Optimal Online Nonlinear/Non-Gaussian State Estimation

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***Abstract*— This simulation aims to test the optimal estimation of non-Gaussian / non-linear state observation models by different nonlinear methods. This report is divided into five parts, and each part will be discussed separately.**

I. Introduction

For the given discrete-time system

$$x_k = \frac{x_{k-1}}{2} + \frac{25x_{k-1}}{1+x_{k-1}^2} + 8cos(1.2k) + v_{k-1} \tag{1}$$

$$z_k = \frac{x_k^2}{20} + n_k \tag{2}$$

where v and n are independent zero mean Gaussian white noises with variances Q = 10 and R = 1, respectively. This example has been used in many articles. [1]

This simulation will use Monte Carlo simulation to estimate state values and observations at different times. Use the extended Kalman filter(EKF), grid based method(GBM), particle filter(PF), iterated extended Kalman Filter(IEKF) and regularized particle filter(RPF) to estimate the unknown state information x_k.

II. Task 1

Assuming that the initial state follows a standard normal Gaussian distribution, use the Monte Carlo method to estimate the probability density function $p(x_1)$, $p(x_{50})$and $p(x_{10})$. The result is shown in Fig 1.

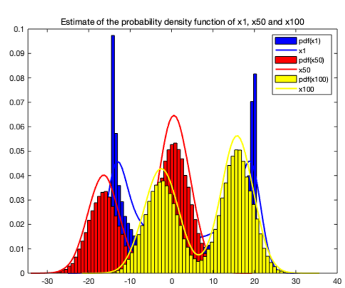

Fig. 1. Estimate of the probability density function of x1, x50 and x100

For reference, the true state and observations are shown in Fig 2. Traditonal Kalman filter is a high efficiency recursive filter, it uses Gaussian distribution to describe state quantities based on Bayesian filtering to obtain an optimal estimate of x_k to minimise mean square error.

Therefore, in the Gaussian environment, Kalman filtering is the best tracking method for linear systems.

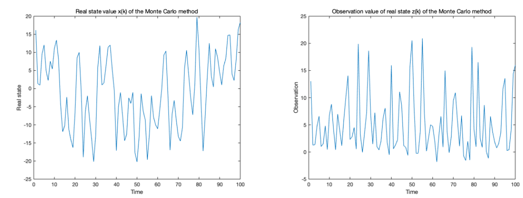

Fig. 2. True State and Observation

But the given system is a nonlinear system, The transmission result of the Gaussian distribution in a nonlinear system will no longer be a Gaussian distribution, so the assumption of Kalman filtering is not satisfied in this system. Therefore, it would be problematic to use the traditional Kalman filter to estimate the unknown state given the measurement data z_k.

III. Task 2

In the given nonlinear / non-Gaussian system, it is necessary to use approximate values. In this section, we consider three approximately nonlinear Bayesian filters:

1) Extended Kalman filter (EKF)
2) Approximate grid-based method;
3) Particle filter

A. *Extended Kalman Filter*

In this simulation, the given state observation model is not a linear system, so it uses tangents to linearize at the operating point. In fact, it is a first-order Taylor expansion at the mean.

In the EKF, the Jacobian matrix using the current estimated state at each step.

Use the EKF, to estimate the unknown state information x_k, $0 \leq k \leq 100$. The result is shown in Fig 3.

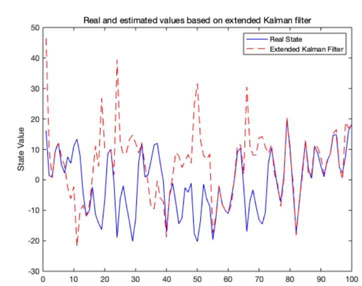

Fig. 3. True and estimated values based on EKF

B. Grid Based Method

For discrete state space variables,in the low-dimensional case, the GBM approximation can model the multimodality of the problem. [1]

Use GBM, to estimate the unknown state information x_k, $0 \leq k \leq 100$. The result is shown in Fig 4.

