GAME THEORY

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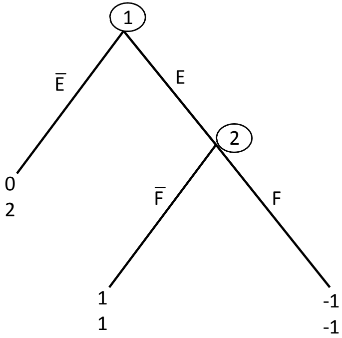
Dynamic Games of Complete Information

- By complete information, we mean that the payoff functions are common knowledge among all the players. We consider two cases:
  1. **Dynamic games with perfect information:** At each move in the game, the player with the move knows the full history of the play of the game thus far
  2. **Dynamic games with imperfect information:** At some move, the player with the move does not know the history of the play of the game

Example: Entry-Deterrence

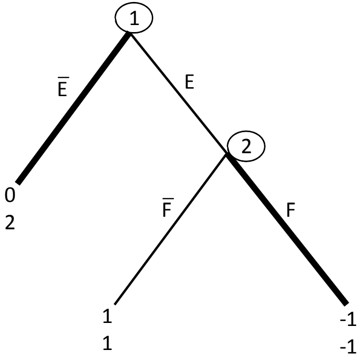


	F	$\bar{F}$
E	$-1, -1$	$1, 1$
$\bar{E}$	$0, 2$	$0, 2$

A normal-form representation

Payer 1 is an entrant; Payer 2 is an incumbent
E = enter, $\bar{E}$ = don't enter; F = fight, $\bar{F}$ = don't fight

Example: Entry-Deterrence



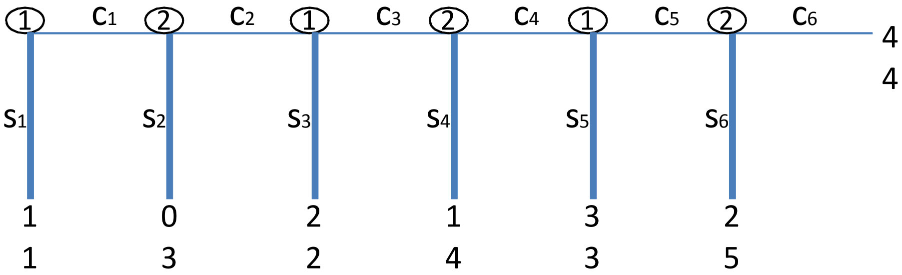
- There are two Nash equilibria: ($\bar{\text{E}}$, F) and (E, $\bar{\text{F}}$)
- **Backward-Induction Outcome:** (E, $\bar{\text{F}}$)
  (in ($\bar{\text{E}}$, F), the threat strategy "fight" for the incumbent is not credible!)

Backward-Induction Outcome: Algorithm

- This algorithm involves going to the end of the tree and working back towards the beginning

1. Start with the last players: Each of the last players chooses one of actions that maximize that player's payoff
2. Turn to the second-to-last players: Taking the last players' choices as determined in the first step, each of the second-to-last players chooses an action that maximizes that player's payoff
3. And so on so forth until the algorithm goes to the beginning of the game

Example: Centipede game



- **Backward-Induction:** $(s_1, s_2, s_3, s_4, s_5, s_6)$
- **Remark.** If player 2 gets to move at the second decision node, can player 2 assume that player 1 is rational?

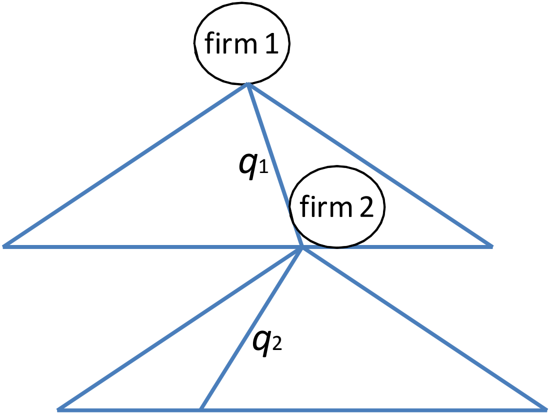
Application: Stackelberg Duopoly

- **players:** $\{$firm 1 (leader), firm 2 (follower)$\}$
- **timing:**
  1. Firm 1 first chooses a quantity $q_1 \geq 0$
  2. Firm 2 observes q_1 and then chooses $q_2 \geq 0$
- **payoff functions:**

