PROBLEM SET 7

Zhiqiang Xie 77892769

Matrix Fixing

1. Can we set a $n \times n$ matrix M of integers into a matrix of all zeros in k operations?
2. Given a sequence of operations with the lenght of k, we can certificate whether it's a yes instance or not in $T(n) = O(k * n) + O(n^2) \in O(n^2)$. Thus, MATRIX-FIXING $\in$ NP.
3. CLIQUE $\leq_p$ MATRIX-FIXING
   - Transformation:
     - Given an arbitrary instance of CLIQUE, $G = (V, E, m)$ where m means a $m-$vertex clique. We construct the instance of MATRIX-FIXING, $F(M, k)$ where $k = |V| - m$ and M is a $|V| \times |V|$ matrix. Then we set the elements of the matrix:
       - $M_{i,i} = 0$ for all $i \in [1, |V|]$
       - $M_{i,j} = 0$ for all $(i, j) \in E$. Note here G is an undirected graph and $(i, j) = (j, i)$
       - $M_{i,j} = 1$ for any entries else.

     The transformation cost $O(n^2)$ time.
   - Proof:

     We now need to prove that G is a yes-instance of CLIQUE if and only if that $F(M, k)$ is a yes-instance of MATRIX-FIXING.
     - ($\Rightarrow$) Suppose that G is a yes instance, thus there are at least m rows and columns of all zeros. Therefore, we can set the residues to be all zeros in $|V| - m = k$ operations.
     - ($\Leftarrow$) Suppose that $F(M, k)$ is a yes-instance. Since every operation (set the $i'th$ row abd the column to zero) is equivalent to connect the corresponding vertex in the graph to any other vertices. Once we can set all entries to be zero we can create a $|V|-$vertex clique by adding at most k vertices to the original clique. Therefore, there must be a $m-$vertex clique inside the original instance G. Thus, it's a yes-instance.

   MATRIX-FIXING problem is NP-complete.
4. Suppose that there is no operation which can change $\frac{1}{k}$ fraction, thus for all operation $\frac{1}{k_i} < \frac{1}{k}$.
   - Therefore, for the optimal k operations, we have $\sum_{i=1}^{k} \frac{1}{k_i} < k * \frac{1}{k} = 1$
   - At that case, no operation sequence with length k can solve this problem.
5. Since we already know that we can find a operation which can change at least $\frac{1}{k}$ fraction of the nonzero entries by greedy method.
   - Thus, here are at most $(1 - \frac{1}{k})$ fraction of nonezero entries left.
   - Every time we do this operation, we can concatenate the residue submatrixs into one with the size of $(n-1) \times (n-1)$. For the new matrix, the optimal number of operations is no more than k.
   - After t greedy operations, we have at most $(1 - \frac{1}{k})^t$ fraction of the nonezero entries. Let $t = 2k \log n$, we have $n^2 * (1 - \frac{1}{k})^{2k \log n} < n^2 * e^{-\frac{1}{k} * 2k \log n} = 1$. Hence, all entries will be zero in at most $2k \log n$ operations.