GAME THEORY

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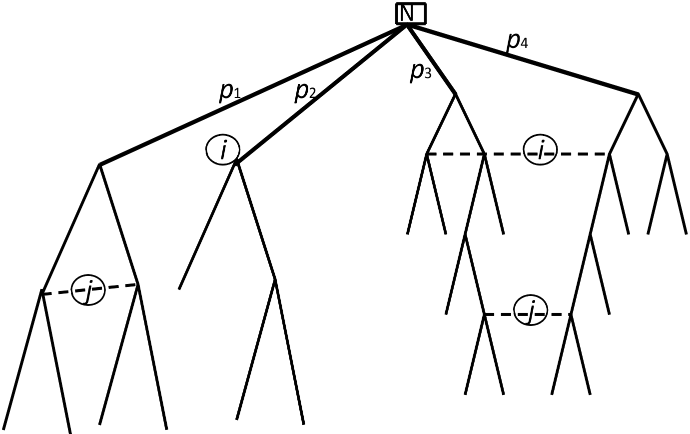
Dynamic Games of Incomplete Information

- We are now studying the last type of games: dynamic games of incomplete information, as well as its solution: Perfect Bayesian (Nash) equilibrium

	Complete Information	**Incomplete Information**
Static	Normal-form games (NE)	Bayesian games (Bayesian NE)
Dynamic	PI games (Subgame perfect NE)	II games (Perfect Bayesian NE)

- By incomplete information, we mean that the payoff functions are not common knowledge among all the players. We consider dynamic games in which (1) the player with the move may not know the history of the play of the game, and (2) some player may not know the other players' payoffs

Dynamic Games of Incomplete Information: Model



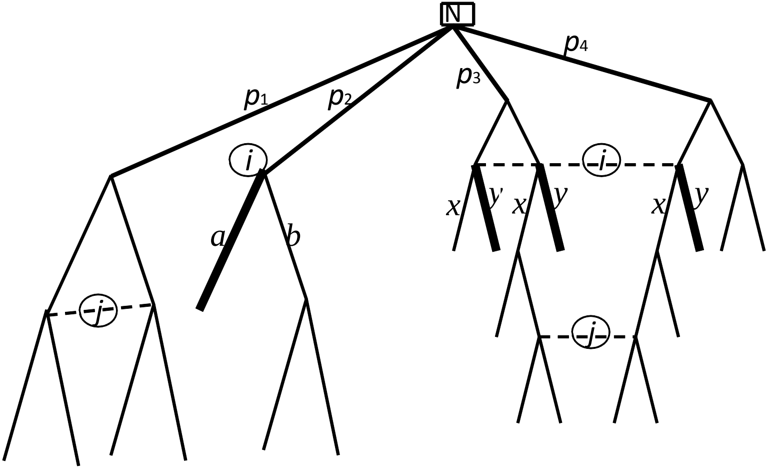
e.g. prisoners' dilemma, Cournot competition

e.g. entry deterrence, sequential bargaining

e.g. Cournot competition under uncertainty, auction, repeated games, signaling …

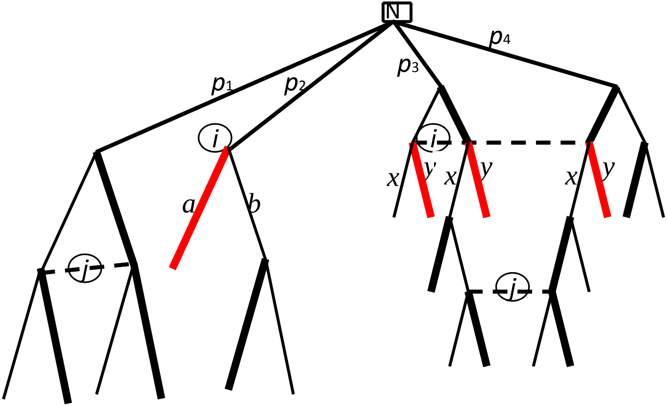
Extensive Game with Imperfect Information (II Game)

