Calcul numérique des solides et structures non linéaires

S. Brisard, D. Duhamel, C. Maurini, S. Neukirch

Third course

Introduction to Non linear elasticity

Some characteristics of non linear mechanics

- In non linear mechanics, stresses and problems can be defined on the deformed or reference configurations while for linear problems everything is defined on the reference configuration

- This leads to several stress tensors

- The stress tensors are generally obtained from the strain by a non linear relation

- Problems can only be solved by iterative process

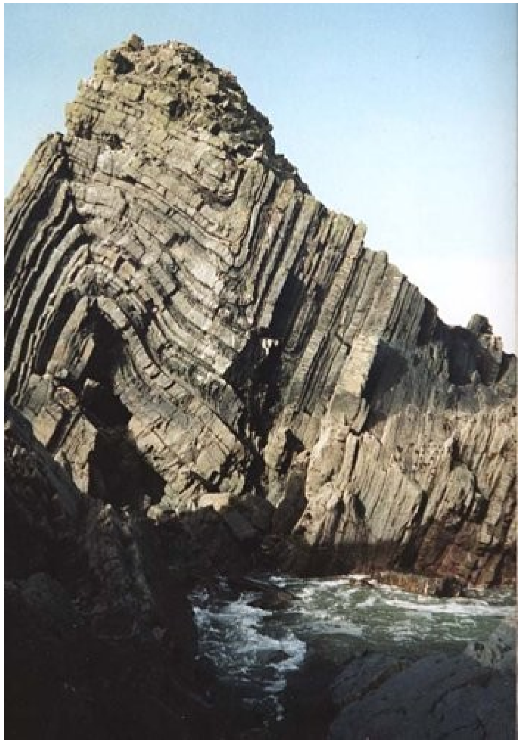
Overview

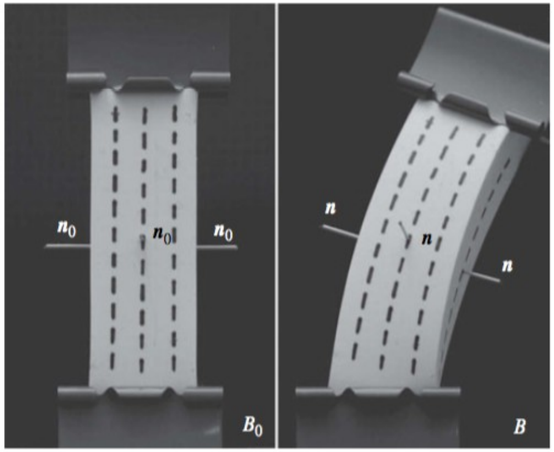
1. Kinematics
2. Stress tensors
3. Derivatives
4. Constitutive laws
5. Hyperelastic materials
6. Minimisation of the potential energy
7. Linearisation
8. FEM implementation
9. References

1. Kinematics

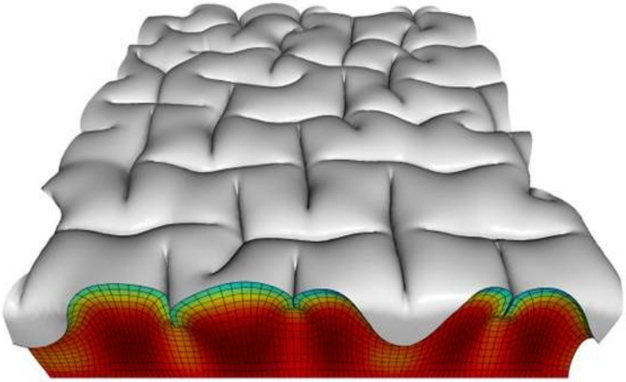
Large deformations

Large strain and large stress

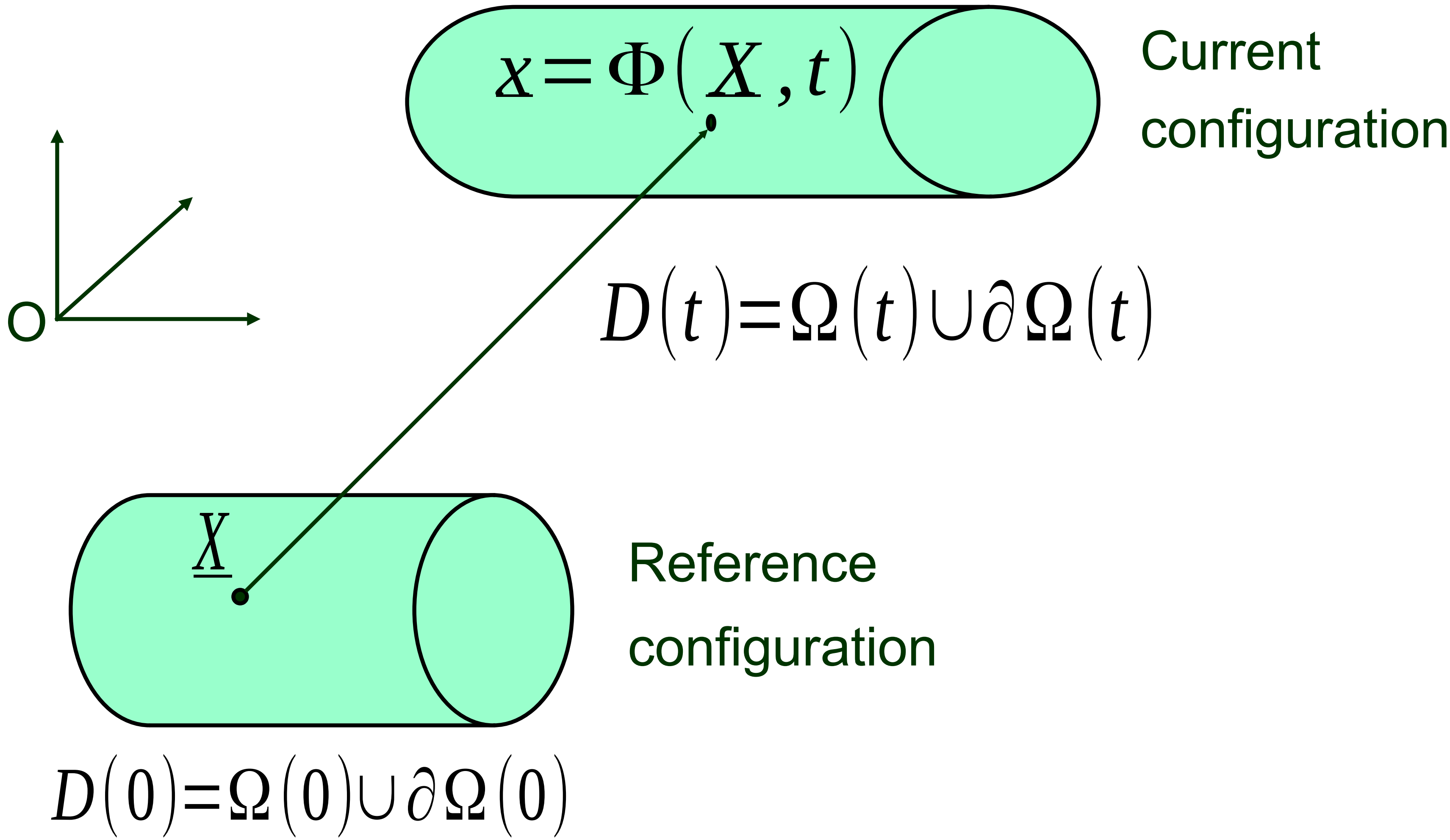




Hyperelastic blocs in large deformations

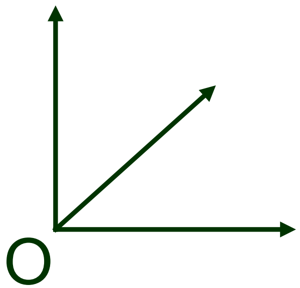


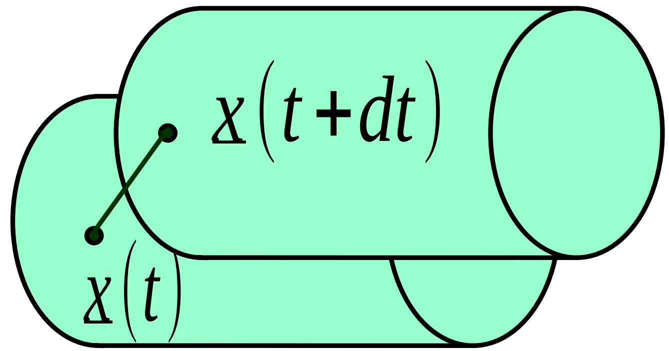
Lagrangian description



Eulerian description

$$D\left(t+dt\right)=\Omega\left(t+dt\right)\cup\partial\Omega\left(t+dt\right)$$

