SPECTRAL METHOD ON AUTOMORPHIC FORMS

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1. The Möbius transform on the Riemann sphere

Let $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere (A surface proposed by Riemann in order to imagine a single-valued domain for a multi-valued analytical function *One-dimensional complex manifold)*. It can be realized as S^2 in $\mathbb{R}^3$, or as $\mathbb{P}^1(\mathbb{C})$(I do not understand what this is).

For $g \in SL_2(\mathbb{C})$, the Möbious transform(when $ad = bc$, This transform degenerates into a constant, and it is generally agreed that the constant function is not a Möbious transform) is defined by

$$g \mapsto g.z = \frac{az+b}{cz+d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

This is a bilinear transformation. One has

$$z = \infty \mapsto \frac{a}{c}, \qquad z = -\frac{d}{c} \mapsto \infty.$$

Proposition 1.1. *The Möbius transforms form a group of conformal maps of the Riemann sphere.*

Consider the fixed points of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{C},\ \textit{and}\ ad - bc = 1 \right\}$,

$$\frac{az+b}{cz+d} = z.$$

One has

$$cz^2 + (d-a)z - b = 0.$$

The solutions are

$$z_1, z_2 = \frac{a - d \pm \sqrt{(d+a)^2 - 4}}{2c}$$

and we know that $a + d = \mathrm{Tr}(g) \neq \pm 2$ then $z_1 \neq z_2$.

We assume that g has two fixed different points $z_1 \neq z_2$. Let $A = \begin{pmatrix} 1 & -z_1 \\ 1 & -z_2 \end{pmatrix}$ so that $z \mapsto A.z$ maps

$$z_1 \mapsto 0, \qquad z_2 \mapsto \infty$$

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which are the south pole and the north pole in S^2, respectively. Then we have

$$\begin{tikzcd} \hat{\mathbb{C}} \arrow[r, "g"] \arrow[d, "A"] & \hat{\mathbb{C}} \arrow[d, "A"] \\ \hat{\mathbb{C}} \arrow[r, "\tilde{g}=AgA^{-1}"] & \hat{\mathbb{C}} \end{tikzcd} \qquad \begin{tikzcd} z_1, z_2 \arrow[r, "g"] & z_1, z_2 \arrow[d, "A"] \\ 0, \infty \arrow[u, "A^{-1}"] \arrow[r, "\tilde{g}=AgA^{-1}"] & 0, \infty \end{tikzcd} .$$

Here

$$\tilde{g} = AgA^{-1} = \begin{pmatrix} \lambda^{1/2} & \\ & \lambda^{-1/2} \end{pmatrix}$$

with

$$\tilde{g}.0 = 0, \quad \tilde{g}.\infty = \infty, \quad \tilde{g}.z = \lambda z$$

where $\lambda^{1/2}$ and $\lambda^{-1/2}$ are eigenvalues of $\tilde{g}$ (hence of g).

- For g with fixed points z_1 and z_2, let $\mathcal{H}_{z_1,z_2}$ be circles passing through z_1 and z_2, called <u>hyperbolic pencil</u>; let $\mathcal{E}_{z_1,z_2}$ be circles orthogonal to those circles in $\mathcal{H}_{z_1,z_2}$, called <u>elliptic pencil</u>.
- Correspondingly, for $\tilde{g}$, the hyperbolic pencil $\mathcal{H}_{0,\infty}$ are circles passing through the north pole and the south pole, i.e. <u>longitude lines</u>; the elliptic pencil $\mathcal{E}_{0,\infty}$ are just <u>latitude lines</u>

We give the classification of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as follows.

1. elliptic.
   If $tr(g) = a + d = \lambda^{1/2} + \lambda^{-1/2}$ satisfies
   $$-2 < tr(g) < 2.$$
   Then we know λ must be of the form $\lambda = e^{i\theta}$, $0 < \theta < 2\pi$. In this case,
   $$\tilde{g} : z \mapsto \lambda z = e^{i\theta} z$$
   Besides the fixed points at 0 and ∞, $\tilde{g}$ moves points along the latitude lines, or equivalently, g moves points along the <u>elliptic pencil</u>. Such g is called elliptic.
2. hyperbolic.
   If $tr(g) = a + d = \lambda^{1/2} + \lambda^{-1/2}$ satisfies
   $$tr(g) > 2, \quad \text{or} \quad tr(g) < -2,$$
   then we know λ must be real and $\lambda = r \neq 1$. In this case,
   $$\tilde{g} : z \mapsto \lambda z = rz$$
   which moves points along the longitude lines; or equivalently, g moves points along the <u>hyperbolic pencil</u>. such g is called <u>hyperbolic</u>.
3. loxodromic.
   If $tr(g) = a + d = \lambda^{1/2} + \lambda^{-1/2}$ is not real, then λ is of the form
   $$\lambda = re^{i\theta}, \quad r \neq 1, 0 < \theta < 2\pi.$$
   In this case, $\tilde{g}$ and hence g is called loxodromic.

4. parabolic.
   If $tr(g) = a + d = \lambda^{1/2} + \lambda^{-1/2}$ and $g \neq id$, we know that g has only one fixed point, and we can choose A so that
$$\tilde{g} = AgA^{-1} = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}.$$
   in this case, $\tilde{g}$ has the only fixed point ∞. Such g is called <u>parabolic</u>.

