Mid term review

L03 Probability

Properties of Probability

- The probability of any event A is between 0 and 1.

$$0 \leq P(A) \leq 1$$

- If outcomes cannot occur simultaneously, then the probability that any one of them will occur is the sum of the outcome probabilities. The sum of the probabilities of all outcomes in the sample space is 1.

$$\sum_{all\ outcomes} P(A) = 1$$

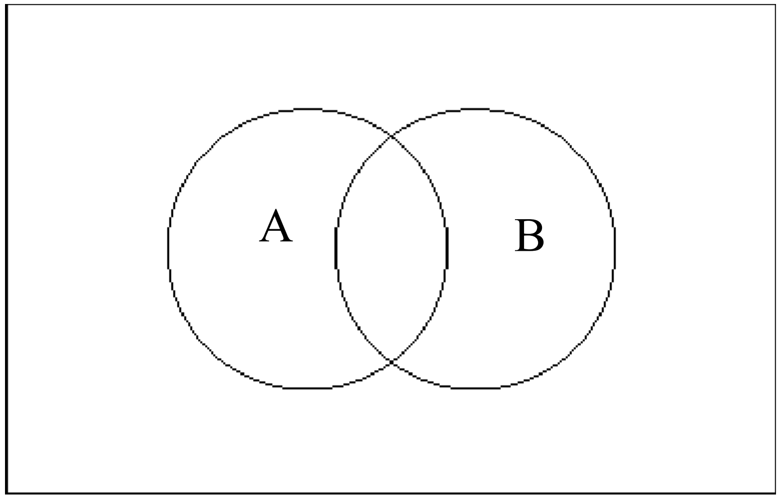
Properties of Probability

General Addition Rule: Let A and B be two events defined in a sample space *S*.

$$P(\text{A or B}) = P(\text{A}) + P(\text{B}) - P(\text{A and B})$$

Note: If two events A and B are mutually exclusive: P(A and B) = 0

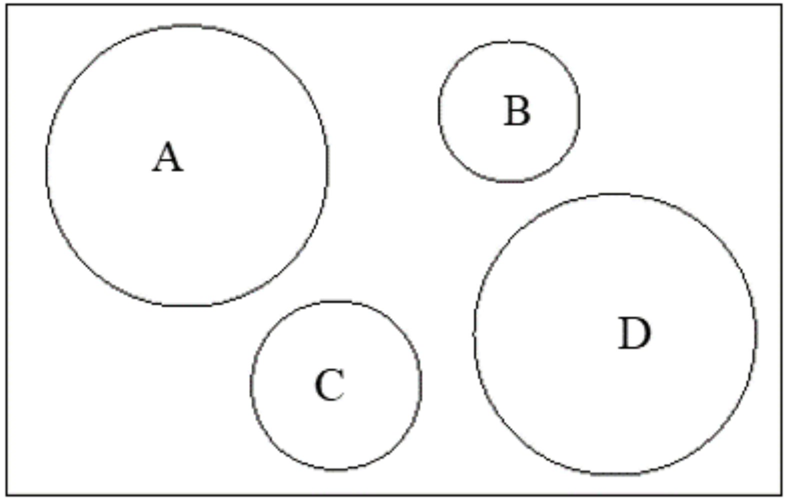
Illustration:



Special Addition Rule: If A and B are mutually exclusive events, then

$$P(\text{A or B}) = P(\text{A}) + P(\text{B})$$

This can be expended to consider more than two mutually exclusive events:



$$P(\text{A or B or C or D}\ldots) = P(\text{A}) + P(\text{B}) + P(\text{C}) + P(\text{C})$$

Properties of Probability

Multiplication Rule for Independent Events

If two events are independent, the probability that *both* events occur is the product of the individual outcome probabilities.

$$P(\text{A and B}) = P(\text{A})P(\text{B})$$

Independent events: Two events are said to be **independent** if the probability that one event occurs is not affected by knowledge of whether the other event has occurred.

Conditional Probability

Conditional probability is defined as the likelihood of an event or outcome occurring, based on the occurrence of a previous event or outcome.

Definition:

Let B be an event with non-zero probability, A is another event. The probability of A given B is

$$P(A|B) = \frac{P(A\ and\ B)}{P(B)} \text{ or } P(A|B) = \frac{P(A \cap B)}{P(B)}$$

General Multiplication Rule

Conditional Probability

$$P(B|A) = \frac{P(AB)}{P(A)}$$

$\Rightarrow$

General Multiplication Rule

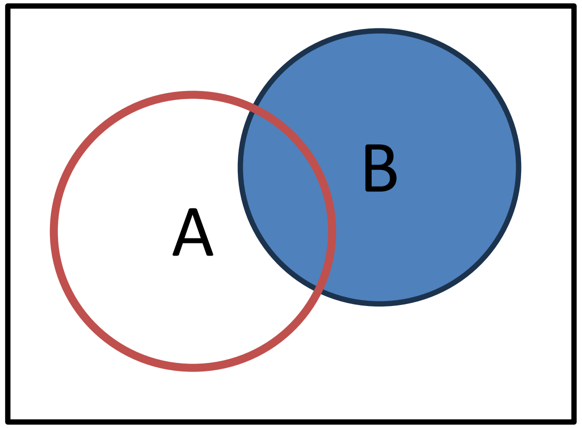
$$P(AB) = P(B|A)P(A)$$

Properties of conditioning probability

1. Complement rule for conditional probability:

$$P(A|B) + P(A^c|B) = 1$$

Recall $P(A) + P(A^C) = 1$

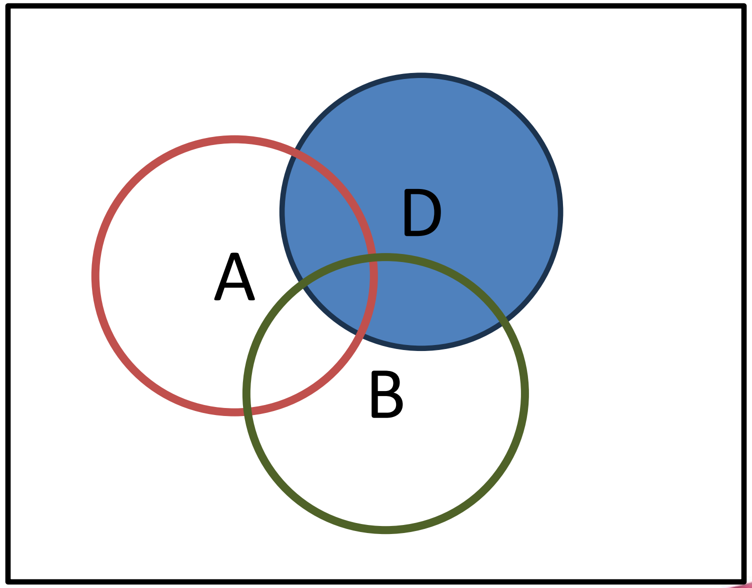


Properties of conditioning probability

2. General Addition Rule for conditional probabilities

The probability of A or B given event D occurred

$$P((A \cup B)|D)=P(A|D)+P(B|D)-P((A \cap B)|D)$$



Recall

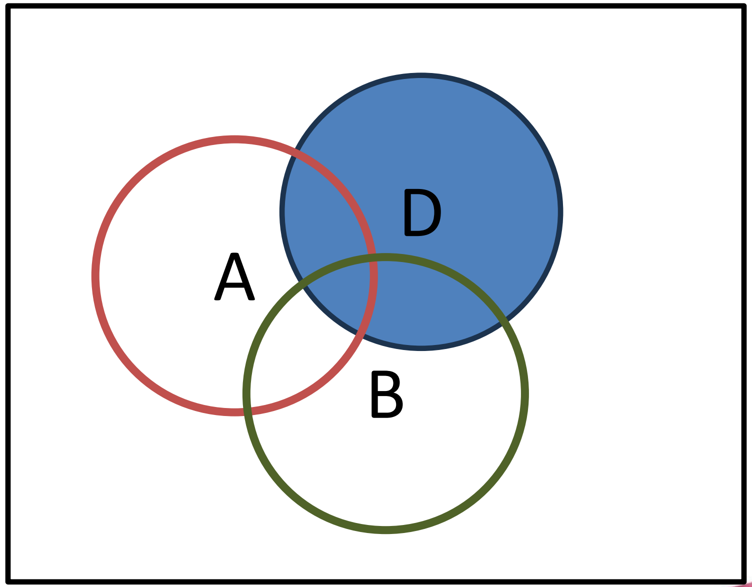
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Properties of conditioning probability