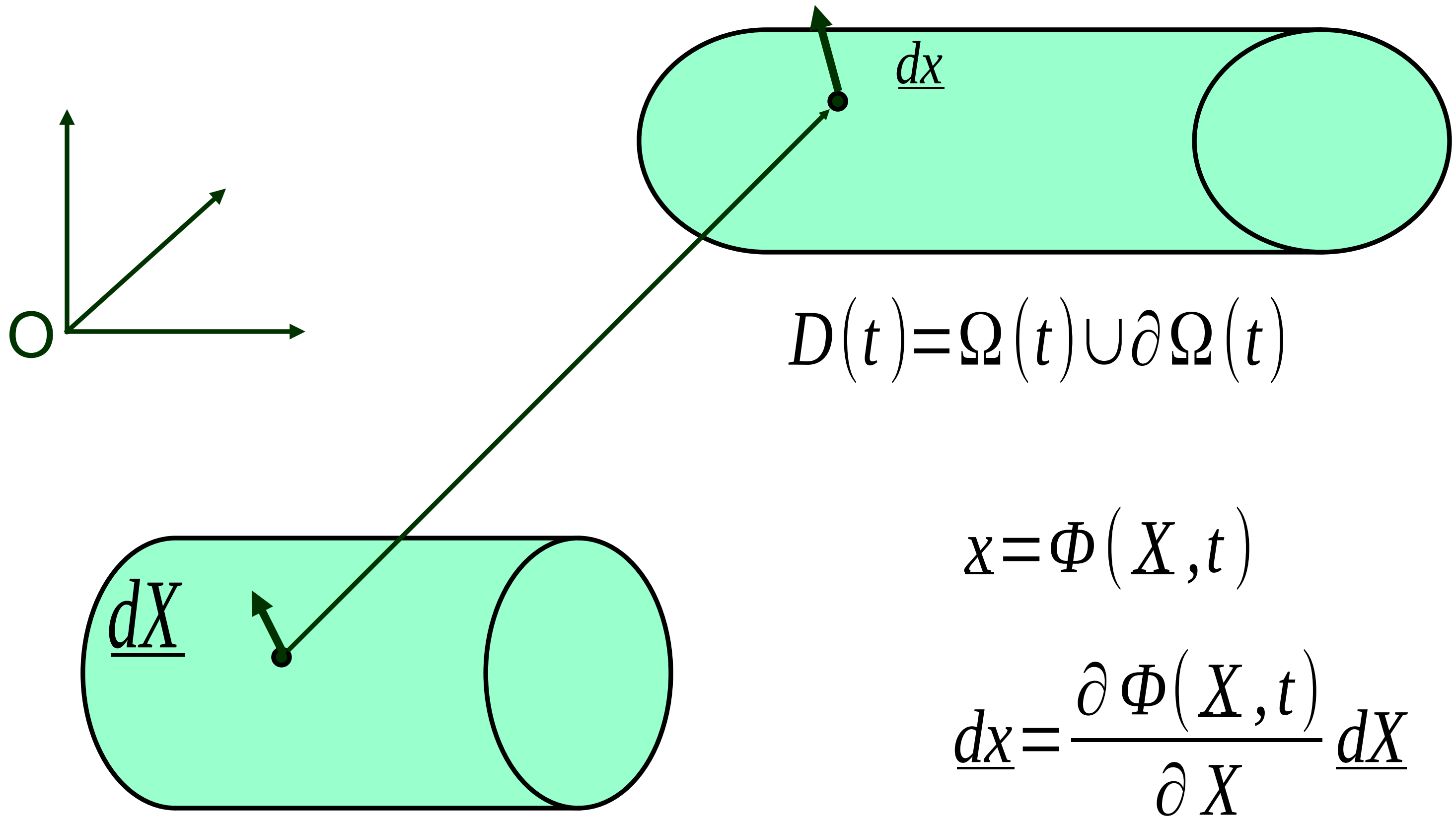




$$D\left(t\right)=\Omega\left(t\right)\cup\partial\Omega\left(t\right)$$

Velocity $\underline{U}\left(x,t\right)$ defined on the current configuration D(t)

Gradient of the transformation (Lagrange)



$$D(t)=\Omega(t)\cup\partial\Omega(t)$$

$$D(0)=\Omega(0)\cup\partial\Omega(0)$$

$$\underline{x}=\Phi\left(\underline{X},t\right)$$

$$\underline{dx}=\frac{\partial\,\Phi\left(\underline{X},t\right)}{\partial\,\underline{X}}\underline{dX}$$

$$\underline{\underline{F}}=\frac{\partial\,\Phi\left(\underline{X},t\right)}{\partial\,\underline{X}}$$ gradient

Strain tensors

$$\underline{x} = \underline{X} + \underline{\xi} \quad \left(= \Phi\left(\underline{X}, t\right)\right)$$

$$\underline{\underline{F}} = \underline{\underline{1}} + \underline{\underline{\nabla \xi}}$$

displacement $\underline{\xi}$

$$\underline{\underline{C}} = {}^t\underline{\underline{F}} \cdot \underline{\underline{F}}$$

Large strain

$$\begin{aligned}
\underline{\underline{e}} &= \frac{1}{2}\left({}^t\underline{\underline{F}} \cdot \underline{\underline{F}} - \underline{\underline{1}}\right) \\
&= \frac{1}{2}\left({}^t\left(\underline{\underline{1}} + \underline{\underline{\nabla \xi}}\right) \cdot \left(\underline{\underline{1}} + \underline{\underline{\nabla \xi}}\right) - \underline{\underline{1}}\right) \\
&= \frac{1}{2}\left(\underline{\underline{\nabla \xi}} + {}^t\underline{\underline{\nabla \xi}} + {}^t\underline{\underline{\nabla \xi}} \cdot \underline{\underline{\nabla \xi}}\right)
\end{aligned}$$

Small strain

$$\underline{\underline{\nabla \xi}} \ll 1$$

$$\underline{\underline{\epsilon}} = \frac{1}{2}\left(\underline{\underline{\nabla \xi}} + {}^t\underline{\underline{\nabla \xi}}\right)$$

Transport of vectors and volumes

$$\underline{dx} = \underline{\underline{F}} . \underline{dX}$$

Vector

$$dV = J\left(\underline{X}, t\right) dV_0$$

Volume

$$J\left(\underline{X}, t\right) = \det\left(\underline{\underline{F}}\left(\underline{X}, t\right)\right)$$

$$\underline{n}\, da = J\, {}^{t}\underline{\underline{F}}^{-1} . \underline{N}\, dA$$

Surface

Time derivative

Lagrangian

- Velovity $$\underline{v}(\underline{X},t)=\frac{\partial\Phi(\underline{X},t)}{\partial t}$$

- Acceleration $$\underline{a}(\underline{X},t)=\frac{\partial^2\Phi(\underline{X},t)}{\partial t^2}$$

Eulerian

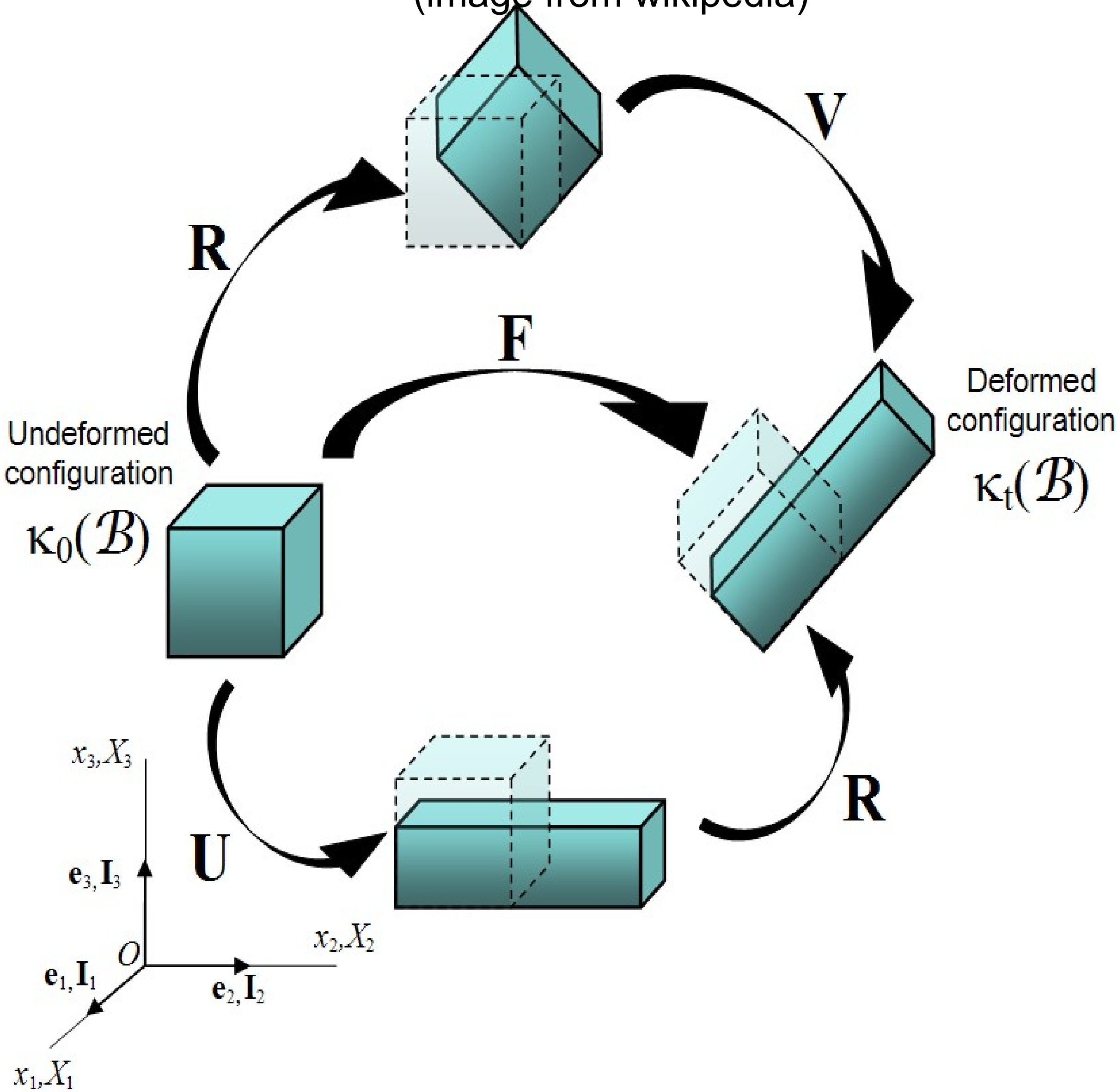
Material derivative

$$\frac{Df(\underline{x},t)}{Dt}=\frac{\partial f(\underline{x},t)}{\partial t}+\underline{\mathrm{grad}}\,f\,.\,\underline{v}$$

$$\frac{D\underline{v}(\underline{x},t)}{Dt}=\frac{\partial\underline{v}(\underline{x},t)}{\partial t}+\underline{\underline{\mathrm{grad}}}\,\underline{v}\,.\,\underline{v}$$

Polar Decomposition

(image from wikipedia)



