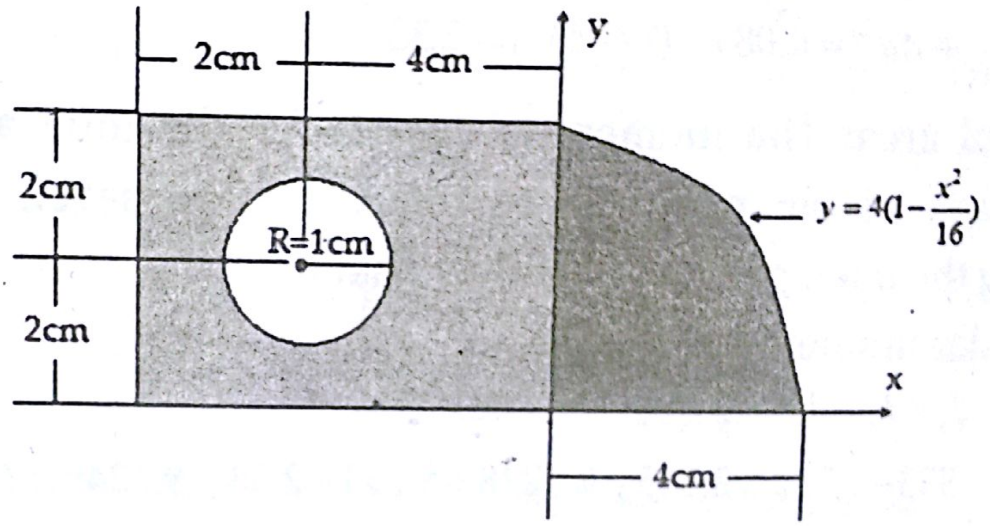
Review Problems on Chapter-6

1. Consider the shaded area below. Then, determine

a) The centroid of the shaded area

b) The rectangular and polar moments of inertia

c) The rectangular and polar radius of gyrations



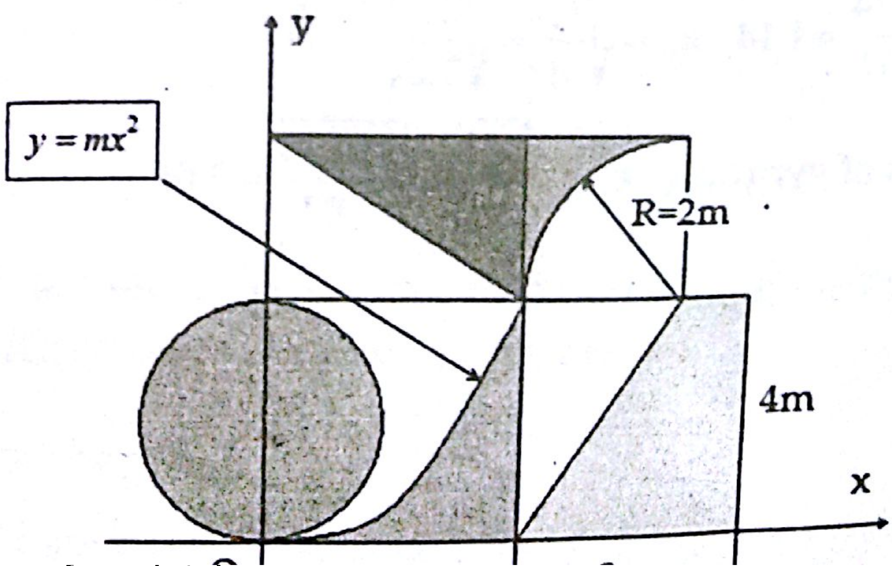
Answer : $I_x = 153.7$, $I_y = 271.1$

2*. Consider the shaded area below. Then, determine

a) The centroid of the shaded area

b) The rectangular and polar moments of inertia

c) The rectangular and polar radius of gyrations

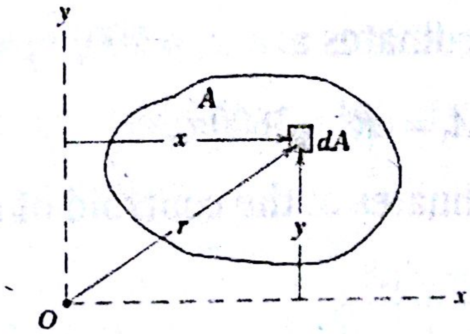


CHAPTER-6

AREA MOMENTS OF INERTIA

6.1 Rectangular and Polar moments of Inertia

Consider the area A of a region in the xy-plane as shown below.



Then, the moments of inertia of the differential element dA are;

i) About x-axis: $dI_x = y^2 dA$

ii) About y-axis: $dI_y = x^2 dA$

Then, if the integration is performed over the whole area A,

i) About x-axis: $I_x = \int y^2 dA$

ii) About y-axis: $I_y = \int x^2 dA$

These two moments are called rectangular moments of inertia

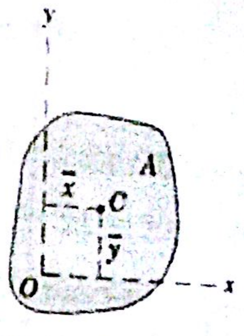
Polar Moment of Inertia:

The moment of inertia of dA about the pole at O, that is about the z-axis, is defined by $I_z = I_O = \int r^2 dA$. But from polar coordinates, we have $r^2 = x^2 + y^2$. Hence, using this relation, we have

$$I_z = \int r^2 dA = \int (x^2 + y^2) dA = \int x^2 dA + \int y^2 dA = I_x + I_y$$

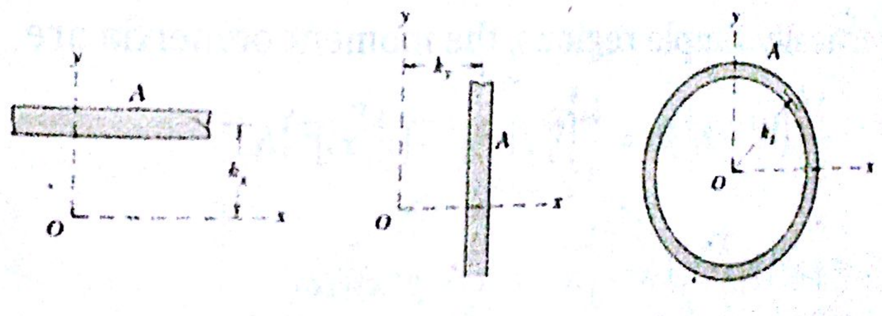
The moment of inertia $I_z = I_x + I_y$ is called polar moment of inertia.

Radius of Gyration: The radius of gyration is the measure of the distribution of the area from the specified axis. A rectangular or polar moment of inertia can be expressed by specifying the radius of gyration and the area. Consider an area A having moments of inertia I_x and I_y with a polar moment of inertia I_z about O as shown.



(a)

Assume a strip of area A at a distance of k_x from the x-axis.



Then the moment of inertia of the strip about the x-axis will be the same as that of the original area if $I_x = k_x^2.A, I_y = k_y^2.A, I_z = k_z^2.A$. Here, the distance k_x, k_y are called radius of gyration of area.

Relation: How k_x, k_y and k_z are related?

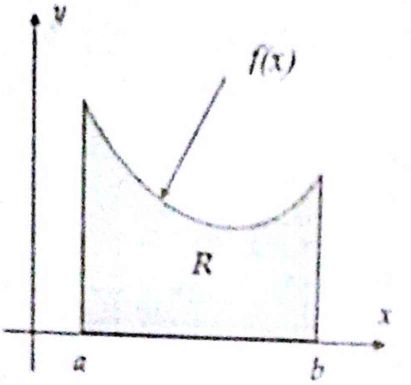
Using the $I_x = k_x^2.A,\ I_y = k_y^2.A,\ I_z = k_z^2.A$, in the relation $I_z = I_x + I_y$, we get the relation among the three radius of gyration as follow.

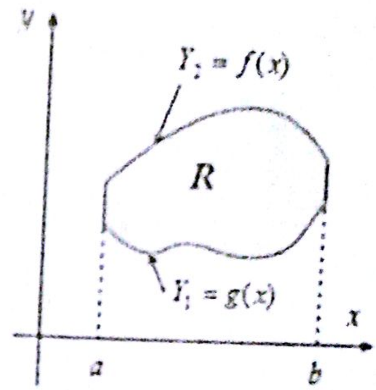
$$I_z = I_x + I_y \Rightarrow Ak_z^2 = Ak_x^2 + Ak_y^2 \Rightarrow k_z^2 = k_x^2 + k_y^2 \Rightarrow k_z = \sqrt{k_x^2 + k_y^2}$$

Computational Formula of Inertia:

Case-I: For Vertically Simple Region

Suppose the region R is above the x-axis, $Y_1 = 0$ under the graph of $Y_2 = f(x)$ between two vertical lines (as in diagram-I) or R is a region bounded between the graphs of f and g as in the diagram-II.





I) $R: \begin{cases} a \le x \le b \\ 0 \le y \le f(x) \end{cases}$ II) $R: \begin{cases} a \le x \le b \\ g(x) \le y \le f(x) \end{cases}$

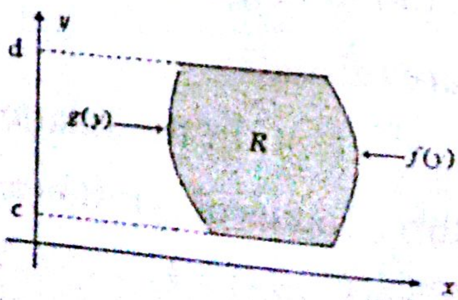
Then, for vertically simple regions, the moment of inertia are

$$\text{I)} \begin{cases} I_x = \dfrac{1}{3}\displaystyle\int_a^b \left(Y_2^3 - Y_1^3\right)dx = \dfrac{1}{3}\int_a^b \left([f(x)]^2 - [g(x)]^2\right)dx, \\ I_y = \displaystyle\int_a^b x^2[Y_2 - Y_1]dx = \int_a^b x^2[f(x) - g(x)]dx \end{cases}$$

Case-II: For Horizontally Simple Region

Suppose R is the region between the graphs of two continuous functions f and g as shown in diagram-III.