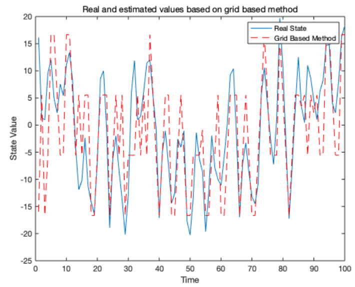

Fig. 4. True and estimated values based on grid based method

C. Particle Filter

Particle filtering refers to the process of finding a set of samples with random weights in the posterior probability by Monte Carlo method to approximate the probability density function, replacing the integral operation with the sample mean, and then obtaining the minimum variance estimate of the system state.

It can be applied to non-linear/non-Gaussian state space models.

Randomly sampled samples (called particles) are extracted from the importance density, and the weights are generated according to the rules of importance sampling.

PF is divided into three stages:
Stage 1: Prediction
At time k, generate N a priori particles

$$x_{k,i}^{-} = f_{k-1}(x_{k-1,i}^{+}, w_{k-1}^{i}) \tag{3}$$

Stage 2: Update
Update the weights based on the new measurement z_k

$$q_i = p(z_k | x_{k,i}^{-}) \tag{4}$$

and then normalise the weights.
Stage 3: Resampling
Based on the posterior particles to caculate the desired statistical measure. And then resampling: generate new a set of particles$\left\{ x_{k,i}^{+}, \frac{1}{N} \right\}$

Use PF, to estimate the unknown state information x_k, $0 \leq k \leq 100$. The result is shown in Fig 5.

IV. Task 3

The root mean square error (RMSE) is widely used in quantitative analysis and is used in various literatures. So this experiment uses the mean square error to measure the difference between the estimated value and the true value. The smaller the mean square error, the smaller the deviation of the estimated value.

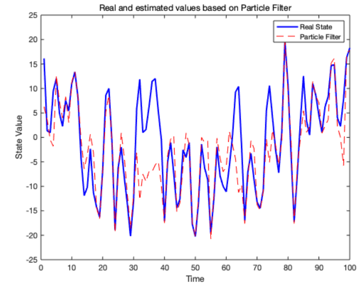

Fig. 5. True and estimated values based on PF

TABLE I
RMSE Values of Extended Kalman Filter, Grid Based Method and Particle Filter

Algorithm	Figure3	RMSE
Extended Kalman Filter	3	15.86
Grid Based Method	4	8.07
Particle Filter	5	6.34

In order to compare and evaluate the three different filtering techniques in TASK2, they were each subjected to 100 Monte Carlo simulations. Their average root mean square error is shown in Table 1.

It can be seen that the RMSE of the EKF is large, which means that its average value rarely approaches the true state value. It shows that the local linearization and approximation by the first-order Taylor expansion at the mean value cannot fully describe the non-Gaussian state process. Although the EKF can sometimes accurately capture the true state, for some double peaks that cannot be accurately approximated, the filter will have a large error in the estimation of the deviation. Therefore, it is the most inaccurate algorithm for approximating the posterior in this equation of state.

Compared with the EKF, the GBM approximates the continuous state space to a set of discrete quantities, which can significantly reduce the root mean square error. It can be seen from Fig 4 that the GBM at low latitudes can basically capture the true state of the non-linear observation model, but for some peaks and troughs with too large differences, the GBM still has the current state The bias estimate is not confident enough.

Different from the above two methods, the PF uses the prior distribution as a function of importance density and samples from it. And in each recursion process, the weight of the next sample is calculated from the weight of the last sample. In this simulation, the number of particles we selected is N = 30. It can be seen that although the RMSE of the PF is smaller than the GBM and EKF, some of its waveforms are still the same as the lower-order particle waveforms. This may be due to the problem of particle weight degradation caused by the number of samples.

In general, the above three methods all use approximate methods to solve the problems of nonlinear systems. Although the PF is flawed in this simulation, its RMSE is the smallest, indicating that among the three methods, the PF is the optiaml method to estimate the true value of a given state equation.

V. TASK 4

In order to improve the three methods proposed in TASK2, two other improved methods were used in this task: Iterated Extended Kalman Filter (IEKF) and Regularized Particle Fiter (RPF).