II Games: Notion of Strategy



Strategy: complete plan of actions

Solution I: Nash Equilibrium

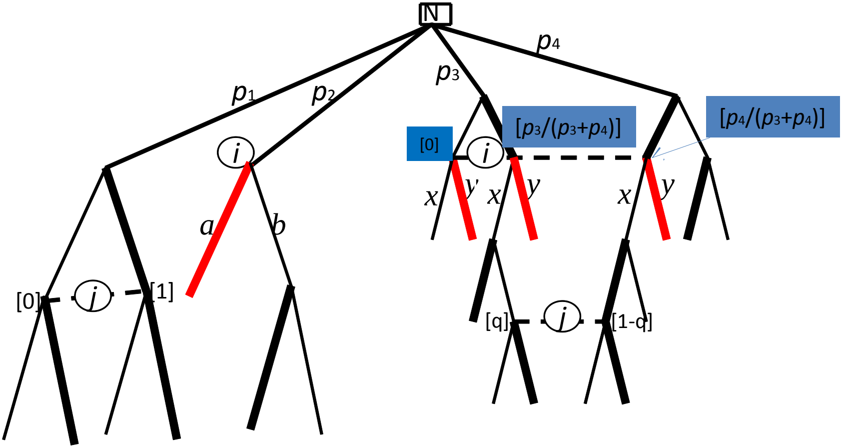


Nash equilibrium: self-enforcing profile of strategies

($i.e.$ $\forall i$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ $\forall s_i \in S_i$)

$\mathcal{Q}$: Why not (Bayesian) Nash equilibrium? Why not SPNE?

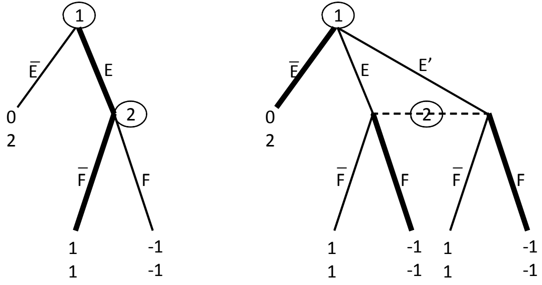
Solution II: PBE – Idea and Method



PBE: sequential rationality + consistent beliefs

Perfect Bayesian Equilibrium: Motivation

- In Game A, NE: ($\bar{E}$, F) & (E, $\bar{F}$); SPNE: (E, $\bar{F}$)
  (SPNE eliminates the noncredible threat strategy "fight" for player 2!)

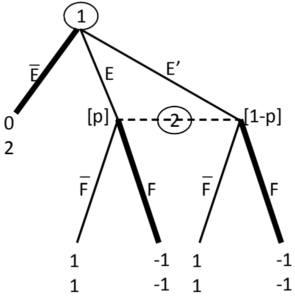


Game A Game B

- In Game B, NE = SPNE: ($\bar{E}$, F), (E, $\bar{F}$) & (E', $\bar{F}$)
  (SPNE does not eliminate the noncredible threat strategy "fight"!?)
- The notion of "perfect Bayesian equilibrium" (PBE) is to rule out this sort of noncredible strategies in Game B

Perfect Bayesian Equilibrium: Motivation

- The main idea of perfect Bayesian equilibrium is to impose the sequential rationality at every information set (both on the equilibrium path and off the equilibrium path)



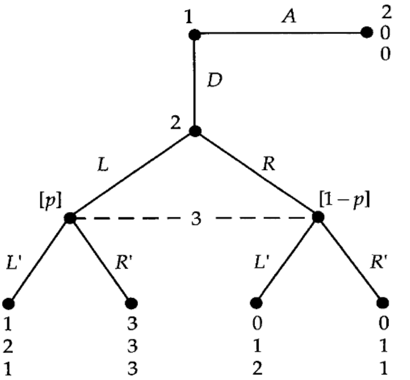
F is not sequentially rational at player 2's information set!

- An information set is *on the equilibrium path* if it can be reached with positive probability if the game is played according to the equilibrium strategies. Otherwise, the information set is *off the equilibrium path*

