Applied Math III
(MATH 2051)
Class 7

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Wronskian (Conti...)

Theorem 3.2.5

Consider the differential equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

whose coefficients p and q are continuous on some open interval I. Choose some point t_0 in I. Let y_1 be the solution of Eq. (2) that also satisfies the initial conditions

$$y(t_0) = 1, \qquad y'(t_0) = 0,$$

and let y_2 be the solution of Eq. (2) that satisfies the initial conditions

$$y(t_0) = 0, \qquad y'(t_0) = 1.$$

Then y_1 and y_2 form a fundamental set of solutions of Eq. (2).

Conti...

First observe that the *existence* of the functions y_1 and y_2 is ensured by the existence part of Theorem 3.2.1. To show that they form a fundamental set of solutions, we need only calculate their Wronskian at t_0:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1.$$

Since their Wronskian is not zero at the point t_0, the functions y_1 and y_2 do form a fundamental set of solutions, thus completing the proof of Theorem 3.2.5.

What if we encounter equations that have complex-valued solutions?

Initial Value Problem

Theorem 3.2.6

Consider again the equation (2),

$$L[y] = y'' + p(t)y' + q(t)y = 0,$$

where p and q are continuous real-valued functions. If $y = u(t) + iv(t)$ is a complex-valued solution of Eq. (2), then its real part u and its imaginary part v are also solutions of this equation.

Conti…

Theorem 3.2.7

(Abel's Theorem)[4]

If y_1 and y_2 are solutions of the differential equation

$$L[y] = y'' + p(t)y' + q(t)y = 0, \tag{22}$$

where p and q are continuous on an open interval I, then the Wronskian $W(y_1, y_2)(t)$ is given by

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)\, dt\right], \tag{23}$$

where c is a certain constant that depends on y_1 and y_2, but not on t. Further, $W(y_1, y_2)(t)$ either is zero for all t in I (if $c = 0$) or else is never zero in I (if $c \neq 0$).

Conti…

EXAMPLE

We know that $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions of the equation

$$2t^2y'' + 3ty' - y = 0, \qquad t > 0. \tag{29}$$

Verify that the Wronskian of y_1 and y_2 is given by Eq. (23). i.e by Abel's Theorem

From the example just cited we know that $W(y_1, y_2)(t) = -(3/2)t^{-3/2}$. To use Eq. (23), we must write the differential equation (29) in the standard form with the coefficient of y'' equal to 1. Thus we obtain

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0,$$

so $p(t) = 3/2t$. Hence

$$W(y_1, y_2)(t) = c\,\exp\left[-\int \frac{3}{2t}\,dt\right] = c\,\exp\left(-\frac{3}{2}\ln t\right)$$
$$= c\,t^{-3/2}. \tag{30}$$

Equation (30) gives the Wronskian of any pair of solutions of Eq. (29). For the particular solutions given in this example, we must choose $c = -3/2$.

Complex Roots of the Characteristic Equation

We continue our discussion of the equation

$$ay'' + by' + cy = 0, \tag{1}$$

where a, b, and c are given real numbers. In Section 3.1 we found that if we seek solutions of the form $y = e^{rt}$, then r must be a root of the characteristic equation

$$ar^2 + br + c = 0. \tag{2}$$

We showed in Section 3.1 that if the roots r_1 and r_2 are real and different, which occurs whenever the discriminant $b^2 - 4ac$ is positive, then the general solution of Eq. (1) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \tag{3}$$

Suppose now that $b^2 - 4ac$ is negative. Then the roots of Eq. (2) are conjugate complex numbers; we denote them by

$$r_1 = \lambda + i\mu, \qquad r_2 = \lambda - i\mu, \tag{4}$$

where λ and μ are real. The corresponding expressions for y are

$$y_1(t) = \exp[(\lambda + i\mu)t], \qquad y_2(t) = \exp[(\lambda - i\mu)t]. \tag{5}$$

Conti…

Our first task is to explore what is meant by these expressions, which involve evaluating the exponential function for a complex exponent. For example, if $\lambda = -1$, $\mu = 2$, and $t = 3$, then from Eq. (5),

$$y_1(3) = e^{-3+6i}. \tag{6}$$

What does it mean to raise the number e to a complex power? The answer is provided by an important relation known as Euler's formula.

