Homework 1: Synthetic Seismograms for displacements, strains, and rotations

Course: Advanced Seismology
Instructor: *Marine Denolle* Due: *October 20*

Learning Goals

By the end of this assignment, you will be able to:

- Implement Aki–Richards-style synthetic seismograms in homogeneous whole space and using moment tensor point source.
- Derive and compute strain from displacement fields (analytical and numerical).
- Generate sources at depth with Brune-type moment-rate functions and a general moment tensor.
- Design surface arrays and study how spacing/aperture impacts gradient estimates.
- Explore strengths and limitations of sensor arrays.

1 Physical Setup

- Homogeneous, isotropic, linear elastic half-space occupying $z \geq 0$ (free surface at $z = 0$).
- Density ρ, P-wave speed α, S-wave speed β.
- Point source at $\boldsymbol{\xi} = (x_s, y_s, z_s)$ with depth $z_s > 0$.
- General (time-dependent) moment tensor $\mathbf{M}(t) = [M_{ij}(t)]$.
- Receivers at the free surface $\mathbf{x} = (x, y, 0)$.

2 Analytical Forms

Let $\mathbf{r} = \mathbf{x} - \boldsymbol{\xi}$, $r = \|\mathbf{r}\|$, and $\mathbf{n} = \mathbf{r}/r$. Define retarded times

$$t_P = \frac{r}{\alpha}, \qquad t_S = \frac{r}{\beta}.$$

We use the representation theorem in the time domain,

$$u_i(\mathbf{x}, t) = \int_{-\infty}^{t} \partial_j G_{ik}(\mathbf{x}, \boldsymbol{\xi}; t - \tau)\, M_{jk}(\tau)\, d\tau, \tag{1}$$

with G_{ik} the displacement Green tensor for a point force. Differentiating the known full-space Green tensor and convolving with $M_{jk}(t)$ yields a decomposition into *far-field* ($\propto r^{-1}$), *intermediate-field* ($\propto r^{-2}$), and *near-field* ($\propto r^{-3}$) parts. Define the tensor–vector maps (for symmetric $\mathbf{A}$):

$$\mathcal{P}_0[\mathbf{A}] := \mathbf{n}\left(\mathbf{n}^\top \mathbf{A}\,\mathbf{n}\right), \tag{2}$$

$$\mathcal{S}_0[\mathbf{A}] := (\mathbf{I} - \mathbf{n}\mathbf{n}^\top)\,\mathbf{A}\,\mathbf{n}, \tag{3}$$

$$\mathcal{P}_1[\mathbf{A}] := 3\,\mathbf{n}\left(\mathbf{n}^\top \mathbf{A}\,\mathbf{n}\right) - 2\,\mathbf{A}\,\mathbf{n}, \tag{4}$$

$$\mathcal{S}_1[\mathbf{A}] := (\mathbf{I} - 3\,\mathbf{n}\mathbf{n}^\top)\,\mathbf{A}\,\mathbf{n} + 2\,\mathbf{n}\left(\mathbf{n}^\top \mathbf{A}\,\mathbf{n}\right), \tag{5}$$

$$\mathcal{N}[\mathbf{A}] := 3\,\mathbf{n}\left(\mathbf{n}^\top \mathbf{A}\,\mathbf{n}\right) - \mathbf{A}\,\mathbf{n} - \left(\mathbf{A}\,\mathbf{n}\right)_\parallel, \tag{6}$$

where $(\mathbf{A}\,\mathbf{n})_\parallel := \left(\mathbf{n}^\top \mathbf{A}\,\mathbf{n}\right)\mathbf{n}$.

