GAME THEORY

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Static Bayesian Games

- static (= one-shot, simultaneous-move)
- games of incomplete information (= *Bayesian games*): at least one player does not know another player's identity, e.g. payoff function
- Examples: Cournot competition under uncertainty, sealed-bid auctions

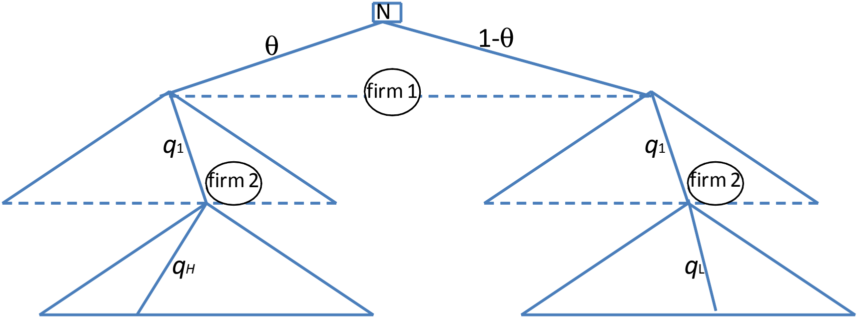
Cournot Competition under Asymmetric Information

- Two firms choose their quantities q_1 and q_2 (of a homogeneous product) simultaneously. The market-clearing price is given by the inverse demand function $P(Q) = a - Q$, where $Q = q_1 + q_2$
- Firm 1's cost function is: $C_1(q_1) = cq_1$, where $c < a$. Firm 2's cost function is:

$$C_2(q_2) = \begin{cases} c_H q_2, & \text{with probability } \theta \\ c_L q_2, & \text{with probability } 1-\theta \end{cases}, \quad \text{where } c_L < c_H < a$$

- Information is asymmetric:
  - firm 2 knows its cost function and firm 1's, but
  - firm 1 knows its cost function and only that firm 2's marginal cost is c_H with probability θ and c_L with probability $1-\theta$
- All of the above is common knowledge

Cournot Competition under Asymmetric Information



$\pi_1(q_1, q_H) = (a - q_1 - q_H)q_1 - cq_1$
$\pi_2(q_1, q_H) = (a - q_1 - q_H)q_H - c_H q_H$

$\pi_1(q_1, q_L) = (a - q_1 - q_L)q_1 - cq_1$
$\pi_2(q_1, q_L) = (a - q_1 - q_L)q_L - c_L q_L$

Note: firm 1's expected profit is: $\theta\pi_1(q_1, q_H) + (1-\theta)\pi_1(q_1, q_L)$

Cournot Competition under Asymmetric Information

- The two firms simultaneously choose $(q_1^*, q_2^*(c_H), q_2^*(c_L))$, where
  1. $q_2^*(c_H)$ solves: $\max_{q_2}[a - q_1^* - q_2 - c_H]q_2$
  2. $q_2^*(c_L)$ solves: $\max_{q_2}[a - q_1^* - q_2 - c_L]q_2$
  3. firm 1 maximizes its expected profit, i.e., q_1^* maximizes: $\theta[a - q_1 - q_2^*(c_H) - c]q_1 + (1-\theta)[a - q_1 - q_2^*(c_L) - c]q_1$
- By FOC, we have: $\begin{cases} q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} \\ q_2^*(c_L) = \frac{a - q_1^* - c_L}{2} \\ q_1^* = \frac{\theta[a - q_2^*(c_H) - c] + (1-\theta)[a - q_2^*(c_L) - c]}{2} \end{cases}$
- The equilibrium of the game is: $\begin{cases} q_1^* = \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3} \\ q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{1-\theta}{6}(c_H - c_L) \\ q_2^*(c_L) = \frac{a - 2c_L + c}{3} - \frac{\theta}{6}(c_H - c_L) \end{cases}$
- In the case of asymmetric information, firm 2 produces more (less) than in the game of complete information when the cost is high (low), because firm 1 produces relatively less (more) in the corresponding game of complete information.

