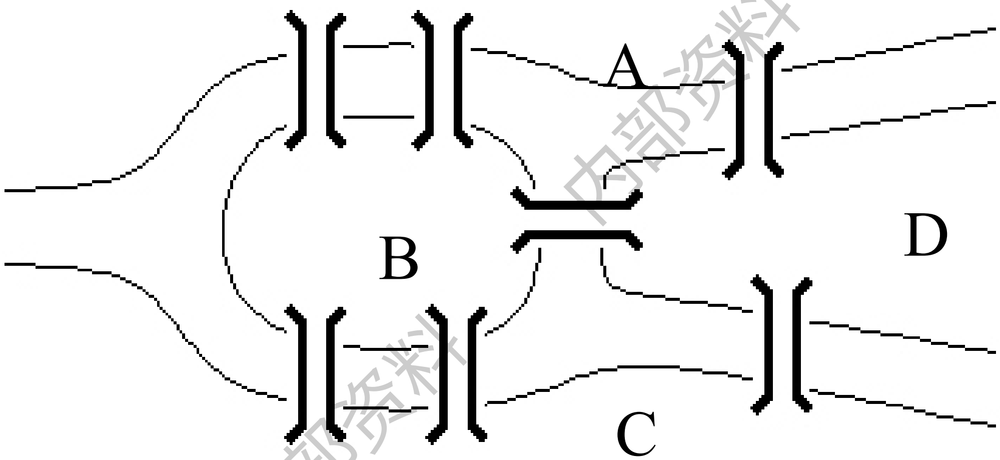
Graph Theory

Rosen 8th ed., ch. 10

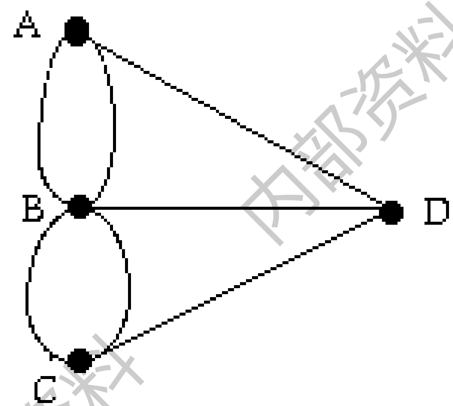
10.5 Euler and Hamilton Paths

Bridges of Königsberg Problem (哥尼斯堡)七桥问题

- Can we walk through town, crossing each bridge exactly once, and return to start?



The original problem



Equivalent multigraph

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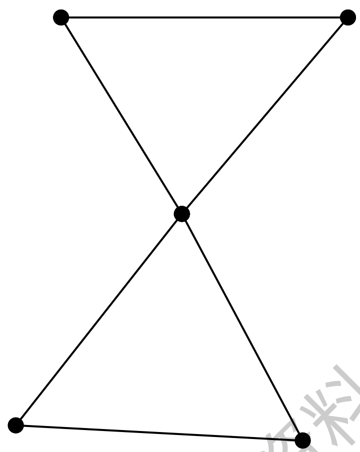
Definition 1:

- An Euler circuit（欧拉回路） in a graph G is simple circuit containing every edge of G.
- An Euler path （欧拉通路）in G is a simple path containing every edge of G.

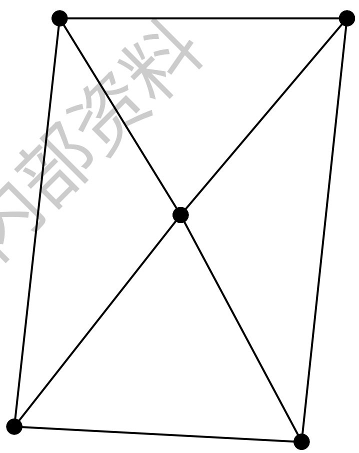
Eulerian graph（欧拉图）

- A walk in a graph is called an ***Euler tour*** if it starts and ends in the same place and uses ***each edge exactly once***.
- A walk in a graph is called an ***Euler trail*** if it uses each edge exactly once.
- If a graph has an Euler tour, it is said to be an ***Eulerian graph***.

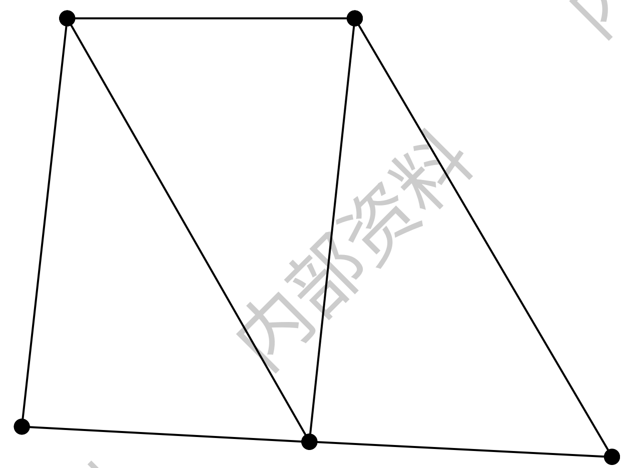
Which of the undirected graphs have an Euler circuit (path)?



G1

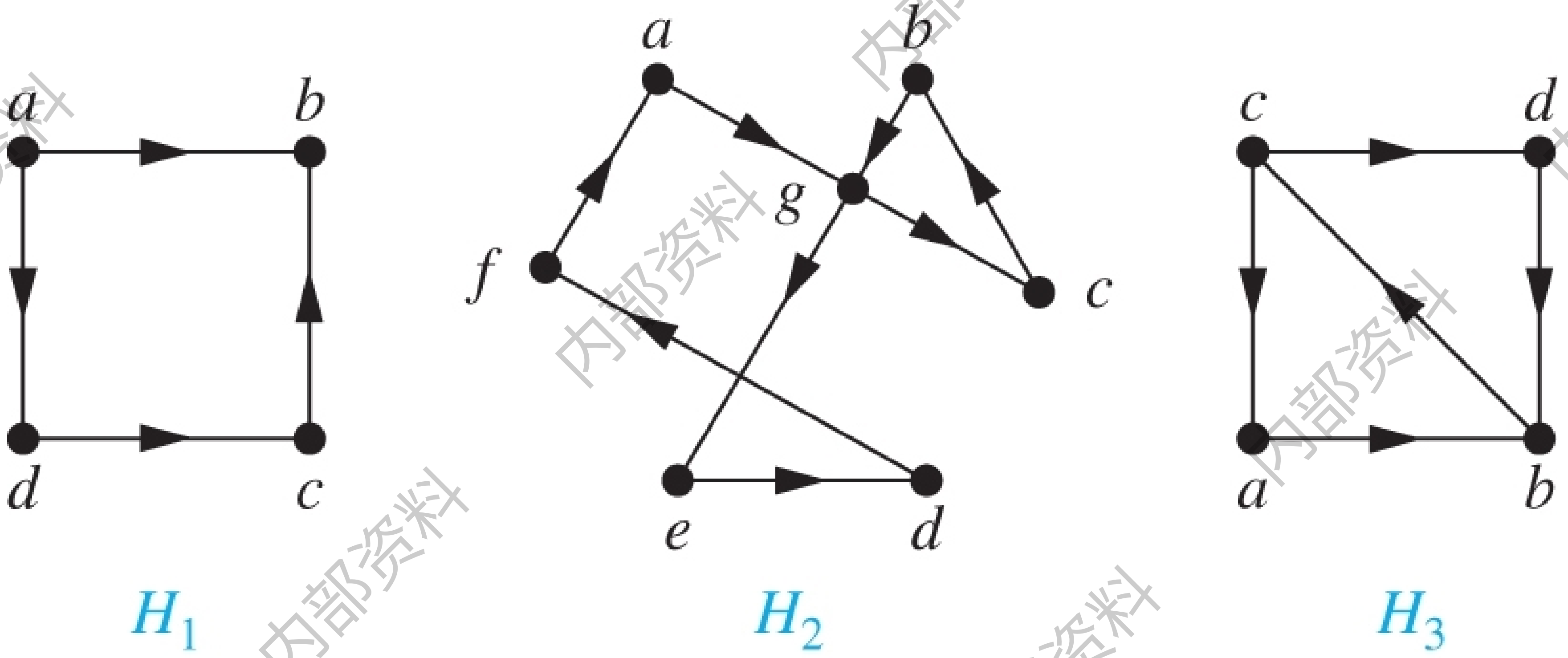


G2

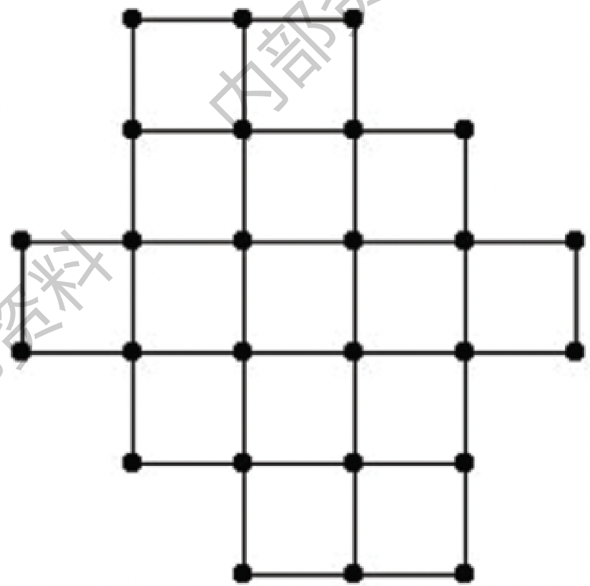


G3

Which of the directed graphs have an Euler circuit (path)?



#1. Determine whether the following graph has an Euler circuit or Euler path.



