Geometrical effects in measurements of magnetoresistance

J. Phys. C: Solid State Phys., Vol. 5, 1972. Printed in Great Britain. © 1972

Geometrical effects in measurements of magnetoresistance

H HØJGAARD JENSEN and HENRIK SMITH
Physics Laboratory I, H C Ørsted Institute, University of Copenhagen, 2100 Denmark

MS received 11 July 1972

Abstract. A duality theorem for a two dimensional conductor in a magnetic field is proved by relating the current and field distribution for a specimen bounded by perfect conductors and insulators to that of a dual system defined by the interchange of conducting and insulating boundaries and reversal of the magnetic field. The theorem is used to consider Hall effect and magnetoresistance of rectangular plates. We also discuss recent attempts to explain the high-field linear magnetoresistance of potassium and other alkali metals in terms of the boundary value problem first considered by Lippmann and Kuhrt. This geometrical effect is shown to be an end effect which is unobservable by the customary experimental arrangement of potential leads. Simple expressions for the Lippmann–Kuhrt magnetoresistance valid for all magnetic fields are derived. A potential problem involving discontinuities in the Hall constant is shown to be closely related to the boundary value problem for a homogeneous plate. The associated linear magnetoresistance requires much higher magnetic fields for its observation than the experimental fields for which a linear magnetoresistance is actually observed.

1. Introduction

In this paper we are concerned with geometrical effects on the measurements of magnetoresistance, that is with effects related to the shape of the specimen and the positions of current and voltage contacts. We shall consider such effects in the context of a general duality theorem of galvanomagnetism, which we prove and apply to rectangular plates. The theorem relates the two dimensional electric current and field distribution of a specimen placed in a magnetic field and bounded by perfect conductors and insulators to the current and field distribution of a 'dual' system, in which the surrounding conductors and insulators are interchanged and the magnetic field reversed. The theorem is analogous to known duality theorems in the theory of networks. The theorem is illustrated by application to the measurement of the Hall effect, for which the ordinary and the dual arrangement are exhibited.

As another example of the use of the theorem we relate the resistance of a rectangular plate to that of the dual system in which length and width are interchanged.

We also wish to clarify the predictions of the Lippmann–Kuhrt theory (Lippmann and Kuhrt 1958) of the transverse magnetoresistance of rectangular plates, since this theory has been considered recently (Babiskin and Siebenmann 1969, Taub *et al* 1971) as a possible explanation of the puzzling high-field linear magnetoresistance† of potassium

† See Taub *et al* (1971) for discussion and references.

and other alkali metals. Babiskin and Siebenmann (1969) reported good agreement with the geometrical theory of Lippmann and Kuhrt (to be referred to as LK). The linear term in the magnetoresistance was consequently ignored as a macroscopic boundary value effect unrelated to the intrinsic scattering mechanisms (which on general grounds are expected to produce a saturating magnetoresistance in high fields, see Lifschitz *et al* 1956, 1957). Taub *et al* (1971) found no general agreement, although in a few cases (H Taub 1971, private communication) the slope of the linear term had the same magnitude as that predicted by the LK theory.

The point we wish to make is that agreement or disagreement with the LK theory is of little significance in these experiments, since this theory does not apply to the usual experimental situation. The reason for this is that the geometrical effect arising from the solution of the LK boundary value problem is an end effect, which escapes the voltage measurement when the voltage contacts are placed at some distance from the ends of the sample. To demonstrate that the geometrical effect is an end effect we first present a simple derivation of the high-field magnetoresistance. In this simple derivation the entire voltage drop occurs at the ends of the sample. Next we use the approach of Lippmann and Kuhrt (1958) to obtain an expression for the variation of the electric field along the side of a rectangular plate, valid for all values of magnetic field (or Hall angle θ) in the situation of experimental interest, when the length of the plate is greater than the width. This expression shows that the extra voltage drop caused by the magnetic field (at fixed current) occurs solely within a distance from the ends of the order of the width. Effectively, then, the LK magnetoresistance is zero for the customary position of voltage probes.

For 'long' samples (length a greater than width b) we also obtain a simple, analytic formula for the LK magnetoresistance, valid for all θ. We show that this formula is reasonably accurate even for a square plate, for which an exact analytic result exists. The duality theorem is used to obtain a formula for the magnetoresistance of short samples ($a < b$) as well.

Finally we demonstrate the close relation between the boundary value problem of a homogeneous plate and that of the inhomogeneous plate considered by Bate *et al* (1961). We show that their step-model discontinuity in the resistivity and Hall constant does give rise to a high-field linear magnetoresistance as hinted by Babiskin and Siebenmann (1971), but that the magnetic field required to enter the linear regime is unrealistically high. The discontinuity problem, which was solved for low fields by Bate *et al* (1961), may be solved very simply for all values of magnetic field as an analogous boundary value problem for a homogeneous plate.

