline{h}_{\beta\nu,\alpha} - 2\underline{h}_{\nu\sigma}\bar{\Gamma}^\sigma_{\alpha\beta}) \\ &= \frac{1}{2}\bar{g}^{00}(\underline{h}_{0\alpha,\beta} - \underline{h}_{\alpha\beta,0} + \underline{h}_{\beta 0,\alpha} - 2\underline{h}_{0\sigma}\bar{\Gamma}^\sigma_{\alpha\beta}) \\ &= -\frac{1}{2}\bar{g}^{00}\underline{h}_{\alpha\beta,0}\end{aligned} \tag{1.95}$$

$$\delta\Gamma^0_{00} = 0 \quad \delta\Gamma^0_{0i} = 0 \quad \delta\Gamma^0_{ij} = \tfrac{\dot{h}_{ij}}{2} + \tfrac{\dot{a}}{a}h_{ij} \tag{1.96}$$

Case:2 $\mu = i$

$$\delta\Gamma^i_{\alpha\beta} = \frac{1}{2}\bar{g}^{ij}(\underline{h}_{j\alpha,\beta} - \underline{h}_{\alpha\beta,j} + \underline{h}_{\beta j,\alpha} - 2\underline{h}_{j\sigma}\bar{\Gamma}^\sigma_{\alpha\beta})$$

$$\delta\Gamma^i_{00} = 0 \quad \delta\Gamma^i_{0k} = \tfrac{\dot{h}^i{}_k}{2} \quad \delta\Gamma^i_{ml} = \tfrac{\delta_{ij}}{2}(h_{jm,l} - h_{ml,j} + h_{lj,m}) \tag{1.97}$$

Perturbed Ricci Tensor

Using eq(1.68)

Case:1 $\alpha = 0$ and $\gamma = 0$

$$\begin{aligned}\delta R_{00} &= (\delta\Gamma^\mu_{00})_{,\mu} - \left(\delta\Gamma^\mu_{0\mu}\right)_{,0} + \delta\Gamma^\nu_{00}\bar{\Gamma}^\mu_{\nu\mu} + \bar{\Gamma}^\nu_{00}\delta\Gamma^\mu_{\nu\mu} - \delta\Gamma^\nu_{0\mu}\bar{\Gamma}^\mu_{\nu 0} - \bar{\Gamma}^\nu_{0\mu}\delta\Gamma^\mu_{\nu 0} \\ &= \left(\delta\Gamma^0_{00}\right)_{,0} + \left(\delta\Gamma^i_{00}\right)_{,i} - \left(\delta\Gamma^0_{00}\right)_{,0} - \left(\delta\Gamma^i_{0i}\right)_{,0} + \delta\Gamma^0_{00}\bar{\Gamma}^\mu_{0\mu} + \delta\Gamma^i_{00}\bar{\Gamma}^\mu_{i\mu} - \delta\Gamma^0_{0\mu}\bar{\Gamma}^\mu_{00} \\ &- \delta\Gamma^i_{0\mu}\bar{\Gamma}^\mu_{i0} + \bar{\Gamma}^0_{00}\delta\Gamma^\mu_{0\mu} + \bar{\Gamma}^i_{00}\delta\Gamma^\mu_{i\mu} - \bar{\Gamma}^0_{0\mu}\delta\Gamma^\mu_{00} - \bar{\Gamma}^i_{0\mu}\delta\Gamma^\mu_{i0} \\ &= \left(\delta\Gamma^i_{0i}\right)_{,0} - \delta\Gamma^i_{0j}\bar{\Gamma}^j_{i0} + \bar{\Gamma}^0_{00}\delta\Gamma^i_{0i} - \bar{\Gamma}^i_{0j}\delta\Gamma^j_{i0} \\ &= -\frac{\ddot{h}^i{}_i}{2} - \frac{\dot{a}}{a}\frac{\dot{h}^i{}_i}{2}\end{aligned} \tag{1.98}$$

Case:2 $\alpha = 0$ and $\gamma = k$

$$\begin{aligned}\delta R_{0k} &= (\delta\Gamma^\mu_{0k})_{,\mu} - \left(\delta\Gamma^\mu_{0\mu}\right)_{,k} + \delta\Gamma^\nu_{0k}\bar{\Gamma}^\mu_{\nu\mu} + \bar{\Gamma}^\nu_{0k}\delta\Gamma^\mu_{\nu\mu} - \delta\Gamma^\nu_{0\mu}\bar{\Gamma}^\mu_{\nu k} - \bar{\Gamma}^\nu_{0\mu}\delta\Gamma^\mu_{\nu k} \\ &= \left(\delta\Gamma^i_{0k}\right)_{,i} - \left(\delta\Gamma^i_{0i}\right)_{,k} + \delta\Gamma^i_{0k}\bar{\Gamma}^\mu_{i\mu} + \bar{\Gamma}^i_{0k}\delta\Gamma^\mu_{i\mu} - \delta\Gamma^i_{0j}\bar{\Gamma}^j_{ik} - \bar{\Gamma}^i_{0j}\delta\Gamma^j_{ik} \\ &= \left(\delta\Gamma^i_{0k}\right)_{,i} - \left(\delta\Gamma^i_{0i}\right)_{,k} + \bar{\Gamma}^i_{0k}\delta\Gamma^j_{ij} - \bar{\Gamma}^i_{0j}\delta\Gamma^j_{ik} \\ &= \frac{\dot{h}^i{}_{k,i}}{2} - \frac{\dot{h}^i{}_{i,k}}{2}\end{aligned} \tag{1.99}$$

Case:3 $\alpha = j$ and $\gamma = k$

$$\begin{aligned}\delta R_{jk} &= \left(\delta\Gamma^\mu_{jk}\right)_{,\mu} - \left(\delta\Gamma^\mu_{j\mu}\right)_{,k} + \delta\Gamma^\nu_{jk}\bar{\Gamma}^\mu_{\nu\mu} + \bar{\Gamma}^\nu_{jk}\delta\Gamma^\mu_{\nu\mu} - \delta\Gamma^\nu_{j\mu}\bar{\Gamma}^\mu_{\nu k} - \bar{\Gamma}^\nu_{j\mu}\delta\Gamma^\mu_{\nu k} \\ &= \left(\delta\Gamma^0_{jk}\right)_{,0} + \left(\delta\Gamma^i_{jk}\right)_{,i} - \left(\delta\Gamma^i_{ji}\right)_{,k} + \delta\Gamma^0_{jk}\bar{\Gamma}^\mu_{0\mu} + \bar{\Gamma}^0_{jk}\delta\Gamma^\mu_{0\mu} \\ &- \delta\Gamma^0_{j\mu}\bar{\Gamma}^\mu_{0k} - \delta\Gamma^i_{j\mu}\bar{\Gamma}^\mu_{ik} - \bar{\Gamma}^0_{j\mu}\delta\Gamma^\mu_{0k} - \bar{\Gamma}^i_{j\mu}\delta\Gamma^\mu_{ik} \\ &= \left(\delta\Gamma^0_{jk}\right)_{,0} + \left(\delta\Gamma^i_{jk}\right)_{,i} - \left(\delta\Gamma^i_{ji}\right)_{,k} + \delta\Gamma^0_{jk}\bar{\Gamma}^\mu_{0\mu} + \bar{\Gamma}^0_{jk}\delta\Gamma^i_{0i} - \delta\Gamma^0_{ji}\bar{\Gamma}^i_{0k} \\ &- \delta\Gamma^i_{j0}\bar{\Gamma}^0_{ik} - \bar{\Gamma}^0_{ji}\delta\Gamma^i_{0k} - \bar{\Gamma}^i_{j0}\delta\Gamma^0_{ik} \\ &= \frac{\ddot{h}_{jk}}{2} + \frac{\ddot{a}}{a}h_{jk} + \frac{\dot{a}^2}{a^2}h_{jk} + \frac{\dot{a}}{a}\dot{h}_{jk} + \frac{1}{2}\delta_{jk}\frac{\dot{a}}{a}\dot{h}_i{}^i + \frac{\delta^{il}}{2}(h_{kl,ji} + h_{ji,kl} - h_{jk,li} - h_{il,jk})\end{aligned} \tag{1.100}$$

Perturbed Ricci Scalar

Using eq(1.69)

$$\begin{aligned}\delta R &= \bar{g}^{\mu\nu}\delta R_{\mu\nu} - \bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}\underline{h}_{\rho\sigma}\bar{R}_{\mu\nu} \\ &= \bar{g}^{00}\delta R_{00} + \bar{g}^{ij}\delta R_{ij} - \bar{g}^{im}\bar{g}^{jl}a^2 h_{ml}\bar{R}_{ij} \\ &= \frac{1}{a^2}\left(\ddot{h}^i{}_i + 3\frac{\dot{a}}{a}\dot{h}^i{}_i + h_{ij}{}^{,i,j} - (h^i{}_i)_{,j}{}^{,j}\right) \end{aligned} \tag{1.101}$$

Perturbed Einstein equation

Using eq(1.80)

$$\delta G_{\mu\nu} = \delta R_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\delta R - \frac{1}{2}a^2 h_{\mu\nu}\bar{R}$$

case:1 $\mu = 0$ and $\nu = 0$

$$\begin{aligned}\delta G_{00} &= \delta R_{00} - \frac{1}{2}\bar{g}_{00}\delta R - \frac{1}{2}a^2 h_{00}\bar{R} \\ &= -\frac{\ddot{h}^i{}_i}{2} - \frac{\dot{a}}{a}\frac{\dot{h}^i{}_i}{2} + \frac{1}{2}a^2\frac{1}{a^2}\left(\ddot{h}^i{}_i + 3\frac{\dot{a}}{a}\dot{h}^i{}_i + h_{ij}{}^{,i,j} - (h^i{}_i)_{,j}{}^{,j}\right) \\ &= \frac{\dot{a}}{a}\dot{h}^i{}_i + \frac{1}{2}(h_{ij}{}^{,i,j} - (h^i{}_i)_{,j}{}^{,j}) \end{aligned} \tag{1.102}$$

$$\delta G^0{}_0 = -\frac{1}{a^2}\delta G_{00} \tag{1.103}$$

case:2 $\mu = 0$ and $\nu = k$

$$\begin{aligned}\delta G_{0k} &= \delta R_{0k} - \frac{1}{2}\bar{g}_{0k}\delta R - \frac{1}{2}a^2 h_{0k}\bar{R} \\ &= \frac{\dot{h}^i{}_{k,i}}{2} - \frac{\dot{h}^i{}_{i,k}}{2} \end{aligned} \tag{1.104}$$