Static Bayesian Game: Example

The Cournot competition under asymmetric information can be viewed as a static Bayesian game: Firm 2 has two payoff functions (which depend on its marginal costs c_L & c_H):

$$\begin{cases} u_2(q_1, q_2; c_L) = [(a - q_1 - q_2) - c_L]q_2 \\ u_2(q_1, q_2; c_H) = [(a - q_1 - q_2) - c_H]q_2 \end{cases}.$$

Firm 1 has only one expected payoff function:

$$u_1(q_1, q_2; c) = E_{q_2}[(a - q_1 - q_2) - c]q_1.$$

We say firm 2's type space is: $T_2 = \{c_L, c_H\}$ and 1's type space is: $T_1 = \{c\}$.

Normal-Form Representation of Static Bayesian Games

- The normal form representation of a static Bayesian game specifies:

$$G = \{A_1, \cdots, A_n; T_1, \cdots, T_n; p_1, \cdots, p_n; u_1, \cdots, u_n\}$$

  - players: $i = 1, ...n$
  - action spaces: $A_1, \cdots, A_n$
  - type spaces: $T_1, \cdots, T_n$
  - player i's belief: $p_i(t_{-i}|t_i)$ (which describes i's uncertainty about the other players' possible types, t_{-i}, given i's own type, t_i)
  - player i's payoff functions: $u_i(a_1, \cdots, a_n; t_1, \cdots, t_n)$

- In many applications, we assume that there exists a prior joint probability distribution over the type profiles: $p(t_1, \cdots, t_n)$ (with $p(t) > 0$ for all $t \in T_1 \times T_1 \times \cdots \times T_n$) and each player can then derive $p_i(t_{-i}|t_i)$ by Bayes' rule:

$$p_i(t_{-i}|t_i) = \frac{p(t_{-i}, t_i)}{p(t_i)} = \frac{p(t_{-i}, t_i)}{\sum_{t_{-i} \in T_{-i}} p(t_{-i}, t_i)}.$$

Normal-Form Representation: Example

- The normal form representation of the Cournot Competition under Asymmetric Information:

$$G = \{A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2\}$$

  - players: $i = 1, 2$
  - action spaces: $A_1 = A_2 = [0, +\infty)$
  - type spaces: $T_1 = \{c\}$, $T_2 = \{c_L, c_H\}$
  - player i's belief: $p_1(c_H|c) = \theta$ and $p_1(c_L|c) = 1 - \theta$, $p_2(c|c_L) = 1$, $p_2(c|c_H) = 1$
  - player i's payoff functions: $u_1(q_1, q_2; c, c_L) = [(a - q_1 - q_2) - c]q_1 = u_1(q_1, q_2; c, c_H)$ and

$$\begin{cases} u_2(q_1, q_2; c, c_L) = [(a - q_1 - q_2) - c_L]q_2 \\ u_2(q_1, q_2; c, c_H) = [(a - q_1 - q_2) - c_H]q_2 \end{cases}.$$

	t_{21}	t_{22}
t_{11}	see t_{11}, t_{21} below	see t_{11}, t_{22} below
t_{12}	see t_{12}, t_{21} below	see t_{12}, t_{22} below

t_{11}, t_{21}:

	x	y
a	3,3	0,4
b	4,0	1,1

[p_{11}=0.4]

t_{11}, t_{22}:

	x	y
a	3,2	0,0
b	4,0	1,1

[p_{12}=0.3]

t_{12}, t_{21}:

	x	y
a	2,3	0,4
b	0,0	1,1

[p_{21}=0.2]

t_{12}, t_{22}:

	x	y
a	2,2	0,0
b	0,0	1,1

[p_{22}=0.1]

Normal-Form Representation: Another Example

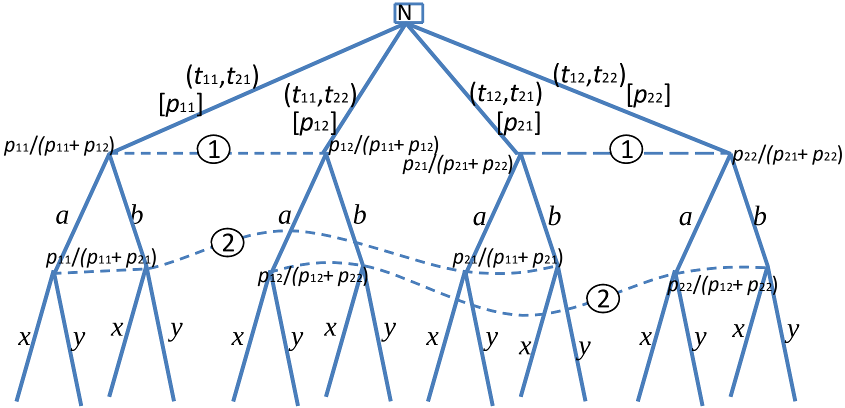
Normal-Form Representation of Static Bayesian Games