2. A duality theorem in galvanomagnetism

In this section we show the existence of a duality theorem for two dimensional electric current and field distributions.

We consider a homogeneous, plane specimen of a conducting material placed in a static, homogeneous magnetic field $\boldsymbol{B}$ which is normal to the plane (see figure 1). The specimen is bounded partly by terminals C_i, assumed to be perfect conductors, partly by nonconducting boundaries N_α. Arrangements of this type are used for measuring magnetoconductivity and Hall effect.

When the terminals are connected to external sources of DC current or potential a two dimensional electric field $\boldsymbol{E}(\boldsymbol{r})$ and a two dimensional current density $\boldsymbol{J}(\boldsymbol{r})$ are established in the plane of the specimen. They obey the Maxwell equations (s and n denote tangential and normal components, respectively, see figure 1):

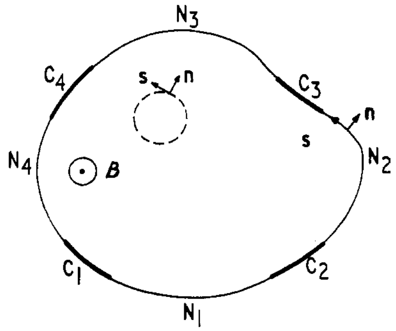

Figure 1. The figure shows a specimen in a magnetic field ***B*** surrounded by conducting (C_i) and nonconducting (N_α) boundaries. The normal (**n**) and tangential (**s**) vectors are indicated.

$$\oint E_s \, ds = 0 \tag{1}$$

$$\oint J_n ds = 0 \tag{2}$$

for any closed curve in the planar specimen.† Furthermore they are subject to the boundary conditions

$$E_s = 0 \quad \text{on} \quad \mathrm{C} \tag{3}$$

$$J_n = 0 \quad \text{on} \quad \mathrm{N}. \tag{4}$$

Finally they are connected by Ohm's law

$$\boldsymbol{E} = \boldsymbol{\rho}_2 \boldsymbol{J} \tag{5}$$

where $\boldsymbol{\rho}_2$ is the (2×2) resistivity tensor of the material in question. In an x, y coordinate system (z axis normal to the plane):

$$\boldsymbol{\rho}_2 = \begin{Bmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{Bmatrix}. \tag{6}$$

The resistivity depends on the magnetic field strength B. We shall disregard the magnetic field due to the current in the specimen. Thus $\boldsymbol{\rho}_2$ is independent of position.

If a solution to equations 1–5 has been found, the potential differences ΔV_α across the nonconducting boundaries N_α and the currents I_i to the terminals C_i are determined by

$$\Delta V_\alpha = \int_{N_\alpha} E_s ds \tag{7}$$

$$I_i = d \int_{C_i} J_n ds \tag{8}$$

† We assume that all conductors are along the boundary: otherwise (2) need not be true for all curves. Arrangements like the Corbino disc are thus not included in our treatment. It is further assumed that the specimen is singly connected: otherwise (2) might fail in the dual problem.

where d is the thickness of the specimen in the z direction. By the assumption (5) of Ohm's law the potential differences are linearly related to the currents:

$$\Delta V_\alpha = \sum_i R_{\alpha i} I_i. \tag{9}$$

Since $\Sigma_i I_i = 0$ the resistance matrix $R_{\alpha i}$ is to some extent arbitrary. We proceed under the assumption that a definite choice, which also respects the condition $\Sigma_\alpha \Delta V_\alpha = 0$, has been made.

We now choose an orientation in the plane of our specimen so that the angle from $\mathbf{n}$ to $\mathbf{s}$ is $+\frac{1}{2}\pi$. Let $\hat{A}$ be a vector obtained by rotating any vector A in the plane through $+\frac{1}{2}\pi$. Since our fields are two dimensional we can rewrite equations 1–5 in terms of $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{J}}$. Then (5) takes the form

$$\hat{\boldsymbol{J}} = \frac{1}{\|\rho_2\|} \cdot \rho_2^{\mathrm{T}} \hat{\boldsymbol{E}} \tag{10}$$

where $\|\rho_2\|$ is the determinant and ρ_2^{T} the transpose of ρ_2.

According to Onsager's relation

$$\rho_2^{\mathrm{T}}(\boldsymbol{B}) = \rho_2(-\boldsymbol{B}) \tag{11}$$

which implies that $\|\rho_2\|$ is unchanged upon reversal of $\boldsymbol{B}$.

It is now seen from (10) and (11) together with the definition of $\hat{A}$ that equations 1–5 are invariant under a certain duality transformation. We can express this more precisely, as follows.