$$\delta G^0{}_k = -\frac{1}{a^2}\delta G_{0k} \tag{1.105}$$

case:3 $\mu = j$ and $\nu = k$

$$\begin{aligned}
\delta G_{jk} &= \delta R_{jk} - \frac{1}{2}\bar{g}_{jk}\delta R - \frac{1}{2}a^2 h_{jk}\bar{R} \\
&= \frac{\ddot{h}_{jk}}{2} + \frac{\ddot{a}}{a}h_{jk} + \frac{\dot{a}^2}{a^2}h_{jk} + \frac{\dot{a}}{a}\dot{h}_{jk} + \frac{1}{2}\delta_{jk}\frac{\dot{a}}{a}\dot{h}_i{}^i \\
&+ \frac{\delta^{il}}{2}(h_{kl,ji} + h_{ji,kl} - h_{jk,li} - h_{il,jk}) \\
&- \frac{1}{2}\delta_{jk}a^2\frac{1}{a^2}\left(\ddot{h}^i{}_i + 3\frac{\dot{a}}{a}\dot{h}^i{}_i + h_{ij}{}^{,i,j} - (h^i{}_i)_{,l}{}^{,l}\right) - \frac{1}{2}a^2 h_{jk}\frac{6}{a^2}\frac{\ddot{a}}{a} \\
&= \frac{\ddot{h}_{jk}}{2} - 2\frac{\ddot{a}}{a}h_{jk} + \frac{\dot{a}^2}{a^2}h_{jk} + \frac{\dot{a}}{a}\dot{h}_{jk} + \frac{1}{2}\delta_{jk}\frac{\dot{a}}{a}\dot{h}_i{}^i \\
&- \frac{1}{2}\delta_{jk}\ddot{h}^i{}_i - \frac{3}{2}\delta_{jk}\frac{\dot{a}}{a}\dot{h}^i{}_i + \frac{\delta^{il}}{2}(h_{kl,ji} + h_{ji,kl} - h_{jk,li} - h_{il,jk}) \\
&- \frac{1}{2}\delta_{jk}\left(h_{ij}{}^{,i,j} - (h^i{}_i)_{,l}{}^{,l}\right) \qquad (1.106) \\
\delta G^j{}_k &= -\frac{\delta^{ij}}{a^2}\left(\delta G_{ik} - h_{ik}\left(\frac{-2\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\right) \\
&= \frac{\ddot{h}^j{}_k}{2a^2} + \frac{\dot{a}}{a^3}\dot{h}^j{}_k + \frac{1}{2a^2}\delta^j{}_k\frac{\dot{a}}{a}\dot{h}_i{}^i \\
&- \frac{1}{2a^2}\delta^j{}_k\ddot{h}^i{}_i - \frac{3}{2a^2}\delta^j{}_k\frac{\dot{a}}{a}\dot{h}^i{}_i \\
&+ \frac{\delta^{ij}}{a^2}\frac{\delta^{ml}}{2}(h_{kl,im} + h_{im,kl} - h_{ik,lm} - h_{ml,ik}) \\
&- \frac{1}{2a^2}\delta^j{}_k\left(h_{il}{}^{,i,l} - (h^i{}_i)_{,l}{}^{,l}\right) \\
&= \frac{\ddot{h}^j{}_k}{2a^2} + \frac{\dot{a}}{a^3}\dot{h}^j{}_k + \frac{1}{2a^2}\delta^j{}_k\frac{\dot{a}}{a}\dot{h}_i{}^i \\
&- \frac{1}{2a^2}\delta^j{}_k\ddot{h}^i{}_i - \frac{3}{2a^2}\delta^j{}_k\frac{\dot{a}}{a}\dot{h}^i{}_i \\
&+ \frac{1}{2a^2}(h_k{}^{m,j}{}_m + h^j{}_{m,k}{}^{,m} - h^j{}_{k,m}{}^{,m} - h_m{}^{m,j}{}_{,k}) \\
&- \frac{1}{2a^2}\delta^j{}_k\left(h_{il}{}^{,i,l} - (h^i{}_i)_{,l}{}^{,l}\right) \qquad (1.107)
\end{aligned}$$

C1: $j = k$

$$\begin{aligned}
\delta G^j{}_j &= \frac{\ddot{h}^j{}_j}{2a^2} + \frac{\dot{a}}{a^3}\dot{h}^j{}_j + \frac{1}{2a^2}3\frac{\dot{a}}{a}\dot{h}_i{}^i \\
&- \frac{1}{2a^2}3\ddot{h}^i{}_i - \frac{3}{2a^2}3\frac{\dot{a}}{a}\dot{h}^i{}_i \\
&+ \frac{1}{2a^2}(h_j{}^{m,j}{}_m + h^j{}_{m,j}{}^{,m} - h^j{}_{j,m}{}^{,m} - h_m{}^{m,j}{}_{,j}) \\
&- \frac{1}{2a^2}3\left(h_{il}{}^{,i,l} - (h^i{}_i)_{,l}{}^{,l}\right) \\
&= -\frac{1}{2a^2}\left(2\ddot{h}^j{}_j + 4\frac{\dot{a}}{a}\dot{h}^j{}_j + \left(h_{il}{}^{,i,l} - (h^i{}_i)_{,l}{}^{,l}\right)\right) \qquad (1.108)
\end{aligned}$$

C2: $j \neq k$

$$\delta G^j{}_k = \frac{\ddot{h}^j{}_k}{2a^2} + \frac{\dot{a}}{a^3}\dot{h}^j{}_k + \frac{1}{2a^2}(h_k{}^{m,j}{}_m + h^j{}_{m,k}{}^{,m} - h^j{}_{k,m}{}^{,m} - h_m{}^{m,j}{}_{,k}) \tag{1.109}$$

In K-Space: Scalar mode of h_{ij} can be written as follows as given in [11]:

$$h_{ij} = \int d^3k e^{\iota k_l x^l} \left\{ \frac{k_i k_j}{k^2} h(\vec{k},\tau) + \left(\frac{k_i k_j}{k^2} - \frac{1}{3}\delta_{ij} \right) 6\eta(\vec{k},\tau) \right\} \tag{1.110}$$

so components of $\delta G^\mu{}_\nu$ in k space can be calculated via putting this expression so calculated perturbation in components of $\delta G^\mu{}_\nu$ are as follows:

$$\delta G^0{}_0 = -\frac{1}{a^2}\left(\frac{\dot{a}}{a}\dot{h} - 2k^2\eta\right) \tag{1.111}$$

$$\delta G^0{}_k = -\frac{1}{a^2} 2\iota k_k \dot{\eta} \tag{1.112}$$

$$\delta G^j{}_j = -\frac{1}{a^2}\left(\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} - 2k^2\eta\right) \tag{1.113}$$

$$\delta G^j{}_k = \frac{k^j k_k}{2a^2}\left(\ddot{h} + 6\ddot{\eta} + 2\frac{\dot{a}}{a}(\dot{h} + 6\dot{\eta}) - 2k^2\eta\right) \tag{1.114}$$

δT^0_0 and δT^i_i are trace part so they are going to remain same. for remaining two following definition is being used as mentioned in [11].

$$(\bar{\rho} + \bar{p})\theta = \iota k^j \delta T^0_j \tag{1.115}$$

$$(\bar{\rho} + \bar{p})\sigma = -(\frac{k_i k^j}{k^2} - \frac{1}{3}\delta^j_i)\Sigma^i{}_j \tag{1.116}$$

where δT^0_j and $\Sigma^i{}_j$ are components in k space

Perturbed Continuity Equation

using eq(1.82)

C1: $\nu = 0$

$$\begin{aligned} 0 =& \delta T^\mu{}_{0,\mu} + \bar{\Gamma}^\mu_{\alpha\mu}\delta T^\alpha{}_0 + \delta\Gamma^\mu_{\alpha\mu}\bar{T}^\alpha{}_0 - \bar{\Gamma}^\alpha_{0\mu}\delta T^\mu{}_\alpha - \delta\Gamma^\alpha{}_{0\mu}\bar{T}^\mu{}_\alpha \\ 0 =& \delta T^0{}_{0,0} + \delta T^i{}_{0,i} + \bar{\Gamma}^\mu_{0\mu}\delta T^0{}_0 + \bar{\Gamma}^j_{ij}\delta T^i{}_0 + \delta\Gamma^\mu_{0\mu}\bar{T}^0{}_0 \\ & - \bar{\Gamma}^0_{00}\delta T^0{}_0 - \bar{\Gamma}^i_{0j}\delta T^j{}_i - \delta\Gamma^i{}_{0j}\bar{T}^j{}_i \\ 0 =& \delta T^0{}_{0,0} + \delta T^i{}_{0,i} + 3\frac{\dot{a}}{a}\delta T^0{}_0 - \frac{\dot{a}}{a}\delta T^i{}_i - \frac{\dot{h}^i{}_i}{2}(\bar{p} + \bar{\rho}) \end{aligned} \tag{1.117}$$

C2: $\nu = i$

$$\begin{aligned} \delta T^\mu{}_{i,\mu} + \bar{\Gamma}^\mu_{\alpha\mu}\delta T^\alpha{}_i + \delta\Gamma^\mu_{\alpha\mu}\bar{T}^\alpha{}_i - \bar{\Gamma}^\alpha_{i\mu}\delta T^\mu{}_\alpha - \delta\Gamma^\alpha{}_{i\mu}\bar{T}^\mu{}_\alpha = 0 \\ \delta T^0{}_{i,0} + \delta T^j{}_{i,j} + 2\frac{\dot{a}}{a}\delta T^0{}_i = 0 \end{aligned} \tag{1.118}$$

Chapter 2

Exploring the Observable Universe

Every theoretical model purposed can be considered of importance only if it could be verified through experiments. This chapter is devoted to how measurements are being done to verify cosmological models.

2.1 Frame of reference:

While doing measurements first thing that need to be keep in mind is from which frame measurements are being made .

2.1.1 Comoving Coordinate System:

Metric for our universe can be written as follows if universe is considered spatially isotropic and homogeneous:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-\kappa r^2} + r^2 d\Omega^2\right) \tag{2.1}$$

The coordinated system in which Metric of Universe takes the above form(FLRW) is known as Comoving Coordinated system.Only observer of this coordinate system(Comoving Observer) consider the universe as maximally symmetric in space. A Comoving Coordinate system have the following properties associated with it:

- Each point of a system in space-time carries a clock and fixed set of spatial coordinates.
- Clocks are considered to be in free fall

2.2 Some Useful Definitions:

In this section , an attempt is made to define some terminology related to observation which is being used in this thesis.

2.2.1 Cosmological redshift:

Light from any source can be said most prominent source of information when it comes from object at large distances(Mega Parsec).Through this light only we can interpret things so there is need to understand how to make interpretation using it. Lets say a light is emitted at a source at radial distance r ($d\theta = 0 = d\phi$) from observer which is taken at origin at time t emit a crest of wave which observer is receiving at time t_0 (present epoch).for light

$$ds^2 = 0 = -dt^2 + \frac{a^2(t)dr^2}{1-\kappa r^2}$$
$$\frac{dr}{\sqrt{1-\kappa r^2}} = -\frac{dt}{a(t)}$$

here negative sign signify that light is approaching the observer i.e.radial distance r decreases as time t increases.

$$\int_r^0 \frac{1}{\sqrt{1-\kappa r^2}}\,dr = -\int_t^{t_0} \frac{1}{a(t)}\,dt \tag{2.2}$$

Now lets say next crest is emitted by it at time $t+\Delta t_1$ and received by observer at time $t_0+\Delta t_2$.

$$\int_r^0 \frac{1}{\sqrt{1-\kappa r^2}}\,dr = -\int_{t+\Delta t_1}^{t_0+\Delta t_2} \frac{1}{a(t)}\,dt \tag{2.3}$$

using eq(2.2) and eq(2.3)

$$\int_t^{t_0} \frac{1}{a(t)}\,dt = \int_{t+\Delta t_1}^{t_0+\Delta t_2} \frac{1}{a(t)}\,dt$$
$$\int_t^{t_0} \frac{1}{a(t)}\,dt = \int_t^{t_0} \frac{1}{a(t)}\,dt - \int_t^{t+\Delta t_1} \frac{1}{a(t)}\,dt + \int_{t_0}^{t_0+\Delta t_2} \frac{1}{a(t)}\,dt$$
$$\int_t^{t+\Delta t_1} \frac{1}{a(t)}\,dt = \int_{t_0}^{t_0+\Delta t_2} \frac{1}{a(t)}\,dt \tag{2.4}$$

Since time of emission and reception is larger than time between two crest(which is basically time period of light emitted) so we can write

$$\frac{\Delta t_1}{a(t)} = \frac{\Delta t_2}{a(t_0)}$$
$$\frac{\Delta t_2}{\Delta t_1} = \frac{a(t_0)}{a(t)} \tag{2.5}$$

Time interval between two signal changes by factor of $\frac{a(t_0)}{a(t)}$ which could be rewritten as-

$$\frac{\lambda_2}{\lambda_1} = \frac{a(t_0)}{a(t)} \tag{2.6}$$
$$1 + z = \frac{a(t_0)}{a(t)} \tag{2.7}$$

where $z = \frac{\lambda_1 - \lambda_2}{\lambda_1}$ is parameter which tell about how wavelength changes as light move away from source in expanding universe . This is known as cosmological redshift.