Euler's Formula. To assign a meaning to the expressions in Eqs. (5), we need to give a definition of the complex exponential function. Of course, we want the definition to reduce to the familiar real exponential function when the exponent is real. There are several ways to discover how this extension of the exponential function should be defined. Here we use a method based on infinite series; an alternative is outlined in Problem 28.

Recall from calculus that the Taylor series for e^t about $t = 0$ is

$$e^t = \sum_{n=0}^{\infty} \frac{t^n}{n!}, \qquad -\infty < t < \infty. \tag{7}$$

Conti...

If we now assume that we can substitute it for t in Eq. (7), then we have

$$
\begin{aligned}
e^{it} &= \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!} + i \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^{2n-1}}{(2n-1)!}, \qquad (8)
\end{aligned}
$$

where we have separated the sum into its real and imaginary parts, making use of the fact that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so forth. The first series in Eq. (8) is precisely the Taylor series for $\cos t$ about $t = 0$, and the second is the Taylor series for $\sin t$ about $t = 0$. Thus we have

$$e^{it} = \cos t + i \sin t. \qquad (9)$$

Equation (9) is known as Euler's formula and is an extremely important mathematical relationship.

Conti…

There are some variations of Euler's formula that are also worth noting. If we replace t by $-t$ in Eq. (9) and recall that $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$, then we have

$$e^{-it} = \cos t - i \sin t. \tag{10}$$

Further, if t is replaced by μt in Eq. (9), then we obtain a generalized version of Euler's formula, namely,

$$e^{i\mu t} = \cos \mu t + i \sin \mu t. \tag{11}$$

Next, we want to extend the definition of the exponential function to arbitrary complex exponents of the form $(\lambda + i\mu)t$. Since we want the usual properties of the exponential function to hold for complex exponents, we certainly want $\exp[(\lambda + i\mu)t]$ to satisfy

$$e^{(\lambda+i\mu)t} = e^{\lambda t} e^{i\mu t}. \tag{12}$$

Conti…

Then, substituting for $e^{i\mu t}$ from Eq. (11), we obtain

$$\begin{aligned} e^{(\lambda+i\mu)t} &= e^{\lambda t}(\cos \mu t + i \sin \mu t) \\ &= e^{\lambda t}\cos \mu t + ie^{\lambda t}\sin \mu t. \end{aligned} \tag{13}$$

EXAMPLE

Find the general solution of the differential equation

$$y'' + y' + 9.25y = 0, \tag{15}$$

Also find the solution that satisfies the initial conditions

$$y(0) = 2, \qquad y'(0) = 8, \tag{16}$$

and draw its graph.

The characteristic equation for Eq. (15) is

$$r^2 + r + 9.25 = 0$$

so its roots are

$$r_1 = -\tfrac{1}{2} + 3i, \qquad r_2 = -\tfrac{1}{2} - 3i.$$

Conti…

Therefore, two solutions of Eq. (15) are

$$y_1(t) = \exp\left[\left(-\tfrac{1}{2} + 3i\right)t\right] = e^{-t/2}(\cos 3t + i\sin 3t) \tag{17}$$

and

$$y_2(t) = \exp\left[\left(-\tfrac{1}{2} - 3i\right)t\right] = e^{-t/2}(\cos 3t - i\sin 3t). \tag{18}$$

You can verify that the Wronskian $W(y_1, y_2)(t) = -6ie^{-t}$, which is not zero, so the general solution of Eq. (15) can be expressed as a linear combination of $y_1(t)$ and $y_2(t)$ with arbitrary coefficients.