Separate deformation from rigid-body rotation

Unique decomposition of deformation gradient

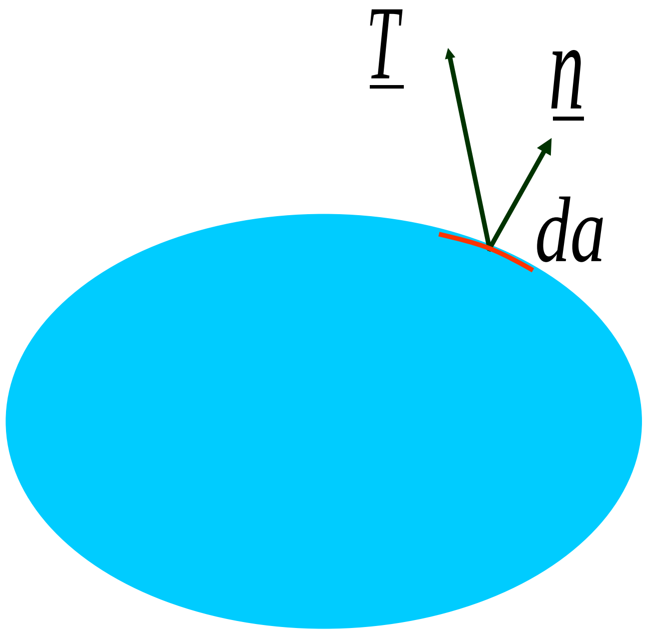
$$\underline{\underline{F}} = \underline{\underline{R}} \cdot \underline{\underline{U}} = \underline{\underline{V}} \cdot \underline{\underline{R}}$$

$\underline{\underline{R}}$: orthogonal tensor (rigid-body rotation)
$\underline{\underline{U}}$, $\underline{\underline{V}}$: right- and left-stretch tensor (symmetric)
$\underline{\underline{U}}$ and $\underline{\underline{V}}$ have the same eigenvalues but different eigenvectors

2. Stress tensors

Stress tensor : a tool to compute a force vector from a surface element

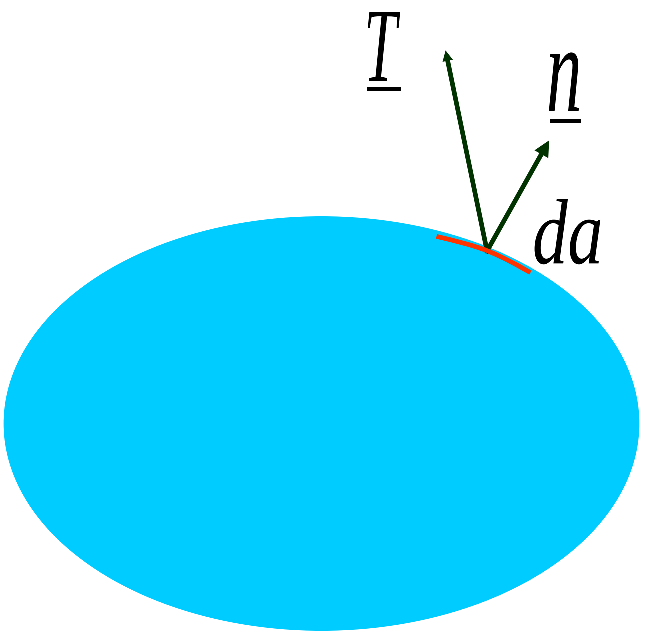
Force of the exterior domain on the surface element da



The force and surface element can be in the reference or current configurations

=> different stress tensors

Cauchy stress tensor $\underline{\underline{\sigma}}$



Equilibrium equations

$$\operatorname{div} \underline{\underline{\sigma}} + \underline{b} = 0 \quad in \quad \Omega$$

$$\underline{\underline{\sigma}} \,.\, \underline{n} = \underline{g} \quad on \quad \partial \Omega_g$$

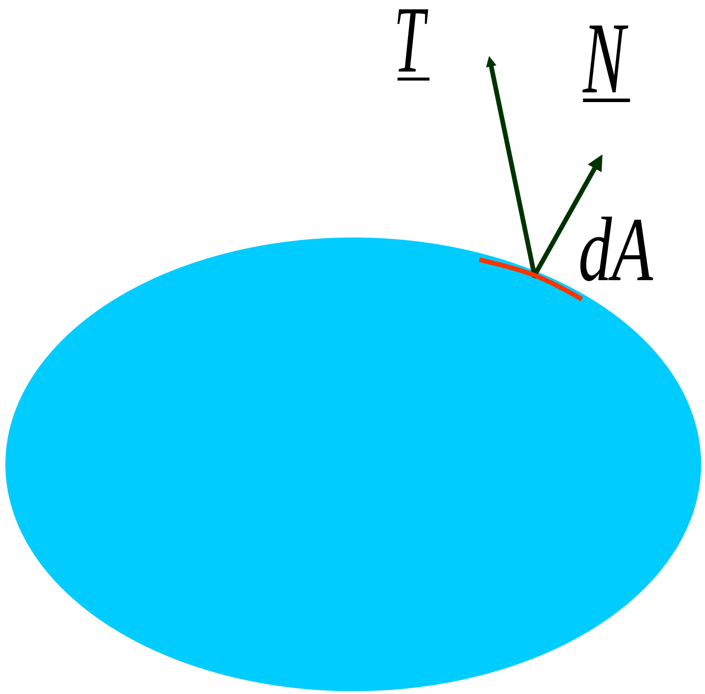
One must note that the problem
is written on the current configuration
which is unknown

Force and normal in the
current configuration

$$\underline{T}\, da = \underline{\underline{\sigma}} \,.\, \underline{n}\, da$$

$\underline{\underline{\sigma}}$ symmetric

First Piola-Kirchhoff stress tensor $\underline{\underline{P}}$



Equilibrium equations

$$\mathrm{DIV}\ \underline{\underline{P}} + \underline{b}_0 = 0 \quad \text{in} \quad \Omega_0$$
$$\underline{\underline{P}} \,.\, \underline{N} = \underline{g}_0 \quad on \quad \partial_g \Omega_0$$

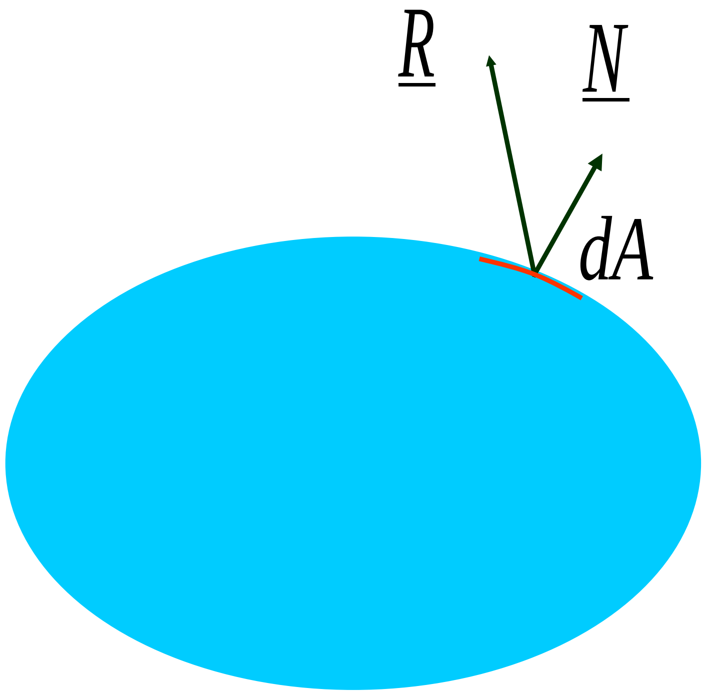
The reference configuration is known

Force in the current configuration and normal in the reference configuration

$$\underline{T}\, da = \underline{\underline{P}} \,.\, \underline{N}\, dA$$

$\underline{\underline{P}}$ Not symmetric

Second Piola-Kirchhoff stress tensor $\underline{\underline{S}}$



Force and normal in the reference configuration

$$\underline{R}\,dA = \underline{\underline{S}}\,.\,\underline{N}\,dA$$

$\underline{\underline{S}}$ symmetric

Relations between these tensors

$$\begin{cases} \underline{dx} = \underline{\underline{F}} \,.\, \underline{dX} \\ \underline{n}\, da = J\, {}^t\underline{\underline{F}}^{-1} \,.\, \underline{N}\, dA \end{cases}$$

Transport relations

$$\begin{cases} \underline{R}\, dA = \underline{\underline{S}} \,.\, \underline{N}\, dA = \underline{\underline{F}}^{-1} \,.\, \underline{T}\, da = J\, \underline{\underline{F}}^{-1} \,.\, \underline{\underline{\sigma}} \,.\, {}^t\underline{\underline{F}}^{-1}\, \underline{N}\, dA \\ \boxed{\underline{\underline{S}} = J\, \underline{\underline{F}}^{-1} \,.\, \underline{\underline{\sigma}} \,.\, {}^t\underline{\underline{F}}^{-1}} \end{cases}$$

$$\begin{cases} \underline{\underline{P}} \,.\, \underline{N}\, dA = \underline{T}\, da = \underline{\underline{F}} \,.\, \underline{R}\, dA = \underline{\underline{F}} \,.\, \underline{\underline{S}} \,.\, \underline{N}\, dA \\ \boxed{\underline{\underline{P}} = \underline{\underline{F}} \,.\, \underline{\underline{S}}} \end{cases}$$

Tensor	**Symbol**	**force**	**aera**	**symmetry**
Cauchy	$\underline{\underline{\sigma}}$	Current configuration	Current configuration	Yes
First Piola-Kirchhoff stress tensor (also Boussinesq tensor)	$\underline{\underline{P}}$	Current configuration	Reference configuration	No
Second Piola-Kirchhoff stress tensor (also Piola tensor)	$\underline{\underline{S}}$	Reference configuration	Reference configuration	Yes

3. Derivatives

For a function

$$f(x+dx)=f(x)+\frac{df}{dx}.dx+O\left(|dx|^2\right)$$

Almost the same for a tensor:
The derivative with respect to a tensor is defined such that

$$f(\underline{\underline{e}}+d\,\underline{\underline{e}})=f(\underline{\underline{e}})+{}^t\!\left(\frac{\partial f}{\partial \underline{\underline{e}}}\right):d\,\underline{\underline{e}}+O\left(|\underline{\underline{e}}|^2\right)$$

$\dfrac{\partial f}{\partial \underline{\underline{e}}}$ is the derivative of the function *f* with respect to the tensor $\underline{\underline{e}}$

