Graph Theory

Rosen 8th ed., ch. 10

10.1 图的概念/Introduction of Graph

10.2 图的术语/Graph Terminology

10.3 图的表示与同构/

Representing Graph and Graph Isomorphism

10.4 连通性/Connectivity

10.5 欧拉道路与哈密尔顿道路/

Euler and Hamilton Paths

10.6 最短道路问题/Shortest Path Problem

10.7 平面图/Planar Graphs

10.8 图的着色/Graph Coloring

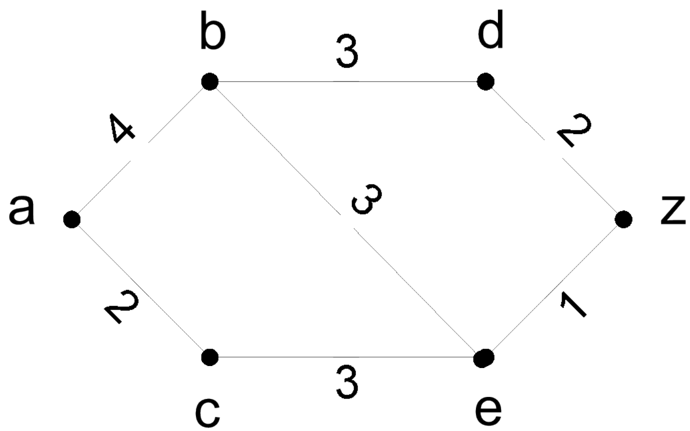
Transport networks传输网流量问题

10.6 Shortest Path Problems 最短通路

- A ***weighed graph*** （带权图）is one in which weights (numbers) are assigned to all edges connecting each two vertices.
- Such numbers can represent, for instance, traveling distance, monthly cost, or traveling time between two vertices.
- The **length** of a path in a weighed graph is the sum of the weights of the edges of this path. (This use of term *length* is different from the use of *length* to denote the number of edges in a path in a graph without weights.)

Example

- Find the length of a shortest path between *a* and *z* in the weighted graph shown below.



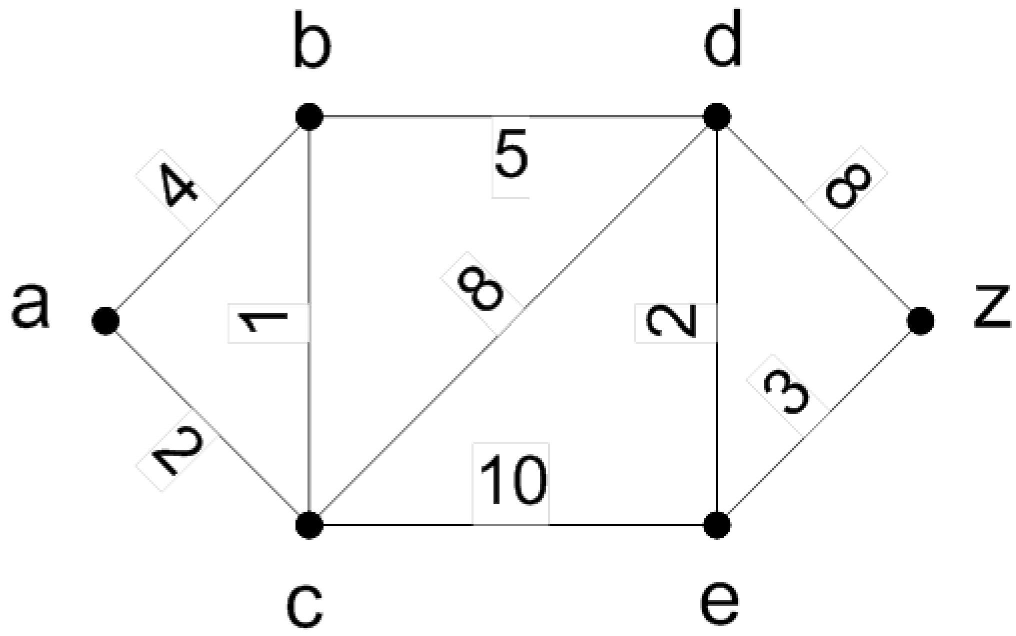
Dijkstra Algorithm
迪克斯特拉

- The algorithm is to find the shortest way from v_1 to v_n,at the same time,it gets the shortest way from v_1 to each other vertices in the graph.

Shortest Path (Dijkstra's)Algorithm

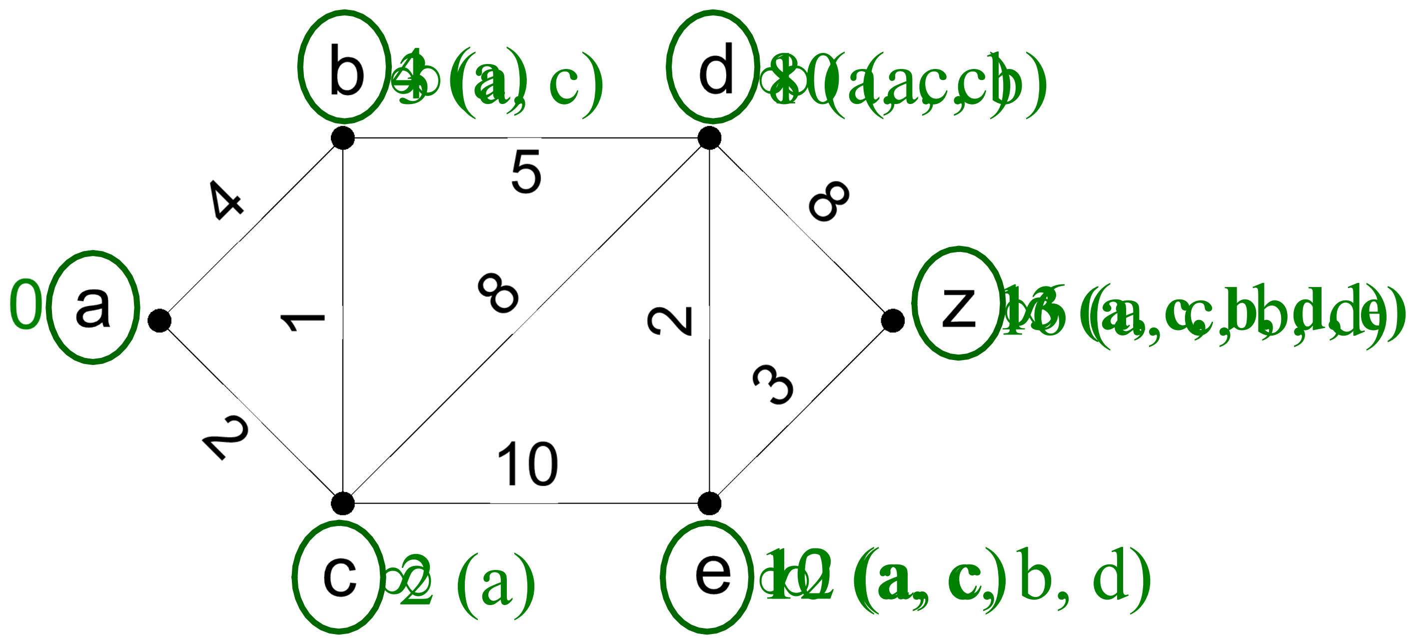
- **procedure** Dijkstra(G: weighted connected simple graph, with all weights positive)
  {G has vertices $a = v_0, v_1, \ldots, v_n = z$ and weights $w(v_i, v_j)$ where $w(v_i, v_j)=\infty$ if $\{v_i, v_j\}$ is not an edge in G}
  for i:=1 to n
  $\quad L(v_i) := \infty$
  $L(a) := 0$
  $S := \phi$
  {the labels are now initialized so that the label of a is 0 and all other labels are ∞, and S is the empty set}
  while $z \notin S$
  begin
  $\quad u :=$ *a* vertex not in S with $L(u)$ minimal
  $\quad S := S \cup \{u\}$
  $\quad$**for** all vertices v not in S
  $\quad\quad$**if** $L(u) + w(u, \text{v}) < L(v)$ **then** $L(v) := L(u) + w(u, \text{v})$
  $\quad\quad${this adds a vertex to S with minimal label and updates the labels of vertices not in S}
  end {$L(z)$ = length of a shortest path from a to z}

Dijkstra's Algorithm Example



Dijkstra’s Algorithm Example

S={ a }
S={ a,c }
S={ a,c,b }
S={ a,c,b,d }
S={ a,c,b,d,e }



Dijkstra’s Algorithm Complexity

- Theorem 1: Dijkstra’s algorithm finds the length of a shortest path between two vertices in a connected simple undirected weighted graph.

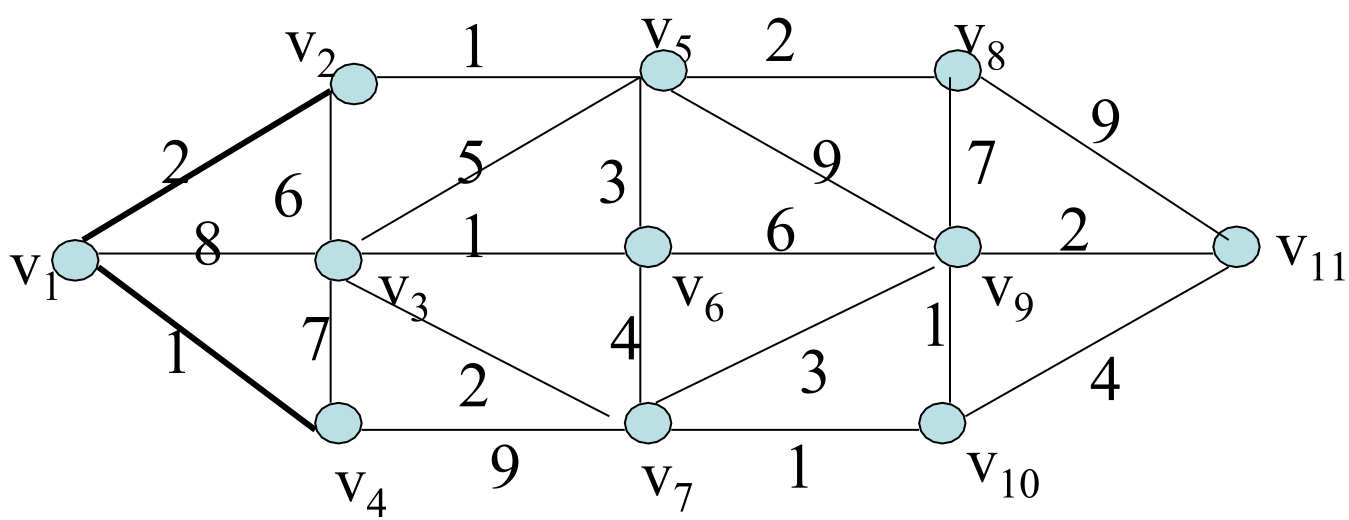
- Theorem 2: Dijkstra’s algorithm uses $O(n^2)$ operations (additions and comparisons) to find the length of a shortest path between two vertices in a connected simple undirected weighted graph with n vertices.