III)



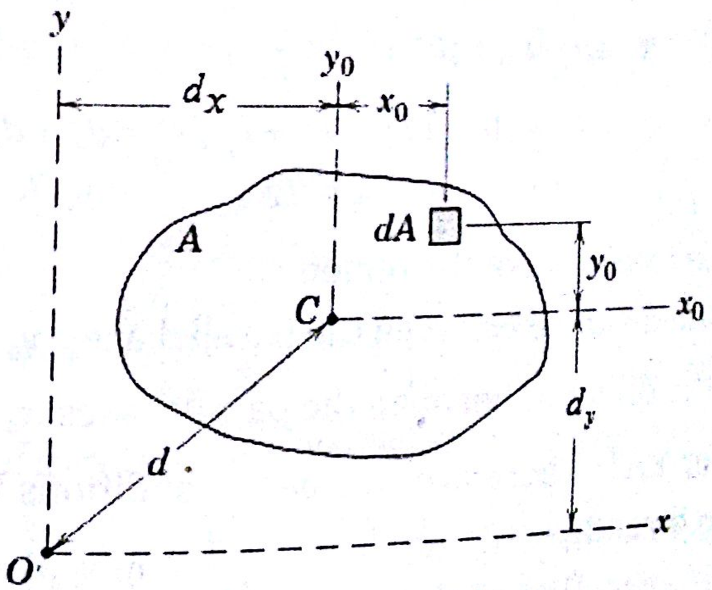
$R: \begin{cases} c \le x \le d \\ g(y) \le x \le f(y) \end{cases}$

Then, for horizontally simple regions, the moment of inertia of the area is given by the formulas;

$$\text{II)}\begin{cases} I_x = \int_c^d y^2[X_2 - X_1]dy = \int_c^d y^2[f(y) - g(y)]dy \\ I_y = \frac{1}{3}\int_c^d \left(X_2^3 - X_1^3\right)dy = \frac{1}{3}\int_c^d \left([f(y)]^2 - [g(y)]^2\right)dy \end{cases}$$

Transfer of Axis:

Centroidal Axis: An axis that passes through the centroid of the area is called centroidal axis. Any axis that does not pass through the centroid of the area is called non-centroidal axis. Now, let's see how can we determine the moments of inertia of an area about a non-centridal axis parallel to the centridal axis. Consider the area with centriod at C where x_0 and y_0 are centrioidal axes.



Suppose the non-centroidal axis x is parallel to x_0 and that of y is parallel to y_0. Then, from the definition of inertia,

$$I_x = \int (y_0 + d_y)^2 dA = \int (y_0^2 + 2y_0 d_y + d_y^2) dA$$
$$= \int y_0^2 dA + \int 2y_0 d_y dA + \int d_y^2 dA, \text{ (Notice that} \int 2y_0 d_y dA = 0)$$
$$= \int y_0^2 dA + d_y^2 \int dA$$
$$= I_{x_0} + Ad_y^2$$

Similarly, we have $I_y = I_{y_0} + Ad_x^2$. Thus, here is the generalization.

Parallel Axis Theorem (PAT):

Given, the moments of inertia of an area I_{x_0} and I_{y_0} about the centroidal axes x_0 and y_0 respectively. Then, the moments of inertia of the same area about non-cenrtroidal parallel axes are given by:

$$I_x = I_{x_0} + Ad_y^2$$
$$I_y = I_{y_0} + d_x^2 A$$
$$I_z = I_{z_0} + Ar^2, \ (I_{z_0} = I_{x_0} + I_{y_0}, \ r^2 = d_x^2 + d_y^2)$$

Where

A is the total area of the region

d_x is the distance between the parallel axes y_0 and y

d_y is the distance between the parallel axes x_0 and x

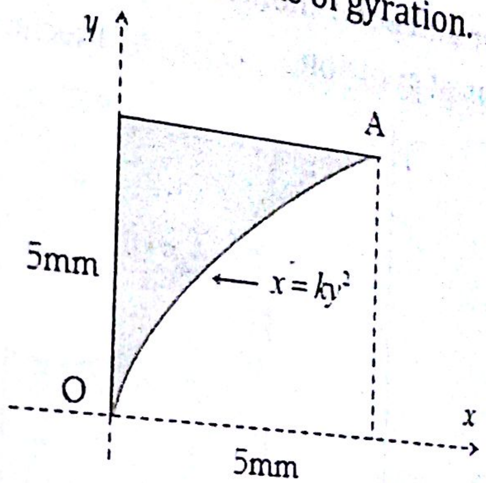
Conditions for PAT: There are two basic conditions to apply parallel axis theorem directly.

i) *Centroidal axes*: One of the pairs of axes must pass through the centroid of the area.

ii) *Parallelism*: One of the other pair of axes must be parallel to the respective centroidal axes.

Examples:

1. Determine the moments of inertia of the shaded area with respect to the x and y-axes and give the radius of gyration.



Solution: First, find the value of k. At point A, we have $x=5, y=5$.

Then, $x = ky^2 \Rightarrow 5 = k(5)^2 \Rightarrow 5 = 25k \Rightarrow k = \frac{1}{5}$. Besides, the region is horizontally simple bounded from the left by y-axis, $X_1 = g(y) = 0$ and from the right by $X_2 = f(y) = \frac{1}{5}y^2$. Then, by the general formula,

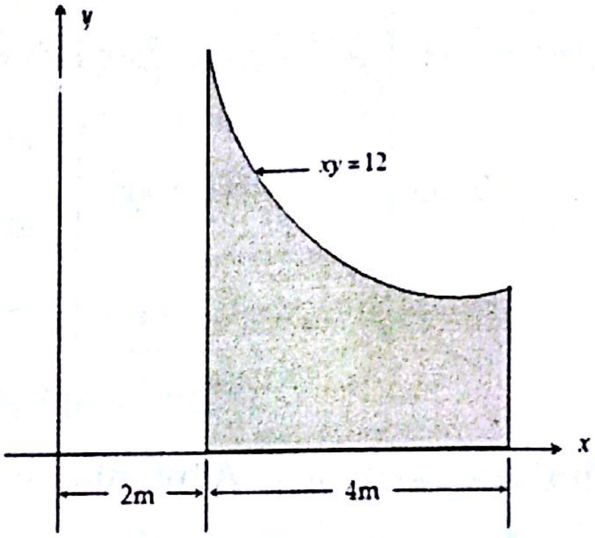
i) $I_x = \int_0^5 y^2[X_2 - X_1]dy = \int_0^5 y^2[f(y) - g(y)]dy = \frac{1}{5}\int_0^5 y^4 dy = \frac{y^5}{25}\Big|_{y=0}^{y=5} = 125\text{mm}^4$

ii) $I_y = \frac{1}{3}\int_0^5 (X_2^3 - X_1^3)dy = \frac{1}{3}\int_0^5 (\frac{y^2}{5})^3 dy = \frac{1}{3}\int_0^5 \frac{y^6}{5}dy = \frac{y^7}{105}\Big|_{y=0}^{y=5} = \frac{625}{21}\text{mm}^4$

For the radius of gyration, first find the area of the region.

The area is $A = \int_0^5 \frac{1}{5}y^2 dy = \frac{y^3}{15}\Big|_{y=0}^{y=5} = \frac{25}{3}$. So, the radius of gyrations are given by $k_x = \sqrt{\frac{I_x}{A}} = \sqrt{15}$, $k_y = \sqrt{\frac{I_y}{A}} = 5\sqrt{21}$.

area below, determine
centroid of the area
b) The rectangular and polar moments of inertia about x and y-axes
c) The polar radius of gyration



Solution:

a) We need the centroid. First, determine the area of the region. Since the region is vertically simple, its area is calculated as follow.

That is $A=\int[f(x)-g(x)]dx=\int_2^6 \frac{12}{x}dx=12\ln x\Big|_{x=2}^{x=6}=12(\ln 6-\ln 2)=12\ln 3$

Then, from the centroid formula, we have

i) $A\bar{x}=\int x[f(x)-g(x)]dx \Rightarrow (12\ln 3)\bar{x}=\int_2^6 x(\frac{12}{x})dx=\int_2^6 12dx \Rightarrow \bar{x}=\frac{2}{\ln 3}$

ii) $A\bar{y}=\frac{1}{2}\int[f(x)]^2dx=\frac{1}{2}\int_2^6(\frac{12}{x})^2dx \Rightarrow (12\ln 3)\bar{y}=18 \Rightarrow \bar{y}=\frac{3}{2\ln 3}$

b) The moment of inertia about x and y axes

Rectangular moments of inertia:

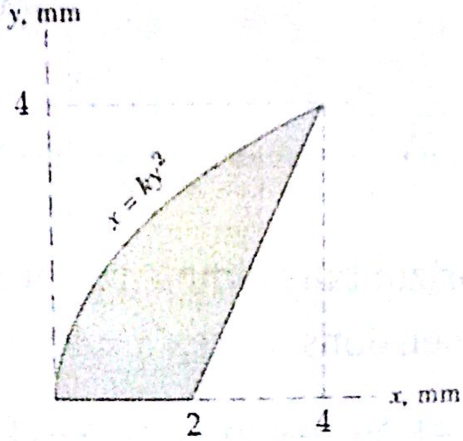
i) $I_x=\frac{1}{3}\int_2^6(y_2^3-y_1^3)dx=\frac{1}{3}\int_2^6(\frac{12}{x})^3]dx=\frac{1}{3}\int_2^6\frac{12^3}{x^3}dx=-\frac{1}{6}(\frac{12^3}{x^2})\Big|_{x=2}^{x=6}=64$

ii) $I_y=\int_2^6 x^2(y_2-y_1)dx=\int_2^6 x^2(\frac{12}{x})dx=\int_2^6 12xdx=192$

Polar moment of inertia: $I_z = I_x + I_y = 64 + 192 = 256$

c) The polar radius of gyration: $k_z = \sqrt{\dfrac{I_z}{A}} = \sqrt{\dfrac{256}{12\ln 3}} = 4.4$.

3. Determine the moment of inertia of the area about the y-axis.



Solution: Let's use horizontally simple region.

First, find the value of k using the given dimensions.