Full-field displacement seismograms

With $\dot{(\,)}$ and $\ddot{(\,)}$ the first and second time derivatives, the **total** displacement in an infinite homogeneous elastic medium is:

$$\begin{aligned}\boldsymbol{u}(\mathbf{x},t) = \frac{1}{4\pi\rho}\Bigg\{ & \underbrace{\frac{1}{\alpha^3 r}\,\mathcal{P}_0\big[\ddot{\mathbf{M}}(t-t_P)\big]}_{\text{P far field } (r^{-1})} + \underbrace{\frac{1}{\beta^3 r}\,\mathcal{S}_0\big[\ddot{\mathbf{M}}(t-t_S)\big]}_{\text{S far field } (r^{-1})} \\ & + \underbrace{\frac{1}{\alpha^2 r^2}\,\mathcal{P}_1\big[\dot{\mathbf{M}}(t-t_P)\big]}_{\text{P intermediate } (r^{-2})} + \underbrace{\frac{1}{\beta^2 r^2}\,\mathcal{S}_1\big[\dot{\mathbf{M}}(t-t_S)\big]}_{\text{S intermediate } (r^{-2})} \\ & + \underbrace{\frac{1}{r^3}\,\mathcal{N}\big[\mathbf{M}(t-t_S)\big]}_{\text{Near field } (r^{-3})} \Bigg\}. \end{aligned} \tag{7}$$

2.1 Strain and Rotation from Displacements

Definition of strain tensor component:

$$\varepsilon_{ij}(\mathbf{x},t) = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \tag{8}$$

Definition of rotation tensor component:

$$\Omega_{ij}(\mathbf{x},t) = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right), \tag{9}$$

$$\boldsymbol{\omega}(\mathbf{x},t) = \tfrac{1}{2}\,\nabla\times\boldsymbol{u}(\mathbf{x},t) = \left(\Omega_{32},\,\Omega_{13},\,\Omega_{21}\right)^\top. \tag{10}$$

Thus the displacement gradient decomposes as $\partial_j u_i = \varepsilon_{ij} + \Omega_{ij}$.

Theoretical evaluation. Obtain the *theoretical* strain $\varepsilon^{(\mathrm{th})}$ and rotation $\boldsymbol{\omega}^{(\mathrm{th})}$ by differentiating (7) with respect to space. Useful identities:

$$\frac{\partial}{\partial x_\ell}\left(\frac{1}{r}\right) = -\frac{n_\ell}{r^2}, \qquad \frac{\partial n_i}{\partial x_\ell} = \frac{\delta_{i\ell} - n_i n_\ell}{r}, \qquad \partial_\ell t_{P,S} = \frac{n_\ell}{c_{P,S}}. \tag{11}$$

Sanity checks: In the far field, the P contribution is (ideally) irrotational, $\nabla\times\boldsymbol{u}^{(P)} \approx \mathbf{0}$; rotation arises from S radiation and non-radiative/induction terms.

2.2 Source Time function

Two convenient forms for the moment rate $\dot{m}(t)$ are:

Parameterization with time constant τ:

$$\dot{m}(t) = \frac{M_0}{\tau^2}\, t\, e^{-t/\tau}\, H(t), \qquad f_c = \frac{1}{2\pi\tau}. \tag{12}$$

Parameterization with corner frequency f_c:

$$\dot{m}(t) = M_0\, (2\pi f_c)^2\, t\, e^{-2\pi f_c t}\, H(t). \tag{13}$$

The cumulative moment is

$$m(t) = \int_0^t \dot{m}(s)\, ds = M_0 \left[1 - \left(1 + \frac{t}{\tau}\right) e^{-t/\tau}\right] H(t), \qquad m(\infty) = M_0. \tag{14}$$

Let the source time dependence be $\mathbf{M}(t) = \dot{m}(t)\,\hat{\mathbf{M}}$ (constant mechanism tensor $\hat{\mathbf{M}}$). Then

$$\ddot{\mathbf{M}}(t) = \frac{d^2}{dt^2}\dot{m}(t)\;\hat{\mathbf{M}}. \tag{15}$$

3 Exercise

Part A — Theoretical DisplacementSeismograms

1. Choose elastic constants: ρ, α, β (defaults below).
2. Implement $\dot{m}(t)$ (Brune-type; parameterize by M_0 and f_c or τ).
3. Implement a general mechanism $\hat{\mathbf{M}}$ (accept strike/dip/rake to build a double couple, or provide 6 independent M_{ij}).
4. Using the time-domain far-field formulas above, synthesize $\boldsymbol{u}(t)$ at each station.
5. Generate 3-component seismograms at surface stations.