However, the initial value problem (15), (16) has only real coefficients, and it is often desirable to express the solution of such a problem in terms of real-valued functions. To do this we can make use of Theorem 3.2.6, which states that the real and imaginary parts of a complex-valued solution of Eq. (15) are also solutions of Eq. (15). Thus, starting from either $y_1(t)$ or $y_2(t)$, we obtain

$$u(t) = e^{-t/2}\cos 3t, \qquad v(t) = e^{-t/2}\sin 3t \tag{19}$$

Conti…

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$$u(t) = e^{-t/2}\cos 3t, \qquad v(t) = e^{-t/2}\sin 3t \tag{19}$$

as real-valued solutions[5] of Eq. (15). On calculating the Wronskian of $u(t)$ and $v(t)$, we find that $W(u, v)(t) = 3e^{-t}$, which is not zero; thus $u(t)$ and $v(t)$ form a fundamental set of solutions, and the general solution of Eq. (15) can be written as

$$y = c_1u(t) + c_2v(t) = e^{-t/2}(c_1\cos 3t + c_2\sin 3t), \tag{20}$$

where c_1 and c_2 are arbitrary constants.

To satisfy the initial conditions (16), we first substitute $t = 0$ and $y = 2$ in Eq. (20) with the result that $c_1 = 2$. Then, by differentiating Eq. (20), setting $t = 0$, and setting $y' = 8$, we obtain $-\frac{1}{2}c_1 + 3c_2 = 8$, so that $c_2 = 3$. Thus the solution of the initial value problem (15), (16) is

$$y = e^{-t/2}(2\cos 3t + 3\sin 3t). \tag{21}$$

The graph of this solution is shown in Figure 3.3.1.

From the graph we see that the solution of this problem is a decaying oscillation. The sine and cosine factors control the oscillatory nature of the solution, and the negative exponential factor in each term causes the magnitude of the oscillations to diminish as time increases.