$$z = v = Hd \tag{2.8}$$

2.2.2 Hubble Constant:

$$H_0 = \left(\frac{a'}{a}\right)|_{a=a_0} = \left(\frac{\dot{a}}{a^2}\right)|_{a=a_0} \tag{2.9}$$

2.2.3 Hubble Length:

It is defined as

$$L_H = \frac{1}{H} \tag{2.10}$$

2.2.4 Hubble Time:

It is defined as

$$L_H = \frac{1}{H} \tag{2.11}$$

2.2.5 Comoving Distance:

It is define as radial distance between two comoving objects which remains constant over time.

$$d_c = \int_0^r \frac{1}{\sqrt{1-\kappa r^2}}\, dr \tag{2.12}$$

$$d_c = \begin{cases} \frac{\sin^{-1}\sqrt{\kappa}r}{\sqrt{(\kappa)}} & \kappa > 0 \\ r & \kappa = 0 \\ \frac{\sinh^{-1}\sqrt{-\kappa}r}{\sqrt{(-\kappa)}} & \kappa < 0 \end{cases} \tag{2.13}$$

Since r can't be measured from observation we need to find a way the again come through light.from eq(2.2)

$$\int_r^0 \frac{1}{\sqrt{1-\kappa r^2}}\, dr = -\int_t^{t_0} \frac{1}{a(t)}\, dt$$

$$\int_0^r \frac{1}{\sqrt{1-\kappa r^2}}\, dr = \int_t^{t_0} \frac{1}{a(t)}\, dt$$

$$d_c = \int_t^{t_0} \frac{1}{\dot{a}(t)a(t)}\, da$$

$$\text{defining} \frac{\dot{a}(t)}{a(t)} = H(t)$$

$$d_c = \int_{a(t)}^1 \frac{1}{a^2(t)H(t)}\, da$$

$$\text{putting} z = a^{-1} - 1$$

$$d_c = \int_0^z \frac{1}{H(z)}\, dz \tag{2.14}$$

2.2.6 Proper Distance:

It is radial distance between two comoving objects at a instant of time.

$$d_p(t) = \frac{a(t)}{a(t_0)} d_c \tag{2.15}$$

It is basically comoving distance multiplied by $\frac{a(t)}{a(t_0)}$

2.2.7 Proper Velocity:

Time derivative of proper distance gives proper velocity.

$$v_p(t) = \dot{d}_p(t) = \frac{\dot{a}(t)}{a(t_0)} d_c$$

here $\frac{\dot{a}(t)}{a(t_0)} = H(t)$ is hubble parameter.

$$v_p(t) = H(t) d_c \tag{2.16}$$

this eq(2.16) is known as Hubble's law,which is explained in 2.3.

2.2.8 Luminosity Distance:

To measure distance of stellar objects which are far away(parallax method can't be used) usually **standard candles**(e.g. Cepheid Variables , Type 1 Supernovae etc.) are being used. These are objects whose luminosity is known. From luminosity We can measure the distance which is known as luminosity distance defined as:

$$d_L = (1+z) d_p \tag{2.17}$$

2.2.9 Density Parameter:

Density parameter is defined as

$$\Omega = \frac{\rho}{\rho_c} \tag{2.18}$$

where $\rho_c = 3H_0^2 M_P^2$

2.3 Hubble's Law:

In 1929,Hubble published his first paper which claimed linear relationship between distance and redshift of galaxies indicating an expanding universe. Distance from galaxies was being measured using Cepheid Variables which are standard candles. It was observed by Hubble that as we go to longer distances galaxies appear to recess faster there is linear relation between velocity and distance.

$$v \alpha d \tag{2.19}$$

$$v = H_0 d \tag{2.20}$$

here, H_0 is constant of proportionality .This relation is same as eq(2.16) at present epoch. Calculating H_0 is one of important aspect of modelling of universe . There are two ways we can measure Hubble constant,one from local measurements another from early universe measurement

2.4 Cosmic Microwave Background(CMB):

As we go back in time temperature becomes higher and higher . After inflationary expansion, universe was hot ball in which baryons and radiation(photons) were in thermal equilibrium due to Thomson scattering(elastic scattering which is enough keep atom ionized) and coulombic interaction. They can be considered as forming a fluid named as photon-baryon fluid. Now as time passes universe cool down and there comes a stage when photons do not have enough energy to keep atom ionized and atoms form,this epoch is known as recombination.This phenomena can be understood using Saha equation(eq2.21 for ionization of hydrogen atom) which tells about fraction of atom ionized at a particular temperature T.

$$\frac{x_e^2}{1-x_e} = (n_H + n_p)^{-1}\left(\frac{m_e k_B T}{2\pi\hbar^2}\right)^{\frac{3}{2}} exp\left(-\frac{E_I}{k_B T}\right) \tag{2.21}$$

where x_e is fraction of ionized atoms, n_H is number density of neutral hydrogen atoms , n_p is number density of ionized hydrogen atoms ,m_e is mass of electron and E_I is ionization energy. From this formula it is clear that till the time temperature is higher than ionization energy atoms will be in ionized state and when temperature fall below this ionization energy then we will get large fraction of atom in neutral state. So it is rapid transition from everything in ionized state to everything in neutral. Temperature at which this occur is approximately 3000 K. so using $(T\alpha\frac{1}{a})$ it we can calculate redshift at which recombination occurred $z \approx 1100$.

Now these neutral atoms do not scatter photons that much and these photons travel unimpeded through the space. We are receiving this radiation at temperature 2.73 K, known as Cosmic Microwave Background.

2.4.1 Properties of CMB

1. **Temperature Anisotropy:** There is anisotropy at level of 1 part per 10^5 in CMB.This anisotropy is considered to be due to quantum mechanical fluctuations in space which were brought to cosmic scale by inflation. Due to these fluctuations,there were regions where density was higher(overdense) compare to others which were further increased due gravitational attraction.Overdense regions tend to attract matter more than underdense (In terms of general theory of relativity more energy density will tend to curve space time more resulting into free falling of mass towards it) so baryon photon fluid get attracted towards overdense region(where dark matter is in falling) but since there is radiation pressure due to photons it resist the collapse resulting push fluid back then again gravity come into seen try to pull it and this process goes on. This process set up oscillations,known as Acoustic oscillations.This gravitational instability led to structure observed today.As photon decoupled from baryons their temperature is going to be different ,one which are coming from region which is at extrema of compressed state is going to have relatively higher temperature then one which are rarefied ..Imprint of these fluctuations can be seen in temperature of CMB.These fluctuations are small so we can consider them as perturbation to background which is uniform with temp $T = 2.73K$.

   Since fluctuations are at many points in space so it results into superposition of these oscillations. To know which frequency dominate there is need to go into Fourier space .Since we get a spherical surface in CMB as shown in fig:2.1 so instead of plane wave superposition of spherical harmonics $Y_{lm}(\theta, \phi)$ is going to produce pattern seen. Temperature fluctuation in a direction $\hat{n}$ $A(\hat{n}) = \frac{\Delta T}{T}$ is given as:

$$A(\hat{n}) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm} Y_{lm}(\hat{n}) \tag{2.22}$$

   here a_{lm} represents set of random coefficients given as:

$$a_{lm} = \int A(\hat{n}) Y_{lm}^*(\hat{n}), \tag{2.23}$$

$$m = 0, \pm 1, ..., \pm l; l = 1, 2, 3, ...$$

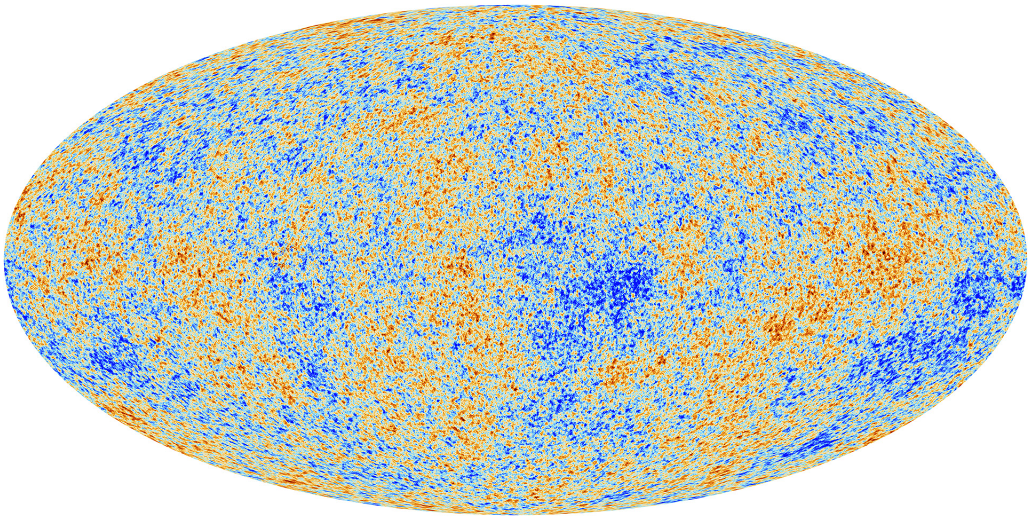

Figure 2.1: Whole sky CMB map (credit: `https://www.esa.int/Science_Exploration/Space_Science/Planck/Planck_and_the_cosmic_microwave_background`) showing contrast in temperature at microkelvin level,red here show higher temperature and as we move toward blue temperature decreases

Y_{lm} is given as:

$$Y_{lm}(\hat{n}) = Y_{lm}(\theta, \phi) = \begin{cases} \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}P_{lm}(cos\theta)exp(\iota m\phi) & \text{m>0} \\ (-1)^m Y^*_{lm}(\theta, \phi) & \text{m<0} \end{cases} \tag{2.24}$$

where

$$P_{lm}(x) = (-1)^m(1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x), \tag{2.25}$$

$$P_l(x) = \frac{1}{2^l l!}\frac{d}{dx^l}(x^2-1)^l \tag{2.26}$$

$$m = 0, 1, ..., l; l = 1, 2, 3, ...$$

Power spectrum of these Fourier fluctuation is given by

$$\langle a^*_{lm}a_{l'm'}\rangle = \delta_{ll'}\delta mm'C_l \tag{2.27}$$

we can define Angular wavelength as $\theta = \frac{2\pi}{l}$

Since we have one sky that implies that number of sample available for a given l is limited. Higher the l larger division of sky so lower the error so for given l variance is given by $\int d^2l\frac{C_l}{(2\pi)^2}$ so power spectrum is:

$$\Delta^2_T = \frac{l(l+1)}{2\pi}C_lT^2 \tag{2.28}$$

Thus,Power spectrum obtained we see different peaks which are interpreted as oscillations as shown in fig: 2.2.

2. **Polarization** : CMB is partially polarized due to Thomson scattering.Polarization and temperature anisotropy are partially correlated since presence of quadruple temperature anisotropy lead to polarization .The Power spectrum of it could also be obtained and comparing it with temperature anisotropy lead to better estimation of parameter.