3. Specific instance of the general multiplication rule for conditional probabilities

The conditional probability of A and B occurring given D in terms of other conditional probabilities

$$P((A \cap B)|D) = P(A|(B \cap D)) \times P(B|D)$$

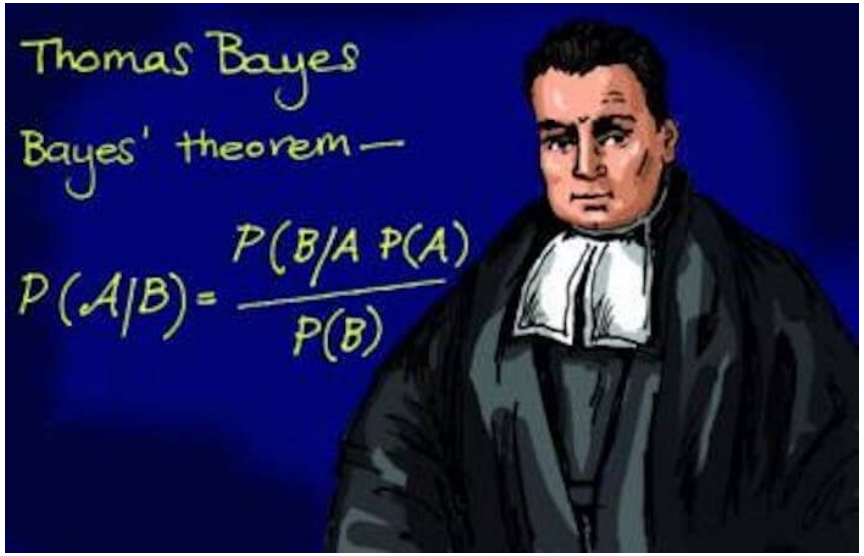


It relates the conditional probability of the intersection of events **A and B given D** to the product of the conditional probabilities of **A given B and D, and B given D**.

Bayes’ theorem

Bayes’ theorem describes the probability of occurrence of an event related to any condition. It is also considered for the case of conditional probability.

(Thomas Bayes 1701-1761)



Bayes' Formula

$\Rightarrow$ Express P(A|B) in terms of P(B|A)

The derivation is very simple, we just use $A \cap B = B \cap A \Rightarrow P(A \cap B) = P(B \cap A)$

and the definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \Rightarrow P(A \cap B) = P(A|B)P(B)$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \Rightarrow P(A \cap B) = P(B|A)P(A)$$

Bayes' formula (Bayes' theorem)

$$\boxed{P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}}$$

We can then use Bayes' theorem, provided we also know $P(A)$ and $P(B)$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

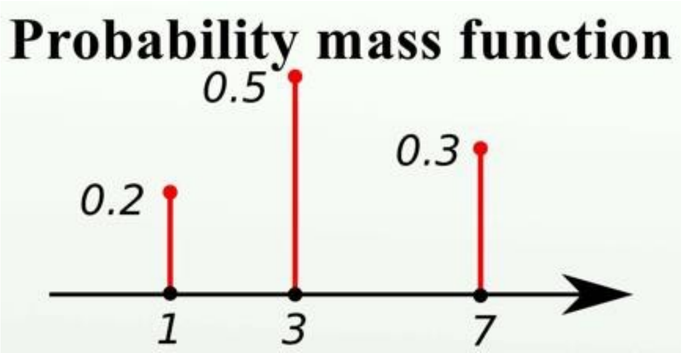
L04
Random Variables – Discrete

Probability Mass Function for Discrete rv

➢ **Probability Mass Function and Distribution Function**

Probability Mass Function (PMF)

The probability MASS function of a DISCRETE rv X, denoted by $p(x)$, is a function that gives us the probability of occurrence for each possible value x of X. It is valid for all possible values x of X.



PMF

A discrete random variable can be characterized by a probability mass function (PMF)

$$p(x_i) = P\{X = x_i\}$$

✓ <u>**Conditions for a PMF:**</u>

- $0 < p(x) \leq 1$, for all x in the range of X.
- $\sum_{x \in \chi} p(x) = 1$

Question:

Is the function $p(x)=x/6$, for x in $\chi=\{1,2,3\}$, a valid PMF?

Expectation

- The expectation (expected value, mean) of a random variable X is denoted by $E[X]$.
- In the discrete case, where X takes on the possible values $x_1, x_2, \ldots, x_n$ with probability $p(x_1), p(x_2), \ldots\ldots, p(x_n)$
- The expectation is the **weighted average** of all possible values.

Expectation

- *The expectation of a* $\boxed{\textit{discrete random variable}}$ *is defined as*

$$E[X] = \sum_i x_i p(x_i)$$

Law of the Unconscious Statistician

✓ Law of the Unconscious Statistician (LOTUS):

If X is discrete:

$$E[g(X)] = \sum_i g(x_i)p(x_i)$$

The Law of the Unconscious Statistician is particularly useful because it allows us to compute expectations without having to derive the distribution of g(X).

Linearity of Expectation

The expectation of linear combination equals to the linear combination of the expectation

LINEARITY

For any constants a, b

$$E[aX + b] = aE[X] + b$$

LINEARITY – MULTI

For any constants $a_1, a_2, \ldots, a_K$ and b

$$E[a_1 X_1 + a_2 X_2 + \cdots a_K X_K + b] = a_1 E[X_1] + a_2 E[X_2] + \cdots + a_K E[X_K] + b$$

This linearity property applies regardless of whether the random variables X and Y are independent or dependent.

Expectation of Product given Independence

If X and Y are independent, then

$$E[XY] = E[X]E[Y]$$

Variance

If X is a random variable with mean , the variance of X, denoted by Var(X), is defined by

$$\boxed{Var(X) = E[(X-\mu)^2]}$$

Note that

$$\begin{aligned} E[(X-\mu)^2] &= E[X^2 - 2\mu X + \mu^2] \\ &= E[X^2] - 2\mu E[X] + \mu^2 \\ &= E[X^2] - 2\mu\mu + \mu^2 \\ &= E[X^2] - \mu^2 \end{aligned}$$

So, alternatively,

$$\boxed{Var(X) = E[X^2] - (E[X])^2}$$

You may choose the one convenient for computation.

Property of Variance

For any constants b

$$\boxed{Var(X+b) = Var(X)}$$

For any constants a

$$\boxed{Var(aX) = a^2 Var(X)}$$

For any constants a and b,

$$\boxed{Var(aX+b) = a^2 Var(X)}$$

Property of Variance

If X and Y are $\boxed{\text{independent}}$, then

$$Var(X + Y) = Var(X) + Var(Y)$$

However, if X and Y are not independent, you must also consider the covariance

Binomial Probability Distribution

Binomial Probability Distribution

Binomial Probability Experiment:

An experiment that is made up of repeated trials that possess the following properties:

1. There are n repeated independent trials.
2. Each **trial** has two possible outcomes (success, failure).
3. $P(\text{success}) = p$, $P(\text{failure}) = q$, and $p + q = 1$
4. The **binomial random variable** x is the count of the number of successful trials that occur; x may take on any integer value from zero to n.