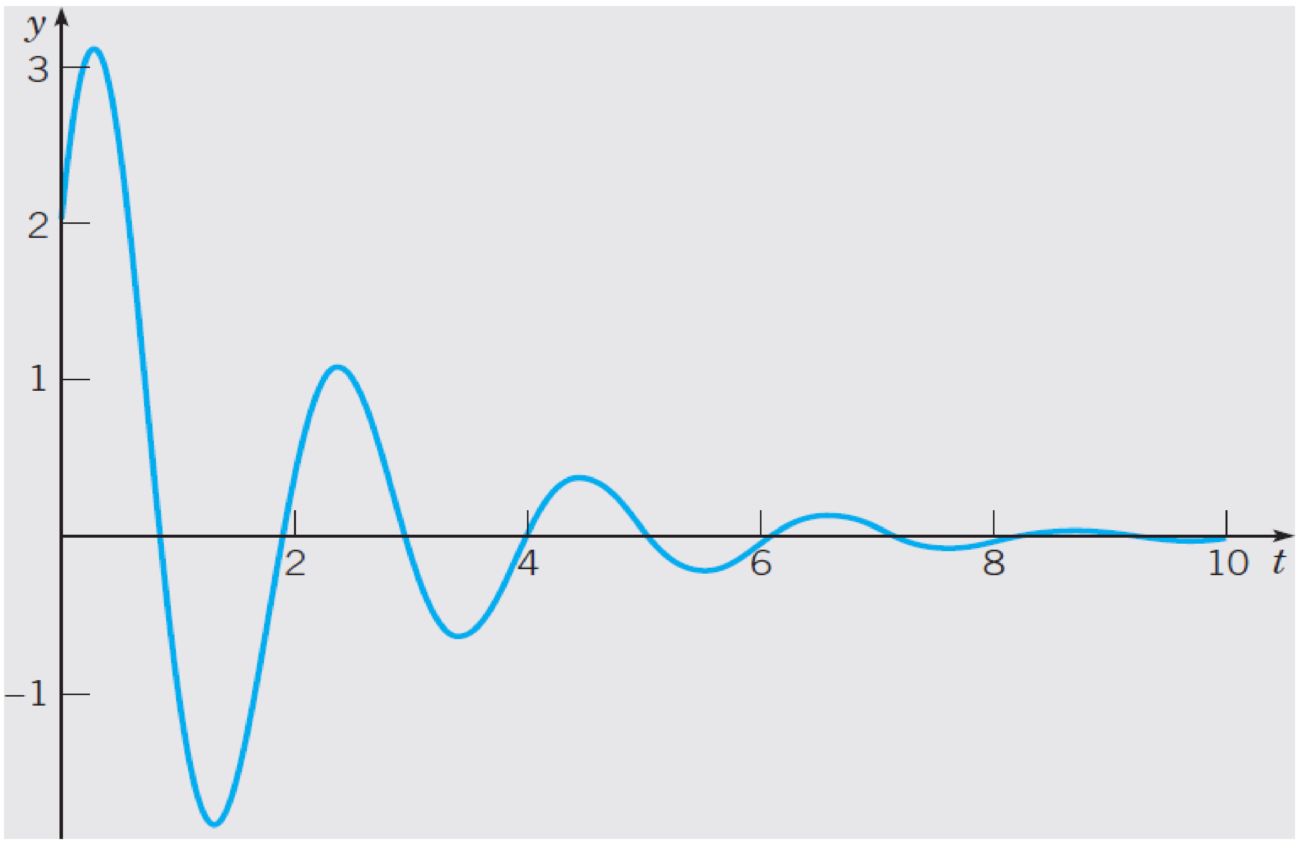

FIGURE 3.3.1 Solution of the initial value problem (15), (16): $y'' + y' + 9.25y = 0, \quad y(0) = 2, \quad y'(0) = 8.$

Repeated Roots; Reduction of Order

In earlier sections we showed how to solve the equation

$$ay'' + by' + cy = 0 \tag{1}$$

when the roots of the characteristic equation

$$ar^2 + br + c = 0 \tag{2}$$

either are real and different or are complex conjugates. Now we consider the third possibility, namely, that the two roots r_1 and r_2 are equal. This case is transitional between the other two and occurs when the discriminant $b^2 - 4ac$ is zero. Then it follows from the quadratic formula that

$$r_1 = r_2 = -b/2a. \tag{3}$$

The difficulty is immediately apparent; both roots yield the same solution

$$y_1(t) = e^{-bt/2a} \tag{4}$$

of the differential equation (1), and it is not obvious how to find a second solution.

EXAMPLE

Solve the differential equation

$$y'' + 4y' + 4y = 0. \tag{5}$$

The characteristic equation is

$$r^2 + 4r + 4 = (r+2)^2 = 0,$$

so $r_1 = r_2 = -2$. Therefore, one solution of Eq. (5) is $y_1(t) = e^{-2t}$. To find the general solution of Eq. (5), we need a second solution that is not a constant multiple of y_1

How?

Eq. (1) satisfy $b^2 - 4ac = 0$, in which case

$$y_1(t) = e^{-bt/2a}$$

The second solution is given by $\quad y_2(t) = te^{-bt/2a}.$

Hence, $\quad y = c_1 e^{-bt/2a} + c_2 t e^{-bt/2a}.\quad$ **is the general solution**

EXAMPLE

Find the solution of the initial value problem

$$y'' - y' + 0.25y = 0, \qquad y(0) = 2, \quad y'(0) = \tfrac{1}{3}. \tag{21}$$

The characteristic equation is

$$r^2 - r + 0.25 = 0,$$

so the roots are $r_1 = r_2 = 1/2$. Thus the general solution of the differential equation is

$$y = c_1 e^{t/2} + c_2 t e^{t/2}. \tag{22}$$

The first initial condition requires that

$$y(0) = c_1 = 2.$$

To satisfy the second initial condition, we first differentiate Eq. (22) and then set $t = 0$. This gives