With each arrangement like figure 1 we associate a dual arrangement defined by (i) reversing the magnetic field $\boldsymbol{B}$ and (ii) making conducting parts of the boundary non-conducting and vice versa. If $(\boldsymbol{E}(\boldsymbol{r}), \boldsymbol{J}(\boldsymbol{r}))$ is a solution to the original problem, a solution to the dual problem is:

$$\boldsymbol{E}^{\mathrm{D}}(r) = p\hat{\boldsymbol{J}}(\boldsymbol{r}) \tag{12}$$

$$\boldsymbol{J}^{\mathrm{D}}(r) = q\hat{\boldsymbol{E}}(\boldsymbol{r}) \tag{13}$$

provided

$$\frac{p}{q} = \|\rho_2\|. \tag{14}$$

As a special application of this theorem we derive a relation between a resistance matrix of the original problem and an admittance matrix of the dual problem. The potential differences and currents of the dual problem are found from (12) and (13) together with (7) and (8). (Note the necessary change of subscripts compared with (7) and (8).)

$$\Delta V_i^{\mathrm{D}} \overset{\mathrm{def}}{=} \int_{\mathrm{C}_i} E_{\mathrm{s}}^{\mathrm{D}} \mathrm{d}s = p \int_{\mathrm{C}_i} J_{\mathrm{n}} \mathrm{d}s = \frac{p}{d} I_i \tag{15}$$

$$I_\alpha^{\mathrm{D}} \overset{\mathrm{def}}{=} d \int_{\mathrm{N}_\alpha} J_{\mathrm{n}}^{\mathrm{D}} \mathrm{d}s = -dq \int_{\mathrm{N}_\alpha} E_{\mathrm{s}} \mathrm{d}s = -dq \Delta V_\alpha. \tag{16}$$

Inserting this in (9) we find for the dual problem, using (14):

$$I_\alpha^{\mathrm{D}} = -\frac{d^2}{\|\rho_2\|} \sum_i R_{\alpha i} \Delta V_i^{\mathrm{D}}. \tag{17}$$

This shows that the dual problem can be described by an admittance matrix:

$$G^{\mathrm{D}}_{\alpha i} = -\frac{d^2}{\|\rho_2\|} R_{\alpha i}. \tag{18}$$

As an example serving to illustrate the duality theorem we finally apply it to the measurement of Hall effect.

Figure 2(O) shows an ordinary arrangement for DC measurement of Hall effect, while figure 2(D) gives the dual arrangement. According to the duality theorem the latter yields in principle the same information as the former.

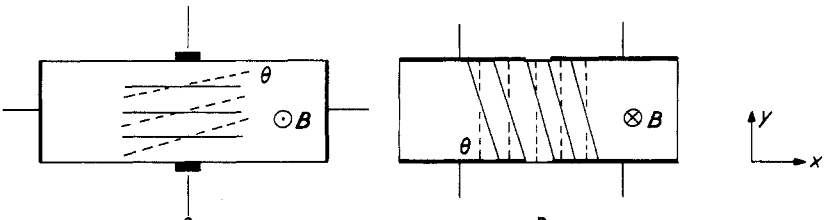

Figure 2. The ordinary (O) and dual (D) arrangement for measuring Hall effect. Full lines, current lines; broken lines, field lines.

If, for example, we have a long, isotropic specimen we can find the Hall angle θ by measuring the potential difference between the two current electrodes and between the two Hall probes under conditions where the currents to the latter are vanishingly small; in fact:

$$\frac{E_y}{E_x} = \tan\theta \qquad \text{if} \qquad J_y = 0. \tag{19}$$

Transforming this to the dual arrangement by means of (12) and (13) we find:

$$\frac{J^{\mathrm{D}}_x}{J^{\mathrm{D}}_y} = -\tan\theta \qquad \text{if} \qquad E^{\mathrm{D}}_x = 0. \tag{20}$$

With the arrangement in figure 2(D) we thus have to measure two *currents*, one from the upper pair of electrodes to the lower pair and one from the left-hand pair to the right-hand pair. This should be done under conditions where the upper pair of electrodes have the same potential (and similarly for the lower pair).

Of course the Hall coefficient and the magnetoresistivity can also be measured in both arrangements.

3. Magnetoresistance of rectangular plates

In this section we consider the magnetoresistance of rectangular plates from several different points of view. First we make use of the duality theorem of the previous section.