- According to Harsanyi (1967), we will assume that the timing of a static Bayesian game is as follows:
  1. Nature selects a type profile $t = (t_1, \cdots, t_n)$
  2. Nature informs each player i about his type (but not any other player's)
  3. The players simultaneously choose their actions
  4. Payoffs $u_i(a_1, \cdots, a_n; t)$ are received
- Harsanyi's approach transforms games of incomplete information into games of imperfect information

Extensive-Form Representation: Example

Consider the static two-person Bayesian game G, where

- $A_1 = \{a, b\}$ and $A_2 = \{x, y\}$; $T_1 = \{t_{11}, t_{12}\}$ and $T_2 = \{t_{21}, t_{22}\}$
- a prior joint probability distribution $p = (p_{11}, p_{12}, p_{21}, p_{22})$



Solution Concept: Bayesian Nash Equilibrium

- We use the model of "static Bayesian game" to represent static games with incomplete information (e.g., asymmetric information among players). The solution concept for static Bayesian games is: "Bayesian Nash equilibrium."
- The notion of Bayesian Nash equilibrium in a static Bayesian game can be defined by the following two ideas:
  1. Nash equilibrium in the agent-normal-form game, or
     [textbook adopts 1st idea to define Bayesian NE]
  2. Nash equilibrium in the (associated) normal-form game.
     [definition based on 2nd idea is often useful in finding Bayesian NE]

Bayesian Nash Equilibrium

Definition (Strategy)

A strategy for player i is a function $s_i(t_i)$, i.e., $s_i : T_i \to A_i$. For given type t_i in T_i, $s_i(t_i)$ specifies the action that type t_i would choose.

Definition (Bayesian Nash Equilibrium)

The strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a (pure-strategy) *Bayesian Nash equilibrium* if for each player i and for each of i's type t_i, $s_i^*(t_i)$ solves:

$$\max_{a_i \in A_i} E_{t_{-i}} u_i(s_{-i}^*(t_{-i}), a_i; t_i),$$

where $E_{t_{-i}} u_i(s_{-i}^*(t_{-i}), a_i; t_i) = \sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) u_i(s_{-i}^*(t_{-i}), a_i; t)$.

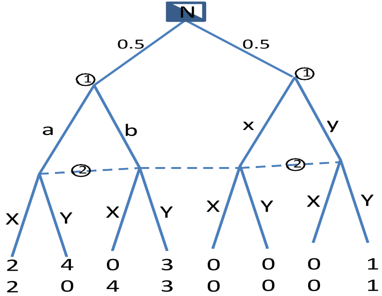
In the previous example, the quantities $(q_2^*(c_H), q_2^*(c_L))$ (which depend on marginal costs c_L & c_H) is a strategy for firm 2, while q_1^* is a strategy for firm 1. The strategy profile $(q_1^*, (q_2^*(c_H), q_2^*(c_L)))$ is a Bayesian Nash equilibrium.

BNE: Another Example

Player 1 knows which of the following two games is played and Player 2 knows only that each game is played with equal probabilities.

	X	Y
a	2, 2	4, 0
b	0, 4	3, 3

	X	Y
x	0, 0	0, 0
y	0, 0	1, 1



Extensive-form Representation

BNE: An Example

By using Definition (on p.12), we can find BNE in following agent-normal-form representation:

player 2 chooses X

	x	y
a	2, 0, 1	2, 0, 1
b	0, 0, 2	0, 0, 2

player 2 chooses Y

	x	y
a	4, 0, 0	4, 1, $\frac{1}{2}$
b	3, 0, $\frac{3}{2}$	3, 1, 2

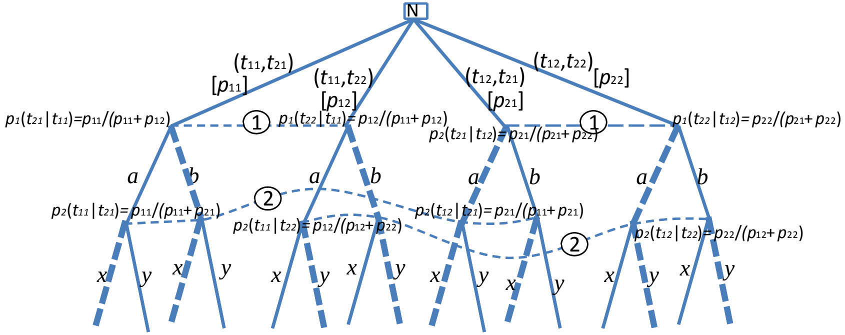
where the agent of player 1 at LHS chooses the row, the agent of player 1 at RHS chooses the column, and player 2 chooses the matrix.

