GAME THEORY

Qiao Yongchuan

School of Management and Economics
UESTC

Topics

	Complete Information	Incomplete Information
Static	Normal-form games (NE)	Bayesian games (Bayesian NE)
Dynamic	PI games (Subgame perfect NE)	II games (Perfect Bayesian NE)

NE = Nash equilibrium

PI games = Extensive games with perfect information

II games = Extensive games with imperfect information

Static Games of Complete Information

- static (one-shot, simultaneous-move)
- complete information: each player's payoff function is common knowledge among all the players
- static games of complete information: normal-form game

Example: The Prisoners' Dilemma

- Two suspects are arrested and charged with a crime. The suspects are held in separate cells and told "If only one of you confesses and testifies against your partner, the person who confesses will go free while the person does not confess will surely be convicted and sent to prison for 9 years. If both of you confess, you will both be convicted and sent to prison for 6 years. Finally, if neither of you confesses, both of you will be convicted of a minor offence and sentenced to 1 year in jail".
- This problem can be represented the following bi-matrix:

	Don't confess	confess
Don't confess	-1, -1	-9, 0
confess	0, -9	-6, -6

- In each entry of the bi-matrix, the first number is the payoff of the row player (Prisoner 1) and the second is that of the column player (Prisoner 2). Both player have a dominant strategy "confess" in the sense that no matter what the other to do, confess is the best choice.

Other Examples

1. **Coordination:**

	Left	right
Left	2, 2	0, 0
right	0, 0	1, 1

2. **Battle of Sexes:**

	opera	fight
opera	2, 1	0, 0
fight	0, 0	1, 2

3. **Matching Pennies:**

	heads	tails
heads	-1, 1	1, -1
tails	1, -1	-1, 1

4. **Hawk-Dove:**

	dove	hawk
dove	3, 3	1, 4
hawk	4, 1	0, 0

Normal-Form Representation of Games

- The normal-form representation of a game:

$$G =< N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} >,$$

  - $N = \{1, 2, \cdots, n\}$: set of players (with typical player $i \in N$)
  - S_i: set of strategies for player i (with typical strategy $s_i \in S_i$)
  - u_i: player i's payoff function
    ($u_i(s_1, \cdots, s_n)$ is the payoff to player i if player $j = 1, 2, ..., n$ plays strategy s_j; the payoff of a player depends not only on his own strategy, but also on the strategies of others!)

- Useful notations:
  - $s = (s_1, \ldots, s_{i-1}, s_i, s_{i+1}, \ldots, s_n) = (s_i, s_{-i})$
  - $s_{-i} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$
  - $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$
  - $S = S_1 \times \cdots \times S_n = S_i \times S_{-i}$ — Cartesian product
    (cf. http://mathworld.wolfram.com/CartesianProduct.html)

Normal-Form Representation of Games

The Prisoners' Dilemma: $G = < N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N} >$,

- $N = \{1, 2\}$ (where 1 = Prisoner 1 and 2 = Prisoner 2)
- $S_1 = S_2 = \{\text{c}, \bar{\text{c}}\}$ (where c = confess and $\bar{\text{c}}$ = don't confess)
- $u_1(\bar{\text{c}}, \bar{\text{c}}) = u_2(\bar{\text{c}}, \bar{\text{c}})$=-1, $u_1(\text{c}, \text{c}) = u_2(\text{c,c})$=-6
  $u_1(\bar{\text{c}}\text{,c})$ = -9; $u_2(\bar{\text{c}}\text{,c})$ = 0, $u_1(\text{c}, \bar{\text{c}})$=0; $u_2(\text{c}, \bar{\text{c}})$=-9
- $$\begin{array}{c|c|c|} & \bar{\text{c}} & \text{c} \\ \hline \bar{\text{c}} & u_1(\bar{\text{c}}, \bar{\text{c}}),\ u_2(\bar{\text{c}}, \bar{\text{c}}) & u_1(\bar{\text{c}}, \text{c}),\ u_2(\bar{\text{c}}, \text{c}) \\ \hline \text{c} & u_1(\text{c}, \bar{\text{c}}),\ u_2(\text{c}, \bar{\text{c}}) & u_1(\text{c}, \text{c}),\ u_2(\text{c}, \text{c}) \\ \hline \end{array} = \begin{array}{c|c|c|} & \bar{\text{c}} & \text{c} \\ \hline \bar{\text{c}} & -1, -1 & -9, 0 \\ \hline \text{c} & 0, -9 & -6, -6 \\ \hline \end{array}$$