Applications

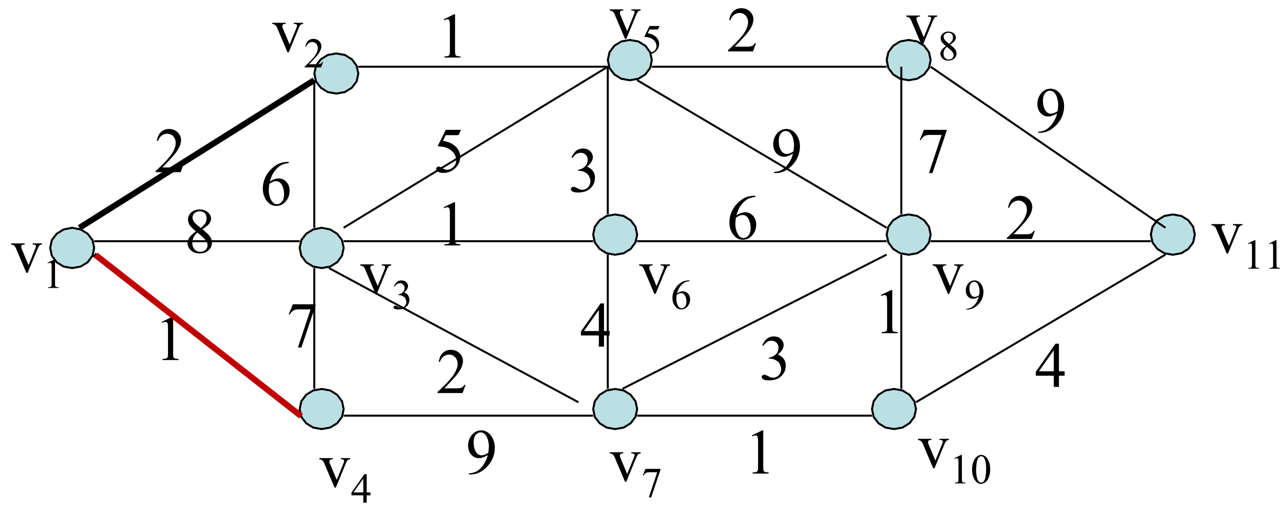
- Many problems in the real life are related to the shortest path problem. Such as
  - Pipe lines
  - Electric power grid
  - Communicating network
  - Railway stations

Example

- Let L_i represent the length from v_1 to v_i
- Let d_{ij} represent the length of the edge (v_i, v_j)
- Find the shortest path from v_1 to v_{11}

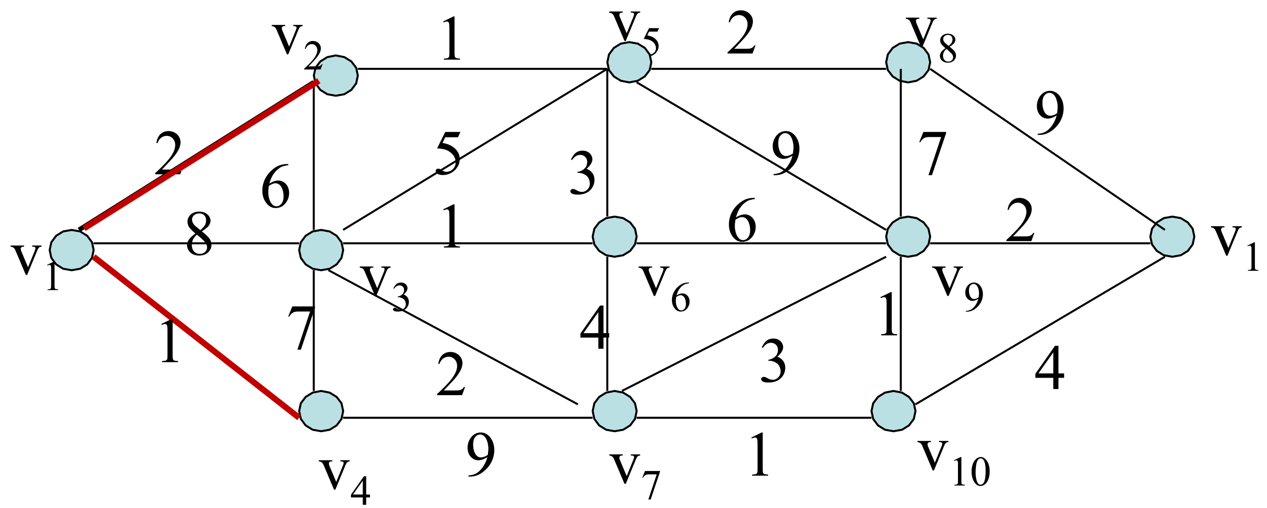


Step 1



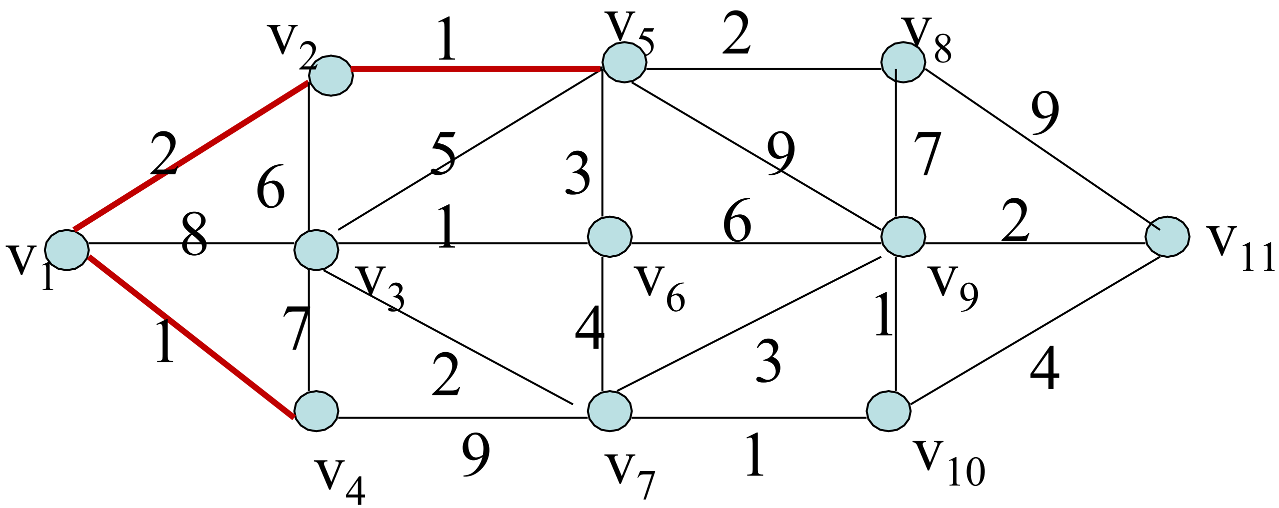
- Find the set of adjacent vertices of v_1,which are $\{v_2, v_3, v_4\}$
- Find the length from v_1 to the vertices in the set.
  - $L_2=d_{12}=2$ $\quad$ $L_3=d_{13}=8$ $\quad$ $L_4=d_{14}=1$
- Find the shortest length of l_2,l_3,l_4
  - $\text{Min}\{L_2,L_3,L_4\}= L_4=1$
- Connect v_1 to v_4

Step 2



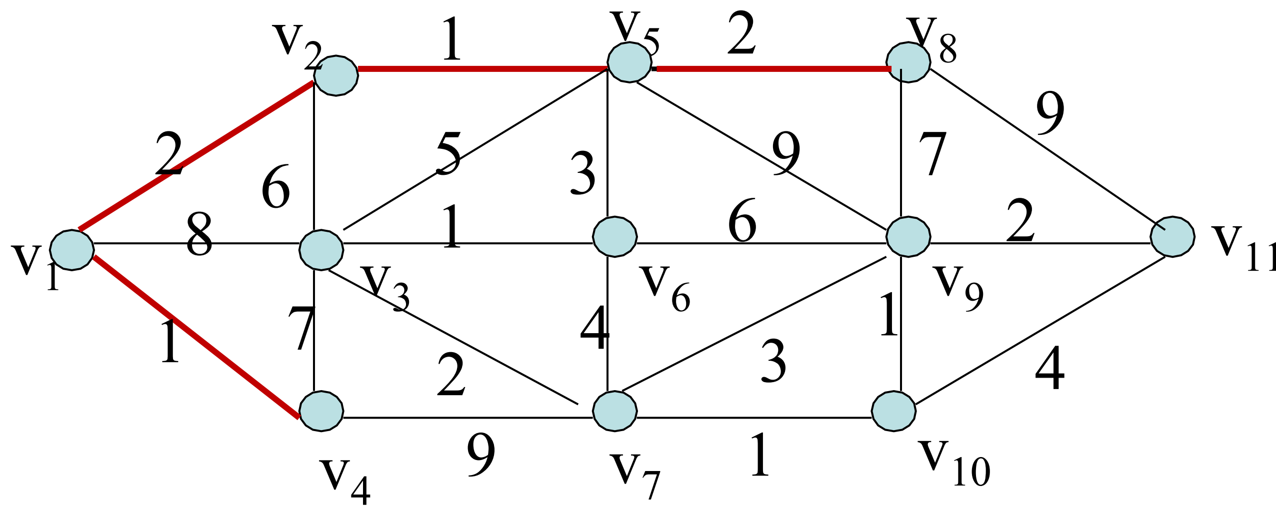
- Find the set of adjacent vertices of $\{v_1, v_4\}$ which are $\{v_2, v_3, v_7\}$
- Find the length from v_1 to the vertices in the set.
  - $L_2=2$ $L_3=8$ $L_7= L_4 + d_{47}=1+9=10$
- Find the shortest length of l_2,l_3 ,l_7
  - $\text{Min}\{L_2,L_3 ,L_7\}= L_2=2$
- Connect v_1 to v_2