BNE: (a,x,X), (a,y,X)

Bayesian Nash Equilibrium: Summary

- Bayesian game: $G =< \{A_i\}, \{T_i\}, \{p_i\}, \{u_i\} >$
- Strategy for player i is a function $s_i : T_i \longrightarrow A_i$ such that it specifies an action $s_i(t_i) \in A_i$ for each type t_i.
- Bayesian Nash equilibrium $s^* = (s_1^*, \ldots, s_n^*)$: $\forall i\ \forall t_i$, $s_i^*(t_i)$ solves:

$$\max_{a_i \in A_i} E_{t_{-i}} u_i(s^*_{-i}(t_{-i}), a_i; t_i) = \sum_{t_{-i} \in T_{-i}} p_i(t_{-i}|t_i) u_i(s^*_{-i}(t_{-i}), a_i; t).$$



$s^*=((s_1^*(t_{11})=b,\ s_1^*(t_{12})=a),\ (s_2^*(t_{21})=x,\ s_2^*(t_{22})=y)$

Bayesian Nash Equilibrium: An Alternative Definition

Definition (BNE)

The strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is a (pure-strategy) *Bayesian Nash equilibrium* if for each player i, s_i^* solves:

$$\max_{s_i \in S_i} \ E_t u_i(s_i, s_{-i}^*; t),$$

where $E_t u_i(s_i, s_{-i}^*; t) = \sum_{t \in T} p(t) u_i(s_i(t_i), s_{-i}^*(t_{-i}); t)$.

In the previous example, by using this alternative definition, we can find these BNE: (a,x,X) and (a,y,X) in following normal-form representation:

	X	Y
ax	$1, 1$	$2, 0$
ay	$1, 1$	$\frac{5}{2}, \frac{1}{2}$
bx	$0, 2$	$\frac{3}{2}, \frac{3}{2}$
by	$0, 2$	$2, 2$

Remark. This alternative definition is often useful in finding BNE because the agent-normal-form game may have too many agents as players.

Appendix I: BNE vs. NE*

- Consider a static Bayesian game:

$$G = \{A_1, ..., A_n; T_1, ..., T_n; p_1, , ..., p_n; u_1, ..., u_n\}$$

- Define the *agent-normal-form game* of $\overline{G}$ as follows:

$$\overline{G} = (\overline{N}, \{\overline{S}_j\}, \{\overline{u}_j\}),$$

where $\overline{N} = \{t_i | \ i \in N \text{ and } t_i \in T_i\}$, $\overline{S}_{t_i} = A_i$, and

$$\text{for } \overline{s} \in \times_{t_i \in \overline{N}} \overline{S}_{t_i}, \ \overline{u}_{t_i}(\overline{s}) = E_{t_{-i}} u_i(\overline{s}_{-i}(t_{-i}), \overline{s}_i(t_i); t_i)$$

- **Claim 1:** *A strategy profile $s^* = (s_1^*, ..., s_n^*)$ is a Bayesian Nash equilibrium in G (on p.12) if, and only if, it is a Nash equilibrium in $\overline{G}$.* (i.e. a Bayesian NE is simply a NE in the agent-normal-form game)
- To find Bayesian Nash equilibria, by Claim 1, we only need to find Nash equilibria in the agent-normal-form game.