Strictly Dominated Strategies

	x	y
a	1, –	1, –
b	1, –	0, –
c	0, –	0, –

- a strictly dominates c. Clearly, player 1 should not play the strictly dominated strategy c.
- Intuitively, rational players do not play strictly dominated strategies because there is no belief that a player could hold such that it would be optimal to play such a strategy.
- **Remark.** A rational player may not exclude playing a weakly dominated strategy! For example, a weakly dominates b and, if player 1 believes that player 2 is playing x, player 1 can play the weakly dominated strategy b.

Definition (Strictly Dominated Strategy)

A strategy s_i' of player i is *strictly dominated* by another strategy s_i'' of player i, if

$$u_i(s_i', s_{-i}) < u_i(s_i'', s_{-i}), \ \forall\, s_{-i} \in S_{-i},$$

i.e., for each feasible combination of the other players' strategies, i's payoff from playing s_i' is strictly less than player i's payoff from playing s_i''.

- A rational player never plays a strictly dominated strategy!
- In the Prisoner's Dilemma:

	$\bar{c}$	c
$\bar{c}$	-1, -1	-9, 0
c	0, -9	-6, -6

, for example, strategy "don't confess" is strictly dominated by "confess".

Iterated Elimination of Strictly Dominated Strategies

- Consider the game:

	Left	Middle	Right
Up	1,0	1,2	0,1
Down	0,3	0,1	2,0

1. For player 2, Right is strictly dominated by Middle. After eliminating Right, we obtain the reduced game:

	Left	Middle
Up	1,0	1,2
Down	0,3	0,1

2. Now, Down is strictly dominated by Up for player 1. After eliminating Down, we obtain the reduced game:

	Left	Middle
Up	1,0	1,2

3. Left is strictly dominated by Middle for player 2. After eliminating Left, we finally obtain the outcome:

	Middle
Up	1,2

Iterated Elimination of Strictly Dominated Strategies

- Iterated elimination of strictly dominated strategies (IESDS) one of the most basic principles in game theory. To analyze complex games, it is very useful to perform IESDS at first place.
- The notion of IESDS aims to be weak; it determines not what actions should actually be taken, but what actions could be ruled out with confidence. Some game may have no strictly dominated strategies.

Nash Equilibrium

To determine a solution for the question "what strategies should actually be taken?", it is necessary that each player be willing to choose the strategy suggested by the solution.

Definition (Nash equilibrium)

A strategy profile $s^* = (s_1^*, s_2^*, \cdots, s_n^*)$ is a *Nash equilibrium* if for each player i, s_i^* is a best response to s_{-i}^*, i.e., $\forall\, i = 1, \ldots, n$

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \;\forall s_i \in S_i,$$

or equivalently,

$$u_i(s_i^*, s_{-i}^*) = \max_{s_i \in S_i} \; u_i(s_i, s_{-i}^*).$$

That is, at Nash equilibrium, no player wants to unilaterally deviate from his strategy as long as the other players keep to theirs. (Unless explicitly noted otherwise, an equilibrium is referred to as a pure-strategy equilibrium.)

The concept of equilibrium

Nash Equilibrium: Interpretations

- **Interpretations:**
  1. Each player is doing the best, given others' actions
  2. No player can gain from a unilateral deviation, given others' actions
  3. Each player is rational (i.e. maximizes his/her own utility) and holds the correct belief about the opponents' actual actions

- **Remark:** Logically speaking, Nash equilibrium is not necessarily achieved because players are rational, but rather players are rational because an equilibrium prevails (from an epistemic point of view, why does each player know the opponents' using strategies?)

Nash Equilibrium: Justifications

- **Justifications**
  1. As a recommended play (by game theorist)
  2. As a self-enforcing agreement
  3. As a steady state of learning or evolution

- The notion of Nash equilibrium is probably the single most fundamental concept in game theory! It is also the most commonly used solution concept in game theory and economics.

Find Nash Equilibria

1. We can simply check each pair of strategies by verifying if no player wants to unilaterally deviate from his strategy.

Example 1:

	a	b
a	2, 2	0, 0
b	0, 0	1, 1

Nash equilibria: (a, a), (b, b). A game may have multiple Nash equilibria!