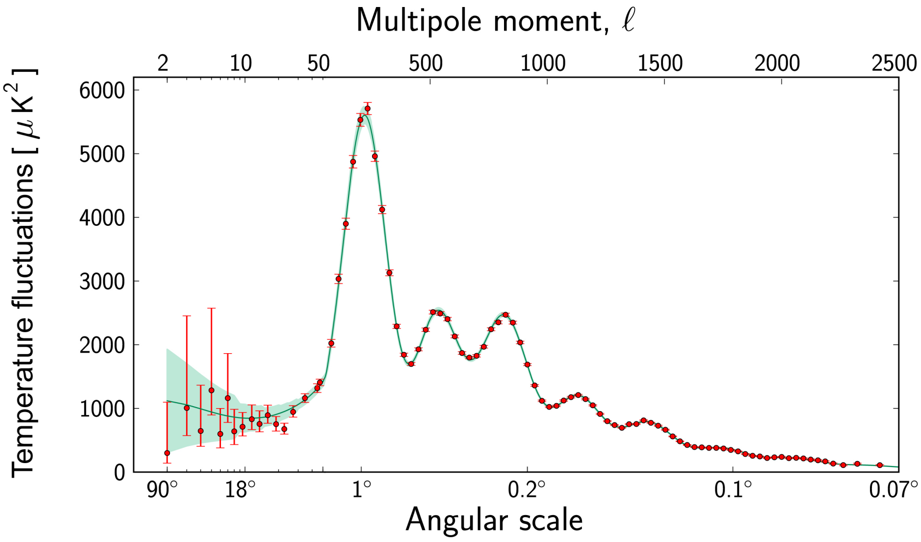

Figure 2.2: Angular Power spectrum of CMB (credit:https://www.esa.int/ESA_Multimedia/Images/2013/03/Planck_Power_Spectrum)

2.5 Ways of measuring Hubble constant:

2.5.1 from Cosmic Microwave Background

: Since oscillations in CMB resemble sound in cavity. We refer to them as sound.Sound can travel a certain distance by recombination. This distance is known as Sound Horizon.Sound horizon is given by:

$$r_s = \int_z^\infty \frac{c_s dz}{H(z)} \tag{2.29}$$

where c_s is speed of sound in baryon photon fluid .If we consider perturbation to be adiabatic then it is given by

$$\begin{aligned} c_s &= \sqrt{\frac{\delta P}{\delta \rho}} \\ c_s &= \frac{1}{\sqrt{3\left(1+\frac{\delta\rho_b}{\delta\rho_r}\right)}} \\ c_s &= \frac{1}{\sqrt{3\left(1+\frac{3\rho_b}{4\rho_r}\right)}} \end{aligned} \tag{2.30}$$

where ρ_b and ρ_r stands for baryon density and radiation density at recombination.

There is angular scale set up by oscillation which is just angular size of sound horizon given by

$$\theta_s = \frac{r_s}{D_A} \tag{2.31}$$

where D_A is comoving angular diameter distance which just comoving distance.D_A is given by:

$$D_A = \int_0^z \frac{dz}{H(z)} \tag{2.32}$$

from location of first acoustic peak(correspond to oscillation which just compressed once by recombination) we can find about θ_s which act as standard ruler.Now via fitting different model for $H(z)$ we can find which one reproduce θ_s.

For ΛCDM ,Hubble parameter is given as:

$$H(z) = H_0\sqrt{\Omega_{m0}(1+z)^3 + 1 - \Omega_{m0}} \tag{2.33}$$

2.5.2 Standard Candles

Standard candles are one with known luminosity. In case we know luminosity(L) we can tell about luminosity distance (d_L) of star from us via measuring flux:

$$F = \frac{L}{4\pi d_L^2} \tag{2.34}$$

we can infer about speed of galaxy using spectroscopy. Here we need to take into account different peculiar velocity also which is due to local gravitational field. Some standard candles are as follows:

Cepheid Variable

First Discovered by Henrietta Swan Leavitt, Cepheid Variable are variable star with known period luminosity relation so via calculating period we can infer about luminosity of star.

Type 1A Supernovae

White dwarf star is one which stops collapse under gravity using electron degeneracy pressure but this pressure can provide support till a certain mass only known Chandrasekhar mass ($M \approx 1.4M_\odot$).Since mass and Luminosity are related. We can infer about luminosity and From that we can find luminosity distance.

Chapter 3

Big Bang Model

Hubble's observation of recession of galaxies in 1929 made it clear that universe was not static but expanding. The most prevailing model to explain the origin and evolution of the universe is the Big Bang model. Cosmic Microwave Background was the first evidence in favour of that. According to this model, our universe started about 13.8 billion years ago from planck scale and evolved into today's universe. How the universe evolves depends on the contents of universe.An approach that is being used to know the content of universe is that make a model with the content of universe and then verify it using observations such as Cosmic microwave background, Large Scale structures etc. This chapter is devoted to the study of different models.

3.1 ΛCDM Model

The model of the universe that has accumulated support from a large section of scientists is ΛCDM Model. This model assumes the universe to be made up of matter (visible and dark), radiation, and cosmological constant.The universe is taken to be spatially flat. In 1.3 a study of flat space is being done throughout, using general theory of relativity for a maximally symmetric universe. for this model, Friedmann equation can be written as follows using eq(1.45):

$$M_P^2\left(\frac{\dot{a}}{a}\right) = a^2\left(\rho_m + \rho_{dm} + \rho_r + \rho_\Lambda\right) \tag{3.1}$$

This equation can be solved if we assume dominance of one component at a time. Early universe,first dominated by radiation. As universe expanded matter takes over. This model suggests that ultimately Λ takes over, since it does not dilute with expansion.

3.1.1 Cosmological Constant Λ

In ΛCDM Model, Cosmological constant is one which is responsible for the acceleration of universe with pressure equal to negative of energy density. It is the most popular form of dark energy. This was introduced by Einstein to get a static universe.

Physical interpretation of this cosmological constant is somewhat puzzling till date. One way to address this issue is by taking Cosmological constant as Vacuum energy. According to quantum mechanics, Vacuum can't be just nothing because that will imply that we have knowledge of position and momentum(which is zero) simultaneously which is forbidden by Heisenberg Uncertainty Principle. So if we add up even zero point energy associated that will lead to infinite energy.In case we consider quantum gravity there is going to be cutoff at Planck scale so vacuum energy will be given as $\rho_\Lambda \approx M_P^4$ .Value of it turns out to be approximately $10^{76}GeV^4$. Meanwhile, observed value of energy density associated with cosmological constant is $10^{-47}GeV^4$.So there is a huge discrepancy between observed and calculated value. So there is a need for fine tuning to get exactly the same value.

3.1.2 Trouble with ΛCDM Model

Despite having trouble in explaining what actually cosmological constant is, ΛCDM fits well to most of the observations except a few. Following are trouble associated with ΛCDM Model-

- Hubble Tension: There is trouble in the measurement of Hubble constant for which late universe value($z = 0$) is at tension with early universe ($z \approx 1100$).
- Large Scale Structure Tension: Observations using weak gravitational lensing suggest lower matter clustering than value evaluated via fitting ΛCDM model to CMB data.

This indicates problem with ΛCDM model and there is a need for physics beyond this model to explain this discrepancy. Well, One need to keep in mind that these troubles could be systematic and get resolved if we made more precise measurement. There are many attempts to solve Hubble tension. There are models which try to solve it either by introducing a change in the contents of universe or via modifying gravity.

3.2 Chameleon Early Dark Energy

Since we get a small value of H_0 from CMB compared to local measurements so by increasing energy density near the time of recombination we can ease Hubble tension. Since this increase in energy density is going to decrease the sound horizon since now expansion is faster, so there is less time for the sound to propagate. Models which work on this principle are known as **early universe solutions**.

In early dark energy models, we introduce a scalar field which has negligible contribution at all times just contributes significantly at a time near recombination. This model eases Hubble tension significantly but increases sigma tension so to resolve both tension and to answer **Why then problem** (why should the scalar field have significant contribution at a specific time) conformal coupling between components of universe is introduced. Let us try to explore a Chameleon Early Dark Energy introduced in [10]. According to this model, the contents of the universe are Matter(ordinary, dark), Radiation, Cosmological constant and scalar field. Dark matter and scalar field are conformally coupled to each other. Action for such a model is given by:

$$S = \int d^4x\sqrt{-g}\left[\frac{M_P^2}{2}R - \frac{1}{2}\phi_{,\mu}\phi^{,\mu} - V(\phi)\right] + S_{dm}(\phi_{dm}, \tilde{g}_{\mu\nu}) + S_m(\phi_m, g_{\mu\nu}) \tag{3.2}$$

here m stands for all matter species baryons,photons and neutrinos.dm for dark matter,ϕ_m and ϕ_{dm} are field associated with matter and dark matter respectively. Dark matter is considered cold, but we assume certain pressure associated with matter since it contains neutrino and photons (baryon photon fluid) which will give nonzero pressure. $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ are related as:

$$\tilde{g}_{\mu\nu} = A^2(\phi)g_{\mu\nu} \tag{3.3}$$

The frame in which metric is given by $\tilde{g}_{\mu\nu}$ is known as Jordan Frame and metric is given by $g_{\mu\nu}$ for the Einstein frame. The transformation between these two frames is discussed in 1.5. Dark matter is at rest in frame with metric $\tilde{g}_{\mu\nu}$ so all its physical values for dark matter are defined in that frame only. Scalar field and dark matter are interacting via $A(\phi)$.

3.2.1 Dynamics of Background:

Let's explore the dynamics of background (here, background implies an isotropic and homogeneous universe with no fluctuation).

- Fridemann Equation:

  for a universe containing matter(dark matter, baryonic matter), cosmological constant and scalar field.

$$3M_P^2\frac{\dot{a}^2}{a^2} = a^2(\rho_\phi + \rho_{dme} + \rho_m + \rho_\Lambda)$$

$$3M_P^2\frac{\dot{a}^2}{a^2} = a^2(\frac{\dot{\phi}^2}{2a^2} + V(\phi) + A^4\tilde{\rho}_{dm} + \rho_m + \rho_\Lambda) \tag{3.4}$$

- Continuity Equation:

  for dark matter:

  case: $\nu = 0$

$$\begin{aligned} T^{\mu}{}_{0;\mu} &= \frac{A_{,0}}{A}T \\ T^{\mu}{}_{0,\mu} + \Gamma^{\mu}{}_{\alpha\mu}T^{\alpha}{}_{0} - \Gamma^{\alpha}{}_{0\mu}T^{\mu}{}_{\alpha} &= \frac{A_{,0}}{A}T \\ T^{0}{}_{0,0} + \Gamma^{\mu}{}_{0\mu}T^{0}{}_{0} - \Gamma^{0}{}_{00}T^{0}{}_{0} &= \frac{A_{,0}}{A}T \\ (A^4\tilde{\rho}_{dm})_{,0} + 3\frac{\dot{a}}{a}A^4\tilde{\rho}_{dm} &= \frac{A_{,0}}{A}A^4\tilde{\rho}_{dm} \\ 4A^3A_{,0}\tilde{\rho}_{dm} + A^4((\tilde{\rho}_{dm})_{,0} + 3\frac{\dot{a}}{a}\tilde{\rho}_{dm}) &= \frac{A_{,0}}{A}A^4\tilde{\rho}_{dm} \\ \dot{\tilde{\rho}}_{dm} &= -3\tilde{\rho}_{dm}\left(\frac{\dot{a}}{a} + \frac{A_{,\phi}}{A}\dot{\phi}\right) \end{aligned} \tag{3.5}$$

  Solving this equation

$$\frac{d\tilde{\rho}_{dm}}{\tilde{\rho}_{dm}} = -3\left(\frac{da}{a} + \frac{dA}{A}\right)$$

$$\tilde{\rho}_{dm} = Ca^{-3}A^{-3}$$

  where C is constant taking condition at $a = a_0$, we have $\tilde{\rho}_{dm} = \tilde{\rho}^0_{dm}$ and $A = A_0$

$$\tilde{\rho}_{dm} = \tilde{\rho}^0_{dm}\left(\frac{A_0}{A}\right)^3 a^{-3} \tag{3.6}$$

$$\rho_{dm} = \tilde{\rho}_{dm}A^3 \tag{3.7}$$

$$\rho_{dm} = \rho^0_{dm}a^{-3} \tag{3.8}$$

  here ρ_{dm} is just a variable which appear to evolve as a^{-3} .