Binomial Probability Function

For a binomial experiment, let p represent the probability of a "success" and q represent the probability of a "failure" on a single trial; then $P(x)$, the probability that there will be exactly x successes on n trials is

$$P(x) = \binom{n}{x}(p^x)(q^{n-x}), \text{ for } x = 0,\ 1,\ 2,\ \ldots,\ \text{or } n$$

Note:

1. The number of ways that exactly x successes can occur in n trials: $\binom{n}{x}$
2. The probability of exactly x successes: p^x
3. The probability that failure will occur on the remaining $(n - x)$ trials: q^{n-x}

Mean and Standard Deviation of the Binomial Distribution

The **mean (expectation)** and **standard deviation** of a theoretical binomial distribution can be found by using the following two formulas:

$$\mu = np$$
$$\sigma = \sqrt{npq}$$

n: number of trials
p: the probability of success
q: the probability of failure (q =1-p)

Note:

1. Mean is intuitive: number of trials multiplied by the probability of a success.
2. The variance of a binomial probability distribution is:

$$\sigma^2 = \left(\sqrt{npq}\right)^2 = npq$$

Binomial Random Variable

The two random variables have the same probability of success

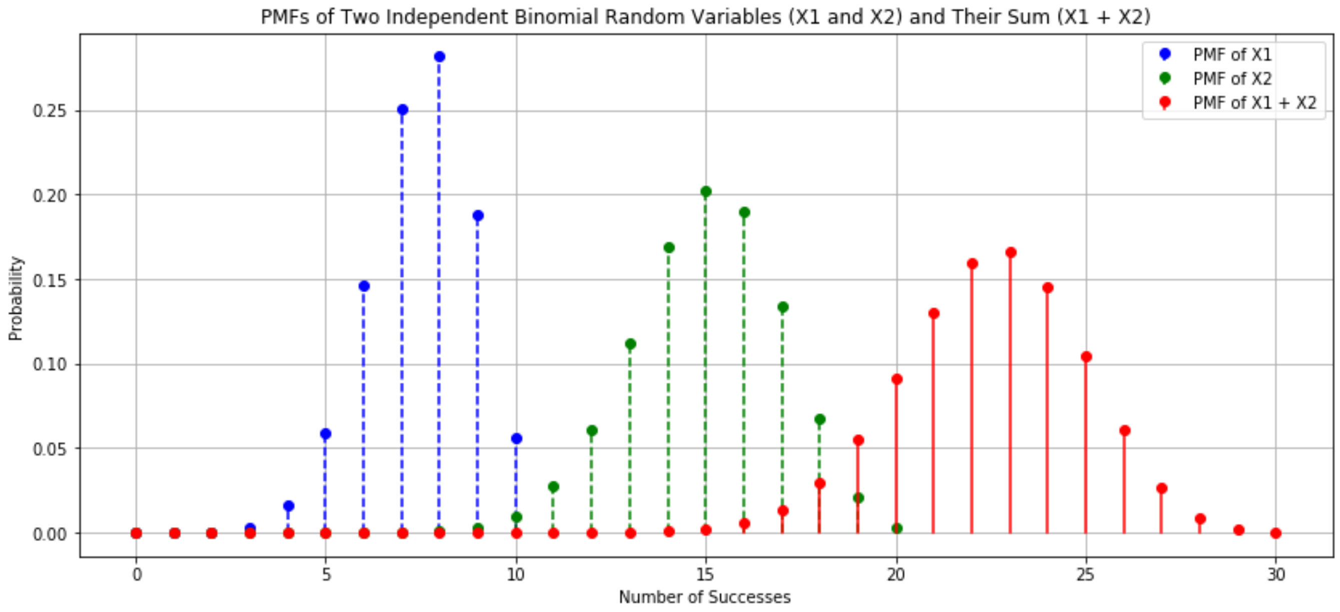
If

$$X_1 \sim \text{Binomial}(n_1, p), \quad X_2 \sim \text{Binomial}(n_2, p)$$

and X_1 and X_2 are $\boxed{\text{independent}}$, then

$$X_1 + X_2 \sim \text{Binomial}(n_1 + n_2, p)$$

Let's assume $n1$=10, n2 = 20 and $p1$=$p2$=0.75



Poisson Distribution

Poisson Distribution

Another discrete distribution that we will consider in this section is the **Poisson distribution**. It can be used to determine the probability of counts of the occurrence of an event over time (or space).

- Notation:

$$X \sim Poisson(\lambda),$$

where $\lambda \in (0, \infty)$ is the rate of occurrences of an event per unit time (or space) or the average number of occurrences of the event per unit time (or space).

Here are some typical examples for Poisson distributions.

1. The number of traffic accidents occurring on a highway in a day.
2. The number of people joining a line in an hour.
3. The number of typos per page of an essay.

Poisson Process

Given these assumptions, the formula:

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$

the probability of observing exactly *k* events in the interval, where *λ* is the expected number of events

$e^{-\lambda}$ is the probability that no events occur, *e* is Euler's number (~2.71828).

λ^k represents the weight of the probability for *k* occurrences.

The division by *k*! accounts for the fact that the order of occurrences does not matter (it's the combination, not the permutation).

Poisson Distribution

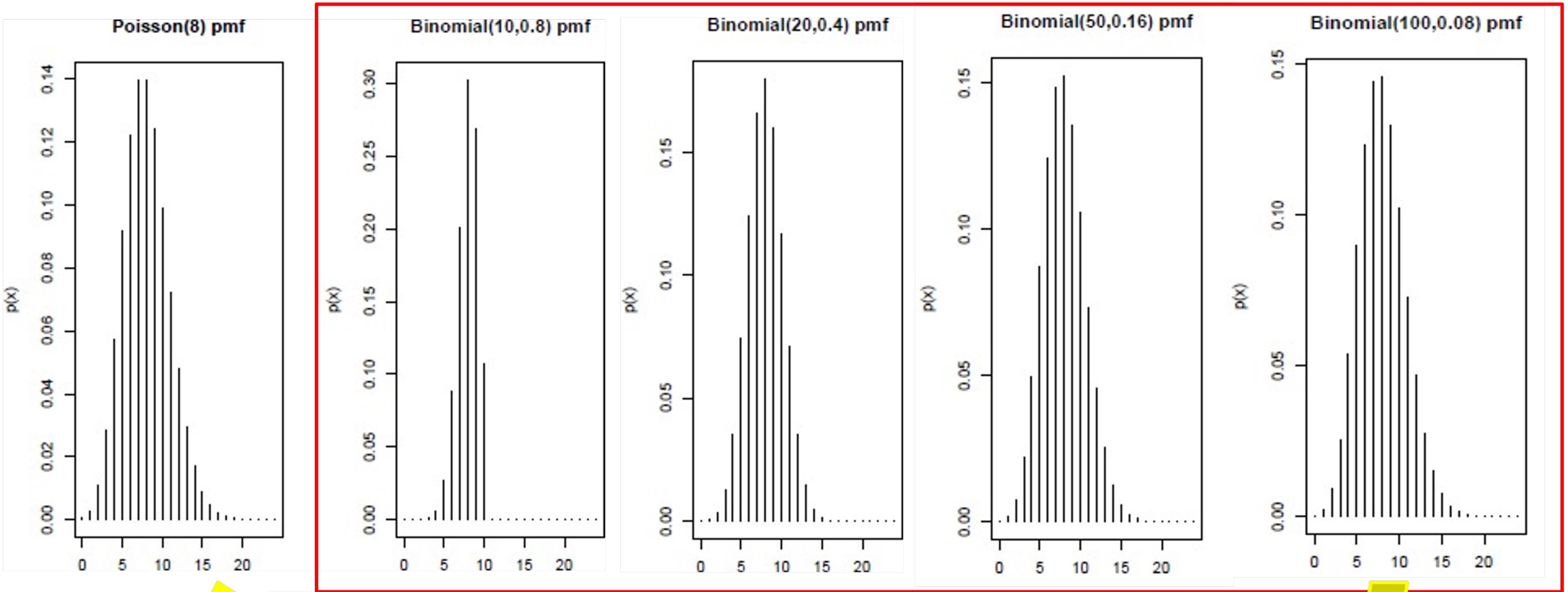
- Population mean and population variance:
  $E(X) = \lambda,\ Var(X) = \lambda.$

λ represents the average rate of occurrence of the events in the given interval.

This rate λ also describes the variance in the number of events.

The Connection Between the Poisson and Binomial Distributions

- The Poisson distribution is actually a limiting case of a Binomial distribution when the number of trials, n, gets very large and p, the probability of success, is small.