Theorem

充要条件

- Theorem 1： A connected multigraph with at least two vertices has an Euler circuit if and only if each of its vertices has even degree.
- Theorem 2： A connected multigraph has an Euler path but not an Euler circuit if and only if it has exactly two vertices of odd degree.

Proof of $\Downarrow$

- Assume G has an Euler path T from node a to node b (a and b not necessarily distinct).
- For every node besides a and b, T uses an edge to exit for each edge it uses to enter. Thus, the degree of the node is even.

1. If $a = b$, then a also has even degree.

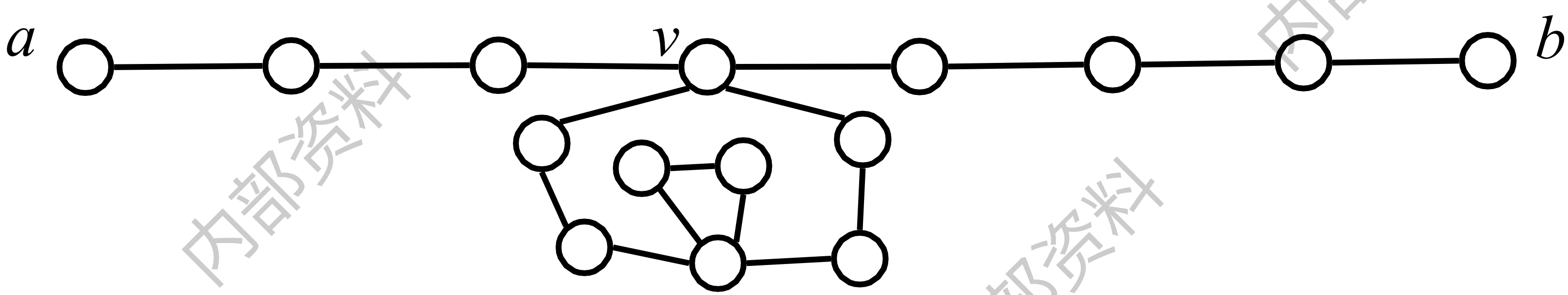
2. If $a \neq b$, then a and b both have odd degree.

Proof of ⇑

- Assume G is connected. If there are no odd-degree nodes, pick any $a = b$.
- If there are two odd-degree nodes, call these nodes a and b.
- Start at a. Take a walk w_1 until you get stuck. You must be at b.

Proof of ⇑

- If no vertex along w_1 has an unused edge, we are done.
- Otherwise, call this vertex v. Walk from v until you get stuck. You must be back at v.



Incorporate this walk from v into w_1.

Euler Circuit Algorithm
欧拉回路算法

- Begin with any arbitrary node.
- Construct a simple path from it till you get back to start.
- Repeat for each remaining subgraph, splicing results back into original cycle.

Algorithm 1

ALGORITHMI Constructing Euler Circuits.

procedure Euler (G:connected multigraph with all vertices of even degree)

circuit := a circuit in G beginning at an arbitrarily chosen vertex with edges successively added to form a path that returns to this vertex

H:= *G* with the edges of this circuit removed

while H has edges

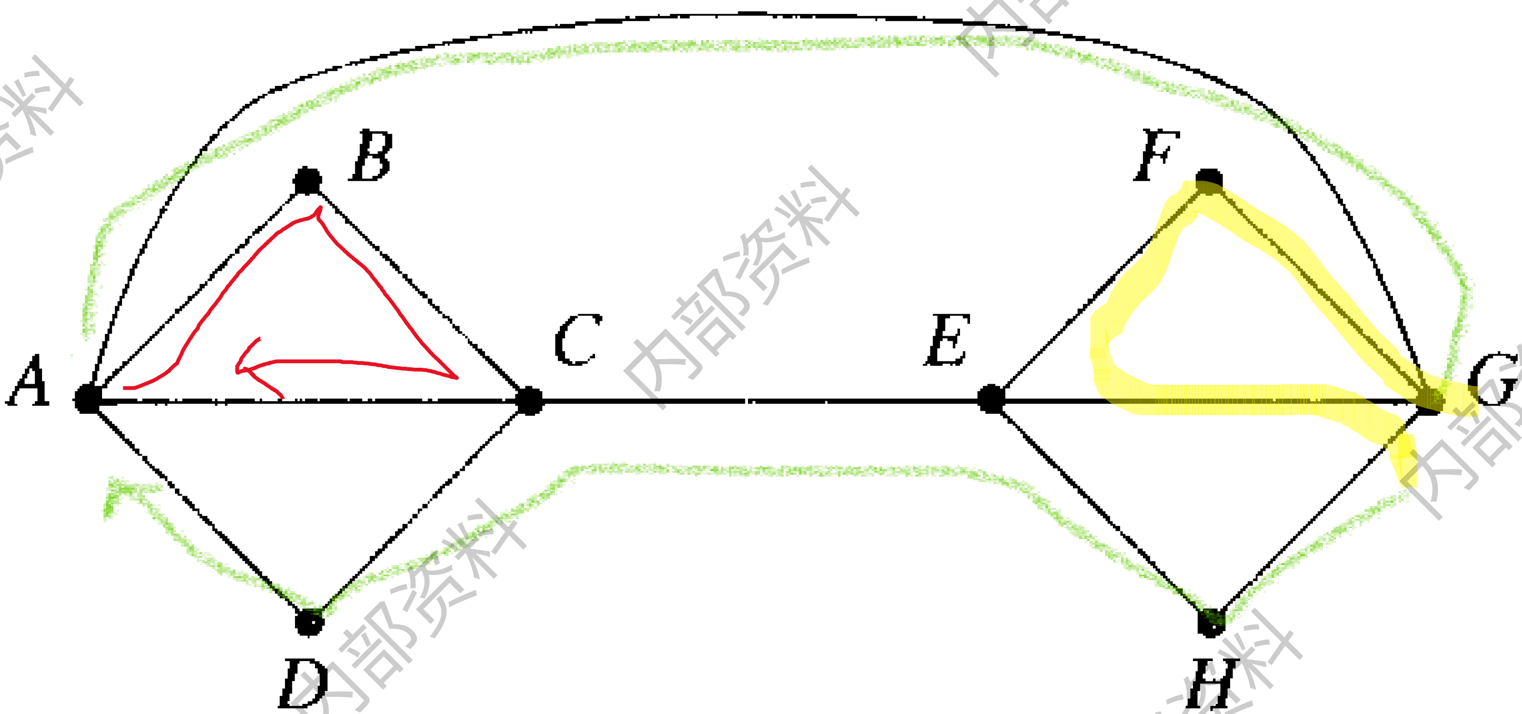
subcircuit := a circuit in H beginning at a vertex in H that also is an endpoint of an edge of circuit

H:= *H* with edges of subcircuit and all isolated vertices Removed

circuit : = *circuit* with subcircuit inserted at the appropriatevertex

return *circuit* (circuit is an Euler circuit)

example



FLEURY's Algorithm（避桥法）

- Let $G = (V, E)$ be a connected graph with each vertex of even degree.
- ***Step 1***: Select an edge e_l that is not a bridge in G. Let its vertices be v_l, v_2. Let π be specified by $V_\pi : v_l, v_2$ and $E_\pi : e_l$. Remove e_l from E and v_l and v_2 from V to create G_1.

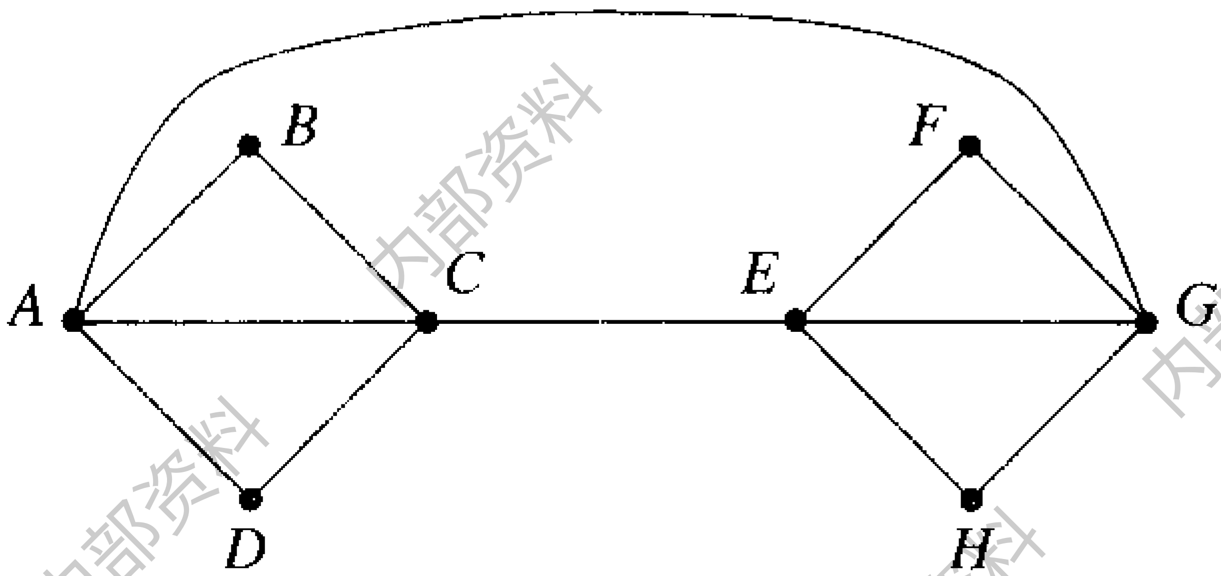
FLEURY's Algorithm

- ***Step* 2:** Suppose that $V_\pi : v_l, v_2, \ldots, v_k$ and $E_\pi : e_l, e_2, \ldots, e_{k-1}$ have been constructed so far, and that all of these edges and isolated vertices have been removed from V and E to form $G_{k\text{-}l}$. Since v_k has even degree and $e_{k\text{-}l}$ ends there, there must be an edge e_k in $G_{k\text{-}l}$ that also has v_k as a vertex. If there is more than one such edge, select one that is not a bridge for G_{k-1}. Denote the vertex of e_k other than v by v_{k+l} and extend V_π and E_π to $V_\pi : v_l, v_2, \ldots, v_k, v_{k+1}$ and $E_\pi : e_l, e_2, \ldots, e_{k-1}, e_k$