3.1. The duality theorem applied

In figure 3 we show a plane two-pole and its dual, both in principle applicable to measurement of magnetoresistance. The two resistances are denoted R and R^{D} respectively, and it should be remembered that according to Onsager, the resistance is an even

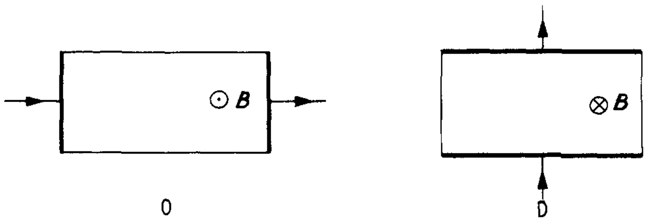

Figure 3. A rectangular plate (O) and its dual (D). The current leads are equipotentials (heavily drawn).

function of B. The reversal of $\boldsymbol{B}$ in figure 3(D) is thus not necessary. From (18) or, simpler, from (14)–(16) we derive the reciprocal relation:

$$R\,R^{\mathrm{D}} = \frac{\|\rho_2\|}{d^2}. \tag{21}$$

Note that this product depends *only* on the material and the thickness, not on the form and other dimensions of the specimen. In particular, if the material is isotropic or has at least a threefold symmetry axis perpendicular to the plane, we have:

$$\rho_2 = \rho(B^2)\begin{Bmatrix} 1 & -\tan\theta \\ \tan\theta & 1 \end{Bmatrix}$$

where θ is the Hall angle. Thus

$$\|\rho_2\| = \frac{\rho^2}{\cos^2\theta}. \tag{22}$$

A further specialization is obtained if $R^{\mathrm{D}} = R$. This is, for example, the case if the specimen is a square with terminals covering two opposite sides or a circle with terminals covering two opposite quadrants. Then the resistance is determined by (21):

$$R = \frac{\rho(B^2)}{d\cos\theta}. \tag{23}$$

For the square this result has been derived in a different way by Lippmann and Kuhrt (1958).

3.2 High field vortex solution

We now proceed to indicate a simple method of solving a two dimensional potential problem for the current and the electric field in the presence of a large magnetic field. The geometry considered is again that of a rectangular plate in a magnetic field transverse to both current and electric field (see figure 4). The current enters the sample at AB and leaves at CD. As a consequence of the two dimensional geometry both the current and the electric field may be derived from a potential satisfying Laplace's equation. The finite magnetoresistance† arises from the need to satisfy the boundary conditions (i) that the current leads AB and CD are equipotentials and (ii) that the current density at the sides AD and BC is parallel to these sides. The angle between the current density and the electric field is everywhere constant, equal to the Hall angle θ.

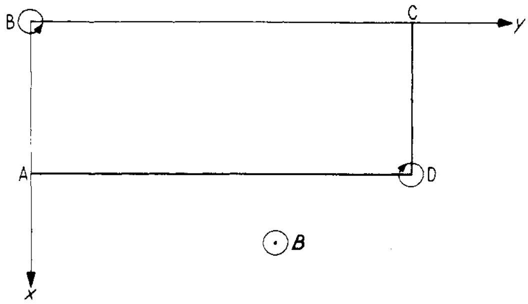

Figure 4. The rectangular plate in a transverse magnetic field. The current enters at AB and leaves at CD. The voltage is measured along AD. Electric field vortices of equal strength but opposite sign are placed at B and D.

To derive the high-field asymptotic value of the resistance, which is the voltage difference between the equipotentials AB and CD divided by the total current, we consider the electric field configuration with the Hall angle $\theta = \frac{1}{2}\pi$. In this case the electric field $\boldsymbol{E}$ is normal to *all* four sides. The potential problem is solved in the immediate vicinity of the two diametrically opposed corners B and D by placing (field) vortices of equal strength Γ but opposite sign at B and D. In polar coordinates the electric field associated with these vortices is

$$\boldsymbol{E} = \pm \frac{\Gamma}{2\pi r} \boldsymbol{e}_\phi \tag{24}$$

with $\boldsymbol{e}_\phi$ being the tangential unit vector and r the distance from B(+) or D(−). In order to find the resistance it is sufficient to know the electric field and current in the immediate vicinity of B and D‡. The magnitude V of the potential drop from the equipotential AB to the equipotential BC is determined by the strength Γ of the vortex according to

$$\begin{aligned} V &= \int_0^{\pi/2} \frac{\Gamma}{2\pi r} r \mathrm{d}\phi \\ &= \tfrac{1}{4}\Gamma. \end{aligned} \tag{25}$$

In high fields the current density $\boldsymbol{J}$ becomes asymptotically

$$\boldsymbol{J} \simeq -\frac{1}{R_{\mathrm{H}} B} \hat{\boldsymbol{E}} \tag{26}$$

where R_{H} is the Hall constant, B the magnetic field, and $\hat{}$ denotes rotation through $\frac{1}{2}\pi$.

† We neglect for convenience any intrinsic dependence of the conductivity σ on the magnetic field although it may be readily kept.

‡ To solve the potential problem everywhere inside ABCD we must place a vortex symmetrically with B on the opposite side of CD in order to compensate for the effect of the vortex at B on the side CD and so on, this giving rise to an infinite two dimensional lattice of field vortices (method of images).