for scalar field:

$$
\begin{aligned}
(T^{\mu}{}_{0})^{\phi}_{;\mu} &= (T^{0}{}_{0})^{\phi}_{,0} + \Gamma^{\mu}{}_{0\mu}(T^{0}{}_{0})^{\phi} - \Gamma^{0}{}_{00} \\
&\quad (T^{0}{}_{0})^{\phi} - \Gamma^{i}{}_{0j}(T^{j}{}_{i})^{\phi} \\
&= (T^{0}{}_{0})^{\phi}_{,0} + 3\frac{\dot{a}}{a}(T^{0}{}_{0})^{\phi} - \frac{\dot{a}}{a}\delta^{i}{}_{j}(T^{j}{}_{i})^{\phi} \\
&= -\left(\frac{\dot{\phi}^2}{2a^2} + V(\phi)\right)_{,0} - 3\frac{\dot{a}}{a}\left(\frac{\dot{\phi}^2}{2a^2} + V(\phi)\right) \\
&\quad - 3\frac{\dot{a}}{a}\left(\frac{\dot{\phi}^2}{2a^2} - V(\phi)\right) \\
&= -\frac{2\dot{\phi}\ddot{\phi}}{2a^2} + \frac{2\dot{\phi}^2\dot{a}}{2a^3} - V_{,\phi}\dot{\phi} - 3\frac{\dot{a}}{a}\left(\frac{\dot{\phi}^2}{a^2}\right) \\
&= -\frac{\dot{\phi}\ddot{\phi}}{a^2} - 2\frac{\dot{a}}{a^3}\dot{\phi}^2 - V_{,\phi}\dot{\phi} \qquad (3.9)
\end{aligned}
$$

$$
\begin{aligned}
(T^{\mu}{}_{0})^{dm}_{;\mu} + (T^{\mu}{}_{0})^{\phi}_{;\mu} &= 0 \\
-\frac{\dot{\phi}\ddot{\phi}}{a^2} - 2\frac{\dot{a}}{a^3}\dot{\phi}^2 - V_{,\phi}\dot{\phi} - \frac{A_{,\phi}}{A}A^4\dot{\phi}\tilde{\rho}_{dm} &= 0 \\
\ddot{\phi} + 2\frac{\dot{a}}{a}\dot{\phi} = -a^2V_{,\phi} - a^2A_{,\phi}A^3\dot{\phi}\tilde{\rho}_{dm} & \qquad (3.10) \\
\ddot{\phi} + 2\frac{\dot{a}}{a}\dot{\phi} = -a^2(V + A^4\tilde{\rho}_{dm})_{,\phi} & \\
\ddot{\phi} + 2\frac{\dot{a}}{a}\dot{\phi} = -a^2(V + A\rho_{dm})_{,\phi} & \qquad (3.11)
\end{aligned}
$$

So here we can define $V_{eff} = V + A\rho_{dm}$,scalar field moves in this potential as in case of uncoupled scalar field ϕ move in potential $V(\phi)$. This manifests that evolution of scalar field is going to be affected by energy density of dark matter.

Matter

Since components of matter are coupled neither to any other component of universe or each other , they are going to evolve as they do in the case of a single component universe

Cosmological Constant

It is going to remain constant as in the case of a single component universe.

In terms of density parameter, the Fridemann equation can be written as for the present epoch:

$$
\Omega_m^0 + A(\phi_0)\Omega_{dm}^0 + \Omega_{\phi}^0 + \Omega_{\Lambda}^0 = 1 \qquad (3.12)
$$

3.2.2 Perturbation Dynamics

We are considering that fluctuation in isotropic and homogeneous universe are small so we can take them as linear perturbation. For structure formation we need to care about scalar perturbation only as already being discussed in 1.6.

$A(\phi)$ is perturbed as:

$$
A(\phi) = \bar{A} + \delta\phi\bar{A}_{,\phi} \qquad (3.13)
$$

ρ is perturbed as:

$$\rho = \bar{\rho} + \delta\bar{\rho} \tag{3.14}$$

Perturbation in continuity equation for dark matter:

$$\begin{aligned}
T^{\mu}{}_{\nu;\mu} &= \frac{A_{,\nu}}{A}T \\
T^{\mu}{}_{\nu,\mu} + \Gamma^{\mu}{}_{\alpha\mu}T^{\alpha}{}_{\nu} - \Gamma^{\alpha}{}_{\nu\mu}T^{\mu}{}_{\alpha} &= \frac{A_{,\nu}}{A}T \\
(\bar{T}^{\mu}{}_{\nu} + \delta T^{\mu}{}_{\nu})_{,\mu} + (\bar{\Gamma}^{\mu}{}_{\alpha\mu} + \delta\Gamma^{\mu}{}_{\alpha\mu})(\bar{T}^{\alpha}{}_{\nu} + \delta T^{\alpha}{}_{\nu}) - (\bar{\Gamma}^{\alpha}{}_{\nu\mu} + \delta\Gamma^{\alpha}{}_{\nu\mu})(\bar{T}^{\mu}{}_{\alpha} + \delta T^{\mu}{}_{\alpha}) & \\
&= \frac{(\bar{A} + \delta\phi\bar{A}_{,\phi})_{,\nu}}{\bar{A} + \delta\phi\bar{A}_{,\phi}}(\bar{T} + \delta T) \\
\bar{T}^{\mu}{}_{\nu;\mu} + \delta T^{\mu}{}_{\nu,\mu} + \bar{\Gamma}^{\mu}{}_{\alpha\mu}\delta T^{\alpha}{}_{\nu} + \delta\Gamma^{\mu}{}_{\alpha\mu}\bar{T}^{\alpha}{}_{\nu} - \bar{\Gamma}^{\alpha}{}_{\nu\mu}\delta T^{\mu}{}_{\alpha} - \delta\Gamma^{\alpha}{}_{\nu\mu}\bar{T}^{\mu}{}_{\alpha} & \\
&= \frac{(\bar{A}_{,\nu} + \delta\phi(\bar{A}_{,\phi})_{,\nu} + (\delta\phi)_{,\nu}\bar{A}_{,\phi})(1 - \frac{\delta\phi\bar{A}_{,\phi}}{\bar{A}})}{\bar{A}}(\bar{T} + \delta T) \\
&= \frac{\bar{A}_{,\nu}}{\bar{A}}\bar{T} + \frac{(\delta\phi(\bar{A}_{,\phi})_{,\nu} + (\delta\phi)_{,\nu}\bar{A}_{,\phi})}{\bar{A}}\bar{T} - \bar{A}_{,\nu}\frac{\delta\phi\bar{A}_{,\phi}}{\bar{A}^2}\bar{T} + \frac{\bar{A}_{,\nu}}{\bar{A}}\delta T \\
\delta T^{\mu}{}_{\nu,\mu} + \bar{\Gamma}^{\mu}{}_{\alpha\mu}\delta T^{\alpha}{}_{\nu} + \delta\Gamma^{\mu}{}_{\alpha\mu}\bar{T}^{\alpha}{}_{\nu} - \bar{\Gamma}^{\alpha}{}_{\nu\mu}\delta T^{\mu}{}_{\alpha} - \delta\Gamma^{\alpha}{}_{\nu\mu}\bar{T}^{\mu}{}_{\alpha} & \\
&= \frac{(\delta\phi(\bar{A}_{,\phi})_{,\nu} + (\delta\phi)_{,\nu}\bar{A}_{,\phi})}{\bar{A}}\bar{T} - \bar{A}_{,\nu}\frac{\delta\phi\bar{A}_{,\phi}}{\bar{A}^2}\bar{T} + \frac{\bar{A}_{,\nu}}{\bar{A}}\delta T
\end{aligned} \tag{3.15}$$

- for dark matter with $\nu = 0$

$$\begin{aligned}
\delta T^{\mu}{}_{0,\mu} + \bar{\Gamma}^{\mu}{}_{\alpha\mu}\delta T^{\alpha}{}_{0} + \delta\Gamma^{\mu}{}_{\alpha\mu}\bar{T}^{\alpha}{}_{0} - \bar{\Gamma}^{\alpha}{}_{0\mu}\delta T^{\mu}{}_{\alpha} - \delta\Gamma^{\alpha}{}_{0\mu}\bar{T}^{\mu}{}_{\alpha} & \\
= \frac{(\delta\phi(\bar{A}_{,\phi})_{,0} + (\delta\phi)_{,0}\bar{A}_{,\phi})}{\bar{A}}\bar{T} - \bar{A}_{,0}\frac{\delta\phi\bar{A}_{,\phi}}{\bar{A}^2}\bar{T} + \frac{\bar{A}_{,0}}{\bar{A}}\delta T &
\end{aligned}$$

$$\begin{aligned}
\delta T^{0}{}_{0,0} + \delta T^{i}{}_{0,i} + 3\frac{\dot{a}}{a}\delta T^{0}{}_{0} - \frac{\dot{a}}{a}\delta T^{i}{}_{i} - \frac{\dot{h}^{i}{}_{i}}{2}\bar{\rho}_{dm} & \\
= \frac{(\delta\phi\bar{A}_{,\phi\phi}\dot{\phi} + \dot{\delta\phi}\bar{A}_{,\phi})}{\bar{A}}(-\bar{\rho}_{dm}) + \dot{\phi}\frac{\delta\phi(\bar{A}_{,\phi})^2}{\bar{A}^2}\bar{\rho}_{dm} + \frac{\bar{A}_{,\phi}}{\bar{A}}\dot{\phi}\delta T &
\end{aligned} \tag{3.16}$$

in k space

$$\delta T^0{}_{0,0} + \imath k_i \delta T^i{}_0 + 3\frac{\dot{a}}{a}\delta T^0{}_0 - \frac{\dot{a}}{a}\delta T^i{}_i - \frac{\dot{h}}{2}\bar{\rho}_{dm}$$
$$= \frac{(\delta\phi \bar{A}_{,\phi\phi}\dot{\phi} + \dot{\delta}\phi \bar{A}_{,\phi})}{\bar{A}}(-\bar{\rho}_{dm}) + \dot{\phi}\frac{\delta\phi(\bar{A}_{,\phi})^2}{\bar{A}^2}\bar{\rho}_{dm} + \frac{\bar{A}_{,\phi}}{\bar{A}}\dot{\phi}\delta T$$

$$-(\delta\rho_{dm})_{,0} - \imath k_i \bar{\rho}_{dm} v^i - 3\frac{\dot{a}}{a}\delta\rho_{dm} - \frac{\dot{a}}{a}(0) - \frac{\dot{h}}{2}\bar{\rho}_{dm}$$
$$= -\frac{\delta\phi \bar{A}_{,\phi\phi}\dot{\phi}}{\bar{A}}\bar{\rho}_{dm} - \frac{\dot{\delta}\phi \bar{A}_{,\phi}}{\bar{A}}\bar{\rho}_{dm} + \dot{\phi}\frac{\delta\phi(\bar{A}_{,\phi})^2}{\bar{A}^2}\bar{\rho}_{dm} - \frac{\bar{A}_{,\phi}}{\bar{A}}\dot{\phi}\delta\rho_{dm}$$

$$(\bar{A}^4\tilde{\delta}_{dm}\tilde{\rho}_{dm} + 4\bar{A}^3 A_{,\phi}\delta\phi\tilde{\rho}_{dm})_{,0} + \bar{A}^4\tilde{\rho}_{dm}\theta_{dm} + 3\frac{\dot{a}}{a}(\delta_{dm}\bar{A}^4\tilde{\rho}_{dm} + 4\bar{A}^3\bar{A}_{,\phi}\delta\phi\tilde{\rho}_{dm}) + \frac{\dot{h}}{2}\bar{A}^4\tilde{\rho}_{dm}$$
$$= \frac{\delta\phi\bar{A}_{,\phi\phi}\dot{\phi}}{\bar{A}}\bar{A}^4\tilde{\rho}_{dm} + \frac{\dot{\delta}\phi\bar{A}_{,\phi}}{\bar{A}}\bar{A}^4\tilde{\rho}_{dm} - \dot{\phi}\frac{\delta\phi(\bar{A}_{,\phi})^2}{\bar{A}^2}\bar{A}^4\tilde{\rho}_{dm} + \frac{\bar{A}_{,\phi}}{\bar{A}}\dot{\phi}(\bar{A}^4\tilde{\delta}_{dm}\tilde{\rho}_{dm} + 4\bar{A}^3\bar{A}_{,\phi}\delta\phi\tilde{\rho}_{dm})$$