Consider Poisson(λ) and Binomial(n, p) with

$$\boxed{\lambda = np}$$

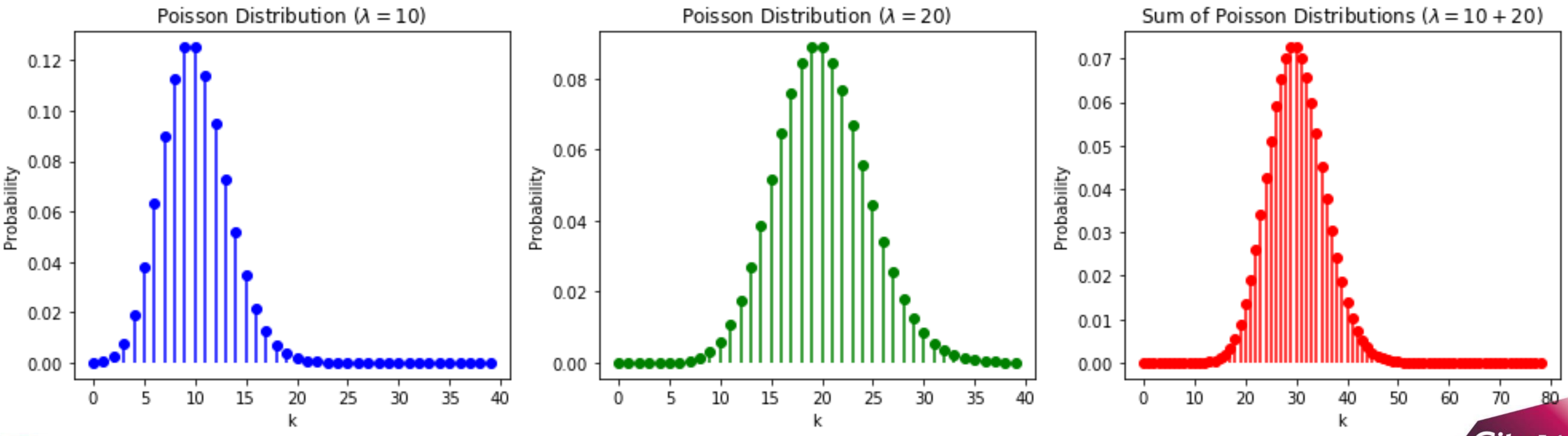
Poisson(λ) and Binomial(n, p) becomes "close" when n becomes large and p becomes small, but keep $\lambda = np$.

A rule of thumb:
- n **≥100 and np≤10**;
- Poisson distribution taking **λ=np**

Poisson Random Variable

If $X_1 \sim \text{Poisson}(\lambda_1)$ and $X_2 \sim \text{Poisson}(\lambda_2)$, and X_1 and X_2 are independent, then;

$$X_1 + X_2 \sim \text{Poisson}(\lambda_1 + \lambda_2)$$



L05
Random Variables – Continuous

Continuous Random Variable

- **Probability Density Function (PDF) and Distribution Function**

The probability DENSITY function of a CONTINUOUS rv X, denoted by $f(x)$, is **a function that gives us a value for the measure of how likely it is that X is near to x**. It is valid for all possible values x of X.

✓ **Conditions for a pdf:**
- $0 < f(x)$, for all x in the range of X.
- $\int_{-\infty}^{+\infty} f(x)dx = 1$

$$\mathrm{P}\{a < x \leq b\} = \int_a^b f(x)dx$$

The probabilities are given by integrating $f(x)$ over a particular interval.

- Probability that X takes values belonging to A:

$$P(x \in A) = \int_{x \in A} f(x)dx$$

Cumulative Distribution Function

- **CDF: the cumulative distribution function *F* of a continuous random variable can be expressed in terms of the PDF *f*(*x*) by**

$$F(a) = \mathrm{P}\{-\infty < X \le a\} = \int_{-\infty}^{a} f(x)\,dx$$

- **Differentiate the CDF**

$$\frac{d}{da} F(a) = \mathrm{f(a)}$$

Rules for Integrals

Power Rule

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$$

$$\int x^{-1} dx = \ln|x| + C$$

Exponential

$$\int e^x dx = e^x + C$$

$$\int a^x dx = \frac{a^x}{\ln a} + C$$

Constant Multiples

$$\int kf(x)dx = k\int f(x)dx$$

Absolute Value

$$\int |x| dx = \frac{x|x|}{2} + C$$

Sums and Differences

$$\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$$

$$\int [f(x) - g(x)]dx = \int f(x)dx - \int g(x)dx$$

$$\int u\,dv = uv - \int v\,du \quad \text{(Integration by parts)}$$

$$u = x; v = -e^{-x}$$

$$dv = e^{-x}dx; du = dx$$

Mean (Expectation)

Mean

If X is a continuous rv with its pdf $f(x)$, then the mean (expectation, expected value) of X is defined as

$$\boxed{E(X) = \int_{-\infty}^{\infty} [x\, f(x)]\, dx, \qquad \text{if it exists.}}$$

- The population mean is usually denoted by μ or μ_X.
- It is a common measure of central location of the random variable X.

Law of the Unconscious Statistician (LOTUS)

LOTUS

- *If X is continuous*

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)dx$$

It provides a method for calculating the **expected value of a function of a random variable without directly knowing the probability distribution of the random variable itself**.

Linear of Expectation

The expectation of linear combination equals to the linear combination of the expectation

LINEARITY

For any constants a, b

$$E[aX + b] = aE[X] + b$$

LINEARITY – MULTI

For any constants $a_1, a_2, \dots, a_K$ and b

$$E[a_1X_1 + a_2X_2 + \cdots a_KX_K + b] = a_1E[X_1] + a_2E[X_2] + \cdots + a_KE[X_K] + b$$

Independence of Random Variables

INDEPENDENCE

Random variables X and Y are independent means for all x, y

$$P(X \leq x, Y \leq y) = P(X \leq x)P(Y \leq y)$$

EXPECTATION OF PRODUCT GIVEN INDEPENDENCE

If X and Y are independent, then

$$E[XY] = E[X]E[Y]$$

Variance

Variance

If X is a continuous rv with its pdf $f(x)$, then the population variance of X is defined as

$$\boxed{Var(X) = \int_{-\infty}^{\infty} [(x-\mu)^2 f(x)]\, dx, \qquad if\ it\ exists.}$$

- The population variance is usually denoted by σ^2 or σ_X^2.
- The positive square root of σ^2, denoted by σ, is called the population standard deviation (sd) of X.
- Both σ^2 and σ are common measures of spread of the random variable X.
- Population variance (sd) is **different** from the sample variance (sd), the variance (sd) of data.
    - Population variance (sd) is determined by the distribution of the random variable, while sample variance (sd) is determined by the collection of the actual observations of the random variable.
    - Thus, population variance (sd) is **fixed** (even it is often unknown in practice) but sample variance (sd) is **different** when different data are used.

It can be showed that:

$$\boxed{Var(X) = E(X^2) - [E(X)]^2}$$

Property of Variance

If X and Y are independent, then

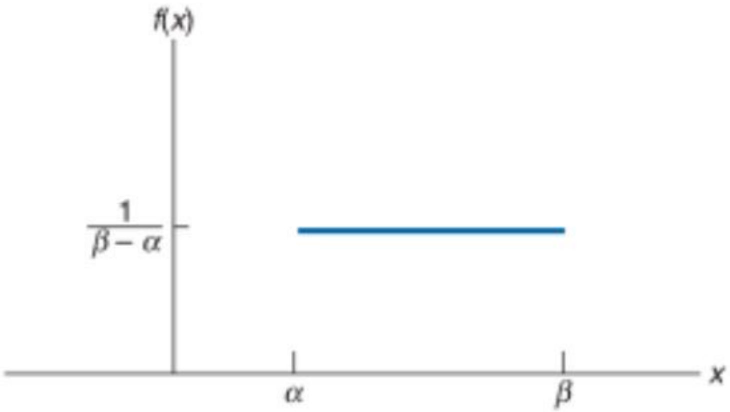
$$Var(X+Y) = Var(X) + Var(Y)$$

Uniform Random Variables

A random variable X is uniformly distributed over the interval $[\alpha, \beta]$ if its probability density function is given by

$$f(x) = \begin{cases} \frac{1}{\beta-\alpha} & \text{if } \alpha \le x \le \beta \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $\int_{-\infty}^{\infty} f(x)dx = 1$



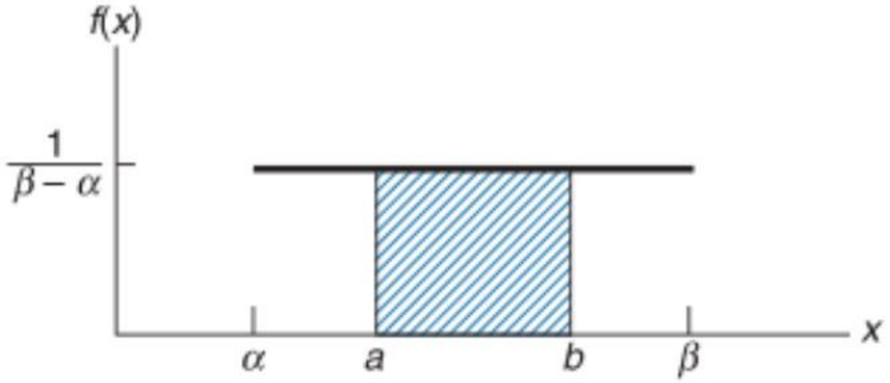
Graph of $f(x)$ for a uniform $[\alpha, \beta]$.