Examples

Derivative of a square (if $\underline{\underline{e}}$ symmetric)

$$\left(\underline{\underline{e}}+d\,\underline{\underline{e}}\right):\left(\underline{\underline{e}}+d\,\underline{\underline{e}}\right)=\underline{\underline{e}}:\underline{\underline{e}}+2\,\underline{\underline{e}}:d\,\underline{\underline{e}}+d\,\underline{\underline{e}}:d\,\underline{\underline{e}}$$

$$\frac{\partial f}{\partial \underline{\underline{e}}}=2\,\underline{\underline{e}}$$

Derivative of a determinant

$$\begin{aligned}
\det\left(\underline{\underline{a}}+d\,\underline{\underline{a}}\right) &= \det\left(\underline{\underline{a}}.\left(\underline{\underline{1}}+\underline{\underline{a}}^{-1}.d\,\underline{\underline{a}}\right)\right)\\
&= \det\left(\underline{\underline{a}}\right).\det\left(\underline{\underline{1}}+\underline{\underline{a}}^{-1}.d\,\underline{\underline{a}}\right)\\
&= \det\left(\underline{\underline{a}}\right).\left(1+Tr\left(\underline{\underline{a}}^{-1}.d\,\underline{\underline{a}}\right)+o\left(|d\,\underline{\underline{a}}|^2\right)\right)\\
&= \det\left(\underline{\underline{a}}\right)+\det\left(\underline{\underline{a}}\right).\left(\underline{\underline{a}}^{-1}:d\,\underline{\underline{a}}\right)+o\left(|d\,\underline{\underline{a}}|^2\right)
\end{aligned}$$

$$\frac{\partial \det\left(\underline{\underline{a}}\right)}{\partial \underline{\underline{a}}}=\det\left(\underline{\underline{a}}\right).{}^{t}\underline{\underline{a}}^{-1}$$

Derivatives of the invariants of a second order tensor

The three invariants of a second order tensor A are

$$
\begin{aligned}
I_1(\underline{\underline{A}}) &= tr(\underline{\underline{A}}) \\
I_2(\underline{\underline{A}}) &= \frac{1}{2}\left[\left(tr(\underline{\underline{A}})\right)^2 - Tr(\underline{\underline{A}}^2)\right] \\
I_3(\underline{\underline{A}}) &= det(\underline{\underline{A}})
\end{aligned}
$$

Their derivatives are

$$
\begin{aligned}
\frac{\partial I_1}{\partial \underline{\underline{A}}} &= \underline{\underline{1}} \\
\frac{\partial I_2}{\partial \underline{\underline{A}}} &= I_1 \underline{\underline{1}} - {}^t\underline{\underline{A}} \\
\frac{\partial I_3}{\partial \underline{\underline{A}}} &= det(A)\, {}^t\underline{\underline{A}}^{-1}
\end{aligned}
$$

4. Constitutive laws

* First principle of thermodynamics

$$\dot{E}_c = \frac{d}{dt}\left(\int_\Omega \frac{\rho}{2}\,\underline{v}\,.\,\underline{v}\,dx + \int_\Omega \rho\, e\, dx\right) = \dot{W} + \dot{Q} \qquad (1)$$

$\dot{E}_c$ *rate of variation of total energy*

$\dot{W}$ *rate of work done on the system*

$\dot{Q}$ *rate of heat recieved by the system*

e *internal energy*

* Theorem of kinetic energy

$$\frac{d}{dt}\left(\int_\Omega \frac{\rho}{2}\,\underline{v}\,.\,\underline{v}\,dx\right) = P_e(\underline{v}) + P_i(\underline{v}) = \dot{W} + P_i(\underline{v}) \qquad (2)$$

because $P_e = \dot{W}$

* One gets

$$-\frac{d}{dt}\left(\int_\Omega \rho\, e\, dx\right) + \dot{Q} = P_i(\underline{v}) \qquad (2) - (1)$$

* Second principle of thermodynamics: Clausius Duhem inequality

$$\int_{\Omega} \rho\, T\, \dot{s}\, dx - \dot{Q} + \int_{\Omega} T\, \underline{q} \,.\, \underline{grad}\left(\frac{1}{T}\right) \geq 0$$

* With the last relation of the precedent slide

$$-\frac{d}{dt}\left(\int_{\Omega} \rho\, e\, dx\right) + \int_{\Omega} \rho\, T\, \dot{s}\, dx - P_i(\underline{v}) + \int_{\Omega} T\, \underline{q} \,.\, \underline{grad}\left(\frac{1}{T}\right) \geq 0$$

If the temperature is uniform and

- Introducing the free energy $\psi = e - Ts$
- Noting that $P_i(\underline{v}) = -\int_{\Omega} \underline{\underline{\sigma}} : \underline{\underline{d}}\, dx$ $\qquad (d = sym(\nabla\, \dot{x}))$

One gets

$$\boxed{-\rho\, \dot{\psi} + \underline{\underline{\sigma}} : \underline{\underline{d}} \geq 0}$$

One also has

$$\int_{\Omega} \left(\underline{\underline{\sigma}} : \underline{\underline{d}}\right) dx = \int_{\Omega_0} \left(\underline{\underline{S}} : \dot{\underline{\underline{e}}}\right) dx_0$$

because

$$\begin{aligned}
\underline{\underline{\sigma}} : \underline{\underline{d}} &= \underline{\underline{\sigma}} : {}^t\!\left(\frac{\partial \underline{U}}{\partial \underline{x}}\right) \\
&= \underline{\underline{\sigma}} : {}^t\!\left(\frac{\partial \underline{U}}{\partial \underline{X}} \cdot \frac{\partial \underline{X}}{\partial \underline{x}}\right) \\
&= Tr\left(\underline{\underline{\sigma}} \cdot {}^t\underline{\underline{F}}^{-1} \cdot {}^t\dot{\underline{\underline{F}}}\right) \\
&= Tr\left(\underline{\underline{F}}^{-1} \cdot \underline{\underline{\sigma}} \cdot {}^t\underline{\underline{F}}^{-1} \cdot {}^t\dot{\underline{\underline{F}}} \cdot \underline{\underline{F}}\right) \\
&= \left(\underline{\underline{F}}^{-1} \cdot \underline{\underline{\sigma}} \cdot {}^t\underline{\underline{F}}^{-1}\right) : \left({}^t\dot{\underline{\underline{F}}} \cdot \underline{\underline{F}}\right) \\
&= \left(\underline{\underline{F}}^{-1} \cdot \underline{\underline{\sigma}} \cdot {}^t\underline{\underline{F}}^{-1}\right) : \left({}^t\dot{\underline{\underline{F}}} \cdot \underline{\underline{F}}\right)_{sym} \\
&= \frac{1}{J}\left(J\, \underline{\underline{F}}^{-1} \cdot \underline{\underline{\sigma}} \cdot {}^t\underline{\underline{F}}^{-1}\right) : \dot{\underline{\underline{e}}} \\
&= \frac{1}{J} \underline{\underline{S}} : \dot{\underline{\underline{e}}}
\end{aligned}$$

The Clausius Duhem inequality is then (uniform temperature)

$$\boxed{-\rho_0 \dot{\psi} + \underline{\underline{S}} : \dot{\underline{\underline{e}}} \geq 0}$$

$$-\rho_0 \dot{\psi} + \underline{\underline{S}} : \dot{\underline{\underline{e}}} \geq 0$$

With the free energy ψ function of the deformation tensor $\underline{\underline{e}}$

$$\left(-\rho_0 \frac{\partial \psi}{\partial \underline{\underline{e}}} + \underline{\underline{S}} \right) : \dot{\underline{\underline{e}}} \geq 0$$

This is possible only if (in case of no internal kinematical condition)

$$\boxed{\underline{\underline{S}} = \rho_0 \frac{\partial \psi}{\partial \underline{\underline{e}}}}$$

If isothermal and isentropic one also has $\left(by\ denoting\ W = \rho_0 \psi \right)$

$$\underline{\underline{S}} = \frac{\partial W}{\partial \underline{\underline{e}}}$$

One also has

$$\underline{\underline{P}} = \underline{\underline{F}} \cdot \underline{\underline{S}} = \underline{\underline{F}} \cdot \frac{\partial W}{\partial \underline{\underline{e}}} = \frac{\partial W}{\partial \underline{\underline{F}}}$$

The energy should satisfy the growth conditions

$$W\left(\underline{\underline{e}}\right) \rightarrow \infty \quad \text{when} \quad J = \det \underline{\underline{F}} \rightarrow 0^{+}$$

$$W\left(\underline{\underline{e}}\right) \rightarrow \infty \quad \text{when} \quad J = \det \underline{\underline{F}} \rightarrow \infty$$

$$w\left(\underline{\underline{e}}\right) = \frac{\mu}{2}\left({}^{t}\underline{\underline{F}} : \underline{\underline{F}} - 3\right) - \mu \ln J + \frac{\lambda}{2}\left(\ln J\right)^{2}$$ good

$$w\left(\underline{\underline{e}}\right) = \frac{\lambda}{2} tr\left(\underline{\underline{e}}\right)^{2} + \mu\, \underline{\underline{e}} : \underline{\underline{e}}$$ Not so good

Case of isotropic materials

The deformation energy can be written according to the invariants

$$W\left(\underline{\underline{e}}\right)=W\left(I_1\left(\underline{\underline{C}}\right),I_2\left(\underline{\underline{C}}\right),I_3\left(\underline{\underline{C}}\right)\right)$$

(with $\underline{\underline{C}}={}^t\underline{\underline{F}}\,.\,\underline{\underline{F}}=2\left(\underline{\underline{e}}+\underline{\underline{1}}\right)$)

$$\begin{aligned}
I_1\left(\underline{\underline{C}}\right) &= tr\left(\underline{\underline{C}}\right) = \lambda_1^2+\lambda_2^2+\lambda_3^2\\
I_2\left(\underline{\underline{C}}\right) &= \frac{1}{2}\left[\left(tr\left(\underline{\underline{C}}\right)\right)^2-Tr\left(\underline{\underline{C}}^2\right)\right] = \lambda_1^2\lambda_2^2+\lambda_1^2\lambda_3^2+\lambda_2^2\lambda_3^2\\
I_3\left(\underline{\underline{C}}\right) &= det\left(\underline{\underline{C}}\right) = \lambda_1^2\lambda_2^2\lambda_3^2
\end{aligned}$$