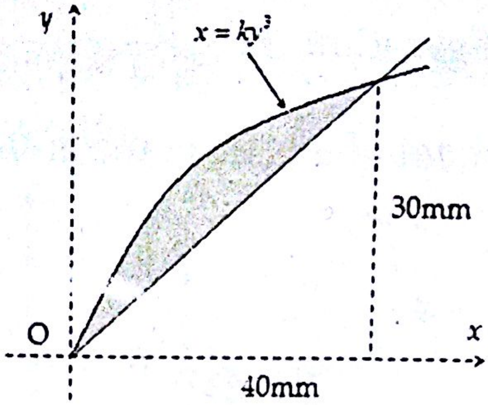
Here, $x = 4$, $y = 4$. So, using the given function,

$x = ky^2 \Rightarrow 4 = k(4)^2 \Rightarrow 4 = 16k \Rightarrow k = \dfrac{1}{4}$. The region is bounded on the left side by the curve $x_1 = f(y) = \dfrac{1}{4}y^2$. Besides, the equation of the right side boundary of the region is obtained using slope intercept form to be $y = 2x - 4$. From this $x_2 = \dfrac{y+4}{2}$.

$$I_y = \int_R \frac{1}{3}(x_2^3 - x_1^3)\,dA = \frac{1}{3}\int_R [(2+\frac{y}{2})^3 - (\frac{1}{4}y^2)^3\,dy = \frac{1}{3}\int_0^4 (2+\frac{y}{2})^3 dy - \frac{1}{3}\int_0^4 \frac{y^6}{64})dy$$

$$= \frac{1}{6}(2+\frac{y}{2})^4\Big|_{y=0}^{y=4} - \frac{y^7}{21(64)}\Big|_{y=0}^{y=4} = 40 - 12.2 = 27.8mm^4$$

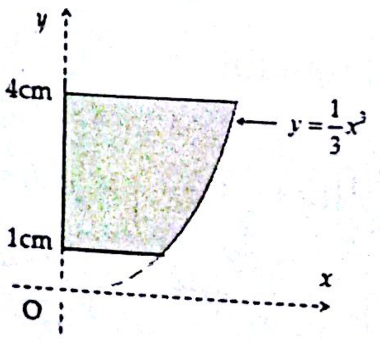
4. Determine the moment of inertia of the area about the y-axis.



Solution: Let's use horizontally simple region. First, find the value of k using the given dimensions. Here, $x = 40,\ y = 30$. So, using the given function, $x = ky^3 \Rightarrow 40 = k(30)^3 \Rightarrow 40 = (30)^3 k \Rightarrow k = \dfrac{4}{2700}$. The region is bounded from the left by the curve $x = f(y) = \dfrac{4}{2700}y^3$. Besides, the equation of the right boundary is to be $y = \dfrac{3}{4}x$. From this $x = \dfrac{4}{3}y$.

$$I_x = \int_0^{30} y^2[X_2 - X_1]dy = \int_0^{30} y^2\left(\frac{4}{3}y - \frac{4}{2700}y^3\right)dy = \frac{y^4}{3} - \frac{y^6}{4050}\Bigg|_{y=0}^{y=30} = 9\times10^4\,mm^4$$

5. Determine the moment of inertia of the area about the y-axis.



Solution: Let's use horizontally simple region.

$$I_y = \int_1^4 \frac{1}{3}x^3 dy = \frac{1}{3}\int_1^4 (\sqrt{3y})^3 dy = \frac{1}{3}\int_1^4 y^{\frac{3}{2}} dy = \frac{2\sqrt{3}}{5}y^{\frac{5}{2}}\Bigg|_{y=1}^{y=4} = \frac{62\sqrt{3}}{5} = 21.5\text{mm}^4$$

6.2 COMPOSITE AREAS

In many situations, we encounter to calculate the moment of inertia of an area composed of two or more distinct parts of simple and basic geometric shapes. The union of two or more non-overlapping regions is called composite area. Then, the moment of a composite area about a particular axis is the algebraic sum of the moments of inertia of its component parts about the same axis.

Procedures to find moment of inertia for composite area:

Step-1: Determination of moments of different parts.

Divide the composite area into different parts. For each part, select appropriate coordinate axes and compute the moments.

Step-2: Transferring into a common axis

Check if transferring is needed or not for each moment. Of course, some of them may not need transferring but for these where transferring is needed use Parallel Axis Theorem to transfer into the common axis about which the moment of the area is asked. For some parts, repeated application of parallel axis theorem may be needed.

Step-3: Summarization of results. To simplify the calculation, use the table of the following form. When you tabulate values make the moments of the open parts of the area as negative.

Parts	Area Of parts	Distance between centoid of parts and the COMMON axis		Transferring element		Moments of parts about centroidal axis	
		d_x	d_y	Ad_x^2	Ad_y^2	$\bar{I}_x$	$\bar{I}_y$
1							
2							
3							
4							
SUMS	$\sum A_i$			$\sum Ad_x^2$	$\sum Ad_y^2$	$\sum \bar{I}_x$	$\sum \bar{I}_y$

Conclusion: Using the sum of the four columns,

i) Rectangular and polar moments of inertia;

$$I_x = \sum \overline{I}_x + \sum A.d_y^2, \quad I_y = \sum \overline{I}_Y + \sum A.d_x^2, I_z = I_x + I_y$$

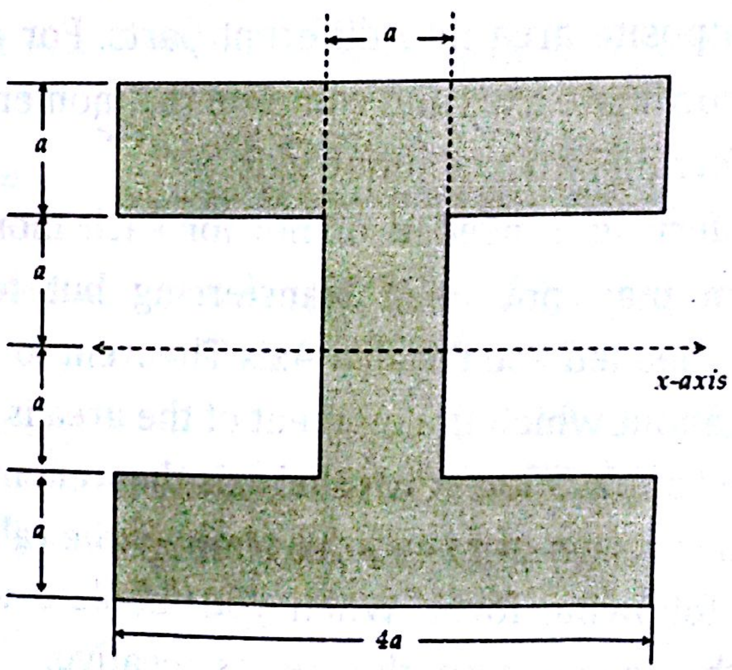
ii) Rectangular and polar radius of gyrations;

$$k_x = \sqrt{\frac{I_x}{A}}, \quad k_y = \sqrt{\frac{I_y}{A}}, k_z = \sqrt{\frac{I_z}{A}}$$

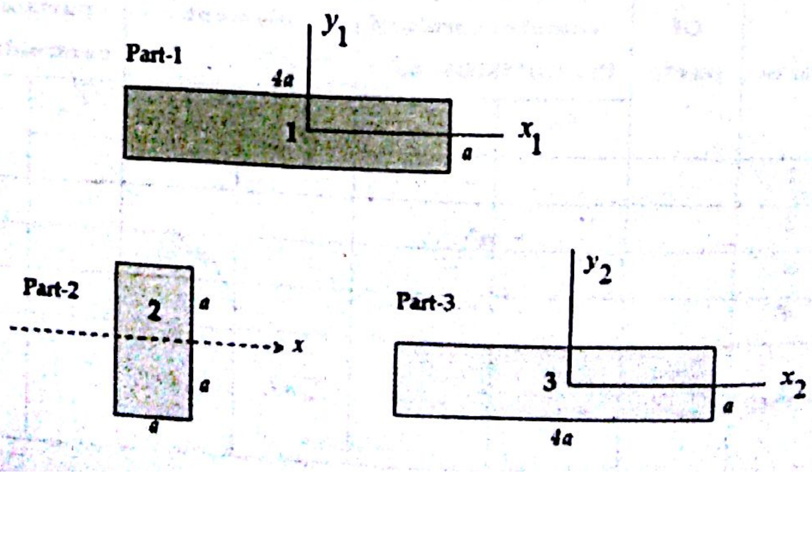
Examples:

1. Consider a composite area with $a = \sqrt{3}mm$ as shown below.

Determine the moment of inertia of the shaded area about the x-axis,



Solution: Let's divide the region into three parts.



Part-1: It is a rectangle with base $b=4a$ and height $h=a$. From moment of inertia of area of a rectangle, the moment of inertia of the rectangular area of part-1 about x_1-axis is $\bar{I}_{x_1}=\frac{bh^3}{12}=\frac{4a^4}{12}=\frac{a^4}{3}$.

Besides, its area is $A_1=(4a)(a)=4a^2$.

Then, by Parallel Axis Theorem, it is transferred to x-axis as

$$I_{1x}=\bar{I}_{x_1}+Ay^2=\frac{a^4}{3}+4a^2(\frac{3a}{2})^2=\frac{a^4}{3}+9a^4=\frac{28a^4}{3}$$

Part-2: It is also a rectangle with base $b=a$ and height $h=2a$. The moment for this part is calculated directly about x-axis.

So, using similar formula, we have $I_{2x}=\frac{bh^3}{12}=\frac{8a^4}{12}=\frac{2a^4}{3}$.

Part-3: It is the same with part-1. That is $\bar{I}_{x_2}=\frac{bh^3}{12}=\frac{4a^4}{12}=\frac{a^4}{3}$.

Then, by Parallel Axis Theorem, it is transferred to x-axis as

$$I_{3x}=\bar{I}_{x_2}+Ay^2=\frac{a^4}{3}+4a^2(\frac{3a}{2})^2=\frac{a^4}{3}+9a^4=\frac{28a^4}{3}$$

Therefore, the total moment of inertia of the entire composite area is determined by taking the sum of the three moments.