The total current which enters the sample at B (and leaves at D) is therefore

$$
\begin{aligned}
I &= d\int_0^{\pi/2} J_{\mathrm{n}} r \mathrm{d}\phi \\
&= \frac{d}{R_{\mathrm{H}}B}\int_0^{\pi/2} E_{\phi} r \mathrm{d}\phi \\
&= \frac{d}{R_{\mathrm{H}}B} V.
\end{aligned}
\tag{27}
$$

These results can also be expressed in terms of the natural high-field expansion parameter δ, which is the cotangent of the Hall angle. In fact

$$\delta = (\tan\theta)^{-1} = \frac{1}{\sigma R_{\mathrm{H}}B} \tag{28}$$

where σ is the conductivity. The resistance becomes

$$R = \frac{V}{I} = \frac{1}{\sigma d\delta} = \frac{1}{\sigma d}\tan\theta. \tag{29}$$

The result (29) is seen to be the leading term in the high-field expansion (LK40). The resistance in zero field is obviously

$$R_0 = \frac{1}{\sigma d}\frac{a}{b} \tag{30}$$

where a/b is the length to width ratio of the sample. If for the moment we assume that by carrying the expansion (29) further we get the term of order $(\tan\theta)^0$ to be R_0 (which is approximately true for long samples $a > b$, see § 3.3, equation 41), we obtain a magnetoresistance $\Delta R = R - R_0$ given by

$$\frac{\Delta R}{R_0} = \frac{R - R_0}{R_0} \approx \frac{b}{a}\tan\theta \tag{31}$$

in the limit $\tan\theta \gg 1$.

This factor b/a thus plays the rôle of the so-called Kohler slope S. Note that the specific geometry dependence of S only arises because of the division by $R_0 = (1/\sigma d)(a/b)$. The magnetoresistance $\Delta R = (1/\sigma d)\tan\theta$ is independent of a and b. The next section shows that ΔR to a good approximation is independent of both a and b for *all* values of θ, as long as $a > b$.

3.3. *Magnetoresistance of 'long' plates*

In the present subsection we wish to obtain the variation of the electric field along the side of a 'long' plate ($a > b$) in order to demonstrate that the main contribution to the 'geometrical' magnetoresistance comes from the ends of the plate. In addition, we derive an accurate analytic expression for the general result (LK22) of Lippmann and Kuhrt in the case $a > b$. The small parameter in our approximation is the elliptic modulus k given by

$$k = 4\exp(-\pi a/b). \tag{32}$$

Our approximation therefore turns out to be quite good for even very short plates ($a \approx b$) and extremely accurate for the usual sample dimensions.

The method of conformal mapping employed by Lippmann and Kuhrt (1958), leads to the following general expression for the resistance, valid for any length to width ratio a/b and Hall angle θ:

$$R = \frac{1}{\sigma d \cos\theta} \frac{I(1/k, 1)}{I(1, -1)} \tag{33}$$

with

$$I(\xi_2, \xi_1) = \int_{\xi_1}^{\xi_2} \frac{d\xi}{|(\xi - 1)(1/k + \xi)|^{1/2-\theta/\pi} |(\xi + 1)(1/k - \xi)|^{1/2+\theta/\pi}} \tag{34}$$

(see LK22). The dependence of the elliptic modulus k on a/b is in general more complicated than (32), which, however, suffices for our purpose, as we are concerned with 'long' plates only.

In the limit $k \ll 1$ we replace in the integrand of $I(1, -1)$ the factors $(1/k \pm \xi)$ by $1/k$, since $|\xi| < 1$. The remaining integral may then be done exactly with the result

$$I(1, -1) = \frac{k\pi}{\cos\theta}. \tag{35}$$

All the dependence of R on θ is, therefore, contained in $I(1/k, 1)$. It is now convenient to transform from the ξ coordinate to the real-space coordinate y according to

$$\alpha \stackrel{\text{def}}{=} \frac{y}{b} = \frac{1}{\pi} \int_1^{\xi} \frac{d\xi'}{\sqrt{(\xi'^2 - 1)}\sqrt{(1 - k^2\xi'^2)}}. \tag{36}$$

The reduced length $\alpha = y/b$ measures the distance from A in figure 4 along the side AD. The reduced length a/b corresponds to $\xi = 1/k$ (cf LK14).

Equations 33–36 combine to give

$$R\sigma d = \int_0^{a/b} \left(\frac{\xi(\alpha) - 1}{\xi(\alpha) + 1}\right)^{\theta/\pi} \left(\frac{1 + k\xi(\alpha)}{1 - k\xi(\alpha)}\right)^{\theta/\pi} d\alpha. \tag{37}$$

The integrand in (37) is just the component of the electric field along the side AD normalized to unity in zero field ($\theta = 0$). The zero-field resistance (30) is obtained for $\theta = 0$. Note that the integrand is approximately unity when $1 \ll \xi(\alpha) \ll 1/k$, regardless of the value of θ.