The second Piola-Kirchoff tensor is then

$$\begin{aligned}
\underline{\underline{S}}\left(\underline{\underline{e}}\right) = \frac{\partial W}{\partial \underline{\underline{e}}} = 2\frac{\partial W}{\partial \underline{\underline{C}}} &= 2\frac{\partial W}{\partial I_1}\frac{\partial I_1}{\partial \underline{\underline{C}}}+2\frac{\partial W}{\partial I_2}\frac{\partial I_2}{\partial \underline{\underline{C}}}+2\frac{\partial W}{\partial I_3}\frac{\partial I_3}{\partial \underline{\underline{C}}}\\
&= 2\left(\frac{\partial W}{\partial I_1}+I_1\frac{\partial W}{\partial I_2}\right)\underline{\underline{1}}-2\underline{\underline{C}}\frac{\partial W}{\partial I_2}+2\frac{\partial W}{\partial I_3}\,I_3\underline{\underline{C}}^{-1}
\end{aligned}$$

Case of incompressible materials

In this case there is no change of volume

During a deformation the material should satisfy the constraint

$$J^2 = I_3 = \left(\det \underline{\underline{F}}\right)^2 = \det \underline{\underline{C}} = 1$$

A general displacement is such that

$$\frac{\partial \det C}{\partial \underline{\underline{C}}} : \dot{\underline{\underline{C}}} = det\left(\underline{\underline{C}}\right)\underline{\underline{C}}^{-1} : \dot{\underline{\underline{C}}} = \underline{\underline{C}}^{-1} : \dot{\underline{\underline{C}}} = 0$$

Remember the Clausius Duhem inequality

$$\left(-\frac{\partial W}{\partial \underline{\underline{e}}}+\underline{\underline{S}}\right):\dot{\underline{\underline{e}}}\geq 0$$

But in this case for all displacements satisfying the internal constraint

$$\underline{\underline{C}}^{-1}:\dot{\underline{\underline{C}}}=0 \qquad \text{(with } \underline{\underline{C}}={}^{t}\underline{\underline{F}}\,.\,\underline{\underline{F}}=2\left(\underline{\underline{e}}+\underline{\underline{1}}\right)\text{)}$$

$\underline{\underline{S}}-\dfrac{\partial W}{\partial \underline{\underline{e}}}$ can be decomposed as

$$\underline{\underline{S}}-\frac{\partial W}{\partial \underline{\underline{e}}}=\underline{\underline{S}}_0-p\,\underline{\underline{C}}^{-1} \qquad \text{with} \qquad \underline{\underline{S}}_0:\underline{\underline{C}}^{-1}=0$$

Taking $\dot{\underline{\underline{e}}}=-\underline{\underline{S}}_0$ one sees that $\underline{\underline{S}}_0=0$ and

$$\boxed{\underline{\underline{S}}=\frac{\partial W}{\partial \underline{\underline{e}}}-p\,\underline{\underline{C}}^{-1}}$$

5. Hyperelastic materials

Some properties

- Stress-strain relationship derives from a strain energy density function
- Stress is independent of history and depends only of the final state
- Different strain energy densities are possible
- Often comes with incompressibility ($J = 1$)
- Example: rubber, biological tissues

Kirchoff-Saint Venant model

Simplest generalisation of linear elastic materials

$$w(\underline{\underline{e}})=\frac{\lambda}{2}\left(Tr(\underline{\underline{e}})\right)^2+\mu\, Tr(\underline{\underline{e}}.\underline{\underline{e}})=\frac{1}{8}(\lambda+2\mu)(I_1-3)^2-\frac{\mu}{2}(I_2-2I_1+3)$$

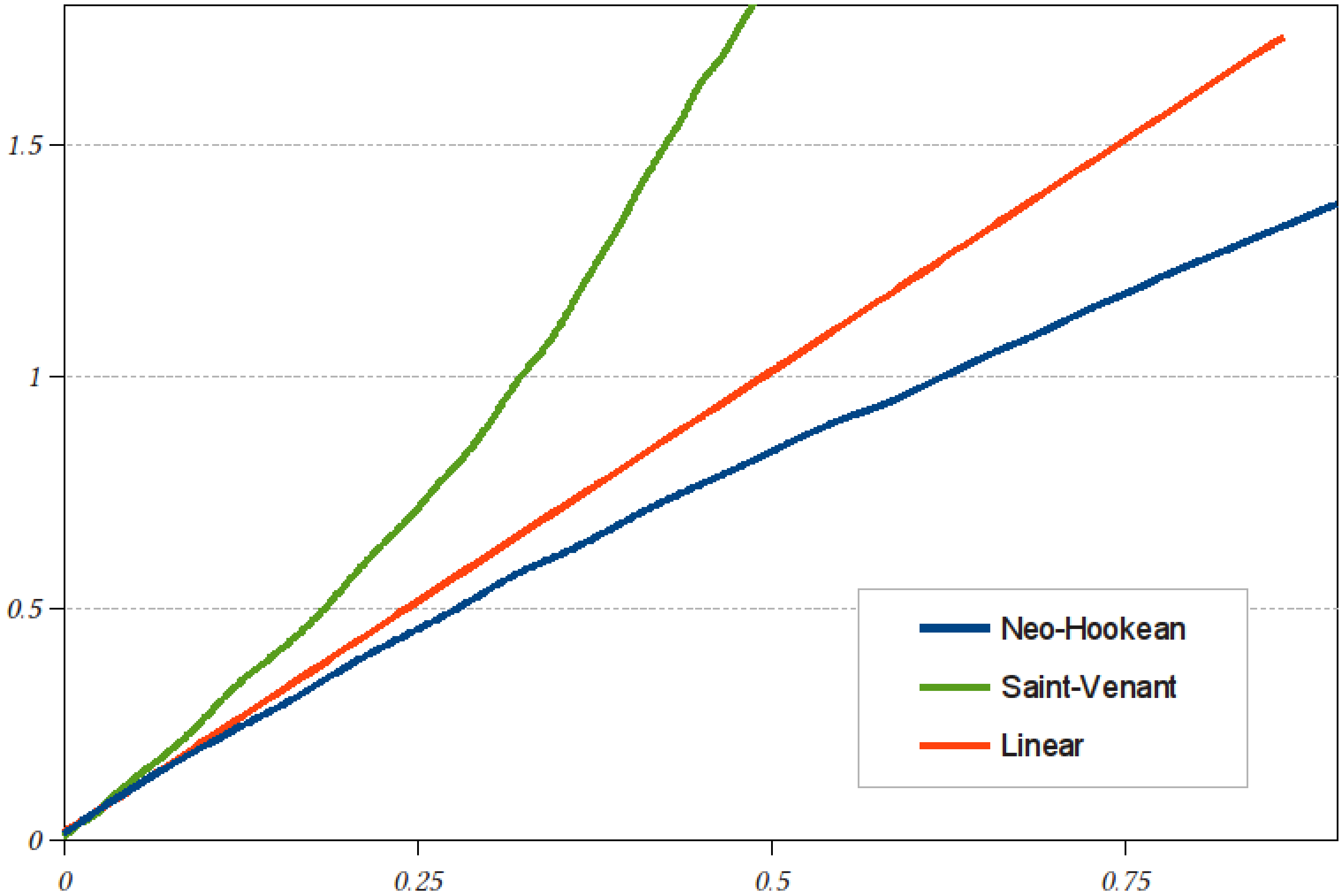
$$\underline{\underline{S}}(\underline{\underline{e}})=\lambda\, tr(\underline{\underline{e}})\underline{\underline{1}}+2\mu\,\underline{\underline{e}}$$

Do not use for really large deformation as the energy is finite for $\det \underline{\underline{F}} \rightarrow 0^+$
(but large rotation possible)

$$\begin{aligned}
\underline{\underline{\sigma}} &= \frac{1}{J}\underline{\underline{F}}.\underline{\underline{S}}.{}^t\underline{\underline{F}}\\
&= \frac{1}{J}\left(\lambda\, Tr(\underline{\underline{e}})\,\underline{\underline{F}}.{}^t\underline{\underline{F}}+2\mu\,\underline{\underline{F}}.\underline{\underline{e}}.{}^t\underline{\underline{F}}\right)\\
&= \frac{1}{J}\left(\lambda\, Tr(\underline{\underline{e}})\,\underline{\underline{B}}+\mu\left(\underline{\underline{B}}.\underline{\underline{B}}-\underline{\underline{B}}\right)\right)
\end{aligned}$$

with $\underline{\underline{B}}=\underline{\underline{F}}.{}^t\underline{\underline{F}}$

Example of stress strain curve



Incompressible Neo-Hook model

Elastic energy

$$w\left(\underline{\underline{e}}\right)=\frac{\mu}{2}\left(I_1-3\right)$$

Second Piola-Kirchhoff stress tensor

$$\underline{\underline{S}}=2\frac{\partial W}{\partial I_1}\underline{\underline{1}}-p\,\underline{\underline{C}}^{-1}=\mu\,\underline{\underline{I}}-p\,\underline{\underline{C}}^{-1}$$

Cauchy stress tensor

$$\underline{\underline{\sigma}}=\frac{1}{J}\,\underline{\underline{F}}\cdot\underline{\underline{S}}\cdot{}^t\underline{\underline{F}}$$

$$\underline{\underline{\sigma}}=-p\,\underline{\underline{I}}+\mu\,\underline{\underline{F}}\cdot{}^t\underline{\underline{F}}$$