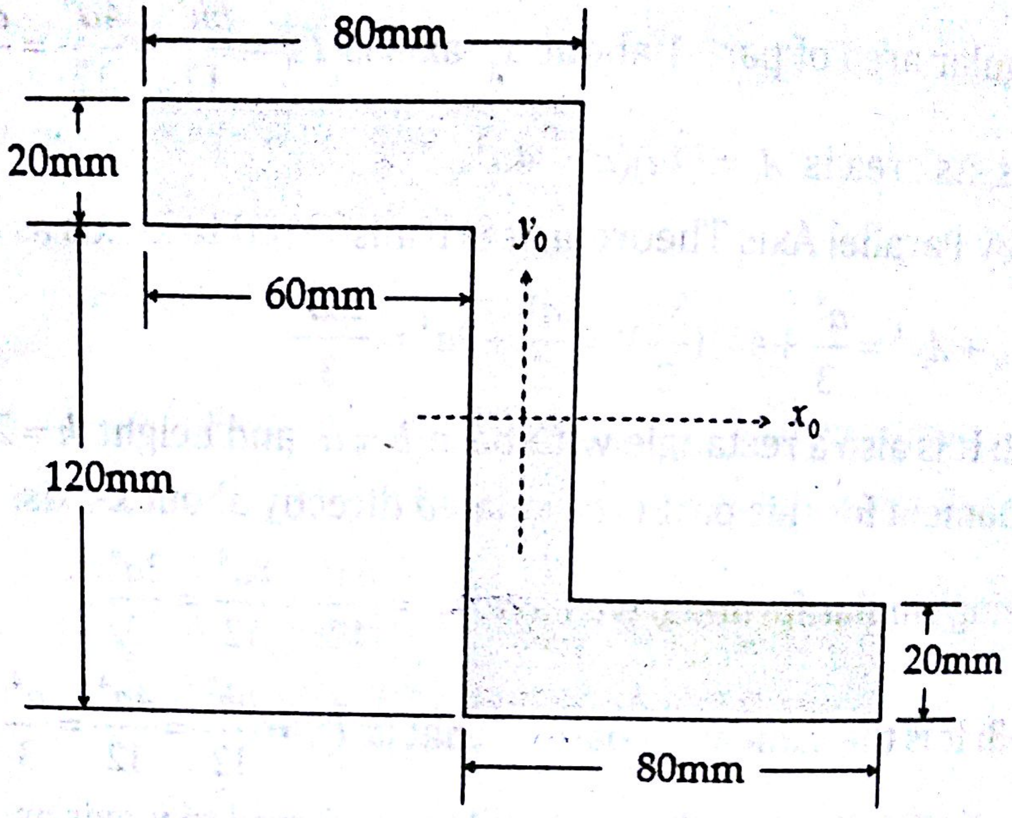
That is $I_x=I_{1x}+I_{2x}+I_{3x}=\frac{28a^4}{3}+\frac{2a^4}{3}+\frac{28a^4}{3}=\frac{58a^4}{3}$

In particular, for $a=\sqrt{3}\text{mm}$, we have

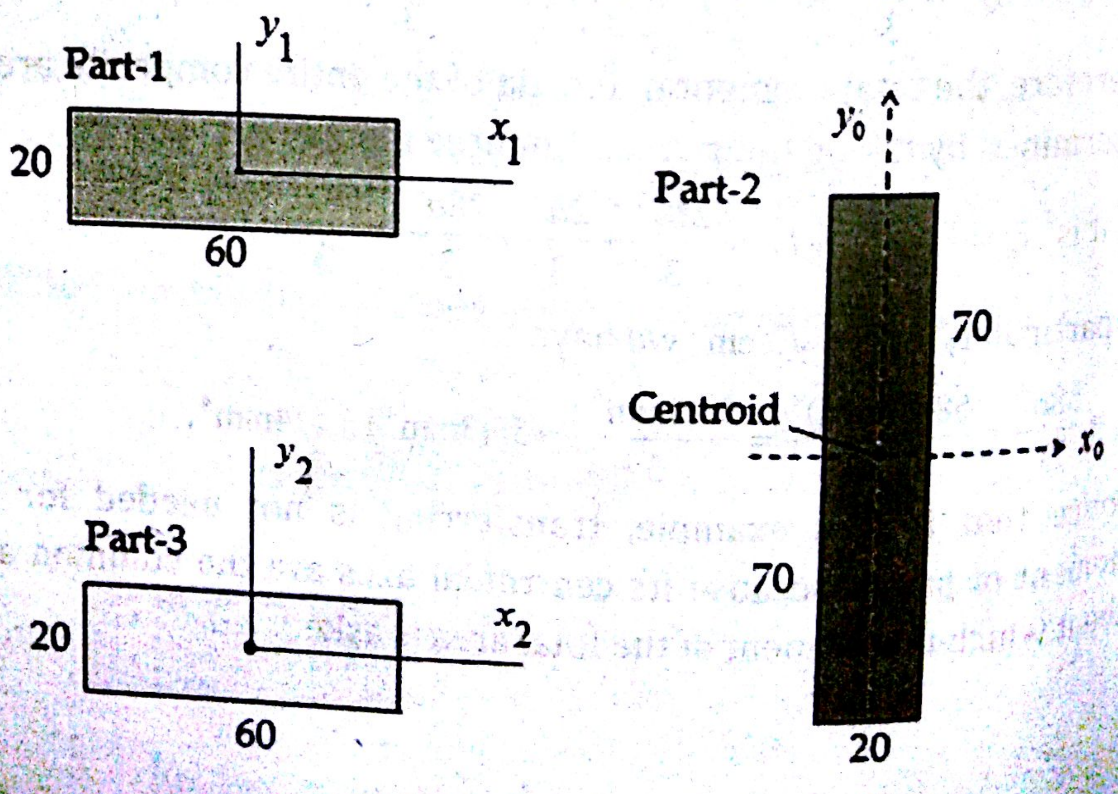
$$I_x=\frac{58a^4}{3}=\frac{58(\sqrt{3}\text{mm})^4}{3}=\frac{58(9\text{mm}^4)}{3}=58(3\text{mm}^4)=174\text{mm}^4.$$

Notice that in this example, transferring is not needed for the moment of part-2 because its centroidal axes are the common axes about which the moment of the total area is asked.

2. Determine the moments of inertia of the Z-section about the centroidal axes x_0 and y_0.



Solution: Let's divide the region into three parts.



Part-1: It is a rectangle with base $b=60\text{mm}$ and height $h=20\text{mm}$. From moment of inertia of a rectangle, using parallel axis theorem, the moments of inertia of part-1 about x_0 and y_0 axes are

i) $$I_{1x_0}=\bar{I}_{x_1}+Ad^2=\frac{bh^3}{12}+bhd^2=\frac{(60)(20)^3}{12}+(60)(20)(60)^2$$

$$=4\times10^4\text{mm}^4+432\times10^4\text{mm}^4=436\times10^4\text{mm}^4$$

ii) $$I_{1y_0}=\bar{I}_{y_1}+Ad^2=\frac{hb^3}{12}+bhd^2=\frac{(20)(60)^3}{12}+(60)(20)(40)^2$$

$$=36\times10^4\text{mm}^4+192\times10^4\text{mm}^4=228\times10^4\text{mm}^4$$

Part-2: It is also a rectangle with base $b=20\text{mm}$ and height $h=140\text{mm}$. The moment for this part is calculated directly about the centroidal x_0 and y_0 axes.

i) $$I_{2x_0}=\frac{bh^3}{12}=\frac{(20)(140)^3}{12}=457.3\times10^4\text{mm}^4$$

ii) $$I_{2y_0}=\frac{hb^3}{12}=\frac{(140)(20)^3}{12}=9.3\times10^4\text{mm}^4$$

Part-3: It is a rectangle with base $b=60\text{mm}$ and height $h=20\text{mm}$. From moment of inertia of a rectangle, using parallel axis theorem, the moments of inertia of part-3 about x_0 and y_0 axes are

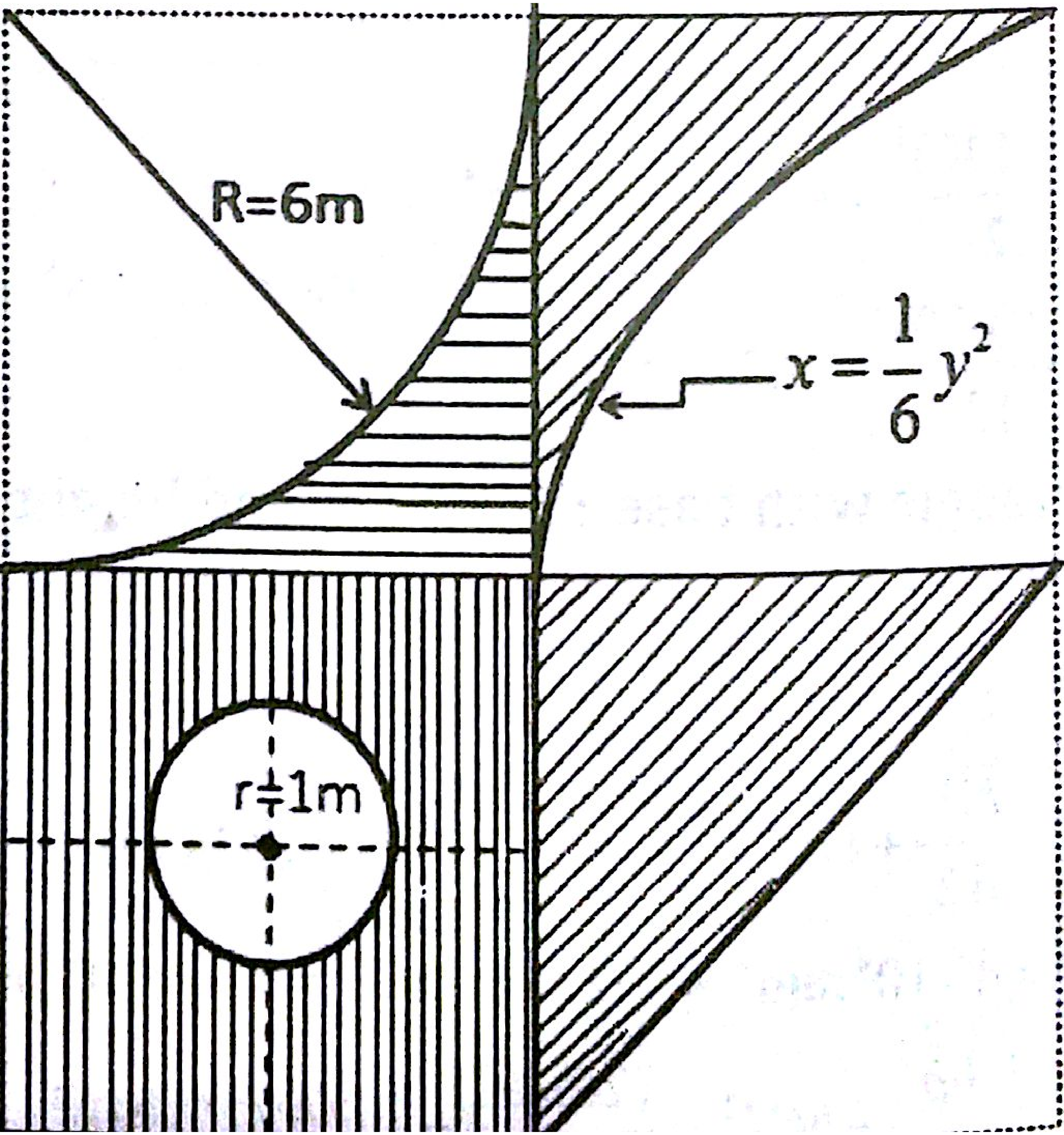
i) $$I_{3x_0}=\bar{I}_{x_2}+Ad^2=\frac{bh^3}{12}+bhd^2=\frac{(60)(20)^3}{12}+(60)(20)(60)^2$$

$$=4\times10^4\text{mm}^4+432\times10^4\text{mm}^4=436\times10^4\text{mm}^4$$

ii) $$I_{2y_0}=\bar{I}_{y_2}+Ad^2=\frac{hb^3}{12}+bhd^2=\frac{(20)(60)^3}{12}+(60)(20)(40)^2$$