Step 3



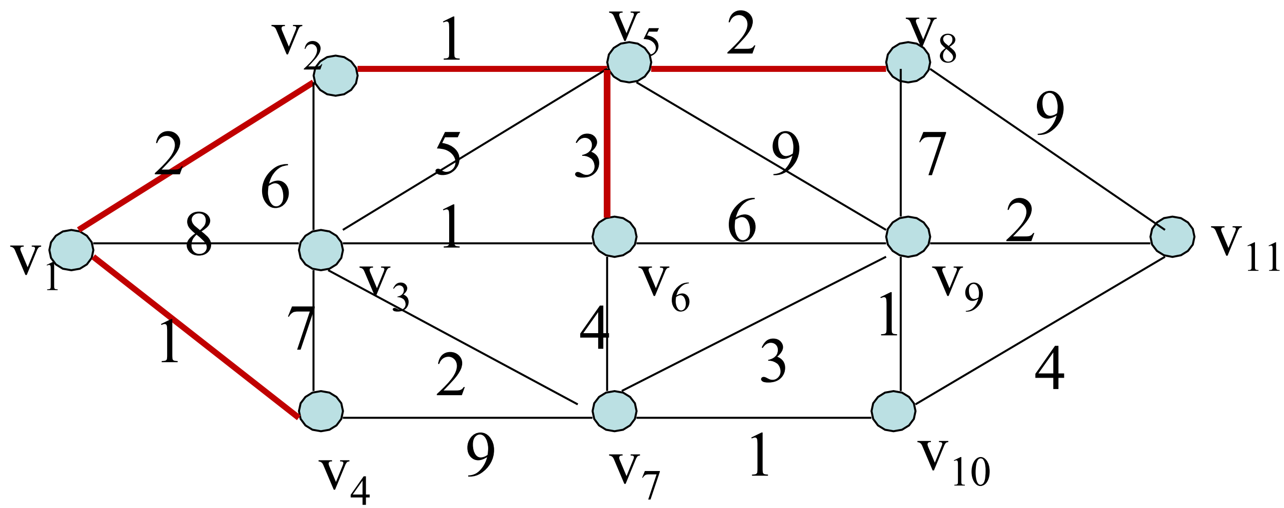
- Find the set of adjacent vertices of $\{v_1, v_2, v_4\}$ which are $\{v_3, v_5, v_7\}$
- Find the length from v_1 to the vertices in the set.
  - $L_3= \min\{8,l_2+d_{23},l_4+d_{43}\}=\{8,8,8\}=8$
  - $L_5= L_2 + d_{25}=3$
  - $L_7= \mathrm{L}_4 + d_{47}=10$
- $\min\{\mathrm{L}_3,\mathrm{L}_5,\mathrm{L}_7\}= \mathrm{L}_5=3$
- Connect v_2 to v_5

Step 4



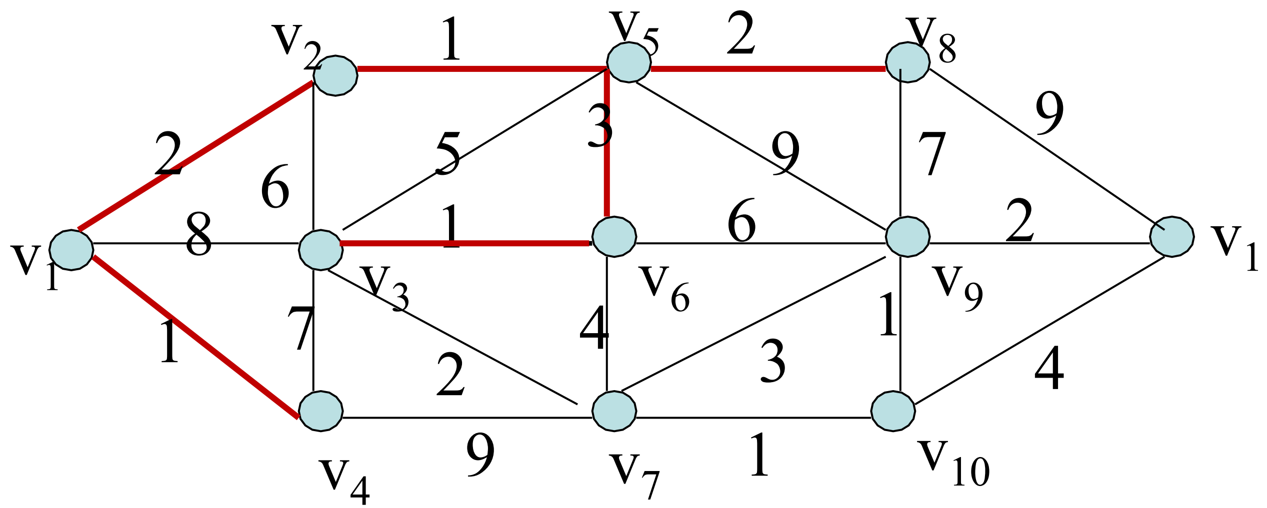
- Find the set of adjacent vertices of $\{v_1, v_2, v_4, v_5\}$ which are $\{v_3, v_6, v_7, v_8, v_9\}$
- Find the length from v_1 to the vertices in the set.
  - $L_3 = 8$ $\quad$ $L_6 = L_5 + d_{56} = 6$ $\quad$ $L_7 = 10$
  - $L_8 = L_5 + d_{58} = 5$ $\quad$ $L_9 = L_5 + d_{59} = 12$
- $\mathrm{Min}\{L_3, L_6, L_7, L_8, L_9\} = L_8 = 5$
- Connect v_5 to v_8

Step 5



- Find the set of adjacent vertices of {v_1, v_2, v_4 , v_5, v_8} which are {v_3 , v_6, v_7, v_9 , v_{11}}
- Find the length from v_1 to the vertices in the set.
  - L_3=8 L_6= $L_5 + d_{56}$=6 L_7=10
  - L_9= $L_8 + d_{89}$=12 L_{11}= $L_8 + d_{8,11}$= 14
- Min{L_3 ,L_6,L_7 ,L_9 ,L_{11}}= L_6=6
- Connect v_5 to v_6

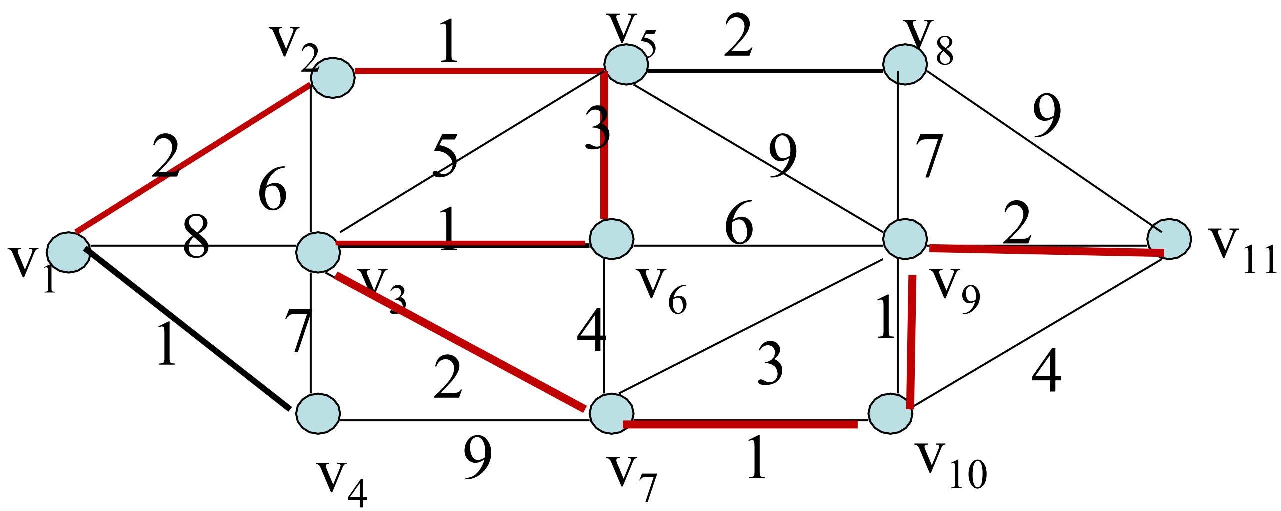
Step 6



- Find the set of adjacent vertices of $\{v_1, v_2, v_4, v_5, v_6, v_8\}$ which are $\{v_3, v_7, v_9, v_{11}\}$
- Find the length from v_1 to the vertices in the set.
  - $L_3 = \min\{8, L_6 + d_{36}\} = \{8, 7\} = 7$
  - $L_7 = \min\{L_4 + d_{47}, L_6 + d_{67}\} = \min\{10, 10\} = 10$
  - $L_9 = \min\{L_8 + d_{89}, L_6 + d_{69}, L_5 + d_{59}\} = \min\{12, 12, 12\} = 12$
  - $L_{11} = 14$
- $\min\{l_3, l_7, l_9, l_{11}\} = l_3 = 7$
- Connect v_6 to v_3

Step 7

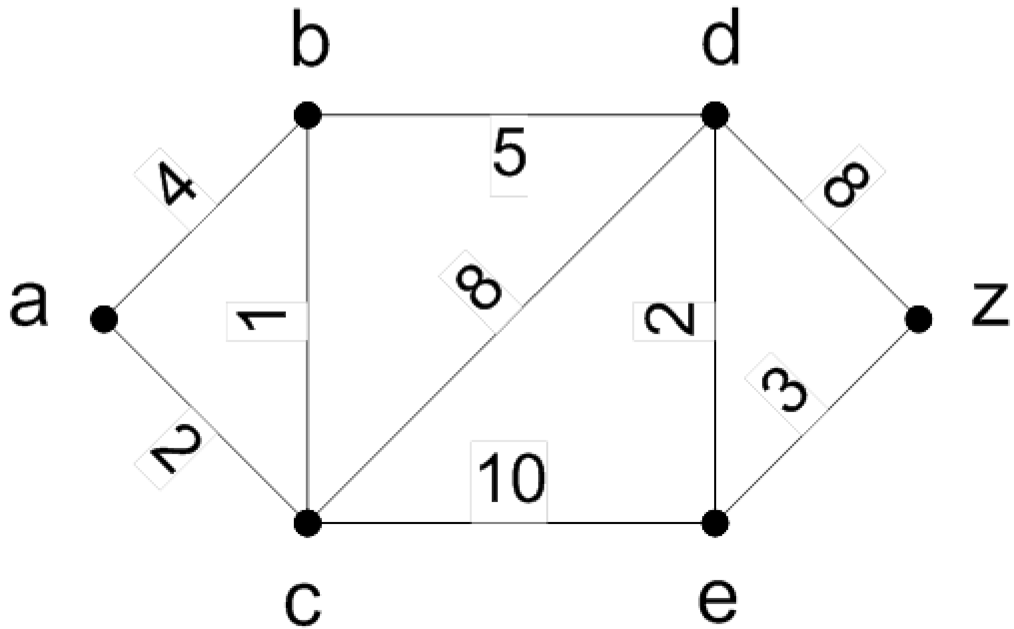
- Continue to
  - connect v_3 to v_7
  - connect v_7 to v_{10}
  - connect v_{10} to v_9
  - connect v_9 to v_{11}
- Then the path from v_1 to v_{11} is the answer.
- At the same time ,we also get the shortest path from v_1 to other vertices in the graph.


v1
v2
v3
v4
v5
v6
v7
v8
v9
v10
v11
2
8
1
6
7
1
5
1
2
9
3
4
2
6
9
3
1
7
1
9
2
4

Problem

- There are two types of such problems
  - Determining the shortest path from a vertex to an assigned vertex.
  - Determining the shortest path of any two vertices in the graph.