Perfect Bayesian Equilibrium: Definition

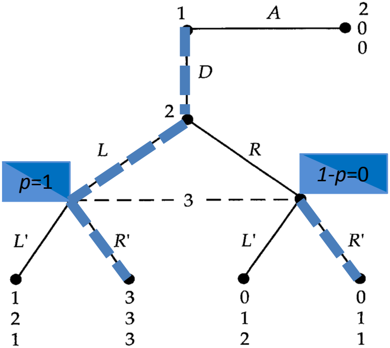
- A *perfect Bayesian equilibrium* (PBE) is a strategy profile $s = (s_1, \cdots, s_n)$ and a belief assessment profile $\beta = (\beta_1, \cdots, \beta_n)$, where the belief assessment β_i specifies player i's belief at each of his information sets, such that for every player i,
  1. **Requirement 1: [Sequential Rationality]**
     At each of his information sets, s_i is a best response to s_{-i}, given his belief β_i at that information set
  2. **Requirement 2: [Belief Consistency]**
     At information sets on the equilibrium path, his belief β_i is derived from Bayes' rule using the strategy profile s
  3. **Requirement 3: [Belief Consistency$^+$]**
     At information sets off the equilibrium path, his belief β_i is derived from Bayes' rule using the strategy profile s where possible
- *Weak perfect Bayesian equilibrium* (WPBE): Requirements 1 & 2
- **Remark.** Sequential rationality implies that no player uses strictly dominated actions at an information set!

Perfect Bayesian Equilibrium: Example

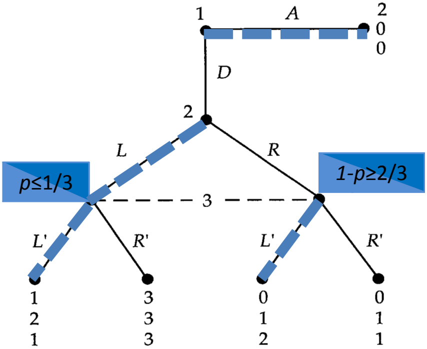


- (B)NE: $(D, L, R'), (A, L, L'), (A, R, L'), (A, R, R')$
- PBE: $[(D, L, R'),\ p = 1]$ (SPNE: (D, L, R'))
  (WPBE: $[(D, L, R');\ p = 1]$ & $[(A, L, L');\ p \leq \frac{1}{3}]$)

Perfect Bayesian Equilibrium: Example



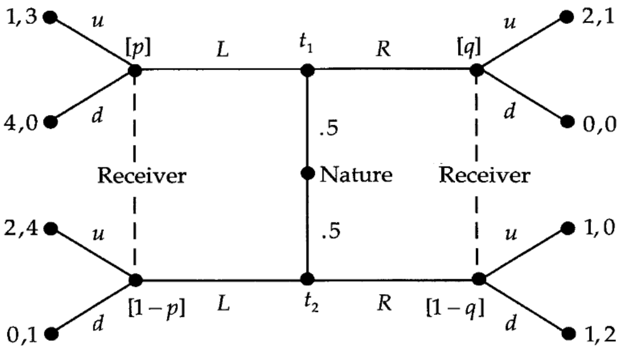
PBE: [(D,L,R'), p=1]



WPBE: [(A,L,L'), $p \leq 1/3$]

(by WPBE, $p+2(1-p) \geq 3p+(1-p)$)

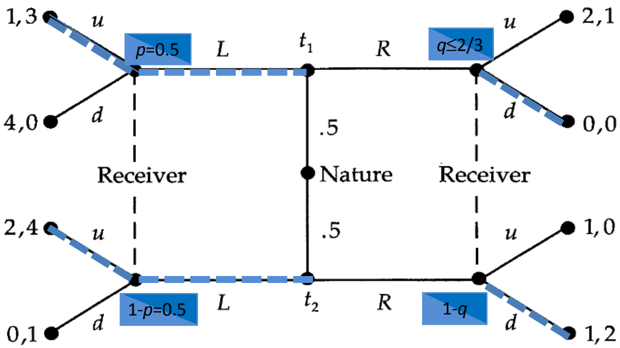
Applications: es



- **Note:** u strictly dominates d at the left information set.
- Therefore, in a PBE receiver must play u at the left information set and, thus, t_2 must play L.

Signaling Games: Pooling PBE

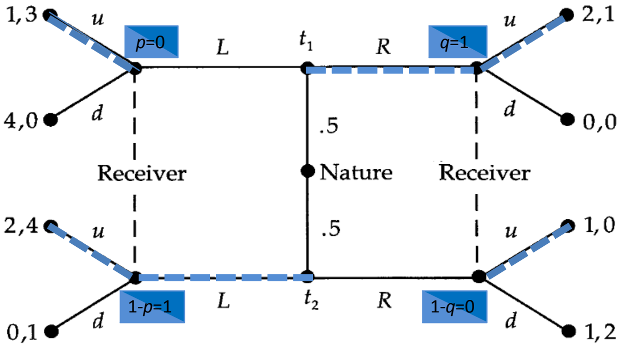
- Pooling on L: $\left[\left(\left(L, L\right), \left(u, d\right)\right); p = 0.5, q \leq \frac{2}{3}\right]$
  (note: Pooling on R is impossible)



(To detemine q, we note: $q \times 0 + (1-q) \times 2 \geq q \times 1 + (1-q) \times 0$.)