$$\pi_i(q_i, q_j) = P(Q)q_i - cq_i,$$

where $P(Q) = a - Q$, $Q = q_1 + q_2$, and c is the constant marginal cost of production.

Application: Stackelberg Duopoly



$$\pi_1(q_1,q_2)=(a-q_1-q_2)q_1-cq_1$$
$$\pi_2(q_1,q_2)=(a-q_1-q_2)q_2-cq_2$$

- **Backwards-Induction Outcome:**
  The best response function $R_2(q_1)$ for firm 2 to quantity q_1 solves:

  $$\max_{q_2 \geq 0} \pi_2(q_1, q_2) = (a - q_1 - q_2)q_2 - cq_2$$

  which yields $R_2(q_1) = \frac{1}{2}(a - c - q_1)$ [which spcifies a contingent action plan of how firm 2 plays for each (on- and off-equilibrium) play q_1 by firm 1]
- Firm 1 knows $R_2(q_1)$ and therefore solves:

  $$\max_{q_1 \geq 0} \pi_1(q_1, R_2(q_1)) = \frac{1}{2}(a - c - q_1)q_1$$

  By FOC, $a - q_1 - c - q_1 = 0$ which yields

  $$q_1^* = \frac{a - c}{2},$$

  and, thus

  $$q_2^* = R_2(q_1^*) = \frac{a - q_1^* - c}{2} = \frac{a - c}{4}$$

Application: Stackelberg Duopoly

- The market price is: $P^* = a - \frac{3(a-c)}{4} = c + \frac{a-c}{4}$.
  The profits for firms 1 and 2 are:

$$(\pi_1^*, \pi_2^*) = \left(\frac{(a-c)^2}{8}, \frac{(a-c)^2}{16} \right).$$

- In the Cournot duopoly, the market price is:
  $P^{**} = a - \frac{2(a-c)}{3} = c + \frac{a-c}{3}$.
  The profits for firms 1 and 2 are:

$$(\pi_1^{**}, \pi_2^{**}) = \left(\frac{(a-c)^2}{9}, \frac{(a-c)^2}{9} \right).$$

- The Stackelberg leader has a first-move advantage:

$$\pi_1^* > \pi_1^{**} = \pi_2^{**} > \pi_2^*$$

Backwards-Induction Outcome

1. At the second stage, player 2 observes the action (say a_1) chosen by player 1 at the first stage, then chooses an action by solving

$$\max_{a_2 \in A_2} u_2(a_1, a_2).$$

Assume this optimization problem has a unique solution, denoted by $R_2(a_1)$. This is player 2's best response to player 1's action.

2. Knowing the player 2's best response, player 1 chooses a_1^* by solving

$$\max_{a_1 \in A_1} u_1(a_1, R_2(a_1)).$$

We call $(a_1^*, R_2(a_1^*))$ the *backwards-induction outcome*.

- If players choose their actions simultaneously, then NE (a_1^{**}, a_2^{**}) is the intersection of the two best responses: $\begin{cases} a_1^{**} = R_1(a_2^{**}) \\ a_2^{**} = R_2(a_1^{**}) \end{cases}$.

In the backwards-induction outcome, a_1^* may not maximize $u_1(a_1, a_2^*)$!