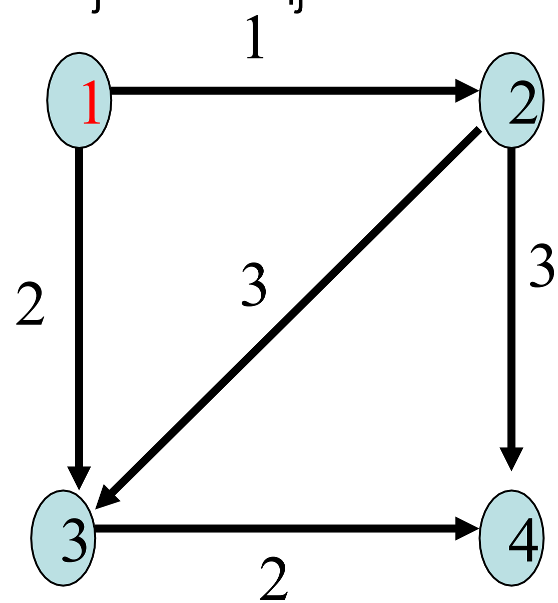
Finding the shortest path between any two vertices



Finding the shortest path between any two vertices

- Distance Matrix: Let G is a graph with n vertices. The distance matrix of G is $D=(d_{ij})_{n\times n}$
  - d_{ij} represent the the weights of the edge (v_i,v_j)
  - If there's no edge between v_i and v_j then $d_{ij}=\propto$

$$D=\begin{bmatrix} \infty & 1 & 2 & \infty \\ \infty & \infty & 3 & 3 \\ \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \end{bmatrix}$$

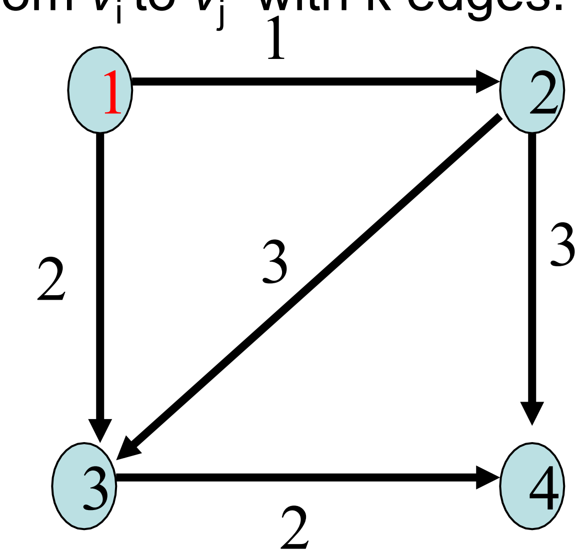


Finding the shortest path between any two vertices

- Let $D^2=D^*D=(d_{ij}^2)_{n\times n}$
- $d_{ij}^2=\min\{d_{i1}+d_{1j}, d_{i2}+d_{2j}, \ldots, d_{in}+d_{nj}\}$
  - d_{ij}^2 is the shortest length of the path from v_i to v_j with two edges.
- As the same $D^k=D^{k-1}*D=(d_{ij}^k)_{n\times n}$
  - d_{ij}^k is the shortest length of the path from v_i to v_j with k edges.

$$D^2=\begin{bmatrix}\infty & 1 & 2 & \infty\\ \infty & \infty & 3 & 3\\ \infty & \infty & \infty & 2\\ \infty & \infty & \infty & \infty\end{bmatrix}*\begin{bmatrix}\infty & 1 & 2 & \infty\\ \infty & \infty & 3 & 3\\ \infty & \infty & \infty & 2\\ \infty & \infty & \infty & \infty\end{bmatrix}$$

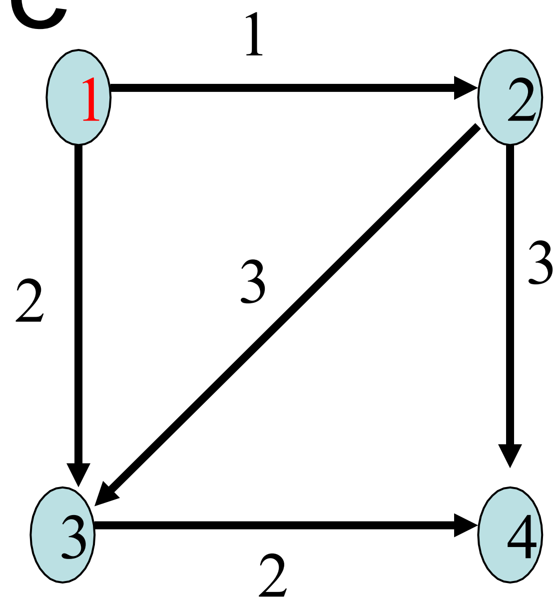
min{∞+ ∞, ∞+3，3+2，3+ ∞}



Define ⊕

- Let $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$
- $C = A \oplus B = (c_{ij})_{n\times n}$
  - $c_{ij} = \min(a_{ij}, b_{ij})$
- $P = D \oplus D^2 \oplus D^3 \oplus \ldots \oplus D^n$
  - $p_{ij} = (p_{ij})_{n\times n}$
  - p_{ij} represent the shortest length from v_i to v_j

Example



$$D = \begin{bmatrix} \infty & 1 & 2 & \infty \\ \infty & \infty & 3 & 3 \\ \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \end{bmatrix}$$

$$D^2 = \begin{bmatrix} \infty & 1 & 2 & \infty \\ \infty & \infty & 3 & 3 \\ \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \end{bmatrix} * \begin{bmatrix} \infty & 1 & 2 & \infty \\ \infty & \infty & 3 & 3 \\ \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \end{bmatrix} = \begin{bmatrix} \infty & \infty & 4 & 4 \\ \infty & \infty & \infty & 5 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix}$$

$$D^3 = \begin{bmatrix} \infty & \infty & 4 & 4 \\ \infty & \infty & \infty & 5 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix} * \begin{bmatrix} \infty & 1 & 2 & \infty \\ \infty & \infty & 3 & 3 \\ \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \end{bmatrix} = \begin{bmatrix} \infty & \infty & \infty & 6 \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix}$$

- $D_4=(\propto)_{4\times4}$
- $P=D \oplus D^2 \oplus D^3 \oplus D^4$

$$\begin{bmatrix} \infty & 1 & 2 & \infty \\ \infty & \infty & 3 & \boxed{3} \\ \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \end{bmatrix} \oplus \begin{bmatrix} \infty & \infty & 4 & 4 \\ \infty & \infty & \infty & \boxed{5} \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix} \oplus \begin{bmatrix} \infty & \infty & \infty & 6 \\ \infty & \infty & \infty & \boxed{\infty} \\ \infty & \infty & \infty & \infty \\ \infty & \infty & \infty & \infty \end{bmatrix} \oplus D_4$$

-

$$P = \begin{bmatrix} \infty & 1 & 2 & 4 \\ \infty & \infty & 3 & \boxed{3} \\ \infty & \infty & \infty & 2 \\ \infty & \infty & \infty & \infty \end{bmatrix}$$

- note: ex.21 floyd

Floyd algorithm

procedure Floyd (G:weighted simple graph)

{ G has vertices $v_1, v_2, \ldots, v_n$ and weights w (v_i, v_j) with w (v_i, v_j) =∞ if $\{v_i, v_j\}$ is not an edge}

```
for i: =1 to n
   for j: =1 to n
        d (v_i, v_j) : =w (v_i, v_j)
for i: =1 to n
   for j: =1 to n
       for k: =1  to  n
           If d (v_j, v_i) +d (v_i, v_k) <d(v_j, v_k)
           then d (v_j, v_k) : =d (v_j, v_i) +d (v_i, v_k)
```

return $[d(v_i, v_j)]$ $\{d(v_i, v_j)$ is the length of a shortest path between v_i and v_j for $1 \le i \le n, 1 \le j \le n\}$

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The traveling salesman problem