$$y'(0) = \tfrac{1}{2}c_1 + c_2 = \tfrac{1}{3},$$

so $c_2 = -2/3$. Thus the solution of the initial value problem is

$$y = 2e^{t/2} - \tfrac{2}{3}te^{t/2}. \tag{23}$$

Conti…

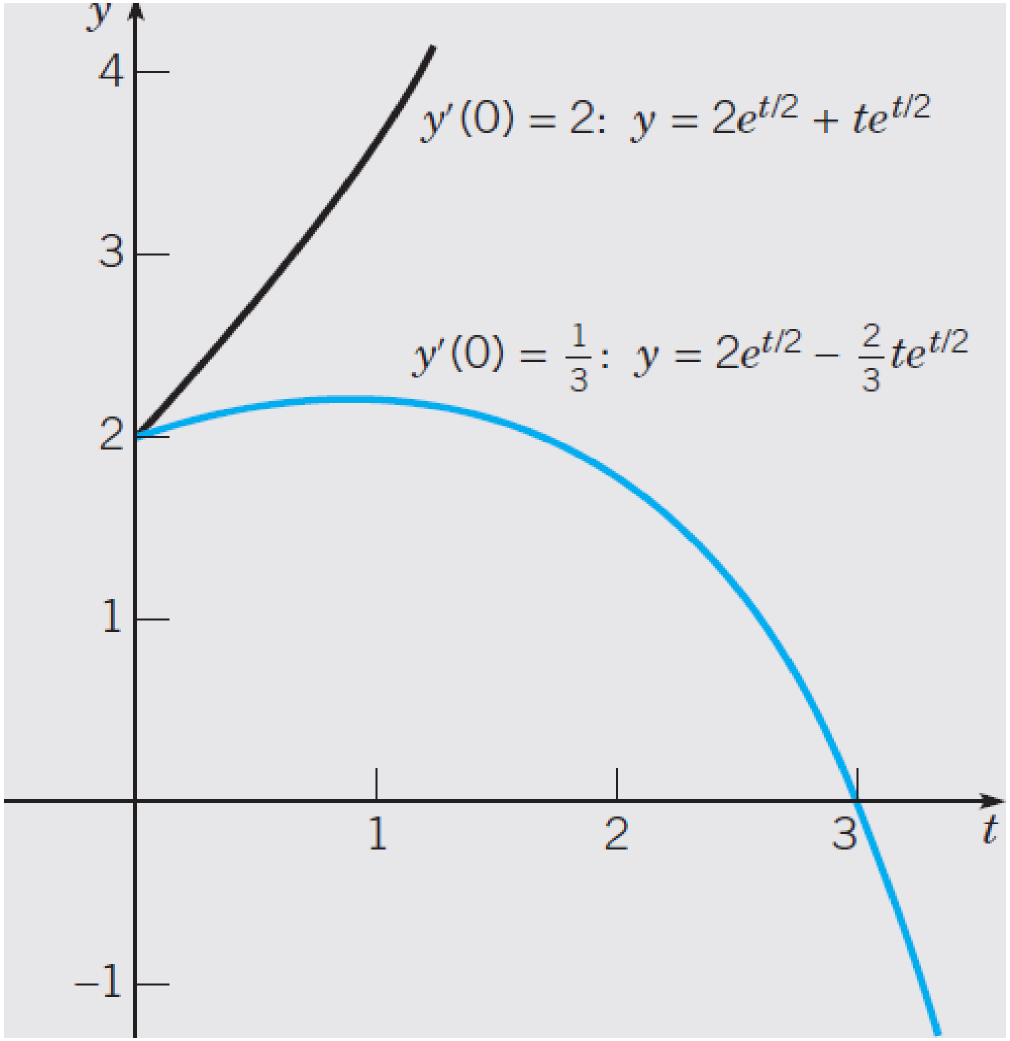

FIGURE 3.4.2 Solutions of $y'' - y' + 0.25y = 0$, $y(0) = 2$, with $y'(0) = 1/3$ (blue curve) and with $y'(0) = 2$ (black curve), respectively.

Thank You!