Application: Bank Runs

- Two investors have each deposited \$50K with a bank. The bank has invested these deposits in a long-term project.
- If the bank is forced to stop at date 1 its investment before the project matures, a total of \$80K can be recovered.
- If the bank allows the investment to reach maturity at date 2, the project will pay out a total of \$200K.
- The payoffs to the two investors are as follows:

	withdraw	don't
withdraw	40, 40	50, 30
don't	30, 50	next stage

date 1

	withdraw	don't
withdraw	100, 100	150, 50
don't	50, 150	100, 100

date 2

Subgame-Perfect Outcome

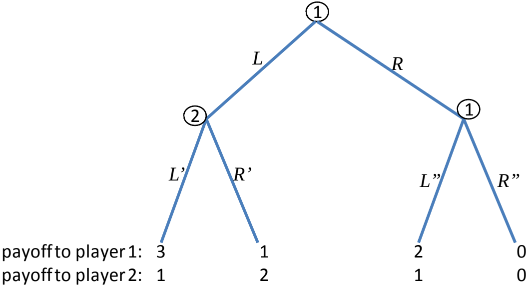
- We work backwards:
  1. At date 2: "withdraw" strictly dominates "don't." The unique Nash equilibrium is: both withdraw and each obtains \$100K.
  2. At date 1: They play the game:

	withdraw	don't
withdraw	$40, 40$	$50, 30$
don't	$30, 50$	$100, 100$

     This game has two pure-strategy Nash equilibria:
     Both investors withdraw, leading to a payoff $(40, 40)$
     Both don't, leading to a payoff $(100, 100)$
- Thus, the original two-stage game has two subgame-perfect outcomes:
  Both withdraw at date 1 to obtain $(40, 40)$. This is a case of bank run!
  Both don't withdraw at date 1, but withdraw at date 2, yielding $(100, 100)$.

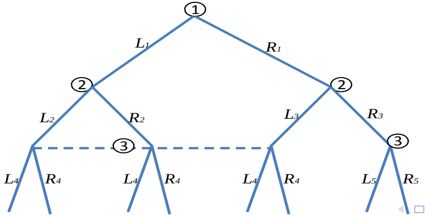
Extensive-Form Representation

- The *extensive-form representation* of a game specifies:
  1. the players in the game
  2a. when each player has the move
  2b. what each player can do at each of his opportunities to move
  2c. what each player knows at each of his opportunities to move
  3. the payoffs received by each player for each combination of moves that could be chosen by the players
- This can be easily represented by a *game tree*:



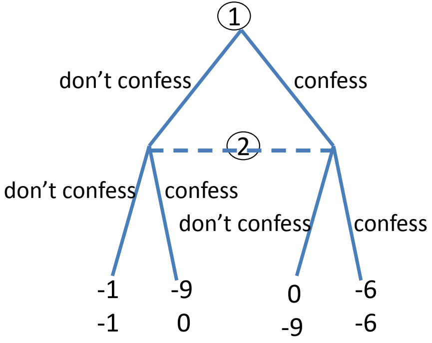
Extensive-Form Representation

- An *information set* for a player is a collection of decision nodes satisfying:
  1. The player needs to move at every node in the information set
  2. When the play of the game reached a node in the information set, the player with the move *does not know which node in the set has (or has not) been reached* (in the case of a singleton of one node, the player knows that the *only* node has been reached) – this implies that the player must have the same set of feasible actions at each decision node in an information set!
- A game is said to be of *perfect information* if every information set is a singleton, and of *imperfect information* if there is at least one nonsingleton information set (represented by a dotted line in the game tree)



Extensive-Form Representation

- A static game can be represented in extensive form:



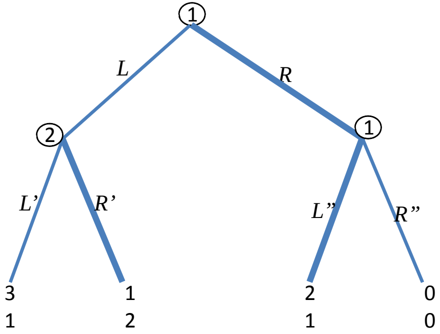
The Prisoners' dilemma

	don't	*confess*
don't	-1,-1	-9,0
confess	0,-9	-6,-6

Definition (Strategy)

A *strategy* for a player is a complete plan of actions – it specifies a feasible action for the player in every contingency (or information set) in which the player might be called on to act.