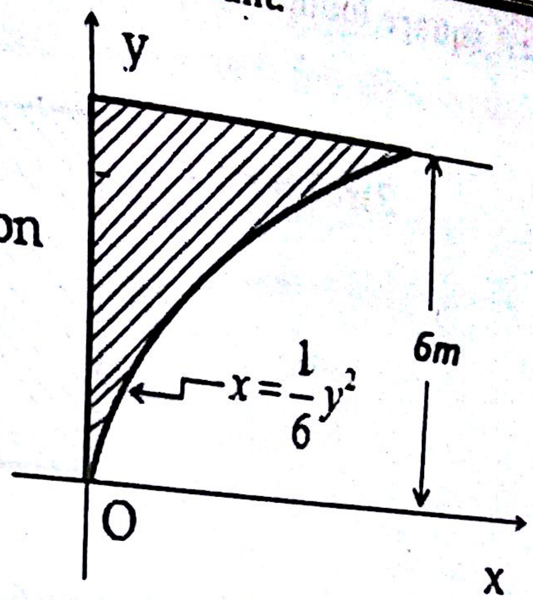
$$=36\times10^4\text{mm}^4+192\times10^4\text{mm}^4=228\times10^4\text{mm}^4$$

Therefore, the total moments of inertia of the entire composite area about the axes x_0 and y_0 is determined by taking the sum of the three moments along the same axis.


R=6m
$x=\frac{1}{6}y^2$
r=1m

Part-1: The parabolic Area in the first quadrant.

Part-1:Parabolic Region



As we see from the diagram, this region is bounded on the right by $x=\frac{1}{6}y^2$ and on the left by $x=0$ on the interval $0\le y\le 6$.

So, let's find the area A of this region.

That is $A=\int_R[f(y)-g(y)dy=\int_0^6\frac{1}{6}y^2dy=\frac{1}{18}y^3\Big|_{y=0}^{y=6}=\frac{216}{18}=12m^2$

Besides, the coordinates of the centroid are

i) $A.\bar{x}=\frac{1}{2}\int_c^d([f(y)]^2-[g(y)]^2)dy \Rightarrow 12\bar{x}=\frac{1}{2}\int_0^6\left(\frac{1}{6}y^2\right)^2dy=\frac{1}{72}\int_0^6 y^4dy$

$$\Rightarrow 12\bar{x}=\frac{y^5}{360}\Big|_{y=0}^{y=6}=\frac{216}{10}$$

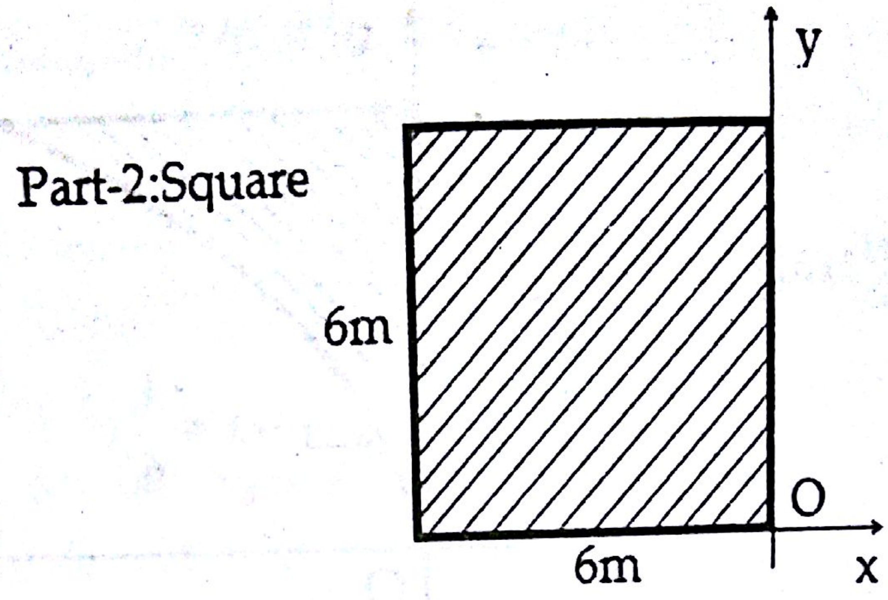
$$\Rightarrow 12\bar{x}=\frac{216}{120}=1.8m$$

ii) $A.\bar{y}=\int_c^d[y(f(y)-g(y)]dy=\int_0^6 y\left(\frac{1}{6}y^2\right)dy$

$$\Rightarrow 12\bar{y}=\int_0^6\frac{y^3}{6}dy=\frac{y^4}{24}\Big|_{y=0}^{y=6}=54$$

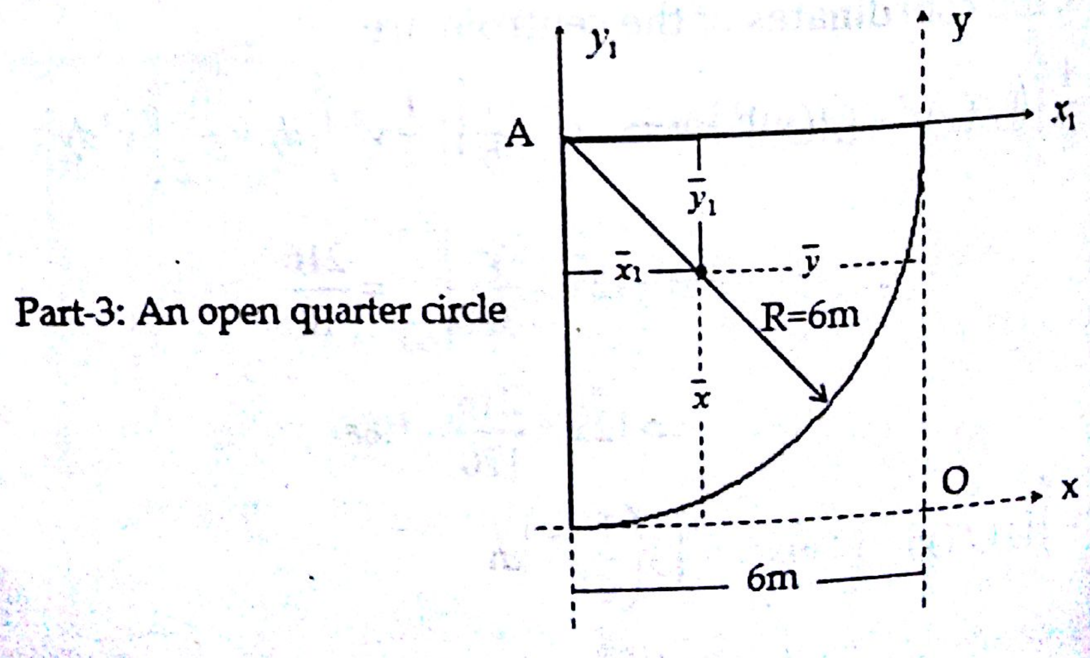
$$\Rightarrow \bar{y}=4.5m$$

Part-2: A square found in the second quadrant as shown.



The area of this square is $A = 6m \times 6m = 36m^2$. Notice that centroid is the geometric center of a region. Just it is a point whose coordinates can be positive, negative or zero depending on the nature of the region under consideration. In our case, as the square region is found in the second quadrant, we expect the x-coordinate of its centroid to be negative. So, the centroid coordinates are $\bar{x} = -3,\ \bar{y} = 3$.

Part-3: A quarter circle of radius $R = 6m$ in the second quadrant



The area of the quarter circle is $A = \dfrac{\pi R^2}{4} = \dfrac{36\pi}{4} = 9\pi m^2$.

Think-Twice: How to get the centroid? It is not obtained directly as we did for the others. Because the center of the quarter circle is at A not at O but we need the centroid relative to O not relative to A. Dear users! What shall we do then?

First, determine the coordinates relative to the new axes x_1 and y_1 which are intersecting at A (the center of the quarter circle). But relative to x_1 and y_1, we have

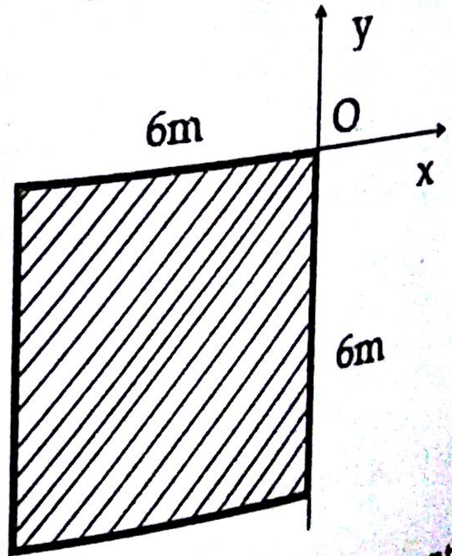
$$\bar{x}_1 = -\frac{4R}{3\pi} = -\frac{24}{3\pi} = -\frac{8}{\pi} = -2.55,$$

$$\bar{y}_1 = \frac{4R}{3\pi} = \frac{24}{3\pi} = \frac{8}{\pi} = 2.55$$

Now, as we see from the diagram, the coordinates of the centroid $C = (\bar{x}, \bar{y})$ relative to the x and y axes are obtained by subtraction.

That is $\begin{cases} \bar{x} = -6 - \bar{x}_1 = -6 - (-2.55) = -6 + 2.55 = -3.45 \\ \bar{y} = 6 - \bar{y}_1 = 6 - 2.55 = 3.45 \end{cases}$

Part-4 A square found in the third quadrant as shown.