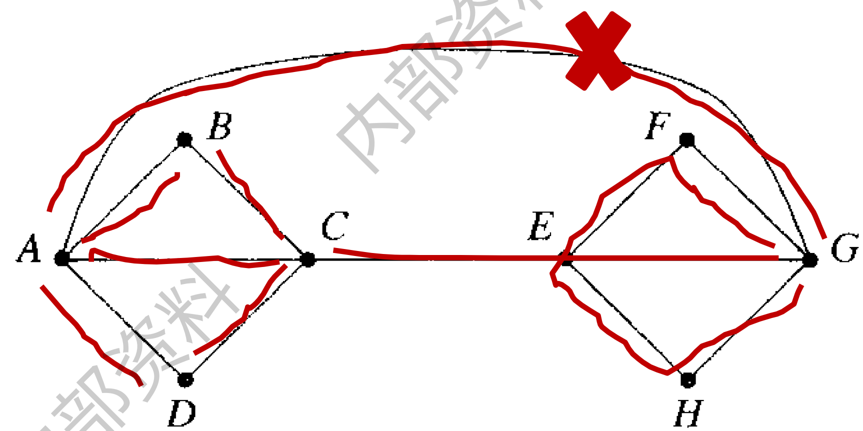
FLEURY's Algorithm

- ***Step 3*:** Repeat step 2 until no edges remains in *E*.

Example

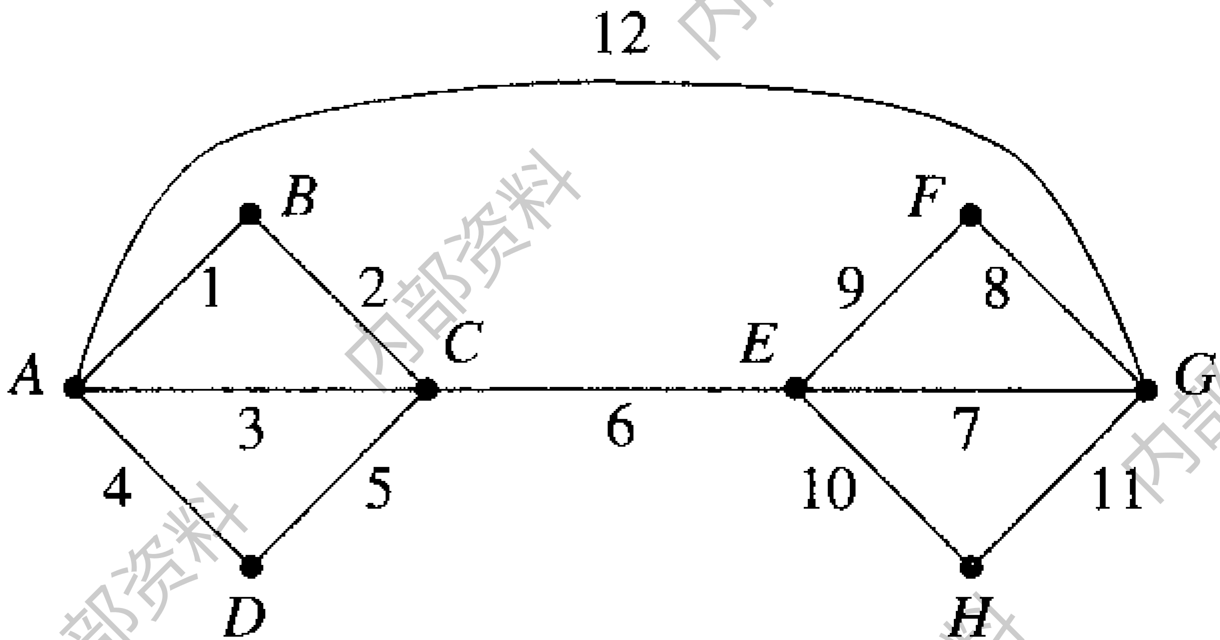


Example



Current Path	Next Edge	Reasoning
$\pi: A$	$\{A, B\}$	No edge from A is a bridge. Choose any one.
$\pi: A, B$	$\{B, C\}$	Only one edge from B remains.
$\pi: A, B, C$	$\{C, A\}$	No edge from C is a bridge. Choose any one.
$\pi: A, B, C, A$	$\{A, D\}$	No edge from A is a bridge. Choose any one.
$\pi: A, B, C, A, D$	$\{D, C\}$	Only one edge from D remains.
$\pi: A, B, C, A, D, C$	$\{C, E\}$	Only one edge from C remains.
$\pi: A, B, C, A, D, C, E$	$\{E, G\}$	No edge from E is a bridge. Choose any one.
$\pi: A, B, C, A, D, C, E, G$	$\{G, F\}$	$\{A, G\}$ is a bridge. Choose $\{G, F\}$ or $\{G, H\}$.
$\pi: A, B, C, A, D, C, E, G, F$	$\{F, E\}$	Only one edge from F remains.
$\pi: A, B, C, A, D, C, E, G, F, E$	$\{E, H\}$	Only one edge from E remains.
$\pi: A, B, C, A, D, C, E, G, F, E, H$	$\{H, G\}$	Only one edge from H remains.
$\pi: A, B, C, A, D, C, E, G, F, E, H, G$	$\{G, A\}$	Only one edge from G remains.
$\pi: A, B, C, A, D, C, E, G, F, E, H, G, A$		