A player's strategy can be viewed as a function which assigns an action to each information set belonging to the player. (E.g., if player i has two information sets where the player has m_1 and m_2 feasible actions, respectively, then player i has a total of $m_1 \times m_2$ strategies.)



	L'	R'
(L,L'')	3,1	1,2
(L,R'')	3,1	1,2
(R,L'')	2,1	2,1
(R,R'')	0,0	0,0

- **Nash equilibrium:** $((R, L''), R')$
- This Nash equilibrium is consistent with the backward-induction outcome

Subgame-Perfect Nash Equilibrium

Definitions (Selten)

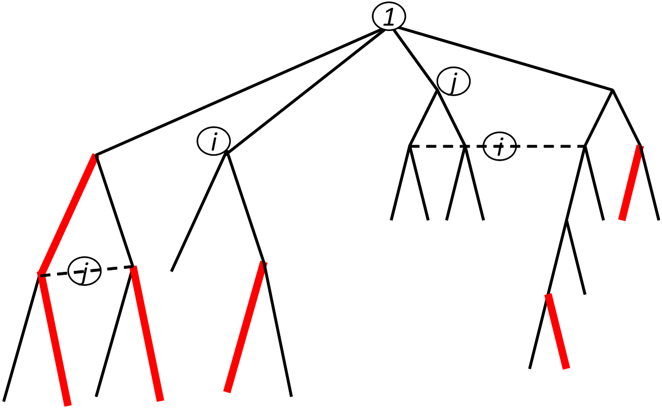
A Nash equilibrium is *subgame-perfect* if the players' strategies constitute a Nash equilibrium in every (proper) subgame.

- where a (proper) *subgame* in an extensive-form game:
  1. begins at a decision node n that is a singleton information set (but is not the game's first decision node)
  2. includes all the decision and terminal nodes following node n in the game tree (but no nodes that do not follow n)
  3. does not cut any information sets (i.e., if a decision node n' follows n in the game tree, then all other nodes in the information set containing n' must also follow n, and so must be included in the subgame).
- Every finite game with perfect information has an SPNE, which can be found by backwards induction! (Zermelo 1912, Kuhn 1953)

Subgame-Perfect Nash Equilibrium: Backward Induction

- Look for subgame-perfect Nash equilibria
  1. Start with the smallest subgames (i.e. the subgames contain no proper subgames: in the case of perfect information, a smallest subgame is one in which only one player takes an action):
     Choose a Nash equilibrium (or an optimal action in the case of perfect information) in each of these subgames for the player(s) who are moving in this subgame
  2. Now consider those second smallest subgames (i.e. the subgames contain no proper subgames other than the smallest subgames):
     Choose a Nash equilibrium (or an optimal action in the case of perfect information) in each of these subgames for the player(s) who are moving in this subgame, knowing 1

     ... ...
  3. Continue until you reach the beginning of the game

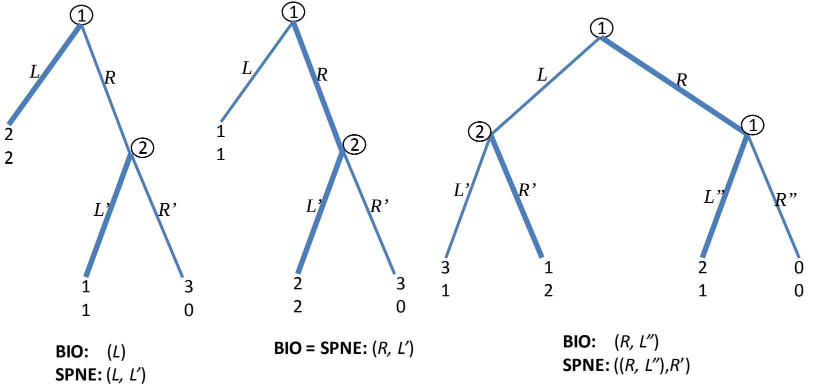


SPNE: Idea

(see, e.g., http://banach.lse.ac.uk/ for finding a Nash equilibrium)

Difference between an Equilibrium and an Outcome

- A backwards-induction (or subgame-perfect) outcome is the *path of play* resulting from an SPNE.