Part B — Theoretical Strain Wavefield

1. Analytically differentiate your time-domain $\boldsymbol{u}$ expressions to obtain $\varepsilon_{ij}^{(\mathrm{th})}(\mathbf{x}, t)$.
2. Provide symbolic steps for P/S far-field terms and state approximations.

Part C — Theoretical Rotation Wavefield

1. Analytically differentiate your time-domain $\boldsymbol{u}$ expressions to obtain $\varepsilon_{ij}^{(\mathrm{th})}(\mathbf{x}, t)$.
2. Provide symbolic steps for P/S far-field terms and state approximations.

Part D — Source at Depth with General Moment Tensor

1. Place the source at depth z_s and arbitrary (x_s, y_s).
2. Use at least two mechanisms: (i) pure double couple (strike/dip/rake), (ii) general full tensor (include small CLVD or volumetric part).
3. Use at least two f_c values to probe bandwidth/resolution.

Part E — Surface Arrays (Geometry Study)

1. Create rectangular grids of stations at $z = 0$ with interstation spacings $\Delta \in \{0.25, 0.5, 1, 2, 4\}\,\lambda_S$, where $\lambda_S = \beta/f_{\text{peak}}$ and $f_{\text{peak}} \approx 1.3 f_c$ for the Brune pulse.
2. Use apertures $A \in \{4\lambda_S, 8\lambda_S, 16\lambda_S\}$.
3. For each geometry, simulate $\boldsymbol{u}(t)$ and compute $\varepsilon^{(\text{th})}(t)$ at all stations.

Part F — Array-Derived Strain

Using only displacement seismograms at grid nodes, estimate spatial gradients via centered differences:

$$\left.\frac{\partial u_i}{\partial x}\right|_{m,n} \approx \frac{u_i(x_{m+1}, y_n, t) - u_i(x_{m-1}, y_n, t)}{2\Delta_x}, \tag{16}$$

$$\left.\frac{\partial u_i}{\partial y}\right|_{m,n} \approx \frac{u_i(x_m, y_{n+1}, t) - u_i(x_m, y_{n-1}, t)}{2\Delta_y}. \tag{17}$$

Form the symmetric strain estimate

$$\varepsilon_{ij}^{(\text{FD})} = \tfrac{1}{2}\left(\partial_j u_i + \partial_i u_j\right) \tag{18}$$

at interior stations (exclude edges, or document one-sided stencils).

Part G — One-Page Discussion: Arrays as Wavefield-Gradient & Rotation Instruments

Limit: 1 page (excluding figures/tables if any). Discuss:

- How array spacing Δ and aperture A control accuracy/bandwidth of *both* strain and rotation estimates.
- Noise amplification in spatial differencing; filtering choices; sensitivity of $\boldsymbol{\omega}$ vs. ε to high-wavenumber noise.
- Physical insights: near-irrotational nature of far-field P; dominance of rotation in S and induction (near-field) terms; free-surface effects on vertical rotation.
- Practical implications for strainmeters, rotational seismometers, and DAS; recommended design rules (e.g., Δ/λ_S targets).

4 Suggested Notebook Structure

I recommend using a single, minimal notebook. Avoid writing too many cells; regroup when appropriate. Separate module imports from defining parameters and functions. Write single functions, single cells per topic;

- `source()`: time-domain $\dot{m}(t)$ and analytic $d^2\dot{m}/dt^2$; mechanism builders (strike/dip/rake $\rightarrow$ $\hat{\mathbf{M}}$; or arbitrary symmetric 6-vector).
- `greens_td()`: functions that return time-domain far-field P/S contributions at retarded times $t - t_{P,S}$ (with optional half-space factors).
- `synth()`: loop over stations and times; assemble $\boldsymbol{u}(t)$ directly from retarded-time evaluations.
- `strain_theory()`: analytical spatial derivatives $\rightarrow \varepsilon^{(\mathrm{th})}(t)$.
- `strain_fd()`: finite-difference gradients $\rightarrow \varepsilon^{(\mathrm{FD})}(t)$.
- `analysis()`: error calculations, parameter sweeps, plotting.