Appendix II: An Alternative Definition of BNE

- The alternative definition of BNE in a static Bayesian game can be viewed as the notion of Nash equilibrium in the normal-form game associated with the static Bayesian game.
- Consider a static Bayesian game:

$$G = \{A_1, ..., A_n; T_1, ..., T_n; p_1, , ..., p_n; u_1, ..., u_n\}$$

Define the *associated normal-form game* of G as follows:

$$\tilde{G} = \{N, S_1, \ldots, S_n, \tilde{u}_1, \ldots, \tilde{u}_n\},$$

where $N = \{1, \ldots, n\}$, $S_i = \{s_i : T_i \to A_i\}$, and

$$\tilde{u}_i (s_1, \ldots, s_n) = \sum_{t=(t_1,...,t_n)\in T} p(t) u_i(s_1(t_1), \ldots, s_n(t_n); t).$$

- **Claim 2.** *A strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ is BNE in G (on p.16) iff it is NE in $\tilde{G}$.*

	t_{-i}^1	t_{-i}^2	...	t_{-i}^ℓ	...	t_{-i}^L
t_i^1	$p\left(t_i^1, t_{-i}^1\right)$	$p\left(t_i^1, t_{-i}^2\right)$	...	$p\left(t_i^1, t_{-i}^\ell\right)$	...	$p\left(t_i^1, t_{-i}^L\right)$
t_i^2	$p\left(t_i^2, t_{-i}^1\right)$	$p\left(t_i^2, t_{-i}^2\right)$	...	$p\left(t_i^2, t_{-i}^\ell\right)$	...	$p\left(t_i^2, t_{-i}^L\right)$
...	...	...	...	...	...	...
t_i^k	$p\left(t_i^k, t_{-i}^1\right)$	$p\left(t_i^k, t_{-i}^2\right)$	...	$p\left(t_i^k, t_{-i}^\ell\right)$	...	$p\left(t_i^k, t_{-i}^L\right)$
...	...	...	...	...	...	...
t_i^K	$p\left(t_i^K, t_{-i}^1\right)$	$p\left(t_i^K, t_{-i}^2\right)$	...	$p\left(t_i^K, t_{-i}^\ell\right)$	...	$p\left(t_i^K, t_{-i}^L\right)$

- Under type profile $t \equiv \left(t_i^k, t_{-i}^\ell\right)$, the players play a normal-form game:

$$G^{[t]} =< N, \{A_i\}_{i\in N}, \left\{u_i^{[t]}\right\}_{i\in N} >$$

where $u_i^{[t]} \equiv u_i(a_1, \cdots, a_n; t)$.

- Given type t_i^k, player i's belief is: $p_i(t_{-i}^\ell | t_i^k) = \frac{p\left(t_i^k, t_{-i}^\ell\right)}{\sum_{\ell=1}^L p\left(t_i^k, t_{-i}^\ell\right)}$ (and, thus, player i thinks the game as: $\sum_{\ell=1}^{L} p_i(t_{-i}^\ell | t_i^k) \circ G^{[t]}$).

Rubinstein's (1989, AER) E-mail game: An example

	A	B
A	M, M	$1, -L$
B	$-L, 1$	$0, 0$

G_a (probability $1 - p$)

	A	B
A	$0, 0$	$1, -L$
B	$-L, 1$	M, M

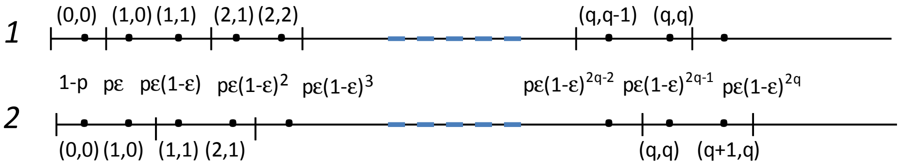
G_b (probability p)

[where the parameters satisfy $L > M > 1$ and $p < \frac{1}{2}$]

- (A, A) is a unique NE in G_a. (A strictly dominates B!)
  (A, A) & (B, B) are two NEs in G_b.

E-mail Game: Information Structure

The true game is known initially only to player 1, but not to player 2. Player 1 can communicate with player 2 via computers if the game is G_b. There is a small probability $\epsilon > 0$ that any given message does not arrive at its intended destination, however. (If a computer receives a message then it automatically sends a confirmation; this is so not only for the original message but also for the confirmation, the confirmation of the confirmation, and so on.)



- 1's signal is q for $(q, q-1)$ and (q, q) if $q \geq 1$ (and 0 for $(0, 0)$); 2's signal is q for (q, q) and $(q+1, q)$ if $q \geq 0$.