From (36) we observe that for $\xi \ll 1/k$ we may approximate $(1 - k^2\xi'^2)$ by 1 and get

$$\xi = \cosh \pi\alpha \qquad \text{for } \xi \ll \frac{1}{k}. \tag{38}$$

Similarly one finds that

$$\xi = \frac{1}{k} \frac{1}{\cosh \pi(a/b - \alpha)} \qquad \text{for } \xi \gg 1. \tag{39}$$

When these expressions are inserted in the integrand $i(\alpha)$ of (37) we obtain

$$i(\alpha) = (\tanh \tfrac{1}{2}\pi\alpha)^{2\theta/\pi} \tag{38a}$$

and

$$i(\alpha) = \left\{\coth \frac{\pi}{2}\left(\frac{a}{b} - \alpha\right)\right\}^{2\theta/\pi} \tag{39a}$$

corresponding to (38) and (39) respectively. This shows that the electric field component parallel to AD and hence the voltage measured along AD is only significantly disturbed from its $\theta = 0$ value over a distance of the order of b from the two ends of the plate. For illustration we have plotted in figure 5 the parallel field component $E_{\parallel}$ as a function of distance along AD.

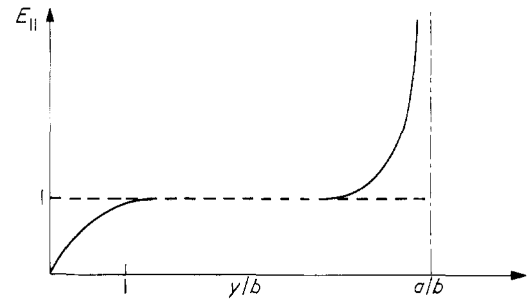

Figure 5. The parallel component of the electric field along the side AD (see figure 4) is plotted (arbitrary units) as a function of distance measured from A. Full curve, $\theta \approx \frac{1}{2}\pi$; broken curve, $\theta = 0$.

We may calculate the contribution to the resistance of this field distortion near the ends by adding and subtracting 1 in (38a) and (39a) in order to be able to extend the integration in (37) to infinity. The result is

$$R\sigma d = \frac{a}{b} + \int_0^{\infty} \left\{(\tanh \tfrac{1}{2}\pi\alpha)^{2\theta/\pi} + (\coth \tfrac{1}{2}\pi\alpha)^{2\theta/\pi} - 2\right\} \mathrm{d}\alpha. \tag{40}$$

By the variable transformation $u = (\tanh \frac{1}{2}\pi\alpha)^2$ the integral in (40) may be evaluated exactly in terms of the digamma function ψ. Using (30) we get

$$\begin{aligned}(R - R_0)\,\sigma d &= -\frac{1}{\pi}\left\{\psi\left(\frac{1}{2} + \frac{\theta}{\pi}\right) + \psi\left(\frac{1}{2} - \frac{\theta}{\pi}\right) - 2\psi\left(\frac{1}{2}\right)\right\} \\ &= \tan\theta - \frac{2}{\pi}\left\{\psi\left(\frac{1}{2} + \frac{\theta}{\pi}\right) - \psi\left(\frac{1}{2}\right)\right\}.\end{aligned} \tag{41}$$

Since the last term of (41) is $-4 \ln 2/\pi$ for $\theta = \frac{1}{2}\pi$, (41) is seen to agree with the high-field expansion (LK47a). Agreement with the low-field expansion (LK46) is apparent from Taylor expansion of (41):

$$(R - R_0)\,\sigma d = \frac{4}{\pi}\sum_{n=1}^{\infty} C_{2n}\left(\frac{2\theta}{\pi}\right)^{2n} \tag{42}$$

with

$$C_{2n} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n+1}}. \tag{43}$$

Since $C_2 = 1{\cdot}0521$ and the higher C_{2n} are even closer to unity, we may replace all the C_{2n} in (42) by 1 and obtain the simple approximate form

$$(R - R_0)\,\sigma\mathrm{d} = \frac{4}{\pi}\frac{\theta^2}{(\pi/2)^2 - \theta^2}. \tag{44}$$

The result (44) differs by at most 5 % from (42), since $C_2 = 1{\cdot}05$, the maximum deviation occurring at small θ. Since our approximation scheme assumed a 'long' plate (small k) it is interesting to see how well (42) and (44) approximate the exact result for a quadratic plate. For a square plate ($a/b = 1$) the resistance obeys (cf equation 23)

$$(R - R_0)\,\sigma\mathrm{d} = \frac{1}{\cos\theta} - 1. \tag{45}$$

Taylor expansion of (45) yields a series like that of (42) with C_{2n} replaced by $B_{2n}\pi^{2n+1}/(2n)!2^{2n+2}$, which in turn is given by the alternating series obtained from (43) by multiplying each term by $(-1)^k$ (the B_{2n} are Euler's numbers). Since $B_2 = 1$ the result (42) for a 'long' plate differs by at most 9 % from the exact result (45) for a quadratic plate, the maximum relative deviation occurring at small θ.