2. We may also underline the payoff to each player's best response to each of the opponent's strategies. A pair of strategies is a Nash equilibrium if both payoffs are underlined.

Example 2:

	L	C	R
T	0,$\underline{4}$	$\underline{4}$,0	5,3
M	$\underline{4}$,0	0,$\underline{4}$	5,3
B	3,5	3,5	$\underline{6}$,$\underline{6}$

Nash equilibrium: (B, R).

- *Remark.* Please always use IESDS to simplify the game at first place. This is often useful in finding equilibria in complex games.

Relationship between IESDS and Nash Equilibrium

- Consider the Prisoners' dilemma, for example:

	$\bar{c}$	c
$\bar{c}$	-1, -1	-9, 0
c	0, -9	-6, -6

Nash equilibrium: (c, c); **IESDS:** (c, c)

- **Observations:**
  1. A Nash equilibrium survives IESDS
     (We can find all Nash equilibria in the reduced game after IESDS.)
  2. A unique outcome resulting from IESDS is a Nash equilibrium
     (The unique outcome is also the unique Nash equilibrium.)

Relationship between IESDS and Nash Equilibrium*

- **Proposition:** *(1) Every Nash equilibrium survives IESDS.*
  (2) If IESDS leads to a unique outcome, then it is a Nash equilibrium.
- **Proof:** (1) Let s^* be a Nash equilibrium. Then, for each player i, s_i^* is a best response to s_{-i}^*, i.e., s_i^* cannot be strictly dominated by any strategy $s_i \in S_i$. Therefore, s^* survives in the 1st round of IESDS. By the same argument, s^* survives in the kth round of IESDS for $k \geq 2$.
- (2) Suppose, on the contrary, that the unique outcome s^* resulting from IESDS is not a Nash equilibrium. Since the game is finite, for some player i, there exists $\mathring{s}_i \in S_i$ such that

$$u_i\left(\mathring{s}_i, s_{-i}^*\right) = \max_{s_i \in S_i} u_i\left(s_i, s_{-i}^*\right) > u_i\left(s_i^*, s_{-i}^*\right). \tag{2.1}$$

Since $\mathring{s}_i$ must be eliminated at some stage $\mathring{k}$, there exists $s_i' \in S_i$ such that

$$u_i\left(s_i', s_{-i}\right) > u_i\left(\mathring{s}_i, s_{-i}\right), \tag{2.2}$$

for each s_{-i} remaining at stage $\mathring{k}$. But since s_{-i}^* survives IESDS, (2.2) implies $u_i\left(s_i', s_{-i}^*\right) > u_i\left(\mathring{s}_i, s_{-i}^*\right)$, which contradicts (2.1). ■

Example: Splitting a pie (or a dollar)

Players 1 and 2 simultaneously demand shares of a pie, s_1 and s_2 in $[0,1]$. If $s_1+s_2 \leq 1$, then each player receives his demanded share; if $s_1+s_2>1$, then both players receive zero. What are Nash equilibria of this game?

- The players of the game are the two players: $N=\{1,2\}$
- Player i's strategy set: $S_i=[0,1]$ with typical strategy $0 \leq s_i \leq 1$
- Player i's payoff function: $u_i\left(s_i, s_{-i}\right)=\begin{cases} s_i, & \text{if } s_1+s_2 \leq 1 \\ 0, & \text{if } s_1+s_2>1 \end{cases}$

Example: Splitting a pie (or a dollar)

- The best response for player 1: $\forall s_2 \in [0,1]$,

$$R_1(s_2) = \begin{cases} 1 - s_2, & \text{if } s_2 < 1 \\ [0,1], & \text{if } s_2 = 1 \end{cases}.$$

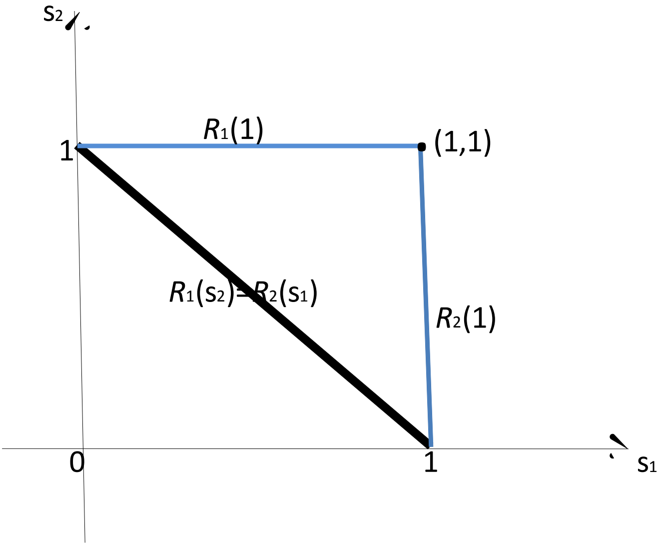
- The best response for player 2: $\forall s_1 \in [0,1]$,