- In the Stackelberg duopoly, BIO is: $(a_1^*, R_2(a_1^*))$ and SPNE is: $(a_1^*, R_2(\cdot))$ (note: $R_2(a_1^*)$ is an action for 2 but not a strategy and $R_2(\cdot)$ is a strategy)

Subgame Perfect Nash Equilibrium vs. Backward Induction Procedure

	BI Procedure	**Equilibrium**
PI Games	BI outcome	SPNE=BI complete outcome
Dynamic Games	SP outcome	SPNE=SP complete outcome

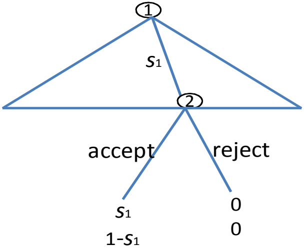
- PI = perfect information
- outcome = realization outcome by the procedure
- complete outcome = strategy profile generated by the procedure
- BI = Backward Induction
- SPNE = Subgame Perfect NE

Application: Sequential Bargaining (1-Period)

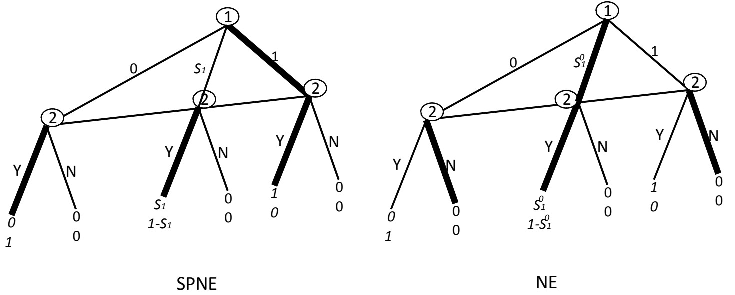
Players 1 and 2 are bargaining over how to split a pie.

- **players:** $\{1, 2\}$
- **timing:**
  1. Player 1 proposes to take a share s_1, leaving $1 - s_1$ for player 2
  2. Player 2 either accepts or rejects the offer
- **payoff functions:** If player 2 accepts the offer, the payoffs are s_1 to 1, and $1 - s_1$ to 2. If player 2 rejects the offer, the payoffs are zero to both

Application: Sequential Bargaining (1-Period)



- **Backward-Induction Outcome:** $(s_1^* = 1, R_2(s_1^*) = \text{accept})$
  [Intuitively, if $s_1 < 1$, player 2 must say yes; if $s_1 = 1$, player 2 can say yes or no. Knowing this, the *only* possibility is $s_1^* = 1$.]
- Note: Any share $s_1 \in [0, 1]$ proposed by player 1 can be supported by a Nash equilibrium. In doing so, player 2 can play the following "strategy": accept the offer iff player 1 proposes to take a share s_1. Nash has no bite!

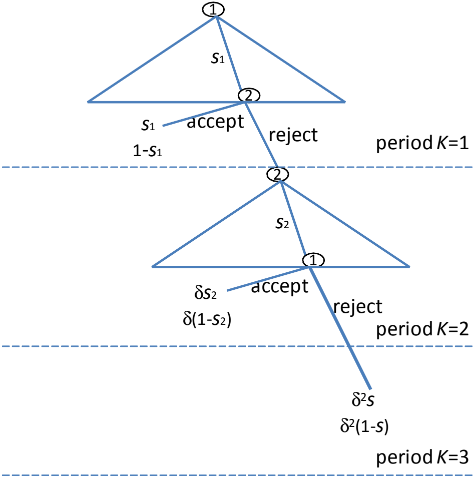


- **SPNE:** $\left(s_1^* = 1, s_2^*\left(s_1\right) = \text{accept for all } s_1 \in [0,1]\right)$.