The uniform distribution can be either continuous or discrete, depending on whether the random variable is continuous or discrete.

Uniform Random Variables

How to calculate probabilities:

$$P\{a < X < b\} = \int_a^b f(x)dx = \frac{1}{\beta - \alpha}\int_a^b 1dx$$
$$= \frac{b-a}{\beta-\alpha}$$



Probabilities of a uniform random variable.

Uniform Random Variables

Mean (middle point)

$$E[X] = \int_{\alpha}^{\beta} \frac{x}{\beta-\alpha}dx = \frac{x^2}{2(\beta-\alpha)}\Big|_{\alpha}^{\beta}$$
$$= \frac{\beta^2-\alpha^2}{2(\beta-\alpha)} = \frac{(\beta+\alpha)(\beta-\alpha)}{2(\beta-\alpha)}$$
$$\boxed{= \frac{\beta+\alpha}{2}}$$

The center or balance point of the distribution.

Variance (square of width)

$$Var(X) = E(X^2) - [E(X)]^2$$
$$= \int_{\alpha}^{\beta} \frac{x^2}{\beta-\alpha}dx - \left(\frac{\beta+\alpha}{2}\right)^2$$
$$= \frac{(\beta^3-\alpha^3)}{3(\beta-\alpha)} - \left(\frac{\beta+\alpha}{2}\right)^2$$
$$\boxed{= \frac{(\beta-\alpha)^2}{12}}$$

The spread or dispersion of the values around the mean. The wider range, the more spread out.

Exponential Random Variables

Exponential random variables are a type of continuous probability distribution that models the time between events occurring in a Poisson process.

An exponential random variable has the following density function:

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & if\, x \geq 0 \\ 0 & if\, x < 0 \end{cases}$$

Expectation: $E[X] = \frac{1}{\lambda}$

Variance: $Var(X) = \frac{1}{\lambda^2}$

Note that the possible values it can take: [0, ∞)

The cumulative distribution function is

$$F(a) = \int_0^a \lambda e^{-\lambda x} dx = 1 - e^{-\lambda a}, \quad a \geq 0$$

Exponential Random Variables

Competition of Two Exponential RV's

If

$$X \sim \text{Exponential}(\lambda_1), \quad Y \sim \text{Exponential}(\lambda_2)$$

And *X*, *Y* are independent, then

$$\boxed{P\{X < Y\} = \frac{\lambda_1}{\lambda_1 + \lambda_2}}$$

If

$$X_1 \sim \text{Exponential}(\lambda_1), \cdots, X_n \sim \text{Exponential}(\lambda_n)$$

And $X_1, \ldots, X_n$ are independent, then

$$\boxed{P\{X_i \text{ is the smallest}\} = \frac{\lambda_i}{\sum_{i=1}^{n} \lambda_i}}$$

Normal Distribution

- The **normal probability distribution** is the most important distribution in all of statistics.

- Many continuous random variables have normal or approximately normal distributions.

Normal Distribution

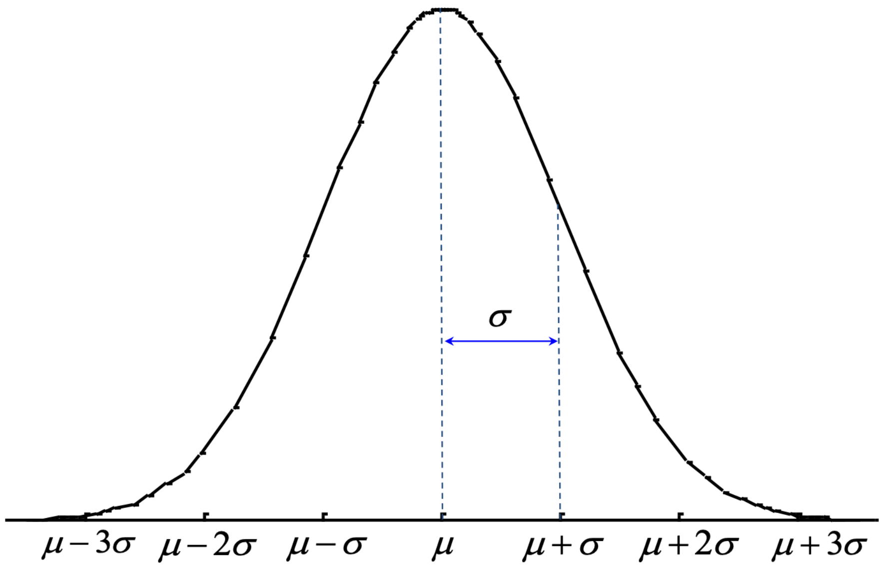
Normal Probability Distribution:

Normal probability distribution function:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

μ: mean
σ: standard deviation

This is the function for the **normal (bell-shaped) curve**.



The probability that *x* lies in some interval is the area under the curve.

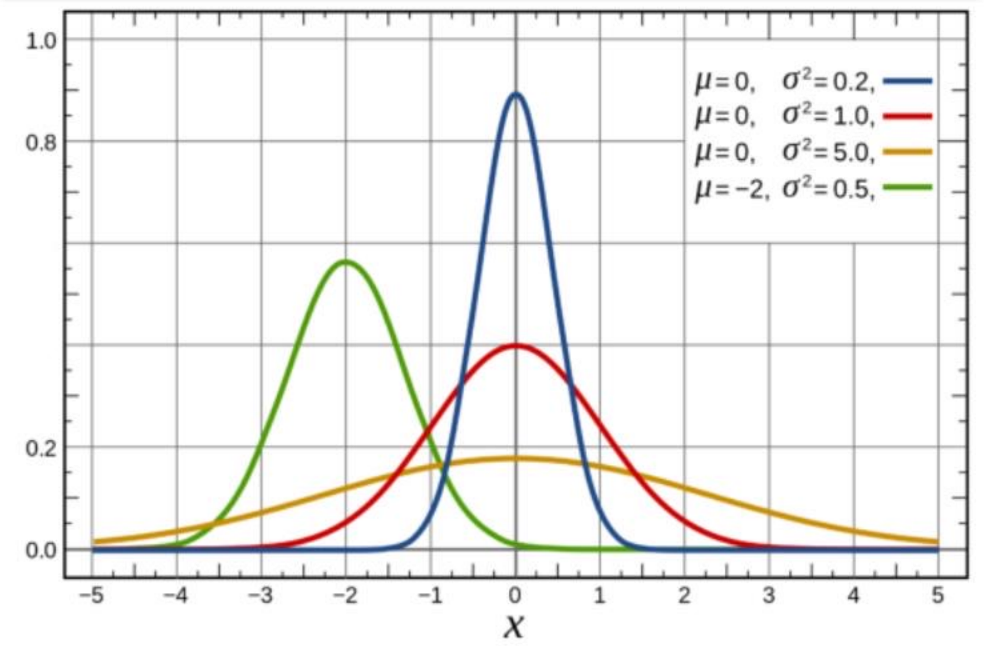
- Notation:

$$X \sim N(\underbrace{\mu}_{a}, \underbrace{\sigma^2}_{b}), \text{ where } \mu \in (-\infty, \infty) \text{ and } \sigma \in (0, \infty).$$