$$R_2(s_1) = \begin{cases} 1 - s_1, & \text{if } s_1 < 1 \\ [0,1], & \text{if } s_1 = 1 \end{cases}.$$

- The set of Nash equilibria consists of intersections of $R_1(\cdot)$ and $R_2(\cdot)$:

$$\{s^* | s_1^* + s_2^* = 1,\ s_1^*, s_2^* \geq 0\} \cup \{(1,1)\}.$$

Example: Splitting a pie



Application: Cournot Duopoly

Two firms choose their quantities q_1 and q_2 (of a homogeneous product) simultaneously. The market-clearing price

$$P(Q) = \begin{cases} a - Q & \text{if } Q < a \\ 0 & \text{if } Q \geq a, \end{cases}$$

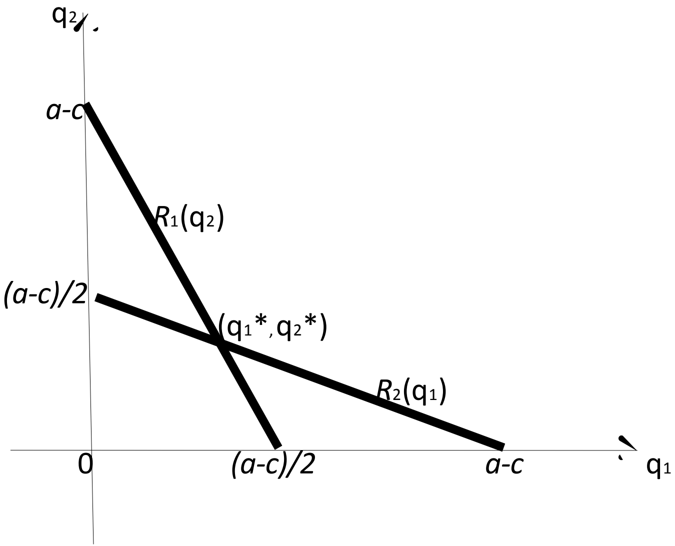
where $Q = q_1 + q_2$. The marginal cost is constant at c, where $0 < c < a$.

- The players of the game are the two firms: $N = \{1, 2\}$
- Firm i's strategy set: $S_i = [0, \infty)$ with typical strategy $q_i \geq 0$
- Firm i's payoff function $u_i\left(s_1, s_2\right)$: $\pi_i\left(q_1, q_2\right) = q_i[a - \left(q_1 + q_2\right) - c]$

Application: Cournot Duopoly

- By FOC, we have $\begin{cases} \frac{\partial \pi_1(q_1,q_2)}{\partial q_1} = a - c - 2q_1 - q_2 = 0 \\ \frac{\partial \pi_2(q_1,q_2)}{\partial q_2} = a - c - 2q_2 - q_1 = 0 \end{cases}$
- We obtain the players' best responses: $\begin{cases} q_1 = \frac{1}{2}(a - c - q_2) \\ q_2 = \frac{1}{2}(a - c - q_1) \end{cases}$
- **Nash equilibrium:** $(q_1^*, q_2^*) = \left(\frac{a-c}{3}, \frac{a-c}{3}\right)$