  NE: For each s_1^0 in $[0,1]$, $\left(s_1^0,\ s_2^0 = \begin{cases} \text{accept}, & \text{if } s_1 = s_1^0 \\ \text{reject}, & \text{if } s_1 \neq s_1^0 \end{cases}\right)$.

- Note: A strategy for player 2 is a function $f : [0,1] \to \{\text{accept, reject}\}$ such that $f\left(s_1\right) \in \{\text{accept, reject}\}\ \forall s_1 \in [0,1]$.

Sequential Bargaining (3-Period)



The players discount payoffs received a period later by a factor $\delta \in (0,1)$

Application: Sequential Bargaining (3-Period)

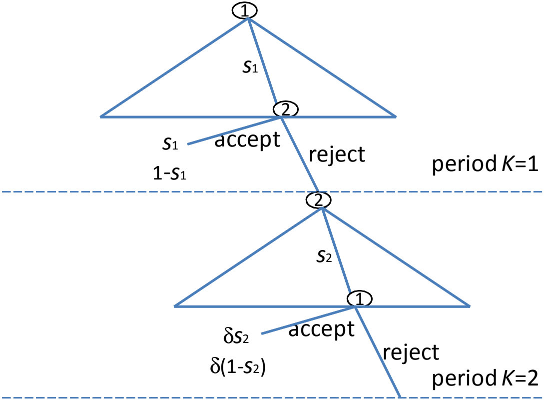
- **Backward-Induction Outcome:** Player 1 makes the following offer $(s_1^*, 1 - s_1^*)$ and player 2 accepts the offer immediately.
  1. In period $K = 2$, player 2 makes the offer:
  $$(s_2^*, 1 - s_2^*) = (\delta s, 1 - \delta s),$$
  and player 1 accepts an offer iff $s_2 \geq \delta s$.
  2. In period $K = 1$, player 1 makes the offer:
  $$(s_1^*, 1 - s_1^*) = (1 - \delta (1 - \delta s), \delta (1 - \delta s)),$$
  and player 2 accepts an offer iff $1 - s_1 \geq \delta (1 - \delta s)$.

Rubinstein's (1982) Alternating-Offer Model

Players 1 and 2 are bargaining over how to split a pie. The players are impatient: they discount payoffs received a period later by a factor δ with $0 < \delta < 1$.

- **players:** $\{1, 2\}$
- **timing:** The players alternate in making offers in periods $K \geq 1$. In odd (even) periods K, player 1 (player 2) makes an offer $(s_K, 1 - s_K)$, and player 2 (player 1) either accepts or rejects the offer. If player 2 (player 1) accepts the offer, then the game ends in period K; otherwise, the game goes to next period $K + 1$.
- **payoff functions:** If the offer is accepted in period K, the payoffs are $(\delta^{K-1} s_K, \delta^{K-1} (1 - s_K))$. In case of the perpetual disagreement, the payoffs are zero to both.

Rubinstein's Model (Infinite-Period)



The first two periods of the alternative-offer bargaining game

Rubinstein's Model: Solution

- **Backward-Induction Outcome:** Player 1 makes the offer $(s^*, 1 - s^*) = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$ and player 2 accepts the offer immediately.

- Suppose that $(s, 1 - s)$ is a backward-induction outcome. Note that the game is 2-period stationary (the subgame starting in period 3 is exactly that in period 1). By the argument in the 3-period case, Player 1 will offer $(f(s), 1 - f(s))$ in the first period and player 2 accepts the offer immediately, where

$$f(s) = 1 - \delta(1 - \delta s).$$

- If the backward-induction outcome is unique, then there must be a unique value $s^* = \frac{1}{1+\delta}$ which satisfies: $f(s) = s$.
  [To see this, let s_H and s_L be the highest and lowest payoffs for player 1 resulting from all the backward-induction outcomes, respectively. Then, $f(s_H) = s_H$ and $f(s_L) = s_L$ (since $f(s)$ is monotonically increasing). Since $f(s) = s$ has a unique solution s^*, it follows $s_H = s_H = s^*$.]