Compressible Neo-Hook model

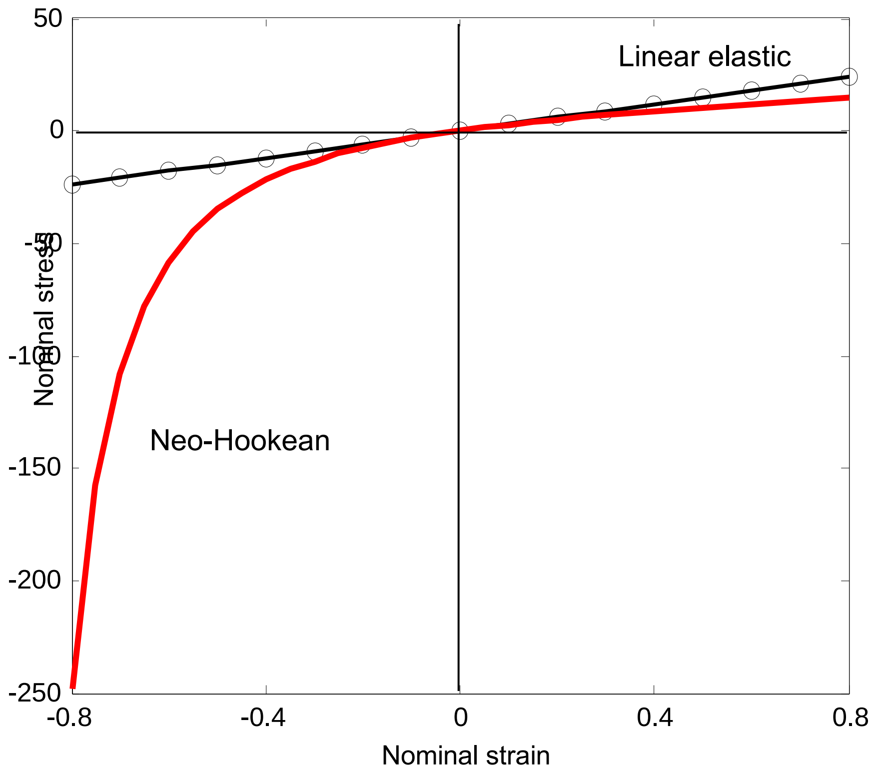
Elastic energy

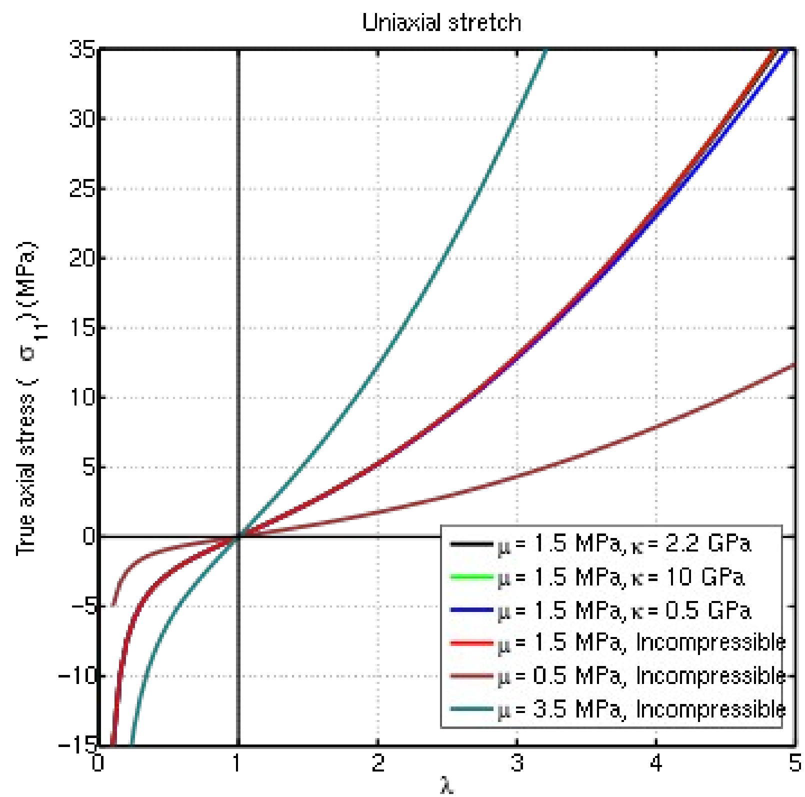
$$w\left(\underline{\underline{e}}\right)=\frac{\mu}{2}\left(I_1-3\right)-\mu\ln J+\frac{\lambda}{2}\left(\ln J\right)^2$$

Exercice : compute the second Piola-Kirchhoff stress tensor and the Cauchy stress tensor

Use for moderate deformation (less than 20%)

Use for plastic and rubber





Mooney-Rivlin model

$$w(\underline{\underline{e}})=\frac{\mu_1}{2}\left(\frac{I_1}{J^{2/3}}-3\right)+\frac{\mu_2}{2}\left(\frac{I_2}{J^{4/3}}-3\right)+\frac{K}{2}(J-1)^2 \quad (Compressible)$$

$$w(\underline{\underline{e}})=\frac{\mu_1}{2}(I_1-3)+\frac{\mu_2}{2}(I_2-3) \quad (Incompressible)$$

$$\underline{\underline{S}}=\left(\frac{\mu_1}{J^{2/3}}+\mu_2\frac{I_1}{J^{4/3}}\right)\underline{\underline{I}}-\frac{\mu_2}{J^{4/3}}\underline{\underline{C}}+\left(KJ(J-1)-\frac{\mu_1 I_1}{3J^{2/3}}-\frac{2\mu_2 I_2}{3J^{4/3}}\right)\underline{\underline{C}}^{-1} \quad (Compressible)$$

$$\underline{\underline{S}}=(\mu_1+\mu_2 I_1)\underline{\underline{I}}-\mu_2\underline{\underline{C}}-p\,\underline{\underline{C}}^{-1} \quad (Incompressible)$$

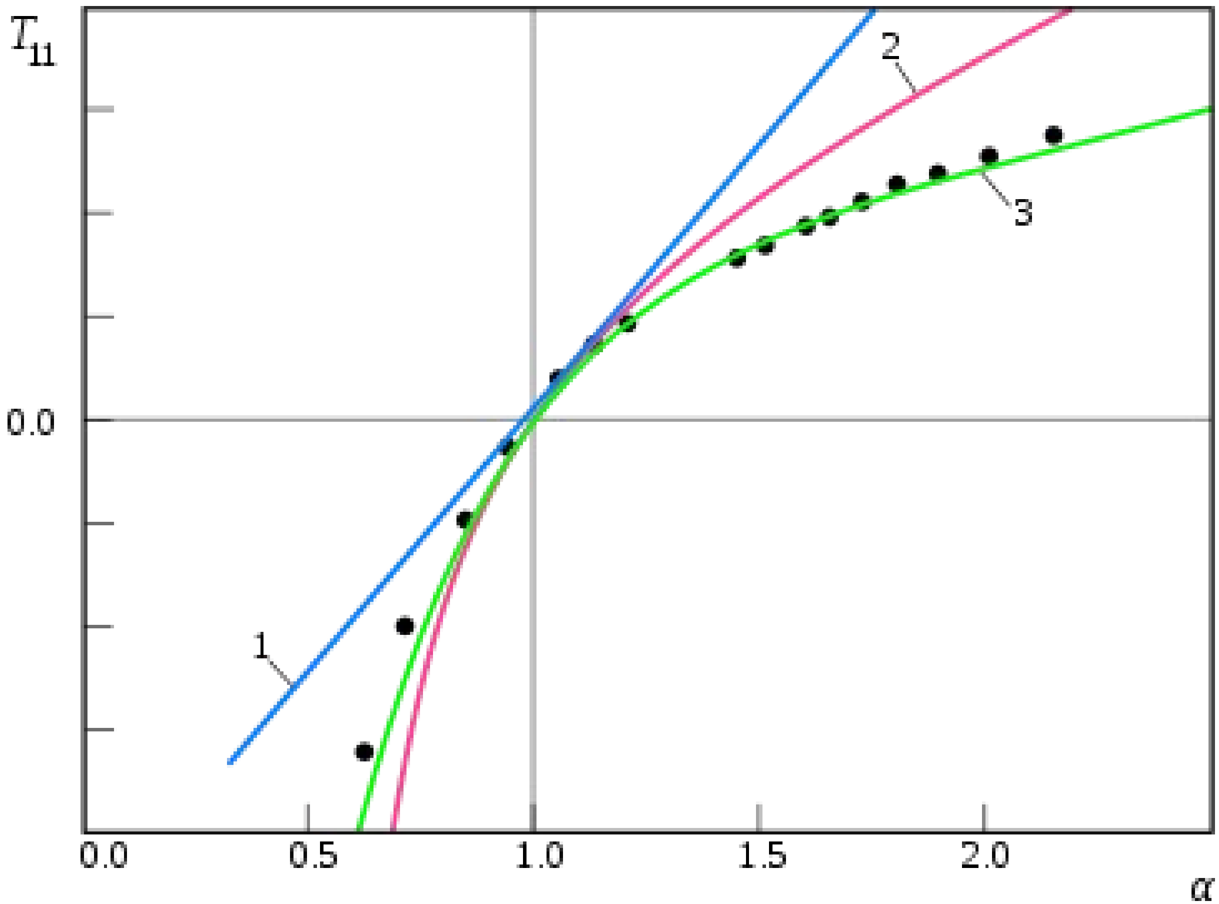
$$\underline{\underline{\sigma}}=\frac{1}{J}\underline{\underline{F}}\,.\,\underline{\underline{S}}\,.\,{}^t\underline{\underline{F}}$$

$$\underline{\underline{\sigma}}=\frac{1}{J}\left(\left(\frac{\mu_1}{J^{2/3}}+\mu_2\frac{I_1}{J^{4/3}}\right)\underline{\underline{B}}-\frac{\mu_2}{J^{4/3}}\underline{\underline{B}}^2+\left(KJ(J-1)-\frac{\mu_1 I_1}{3J^{2/3}}-\frac{2\mu_2 I_2}{3J^{4/3}}\right)\underline{\underline{1}}\right) \quad (Compressible)$$

$$\underline{\underline{\sigma}}=(\mu_1+\mu_2 I_1)\underline{\underline{B}}-\mu_2\underline{\underline{B}}^2-p\,\underline{\underline{I}} \quad (Incompressible)$$

Exercice: check the formulas for $\underline{\underline{S}}$ and $\underline{\underline{\sigma}}$

Example of traction compression curve



Experimental results (dots)
Linear Hooke's law(1, blue line)
Neo-Hookean solid(2, red line) and
Mooney–Rivlin solid models(3, green line)

Other models

Ogden model

$$W(\lambda_1,\lambda_2,\lambda_3)=\sum_{i=1}^{N}\frac{\mu_i}{\alpha_i}\left(\lambda_1^{\alpha_i}+\lambda_2^{\alpha_i}+\lambda_3^{\alpha_i}-3\right)$$