Application: Cournot Duopoly



Application: Cournot Duopoly

- We can solve the Cournot duopoly by IESDS as follows:
  1. The monopoly quantity $q_m = \frac{a-c}{2}$ strictly dominates any strategy $q > q_m$ (i.e. each firm should produce at most $\frac{a-c}{2}$ given that the other firm produces at least 0)
  2. After eliminating $q > q_m$, the quantity $\frac{a-c}{4}$ strictly dominates any strategy $q < \frac{a-c}{4}$ (i.e. each firm should produce at least $\frac{a-c}{4}$ given that the other firm produces at most $\frac{a-c}{2}$)
  3. After eliminating $q < \frac{a-c}{4}$, we have $q \in \left[\frac{a-c}{4}, \frac{a-c}{2}\right]$.
  4. Repeating these arguments, we have $q \in \left[\frac{5(a-c)}{16}, \frac{3(a-c)}{8}\right]$ (intuitively, by (3), each firm should produce at most $\frac{3(a-c)}{8}$ given that the other firm produces at least $\frac{a-c}{4}$, and each firm should produce at least $\frac{5(a-c)}{16}$ given that the other firm produces at most $\frac{3(a-c)}{8}$)
  5. In the limit, these intervals converge to $q^* = \frac{a-c}{3}$

Application: Bertrand Duopoly

Two firms choose their prices p_1 and p_2 (of differentiated products) simultaneously. The quantity that consumers demand from firm i is:

$$q_i\left(p_i, p_j\right) = a - p_i + bp_j$$

where $b > 0$ reflects the extent to which firm i's product is a substitute for firm j's product. The marginal cost c satisfies: $0 < c < a$.

- There are two players: firm 1 and firm 2
- Firm i's strategy set: $S_i = [0, \infty)$ with typical strategy $p_i \geq 0$
- Firm i's payoff function: $\pi_i\left(p_1, p_2\right) = \left(a - p_i + bp_j\right)\left(p_i - c\right)$

Application: Bertrand Duopoly

- The equilibrium price pair (p_1^*, p_2^*) solves:

$$\max_{0 \le p_i < \infty} \pi_i(p_i, p_j^*) = \max_{0 \le p_i < \infty} (a - p_i + bp_j^*)(p_i - c).$$

By FOC, we have

$$\begin{cases} p_1^* = \frac{1}{2}(a + bp_2^* + c) \\ p_2^* = \frac{1}{2}(a + bp_1^* + c) \end{cases}$$

- **Nash equilibrium:** $(p_1^*, p_2^*) = \left(\frac{a+c}{2-b}, \frac{a+c}{2-b}\right)$
(it makes sense only if $b < 2$!)

Example: Matching Pennies

	heads	tails
heads	-1, 1	1, -1
tails	1, -1	-1, 1

- Each player would like to outguess the other player's play and take an advantage over the opponent
- There is no (pure-strategy) Nash equilibrium in this game!
  (There is no (pure-strategy) Nash equilibrium in the Rock-Scissors-Paper!?)

Mixed Strategies

Definition

Suppose $S_i = \{s_{i1}, \ldots, s_{iK}\}$. A *mixed strategy* for player i is a probability distribution $p_i = (p_{i1}, \ldots, p_{iK})$, where $p_{i1} + \cdots + p_{iK} = 1$ and $p_{ik} \geq 0$.

- In Matching Pennies, for example, $S_i = \{\text{Heads}, \text{Tails}\}$, i.e., Heads and Tails are two pure strategies
- A mixed strategy for player i is the probability distribution $(q, 1-q)$, where $q \in [0, 1]$ is the probability of playing Heads and $1 - q$ is the probability of playing Tails
- The mixed strategy $p = (1, 0)$ can be viewed as the pure strategy Heads
  The mixed strategy $p = (0, 1)$ can be viewed as the pure strategy Tails

Mixed-Strategy Nash Equilibrium

Recall that Nash equilibrium is a strategy profile from which no unilateral deviation by any one player can improve this player's payoff. To extend the notion to mixed strategies, we simply require that each player's mixed strategy be a best response to the other players' mixed strategies.

- Let $S_1 = \{s_{11}, s_{12}, \ldots, s_{1J}\}$ and $S_2 = \{s_{21}, s_{22}, \ldots, s_{2K}\}$.
- For player 1's mixed strategy $p_1 = (p_{11}, p_{12}, \ldots, p_{1J})$ and player 2's mixed strategy $p_2 = (p_{21}, p_{22}, \ldots, p_{2K})$, player i's expected payoff is:

$$v_i(p_1, p_2) = \sum_{j=1}^{J} \sum_{k=1}^{K} p_{1j} p_{2k} \; u_i(s_{1j}, s_{2k}).$$

That is, $v_i(p_1, p_2)$ is the weighted sum of payoffs from all the pure strategy profiles, where the weight $p_{1j} p_{2k}$ is the probability of playing (s_{1j}, s_{2k}).