Example



Hamiltonian Paths and Circuits（哈密顿通路与回路）

Hamilton(哈密顿)道路问题

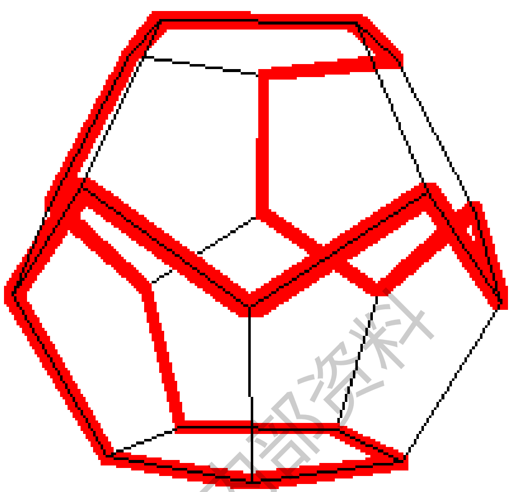
1859年发明的一种游戏。

在一个实心的正十二面体，20个顶点标上世界著名大城市的名字，要求游戏者从某一城市出发，遍历各城市一次，最后回到原地。

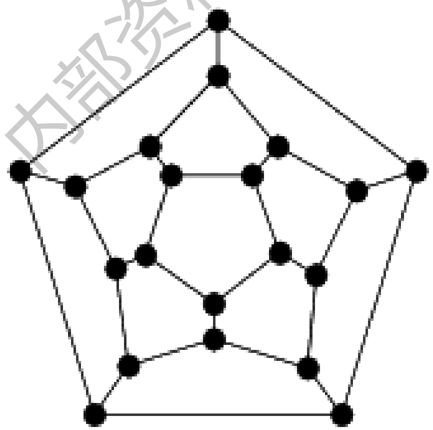
这就是“绕行世界”问题。即找一条经过所有顶点（城市）的基本道路（回路）。

Round-the-World Puzzle

- Can we traverse all the vertices of a dodecahedron, visiting each once?`



Dodecahedron puzzle

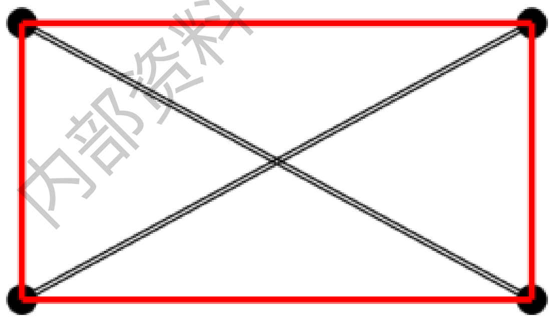


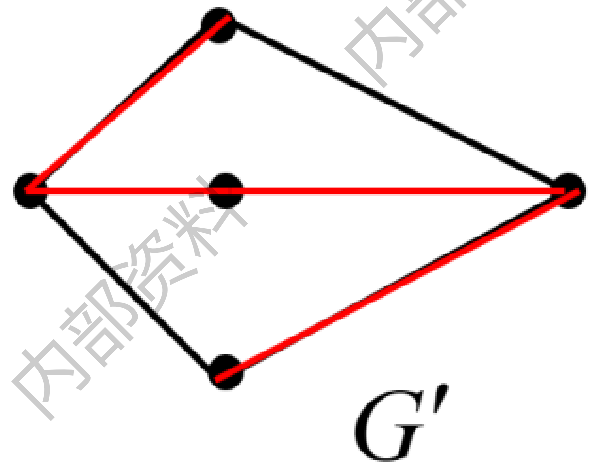
Equivalent graph

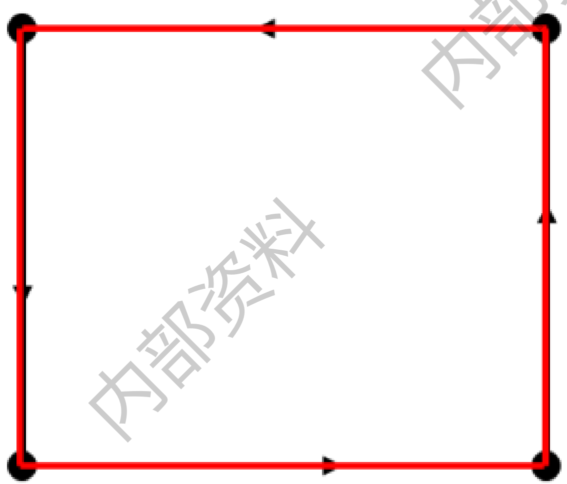
Pegboard version

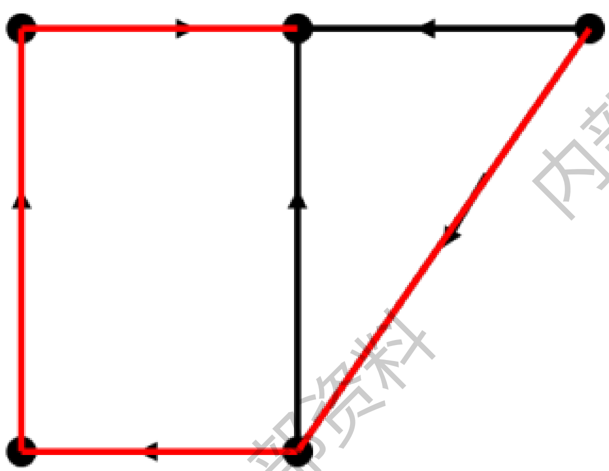
Hamiltonian Graph （哈密顿图）

- A graph has a *Hamiltonian tour* if there is a tour that visits every vertex exactly once (and returns to its starting point).
- A graph with a Hamiltonian tour is called a *Hamiltonian graph*.
- A *Hamiltonian path* is a path that contains each vertex exactly once.
- A ***Hamilton circuit*** is a circuit that traverses each vertex in *G* exactly once.
- A ***Hamilton path*** is a path that traverses each vertex in *G* exactly once




G′

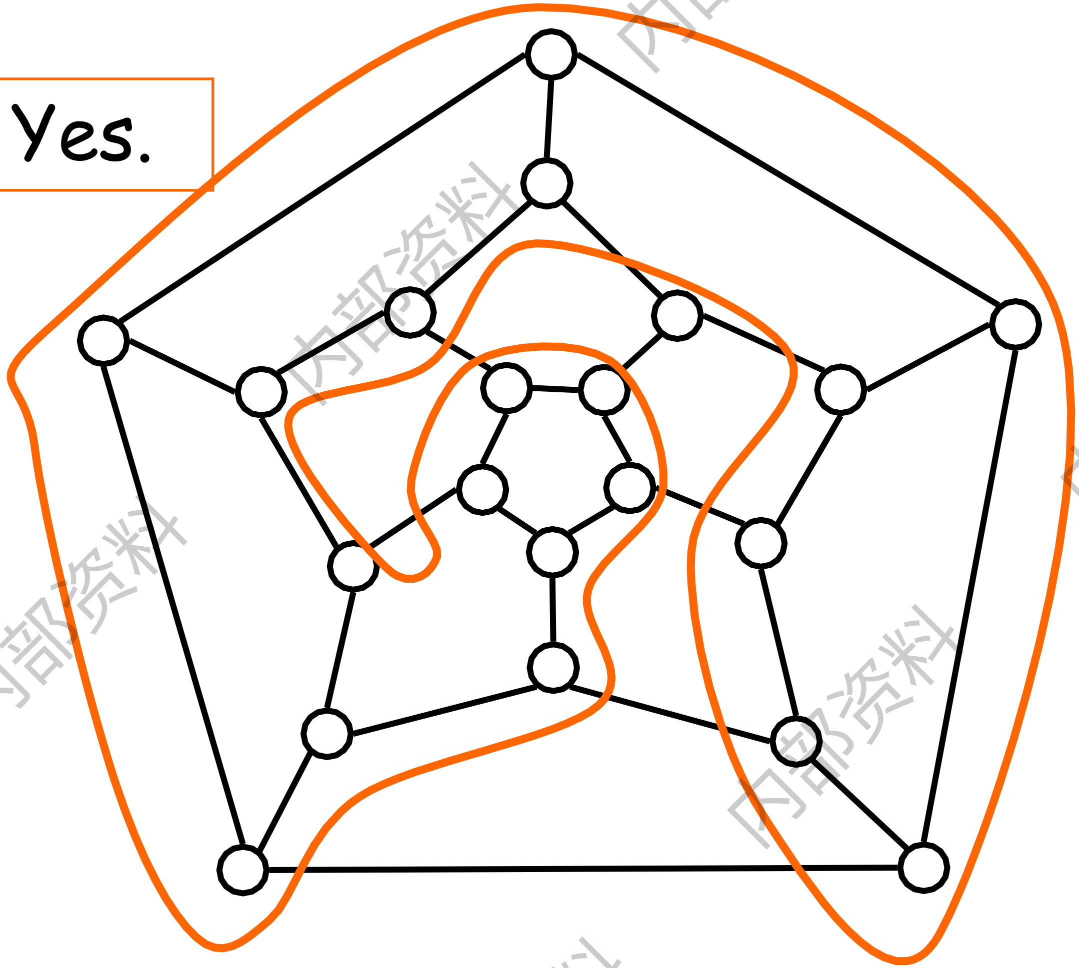




Is it Hamiltonian?

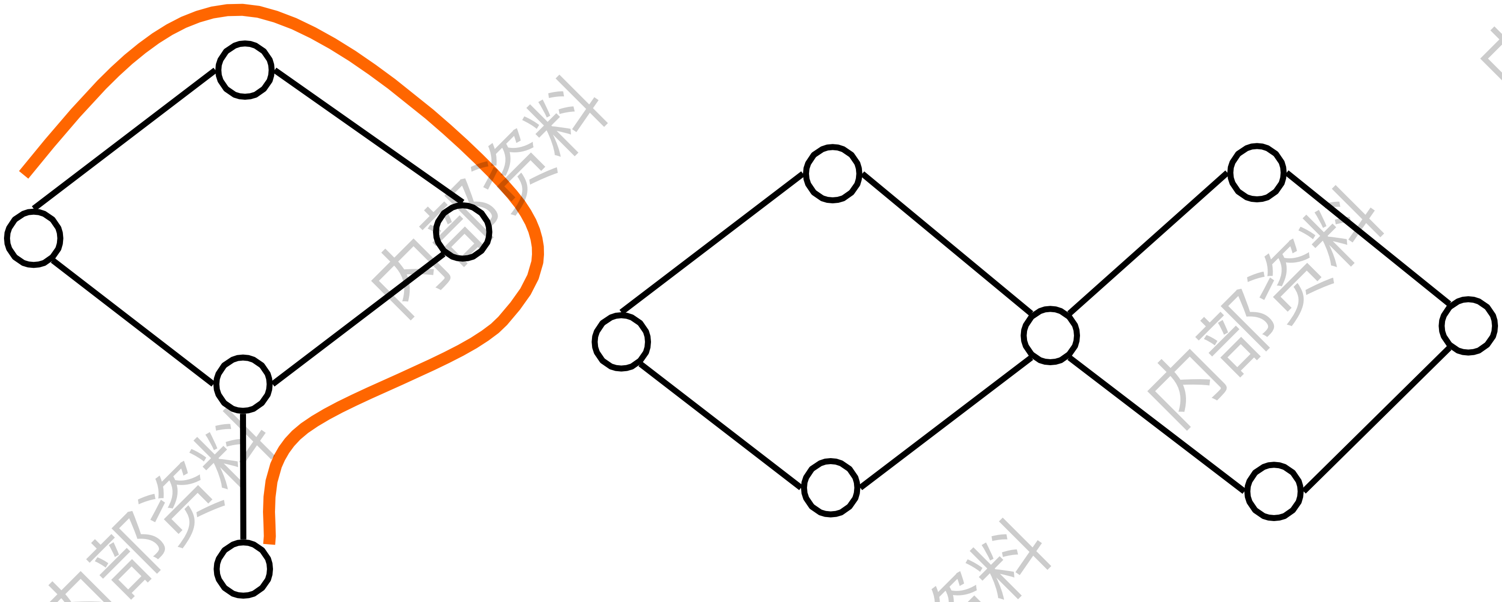
- A graph of the vertices of a dodecahedron.

Yes.

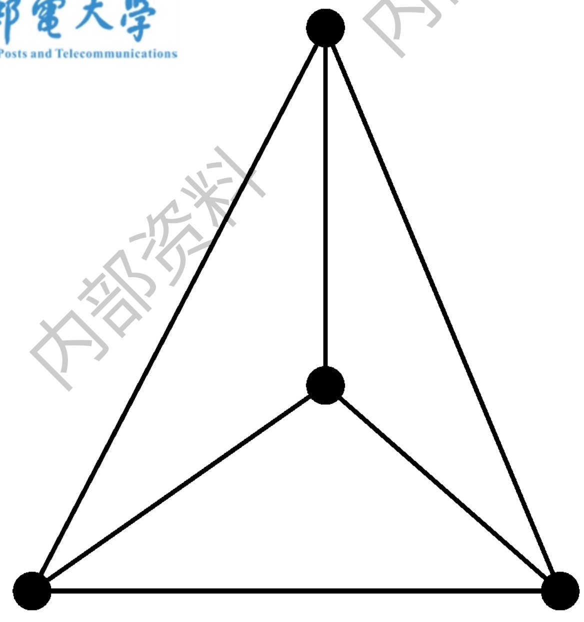


Euler tour
Hamiltonian tour

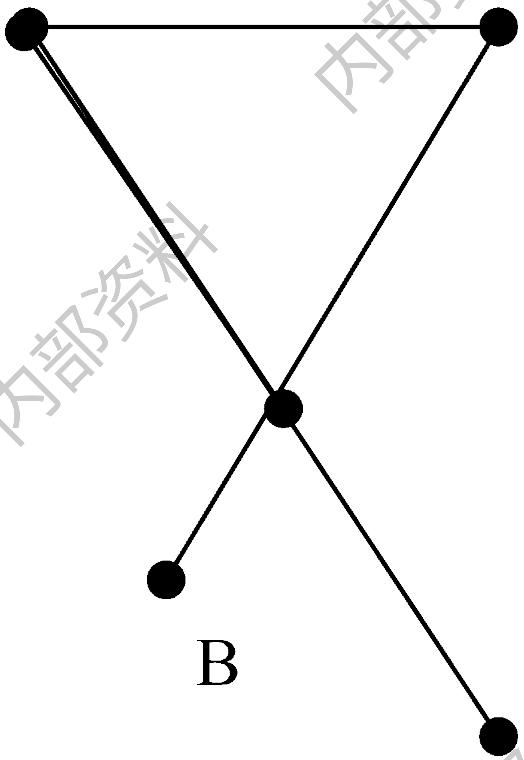
- Left one has a Hamiltonian path, but not a Hamiltonian tour.