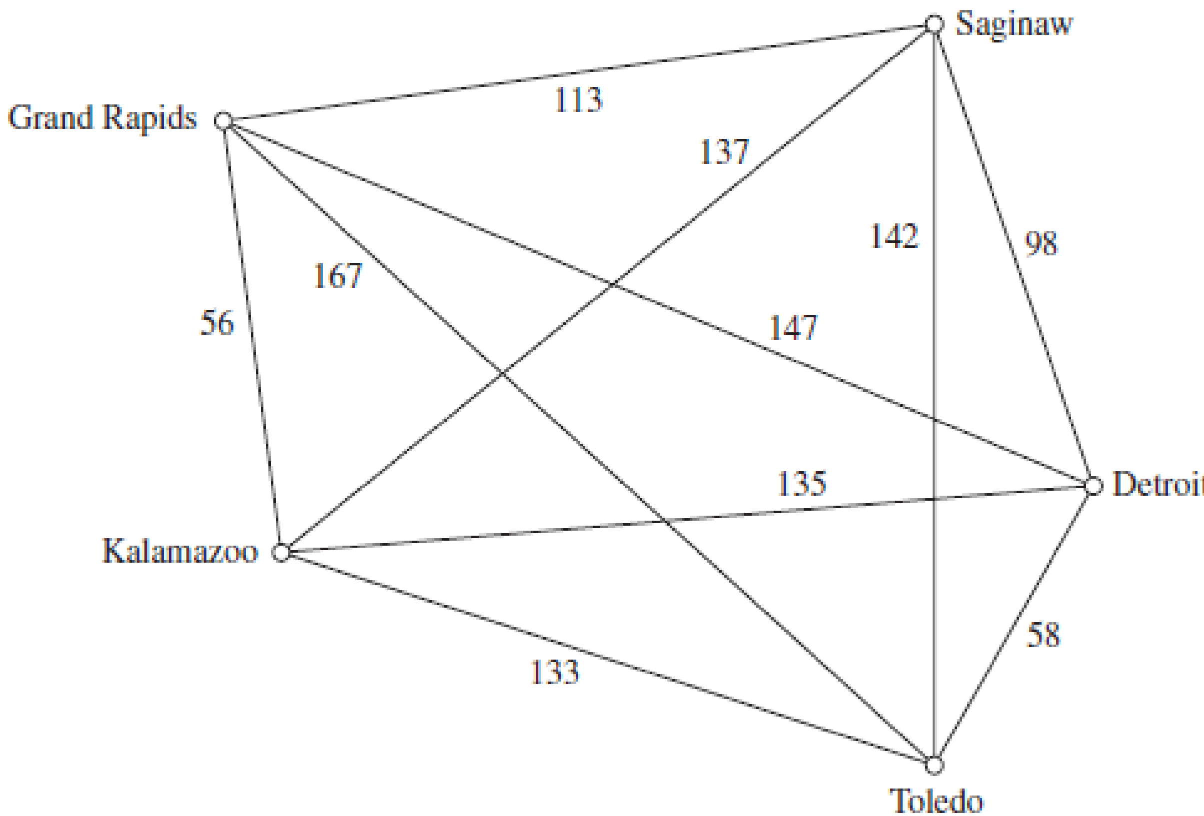


FIGURE 5 **The Graph Showing the Distances between Five Cities.**

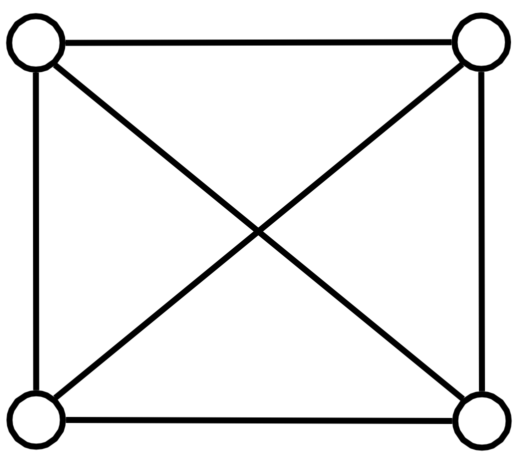
Planar Graphs

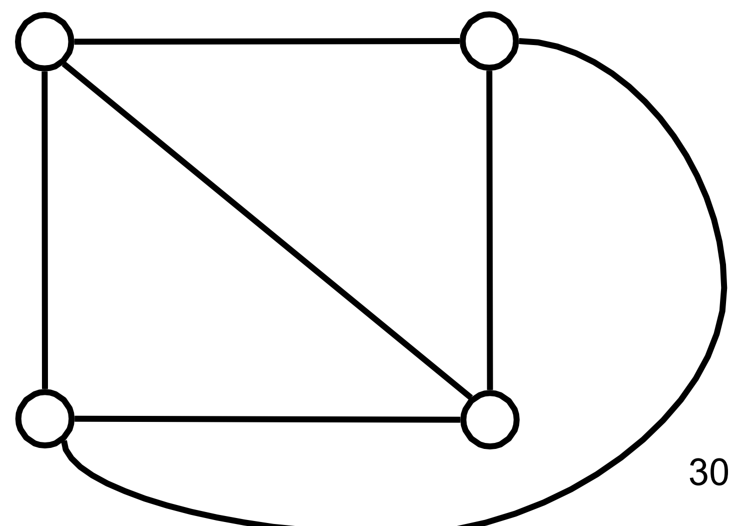
- Definition
- Euler Theorem
- Determine non-planar

Planar Graphs – 平面图

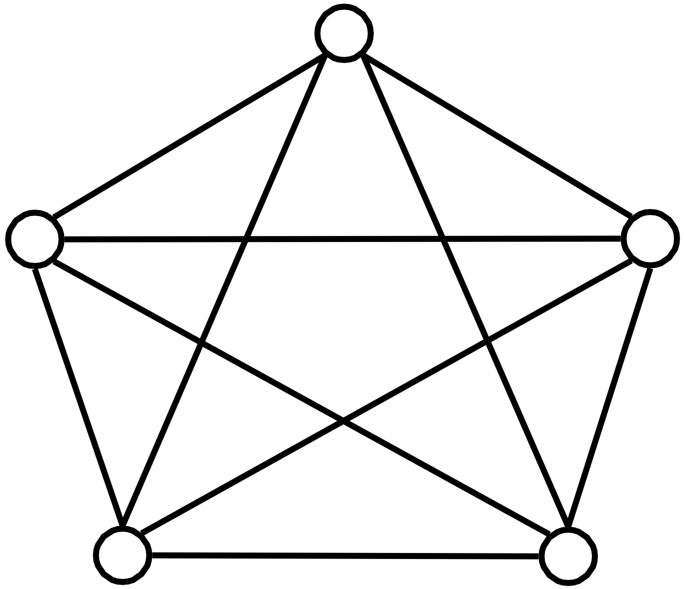
- A graph is called ***planar*** if it can be drawn in the plane in such a way that no two edges cross.
- Example of a planar graph: The clique on 4 nodes.

K_4

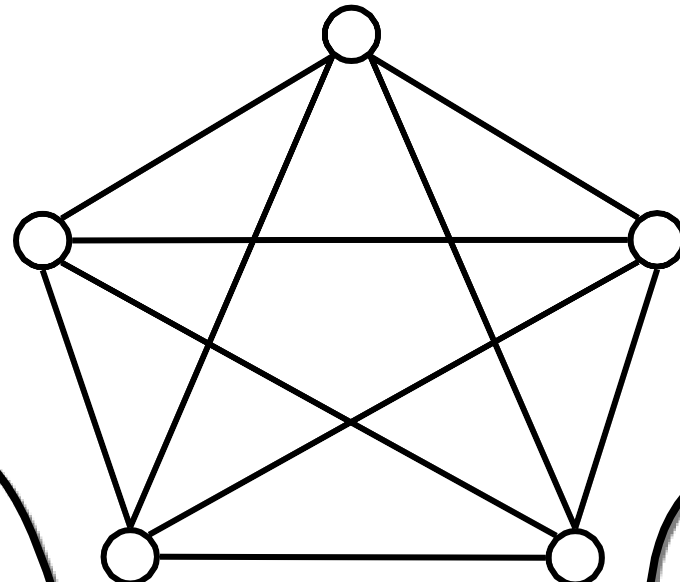


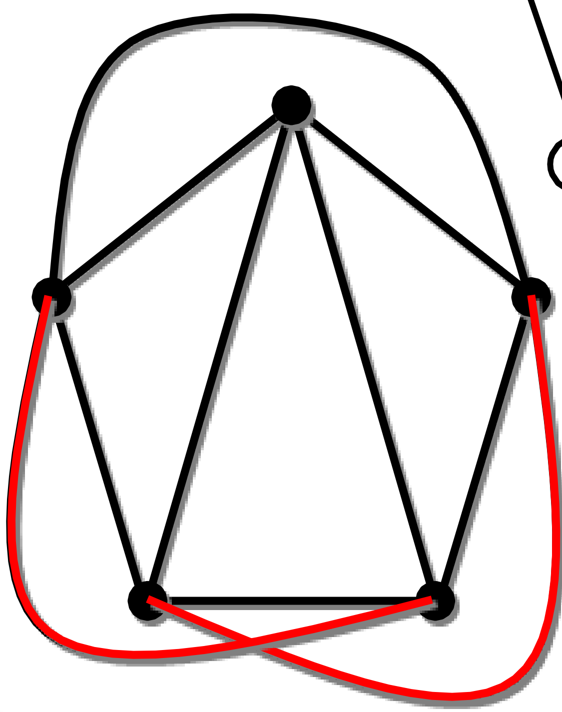


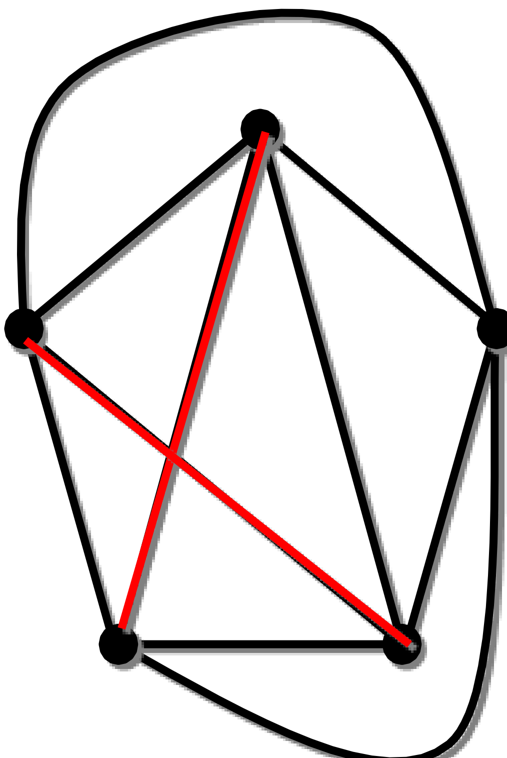
Is K_5 planar?



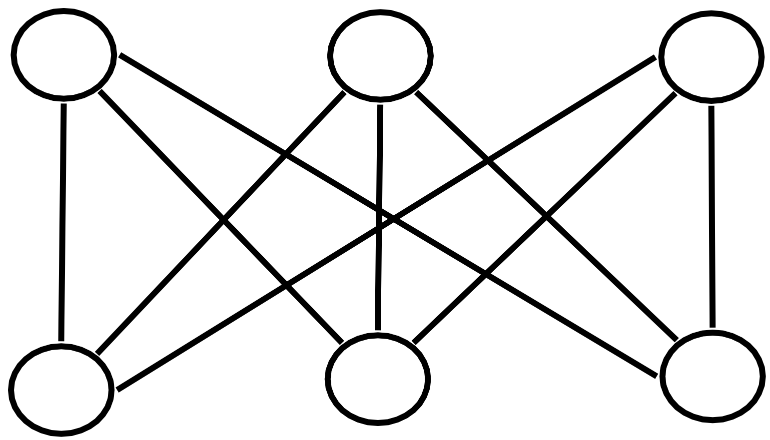
Is K_5 planar?







What about $K_{3,3}$?



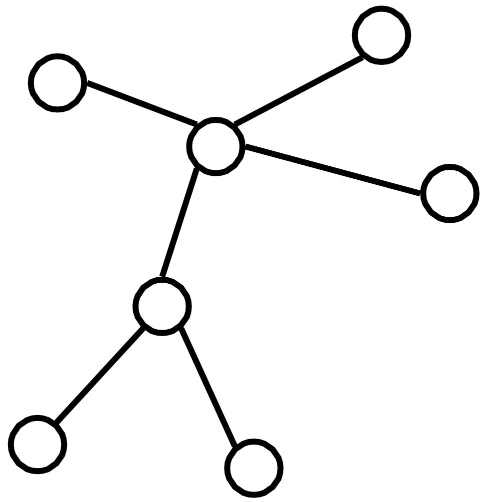
Why Planar?

- The problem of drawing a graph in the plane arises frequently in VLSI layout problems.