Signaling Games: Separating PBE

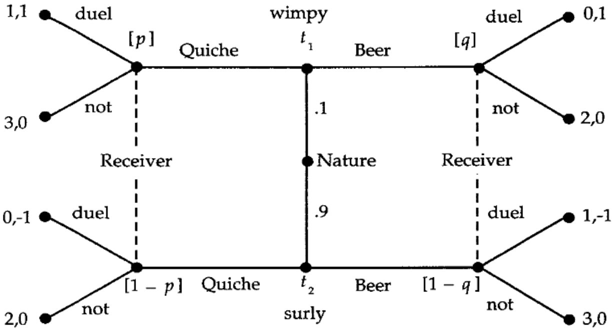
- Separation with t_2 playing L: $[((R,L),(u,u));p=0,q=1]$
  (note: Separation with t_2 playing R is impossible)



Application: Beer and Quiche

- **Cho and Kreps's (1987) Model:**
- The Sender has two types: $t_1 =$"wimpy" (with probability 0.1) and $t_2 =$"surly" (with probability 0.9)
- The Sender chooses whether to have beer or quiche for breakfast, and Receiver chooses whether or not to duel with the Sender
- The wimpy type would prefer to have quiche, the surly type would prefer to have beer, and both types would prefer not to duel with the Receiver
- The Receiver would prefer to duel with the wimpy type, but not duel with the surly type

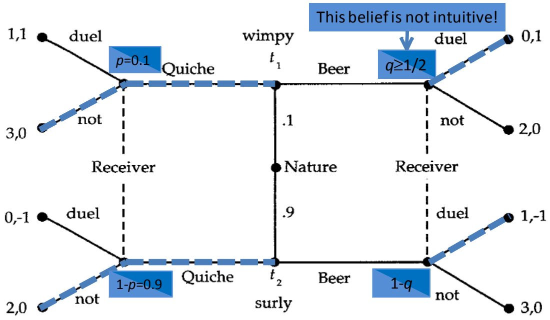
Application: Beer and Quiche



- No separating PBE! ? (**Reason:** In the separating situation, the "wimpy" type has an incentive to mimic the "surly" type and, thus, can avoid "duel".)
- Pooling PBE: $\left[(\text{Quiche, Quiche}), (\text{not, duel}); p = 0.1, q \geq \frac{1}{2}\right]$
  $\left[(\text{Beer, Beer}), (\text{duel, not}); p \geq \frac{1}{2}, q = 0.1\right]$

Beer and Quiche: The Intuitive Criterion*

- In PBE: $\left[(\text{Quiche, Quiche}), (\text{not, duel}); p = 0.1, q \geq \frac{1}{2}\right]$, the belief $q \geq \frac{1}{2}$ is not "intuitive" because the wimpy type has received the highest payoff of 3 and it is unlikely for the type to deviate from the equilibrium



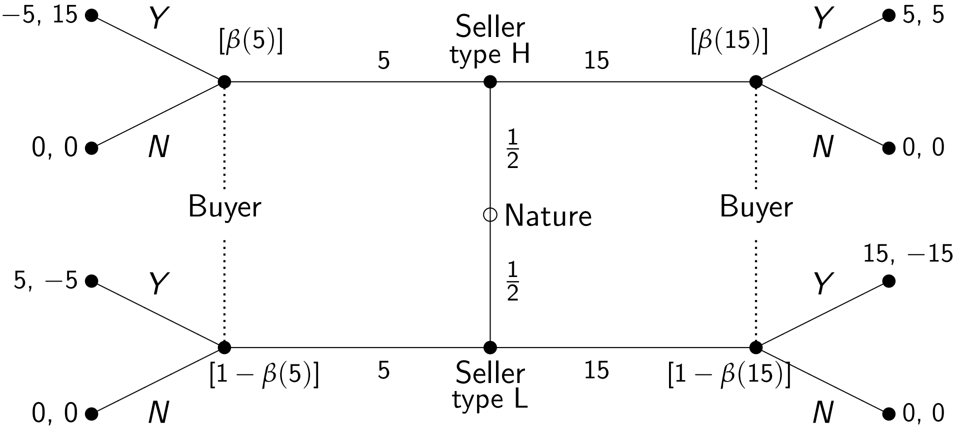
- **The Intuitive Criterion:** putting zero probability on the equilibrium-dominated thing (the second pooling PBE does satisfy this criterion)