A Bayesian game model for this problem is: $(N, \{Q_i\}, \{A_i\}, \{u_i\}, \{p_i\})$ where

- $N \equiv \{1, 2\}$; $Q_i \equiv \{0, 1, 2, ...\}$; $A_i \equiv \{\text{A, B}\}$
- if $(q_1, q_2) = (0, 0)$, then payoffs are given by G_a
  if $(q_1, q_2) \neq (0, 0)$, then payoffs are given by G_b
- For player 1: if $q_1 = 0$, then $p_1(q_2|0) = \begin{cases} 1, & \text{if } q_2 = 0 \\ 0, & \text{otherwise} \end{cases}$;

  if $q_1 = q \neq 0$, then $p_1(q_2|q) = \begin{cases} \frac{1}{2-\epsilon}, & \text{if } \ q_2 = q - 1 \\ \frac{1-\epsilon}{2-\epsilon}, & \text{if } \ q_2 = q \\ 0, & \text{otherwise} \end{cases}$
- For player 2: if $q_2 = 0$, then $p_2(q_1|0) = \begin{cases} \frac{1-p}{1-p+p\epsilon}, & \text{if } \ q_1 = 0 \\ \frac{p\epsilon}{1-p+p\epsilon}, & \text{if } \ q_1 = 1 \\ 0, & \text{otherwise} \end{cases}$

  if $q_2 = q \neq 0$, then $p_2(q_1|q) = \begin{cases} \frac{1}{2-\epsilon}, & \text{if } \ q_1 = q \\ \frac{1-\epsilon}{2-\epsilon}, & \text{if } \ q_1 = q + 1 \\ 0, & \text{otherwise} \end{cases}$

E-mail Game: Solution

- **Claim:** *The electronic mail game has a unique (Bayesian) Nash equilibrium, in which both players always choose* A.
- **Proof:** Clearly, in any equilibrium 1 must choose A when receiving the signal 0. When 2's signal is 0, if 2 chooses A, then 2's expected payoff is at least $(1-p)M/[(1-p)+p\epsilon]$; if 2 chooses B then 2's payoff is at most $[-L(1-p)+p\epsilon M]/[(1-p)+p\epsilon]$. Therefore 2 must also choose A when receiving the signal 0.
- Assume inductively that when the received signal is less than q, 1 and 2 both choose A in any equilibrium. Consider 1's decision when receiving the signal q. In this case 1 believes $(q, q-1)$ with probability $z = 1/(2-\epsilon) > \frac{1}{2}$ and (q, q) with probability $1-z$. If 1 chooses B then 1's expected payoff is at most $z(-L)+(1-z)M$ (since under the induction assumption, 2 chooses A at $(q, q-1)$). Therefore, 1 should choose A which results in a payoff at least 0. Similarly, 2 chooses A when receiving the signal q. $\square$

The Revelation Principle

- A static Bayesian game is called a *direct mechanism* if each player's only action is to submit a claim about his type (i.e. $A_i = T_i$)
- A direct mechanism is *incentive-compatible* if truth-telling (i.e. every player tells the truth: $s_i(t_i) = t_i$) is a Bayesian Nash equilibrium
- **The Revelation Principle (Myerson):** Any Bayesian Nash equilibrium of any Bayesian game can be represented by an incentive-compatible direct mechanism
- **Application of the Revelation Principle:** In auction theory, a seller wishes to design an auction (e.g. 1st price auction, 2nd price auction, etc.) to maximize his expected revenue. By using the revelation principle, the seller can restrict attention to the class of incentive-compatible direct mechanisms

The Revelation Principle: Proof

- Let s^* be an equilibrium of Bayesian game G. Define the direct mechanism $\widetilde{G}$ such that, for every $t \in T$,

$$\widetilde{u}_i\left(t', t\right) \equiv u_i\left(s^*\left(t'\right), t\right), \forall t' \in T.$$

- Consider the truth-telling strategy $\tau_i\left(t_i\right) = t_i$, $\forall t_i$. Therefore, for any individual i and any two types t_i and t_i' in T_i,

$$\begin{aligned}
\sum_{t_{-i}\in T_{-i}} p_i\left(t_{-i}|t_i\right) \widetilde{u}_i\left(\tau\left(t\right), t\right) &= \sum_{t_{-i}\in T_{-i}} p_i\left(t_{-i}|t_i\right) u_i(s^*\left(t\right), t) \\
&\geq \sum_{t_{-i}\in T_{-i}} p_i\left(t_{-i}|t_i\right) u_i(\left(s^*_{-i}\left(t_{-i}\right), s_i^*\left(t_i'\right)\right), t) \\
&= \sum_{t_{-i}\in T_{-i}} p_i\left(t_{-i}|t_i\right) \widetilde{u}_i(\left(\tau_{-i}\left(t_{-i}\right), t_i'\right), t).
\end{aligned}$$