- Right one has an Euler tour, but no Hamiltonian tour.

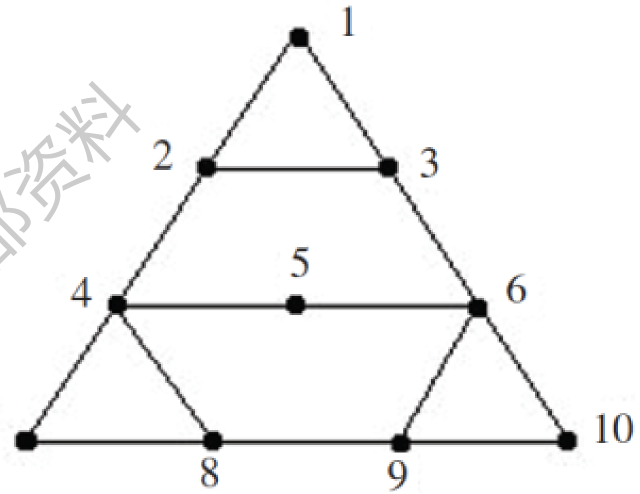


A

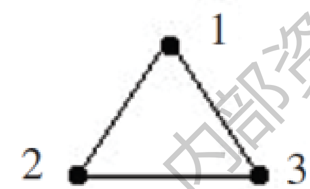


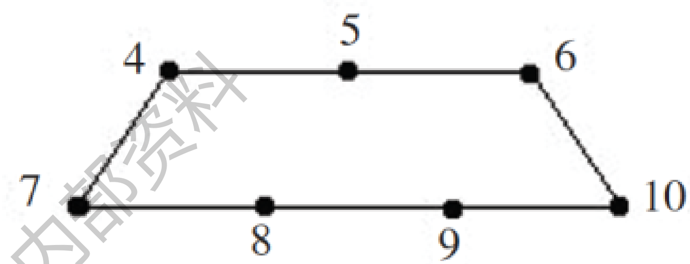
B

Determine whether the following graph has a Hamilton circuit or Hamilton p



Solution:
The graph does not have a Hamilton circuit. Suppose the graph did have a Hamilton circuit. Then the following edges must all be used in such a circuit:
$\{1,2\}, \{1,3\}, \{4,5\}, \{5,6\}, \{4,7\}, \{7,8\}, \{6,10\}, \{9,10\}$7
^9; 10`. If these edges must be used, then the following edges cannot be used: $\{2,4\}, \{3,6\}, \{4,8\}$, and $\{6,9\}$.
If it is impossible to use these four edges, they can be removed from the graph, yielding





But this graph has no Hamilton circuit because it is disconnected. Therefore the original graph has no Hamilton circuit

The graph does have a Hamilton path — for example, 1, 2, 3, 6, 5, 4, 7, 8, 9, 10.

Question?
如何能快速确定哈密顿图

- Given a graph, what is a fast way to determine if it is Hamiltonian? Can we give a characterization of Hamiltonian graphs that is as simple as the one for Eulerian graphs?

No one knows

- There is probably no nice characterization of Hamiltonian graphs the way there was with Eulerian graphs.
  - Deciding if a graph is Hamiltonian is NP–complete.

判断哈密顿图的充要条件？

Partial result

- We now state some partial answers that say if a graph G has "*enough*" edges, a Hamiltonian circuit can be found.
- These are again existence statements; no method for constructing a Hamiltonian circuit is given.

Hamiltonian Path Theorems

Theorem3(Dirac's theorem（狄拉克））:

- If (but not only if) G is connected, simple, has $n \geq 3$ vertices, and $\forall v \deg(v) \geq n/2$, then G has a Hamilton circuit.

Theorem4(Ore's theorem-欧尔定理):

- If G is connected, simple, has $n \geq 3$ nodes, and $\deg(u)+\deg(v) \geq n$ for every pair u,v of non-adjacent nodes, then G has a Hamilton circuit.

哈密尔顿通路

[定理]：设 G＝（ V， E）是 n 个顶点的简单图，如果任何一对不相邻顶点的度之和 ≥ n － 1 ，则 G 中一定有 H 通路。

证明：
1.证连通，如上，任意两点要么直接连通，要么通过一个点连通。

2、用逐步递推构造法证明G中存在H道路：

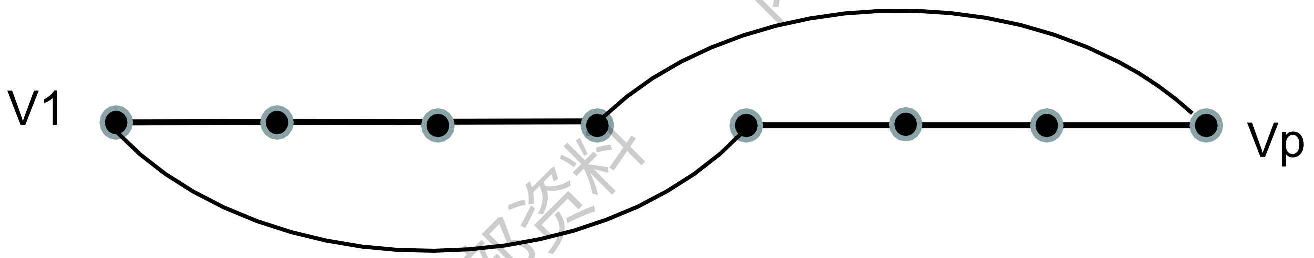
(1) 任取一条边（V_1，V_2），是含 2 个顶点的基本道路。

(2) 如果已有 p 个顶点的基本道路

（V_1，V_2，…，V_p），（ p ≤ n － 1 ）

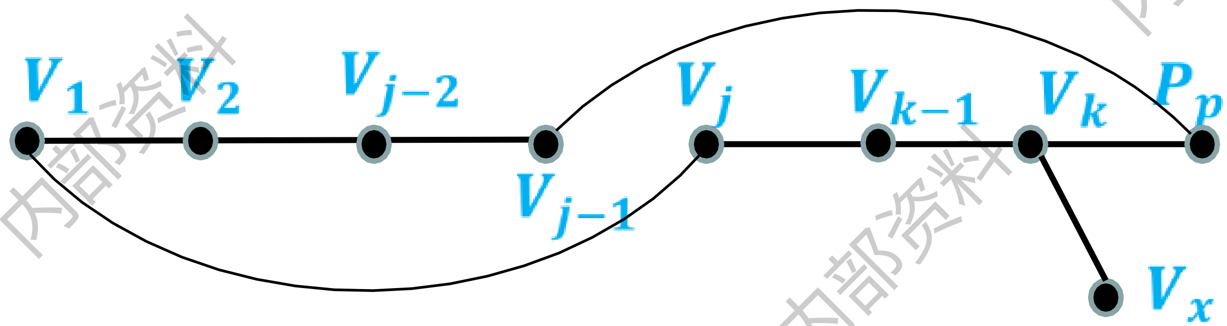
必能构造 p ＋ 1 个顶点的基本道路。

- 如果在V －｛V_1，V_2，…，V_p｝中存在与V_1或V_p相邻的顶点，则基本道路自然可以扩充一个顶点。
- 如果V_1，V_p仅与｛V_1，V_2，…，V_p｝中顶点相邻，则｛V_1，V_2，…，V_p｝必可适当排列，形成回路。


V1
Vp

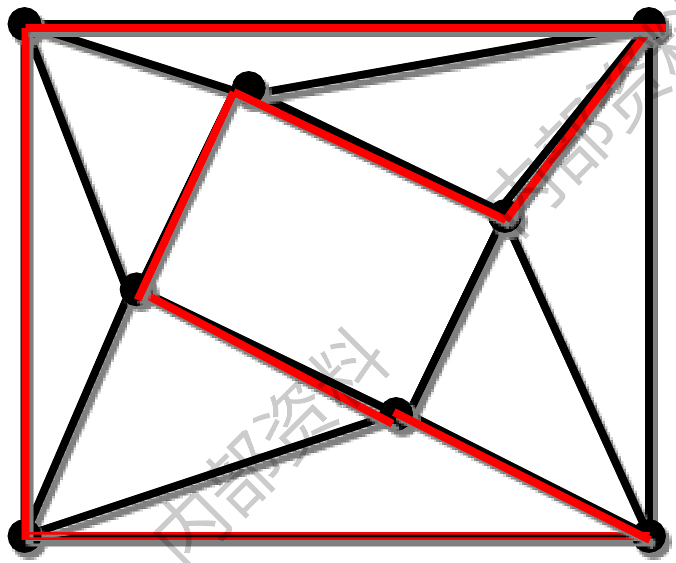
如果V_1与V_p相邻，显然成了环。不然，由于V_1，V_p仅与$\{V_1, V_2, \cdots, V_p\}$中顶点相邻，$V_1$，$V_p$的度$\leqslant p-1$。不妨设$V_1$的度为$k \leqslant p-1$，分别记相邻顶点为$V_{i_1}, V_{i_2}, \cdots, V_{i_k}$，它们前面的顶点（指在基本道路$\{V_1, V_2, \cdots, V_p\}$中的序）为$V_{i_1-1}, V_{i_2-1}, \cdots, V_{i_k-1}$，$V_p$必与$V_{i_1-1}, V_{i_2-1}, \cdots, V_{i_k-1}$中某顶点相邻，否则$V_p$的度$\leqslant p-1-k$，$V_1$与$V_p$的度之和$\leqslant k+p-1-k=p-1<n-1$，与任一对顶点次之和$\geqslant n-1$矛盾。