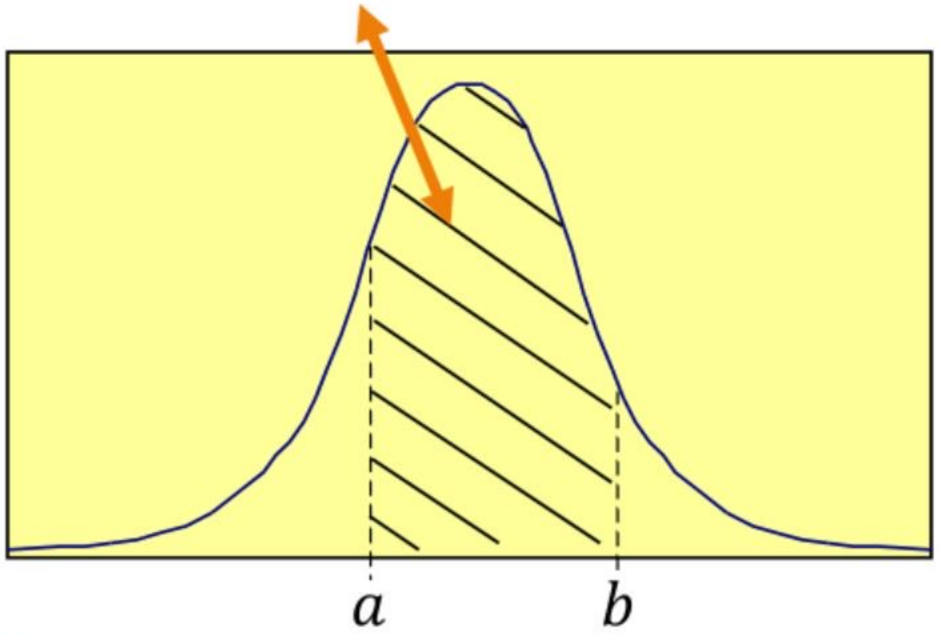
- Population mean and population variance:

$$E(X) = a,\ Var(X) = b.$$

Normal Distribution



- The area under the normal density curve over the interval from *a* to *b* represents the probability that the normal-distributed rv *X* falls into that interval, i.e., P($a \le X \le b$).



Standard Normal Distribution

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty\le x\le\infty$$

$$=\frac{1}{1\times\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-0}{1}\right)^2} \qquad -\infty\le x\le\infty$$

$$=\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(z)^2} \qquad -\infty\le z\le\infty \qquad \left(\frac{x-0}{1}=z\right)$$

Standard normal distribution

The normal distribution with mean 0 and variance 1, i.e. N(0,1).

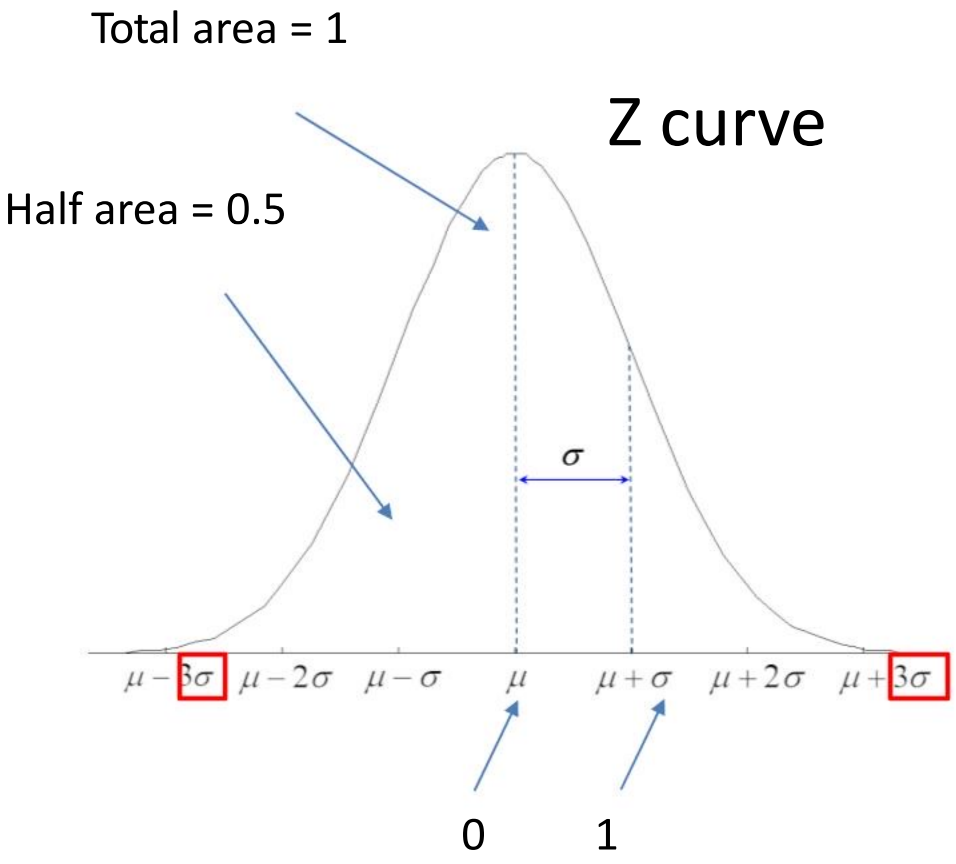
The random variable following the standard normal distribution is often denoted by *Z* in probability and statistics

Any normal probability can be reduced to the probability of N(0,1). If we want to find an area over an interval ***for any normal curve*** by just finding the corresponding area ***under a standard normal density curve***.

Standard Normal Distribution

Properties of the *Standard* Normal Distribution:

- The total area under the curve is 1. The distribution is symmetric.
- The distribution has a mean of 0 and standard deviation of 1.
- The mean divides the area in half, 0.5 on each side.
- Nearly all the area is between the standard variable z = -3 and z = 3.



Standardization

- Standardization is a process used to transform any normal-distributed r.v., say X~N(μ, σ^2) , to a <u>standard</u> normal-distributed r.v.

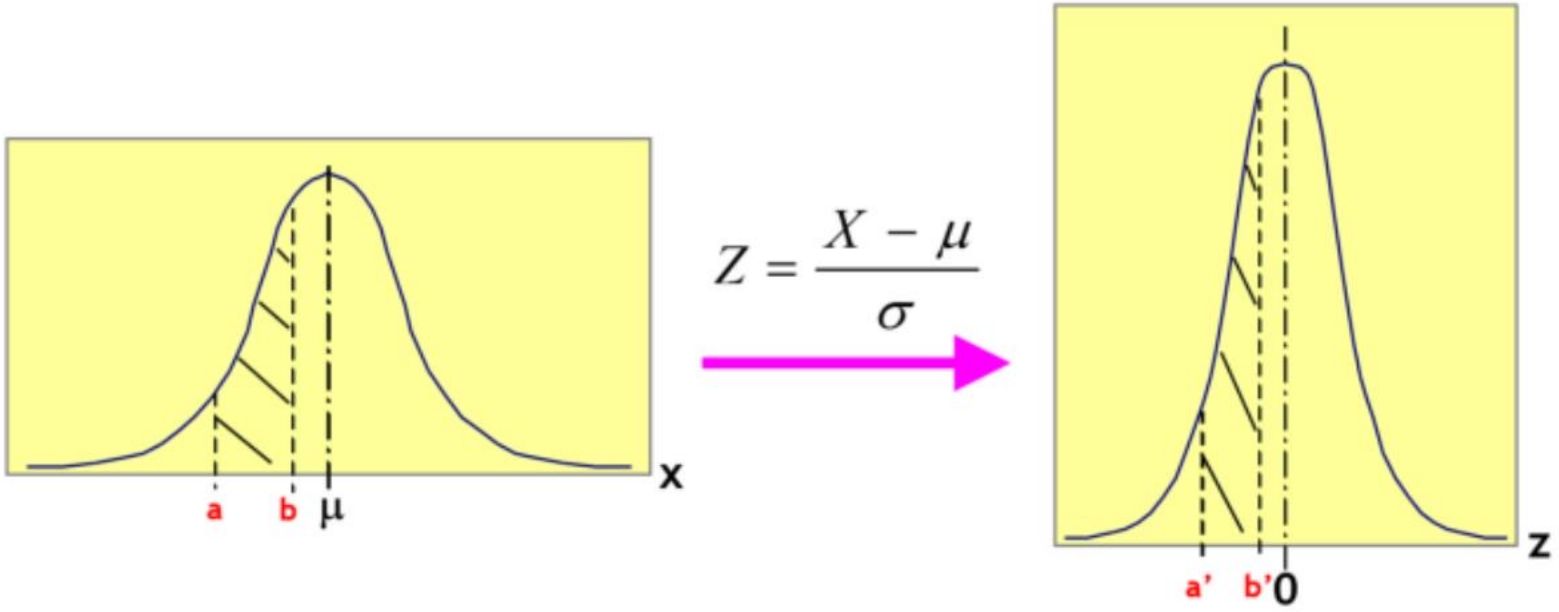
- To be more precise, we have the following result:

If $X \sim N(\mu, \sigma^2)$, and then we standardize X, i.e. subtract μ from X and then divide by σ then

$$\frac{X-\mu}{\sigma} \sim N(0,1)\,.$$

Thus, notationally, we have $\boxed{Z = \frac{X-\mu}{\sigma}}$.