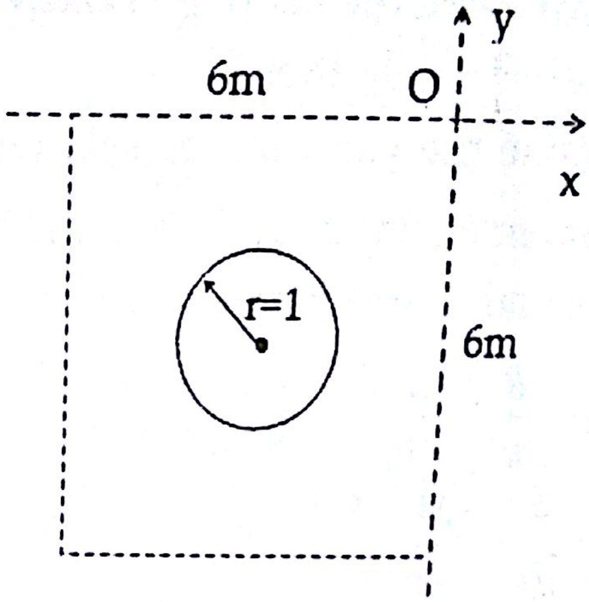


Part-4 Square

Then the area is $A = 36m^2$ and the coordinates of the centroid are $\bar{x} = -3, \bar{y} = -3$. (Why both coordinates are negative?)

Part-5: A circle of radius $r = 1m$ in the third quadrant as shown.

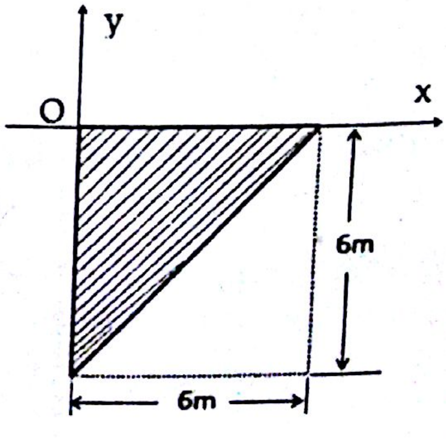
The area of the circle is $A = \pi r^2 = \pi m^2 = \pi m^2$.



Part-5: An open circle in the third quadrant

Notice: For a full circular region, the centroid is always the center of the circle. Wow! That is great. But the circle is found at the center of a square of side $6m$. So, the center of the circle is $C = (-3,-3)$. Hence, the coordinates of the centroid are $\bar{x} = -3, \bar{y} = -3$.

Part-6: A right angle triangle of side $S = 6m$ in the fourth quadrant.



Part-6: A triangle in the IV quadrant

Then, the area of the triangle is $A = \frac{1}{2}(6m)(6m) = 18m^2$ and the coordinates of the centroid are $\bar{x} = \frac{b}{3} = \frac{6}{3} = 2, \bar{y} = -\frac{h}{3} = -\frac{6}{3} = -2$.

Now, let's go back to part (a) of the problem.

a) Centroid of the total area. Consider the following summary table.

Parts	Areas (m^2)	$\bar{x}(m)$	$\bar{y}(m)$	$\bar{x}.A\ (m^3)$	$\bar{y}.A(m^3)$
1	12	1.8	4.5	21.6	54
2	36	−3	3	−108	108
3	-9π	−3.45	3.45	97.5	−97.5
4	36	−3	−3	−108	−108
5	$-\pi$	−3	−3	3π	3π
6	18	2	−2	36	−36
Sums	$\sum_{i=1}^{6} A_i = 70.6$			$\sum_{i=1}^{6} A_i.\bar{x}_i = -51.5$	$\sum_{i=1}^{6} A_i.\bar{y}_i = -70.1$

Therefore, the coordinates of the centroids are

$$\bar{X} = \frac{\sum_{i=1}^{n} A_i.\bar{x}_i}{\sum_{i=1}^{n} A_i} = \frac{-51.5}{70.6} = -0.729\text{m}, \quad \bar{Y} = \frac{\sum_{i=1}^{n} A_i.\bar{y}_i}{\sum_{i=1}^{n} A_i} = \frac{-70.1}{70.6} = -0.993\text{m}$$

b) The rectangular and polar moments of inertia.

To find the moment of inertia of the total area, first determine the moments of inertia of each of the six parts. Please give attention how the inertia of each part is calculated.

Part-1: The parabolic curve

$$I_x = \int_R y^2 dA = \int_R y^2 x dy = \int_0^6 y^2.\frac{1}{6}y^2 dy = \frac{1}{6}\int_0^6 y^4 dy = \left.\frac{y^5}{30}\right|_{y=0}^{y=6} = 259.2$$

$$I_y = \frac{1}{3}\int_R x^2 dA = \frac{1}{3}\int_R x^2.x dy = \frac{1}{3}\int_0^6 \frac{y^6}{216} dy = \left.\frac{y^7}{4536}\right|_{y=0}^{y=6} = 61.7$$

Part-2: For the square, about its centroid,

$$I_{\bar{x}} = I_{\bar{y}} = \frac{bh^3}{12} = \frac{6(6^3)}{12} = 108$$

Part-3: For the quarter circle about its centroid,

$$I_{\bar{x}} = I_{\bar{y}} = R^4\left(\frac{\pi}{16} - \frac{4}{9\pi}\right) = 1296\left(\frac{\pi}{16} - \frac{4}{9\pi}\right) = 70.6$$

Part-4: For the square, about its centroid,

$$I_{\bar{x}} = I_{\bar{y}} = \frac{bh^3}{12} = \frac{6(6^3)}{12} = 108$$

Part-5: For the small open circle, about its centroid,

$$I_{\bar{x}} = I_{\bar{y}} = \frac{\pi r^4}{4} = \frac{\pi}{4} = 0.785$$

Part-6: For the triangle, about its centroid,

$$I_{\bar{x}} = I_{\bar{y}} = \frac{bh^3}{36} = \frac{6(6^3)}{36} = 36$$

Now, let's construct the following summary table.

Parts	Areas	d_x	d_y	$A.d_x^2$	$A.d_y^2$	$\bar{I}x$	$\bar{I}y$
1	12	1.8	4.5	-	-	-	-
2	36	-3	3	324	324	108	108
3	-9π	-3.45	3.45	-336.4	-336.4	-70.6	-70.6
4	36	-3	-3	324	324	108	108
5	$-\pi$	-3	-3	-9π	-9π	-0.785	-0.785
6	18	2	-2	72	72	36	36
Sums	70.6			355.34	355.34	180.62	180.62

Therefore, for the total area, by applying Parallel axes Theorem, we have the following results.

i) Rectangular moments of inertia;

$$I_x = \sum \bar{I}x + \sum A.d_x^2 + I_{x(\text{The parabola})} = 355.34 + 180.62 + 259.2 = 795.16$$

$$I_y = \sum \bar{I}y + \sum A.d_y^2 + I_{y(\text{The parabola})} = 355.34 + 180.62 + 61.7 = 597.66$$

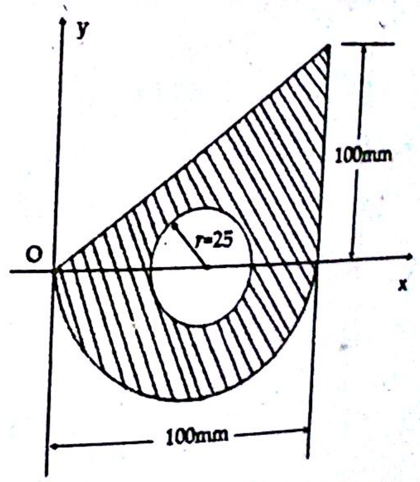
ii) Polar moment of inertia; $I_z = I_x + I_y = 795.16 + 597.66 = 1392.82$

c) i) Rectangular radius of gyration,

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{795.16}{70.6}} = 3.356,\quad k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{597.66}{70.6}} = 2.91$$

c) ii) Polar radius of gyration, $k_z = \sqrt{\frac{I_z}{A}} = \sqrt{\frac{1392.82}{70.6}} = 4.44$

4. For the shaded area, determine the moment of inertia about x-axis



Solution: Let's divide the region into three parts.

Part-1: A triangle of side $b = 300mm$ and height $h = 100mm$.

As we see from the diagram of the problem, the base of the triangle is along x-axis. But we know that for a triangle, the moment of inertia about its base is given by $I_{x_1} = \frac{bh^3}{12} = \frac{(100)(100)^3}{12} = 8.33\times10^6$.

Part-2: The semi circle of radius $R = 50mm$ below the x-axis.

As we see from the diagram, the diameter of the semicircle is along x-axis. But we know that for a semicircle, the moment of inertia about its diameter is given by $I_{x_2} = \pi R^4/8 = \pi(50)^4/8 = 2.45\times10^6$.

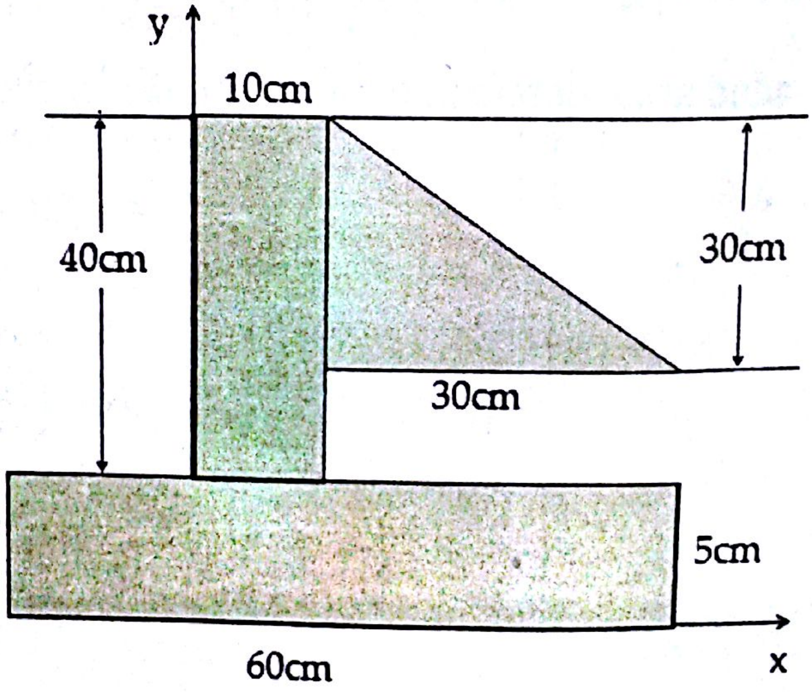
Part-3: An open circle of radius $r = 25mm$. For a circle, the moment of inertia is $I_{x_3} = 0.31\times10^6$. Therefore, the moment of inertia of the total area is the sum of the moments of the individual parts by assuming the moment of inertia of the open part (the hole) as negative.