$$\bar{A}^4\tilde{\delta}_{dm}\left(\dot{\tilde{\rho}}_{dm} + 3\frac{\dot{a}}{a}\tilde{\rho}_{dm}\right) + \bar{A}^4\dot{\tilde{\delta}}_{dm}\tilde{\rho}_{dm} + 4\bar{A}^3\bar{A}_{,\phi}\dot{\phi}\tilde{\delta}_{dm}\tilde{\rho}_{dm} + 12\bar{A}^2(\bar{A}_{\phi})^2\delta\phi\tilde{\rho}_{dm}\dot{\phi}$$
$$+ 4\bar{A}^3\bar{A}_{,\phi\phi}\dot{\phi}\delta\phi\tilde{\rho}_{dm} + 4\bar{A}^3\bar{A}_{,\phi}\dot{\delta}\phi\tilde{\rho}_{dm} + 4\bar{A}^3\bar{A}_{,\phi}\delta\phi\dot{\tilde{\rho}}_{dm} + 12\frac{\dot{a}}{a}(\bar{A}^3\bar{A}_{,\phi}\delta\phi\tilde{\rho}_{dm})$$
$$+ \bar{A}^4\tilde{\rho}_{dm}\theta_{dm} + \frac{\dot{h}}{2}\bar{A}^4\tilde{\rho}_{dm}$$

$$= \bar{A}^4\tilde{\rho}_{dm}\left(\delta\phi\dot{\phi}\left(\frac{\bar{A}_{,\phi\phi}}{\bar{A}} + 3\frac{(\bar{A}_{,\phi})^2}{\bar{A}^2}\right)) + \frac{\dot{\delta}\phi\bar{A}_{,\phi}}{\bar{A}}\right) + \bar{A}_{,\phi}\dot{\phi}\bar{A}^3\tilde{\delta}_{dm}\tilde{\rho}_{dm}$$

$$\bar{A}^4\tilde{\delta}_{dm}\left(-3\tilde{\rho}_{dm}\frac{\bar{A}_{,\phi}}{\bar{A}}\dot{\phi}\right) + \bar{A}^4\dot{\tilde{\delta}}_{dm}\tilde{\rho}_{dm} + 12\bar{A}^2(\bar{A}_{\phi})^2\delta\phi\tilde{\rho}_{dm} + 4\bar{A}^4\tilde{\rho}_{dm}\left(\delta\phi\dot{\phi}\left(\frac{\bar{A}_{,\phi\phi}}{\bar{A}}\right) + \frac{\dot{\delta}\phi\bar{A}_{,\phi}}{\bar{A}}\right)$$
$$+ 4\bar{A}^3 A_{\phi}\delta\phi\left(-3\tilde{\rho}_{dm}\frac{\bar{A}_{,\phi}}{\bar{A}}\dot{\phi}\right) + 3\bar{A}^3\bar{A}_{,\phi}\dot{\phi}\tilde{\delta}_{dm}\tilde{\rho}_{dm} + \bar{A}^4\tilde{\rho}_{dm}\theta_{dm} + \frac{\dot{h}}{2}\bar{A}^4\tilde{\rho}_{dm}$$

$$= \bar{A}^4\tilde{\rho}_{dm}\left(\delta\phi\dot{\phi}\left(\frac{\bar{A}_{,\phi\phi}}{\bar{A}} + 3\frac{(\bar{A}_{,\phi})^2}{\bar{A}^2}\right)) + \frac{\dot{\delta}\phi\bar{A}_{,\phi}}{\bar{A}}\right)$$

$$\dot{\tilde{\delta}}_{dm} = -\dot{\tilde{\theta}}_{dm} - \frac{\dot{h}}{2} - 3\left(\delta\phi\dot{\phi}\left(\frac{\bar{A}_{,\phi\phi}}{\bar{A}} - \frac{(\bar{A}_{,\phi})^2}{\bar{A}^2}\right) + \frac{\dot{\delta}\phi\bar{A}_{,\phi}}{\bar{A}}\right) \tag{3.17}$$

- for dark matter with $\nu = j$

$$\delta T^{\mu}{}_{j,\mu} + \bar{\Gamma}^{\mu}{}_{\alpha\mu}\delta T^{\alpha}{}_{j} + \delta\Gamma^{\mu}{}_{\alpha\mu}\bar{T}^{\alpha}{}_{j} - \bar{\Gamma}^{\alpha}{}_{j\mu}\delta T^{\mu}{}_{\alpha} - \delta\Gamma^{\alpha}{}_{j\mu}\bar{T}^{\mu}{}_{\alpha}$$
$$= \frac{(\delta\phi(\bar{A}_{,\phi})_{,j} + (\delta\phi)_{,j}\bar{A}_{,\phi})}{\bar{A}}\bar{T} - \bar{A}_{,j}\frac{\delta\phi\bar{A}_{,\phi}}{\bar{A}^2}\bar{T} + \frac{\bar{A}_{,j}}{\bar{A}}\delta T$$

$$\delta T^{0}{}_{j,0} + \delta T^{k}{}_{j,k} + 4\frac{\dot{a}}{a}\delta T^{0}{}_{j} = \frac{((\delta\phi)_{,j}\bar{A}_{,\phi})}{\bar{A}}(-\bar{\rho}_{dm})$$

$$(\bar{\rho}_{dm}v_j)_{,0} + 4\frac{\dot{a}}{a}\bar{\rho}_{dm}v_j = \frac{((\delta\phi)_{,j}\bar{A}_{,\phi})}{\bar{A}}(-\bar{\rho}_{dm})$$

$$(\bar{A}^4\tilde{\rho}_{dm}v_j)_{,0} + 4\frac{\dot{a}}{a}\bar{A}^4\tilde{\rho}_{dm}v_j = \frac{((\delta\phi)_{,j}\bar{A}_{,\phi})}{\bar{A}}(-\bar{A}^4\tilde{\rho}_{dm})$$

$$4\frac{\bar{A}_{,\phi}\dot{\phi}v_j}{\bar{A}} + \frac{\dot{\tilde{\rho}}_{dm}}{\tilde{\rho}_{dm}}v_j + \dot{v}_j + 4\frac{\dot{a}}{a}v_j = -\frac{(\delta\phi)_{,j}\bar{A}_{,\phi}}{\bar{A}}$$

using eq(3.5)

$$4\frac{\bar{A}_{,\phi}\dot{\phi}v_j}{A} - 3\left(\frac{\dot{a}}{a} + \frac{\bar{A}_{,\phi}\dot{\phi}}{\bar{A}}\right)v_j + \dot{v}_j + 4\frac{\dot{a}}{a}v_j = -\frac{(\delta\phi)_{,j}\bar{A}_{,\phi}}{\bar{A}}$$

$$\frac{A_{,\phi}\dot{\phi}v_j}{A} + \dot{v}_j + \frac{\dot{a}}{a}v_j = -\frac{(\delta\phi)_{,j}\bar{A}_{,\phi}}{\bar{A}} \tag{3.18}$$

in k space

$$\iota k^j\frac{\bar{A}_{,\phi}\dot{\phi}v_j}{\bar{A}} + \iota k^j\dot{v}_j + \frac{\dot{a}}{a}\iota k^j v_j = -\iota k^j\iota k_j\frac{(\delta\phi)\bar{A}_{,\phi}}{\bar{A}}$$
$$\frac{\bar{A}_{,\phi}\dot{\phi}}{\bar{A}}\tilde{\theta}_{dm} + \dot{\tilde{\theta}}_{dm} + \frac{\dot{a}}{a}\tilde{\theta}_{dm} = k^2\frac{(\delta\phi)\bar{A}_{,\phi}}{\bar{A}}$$
$$\dot{\tilde{\theta}}_{dm} = -\left(\frac{\bar{A}_{,\phi}\dot{\phi}}{\bar{A}} + \frac{\dot{a}}{a}\right)\tilde{\theta}_{dm} + k^2\frac{(\delta\phi)\bar{A}_{,\phi}}{\bar{A}} \tag{3.19}$$

- Scalar field:

$$\delta T^{\mu}{}_{0,\mu}+\bar{\Gamma}^{\mu}{}_{\alpha\mu}\delta T^{\alpha}{}_{0}+\delta\Gamma^{\mu}{}_{\alpha\mu}\bar{T}^{\alpha}{}_{0}-\bar{\Gamma}^{\alpha}{}_{0\mu}\delta T^{\mu}{}_{\alpha}-\delta\Gamma^{\alpha}{}_{0\mu}\bar{T}^{\mu}{}_{\alpha}$$
$$=\delta T^{0}{}_{0,0}+\delta T^{j}{}_{0,j}+\bar{\Gamma}^{\mu}{}_{0\mu}\delta T^{0}{}_{0}+\delta\Gamma^{\mu}{}_{0\mu}\bar{T}^{0}{}_{0}-\bar{\Gamma}^{0}{}_{00}\delta T^{0}{}_{0}-\bar{\Gamma}^{i}{}_{0j}\delta T^{j}{}_{i}-\delta\Gamma^{i}{}_{0j}\bar{T}^{j}{}_{i}$$
$$=\delta T^{0}{}_{0,0}+\delta T^{j}{}_{0,j}+3\frac{\dot{a}}{a}\delta T^{0}{}_{0}-\frac{\dot{a}}{a}\delta T^{i}{}_{i}-\frac{\dot{h}^{i}{}_{i}}{2}(\bar{p}+\bar{\rho})$$
$$=-\left(\frac{\dot{\phi}\dot{\delta\phi}}{a^2}+V_{,\phi}\delta\phi\right)_{,0}+\left(\frac{1}{a^2}\dot{\phi}(\delta\phi)^{,j}\right)_{,j}-3\frac{\dot{a}}{a}\left(\frac{\dot{\phi}\dot{\delta\phi}}{a^2}+V_{,\phi}\delta\phi\right)$$
$$-3\frac{\dot{a}}{a}\left(\frac{\dot{\phi}\dot{\delta\phi}}{a^2}-V_{,\phi}\delta\phi\right)-\frac{\dot{h}^{i}{}_{i}}{2}(\left(\frac{\dot{\phi}^2}{2a^2}+V_{,\phi}+\frac{\dot{\phi}^2}{2a^2}-V_{,\phi}\right)$$
$$=-\left(\frac{\ddot{\phi}\dot{\delta\phi}}{a^2}+\frac{\dot{\phi}\ddot{\delta\phi}}{a^2}-\frac{2\dot{\phi}\dot{\delta\phi}\dot{a}}{a^3}\right)-V_{,\phi\phi}\dot{\phi}\delta\phi-V_{,\phi}\dot{\delta\phi}$$
$$+\frac{1}{a^2}\left(\dot{\phi}(\delta\phi)^{,j}{}_{,j}\right)-6\frac{\dot{a}}{a}\left(\frac{\dot{\phi}\dot{\delta\phi}}{a^2}\right)-\frac{\dot{h}^{i}{}_{i}}{2}\left(\frac{\dot{\phi}^2}{a^2}\right)$$

in K space

$$=-\left(\frac{\ddot{\phi}\dot{\delta\phi}}{a^2}+\frac{\dot{\phi}\ddot{\delta\phi}}{a^2}-\frac{2\dot{\phi}\dot{\delta\phi}\dot{a}}{a^3}\right)-V_{,\phi\phi}\dot{\phi}\delta\phi-V_{,\phi}\dot{\delta\phi}+\frac{1}{a^2}\left(-k^2\dot{\phi}\delta\phi\right)$$
$$-6\frac{\dot{a}}{a}\left(\frac{\dot{\phi}\dot{\delta\phi}}{a^2}\right)-\frac{\dot{h}}{2}\left(\frac{\dot{\phi}^2}{a^2}\right)$$
$$=-\left(\frac{\dot{\phi}}{a^2}\left(\ddot{\delta\phi}-\frac{2\dot{\delta\phi}\dot{a}}{a}+a^2V_{,\phi\phi}\delta\phi+k^2\delta\phi+6\frac{\dot{a}}{a}\dot{\delta\phi}+\frac{\dot{h}}{2}\dot{\phi}\right)+\frac{\ddot{\phi}\dot{\delta\phi}}{a^2}+V_{,\phi}\dot{\delta\phi}\right)$$