Standardization



$$P(a \le X \le b) = P(a' \le Z \le b')$$

According to the standardization, the probability of any normal-distributed r.v. can be converted to the probability of a standard normal-distributed.

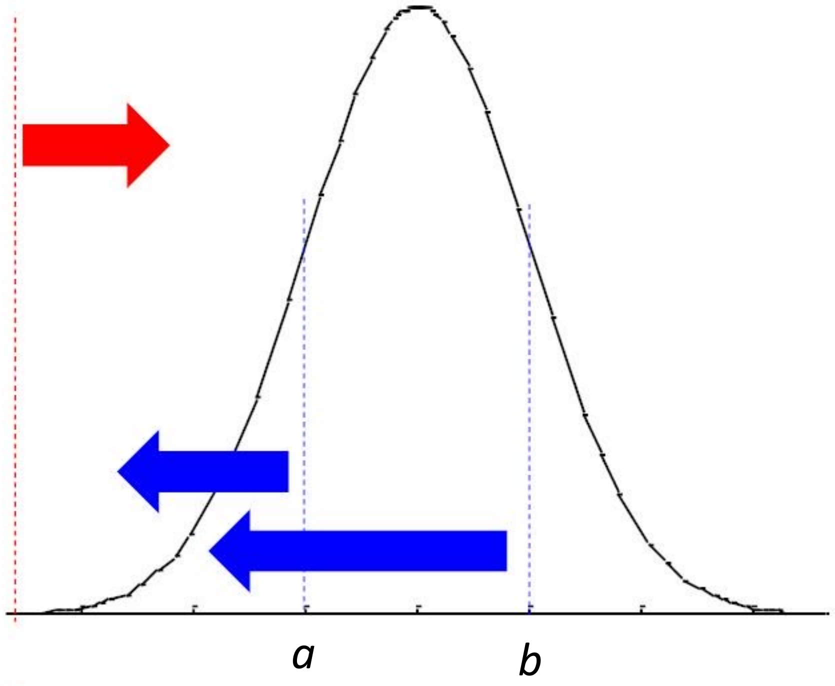
Standardization

Suppose $X \sim N(\mu, \sigma^2)$. Let's define

$$Z = \frac{X-\mu}{\sigma}$$

It is clear that $Z \sim N(0,1)$. Compute

$$\begin{aligned} P\{X<b\} &= P\left\{\frac{X-\mu}{\sigma} < \frac{b-\mu}{\sigma}\right\} \\ &= P\left\{Z < \boxed{\frac{b-\mu}{\sigma}}\right\} \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) \end{aligned}$$



Pick another number a, then

$$P\{X<a\} = \Phi\left(\boxed{\frac{a-\mu}{\sigma}}\right)$$

Note that

$$\{X<b\} = \{X \le a\} \cup \{a<X<b\}$$

And $\{X \le a\} \cap \{a<X<b\} = \emptyset$

So

$$\boxed{\begin{aligned} P\{a<X<b\} &= P\{X<b\} - P\{X<a\} \\ &= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \end{aligned}}$$

Table of Standard Normal (z) Curve Areas

- For any number z^*, from -3.49 to 3.49 and rounded to two decimal places, the Z table gives **<u>the area under the *z* curve and to the left of *z**.</u>**

$$P(z < z^*) = P(z \leq z^*)$$

Where

the letter z is used to represent a random variable whose distribution is the standard normal distribution.

Table of Standard Normal (z) Curve Areas

The table is typically organized with rows representing the tenths and hundredths digit of the Z-score and columns representing the thousandths digit.

The values in the table represent the area to the left of the corresponding Z-score.

This value can be interpreted as the probability that a standard normal random variable is less than or equal to the given Z-score.

To use the table:

Find the correct row and column (see the following example)

The number at the intersection of that row and column is the probability

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
-3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
-3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
-3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
-3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
-3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
-3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
-3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
-3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
-3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
-2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
-2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
-2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
-2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
-2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
-2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
-2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
-2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
-2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
-2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
-1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
-1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
-1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
-1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
-1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
-1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
-1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08691	.08534	.08379	.08226
-1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
-1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
-1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
-0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
-0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
-0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
-0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
-0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
-0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
-0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
-0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
-0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
-0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414

STANDARD NORMAL DISTRIBUTION: Table Values Represent AREA to the LEFT of the Z score.

Z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.50000	.50399	.50798	.51197	.51595	.51994	.52392	.52790	.53188	.53586
0.1	.53983	.54380	.54776	.55172	.55567	.55962	.56356	.56749	.57142	.57535
0.2	.57926	.58317	.58706	.59095	.59483	.59871	.60257	.60642	.61026	.61409
0.3	.61791	.62172	.62552	.62930	.63307	.63683	.64058	.64431	.64803	.65173
0.4	.65542	.65910	.66276	.66640	.67003	.67364	.67724	.68082	.68439	.68793
0.5	.69146	.69497	.69847	.70194	.70540	.70884	.71226	.71566	.71904	.72240
0.6	.72575	.72907	.73237	.73565	.73891	.74215	.74537	.74857	.75175	.75490
0.7	.75804	.76115	.76424	.76730	.77035	.77337	.77637	.77935	.78230	.78524
0.8	.78814	.79103	.79389	.79673	.79955	.80234	.80511	.80785	.81057	.81327
0.9	.81594	.81859	.82121	.82381	.82639	.82894	.83147	.83398	.83646	.83891
1.0	.84134	.84375	.84614	.84849	.85083	.85314	.85543	.85769	.85993	.86214
1.1	.86433	.86650	.86864	.87076	.87286	.87493	.87698	.87900	.88100	.88298
1.2	.88493	.88686	.88877	.89065	.89251	.89435	.89617	.89796	.89973	.90147
1.3	.90320	.90490	.90658	.90824	.90988	.91149	.91309	.91466	.91621	.91774
1.4	.91924	.92073	.92220	.92364	.92507	.92647	.92785	.92922	.93056	.93189
1.5	.93319	.93448	.93574	.93699	.93822	.93943	.94062	.94179	.94295	.94408
1.6	.94520	.94630	.94738	.94845	.94950	.95053	.95154	.95254	.95352	.95449
1.7	.95543	.95637	.95728	.95818	.95907	.95994	.96080	.96164	.96246	.96327
1.8	.96407	.96485	.96562	.96638	.96712	.96784	.96856	.96926	.96995	.97062
1.9	.97128	.97193	.97257	.97320	.97381	.97441	.97500	.97558	.97615	.97670
2.0	.97725	.97778	.97831	.97882	.97932	.97982	.98030	.98077	.98124	.98169
2.1	.98214	.98257	.98300	.98341	.98382	.98422	.98461	.98500	.98537	.98574
2.2	.98610	.98645	.98679	.98713	.98745	.98778	.98809	.98840	.98870	.98899
2.3	.98928	.98956	.98983	.99010	.99036	.99061	.99086	.99111	.99134	.99158
2.4	.99180	.99202	.99224	.99245	.99266	.99286	.99305	.99324	.99343	.99361
2.5	.99379	.99396	.99413	.99430	.99446	.99461	.99477	.99492	.99506	.99520
2.6	.99534	.99547	.99560	.99573	.99585	.99598	.99609	.99621	.99632	.99643
2.7	.99653	.99664	.99674	.99683	.99693	.99702	.99711	.99720	.99728	.99736
2.8	.99744	.99752	.99760	.99767	.99774	.99781	.99788	.99795	.99801	.99807
2.9	.99813	.99819	.99825	.99831	.99836	.99841	.99846	.99851	.99856	.99861
3.0	.99865	.99869	.99874	.99878	.99882	.99886	.99889	.99893	.99896	.99900
3.1	.99903	.99906	.99910	.99913	.99916	.99918	.99921	.99924	.99926	.99929
3.2	.99931	.99934	.99936	.99938	.99940	.99942	.99944	.99946	.99948	.99950
3.3	.99952	.99953	.99955	.99957	.99958	.99960	.99961	.99962	.99964	.99965
3.4	.99966	.99968	.99969	.99970	.99971	.99972	.99973	.99974	.99975	.99976
3.5	.99977	.99978	.99978	.99979	.99980	.99981	.99981	.99982	.99983	.99983
3.6	.99984	.99985	.99985	.99986	.99986	.99987	.99987	.99988	.99988	.99989
3.7	.99989	.99990	.99990	.99990	.99991	.99991	.99992	.99992	.99992	.99992
3.8	.99993	.99993	.99993	.99994	.99994	.99994	.99994	.99995	.99995	.99995
3.9	.99995	.99995	.99996	.99996	.99996	.99996	.99996	.99996	.99997	.99997