That is $I_x = I_{x_1} + I_{x_2} - I_{x_3} = 10.47\times10^6$

5*. Consider the shaded area below. Then, determine

a) The moment of inertia of the total area about the x-axis

b) Give the moment of the total area about the horizontal centroidal axis of the upper rectangle



Solution: Let's divide the region into three parts.

Step-1: Determine the moments of each part.

Part-1: The lower rectangle with base $b = 60$ and height $h = 5$ .Since tha base of this rectangle is along the x-axis, its moment of inertia is directly computed withoit a need of transferring.

Then, its area is $A_1 = 60\times5 = 300$. Its moment of inertia about the base (x-axis) is simply $I_x = \frac{bh^3}{3} = \frac{60(5^3)}{3} = 2500$.

Part-2: The upper rectangle with base $b=10$ and height $h=40$. Then, its area is $A_2=10\times 40=400$. Its moment of inertia about its centroid is $\bar{I}_{x_2}=\frac{bh^3}{12}=\frac{10(40^3)}{12}=5.33\times 10^4$.

Part-3: The triangle of base $b=30$ and height $h=30$.

Then, its area is $A_3=\frac{1}{2}(30)(30)=450$. Besides, the moment about its centroid is given by $\bar{I}_{x_3}=\frac{bh^3}{36}=\frac{30(30)^3}{36}=22{,}500$

Step-2: Check if transfer is needed (if so apply PAT). In many situations of moment computation for composite area, transferring into a common axis is encountered because the centroidal axis need not be the same for all parts. In our case, we are interested the total moment about x-axis. But from the above three parts, only the moment of part-1 is calculated about x-axis. So, transferring is needed for part-2 and part-3. For part-1, $I_1=I_x=\frac{bh^3}{3}=\frac{60(5^3)}{3}=2500$.

For part-2, $I_2=\bar{I}_{x_2}+A_2d^2$ where is d is the distance from the centroid to the x-axis. What is the value of d? Look $d=5+\frac{h}{2}=25$. So, $I_2=\bar{I}_{x_2}+A_2d^2=5.33\times 10^4+(400)(25)^2=30.33\times 10^4$.

For part-3, $I_3=\bar{I}_{x_3}+A_3d^2$ where d is the distance from the centroid of the triangle to the x-axis. What is the value of d? Look $d=5+10+\frac{h}{3}=25$. So, $I_3=\bar{I}_{x_3}+A_3d^2=22{,}500+(450)(25)^2=30.4\times 10^4$.

Therefore, the total moment of inertia about the common axis (x-axis) is $I_x=I_1+I_2+I_3=2500+30.33\times 10^4+30.4\times 10^4=6.1\times 10^5$

Part-2: The upper triangle with base $b=3$ and height $h=6$.

So, the coordinates of its centroid are $\bar{x}_2 = 6+\frac{b}{3} = 7\text{cm}, \bar{y}_2 = \frac{h}{3} = 2\text{cm}$.

Besides, its area is $A_2 = \frac{1}{2}(3)(6) = 9$. Its moment of inertia about its centroid is $I_{\bar{x}_2} = \frac{bh^3}{36} = \frac{3(6)^3}{36} = 18, I_{\bar{y}_2} = \frac{hb^3}{36} = \frac{6(3)^3}{36} = 4.5$.

Part-3: The lower triangle with base $b=9$ and height $h=6$.

So, the coordinates of its centroid are $\bar{x}_3 = \frac{2b}{3} = 6\text{cm}, \bar{y}_3 = -\frac{h}{3} = -2\text{cm}$.

Then, its area is $A_2 = \frac{1}{2}(9)(6) = 27$. Its moment of inertia about its centroid is $I_{\bar{x}_3} = \frac{bh^3}{36} = \frac{9(6)^3}{36} = 54, I_{\bar{y}_3} = \frac{hb^3}{36} = \frac{6(9)^3}{36} = 121.5$.

a) Centroid of total area: The coordinates of the centroid of the total area is given by

i) $$\bar{X} = \frac{A_1\bar{x}_1 + A_2\bar{x}_2 + A_3\bar{x}_3}{A_1 + A_2 + A_3} = \frac{(36)(3)+(9)(7)+(27)(6)}{36+9+27} = 4.625\text{cm}$$

ii) $$\bar{Y} = \frac{A_1.\bar{y}_1 + A_2.\bar{y}_2 + A_3.\bar{y}_3}{A_1 + A_2 + A_3} = \frac{(36)(3)+(9)(2)+(27)(-2)}{36+9+27} = 1\text{cm}$$

b) Moments of inertia

Step-2: Transfer each moment about x-axis using PAT.

Parts	Areas (A)	d_y	d_x	$\bar{I}x + Ad_x^2$	$\bar{I}y + Ad_y^2$
1	36	3	3	432	432
2	9	7	2	54	445.5
3	27	6	−2	162	1093.5
Sums	72			648	1971

Therefore, the total moment of inertia is given by

$$I_x = \sum I_{\bar{x}_i} + \sum A.d_x^2 = 648, \; I_y = \sum I_{\bar{y}_i} + \sum A.d_y^2 = 1971$$

The polar moment of inertia is given by $I_z = I_x + I_y = 648 + 1971 = 2619$

The polar radius of gyration is $k_z = \sqrt{\frac{I_z}{A}} = \sqrt{\frac{2619}{72}} = 6.03$

c) We are asked the moment of inertia about the centroidal axes. But from part (a), the centroidal axe are $x_0 = 4.625$ and $y_0 = 1$. Besides, from part (b), we have $I_x = 648, \; I_y = 1971$. So, by Parallel Axes Theorem, $I_x = I_{x_0} + A.d_x^2, \; I_y = I_{y_0} + A.d_y^2$. Now, the problem is what are the distances d_x and d_y ? The distance d_x is the distance between the x-axis and the horizontal centroidal axis $y_0 = 1$ of the area. It is $d_x = 1$. Similarly, the distance d_y is the distance between the *y*-axis and the vertical centroidal axis $x_0 = 4.625$ of the area. It is $d_y = 4.625$. Hence,

$$I_x = I_{x_0} + A.d_x^2 \Rightarrow I_{x_0} = I_x - A.d_x^2 = 648 - 72 = 576$$

$$I_y = I_{y_0} + A.d_y^2 \Rightarrow I_{y_0} = I_y - A.d_y^2 = 1971 - 1540 = 431$$

d) We are asked the moment of inertia about the non-centroidal axes $x_1 = -0.375$ and $y_1 = -9$. But from part (c), the moments about the centroidal axes are found to be $I_{x_0} = 576, \; I_{y_0} = 431$.

Then, by PAT,

$$I_{x_1} = I_{x_0} + A.d_x^2 = 576 + 72(10)^2 = 7806$$

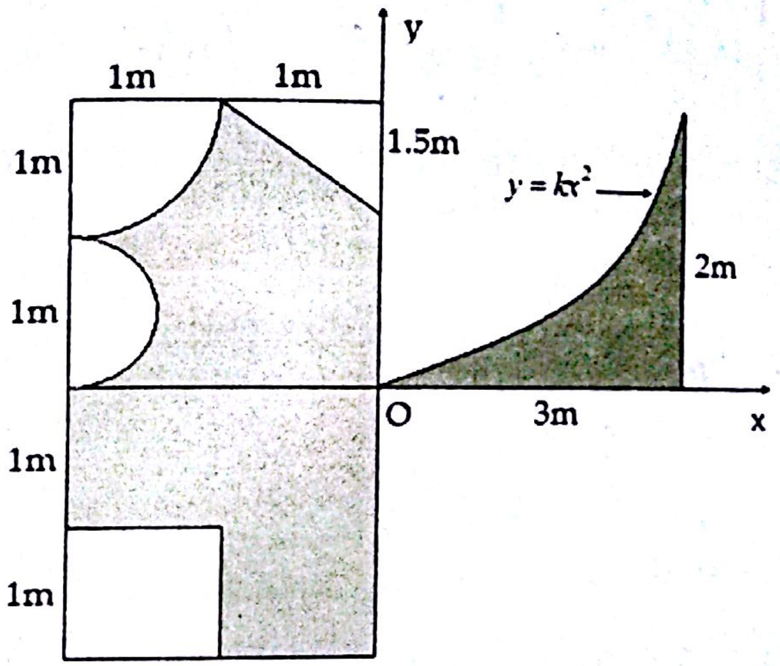
$$I_{y_1} = I_{y_0} + A.d_y^2 = 431 + 72(5)^2 = 2231$$

6*. Consider the shaded area below. Then, determine

a) The centroid of the shaded area

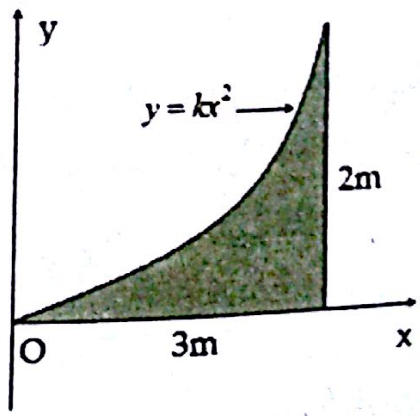
b) The rectangular and polar moments of inertia

c) The rectangular and polar radius of gyrations



Solution: Let's divide the region into seven parts.

Part-1: The parabolic Area in the first quadrant.



As we see from the diagram, this region is bounded on the left by $y = kx^2$ and on the interval $0 \le x \le 3$ where the constant k is to be determined. On the curve when $x = 3$, w have $y = 2$.

So, $y = kx^2 \Rightarrow k(3)^2 = 2 \Rightarrow 9k = 2 \Rightarrow k = 2/9$.

Thus, the area bounded by the curve is obtained using integration as as follow: $A = \int_0^3 \frac{2}{9}x^2 dy = \frac{2}{27}x^3\Big|_{x=0}^{x=3} = 2m^2$.