To a very good approximation, then, the increase in resistance due to the magnetic field is independent of length a (and width b) as might be expected for an end effect. We further note that the nonsaturating magnetoresistance coming from the $\tan\theta$ term in (41) arises from the divergence at $\alpha = 0$ of the $(\coth\frac{1}{2}\pi\alpha)^{2\theta/\pi}$ term in the integrand of (40). If we calculate the voltage difference between points along the side AD (see figure 4) which are at a distance of the order of b from the ends we obtain a very small, saturating magnetoresistance. Thus, if the lower limit in (40) is taken to be 1 (corresponding to the two voltage probes situated on AD a distance b from the endpoints A and D), the dimensionless quantity $(R - R_0)\,\sigma d$ is saturating at its maximum value $1{\cdot}2 \times 10^{-3}$ rather than rising as $\tan\theta$ (proportionally to the magnetic field) at high fields.

Finally we use the duality theorem of § 2 to obtain a simple formula for the magnetoresistance of a rectangular plate with an arbitrary length to width ratio a/b. Since the magnetoresistance of a long plate is essentially the same as that of a square plate we may use for $a \geqslant b$ the expression

$$R\sigma d = \frac{a}{b} + \left(\frac{1}{\cos\theta} - 1\right). \tag{46}$$

This expression is exact for $a = b$ (square plate) as we saw using the duality theorem.

By using the duality theorem in the form (21)–(22) we obtain for $a \leqslant b$ the expression

$$R\sigma d = \frac{1}{\cos^2\theta}\Bigg/\left\{\frac{b}{a} + \left(\frac{1}{\cos\theta} - 1\right)\right\}. \tag{47}$$

Equations 46 and 47 combine to give an approximate, simple result for any length to width ratio a/b. One observes that this resistance when viewed as a function of a/b in addition to being exact is continuous with a continuous derivative at $a/b = 1$.

4. A linear high-field magnetoresistance from a discontinuous Hall constant

As a final point we shall show how the preceding results may be used for establishing a suggestion by Babiskin and Siebenmann (1971) that a step-like discontinuity in the Hall constant gives rise to a magnetoresistance rising linearly with magnetic field at high fields. Such a step model has been considered in detail by Bate *et al* (1961) and solved for low fields by expansion in the dimensionless small parameter

$$\gamma = \frac{(R_{H_1} - R_{H_2})B}{\rho_1 + \rho_2} \tag{48}$$

where R_{H_i} is the Hall constant and ρ_i the resistivity of region i ($i = 1, 2$; see figure 6). The step model considered neglects all end effects (by neglecting the boundary conditions at the ends), but takes into account the proper boundary conditions at the discontinuity.

We are interested in the magnetoresistance (averaged over the direction of the magnetic field) as measured between two voltage contacts situated on either side of the discontinuity at a distance of the order of the width or more from the discontinuity (see figure 6). This magnetoresistance corresponds to the (directionally averaged)

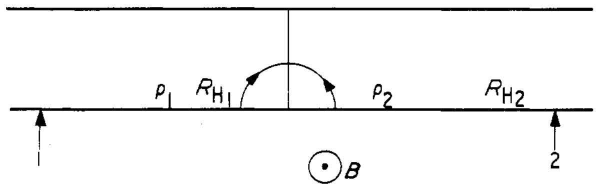

Figure 6. Placement of voltage probes (indicated by arrows) for measuring the magnetoresistance due to a discontinuity in the Hall constant R_H.

