Previously on Math10220...

Binomial Identities

For any integers n, k with $0 \leq k \leq n$, it holds that

$$\binom{n}{k} = \binom{n}{n-k}$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$\binom{n}{k}\binom{k}{r} = \binom{n}{r}\binom{n-r}{n-k}$$

There are many more binomial identities - e.g. upcoming homework!

Binomial Identities

Consider the number of ways of choosing a committee of size k from a set of m men and n women. Count this in two different ways in order to show that

$$\binom{m+n}{k} = \sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r}$$

From this we can immediately prove that

$$\sum_{r=0}^{n} \binom{n}{r}^2 = \binom{2n}{n}$$

Binomial Distribution

Binomial Coefficients appear naturally in *probability*.

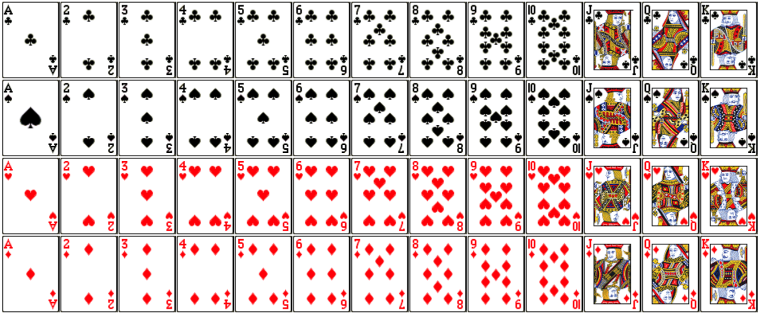
For example, if I flip n coins, what is the probability that I get k heads? There are $\binom{n}{k}$ ways of getting k heads. There are 2^n possible outcomes in total.

Hence the probability is

$$\frac{\binom{n}{k}}{2^n}.$$

For example with 10 coins, the probability of getting five heads is $252/1024 = 0.246$.

Cards



13 different face values
4 of each (four suits)

Poker

In the card game Poker, players are dealt a "hand" of 5 cards from the deck of 52 cards.

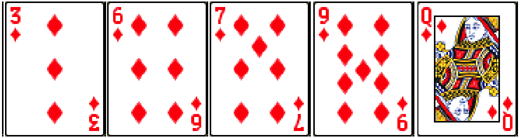
There are then $\binom{52}{5} = 2598960$ potentially different hands.

The winner is the person with the highest ranked hand.

Poker

Straight Flush	Five cards in sequence, all of the same suit.	3♠ 4♠ 5♠ 6♠ 7♠
Four of a Kind	Four cards of the same denomination, one in each suit.	10♦ 10♠ 10♥ 10♣ 4♦
Full House	Three cards of one denomination and two cards of another denomination.	J♥ J♣ 7♥ 7♦ 7♣
Flush	Five cards all of the same suit.	2♥ 6♥ 9♥ Q♥ K♥
Straight	Five cards in sequence of any suit.	3♥ 4♣ 5♦ 6♣ 7♠
Three of a Kind	Three cards of the same denomination and two unmatched cards.	9♣ 9♦ 9♥ 6♣ 2♥
Two Pairs	Two sets of two cards of the same denomination and any fifth card.	4♦ 4♠ J♣ J♥ 9♥
One Pair	Two cards of the same denomination and three unmatched cards.	6♣ 6♥ 3♣ Q♦ 10♦
No Pair	All five cards of different rank and a variety of suits.	A♣ K♦ 4♣ 10♦ 8♥

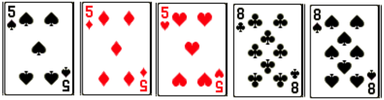
Poker



A *flush* is a hand of five cards, all of the same suit. How many possible flushes are there?

For each suit, there are $\binom{13}{5}$ possible flushes. Hence in total there are $4\binom{13}{5} = 5148$ flushes in total.

Poker



A *full house* is a hand of five cards consisting of three cards of one value, and two cards of the other; e.g. three 5's and two 8's.