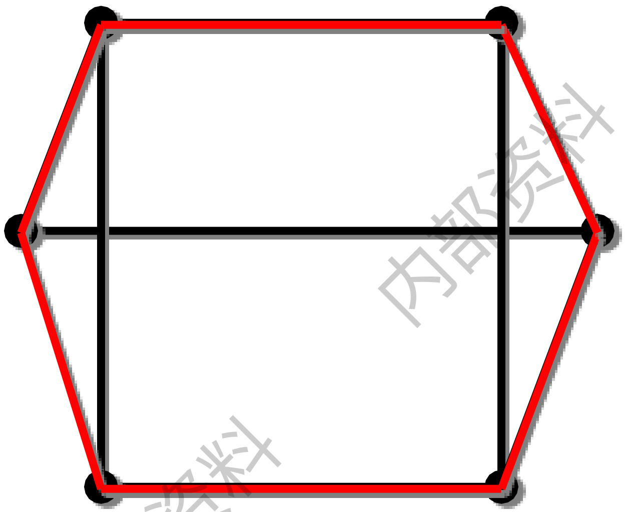
不妨V_p与V_{j-1}相邻，V_1与V_j相邻。将V_1与V_j连起来，V_p与V_{j-1}连起来，并将V_{j-1}到V_j的边去掉，就形成一个环，如下图所示。



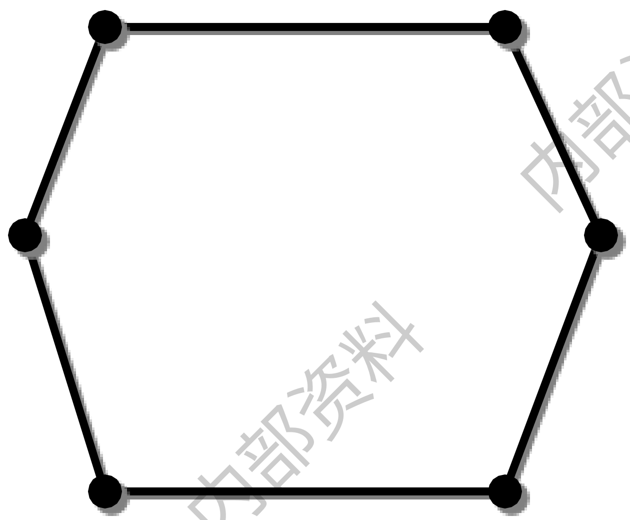
又由G的连通性，总可在V－{V_1，V_2，…，V_p}中找到一个点V_x，与{V_1，V_2，…，V_p}中某一顶相邻，不妨与V_k相邻，$V_k \neq V_1$，$V_k \neq V_p$，连上V_x与V_k的边，去掉V_{k-1}到V_k的边，可以从V_{k-1}为起点，一直走到V_k，再到V_x，这是一条p＋1个顶点的基本道路。

例：下图满足狄拉克定理。
是哈密顿图





狄拉克定理， 给出的是哈密顿图的充分条件，而不是必要条件。（反例）



两个结点度数之和都是 4 不大于 6。

G是哈密顿图

Theorem 2

- Let the number of edges of G be m.
- Then G has a Hamiltonian circuit if
  - $m \geq (n^2-3n+6)/2$.

Proof

- Suppose that u and v are any two vertices of G that are not adjacent.
  - We write deg(u) for the degree of u.
- Let H be the graph produced by eliminating u and v from G along with any edges that have u or v as end points.
- Then H has n-2 vertices and m-deg(u)-deg(v) edges (one fewer edge would have been removed if u and v had been adjacent).

Proof

- The maximum number of edges that *H* could possibly have is ${}_{n-2}C_2$. This happens when there is an edge connecting every distinct pair of vertices. Thus the number of edges of *H* is at most

$${}_{n-2}C_2 = \frac{(n-2)(n-3)}{2} \quad \text{or} \quad \frac{1}{2}(n^2 - 5n + 6).$$

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Proof

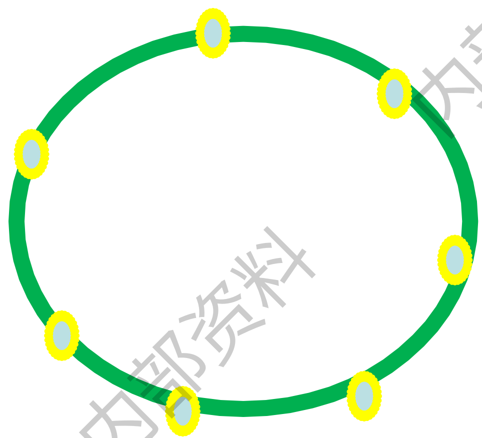
- So
  - m - deg(u) - deg(v) $\leqslant$ $(n^2 - 5n + 6)/2$.
- Therefore
  - deg(u)+deg(v) $\geq m - (n^2 - 5n + 6)/2$.
- By the hypothesis of the theorem,
  - deg(u)+deg(v) $\geq (n^2-3n+6)/2 - (n^2-5n+6)/2 = n$.
- Thus the result follows from Ore‘s Theorem

Note

- The converses of Theorems 3 and 4 given above are not true; that is, the conditions given are sufficient, but not necessary, for the conclusion.

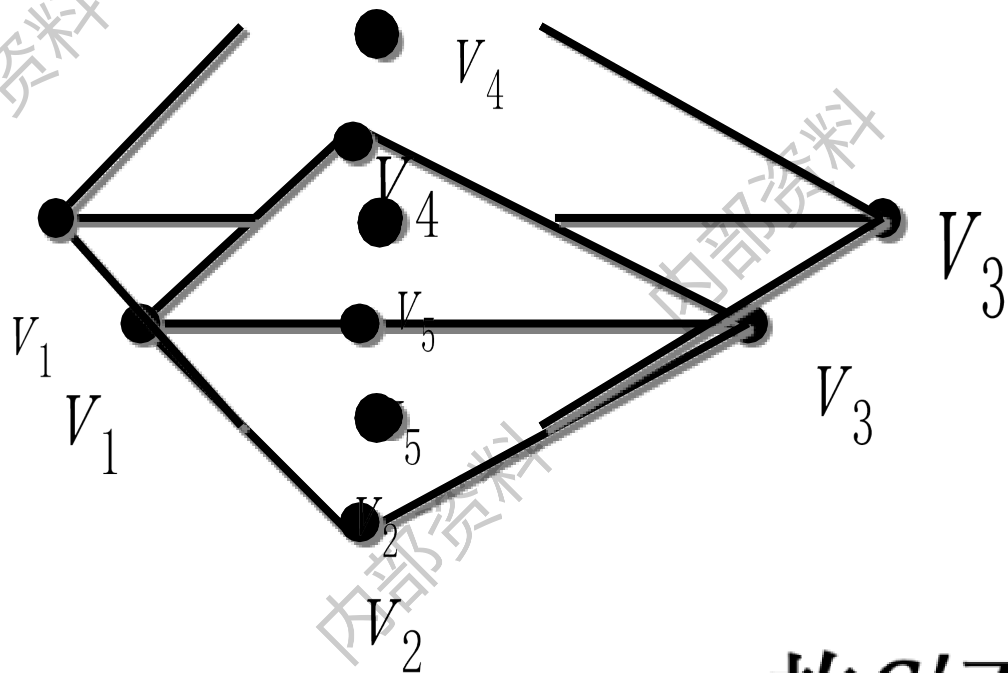
定理

- 哈密顿回路必要条件
- 设无向图$G=(V, E)$，非空子集$V_1 \subseteq V$，则$P(G-V_1) \leq |V_1|$,其中$P(G-V_1)$为图中删除V_1节点所形成的连通分支数。



利用定理 可以判断下图为非哈密顿

$$P(G-V_1) > |V_1|$$



G'