Mixed-Strategy Nash Equilibrium

Definition (Mixed-strategy Nash equilibrium)

A profile (p_1^*, p_2^*) is a *mixed-strategy Nash equilibrium* if

$$\forall i,\ v_i(p_i^*, p_{-i}^*) \geq v_i(p_i, p_{-i}^*)$$

for all probability distributions p_i on S_i.

- A profile p^* is a mixed-strategy Nash equilibrium if, and only if, for each player i, $v_i(p_i^*, p_{-i}^*) \geq v_i(s_{ik}, p_{-i}^*)\ \forall s_{ik} \in S_i$
- If a profile p^* is a mixed-strategy Nash equilibrium, then $v_i(s_{ik}^*, p_{-i}^*) = v_i(p_i^*, p_{-i}^*)$ where $p_{ik}^* > 0$
  (i.e. every pure strategy s_{ik}^* in the support of p_i^* is a best response to p_{-i}^*, thus, it yields a constant expected payoff to p_{-i}^*)

Find Nash Equilibria in Matching Pennies

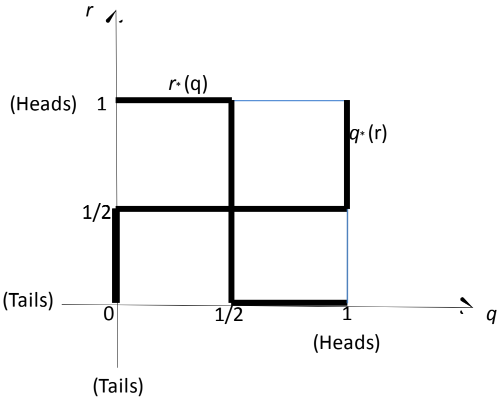
	heads	tails
heads	-1, 1	1, -1
tails	1, -1	-1, 1

  Let $p_1 = (r, 1-r)$ and $p_2 = (q, 1-q)$ where players 1 and 2 play Heads with probability r and q, respectively
- $\begin{cases} v_1(s_{11}, p_2) = q \cdot (-1) + (1-q) \cdot 1 = 1 - 2q \\ v_1(s_{12}, p_2) = q \cdot 1 + (1-q) \cdot (-1) = -1 + 2q \end{cases}$
- 1 chooses Heads, i.e., $r^*(q) = 1 \iff v_1(s_{11}, p_2) > v_1(s_{12}, p_2) \iff q < 1/2$. Hence,

$$r^*(q) = \begin{cases} 1, & \text{if } q < \frac{1}{2} \\ [0,1] & \text{if } q = \frac{1}{2} \\ 0, & \text{if } q > \frac{1}{2} \end{cases}$$

- Similarly, $$q^*(r) = \begin{cases} 0, & \text{if } r < \frac{1}{2} \\ [0,1] & \text{if } r = \frac{1}{2} \\ 1, & \text{if } r > \frac{1}{2} \end{cases}$$

Find Nash Equilibria in Matching Pennies



- **Nash Equilibrium:** $(p_1^*, p_2^*) = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$
  (choosing Tails and Heads with equal probability is the best strategy)

Mixed Nash Equilibrum Strategies: Why?

	heads (1/2)	tails (1/2)
heads (1/2)	-1, 1	1, -1
tails (1/2)	1, -1	-1, 1

Kicker vs. Goalkeeper

- Walker and Wooders, "Minimax Play at Wimbledon," *American Economic Review* (2001)**91**, 1521-1538.

Find Nash Equilibria in Matching Pennies: FOC

	heads	tails
heads	-1, 1	1, -1
tails	1, -1	-1, 1

  Let $p_1 = (r, 1-r)$ and $p_2 = (q, 1-q)$ where players 1 and 2 play Heads with probability r and q, respectively. The payoff functions for players 1 and 2 are as follows:

- $\begin{cases} v_1(p_1, p_2) = -rq + r(1-q) + (1-r)q - (1-r)(1-q) \\ v_2(p_1, p_2) = rq - r(1-q) - (1-r)q + (1-r)(1-q) \end{cases}$

- By FOC, we have $\begin{cases} -q + (1-q) - q + (1-q) = 0 \\ r + r - (1-r) - (1-r) = 0 \end{cases}$.