How many possible full houses are there?

Note that for example three 5's and two 8's is considered different to three 8's and two 5's.

Suppose our hand is $XXXYY$. There are $13 \cdot 12$ choices for (X, Y); choose two from 13 in an *ordered* way. Then there are $\binom{4}{3} = 4$ ways to choose the three X's, and $\binom{4}{2} = 6$ ways to choose the two Y's.

Hence in total there are $13 \cdot 12 \cdot 6 \cdot 4 = 3744$ possible full houses.

Multisets

If we ignore suits, poker hands are an example of multisets; we are choosing five objects from a set of 52, where repetition is allowed and order doesn't matter.

Multisets are like sets where repetition is allowed, or strings where order doesn't matter.

We write them using the notation $\{^* \quad ^*\}$; for example, $\{^*5, 5, 5, 8, 8^*\}$.

We call the number of times an element is repeated its multiplicity; e.g. above, 5 has multiplicity 3, while 8 has multiplicity 2.

We say the above multiset has size five, or five elements *counting multiplicity*.

Multisets

Multisets are the objects we need to consider when trying to answer the question

Question

How many ways can we choose k elements from the set $\{1, 2, \ldots, n\}$, where repetition is allowed but order doesn't matter?

Selections

The number of ways of selecting k objects from a set of size n is:

	Repetition	No repetition
Ordered	n^k	$n(n-1)\cdots(n-k+1) = \frac{n!}{(n-k)!}$
Unordered		$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots(2)(1)} = \frac{n!}{(n-k)!k!}$

Multisets

Let us write out all the multisets consisting of elements of $\{1, 2, 3, 4\}$ of different sizes.

$$\{^*1^*\}, \{^*2^*\}, \{^*3^*\}, \{^*4^*\}$$

$$\{^*1, 1^*\}, \{^*1, 2^*\},\{^*1, 3^*\}, \{^*1, 4^*\}, \{^*2, 2^*\},$$
$$\{^*2, 3^*\}, \{^*2, 4^*\},\{^*3, 3^*\}, \{^*3, 4^*\}, \{^*4, 4^*\}$$

$$\{^*1, 1, 1^*\}, \{^*1, 1, 2^*\},\{^*1, 1, 3^*\}, \{^*1, 1, 4^*\},$$
$$\{^*1, 2, 2^*\}, \{^*1, 2, 3^*\},\{^*1, 2, 4^*\}, \{^*1, 3, 3^*\},$$
$$\{^*1, 3, 4^*\}, \{^*1, 4, 4^*\},\{^*2, 2, 2^*\}, \{^*2, 2, 3^*\},$$
$$\{^*2, 2, 4^*\}, \{^*2, 3, 3^*\},\{^*2, 3, 4^*\}, \{^*2, 4, 4^*\},$$
$$\{^*3, 3, 3^*\}, \{^*3, 3, 4^*\},\{^*3, 4, 4^*\}, \{^*4, 4, 4^*\}$$

Multisets

$$\{^*2,2,2,2^*\}, \{^*1,3,3,3^*\}, \{^*3,3,4,4^*\},\{^*1,1,2,3^*\}, \{^*3,3,3,4^*\},$$
$$\{^*2,2,2,3^*\}, \{^*1,1,1,1^*\}, \{^*1,1,1,2^*\},\{^*1,4,4,4^*\}, \{^*1,1,3,4^*\},$$
$$\{^*2,3,3,4^*\}, \{^*2,2,3,3^*\}, \{^*1,3,3,4^*\},\{^*1,2,2,3^*\}, \{^*3,3,3,3^*\},$$
$$\{^*2,2,2,4^*\}, \{^*2,2,4,4^*\}, \{^*1,1,4,4^*\},\{^*1,2,4,4^*\}, \{^*1,1,1,3^*\},$$
$$\{^*1,1,3,3^*\}, \{^*1,1,2,4^*\}, \{^*1,2,3,3^*\},\{^*1,1,2,2^*\}, \{^*1,3,4,4^*\},$$
$$\{^*4,4,4,4^*\}, \{^*1,2,2,2^*\}, \{^*2,2,3,4^*\},\{^*1,1,1,4^*\}, \{^*2,3,3,3^*\},$$
$$\{^*2,3,4,4^*\}, \{^*2,4,4,4^*\}, \{^*3,4,4,4^*\},\{^*1,2,3,4^*\}, \{^*1,2,2,4^*\}$$