General model

$$W(I_1,I_2,I_3)=\sum_{m+n+k=1}^{\infty}A_{mnk}(I_1-3)^m(I_2-3)^n(I_3-1)^k$$

General incompressible model

$$W(I_1,I_2)=\sum_{m+n=1}^{\infty}A_{mn}(I_1-3)^m(I_2-3)^n$$

6. Minimisation of the potential energy

Consider a solid that has a reference configuration Ω_0 and is submitted

- to body forces $\underline{b}_0$ in Ω_0
- to surface forces $\underline{g}_0$ on $\partial_g\Omega_0$
- and is such that $\underline{u} = \underline{u}_0$ on $\partial_u\Omega_0$

The total potential energy is

$$\begin{aligned} E(\underline{u}) &= E_{\text{int}}(\underline{u}) + E_{\text{ext}}(\underline{u}) \\ &= \int_{\Omega_0} W(\underline{u})\, dx_0 - \int_{\Omega_0} \underline{b}_0 . \underline{u}\, dx_0 - \int_{\partial_g \Omega_0} \underline{g}_0 . \underline{u}\, ds_0 \end{aligned}$$

The solution of the mechanical problem is the displacement minimizing *E($\underline{u}$)* among the functions such that $\underline{u} = \underline{u}_0$ on $\partial_u\Omega_0$

At the minimum of the potential energy one should have

$$E(\underline{u}) \leq E(\underline{u} + h\,\underline{v}) \quad \forall\, \underline{v}, h$$

This implies that the derivative is zero for h=0

$$E'(\underline{u}, \underline{v}) = \frac{d}{dh} E(\underline{u} + h\,\underline{v}) = 0 \quad for\, h = 0$$

With the energy given by

$$E(\underline{u} + h\,\underline{v}) = \int_{\Omega_0} W(\underline{\underline{e}}(\underline{u} + h\,\underline{v}))\, dx_0 - \int_{\Omega_0} \underline{b}_0 . (\underline{u} + h\,\underline{v})\, dx_0 - \int_{\partial_g \Omega_0} \underline{g}_0 . (\underline{u} + \underline{v})\, ds_0$$

One gets the derivative

$$E'(\underline{u}, \underline{v}) = \int_{\Omega_0} \frac{\partial W}{\partial \underline{\underline{e}}}(\underline{\underline{e}}(\underline{u})) : \underline{\underline{e}}(\underline{v})\, dx_0 - \int_{\Omega_0} \underline{b}_0 . \underline{v}\, dx_0 - \int_{\partial_g \Omega_0} \underline{g}_0 . \underline{v}\, ds_0$$

with $\underline{\underline{e}}_d(\underline{v}) = \frac{d}{dh} \underline{\underline{e}}(\underline{u} + h\,\underline{v}) \quad for \quad h = 0$

$$E'(\underline{u},\underline{v})=\int_{\Omega_0}\frac{\partial W}{\partial \underline{\underline{e}}}\left(\underline{\underline{e}}(\underline{u})\right):\underline{\underline{e}}_d(\underline{v})\,dx_0-\int_{\Omega_0}\underline{b}_0\,.\,\underline{v}\,dx_0-\int_{\partial_g\Omega_0}\underline{g}_0\,.\,\underline{v}\,ds_0$$

Is the variational formulation defined for all $\underline{v} = 0$ on $\partial_u\Omega_0$

The non linearity is in the strain energy term and

$$\begin{aligned}
\underline{\underline{e}}_d(\underline{v}) &= \frac{d\,\underline{\underline{e}}(\underline{u}+h\,\underline{v})}{dh}(0)\\
&= \frac{1}{2}\frac{d}{dh}\left({}^t\left(\nabla(\underline{u}+h\,\underline{v})\right).\nabla(\underline{u}+h\,\underline{v})-\underline{\underline{1}}\right)(0)\\
&= \frac{1}{2}\left({}^t(\nabla\,\underline{u}).\nabla\,\underline{v}+{}^t(\nabla\,\underline{v}).\nabla\,\underline{u}\right)\\
&= \frac{1}{2}\left({}^t\underline{\underline{F}}.\nabla\,\underline{v}+{}^t(\nabla\,\underline{v}).\underline{\underline{F}}\right)\\
&= \left({}^t\underline{\underline{F}}.\nabla\,\underline{v}\right)_{\mathrm{sym}}
\end{aligned}$$

$$E'(\underline{u},\underline{v})=\int_{\Omega_0}\frac{\partial W}{\partial \underline{\underline{e}}}\left(\underline{\underline{e}}(\underline{u})\right):\left({}^t\underline{\underline{F}}.\nabla\,\underline{v}\right)_{\mathrm{sym}}dx_0-\int_{\Omega_0}\underline{b}_0\,.\,\underline{v}\,dx_0-\int_{\partial_g\Omega_0}\underline{g}_0\,.\,\underline{v}\,ds_0$$

$$\int_{\Omega_0} \frac{\partial W}{\partial \underline{\underline{e}}}\left(\underline{\underline{e}}(\underline{u})\right) : \left({}^t\underline{\underline{F}} . \nabla \underline{v}\right)_{\text{sym}} dx_0 = \int_{\Omega_0} \underline{b}_0 . \underline{v}\, dx_0 + \int_{\partial_g \Omega_0} \underline{g}_0 . \underline{v}\, ds_0$$

$$a(\underline{u}, \underline{v}) = l(\underline{v})$$

- $a(\underline{u}, \underline{v})$ Non linear in u, linear in v
- $l(\underline{v})$ Linear in v

$$\begin{aligned}\int_{\Omega_0} \frac{\partial W}{\partial \underline{\underline{e}}}\left(\underline{\underline{e}}(\underline{u})\right) : \left({}^t\underline{\underline{F}} . \nabla \underline{v}\right)_{\text{sym}} dx_0 &= \int_{\Omega_0} \underline{\underline{S}}\left(\underline{\underline{e}}(\underline{u})\right) : \left({}^t\underline{\underline{F}} . \nabla \underline{v}\right) dx_0 \\ &= \int_{\Omega_0} \left(\underline{\underline{F}} . \underline{\underline{S}}\left(\underline{\underline{e}}(\underline{u})\right)\right) : \nabla \underline{v}\, dx_0 \\ &= \int_{\Omega_0} \underline{\underline{P}}\left(\underline{\underline{e}}(\underline{u})\right) : {}^t\nabla \underline{v}\, dx_0\end{aligned}$$

And one recovers the equation in the reference configuration

$$\text{DIV}\, \underline{\underline{P}} + \underline{b}_0 = 0$$

$$\underline{\underline{P}} . \underline{N} = \underline{g}_0 \quad on \quad \partial_g \Omega_0$$

7. Linearisation

One defines the residual $R(u) = a(\underline{u}, \underline{v}) - l(\underline{v})$

The objective is to solve $R(u) = a(\underline{u}, \underline{v}) - l(\underline{v}) = 0$

This is a non linear equation

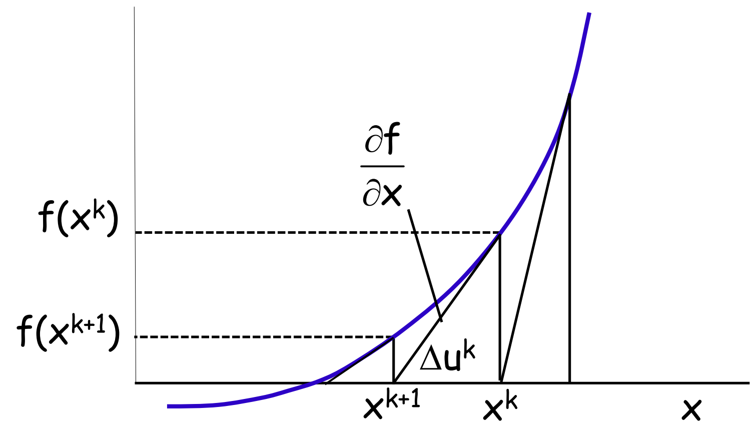
One possibility is to solve by the Newton – Raphson algorithm

For a scalar function around a point x_0

$$f(x) \simeq f(x_0) + f'(x_0)(x - x_0)$$

An approximation of the point such that f(x)=0 is solution of

$$f(x_0) + f'(x_0)(x - x_0) = 0$$



This leads to the iterative process $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$

$$R(u+\Delta u)\simeq R(u)+\frac{\partial R}{\partial \underline{u}}(u).\Delta u+o\left(|\Delta u|^2\right)$$

One solves the approximate equation

$$\frac{\partial R}{\partial \underline{u}}\left(u^k\right).\Delta u^k=-R\left(u^k\right)$$

$$\underline{u}^{k+1}=\underline{u}^k+\Delta \underline{u}^k$$

The problem is to compute $\frac{\partial R}{\partial \underline{u}}$

One notes that $\frac{\partial R}{\partial \underline{u}}=\frac{\partial a(\underline{u},\underline{v})}{\partial \underline{u}}$

(it is possible to have loads and thus l($\underline{v}$) depending on the displacement but this case will not be considered here)

The left-hand side of the variational relation is

$$a(\underline{u},\underline{v})=\int_{\Omega_0}\frac{\partial W}{\partial \underline{\underline{e}}}(\underline{\underline{e}}(\underline{u})):\left({}^t\underline{\underline{F}}.\nabla\,\underline{v}\right)_{\mathrm{sym}}dx_0$$

$$a(\underline{u}+\Delta\,\underline{u},\underline{v})=\int_{\Omega_0}\underline{\underline{S}}(\underline{\underline{e}}(\underline{u}+\Delta\,\underline{u})):\left(\left({}^t\underline{\underline{F}}+{}^t(\Delta\,\underline{\underline{F}})\right).\nabla\,\underline{v}\right)_{\mathrm{sym}}dx_0$$