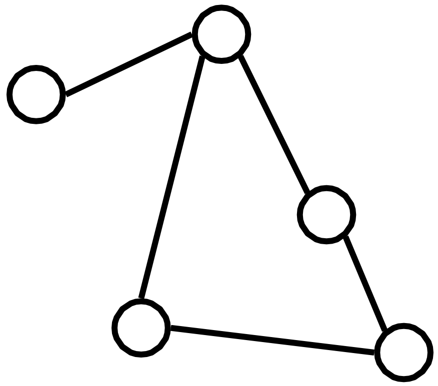
Regions, faces – 面

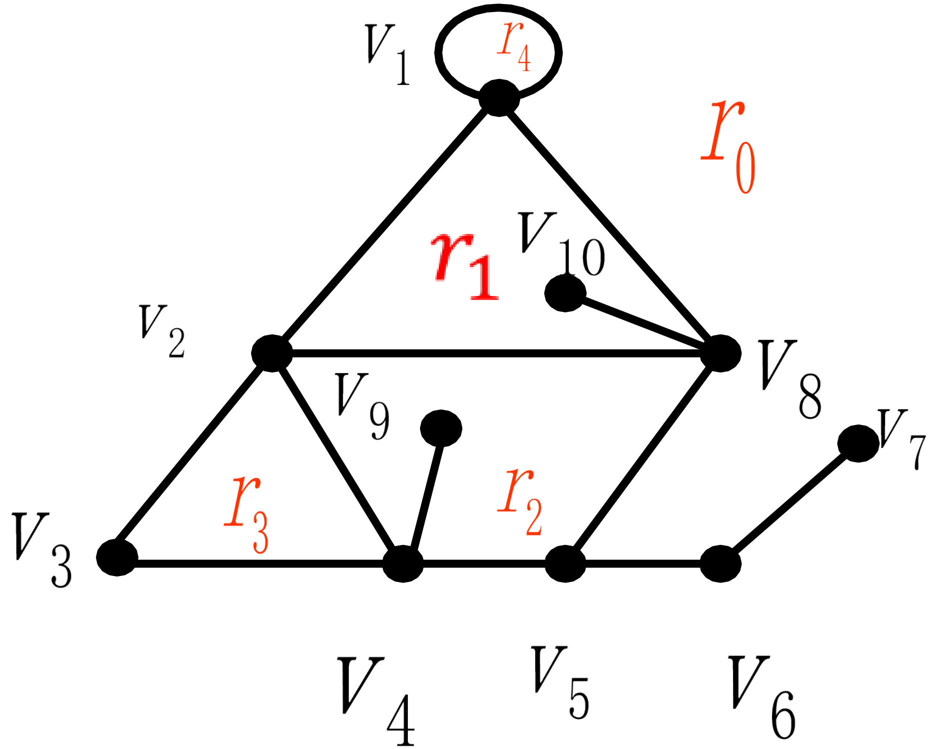
- A plane graph cuts the plane into regions that we call ***faces***.

one face



two faces





r_1由回路
$v_1v_2v_8v_{10}v_8v_1$
所包围，
$\deg(r_1) = 5$

r_0由回路
$v_2v_3v_4v_5v_6v_7v_6v_5v_8v_1v_1v_2$
所包围。$Deg(r_0) = 11$

r_2由$v_2 v_4 v_9 v_4 v_5 v_8 v_2$所包围，

$\deg(r_2) = 6$

r_3由回路$v_2 v_3 v_4$所包围，

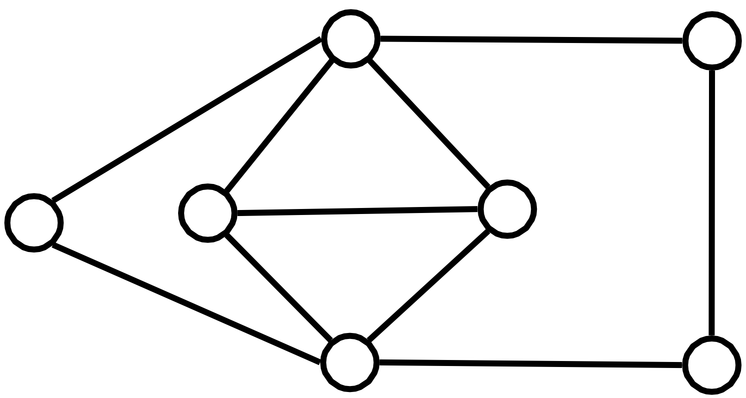
$\deg(r_3) = 3$

r_4由回路$v_1 v_1$所包围，

$\deg(r_4) = 1$

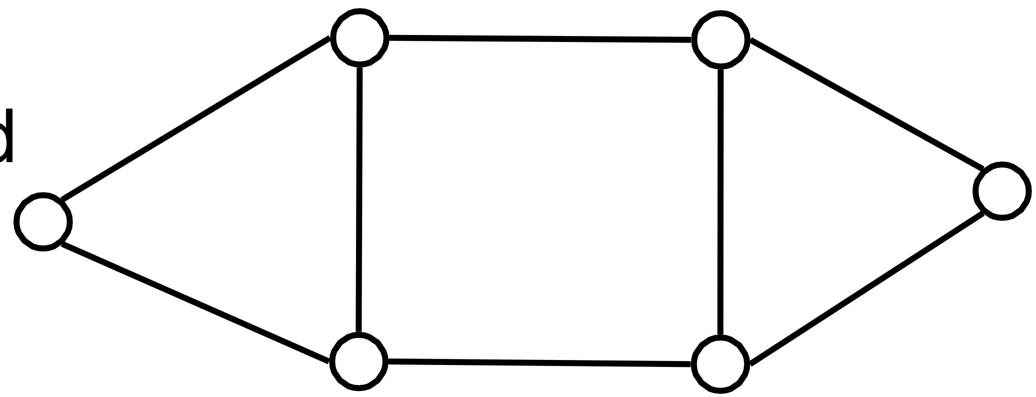
Question

- Can you redraw this graph as a plane graph so as to alter the number of its faces?



Example

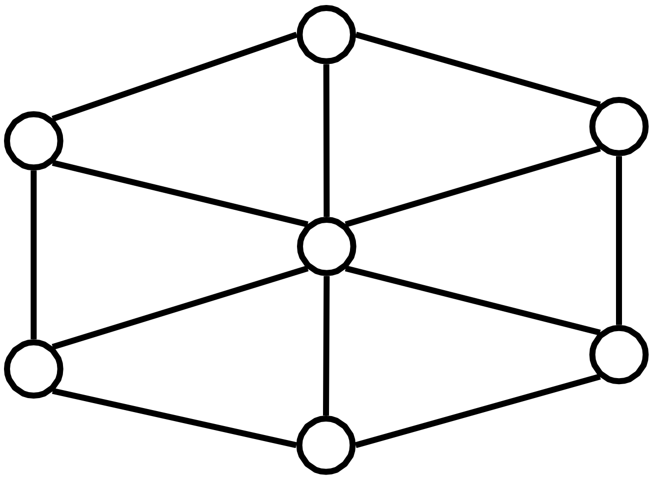
- This graph has
  - 6 vertices
  - 8 edges and
  - 4 faces



vertices – edges + faces = 2

Example

- This graph has
  - 7 vertices
  - 12 edges and
  - 7 faces



- vertices – edges + faces = 2

Euler Theorem

- If *G* is a connected plane graph, then
  - vertices – edges + faces = 2
- Let
  - v = # of vertices
  - e = # of edges
  - r = # of region （faces）

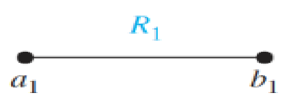
$$v-e+r=2$$

[定理1]（欧拉公式/Euler’s formula）

- Let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + 2$.

proof：induction

(1) $e = 1$



$v = 2$， $r = 1$， $v - e + r = 2 - 1 + 1 = 2$

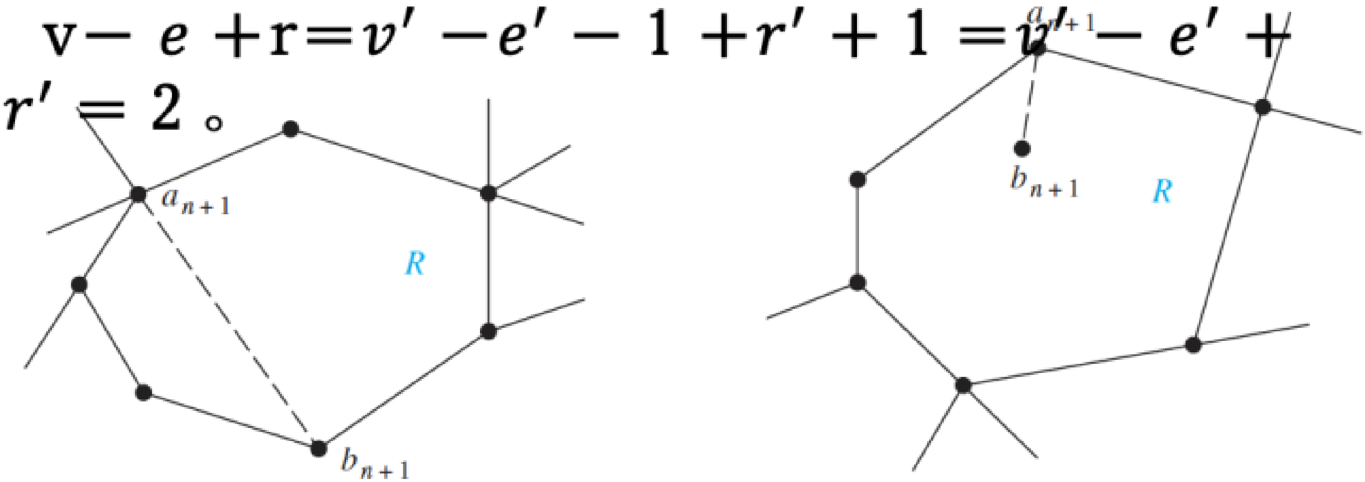
(2)assume， $e = \mathrm{n}$， r-e+v=2

then， while $e = \mathrm{n} + 1$， eliminate one edge， there are two cases:

case a) $v' - e' + r' = 2$，

$\mathrm{v} = v'$， $e = e' + 1$， $\mathrm{r} = r' + 1$，

$\mathrm{v} - e + \mathrm{r} = v' - e' - 1 + r' + 1 = v' - e' + r' = 2$。



(a) (b)

b)

- $v' - e' + r' = 2$，
- $v = v' + 1$， $e = e' + 1$， $r = r' = 1$，
- $v - e + r = v' + 1 - e' - 1 + r' = v' - e' + r' = 2$。

Corollary

- No matter how we redraw a planar graph it will have the same # of regions.
- Proof:
  - $r = 2 - v + e$ is determined by v and e, neither of which change when we redraw the graph.
  - 图重画，不影响$r = 2 - v + e$ 。

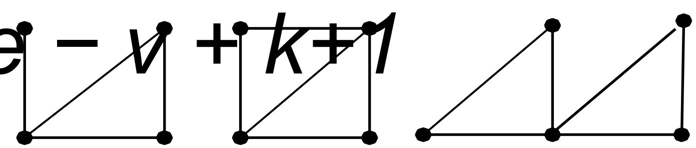
Theorem

k个分支的图

- Let G be a planar graph with k connected component and e edges and v vertices. Let r be the number of regions in a planar representation of G. Then $r = e - v + k+1$.