5 Input Parameters (Default Suggestions)

- $\rho = 2700\ \mathrm{kg\,m^{-3}}$, $\alpha = 6000\ \mathrm{m\,s^{-1}}$, $\beta = 3464\ \mathrm{m\,s^{-1}}$.
- $M_0 = 10^{15}$–10^{17} N m (choose a convenient scale).
- $f_c \in \{1, 2, 4\}$ Hz and $z_s \in \{1, 3, 5\}$ km.
- Record length T such that $T \geq 6 \max(r/\alpha, r/\beta) + 5\tau$.
- Sampling: choose dt such that $f_{\mathrm{Nyq}} \geq 3f_c$; apply gentle tapers and consistent causality.

6 What to Turn In

1. **Report (PDF)** including:
   - Derivation highlights (Part B): steps to obtain strain from time-domain displacement (P/S far-field), and any free-surface factors used.
   - Clear description of half-space treatment (baseline far-field with optional surface factors; full half-space gets bonus).
   - Figures: example 3-C seismograms, theoretical vs. FD strain for selected stations, MSE curves vs. Δ/λ_S and aperture, radiation patterns at the surface.
   - The essay (Part G).
2. **Code repository** with a `README.md` explaining reproducibility and how to run experiments.
3. **Config files** (YAML/JSON) for geometry, elastic constants, and source parameters.

totally—here are **drop-in LaTeX replacements** for the Discussion and Grading Rubric sections. paste these into your document to (1) set the discussion to **exactly one page**, and (2) **reallocate points** to explicitly include rotations.

Part G — Discussion: Arrays as Wavefield-Gradient & Rotation Instruments

Limit: 1 page (including figures/tables if any). Discuss:

- How array spacing Δ and aperture A control accuracy/bandwidth of *both* strain and rotation estimates.
- Noise amplification in spatial differencing; filtering choices; sensitivity of $\boldsymbol{\omega}$ vs. ε to high-wavenumber noise.
- Physical insights: near-irrotational nature of far-field P; dominance of rotation in S and induction (near-field) terms; free-surface effects on vertical rotation.
- Practical implications for strainmeters, rotational seismometers, and DAS; recommended design rules (e.g., Δ/λ_S targets).

7 Grading Rubric (100 pts, rotations included explicitly)

- **Theory & derivation (22 pts):** Full-field displacement (near/intermediate/far) derived and explained; correct, consistent definitions and use of *both* strain and rotation (ε_{ij}, Ω_{ij}, $\boldsymbol{\omega} = \frac{1}{2}\nabla \times \boldsymbol{u}$); discussion of P near-irrotationality and S/near-field rotation.
- **Implementation correctness (26 pts):** Accurate coding of $r^{-1}/r^{-2}/r^{-3}$ displacement terms; analytic time-derivatives of the Brune source; correct assembly of numerical gradients for *both* ε and $\boldsymbol{\omega}$; consistent units and causality. (Optional half-space factors accepted if documented.)
- **Array design & experiments (12 pts):** Multiple spacings/apertures tied to λ_S; clear geometry specification; reproducible parameter sweeps; interior-node handling for FD gradients.
- **Error analysis & plots (20 pts):** MSE curves for *both* strain and rotation vs. Δ/λ_S and A/λ_S; identification of aliasing thresholds; interpretation of the different sensitivity of ε vs. $\boldsymbol{\omega}$ to noise and spacing.
- **One-page discussion (10 pts):** Concise, well-supported conclusions addressing gradient *and* rotation estimation tradeoffs; actionable array-design guidance.
- **Clarity & repo quality (10 pts):** Clean notebooks, no repetitive or extra cells; labeled figures with units; thoughtful comments.

8 Hints & Checks

- **Units sanity:** $\boldsymbol{u}$ in meters. The factor $\ddot{\mathbf{M}}$ (N·m/s^2) over $4\pi\rho c^3 r$ (kg·m^{-2}·s^{-3}) yields meters.
- **Radiation pattern check:** For a pure DC, verify nodal planes in surface amplitude maps.
- **Spatial sampling:** To resolve gradients of dominant wavelength λ, target $\Delta \lesssim \lambda/4$. Expect deterioration as $\Delta/\lambda_S \to 1/2$.
- **Stability:** Centered differences amplify high-k noise; consider light spatial/temporal filtering and discuss the tradeoffs (treat theory & FD inputs consistently).