Application: The Market for "Lemons"

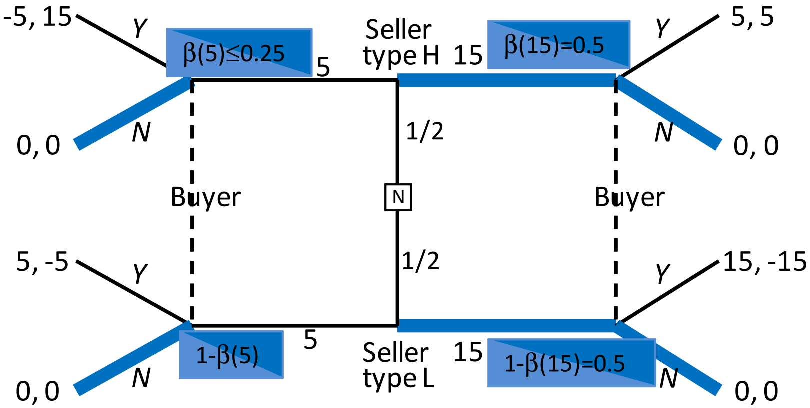
- **Akerlof's (1970) Model:**
A seller wants to sell his used car. The seller knows what is the quality of the car, but the buyer does not. The buyer knows only that the car could be a "good quality" car with probability $\frac{1}{2}$ and a "lemon" with probability $\frac{1}{2}$. If the car is good, the buyer's valuation for it is \$20K and the seller's is \$10K. If it is a "lemon," both buyer's and seller's valuations are \$0
The seller can make two offers (asking price): \$5K and \$15K. Then, the buyer can accept the offer (buy the car) or reject the offer

- Akerlof (1970) showed that: the market collapses ("lemon effect")

Application: The Market for "Lemons"



Application: The Market for "Lemons"



- There is no separating PBE!
- **Pooling PBE:** $\left[\left(\left(15,15\right),\left(N,N\right)\right);\beta\left(5\right)\leq\frac{1}{4},\beta\left(15\right)=\frac{1}{2}\right]$ (No-trade! In other words, "the bad driving out the good.")

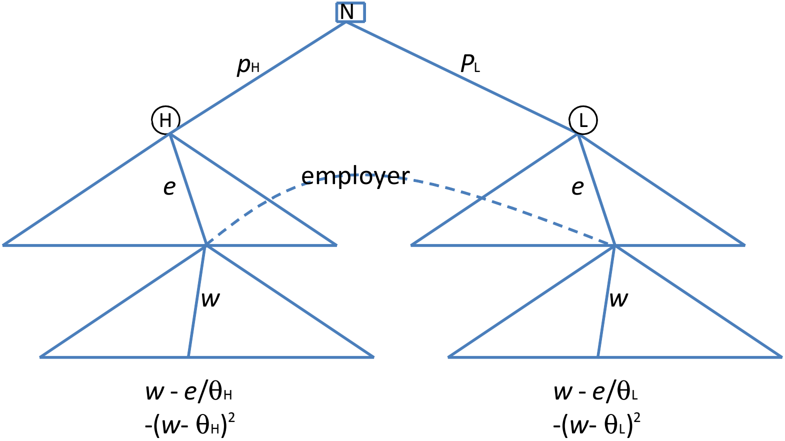
Application: The Market for "Lemons"