Proof:

- Suppose component $G_1 G_2 \ldots\ldots G_k$,
- so $e_i - v_i + 2 = r_i$. => $2k = \Sigma r_i + \Sigma v_i - \Sigma e_i$
- and $r = \Sigma r_i - (k-1)$;(外部面只有一个)
- *so* $2k = r + k - 1 + v - e$ => $r = e - v + k + 1$



corollary 1

- Every <u>connected planar simple graph</u> *G* with e-edges, *v*-node($v \geq 3$) has at most 3*v*-6 edges. $e \leq 3v-6$

Proof:

- $v = 3$， . $e \leq 3v-6$ is true .
- $v \geq 3$: G is simple graph, at least 3 edges per face.
- At most 2 faces per edge

corollary 2

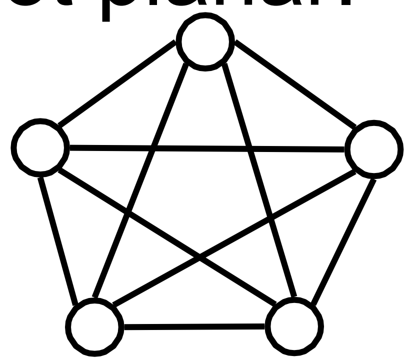
- If G is a connected planar simple graph, then G has a vertex of degree not exceeding five. （$\exists\ \delta(G) \le 5$）
- proof: v=1,or 2, It is clearly true.
- v≥3, by corollary 1, $e \le 3v-6$, $2e \le 6v-12$.
- If $\delta(G)=6$, $2e=\Sigma deg(v)$ ≥6v (by handshaking theroem).
- contradict, so $\delta(G) \le 5$

K_5 is not planar

- A connected planar simple graph on
- $n = 5$ nodes can have at most
- $3n$-6 = 9 edges.

Thus: K_5 is not planar.

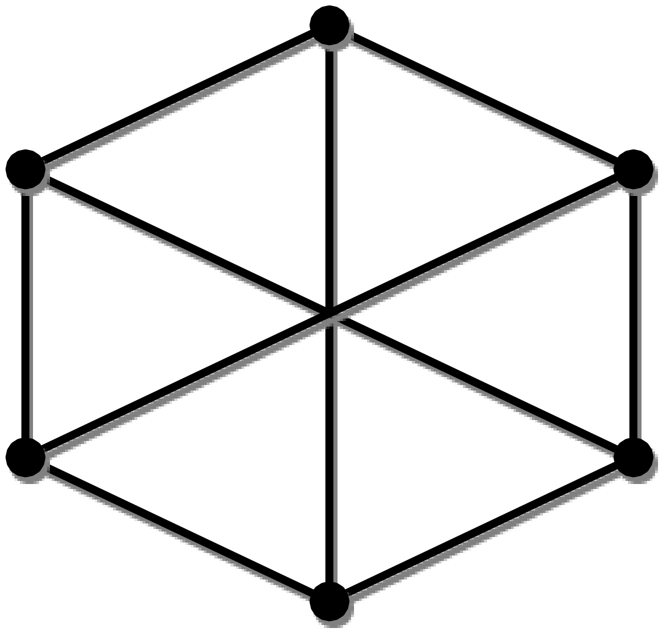
$$n = 5$$



$$e = \binom{5}{2} = 10$$

$$e \le 3v - 6$$

是简单连通平面图的必要条件



$K_{3,3}$

$$e \le 3v - 6$$

$$9 \le 3 \times 6 - 6 = 12$$

$K_{3,3}$ is not planar

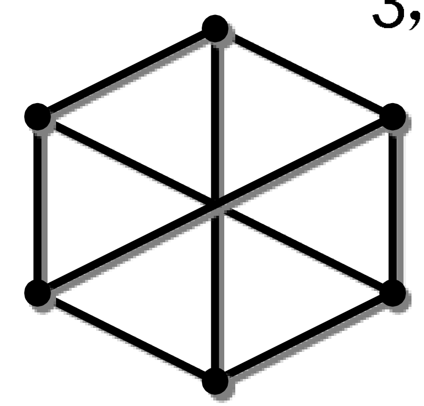
corollary 3

- If a <u>connected planar simple graph</u> has *e* edges and *v* vertices with $v \geq 3$ and no circuit of length three, then $e \leq 2v-4$.

Proof:

$v \geq 3$: G is simple graph, no circuit of length three, so <u>every face has at least 4 edges on its boundary</u>. Thus $2e>=4r$.

$4v-4e+4r=8 \Rightarrow 4v-2e\geq 8 \Rightarrow e\leq 2v-4$.



If a graph is planar, so will be any graph obtained by removing an edge {u, v} and adding a new vertex w together with edges {u, w} and {w, v}. Such an operation is called an elementary subdivision（初等细分）. The graphs G1 = (V1, E1) and G2 = (V2, E2) are called homeomorphic （同胚）if they can be obtained from the same graph by a sequence of elementary subdivisions.

Homeomorphic（同胚）

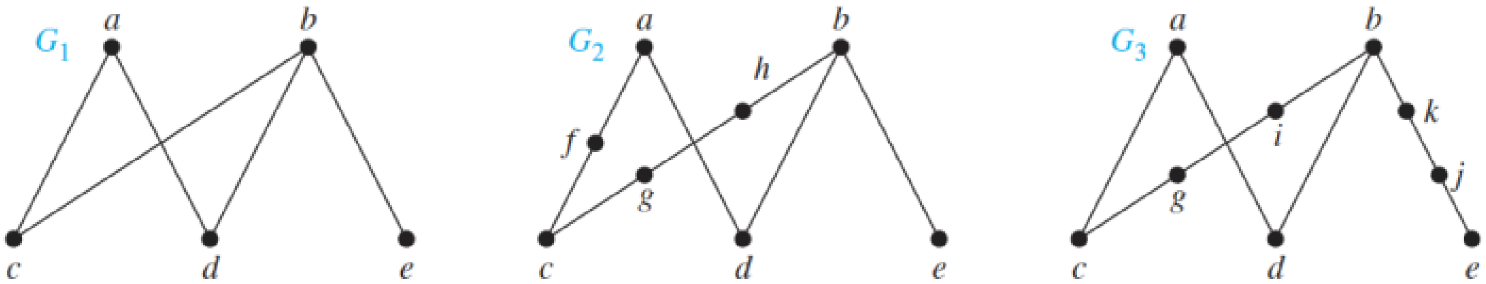
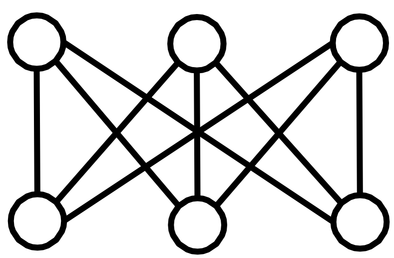


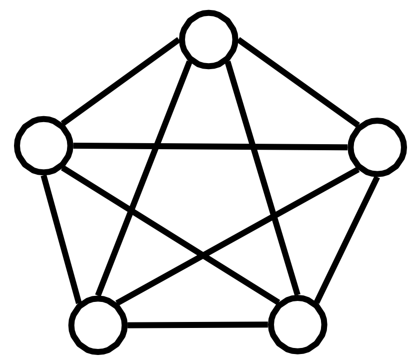
FIGURE 12 **Homeomorphic Graphs.**

Kuratowski's Theorem
库拉托夫斯基定理

A graph is planar if and only if it contains no subgraph obtainable from K_5 or $K_{3,3}$ by replacing edges with paths.



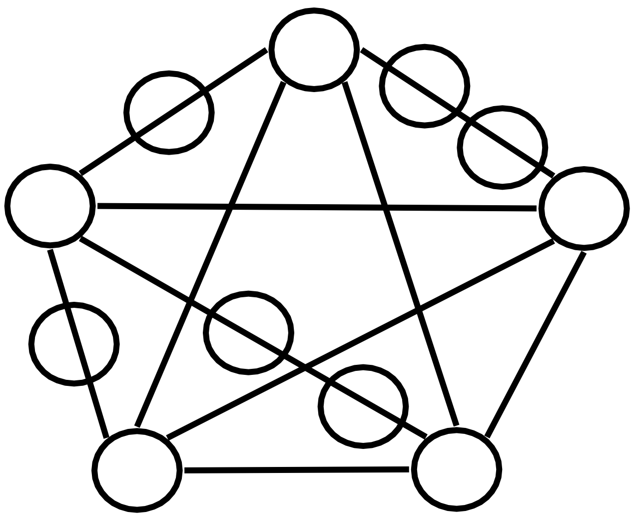
$K_{3,3}$



K_5

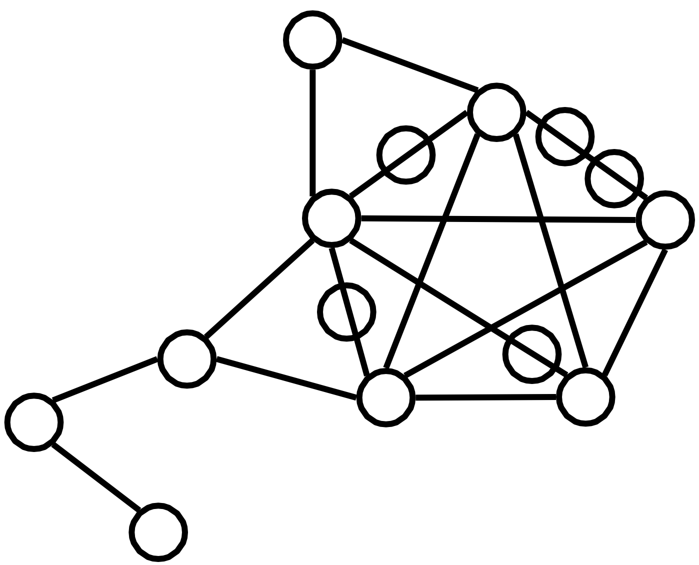
Insight 1

- If we replace edges in a Kuratowski graph by paths of whatever length, they remain non-planar.