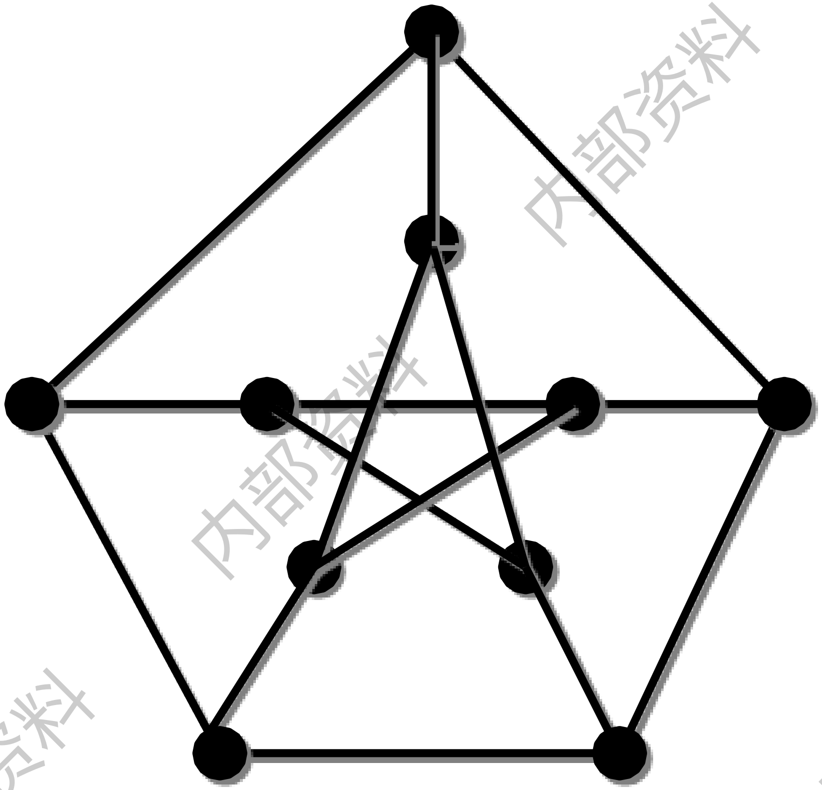
$$V_1=\{v_1, v_3\}$$

$$|P(G-V_1)| = 3$$

$$|V_1| = 2$$

故G'不是哈密顿图

定理 给出的是汉密顿图的必要条件，而不是充分条件。彼得Petersen图满足 $P(G-V_1) \le |V_1|$，但它不是汉密顿图



彼得森Peterssen图

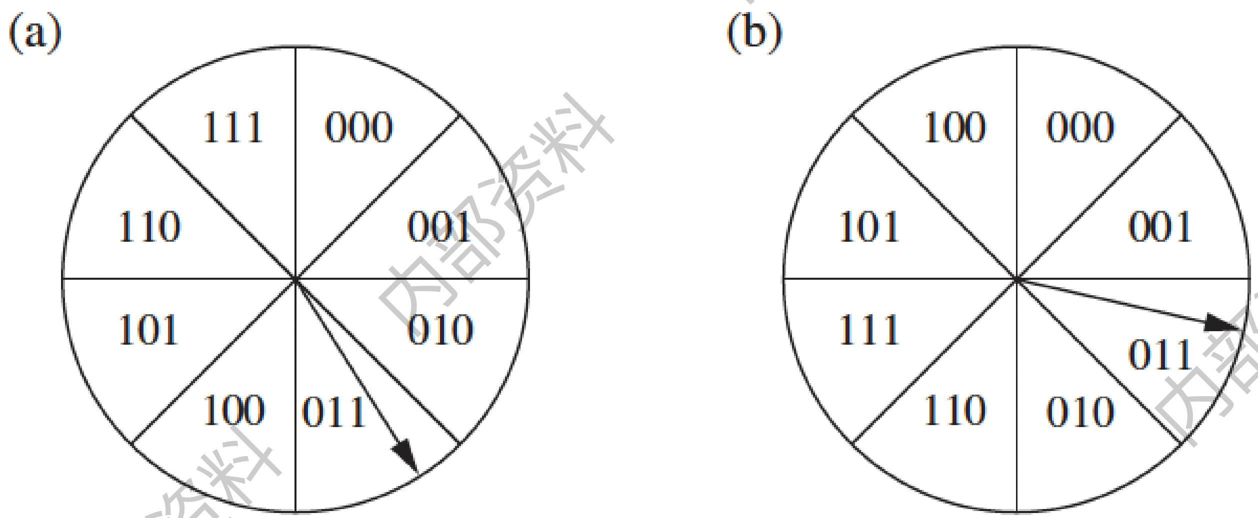
小结

- 哈密顿回路充分条件：
  - ✓ 狄拉克定理（Dirac’s theorem）n个顶点的简单图，任意顶点deg(v) ≥n/2
  - ✓ 欧尔定理（Ore’s theorem）任意不相邻的顶点， deg(u) +deg(v) ≥n
  - ✓ 推论：任意不相邻的顶点， deg(u) +deg(v) ≥n-1，则存在哈密顿通路