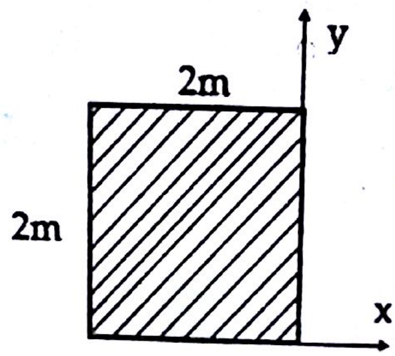
Now, obtain the moment of inertia of the area. That is

$$I_y = \int_R x^2 dA = \int_R x^2 y dx = \int_0^3 x^2 .(\frac{2}{9}x^2)dx = \frac{2}{9}\int_0^3 x^4 dd = \frac{2}{45}x^5\Big|_{x=0}^{x=3} = \frac{54}{5} = 10.8$$

$$I_x = \int_R \frac{1}{3}y^3 dx = \frac{1}{3}\int_R (\frac{2}{9}x^2)^3 .dx = \frac{1}{3}\int_0^3 \frac{8x^6}{9^3}dx = \frac{8x^7}{21(9^3)}\Big|_{x=0}^{x=3} = \frac{8}{7} = 1.143$$

Part-2: A square found in the second quadrant as shown.

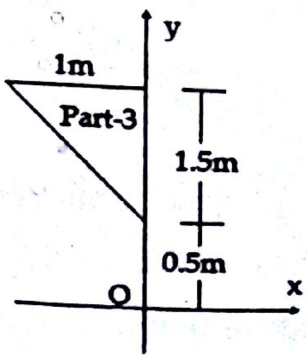
Part-2:Square



The area of this square is $A = 2m \times 2m = 4m^2$. Besides, for the square, its moments about its base (x-axis or y-axis) are,

$$I_x = I_y = \frac{bh^3}{3} = \frac{2(2^3)}{3} = \frac{16}{3} = 5.333$$

Part-3: The upper open triangle. Please observe that this part has a trick in the calculation of the moments.



For a triangle if the base lies on the x-axis and the height lies along the y-axis, the moment are $I_x = \frac{bh^3}{12}, I_y = \frac{hb^3}{12}$.

In this example, one side of the triangle lies along the y-axis but none of the sides coincide with the x-axis. This means the moment of inertia about the y-axis is $I_y = \frac{hb^3}{12} = \frac{(1.5)(1^3)}{12} = \frac{1.5}{12} = \frac{1}{8} = 0.125$.

So, how do we get I_x? Construct a new x_1-axis along the upper base as shown and calculate about the centroid with respect to the new axes. Finally, transfer the result into the x-axis. For the triangle, about its centroid, $I_{\overline{x1}} = \frac{bh^3}{36} = \frac{(1)(1.5)^3}{36} = 0.094$.

Then, by PAT, $I_x = I_{\overline{x1}} + Ad^2 = 0.094 + (0.75)(1.5)^2 = 1.782$.

Part-4: For the quarter circle about its centroid,

$$I_{x_1} = I_{y_1} = \frac{\pi R^4}{16} = \frac{\pi}{16} = 0.1963$$

Then, by PAT,

$$\begin{cases} I_{x1} = I_{\overline{x1}} + Ad^2 \Rightarrow I_{\overline{x1}} = I_{x1} - Ad^2 = 0.1963 - \frac{\pi}{4}(\frac{4}{3\pi})^2 = 0.0548 \\ I_{y1} = I_{\overline{y1}} + Ad^2 \Rightarrow I_{\overline{y1}} = I_{y1} - Ad^2 = 0.1963 - \frac{\pi}{4}(\frac{4}{3\pi})^2 = 0.0548 \end{cases}$$

Using PAT for the second time, we transfer into the x and y axes.

$$\begin{cases} I_x = I_{\overline{x1}} + Ad^2 = 0.0548 + \frac{\pi}{4}(2 - \frac{4}{3\pi})^2 = 0.0548 + 1.95 = 2.005 \\ I_y = I_{\overline{y1}} + Ad^2 = 0.0548 + \frac{\pi}{4}(2 - \frac{4}{3\pi})^2 = 0.0548 + 1.95 = 2.005 \end{cases}$$

Part-5: A semicircle of radius $R = 0.5$. For a semicircle, about its centroid, the moments are $I_{x1} = I_{y1} = \frac{\pi R^4}{8} = \frac{\pi(0.5)^4}{8} = 0.0245$.

Besides, the area of the semicircle is $A = \frac{\pi R^2}{2} = \frac{\pi(0.5)^2}{2} = \frac{\pi}{8} = 0.393$.

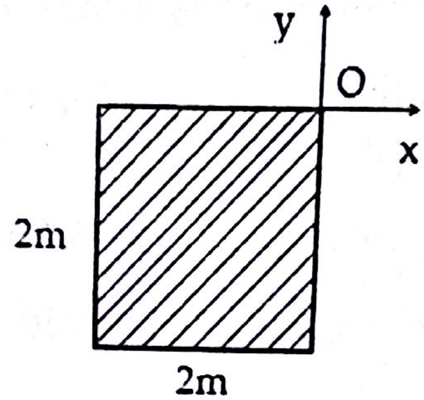
Then, by PAT, $I_x = I_{x1} + Ad^2 = 0.0245 + (0.393)(0.5)^2 = 0.1228$.

Again, apply PAT twice, to get I_y.

$$I_{y1} = I_{\bar{y}} + Ad^2 \Rightarrow I_{\bar{y}} = I_{y1} - Ad^2 = 0.0245-(0.393)(\frac{2}{3\pi})^2 = 0.0068$$

Thus, by PAT, $I_y = I_{\overline{y1}} + Ad^2 = 0.0068+(0.393)(2-\frac{2}{3\pi})^2 = 1.2629$.

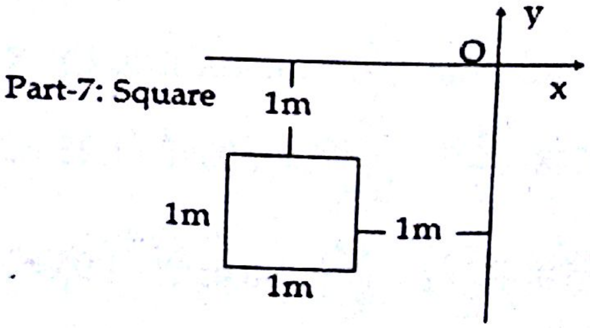
Part-6: A square found in the third quadrant as shown.



The area of this square is $A = 2m \times 2m = 4m^2$. Besides, for the square, its moments about its base (x-axis or y-axis) are,

$$I_x = I_y = \frac{bh^3}{3} = \frac{2(2^3)}{3} = \frac{16}{3} = 5.333$$

Part-7: An open square found in the third quadrant as shown.



The area of this square is $A = 1m \times m = 1m^2$. Besides, for the square, its moments with respect to x_1 and y_1 axes are

$$I_{x1} = I_{y1} = \frac{bh^3}{3} = \frac{1(1^3)}{3} = \frac{1}{3}.$$

Then, by PAT about the centroids,

$$\begin{cases} I_{x1} = I_{\overline{x1}} + Ad^2 \Rightarrow I_{\overline{x1}} = I_{x1} - Ad^2 = \frac{1}{3} - (0.5)^2 = 0.083 \\ I_{y1} = I_{\overline{y1}} + Ad^2 \Rightarrow I_{\overline{y1}} = I_{y1} - Ad^2 = \frac{1}{3} - (0.5)^2 = 0.083 \end{cases}$$

Then, using PAT for the second time,

$$I_x = I_{\overline{x1}} + Ad^2 = 0.083 + (1)(1.5)^2) = 2.33$$

$$I_y = I_{\overline{y1}} + Ad^2 = 0.083 + (1)(1.5)^2) = 2.33$$

For the total area: The moment of inertia for the total are is the algebraic sum of the moments of inertia of the seven parts by considering the open parts as negative. That is

i) Rectangular moments of inertia;

$$I_x = I_1 + I_2 - I_3 - I_4 - I_5 - I_6 + I_7$$
$$= 1.143 + 5.333 - 1.782 - 2.005 - 0.1228 + 5.333 + 2.33 = 9.124$$

$$I_y = I_1 + I_2 - I_3 - I_4 - I_5 - I_6 + I_7$$
$$= 10.8 + 5.333 - 0.125 - 2.005 - 1.2629 + 5.333 + 2.33 = 20.4$$

ii) Polar moment of inertia; $I_z = I_x + I_y = 9.124 + 20.4 = 29.524$

c) Rectangular and polar radius of gyrations;

i) Rectangular radius of gyration,

$$k_x = \sqrt{\frac{I_x}{A}} = \sqrt{\frac{9.124}{7.07}} = 1.14, \quad k_y = \sqrt{\frac{I_y}{A}} = \sqrt{\frac{20.4}{7.07}} = 1.7$$

ii) Polar radius of gyration, $k_z = \sqrt{\frac{I_z}{A}} = \sqrt{\frac{29.524}{7.07}} = 2.04$

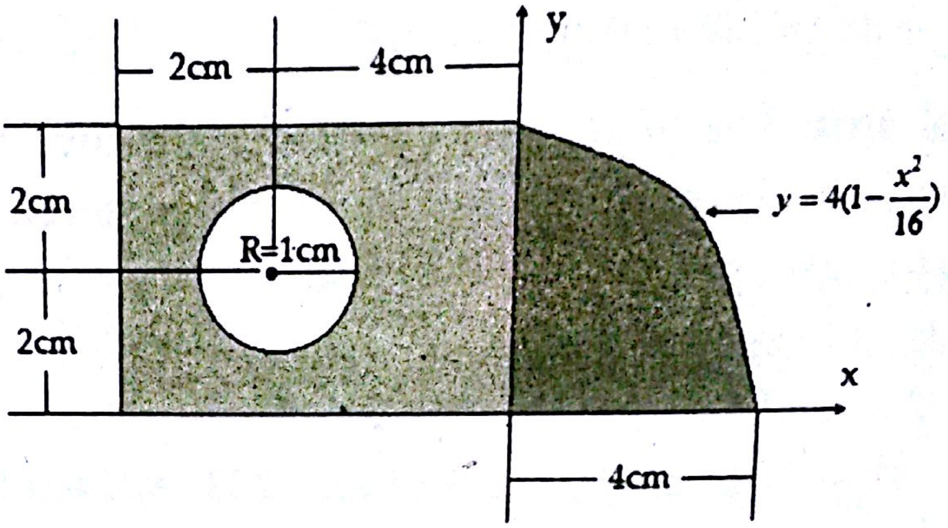
Review Problems on Chapter-6

1. Consider the shaded area below. Then, determine

a) The centroid of the shaded area

b) The rectangular and polar moments of inertia

c) The rectangular and polar radius of gyrations



Answer : $I_x = 153.7, I_y = 271.1$

2*. Consider the shaded area below. Then, determine

a) The centroid of the shaded area

b) The rectangular and polar moments of inertia

c) The rectangular and polar radius of gyrations