1. **Separation with type H playing 5**
   In this case, $\beta(5) = 1$ and $a^*(5) = Y$. Thus, it is not optimal for type H to play 5.
2. **Separation with type L playing 5**
   In this case, it is easy to see $\beta(5) = 0$, $\beta(15) = 1$, $a^*(5) = N$ and $a^*(15) = Y$. Thus, type L should deviate from the original strategy 5.
3. **Pooling on 5**
   In this case, $\beta(5) = \frac{1}{2}$ and $a^*(5) = Y$. Thus, it is not optimal for type H to play 5.
4. **Pooling on 15**
   In this case, $\beta(15) = \frac{1}{2}$ and $a^*(15) = N$. If $\beta(5) > \frac{1}{4}$, then $a^*(5) = Y$. Thus, it is not optimal for type L to play 15. If $\beta(5) \leq \frac{1}{4}$, then $\left[((15, 15), (N, N)), \beta(5) \leq \frac{1}{4}, \beta(15) = \frac{1}{2}\right]$ is a PBE.

Application: Job-Market Signaling

- **Spence's Model of Education:**
  A worker (the sender) knows his productive ability θ, while his employer (the receiver) does not. The timing is as follows:
  1. Nature determines a worker's productive ability, θ, which can be either high (θ_H) with probability p_H or low ($\theta_L < \theta_H$) with probability p_L
  2. The worker learns his ability and then chooses a level of education $e \in [0, +\infty)$
  3. A employer observes the worker's education (but not the worker's ability) and then pays the worker a wage $w \in [0, +\infty)$
  4. The payoffs are $w - \frac{e}{\theta}$ to the worker and $-(\theta - w)^2$ to the employer (under the assumption of perfect competition on the demand side)

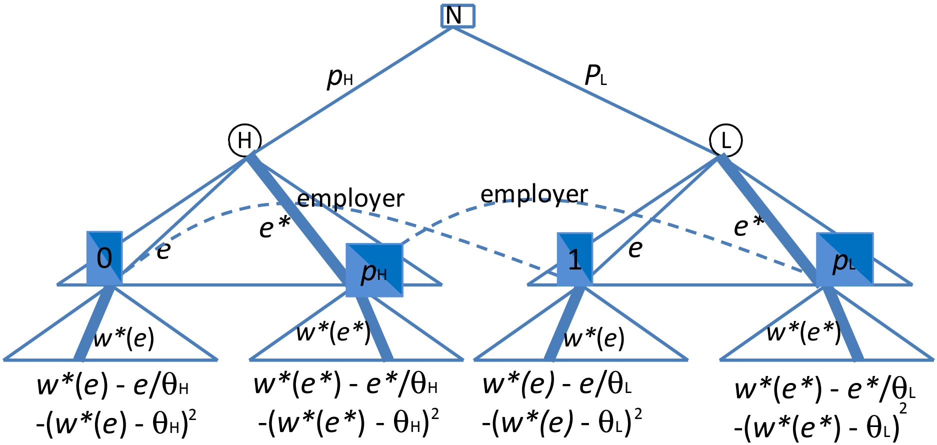
Application: Job-Market Signaling



Application: Job-Market Signaling

- A strategy for the worker is $(e_H,\ e_L)$ which specifies actions for types H and L, where $e_H,\ e_L \in [0, +\infty)$
- A strategy for the employer is a wage schedule $w(\cdot)$ which depends on the observed signal (i.e. the level of education)
- **PBE:** $[(e_H^*,\ e_L^*), w^*(\cdot); \beta^*]$, where $\beta^*(\cdot|e)$ specifies the belief about the worker's types when the observed signal is e

Job-Market Signaling: Pooling Equilibrium



- Both types choose the same level of education: $e_H^* = e_L^* = e^*$
- By the definition of PBE, $w^*(e^*) = p_H\theta_H + p_L\theta_L$. To be a PBE pooling on e^*, the easiest way is to pessimistically believe that any deviation $e \neq e^*$ is from type L.[1] Thus the wage schedule should be:
$$w^*(e) = \begin{cases} p_H\theta_H + p_L\theta_L, & \text{if } e = e^* \\ \theta_L, & \text{if } e \neq e^* \end{cases}$$
- To be a PBE pooling on e^*, each type of workers does not want to deviate from e^*. Thus, $\begin{cases} w^*(e^*) - \frac{e^*}{\theta_H} \geq \theta_L - \frac{e}{\theta_H}, \forall e \neq e^* \\ w^*(e^*) - \frac{e^*}{\theta_L} \geq \theta_L - \frac{e}{\theta_L}, \forall e \neq e^* \end{cases} \iff$
$w^*(e^*) - \frac{e^*}{\theta_L} \geq \theta_L \iff e^* \leq p_H(\theta_H - \theta_L)\theta_L$
- **Pooling PBE:** $[e_H^* = e_L^* = e^*, w^*(\cdot); \beta^*]$ where
$\beta^*(H|e) = \begin{cases} p_H, & \text{if } e = e^* \\ 0, & \text{if } e \neq e^* \end{cases}$ and $0 \leq e^* \leq p_H(\theta_H - \theta_L)\theta_L$