A. Iterated Extended Kalman Filter

For the nonlinear observation model, EKF and IEKF select one of the points $x_{op,k}$ for linearization. The only difference between the two is the choice of the working point for linearization. The working point of each iteration of IEKF is set to the posterior mean of the previous iteration.

Only in the first iteration, let

$$x_{op,k} \leftarrow \breve{x}_k \tag{5}$$

Use iterated extended kalman filter to estimate the unknown state information x_k, $0 \leq k \leq 100$. The result is shown in Fig 6.

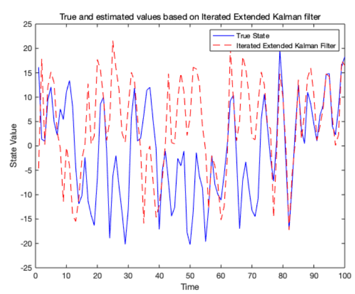

Fig. 6. True and estimated values based on IEKF

Its RMSE is 12.71. It can be seen that the RMSE of the iterated EKF is slightly reduced compared to the EKF, indicating that it has some improvement in the approximation of the true state. As can be seen from the figure, the average value of the EKF is still rarely close to the true value, but it has a significant improvement in the approximation of certain double peaks. It shows that although this method is not accurate in approximating the posterior probability of the state observation model, it still has some merits.

B. Regularized Particle Filter

The common problem in the basic particle filtering algorithm is the degradation phenomenon, because the variance of the particle weights will continue to increase with time iterations. After iteration, only a few particles have a large weight, and a lot of computing power is wasted on particles with almost no weight, which reduces the estimated performance. A resampling mechanism and a reasonable importance density can be introduced to slow down particle degradation.

In the usual particle filtering technique, in the resampling stage, the sample is sampled from a discretely distributed importance density function, so it will cause the particles to collapse.

To solve this problem, Christian Musso et al. in 2001 proposed a new filtering algorithm: regularized particle filter [2]. It converts the discrete distribution importance density function into a continuous distribution importance density function, and samples the particle set from it, so that the problem of particle collapse is improved.

Except for the resampling stage, the regularized particle filter is the same as the sequence importance resampling filter.

Use RPF to estimate the unknown state information x_k, $0 \leq k \leq 100$. The result is shown in Fig 7.

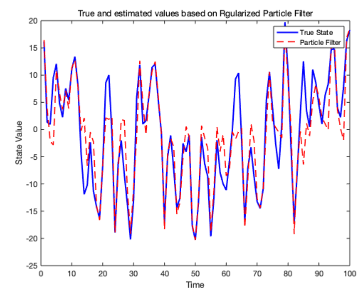

Fig. 7. True and estimated values based on the RPF

Its RMSE is 4.56. In this simulation, we still choose the number of particles N = 30. It can be seen that compared with the previous image of the PF, although the RPF obtains a similar RMSE, it better captures the reality State value, and has significantly improved the problem of particle degradation.

VI. TASK 5

In this simulation we used 5 different methods: EKF, GBM, PF, IEKF, RPF to estimate the state of a given nonlinear / non-Gaussian state observation model. And the root mean square error RMSE is used to measure the accuracy of various methods to estimate the state observation model.

The conclusion of this simulation is that for this particular application, particle filters outperform other applications. But for particle filters, the choice of importance density is the key.

The limitation of this simulation is that it does not explore the impact of diversified importance density functions on the simulation results of particle filter. In future research, more importance density functions will be tested to see if better results can be obtained.

In actual operation, for the particle filter, the choice of importance density will become the key problem. So it is worth discussing that how to choose the right importance density.

REFERENCES

[1] M. S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp, "A tutorial on particle filters for online nonlinear/non-gaussian bayesian tracking," *IEEE Transactions on Signal Processing*, vol. 50, no. 2, pp. 174–188, 2002.

[2] C. Musso, N. Oudjane, and F. Le Gland, *Improving Regularised Particle Filters*. New York, NY: Springer New York, 2001, pp. 247–271.