- That is, $\widetilde{G}$ is an incentive-compatible direct mechanism that is equivalent to the given Bayesian game G with its equilibrium s^*.$\blacksquare$

Sealed-bid Auctions: Complete Information

The bidders $i = 1, 2, ..., n$ simultaneously submit bids ($b_i \geq 0$) for an object. The bidders have valuations $v_1 > v_2 > \cdots > v_n$.

[First-Price Auction] The highest bidder gets the object and pays the price her bids. In case of a tie, each of the highest bidders wins the object with equal probability. Bidder i's payoff function is

$$u_i(b_i, b_{-i}) = \begin{cases} (v_i - b_i)/k, & \text{if } b_i = \max_{j=1}^n b_j \\ 0, & \text{if } b_i \neq \max_{j=1}^n b_j \end{cases}$$

where k is the number of highest bidders.

There are other forms of auction include the Vickrey auction, or second-price auction, where the highest bidder wins but pays only the second-highest bid. In principle, we can formulate these auctions with complete information as normal-form games and analyze their Nash equilibria.

Application: First-Price Sealed-Bid Auction

We now consider a "first-price sealed-bid auction" with **incomplete information**:

- Two bidders ($i = 1, 2$) simultaneously submit bids ($b_i \geq 0$) for an object
- The bidders have valuations v_1 and v_2, which are their private information. It is common knowledge that v_1 and v_2 are independently and uniformly distributed on $[0, 1]$
- The higher bidder gets the good and pays the price she bids. In case of a tie, the winner is determined by a flip of a coin. Bidder i's payoff function is

$$u_i(b_1, b_2; v_1, v_2) = \begin{cases} v_i - b_i, & \text{if } b_i > b_j \\ (v_i - b_i)/2, & \text{if } b_i = b_j \\ 0, & \text{if } b_i < b_j \end{cases}$$

Application: First-Price Sealed-Bid Auction

- Formulate this auction as a static Bayesian game:

$$G = \{A_1, A_2; T_1, T_2; p_1, p_2; u_1, u_2\}$$

  - $A_1 = A_2 = [0, \infty)$ (bids $b_i \in A_i$)
  - $T_1 = T_2 = [0, 1]$ (valuations $v_i \in T_i$)
  - i's belief $p_i(v_j)$ is the uniform distribution on $[0, 1]$
  - i's payoff is:

$$u_i(b_i, b_j; v_i, v_j) = \begin{cases} v_i - b_i, & \text{if } b_i > b_j \\ (v_i - b_i)/2, & \text{if } b_i = b_j \\ 0, & \text{if } b_i < b_j \end{cases}$$

- Player i's strategy is a function $b_i(v_i)$ from $[0, 1]$ into $[0, \infty)$ (i.e., i can sumbit a bid $b_i(v_i)$ which depends on his valuation v_i)

Application: First-Price Sealed-Bid Auction

- A pair of strategies $(b_1(v_1), b_2(v_2))$ is a Bayesian Nash equilibrium if for each $v_i \in [0,1]$, $b_i = b_i(v_i)$ maximizes i's expected payoff:

$$(v_i - b_i)Prob\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i)Prob\{b_i = b_j(v_j)\}$$

- For simplicity we look for equilibria in the form of linear functions:

$$b_1(v_1) = a_1 + c_1 v_1, \qquad b_2(v_2) = a_2 + c_2 v_2,$$

  where $a_i \geq 0$, $c_i > 0$ and $a_i < 1$, $i = 1, 2$.
- Thus, for any given $v_i \in [0,1]$, player i's best response $b_i(v_i)$ maximizes

$$(v_i - b_i)Prob\{b_i > a_j + c_j v_j\},$$

  where $Prob(b_i = a_j + c_j v_j) = 0$.