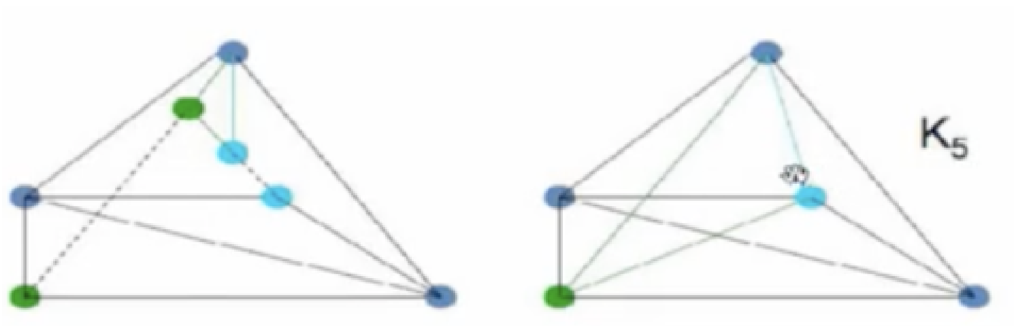


Insight 2

- If a graph G contains a subgraph obtained by starting with K_5 or $K_{3,3}$ and replacing edges with paths, then G is non-planar.



example



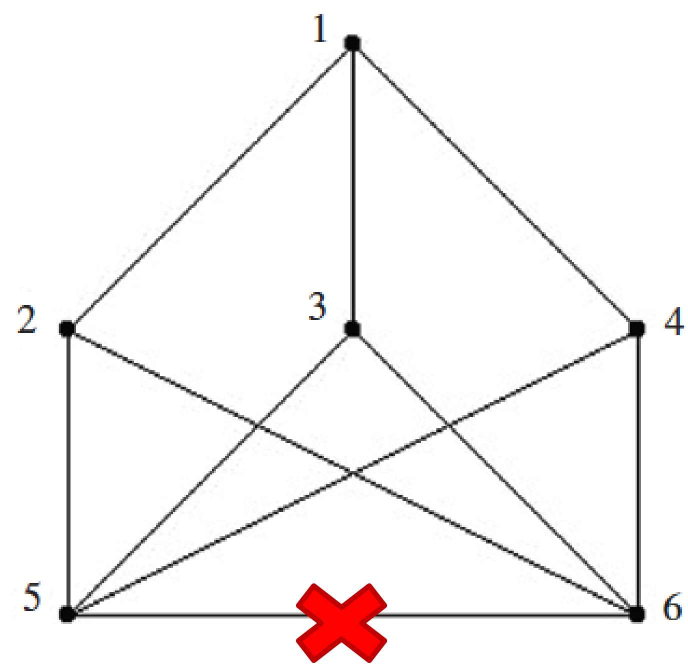

K_5

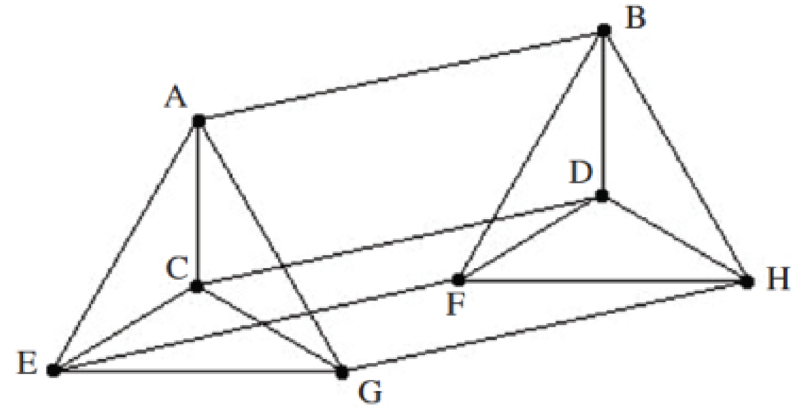

$K_{3,3}$

example

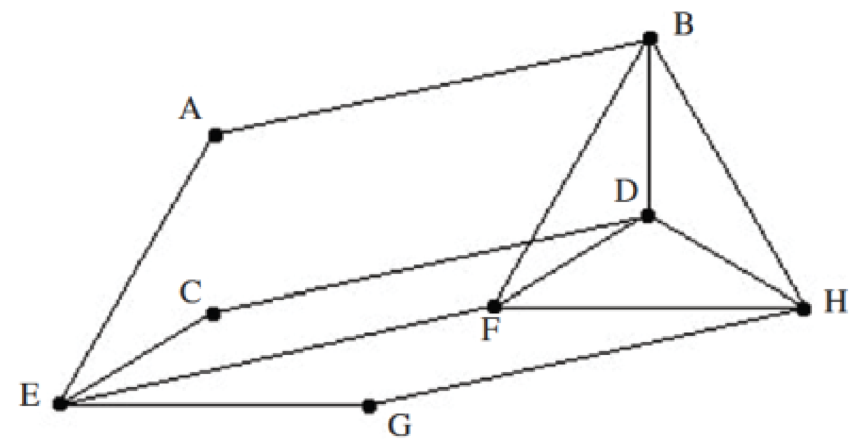
Determine whether the following graph is planar.

Solution:
The graph is not planar.
If the edge {5; 6} is removed, the resulting subgraph is isomorphic to K3;3. (Use {2; 3; 4} and
{1; 5; 6`}as the partition of the vertices of K3;3.

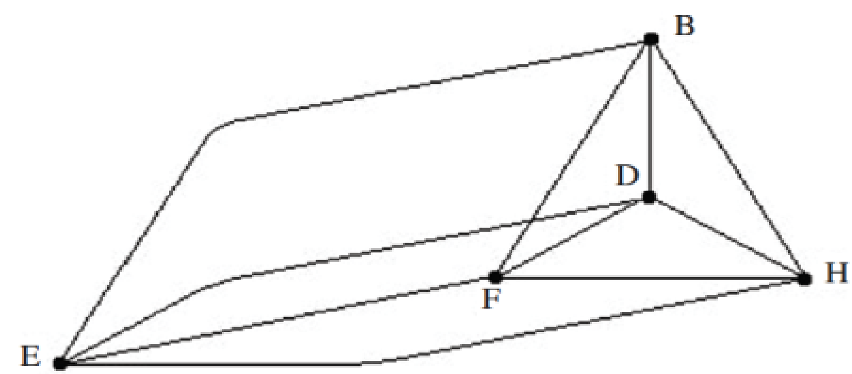




Solution:
The graph is not planar. It contains a subgraph homeomorphic to K5, using vertices E;B;D; F;H. First remove some
edges to obtain the following subgraph:

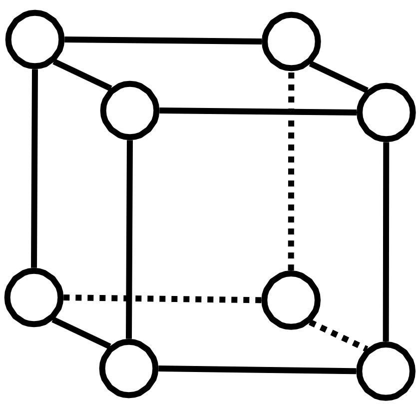


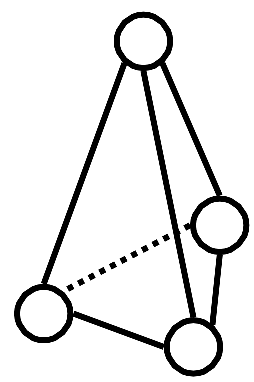
Then use elementary subdivisions at vertices A;C;G to obtain the following graph, K5:



Platonic Solids – 柏拉图体

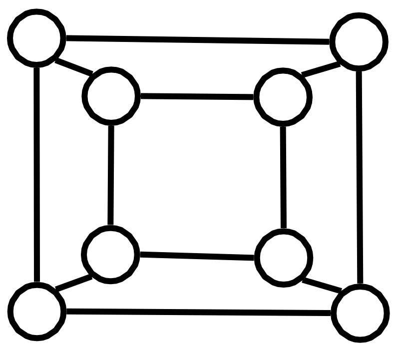
- A ***Platonic solid*** has congruent *regular polygons*(正则多边形) as faces and has the same number of edges meeting at each corner.

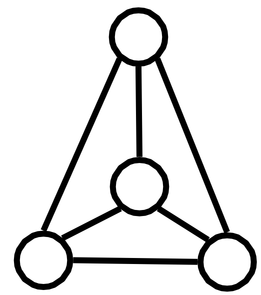




Platonic Solids

- Each one can be flattened into a planar graph:





- with constant degree: ***k*** and
- the same number of edges bounding each face: ***l***

$$\sum_{\text{vertex } x} \begin{array}{l}\text{\# of edges}\\ \text{coming from } x\end{array} = 2e$$

$$\| $$

$$kv \qquad \boxed{kv = 2e}$$

Each edge belongs to 2 faces: $\boxed{fl = 2e}$

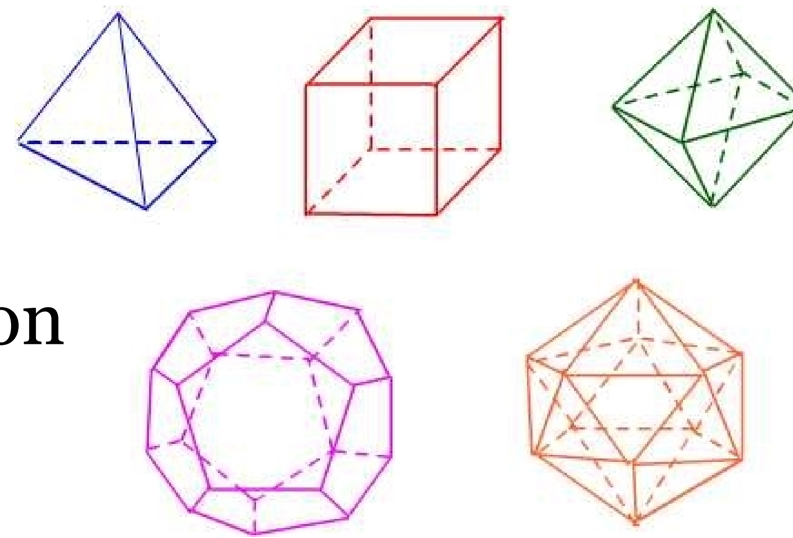
By Euler's formula: $\boxed{v - e + f = 2}$

$$(k-2)(l-2) = \frac{2f-4}{f} \cdot \frac{2v-4}{v} \quad \Rightarrow \quad (k-2)(l-2) < 4$$

and $k,l \geq 3$ for physical reasons

The only solutions

k	l	e	v	f	
3	3	6	4	4	tetrahedron
3	4	12	8	6	cube
4	3	12	6	8	octahedron
3	5	30	20	12	dodecahedron
5	3	30	12	20	icosahedron



作业

- §10.5 8, 10, 16, 26, 34, 48, 58
- §10.6 8, 16, 18, 26
- §10.7 6, 8, 12, 18, 24, 30