Repeated Games

- A *repeated game* is a dynamic game in which the same static game (i.e. stage game) is played at every stage
- After every stage, players observe the actions chosen by all players
- The repeated game has either a finite or an infinite number of repetitions

Repeated Games

- Many economic and social phenomena can be modelled and analyzed using the model of repeated game. The examples include wars and human conflicts (e.g. India–Pakistan, Israel–Arab, North–South Korea, and China-Taiwan); see Robert Aumann's Nobel lecture: War and Peace.
- The model of repeated game is designed to examine the logic of longterm interaction. It captures the idea that a player will take into account the effect of his current behavior on the other players' future behavior, and aims to explain phenomena like cooperation, revenge, and threats.
- Folk Theorem: The cooperative outcomes of stage game G can be supported by the subgame-perfect equilibrium outcomes of its repeated game G^∞.

Finitely Repeated Games

- We assume that the payoff in the finitely repeated game is simply the sum of the payoffs attained in all the stage games
- **Example:** The following prisoners' dilemma game is repeated twice

	don't confess	confess
don't confess	4,4	0,5
confess	5,0	1,1

- The unique **subgame-perfect outcome**: (confess, confess) in the first stage, followed by (confess, confess) in the second stage
- **Proposition:** If the stage game has a unique Nash equilibrium, then the finitely repeated game has a unique subgame-perfect equilibrium: the Nash equilibrium of the stage game is played in every stage

Finitely Repeated Games

- An additional strategy (A) is added for each player to the previous Prisoners' Dilemma game. They play this new stage game twice

	D	C	A
D	4,4	0,5	0,0
C	5,0	1,1	0,0
A	0,0	0,0	3,3

- The stage game has two Nash equilibria: (C, C) and (A, A)
- There is a subgame-perfect outcome of the repeated game in which the cooperation strategy pair (D,D) is played in the first stage [this outcome is supported by the subgame-perfect Nash equilibrium: play (A, A) if the first stage outcome is (D,D); play (C, C) if any of the eight other first-stage outcomes occurs]

Infinitely Repeated Games

- Assume players discount the future at a common rate $\delta \in (0, 1)$
- The present value of the infinite sequence of payoffs $\pi_1, \pi_2, \cdots$ is:

$$\pi_1 + \delta\pi_2 + \delta^2\pi_3 + \cdots = \sum_{t=1}^{\infty} \delta^{t-1}\pi_t.$$

The average payoff of the infinite sequence of payoffs $\pi_1, \pi_2, \cdots$ is:

$$(1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1}\pi_t.$$

Infinitely Repeated Games

- Consider the infinitely repeated Prisoners' dilemma. Clearly, there is an SPNE outcome in which both players play the noncooperative strategy C [indeed, if players play a Nash equilibrium of the stage game in every stage, then this constituents an SPNE in an (in)finitely repeated game]
- We can also support perpetual cooperation as an SPNE outcome when δ is big enough. Consider the following trigger strategy:
  - Play D in the first stage. In the t-th stage ($t \geq 2$), if the outcome of all $t-1$ preceding stages has been (D,D) then play D; otherwise, play C
- If both players adopt this trigger strategy, then the cooperative outcome (D,D) occurs in every stage

- **Claim:** Adopting the trigger strategy is an SPNE in the infinite Prisoners' dilemma if and only if $\delta \geq 1/4$.
- **Proof:** Assume player i adopts the trigger strategy. We seek to show j's best response is also to adopt the trigger strategy at every subgame. We distinguish two cases:
  1. If the outcome in a previous stage is not (D, D), then i continues to play C forever. Thus j's best response is to play C from this stage onwards.
  2. Suppose all the preceding outcomes are (D, D). If j deviates to play C in this stage, then this deviation yields a payoff of 5 in this stage but will trigger C by player i forever from the next stage. Thus j's payoff from this stage onwards is: $5 + \delta \cdot 1 + \delta^2 \cdot 1 + \cdots = 5 + \frac{\delta}{1-\delta}$.
     If j conforms to the trigger strategy, however, then j's payoff from this stage onwards should be: $4 + 4\delta + 4\delta^2 + \cdots = \frac{4}{1-\delta}$.
- Therefore, playing the trigger strategy is optimal (from this stage onwards) iff $\frac{4}{1-\delta} \geq 5 + \frac{\delta}{1-\delta} \iff \delta \geq \frac{1}{4}$. ■