- Hence, $\begin{cases} q^* = \frac{1}{2} \\ r^* = \frac{1}{2} \end{cases}$. **NE:** $(p_1^*, p_2^*) = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.

Find a Nash Equilibrium: Procedures

1. Simplify the game by using IESDS
   (every Nash equilibrium survives IESDS)
2. Calculate the best-response correspondences $r^*(q)$ and $q^*(r)$
3. Take the intersection of correspondences $r^*(q)$ and $q^*(r)$
   (Nash equilibria $=$ intersection of $r^*(q)$ and $q^*(r)$)

- **Nash's (1950) Theorem:**
  Every finite normal-form game has at least one Nash equilibrium, possibly involving mixed strategies.
  (The proof follows directly from Kakutani's fixed point theorem.)

Remark: IESDS under Mixed Strategies

	x	y
a	3, –	0, –
b	0, –	3, –
c	1, –	1, –

	x	y
a	3, –	0, –
b	0, –	3, –
c	2, –	2, –

- In the left game, strategy c can be strictly dominated *only* by a mixed strategy, e.g. $\frac{1}{2}a + \frac{1}{2}b$ (c is not dominated by any pure strategy)
- In the right game, strategy c can be a best response *only* to a mixed strategy, e.g. $\frac{1}{2}x + \frac{1}{2}y$ (c is not a best reply to any pure strategy)
- **Observation:** *A strategy is strictly dominated iff it is a never-best response.*

- We can define a stronger notion of IESDS as follows:
  At each stage of IESDS, we eliminate all the strategies that are strictly dominated by a pure *or mixed strategy*.

Find Nash Equilibria: Another Example

In the following 3-person game, players 1, 2, and 3 pick the row, column, and matrix, respectively:

x

	a	b	c
a	0, 1, 0	3, 3, 1	3, 0, 3
b	3, 3, 1	0, 0, 1	3, 1, 3
c	3, 3, 2	3, 3, 2	2, 1, 3

y

	a	b	c
a	0, 3, 1	3, 1, 2	3, 1, 0
b	2, 1, 2	0, 3, 2	2, 0, 0
c	1, 2, 3	1, 2, 3	3, 0, 0

- There is a unique NE: $\left(\frac{1}{2}\text{a} + \frac{1}{2}\text{b},\ \frac{3}{5}\text{a} + \frac{2}{5}\text{b},\ \text{y}\right)$
- The reasons are as follows:
  1. [for player 2] c is strictly dominated by a,
  2. [for player 3] after eliminating c for 2, x is strictly dominated by y,
  3. [for player 1] after eliminating x for 3, c is strictly dominated by $(0.5 - \epsilon)$a$+ (0.5 + \epsilon)$b, where $\epsilon > 0$ is sufficiently small.

More Elaboration

- **Claim:** If p^* is a mixed NE, then $v_i(s^*_{ik}, p^*_{-i}) = v_i(p^*_i, p^*_{-i})$ for $p^*_{ik} > 0$. (note: $p^*_i = (p^*_{i1}, \ldots, p^*_{iK})$ can be written as: $p^*_{i1}s^*_{i1} + \cdots + p^*_{iK}s^*_{iK}$)
- E.g., consider the mixed NE $p^* = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$ in Matching Pennies:

	heads	tails
heads	-1, 1	1, -1
tails	1, -1	-1, 1

Clearly, given that player 2 plays $\frac{1}{2}$heads$+\frac{1}{2}$tails, player 1's expected payoff is constant (0). Thus, $v_1(s^*_{1k}, p^*_2) = 0 = v_1(p^*_1, p^*_2)$ for s^*_{1k} = heads/tails.

- We can use this claim to show that p^* is a unique NE in Matching Pennies. By Claim, a *totally mixed* NE $p = ((p_{11}, p_{12}), (p_{21}, p_{22}))$ must satisfy

$$\begin{cases} (-1) \times p_{21} + 1 \times p_{22} = 1 \times p_{21} + (-1) \times p_{22} \\ 1 \times p_{11} + (-1) \times p_{12} = (-1) \times p_{21} + 1 \times p_{22} \end{cases}$$

The unique solution of the two equations is: $p^* = ((\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}))$.

Summary

- **Model:** Normal-form representation
- **Solution:** Nash equilibrium/IESDS
- **Applications:** Cournot & Bertrand duopolies
- **Others:** Mixed-strategy Nash equilibrium; Nash's Theorem