This expression should be developed at the first order in $\Delta\underline{u}$

Consider first

$$\begin{aligned}\Delta\,\underline{\underline{F}} &= \Delta\left(\frac{\partial\,\underline{x}}{\partial\,\underline{X}}\right)\\ &= \Delta\left(\frac{\partial(\underline{X}+\underline{u})}{\partial\,\underline{X}}\right)\\ &= \frac{\partial(\underline{X}+\underline{u}+\Delta\,\underline{u})}{\partial\,\underline{X}}-\frac{\partial(\underline{X}+\underline{u})}{\partial\,\underline{X}}\\ &= \frac{\partial\,\Delta\,\underline{u}}{\partial\,\underline{X}}\\ &= \nabla(\Delta\,\underline{u})\end{aligned}$$

$$\Delta \underline{\underline{F}} = \nabla(\Delta \underline{u})$$

$$a(\underline{u}+\Delta \underline{u}, \underline{v}) = \int_{\Omega_0} \underline{\underline{S}}(\underline{\underline{e}}(\underline{u}+\Delta \underline{u})) : \left(\left({}^t\underline{\underline{F}} + {}^t(\Delta \underline{\underline{F}})\right) . \nabla \underline{v}\right)_{\text{sym}} dx_0$$

Consider now

$$\begin{aligned} \underline{\underline{S}}(\underline{\underline{e}}(\underline{u}+\Delta \underline{u})) &= \underline{\underline{S}}(\underline{\underline{e}}(\underline{u})) + \frac{\partial \underline{\underline{S}}}{\partial \underline{\underline{e}}} : \Delta \underline{\underline{e}} + o(|\Delta \underline{\underline{e}}|^2) \\ &= \underline{\underline{S}}(\underline{\underline{e}}(\underline{u})) + \underline{\underline{\underline{\underline{D}}}} : \Delta \underline{\underline{e}} + o(|\Delta \underline{\underline{e}}|^2) \end{aligned}$$

$$\begin{aligned} \Delta \underline{\underline{e}} &= \frac{1}{2}\left({}^t(\Delta \underline{\underline{F}}) . \underline{\underline{F}} + {}^t\underline{\underline{F}} . \Delta F\right) \\ &= sym\left({}^t\underline{\underline{F}} . \nabla(\Delta \underline{u})\right) \end{aligned}$$

$\underline{\underline{\underline{\underline{D}}}}$ are the tangent elastic coefficients

So one gets the development

$$
\begin{aligned}
a(\underline{u}+\Delta\underline{u},\underline{v}) &= \int_{\Omega_0} \underline{\underline{S}}(\underline{\underline{e}}(\underline{u}+\Delta\underline{u})) : \left(\left({}^t\underline{\underline{F}} + {}^t(\Delta\underline{\underline{F}})\right).\nabla\underline{v}\right)_{\text{sym}} dx_0 \\
&= a(\underline{u},\underline{v}) + \int_{\Omega_0} \left(\left(\underline{\underline{\underline{\underline{D}}}} : \Delta\underline{\underline{e}}\right) : \left({}^t\underline{\underline{F}}.\nabla\underline{v}\right)_{\text{sym}} + \underline{\underline{S}} : \left({}^t(\nabla(\Delta\underline{u})).\nabla\underline{v}\right)_{sym}\right) dx_0 + o\left(|\underline{u}|^2\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial R}{\partial \underline{u}}.\Delta\underline{u} &= \frac{\partial a(\underline{u},\underline{v})}{\partial \underline{u}}.\Delta\underline{u} \\
&= \int_{\Omega_0} \left(\left(\underline{\underline{\underline{\underline{D}}}} : \Delta\underline{\underline{e}}\right) : \left({}^t\underline{\underline{F}}.\nabla\underline{v}\right)_{\text{sym}} + \underline{\underline{S}} : \left({}^t(\nabla(\Delta\underline{u})).\nabla\underline{v}\right)_{sym}\right) dx_0
\end{aligned}
$$

The first term is the material tangent stiffness

The second term is geometric stiffness

The step k of the Newton Raphson algorithm consists in solving

$$\frac{\partial R}{\partial \underline{u}}\left(u^k\right).\Delta u^k = -R\left(u^k\right)$$

$$\int_{\Omega_0}\left(\left(\underline{\underline{\underline{\underline{D}}}}:\Delta \underline{\underline{e}}\right):\left({}^t\underline{\underline{F}}.\nabla \underline{v}\right)_{\mathrm{sym}} + \underline{\underline{S}}:\left({}^t\left(\nabla\left(\Delta \underline{u}\right)\right).\nabla \underline{v}\right)_{sym}\right) dx_0 = l\left(\underline{v}\right) - a\left(\underline{u}^k, \underline{v}\right)$$

- The right-hand side is known
- The left-hand side is linear in $\Delta \underline{u}$
- Both sides are linear in $\underline{v}$

In discrete form this can be written as

$${}^t\underline{V}.\underline{\underline{K}}.\Delta \underline{U} = {}^t\underline{V}.\underline{R}$$

- $\underline{V}$ is the vector of nodal values of the function $\underline{v}$
- $\Delta\underline{U}$ is the vector of nodal values of the function $\Delta\underline{u}$
- $\underline{\underline{K}}$ is a matrix resulting from the discretisation of the left-hand side
- $\underline{R}$ is a vector resulting from the discretisation of the right-hand side

One has to compute $\underline{\underline{K}}$ and $\underline{R}$

8. FEM implementation

Displacement gradient and deformation

Displacement gradient in an element with N_e nodes

$$
\begin{aligned}
[\nabla \underline{u}]_{ij} &= \sum_{I=1}^{I=N_e} \frac{\partial N_I(\underline{X}(\underline{\xi}))}{\partial \underline{X}} . \underline{u}_I \\
&= \sum_{I=1}^{I=N_e} \left(\frac{\partial N_I(\underline{\xi})}{\partial \underline{\xi}} . \frac{\partial \underline{\xi}}{\partial \underline{X}} \right) . \underline{u}_I
\end{aligned}
$$

$\frac{\partial N_I(\underline{\xi})}{\partial \underline{\xi}}$ Is easy to compute from the definition of the shape functions

$\frac{\partial \underline{\xi}}{\partial \underline{X}}$ Is the inverse matrix of $\frac{\partial \underline{X}}{\partial \underline{\xi}} = \sum_{I=1}^{I=N_e} \frac{\partial N_I(\underline{\xi})}{\partial \underline{\xi}} \underline{X}_I$

Displacement gradient

$$[F]_{ij} = \delta_{ij} + [\nabla \underline{u}]_{ij} \quad for\, i, j = 1 \ldots 3$$

$$[F]=\begin{bmatrix} 1+u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{2,1} \\ 1+u_{2,2} \\ \vdots \end{bmatrix} \qquad [e]=\begin{bmatrix} u_{1,1}+\frac{1}{2}\left(u_{1,1}u_{1,1}+u_{2,1}u_{2,1}+u_{3,1}u_{3,1}\right) \\ \frac{1}{2}\left(u_{1,2}+u_{2,1}+u_{1,1}u_{1,2}+u_{2,1}u_{2,2}+u_{3,1}u_{3,2}\right) \\ \frac{1}{2}\left(u_{1,3}+u_{3,1}+u_{1,1}u_{1,3}+u_{2,1}u_{2,3}+u_{3,1}u_{3,3}\right) \\ \vdots \end{bmatrix}$$

$$[S]=[D][e]$$

From the matrix of tangent elastic coefficients $[D]$

One also needs $\left({}^t\underline{\underline{F}}\,.\,\nabla\,\underline{v}\right)_{\text{sym}}$

From

$$[F]=\begin{bmatrix} 1+u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{2,1} \\ 1+u_{2,2} \\ \vdots \end{bmatrix}$$

and

$$\begin{aligned}[\nabla\,\underline{v}]_{ij} &= \sum_{I=1}^{I=N_e} \frac{\partial N_I\left(\underline{X}\left(\underline{\xi}\right)\right)}{\partial\,\underline{X}}\,.\,\underline{v}_I \\ &= \sum_{I=1}^{I=N_e} \left(\frac{\partial N_I\left(\underline{\xi}\right)}{\partial\,\underline{\xi}}\,.\,\frac{\partial\,\underline{\xi}}{\partial\,\underline{X}}\right).\,\underline{v}_I\end{aligned}$$

$$\left[\left({}^t\underline{\underline{F}}\,.\,\nabla\,\underline{v}\right)_{\text{sym}}\right]=\left[B_m\right]\left[v\right]$$

Where $[B_m]$ is a matrix and $[v]$ the vector of nodal values of the function $\underline{v}$ in the element

$$\int_{\Omega_0} \left(\left(\underline{\underline{\underline{\underline{D}}}} : \Delta \underline{\underline{e}} \right) : \left({}^t \underline{\underline{F}} . \nabla \underline{v} \right)_{\mathrm{sym}} + \underline{\underline{S}} : \left({}^t \left(\nabla \left(\Delta \underline{u} \right) \right) . \nabla \underline{v} \right)_{sym} \right) dx_0 = l(\underline{v}) - a(\underline{u}^k, \underline{v})$$

The right-hand side can be put under the form

$$R(\underline{u}^k) = a(\underline{u}^k, \underline{v}) - l(\underline{v}) = {}^t[V][dF]^k$$

As

$$[\Delta e] = [B_m][d]$$

With [d] the vector of the nodal values of $\Delta \underline{u}$

Material stiffness

$$\int_{\Omega_0} \left(\left(\underline{\underline{\underline{\underline{D}}}} : \Delta \underline{\underline{e}} \right) : \left({}^t \underline{\underline{F}} . \nabla \underline{v} \right)_{\mathrm{sym}} \right) dx_0 = {}^t[V] \left[\int_{\Omega_0} {}^t[B_m][D][B_m] dx_0 \right] [d]$$

Geometric stiffness

$$\int_{\Omega_0} \left(\underline{\underline{S}} : \left({}^t \left(\nabla \left(\Delta \underline{u} \right) \right) . \nabla \underline{v} \right)_{sym} \right) dx_0 = {}^t[V] \left[\int_{\Omega_0} {}^t[B_g][S][B_g] dx_0 \right] [d]$$

One gets the tangent stiffness matrix

$$[K_T] = \int_{\Omega_0} \left[{}^t[B_m][D][B_m] + {}^t[B_g][S][B_g] \right] dx_0$$

So that the incremental equation is

$${}^t[V][K_T][d_k] = {}^t[V][dF]^k$$

K_T changes at each step

One continues until the norm of [d] is inferior to a given small value

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THE END