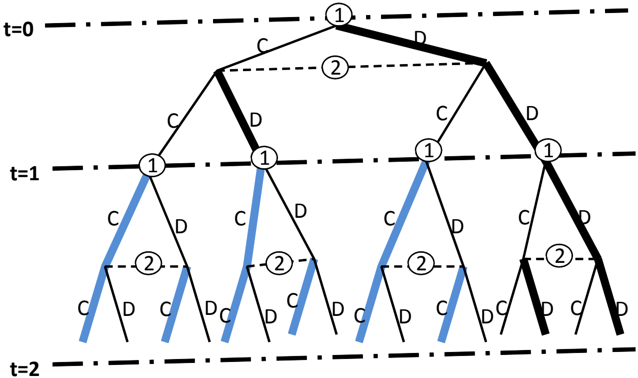


Figure: Repeated Prisoners' Dilemma

Remark. Player i's strategy is a function which specifies an action for player i in every of his information sets.

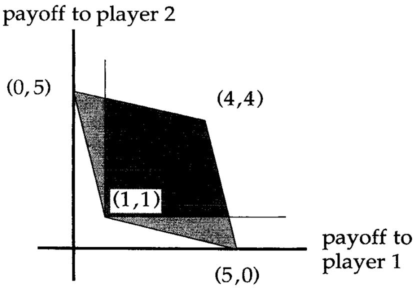
Folk Theorem

Theorem (Friedman 1971)

Let $(e_1, \cdots, e_n)$ denote the payoffs from a Nash equilibrium of stage game G, and let $(x_1, \cdots, x_n)$ denote any other feasible payoffs from G. If $x_i > e_i$ for every i and if discount factor δ is sufficiently close to one, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game $G(\infty, \delta)$ that achieves $(x_1, \cdots, x_n)$ as the average payoff.

Folk Theorem

- In the infinite Prisoners' dilemma, for example, many other payoffs (the shaded region in the figure) can be achieved by SPNE:



$$(e_1, e_2) = (1, 1);\ (x_1, x_2) = (4, 4)$$

Collusion between Cournot Duopolists

- Cournot quantity: $q_C = \frac{a-c}{3}$; Cournot profit to each firm: $\pi_C = \frac{(a-c)^2}{9}$
- monopoly quantity: $q_m = \frac{a-c}{2}$; monopoly profit to each firm: $\frac{\pi_m}{2} = \frac{(a-c)^2}{8}$ (in this case, two firms collude to produce $\frac{q_m}{2}$ each)
- If firm i produces $\frac{q_m}{2}$, then j's best response is to produce $q_d = \frac{3(a-c)}{8}$, which maximizes $q_j(a - q_j - \frac{q_m}{2} - c)$. This profit is: $\pi_d = \frac{9(a-c)^2}{64}$
- Consider the following trigger strategy in the infinitely repeated game for the Cournot stage game with the discount factor δ.
  1. produce half the monopoly quantity, $\frac{q_m}{2}$, in the first period
  2. In the t-th period, produce $\frac{q_m}{2}$ if both firms have produced $\frac{q_m}{2}$ in all the preceding $(t-1)$ periods; otherwise, produce the Cournot quantity q_C
- Similarly to the Prisoners' dilemma case, both playing the trigger strategy is an SPNE provided that

$$\frac{1}{1-\delta}\frac{\pi_m}{2} \geq \pi_d + \frac{\delta}{1-\delta}\pi_C,$$

i.e., $\delta \geq 9/17$.

Summary

- **Model:** Extensive-form representation
- **Solution:** Subgame-perfect Nash equilibrium
  (backward induction/subgame perfection outcomes)
- **Applications:** Stackelberg duopoly, bank run, and sequential bargaining
- **Others:** Repeated games; folk theorem