Using eq:(3.11)

$$=-\left(\frac{\dot{\phi}}{a^2}\left(\ddot{\delta\phi}+a^2V_{,\phi\phi}\delta\phi+k^2\delta\phi+4\frac{\dot{a}}{a}\dot{\delta\phi}+\frac{\dot{h}}{2}\dot{\phi}\right)\right)$$
$$-\left(-a^2\bar{A}_{,\phi}\bar{A}^3\tilde{\rho}_{dm}-a^2V_{,\phi}-2\frac{\dot{a}}{a}\dot{\phi}\right)\frac{\dot{\delta\phi}}{a^2}-V_{,\phi}\dot{\delta\phi}$$
$$=-\left(\frac{\dot{\phi}}{a^2}\left(\ddot{\delta\phi}+a^2V_{,\phi\phi}\delta\phi+k^2\delta\phi+2\frac{\dot{a}}{a}\dot{\delta\phi}+\frac{\dot{h}}{2}\dot{\phi}\right)-\dot{\delta\phi}\bar{A}_{,\phi}\bar{A}^3\tilde{\rho}_{dm}\right) \tag{3.20}$$

Now since dark matter and scalar field together going to give us energy momentum conservation:

$$(\delta T^{\mu}{}_{0})^{dm}_{;\mu} + (\delta T^{\mu}{}_{0})^{\phi}_{;\mu} = 0$$

$$-\bar{A}^4\tilde{\rho}_{dm}\left(\delta\phi\dot{\phi}\left(\frac{\bar{A}_{,\phi\phi}}{\bar{A}} + 3\frac{(\bar{A}_{,\phi})^2}{\bar{A}^2}\right) + \frac{\dot{\delta}\phi\bar{A}_{,\phi}}{\bar{A}}\right) - \bar{A}_{,\phi}\dot{\phi}\bar{A}^3\tilde{\delta}_{dm}\tilde{\rho}_{dm}$$

$$-\left(\frac{\dot{\phi}}{a^2}\left(\ddot{\delta}\phi + a^2V_{,\phi\phi}\delta\phi + k^2\delta\phi + 2\frac{\dot{a}}{a}\dot{\delta}\phi + \frac{\dot{h}}{2}\dot{\phi}\right) - \dot{\delta}\phi\bar{A}_{,\phi}\bar{A}^3\tilde{\rho}_{dm}\right) = 0$$

$$\frac{\dot{\phi}}{a^2}\left(\ddot{\delta}\phi + a^2V_{,\phi\phi}\delta\phi + k^2\delta\phi + 2\frac{\dot{a}}{a}\dot{\delta}\phi + \frac{\dot{h}}{2}\dot{\phi}\right)$$

$$= -\bar{A}^4\tilde{\rho}_{dm}\left(\delta\phi\dot{\phi}\left(\frac{\bar{A}_{,\phi\phi}}{\bar{A}} + 3\frac{(\bar{A}_{,\phi})^2}{\bar{A}^2}\right)\right) - \bar{A}_{,\phi}\dot{\phi}\bar{A}^3\tilde{\delta}_{dm}\tilde{\rho}_{dm}$$

$$\ddot{\delta}\phi + a^2V_{,\phi\phi}\delta\phi + k^2\delta\phi + 2\frac{\dot{a}}{a}\dot{\delta}\phi + \frac{\dot{h}}{2}\dot{\phi} = -a^2\bar{A}^3\tilde{\rho}_{dm}\delta\phi\left(\bar{A}_{,\phi\phi} + 3\frac{(\bar{A}_{,\phi})^2}{\bar{A}}\right) - a^2\bar{A}_{,\phi}\dot{\phi}\bar{A}^3\tilde{\delta}_{dm}\tilde{\rho}_{dm} \tag{3.21}$$

here eq(3.17) ,(3.19) and (3.21) clearly manifest that evolution of perturbation of scalar field is going to influenced by dark matter and vice versa.

for matter, perturbation are going to evolve independently as shown in [11]

Perturbed Einstein equations

using eq(1.79), (1.111) ,(1.112),(1.113) and (1.114)

$$-\frac{M_P^2}{a^2}\left(\frac{\dot{a}}{a}\dot{h} - 2k^2\eta\right) = (\delta T^0{}_0)^m + (\delta T^0{}_0)^{dm} + (\delta T^0{}_0)^{\phi}$$

$$\frac{M_P^2}{a^2}\left(\frac{\dot{a}}{a}\dot{h} - 2k^2\eta\right) = \rho_m\delta_m + 4A_{,\phi}\rho_{dm}\delta\phi + A\rho_{dm}\tilde{\rho}_d m\tilde{\delta}_{dm} + V_{,\phi}\delta\phi + \frac{\dot{\phi}}{a^2}\dot{\delta}\phi \tag{3.22}$$

$$-\frac{M_P^2}{a^2}2\iota k_k\dot{\eta} = (\delta T^0{}_k)^m + (\delta T^0{}_k)^{dm} + (\delta T^0{}_k)^{\phi}$$

$$-\frac{M_P^2}{a^2}2\iota k^k\iota k_k\dot{\eta} = \iota k^k\left[(\rho_m + P_m)v_k^m + A(\rho_{dm})v_k^{dm} - \frac{\iota k_k}{a^2}\dot{\phi}(\delta\phi)\right]$$

$$\frac{M_P^2}{a^2}2k^2\dot{\eta} = (\rho_m + P_m)\theta_m + A(\rho_{dm})\theta_{dm} - \frac{k^2}{a^2}\dot{\phi}(\delta\phi) \tag{3.23}$$

$$-\frac{M_P^2}{a^2}\left(\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} - 2k^2\eta\right) = (\delta T^i{}_i)^m + (\delta T^i{}_i)^{dm} + (\delta T^i{}_i)^{\phi}$$

$$-\frac{M_P^2}{a^2}\left(\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} - 2k^2\eta\right) = 3\left(\delta P_m - V_{,\phi}\delta\phi + \frac{\dot{\phi}}{a^2}\dot{\delta}\phi\right) \tag{3.24}$$

$$M_P^2\frac{k^jk_k}{2a^2}\left(\ddot{h} + 6\ddot{\eta} + 2\frac{\dot{a}}{a}(\dot{h} + 6\dot{\eta}) - 2k^2\eta\right) = (\delta T^j{}_k)^m + (\delta T^j{}_k)^{dm} + (\delta T^j{}_k)^{\phi}$$

$$M_P^2\frac{k^jk_k}{2a^2}\left(\frac{k_jk^k}{k^2} - \frac{1}{3}\delta_j^k\right)\left(\ddot{h} + 6\ddot{\eta} + 2\frac{\dot{a}}{a}(\dot{h} + 6\dot{\eta}) - 2k^2\eta\right) = \left(\frac{k_jk^k}{k^2} - \frac{1}{3}\delta_j^k\right)\Sigma_k^j$$

$$M_P^2\left(\ddot{h} + 6\ddot{\eta} + 2\frac{\dot{a}}{a}(\dot{h} + 6\dot{\eta}) - 2k^2\eta\right) = -3(\rho_m + P_m)\sigma_m \tag{3.25}$$

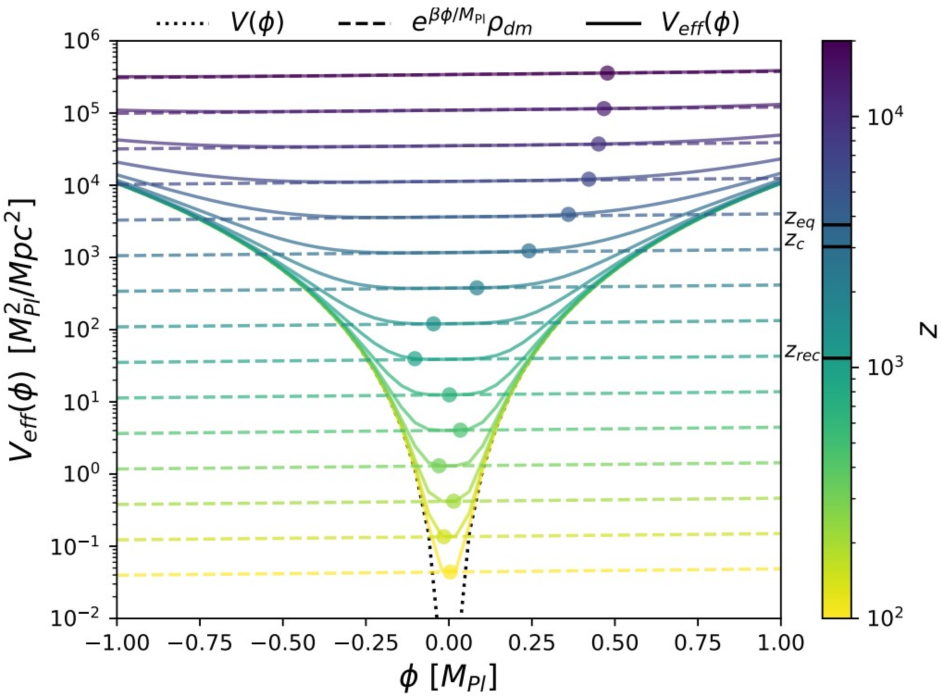

Figure 3.1: Effective potential under which scalar field is rolling.Here y axis is in logarithmic scale. (Credit:Chameleon Early Dark Energy and the Hubble Tension by Tanvi Karwal and et.al)

3.2.3 A Specific Scenario:

Lets try to explore universe with following conditions discussed in [10]:

$$A(\phi) = e^{\frac{\beta\phi}{M_P}} \tag{3.26}$$

$$V(\phi) = \lambda\phi^4 \tag{3.27}$$

here $\lambda = \lambda_{scf} V_0 \phi^4$ where $V_0 = 2X10^{-10} M_P^{-2} H_0^2$. β is coupling constant

$V_{eff} = \lambda\phi^4 + e^{\frac{\beta\phi}{M_P}}\rho_{dm}$ is shown in fig:3.1.Here term coming due to coupling of dark matter and scalar field introduce asymmetry in potential. As shown in figure initial conditions ($\lambda_{scf} =$ 1) are taken in such a way that initially asymmetric term dominates potential which is constant initially if $\phi_i = M_P$. So value of ϕ_i dictate when we are going to get required amount of energy density to resolve Hubble tension. Assuming that $\ddot{\phi}_i = 0$ from eq(3.11) we get, $\dot{\phi}$ =constant which implies that kinetic energy goes as a^{-2} so initially energy density of scalar field goes as a^{-2} as shown in fig:3.2 since potential energy is constant. After some time when native potential term start to dominate, we get Hubble frozen period in which energy density is going to remain essentially constant

For Hubble tension to resolve there is need of fraction energy density of dark energy of order $\mathfrak{O}(10\%)$ at time of matter radiation equality ($z \approx \frac{\Omega_{m0}}{\Omega_{r0}}$) so as we can see in fig:3.1 and 3.2.Here matter radiation equality means total density of ordinary matter and dark matter equals to radiation.