- 哈密顿回路必要条件：$P(G-V_1) \leq |V_1|$

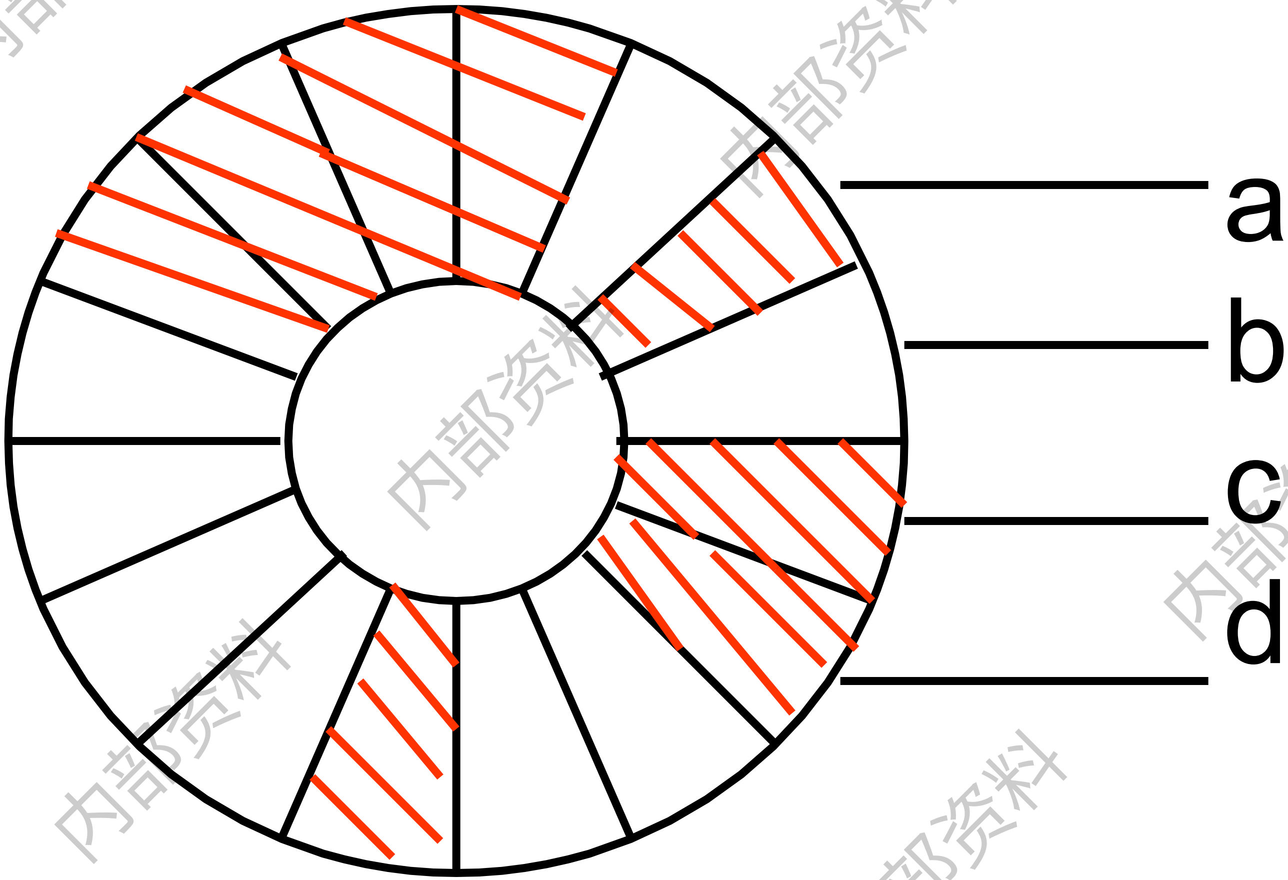
  - ✓ 找哈密顿回路没有好的算法！
  - ✓ 证明不是哈密顿图困难！

Applications of Hamilton Circuits And Euler Circuits



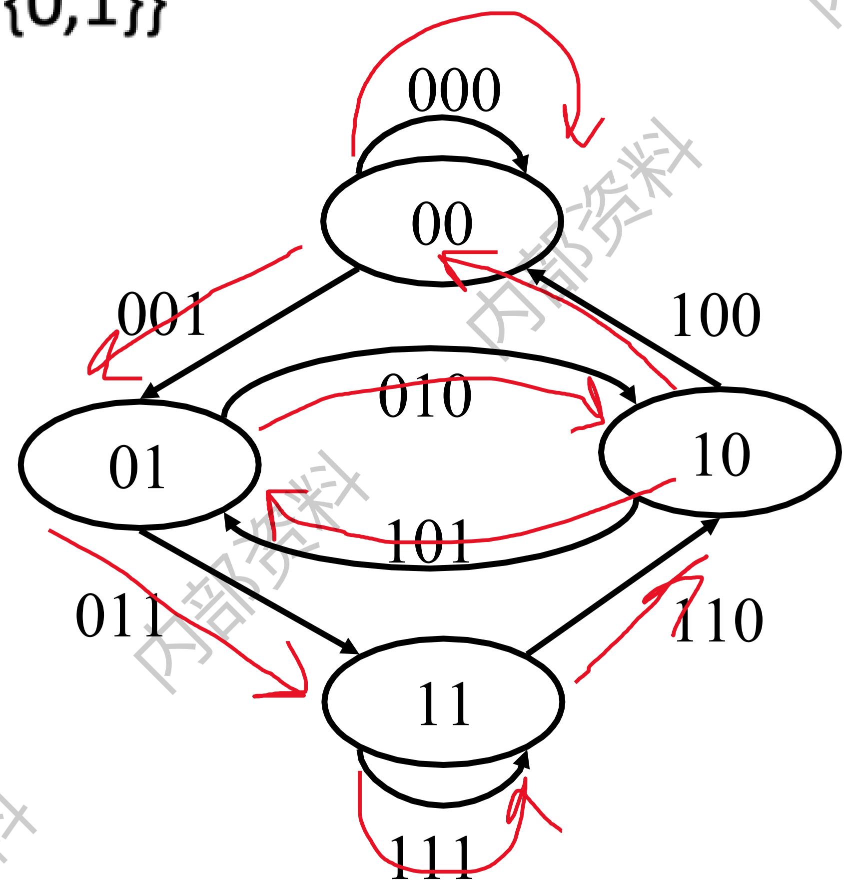
计算机鼓轮的设计

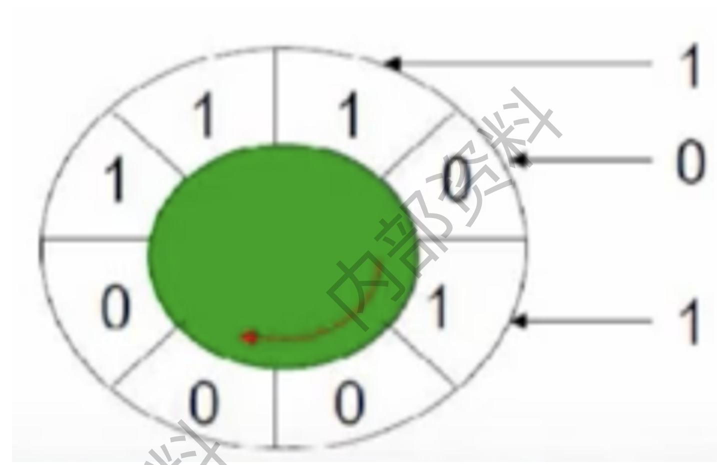
阴影表示导体 1



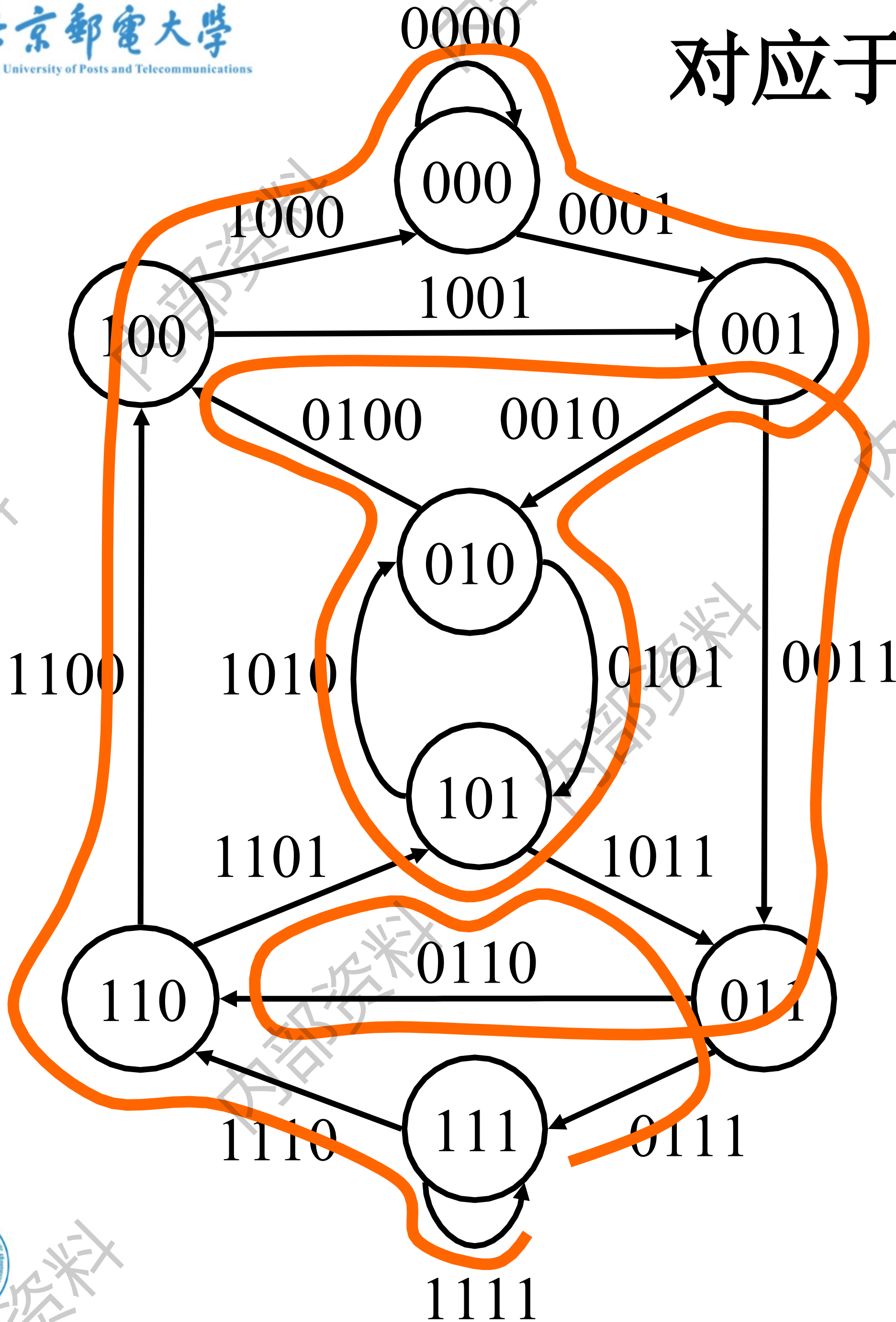
空白表示绝缘体 0

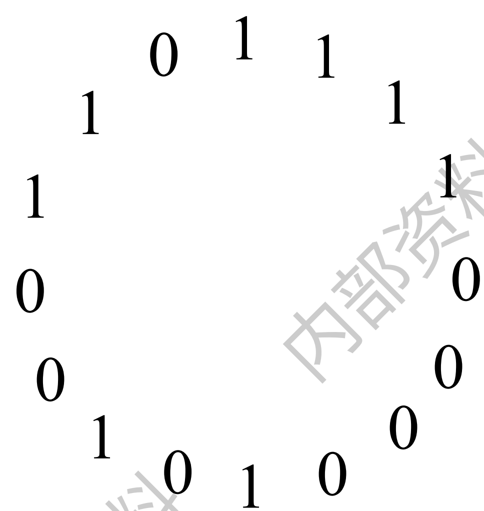
- Gray Codes（格雷码）：
- $G=(V, E)$，V={00,01,10,11}，$E=\{abc=(ab,bc) \mid a,b,c \in\{0,1\}\}$

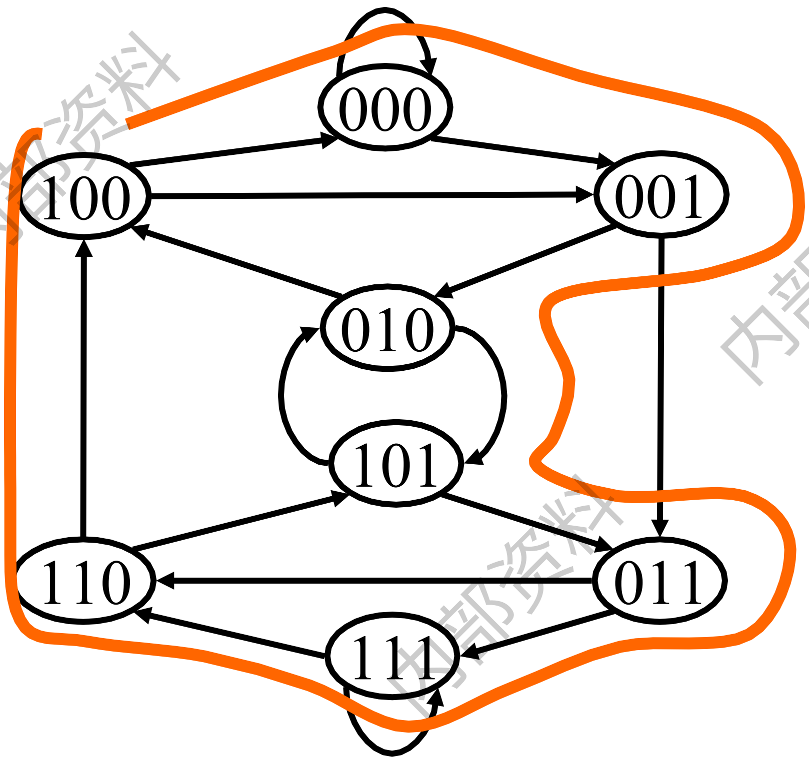




对应于图中的一条欧拉回路.



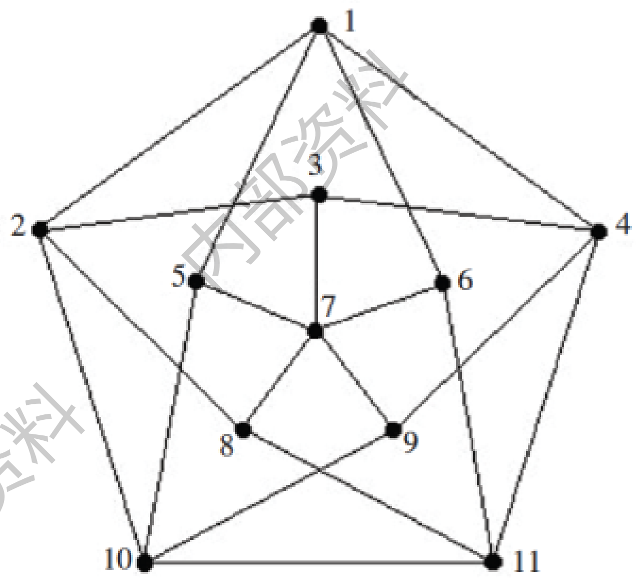




1 0 0
1 0
1 0 1

Example

Find a Hamilton circuit in the graph at the right, called the Grötzsch graph.



作业

- §10.5 8, 10, 16, 26, 34, 48, 58