voltage $(V_{10} - V_{20})$ in Appendix A of Bate *et al* (1961), where it is obtained as a series expansion in γ (see their equation A22). It is straightforward to show that this voltage (or the equivalent magnetoresistance) may be obtained very simply for all values of γ from equation 41 (or its approximate form (44)) by the identifications $\theta = \tan^{-1}\gamma$ and $1/\sigma = \rho_1 + \rho_2$. This is simply because $(V_{10} - V_{20})$ only depends on the difference of the Hall coefficients and the sum of the resistivities in the two regions. We may therefore equally well obtain $(V_{10} - V_{20})$ by letting, say, the right region of figure 6 have zero resistivity and zero Hall coefficient (and hence constant potential everywhere) while letting the left region have resistivity $\rho_1 + \rho_2$ and Hall coefficient $R_{H_1} - R_{H_2}$. Then the boundary value problem resembles that of the ('long') rectangular plate considered in § 3.3. The voltage $(V_{10} - V_{20})$ is therefore given in terms of the Hall angle $\tan^{-1}\gamma$ of the fictitious medium with Hall constant $(R_{H_1} - R_{H_2})$ and resistivity $(\rho_1 + \rho_2)$. If we average over the two opposite directions of the magnetic field, then we find that the increase in resistance as measured between the voltage contacts 1 and 2 of figure 6 (the contacts being a width or more away from the discontinuity) is simply half the increase in resistance one would measure between the (equipotential) ends of a ('long') rectangular plate with Hall angle $\tan^{-1}\gamma$ and resistivity $(\rho_1 + \rho_2)$.

Specifically we conclude that the magnetoresistance measured between the voltage contacts of figure 6 after averaging over the direction of magnetic field is given by the approximate form

$$(R - R_0)\,d = (\rho_1 + \rho_2)\frac{2}{\pi}\frac{(\tan^{-1}\gamma)^2}{(\frac{1}{2}\pi)^2 - (\tan^{-1}\gamma)^2} \tag{49}$$

for all values of γ, or by the equivalent exact form (41) (the exactness being conditioned on the placement of the potential contacts a distance of the order of the width or more from the discontinuity). The expression that results from (41) and (42) is seen to agree precisely with the small γ expansion (A.22) of Bate *et al* (1961). The limit $|\gamma| \gg 1$ produces the result

$$(R - R_0)\,d \simeq \tfrac{1}{2}|R_{H_1} - R_{H_2}|B. \tag{50}$$

The high-field behaviour (50) can also be derived by the same vortex method of solution, which we used in § 3.2 to obtain the high-field magnetoresistance of a rectangular plate. In the limit $|\gamma| \gg 1$ the discontinuity becomes an equipotential† and the current lines are strongly distorted near the discontinuity. The potential problem is treated in each region separated by the placement of one vortex in the lower right corner of region 1 and of another in the lower left corner of region 2 (figure 6). The strengths of the vortices are related by the requirement that the total current is conserved. The linear magnetoresistance is measured between the contacts 1 and 2 when the vortices are placed on the same side as the contacts. If the magnetic field is reversed the vortices move to the opposite side of the sample and the measured magnetoresistance would have no linear component (only a saturating one). This explains how a factor of $\frac{1}{2}$ in (50) arises from the averaging over the direction of the magnetic field.

For $|\gamma| \gg 1$ the magnetoresistance thus rises linearly with the magnetic field. With the usual free electron expressions $R_{H_i} = 1/n_i e$ and $\rho_i = m/n_i e^2\tau$ the linear regime is, however, not reached until the magnetic field is so high that

$$|\gamma| = \omega_c\tau\frac{|n_1 - n_2|}{n_1 + n_2} \gg 1 \tag{51}$$

(ω_c is the cyclotron frequency eB/m). In a metal this is clearly a much stronger condition than the usual high-field condition $\omega_c\tau \gg 1$. A conservative estimate would be to set the relative change in density $|n_1 - n_2|/(n_1 + n_2)$ equal to 5%, which means that the linear regime is not reached unless the magnetic field satisfies $\omega_c\tau \geqslant 200$. This contradicts the experimental observation of a linear magnetoresistance in fields strong enough to make $\omega_c\tau$ equal to ten or more. This consideration excludes discontinuities in the Hall constant as a plausible mechanism for explaining the linear term in the magnetoresistance of the alkali metals.

Acknowledgments

It is a pleasure to thank J Babiskin, L M Falicov, H Taub, and J W Wilkins for useful correspondence and comments. One of the authors (HS) would like to acknowledge the hospitality of the Research Institute for Theoretical Physics, Helsinki, Finland, where part of the present work was performed.

† That the discontinuity becomes an equipotential in this limit is a consequence of the assumption of uniform current flow far away from the discontinuity (Bate *et al* 1961).

References

Babiskin J and Siebenmann P G 1969 *Phys. kondens. Mater.* **9** 113–21
—— 1971 *Phys. Rev. Lett.* **27** 1361–4
Bate R T, Bell J C and Beer A C 1961 *J. appl. Phys.* **32** 806–14
Lippmann H J and Kuhrt R 1958 *Z. Naturf.* **13** 462–74
Lifschitz I M, Azbel M Ya and Kaganov M I 1956 *Sov. Phys.*–JETP **3** 143–145
—— 1957 *Sov. Phys.–JETP* **4** 41–54
Taub H, Schmidt R L, Maxfield B W and Bowers R 1971 *Phys. Rev.* B **4** 1134–52