At matter radiation equality epoch we have potential which is dominated by native potential. Now scalar field moving under a potential ϕ^4 with asymmetry introduced by coupling.Therefore we are going to have oscillations. Due to asymmetry amplitude of odd and even peak is not same which peak is going to be higher than uncoupled case depends on value of β. With that it is also going to decay as a^{-4} due to Hubble friction as discussed in [18].If

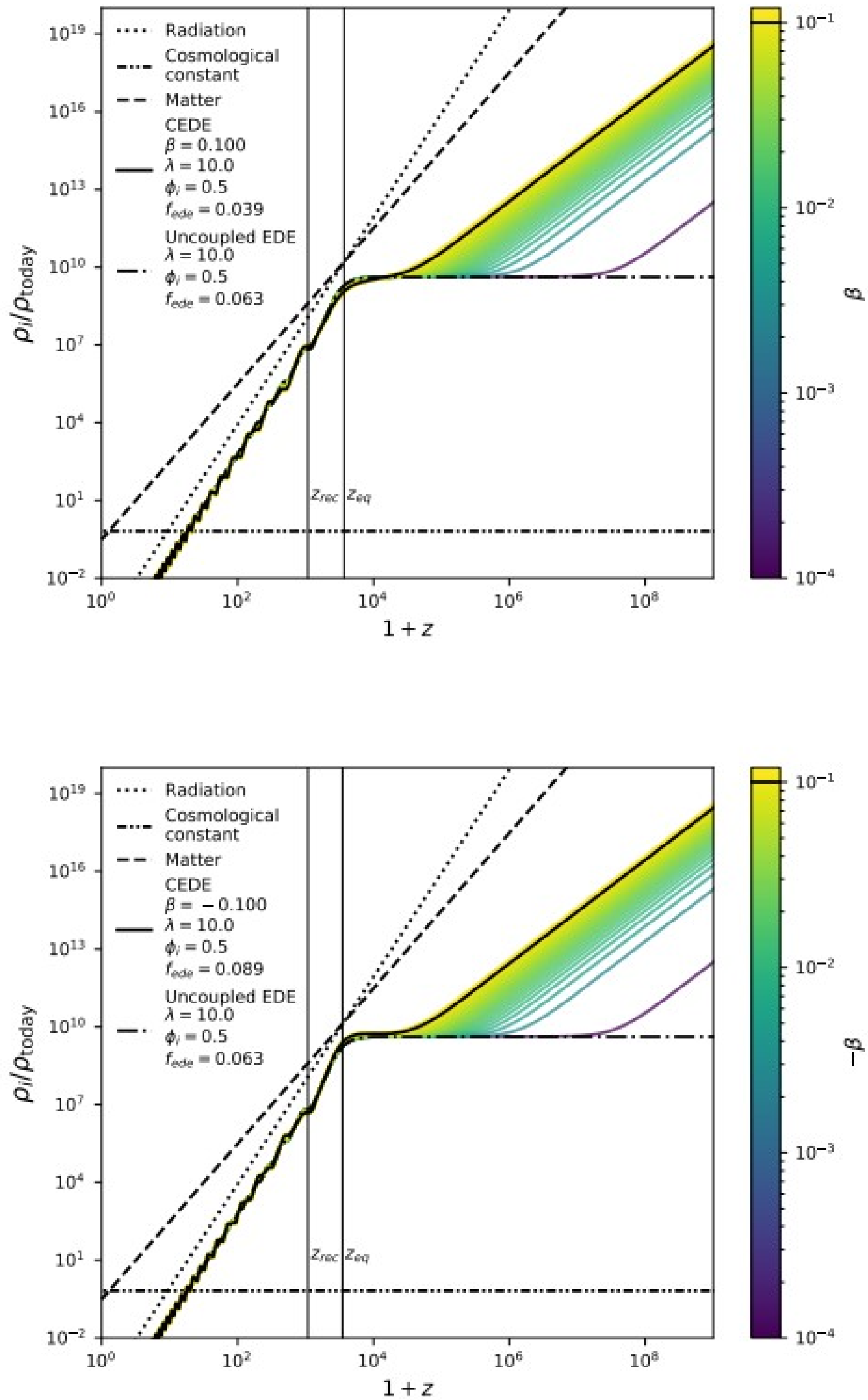

Figure 3.2: Evolution of energy density of different components over time (Credit : Chameleon Early Dark Energy and the Hubble Tension by Tanvi Karwal and et.al)

energy density of scalar field evolve as constant equation of state then we obtain :

$$\frac{\rho_\phi}{\rho_{\phi 0}} = \left(\frac{a}{a_0}\right)^{-3(1+w_\phi)} \tag{3.28}$$

where w_ϕ is given by:

$$w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\frac{\dot{\phi}^2}{2a^2} - V(\phi)}{\frac{\dot{\phi}^2}{2a^2} + V(\phi)} \tag{3.29}$$

So scalar field density roll down over time. Potential which fulfils this requirement are one which are proportional to ϕ^{2n} as shown in [18].For this type of potential we are going to get for constant equation of state average over oscillations (result taken from [18]),

$$w_{osc} = \frac{n-1}{n+1} \tag{3.30}$$

In our case $n = 2$ so w_{osc} will be $\frac{1}{3}$ giving evolution of ρ_ϕ as a^{-4}. So, we get a scalar field which is now rolling with oscillations.

background scalar field evolve as From eq(3.11)

$$\ddot{\phi} + 2\frac{\dot{a}}{a}\dot{\phi} = -a^2 4\lambda\phi^3 - a^2\frac{\beta}{M_P}e^{\frac{\beta\phi}{M_P}}\rho_{dm} \tag{3.31}$$

Calculating minimum value of ϕ possible (taking $\ddot{\phi} = 0$ and $\dot{\phi} = 0$).

$$4\lambda\phi_{min}^3 = -\frac{\beta}{M_P}e^{\frac{\beta\phi_{min}}{M_P}}\rho_{dm} \tag{3.32}$$

ϕ_{min} turn out to be non zero in case of coupling

Initial condition for perturbation in dark matter are taken as adiabatic

$$\begin{aligned} \tilde{\delta}_{dm} &= \frac{3}{4}\delta_g \\ \tilde{\theta}_{dm} &= 0 \end{aligned} \tag{3.33}$$

where δ_g is perturbation in energy density of photons.

Initial condition for scalar field perturbation are taken as:

$$\begin{aligned} \delta\phi &= 0 \\ \dot{\delta\phi} &= 0 \end{aligned} \tag{3.34}$$

Impact of Coupling Constant β

There could be two specific scenarios:

- $\beta < 0$
- $\beta > 0$

If $\phi > 0$ then in case of $\beta < 0$ according to eq(3.31) $|\dot{\phi}_i|$ is going to be small than in case of $\beta > 0$. Value of $|\dot{\phi}_i|$ is higher than case of uncoupled scalar field where there is no coupling term which is dominating initially.

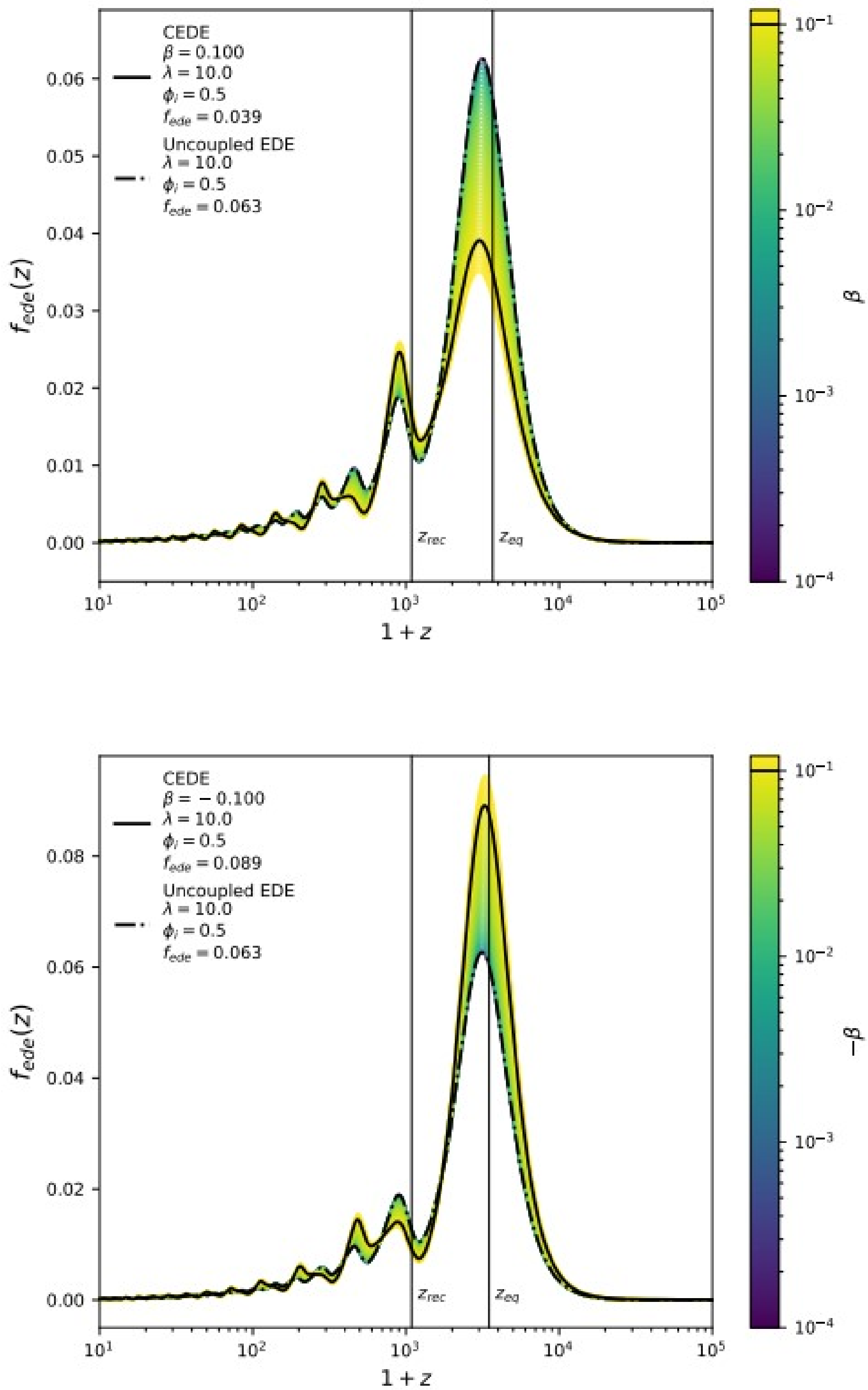

Figure 3.3: fractional dark energy density (Credit : Chameleon Early Dark Energy and the Hubble Tension by Tanvi Karwal and et.al)

Since $\dot{\phi}_i$ is positive in case of $\beta < 0$. That implies that ϕ is going to increase so native potential $(\alpha\phi^4)$ is going to increase which make it higher than uncoupled case at time of matter radiation equality while in case of $\beta > 0$, $\dot{\phi}_i$ is negative so ϕ is going to decrease over time making $(\alpha\phi^4)$ lower than uncoupled case at time of matter radiation equality

At time of matter radiation equality, energy density is dominated by native potential so we get, $f_{\beta<0} > f_{\beta=0} > f_{\beta>0}$. Here f implies fraction of dark energy density.This is visible in 3.3

As already discussed coupling term give rise to asymmetry which means its sign is going to determine which peaks are lower or higher compare to uncoupled case. In case $\beta > 0$,we get potential higher on right side of ϕ_{min} than left side so when ϕ goes to left(right) side it needs to go for higher(lower) values of ϕ so more(less) energy density at even(odd) peaks than uncoupled case.

Since Hubble frozen period comes when native potential becomes dominate factor so higher value of β leads to lower frozen period .

In this way we are able to produce required energy density in small redshift window

Conclusion

Hubble tension associated with ΛCDM model is being studied. The Chameleon Early Dark Energy Model introduced in [10] is being studied to see how it resolves Hubble tension and sigma 8 tension associated with ΛCDM model. This model introduces a scalar field which plays a significant role in energy density budget at the time of matter radiation equality. At the point of time, dark matter becomes a dominant factor, so it could have triggered energy density release. This model introduces modulation of dark matter's energy density, which can help resolve Sigma 8 tension. Dynamics of background and perturbations are also being analysed.

There are other ways present in literature to solve this problem . Answer to which one is correct can come only through observations. We need to keep in mind that may be it turns out to be systematic only. Even in that case we still need to explain what exactly is Λ.

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