[1] It is possible for the employer to use different beliefs, e.g., $p_H \circ \theta_H + p_L \circ \theta_L$ if $e \geq e^*$ and $1 \circ \theta_L$ if $e < e^*$.

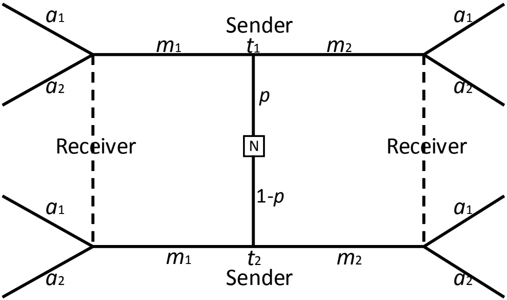
Job-Market Signaling: Separating Equilibrium

- The two types of workers choose different levels of education: $e_H^* > e_L^* = 0$ (the wage paid to L is θ_L, independent of e_L)
- We consider the most pessimistic wage schedule:
$w^*(e) = \begin{cases} \theta_H, & \text{if } e = e_H^* \\ \theta_L, & \text{if } e \neq e_H^* \end{cases}$
- To be a separating PBE, each type should have no incentive to mimic the other. Therefore,

$$\begin{cases} \theta_H - \frac{e_H^*}{\theta_H} \geq \theta_L \\ \theta_L \geq \theta_H - \frac{e_H^*}{\theta_L} \end{cases} \iff (\theta_H - \theta_L)\,\theta_L \leq e_H^* \leq (\theta_H - \theta_L)\,\theta_H$$

- **Separating PBE:** $[e_H^* > e_L^* = 0, w^*(\cdot); \beta^*]$ where
$\beta^*(H|e) = \begin{cases} 1, & \text{if } e = e_H^* \\ 0, & \text{if } e \neq e_H^* \end{cases}$ and $e_H^* \in [(\theta_H - \theta_L)\,\theta_L, (\theta_H - \theta_L)\,\theta_H]$

Signaling Games: A Summary

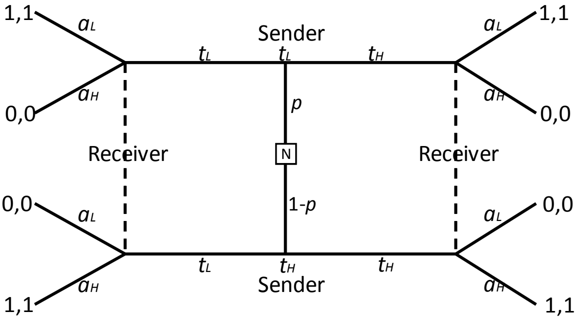


- Nature draw a type t_i for Sender
- Sender observes t_i and then sends a message m_j
- Receiver observes m_j and then chooses an action a_k
- Payoffs are given by $U_S(t_i, m_j, a_k)$ and $U_R(t_i, m_j, a_k)$

Signaling Games: A Summary

- **PBE:** Let $\mu(\cdot|m_j)$ be a probability distribution on types after observing m_j
  1. **Requirement 1: [Sequential rationality]**
     For each message m_j, Receiver chooses $a^*(m_j)$ solving
     $$\max_{a_k} \sum_{t_i} \mu(t_i|m_j)\, U_R(t_i, m_j, a_k)$$
     For each type t_i, Sender chooses $m^*(t_i)$ solving
     $$\max_{m_j} U_S(t_i, m_j, a^*(m_j))$$
  2. **Requirement 2: [Belief Consistency]**
     For each m_j, if there is t_i' such that $m^*(t_i') = m_j$, then $\mu(t_i|m_j)$ is derived from Bayes' rule using Sender's strategy $m^*(t_i)$
- **Applications:** Beer-and-Quiche, Lemons Market, Job-Market Signaling