The pattern begins $1, 4, 10, 20, 35, \ldots$

Multisets

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

Multisets

Pascal's Triangle

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

Multisets

Pascal's Triangle

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

$$\binom{6}{0} \quad \binom{6}{1} \quad \binom{6}{2} \quad \binom{6}{3} \quad \binom{6}{4} \quad \binom{6}{5} \quad \binom{6}{6}$$

$$\binom{7}{0} \quad \binom{7}{1} \quad \binom{7}{2} \quad \binom{7}{3} \quad \binom{7}{4} \quad \binom{7}{5} \quad \binom{7}{6} \quad \binom{7}{7}$$

$$\binom{8}{0} \quad \binom{8}{1} \quad \binom{8}{2} \quad \binom{8}{3} \quad \binom{8}{4} \quad \binom{8}{5} \quad \binom{8}{6} \quad \binom{8}{7} \quad \binom{8}{8}$$

$$\binom{9}{0} \quad \binom{9}{1} \quad \binom{9}{2} \quad \binom{9}{3} \quad \binom{9}{4} \quad \binom{9}{5} \quad \binom{9}{6} \quad \binom{9}{7} \quad \binom{9}{8} \quad \binom{9}{9}$$

$$\binom{10}{0} \quad \binom{10}{1} \quad \binom{10}{2} \quad \binom{10}{3} \quad \binom{10}{4} \quad \binom{10}{5} \quad \binom{10}{6} \quad \binom{10}{7} \quad \binom{10}{8} \quad \binom{10}{9} \quad \binom{10}{10}$$

Multisets

Question

How many ways can we choose k elements of the set $\{1, 2, \ldots, n\}$, where order is not important and repetition is allowed?

It appears as though the answer is as follows.

Guess

$\binom{n+k-1}{k}$

For example, the number of multisets of size 4 from a set of size 4 is equal to the number of subsets of size 4 from a set of size 7.

How can we prove this?

An intermediate step

The number of subsets of $\{1, 2, \ldots, n\}$ of size k is equal to the number of ordered k-tuples whose entries are *strictly increasing*.

The number of multisets containing elements of $\{1, 2, \ldots, n\}$ of size k is equal to the number of ordered k-tuples whose entries are *non-decreasing*.

We want to send a non-decreasing string of length k with elements from $\{1, 2, \ldots, n\}$ to a strictly increasing string, with elements from $\{1, 2, \ldots, n + k - 1\}$.

The trick is to add something to each position.

Counting Multisets

- Take a multiset of k elements from $\{1, 2, \ldots, n\}$;
- Put the elements in non-decreasing order; e.g. $\{^*1, 2, 4, 1^*\} \mapsto (1, 1, 2, 4)$.
- Next, add zero to the first position; add one to the second position etc. e.g. $(1, 1, 2, 4) \mapsto (1, 2, 4, 7)$.
- Finally, send this to a set; this is now a subset $(1, 2, 4, 7) \mapsto \{1, 2, 4, 7\}$.
- Our output is a subset of $\{1, 2, \ldots, n + k - 1\}$ of size k.

Then we just need to check that this is indeed a bijection. Given this, we get our required count of unordered selections with repetition.

Selections

The number of ways of choosing k objects from a set of size n is:

	Repetition	No repetition
Ordered	n^k	$\frac{n!}{(n-k)!}$
Unordered	$\binom{n+k-1}{k}$	$\binom{n}{k}$

For many questions, the difficult part is to recognise which of the four situations we are in.