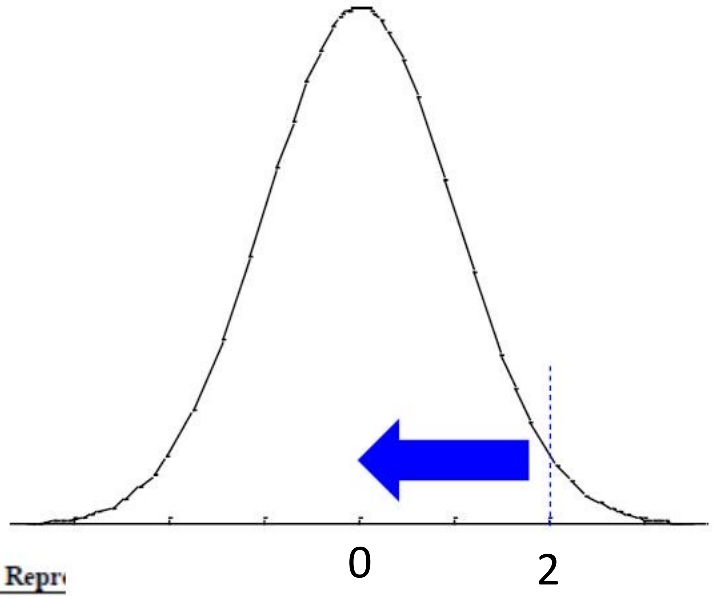
Example

If X is a normal random variable with mean μ = 3 and variance σ^2 = 16, find $P\{X < 11\}$

Solution:

$$P\{X < 11\} = \Phi\left(\boxed{\frac{11-3}{4}}\right) = \Phi(2) \approx .9772$$

Standardization



STANDARD NORMAL DISTRIBUTION: Table Values Repr

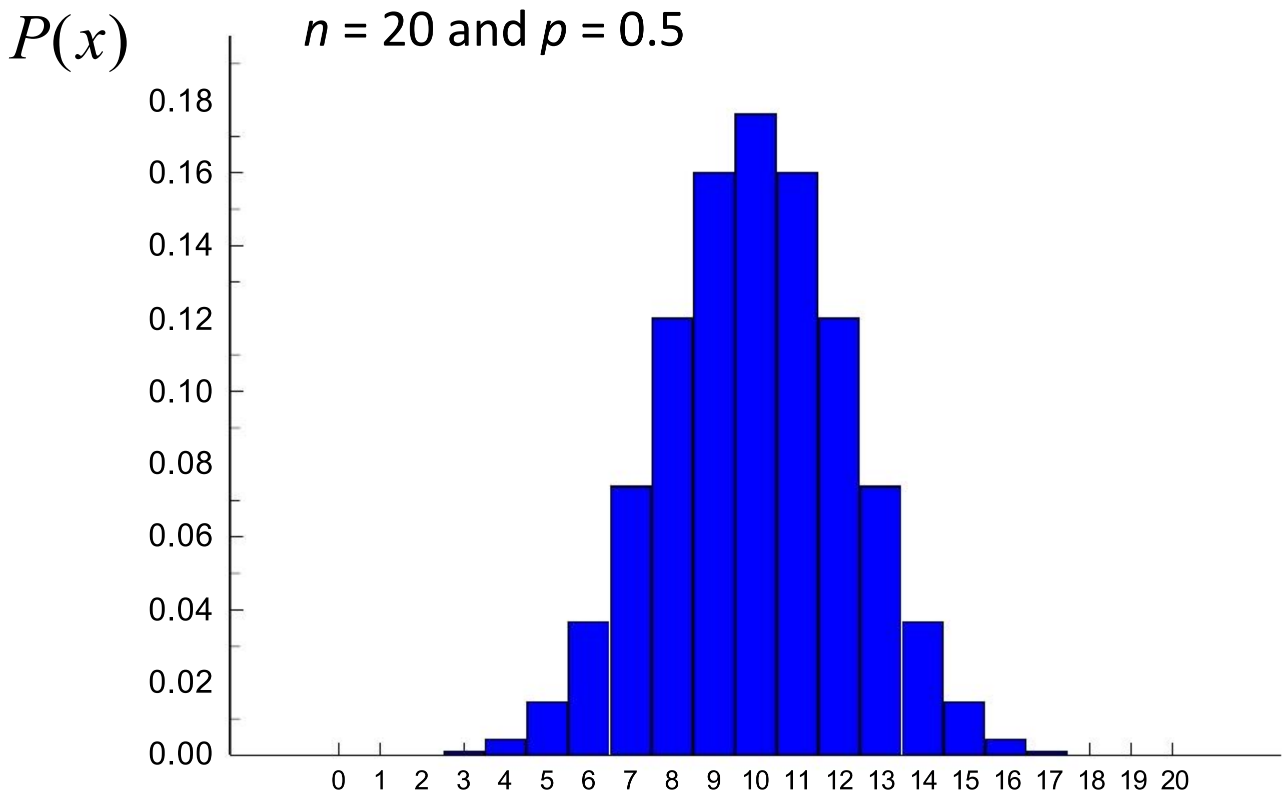
Z	.00	.01	.02	.03	.04	
0.0	.50000	.50399	.50798	.51197	.51595	.5
0.1	.53983	.54380	.54776	.55172	.55567	.5
0.2	.57926	.58317	.58706	.59095	.59483	.5
0.3	.61791	.62172	.62552	.62930	.63307	.6
0.4	.65542	.65910	.66276	.66640	.67003	.6
			⋮			
2.0	.97725	.97778	.97831	.97882	.97932	.9
2.1	.98214	.98257	.98300	.98341	.98382	.9
2.2	.98610	.98645	.98679	.98713	.98745	.9

Normal Approximation of the Binomial

- Binomial probabilities can be reasonably estimated by using the normal probability distribution.

Normal Approximation of the Binomial

Background: Consider the distribution of the binomial variable *x* when



The histogram may be approximated by a *normal* curve.

Normal Approximation of the Binomial

Note:

1. The normal curve has mean and standard deviation from the binomial distribution.

$$\boxed{\begin{aligned}\mu &= np = (20)(0.5) = 10 \\ \sigma &= \sqrt{npq} = \sqrt{(20)(0.5)(0.5)} = \sqrt{5} \approx 2.236\end{aligned}}$$

2. Can approximate the area of the rectangles with the area under the normal curve.
3. The approximation becomes more accurate as *n* becomes larger.

Normal Approximation of the Binomial

Two Problems:

1. As p moves away from 0.5, the binomial distribution is less symmetric, less normal-looking.

Solution: The normal distribution provides a reasonable approximation to a binomial probability distribution whenever the values of np and $n(1 - p)$ both exceed 5.

2. The binomial distribution is *discrete*, and the normal distribution is *continuous*.

Solution: Use the **continuity correction factor**. Add or subtract 0.5 to account for the width of each rectangle.

If $P(X=n)$ use $P(n - 0.5 < X < n + 0.5)$
If $P(X>n)$ use $P(X > n + 0.5)$
If $P(X\leq n)$ use $P(X < n + 0.5)$
If $P(X<n)$ use $P(X < n - 0.5)$
If $P(X \geq n)$ use $P(X > n - 0.5)$