Application: First-Price Sealed-Bid Auction

- Since it is pointless for player i to bid below player j's minimum bid and foolish for i to bid above j's maximum bid, we can restrict $b_i \in [a_j, a_j + c_j]$, so

$$Prob\{b_i > a_j + c_j v_j\} = Prob\{v_j < \frac{b_i - a_j}{c_j}\} = \frac{b_i - a_j}{c_j}$$

- Therefore, i's best response solves:

$$\max_{a_j \leq b_i \leq a_j + c_j} (v_i - b_i) \frac{b_i - a_j}{c_j}$$

- Thus, player i's best response is: $b_i(v_i) = v_i/2$.[1] That is, each bidder's optimal strategy is to bid half her valuation.

[1] Note: $a_j = 0$; otherwise, it would be unreasonable that $b_i(v_i) > 0$ for $v_i = 0$.

Application: Mixed Strategies Revisited

- **Harsanyi's (1973) Purification Theorem:** A mixed-strategy Nash equilibrium in a game of complete information can almost always be interpreted as a pure-strategy Bayesian Nash equilibrium in a closely related game with a little bit of incomplete information

- **Example:**

	opera	fight
opera	2, 1	0, 0
fight	0, 0	1, 2

  In this game, there are two pure-strategy Nash equilibria: (opera, opera), (fight, fight), and a mixed-strategy Nash equilibrium in which player 1 plays opera with probability $2/3$ and player 2 plays fight with probability $2/3$

- We next show that this mixed-strategy Nash equilibrium can be approximated by pure-strategy Bayesian Nash equilibria of properly perturbed games of incomplete information

Application: Mixed Strategies Revisited

- Suppose the players are not completely sure of each other's payoff: if both go to opera, 1's payoff is $2 + t_1$; if both go to fight, 2's payoff is $2 + t_2$, where t_1 is privately known by player 1 and t_2 is privately known by player 2, and t_1 and t_2 are independently drawn from a uniform distribution on $[0, x]$

	opera	fight
opera	$2 + t_1, 1$	$0, 0$
fight	$0, 0$	$1, 2 + t_2$

- That is, we consider the following collection of Bayesian games:

$$G^x = < \{A_i\}, \{T_i^x\}, \{p_i\}, \{u_i\} > ,$$

where

- $A_i = \{\text{opera, fight}\}$
- $T_i^x = [0, x]$
- $p_i -$ iid $U[0, x]$
- $u_i(a_1, a_2; t_1, t_2)$ is given by the above matrix

Application: Mixed Strategies Revisited

- Consider the following simple class of strategies:

$$s_1\left(t_1\right)=\begin{cases}\text{opera,} & \text{if } t_1 \geq t_1^* \\ \text{fight,} & \text{if } t_1 < t_1^*\end{cases}; \quad s\left(t_2\right)=\begin{cases}\text{fight} & \text{if } t_2 \geq t_2^* \\ \text{opera} & \text{if } t_2 < t_2^*\end{cases}$$

where t_1^* and t_2^* are critical value.

- The probability p_x for player 1 to play opera and the probability q_x for player 2 to play fight are as follows:

$$\begin{cases} p_x = 1 - \frac{t_1^*}{x} \\ q_x = 1 - \frac{t_2^*}{x} \end{cases}.$$

Application: Mixed Strategies Revisited

- In order to be a Bayesian Nash equilibrium, the critical values t_1^* and t_2^* must satisfy the following conditions:

$$\begin{cases} (1-q_x)(2+t_1^*) = q_x \\ p_x = (1-p_x)(2+t_2^*) \end{cases}.$$

- Thus, we obtain:

$$t_1^* = t_2^* = \frac{\sqrt{9+4x}-3}{2}.$$

- Therefore, we have

$$\begin{cases} p_x^* = 1 - \frac{\sqrt{9+4x}-3}{2x} \longrightarrow \frac{2}{3} \text{ (as } x \to 0) \\ q_x^* = 1 - \frac{\sqrt{9+4x}-3}{2x} \longrightarrow \frac{2}{3} \text{ (as } x \to 0) \end{cases}.$$

  That is, when the Bayesian game (G^x) converges to the original game of complete information (as $x \longrightarrow 0$), the pure-strategy BNE distribution (p_x^*, q_x^*) converges to the mixed-strategy Nash equilibrium $\left(\frac{2}{3}, \frac{2}{3}\right)$.

Summary

- **Model:** Normal-form representation of a static Bayesian games
- **Solution:** Bayesian Nash equilibrium; the revelation principle
- **Applications:** Cournot duopoly under uncertainty, auction, and purification