Cheap-Talk Games: A special kind of signaling games



Major Feature: The messge m_j has no direct effect on payoffs U_S and U_R

- There is always a pooling (or babbling) equilibrium where Receiver ignores all the messages! If the players' interest are sufficiently aligned, then we may find a separating (partially pooling) equilibrium, e.g. $\left[(t_L, t_H), (a_L, a_H); \mu(t_L|t_L) = 1, \mu(t_L|t_H) = 0\right]$

Cheap-Talk Games: Crawford and Sobel's (1982) Model

- Sender's types is a uniform distribution on the interval $[0, 1]$
- Receiver's action space is the interval $[0, 1]$
- Receiver's payoff function: $U_R(t, a) = -(a - t)^2$
  Sender's payoff function: $U_S(t, a) = -[a - (t + b)]^2$
  (parameter $b > 0$ measures the similarity of the players' preferences)

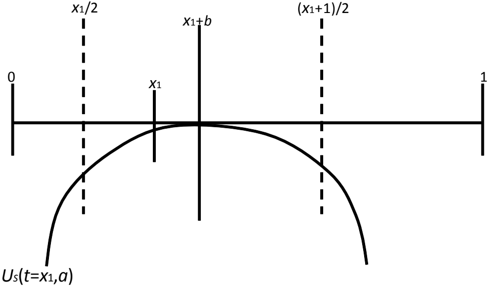
- **Crawford and Sobel's (1982) result:** There exists an integer $n^*(b)$ such that for each $n = 1, 2, ..., n^*(b)$, there is a partially pooling equilibrium in the following form: the type space is divided into the n intervals $[0, x_1), [x_1, x_2), ..., [x_{n-1}, 1]$; all the types in a given interval send the same message, but types in different intervals send different messages. (Note: The babbling equilibrium ($n = 1$) always exists.)

Cheap-Talk Games: Crawford and Sobel's (1982) Model

- For a two-interval partially pooling equilibrium ($n = 2$), the condition is:

$$x_1 + b = \frac{1}{2}\left(\frac{x_1}{2} + \frac{x_1 + 1}{2}\right).$$

That is, $x_1 = \frac{1}{2} - 2b$.



- A two-interval partially pooling equilibrium exists only if $b < \frac{1}{4}$!

Cheap-Talk Games: Crawford and Sobel's (1982) Model

- For an n-interval partially pooling equilibrium, the condition is: for $k = 1, 2, ..., n-1$

$$x_k + b = \frac{1}{2}\left(\frac{x_{k-1} + x_k}{2} + \frac{x_k + x_{k+1}}{2}\right).$$

  That is, $\left(x_{k+1} - x_k\right) = \left(x_k - x_{k-1}\right) + 4b$.

- Let $\left(x_1 - 0\right) = d$. Then, we must have

$$d + \left(d + 4b\right) + ... + \left[d + \left(n-1\right)4b\right] = 1.$$

  That is, $nd + n\left(n-1\right)2b = 1$.

- n^* is the largest integer less than

$$\frac{1}{2}\left[1 + \sqrt{1 + 2/b}\right].$$

Summary

- **Model:** Extensive-form representation of a dynamic games of incomplete information
- **Solution:** Perfect Bayesian (Nash) equilibrium
- **Applications:** Beer and Quiche, Lemons Market, Job-Market Signaling, Cheap-Talk

Final Remarks

	Complete Information	Incomplete Information
Static	Normal-form games (NE)	Bayesian games (Bayesian NE)
Dynamic	PI games (Subgame perfect NE)	II games (Perfect Bayesian NE)

NE = Nash equilibrium

PI games = Extensive games with perfect information

II games = Extensive games with imperfect information

Final Remarks

- From a conceptual perspective, the notion of Nash Equilibrium is the most fundamental solution concept. Other notions: SPNE, BNE and PBE can be viewed as stronger versions of Nash Equilibrium with additional behavioral requirements
- From a technical perspective, the notion of PBE is most generalized solution concept. Other notions: NE, SPNE, and BNE can be derived from the notion of PBE as special cases of dynamic games of incomplete information
  (NE can be viewed as PBE for static games of complete information; SPNE can be viewed as PBE for dynamic games of complete information; BNE can be viewed as PBE for static games of incomplete information)
- Any economic and social problem can be modelled as a game and, then, be analyzed by using an equilibrium notion!