Remark 1. We consider the finite order move, i.e. those g with $g^n = id$ for some $n \in \mathbb{N}$. Note that $id = g^n = (AgA^{-1})^n = \tilde{g}^n$. By the classification above,
$$\tilde{g}^n.z = \lambda^n.z = z, \quad \forall z \in \hat{\mathbb{C}}$$
if and only if $\lambda = e^{\alpha 2\pi i}$ for some $\alpha \in \mathbb{Q}$. Thus the finite orders are those elliptic ones with the rotation angle being rational multiple of $2\pi i$.

2. The hyperbolic geometry - The Poincare Upper half plane with rectangular coordinate

One of the realization of the hyperbolic plane is the Poincare upper half plane with the action of $SL_2(\mathbb{R})$. Let
$$\mathfrak{h} = \{z = x + iy, \quad x, y \in \mathbb{R}, y > 0\}$$
be the Poincare Upper half plane. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and $z \in \mathfrak{h}$, we have the map
$$z \mapsto g.z = \frac{az + b}{cz + d}.$$
which makes $\mathfrak{h}$ to be a $SL_2(\mathbb{R})$-space. the action of $SL_2(\mathbb{R})$ on $\mathfrak{h}$ is transitive and the stablizer at $z = i$ is
$$\{g \in SL_2(\mathbb{R}), \quad g.i = i\} = SO(2)$$
so that one has G-space isomorphism
$$SL_2(\mathbb{R})/SO(2) \to \mathfrak{h}, \quad gSO(2) \mapsto g.i.$$
By Iwasawa decomposition, each $g \in SL_2(\mathbb{R})$ can be uniquely expressed as
$$g = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \kappa_\theta$$
with $\kappa_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SO(2)$. Thus we can take the representative elements in $gSO(2)$ as
$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}$$
so that one has the identification of $SL_2(\mathbb{R})$-spaces,
$$SL_2(\mathbb{R})/SO(2) \to \mathfrak{h}, \quad \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \mapsto \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}.i = x + iy = z$$

Remark 2. Note that for $g \in SL_2(\mathbb{R})$, $g.z = (-g).z$. Thus it is natural to consider the action of $PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm 1\}$. If one consider $PSL_2(\mathbb{R})$, the isotropic subgroup at i is $SO(2)/\{\pm id\}$.

2.1. **Hyperbolic arc density.** We recall the definition of the arc density and geodesic lines. Let $\Omega \subset \widehat{C}$ be a region with arc density $\rho(z)$. for $\gamma(z)$ a curve in Ω, the length of γ is defined as

$$d(\gamma) = \int_\gamma \rho(z)|dz|$$

Here

$$z = x + iy, \quad dz = (1, i)\begin{pmatrix} dx \\ dy \end{pmatrix} = dx + idy, \quad |dz| = \sqrt{dx^2 + dy^2}.$$

For $p, q \in \Omega$, the distance is defined to be

$$d_\Omega(p, q) = \inf_{\substack{\gamma \\ \gamma(0)=p, \gamma(1)=q}} d(\gamma).$$

The hyperbolic line density $\rho(z)|dz|$ should be invariant under conformal homeomorphisms, i.e. for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, one should has

$$\rho(g.z)|d(g.z)| = \rho(z)|dz| \tag{2.1}$$

Here

$$d(g.z) = \frac{d}{dz}\left(\frac{az+b}{cz+d}\right) = \frac{1}{(cz+d)^2}dz$$

and thus $\rho(z)$ satisfies

$$\frac{\rho(g.z)}{|cz+d|^2} = \rho(z). \tag{2.2}$$

Since $SL_2(\mathbb{R})$ acts transitively on $\mathfrak{h}$, the value of $\rho(z)$ at $z \in \mathfrak{h}$ is determined by (2.1). So we assume that $\rho(i) = 1$. For

$$z = x + iy = \begin{pmatrix} y^{1/2} & y^{-1/2}x \\ & y^{-1/2} \end{pmatrix}.i,$$

by (2.2),

$$\rho(z) = \rho\left(\begin{pmatrix} y^{1/2} & y^{-1/2}x \\ & y^{-1/2} \end{pmatrix}.i\right) = \frac{\rho(i)}{|y|} = \frac{1}{|y|}.$$

Thus we conclude that the hyperbolic line density is just

$$ds = \rho(z)|dz| = \frac{|dz|}{|y|} = \frac{\sqrt{dx^2 + dy^2}}{|y|}$$

and thus

$$ds^2 = \frac{dx^2 + dy^2}{y^2}. \tag{2.3}$$

2.2. **The geodesics and distance function.** Let $z_1 = is_1$ and $z_2 = is_2$ be two different points on y-axis. Let γ be the line on y-axis with start point z_1 and end points z_2, i.e.

$$\gamma = \gamma(t) = tz_2 + (1-t)z_1, \qquad 0 \le t \le 1.$$

We have

$$\begin{aligned} d(\gamma) &= \int_\gamma \rho(z)|dz| = \int_0^1 \frac{1}{|\mathrm{Im}(tz_2 + (1-t)z_1)|}|z_2 - z_1||dt| \\ &= \int_0^1 \frac{|s_2 - s_1|}{|t(s_2 - s_1) + s_1|} dt = |\log(t(s_2 - s_1) - + s_1)|_0^1 \\ &= |\log \frac{s_2}{s_1}|. \end{aligned}$$

Next, we prove that

$$d(is_1, is_2) = \left|\log \frac{s_2}{s_1}\right|,$$

or equivalently, y-axis is a geodesic.

In fact, assume that

$$\gamma(t) = x(t) + iy(t), \qquad x(0) = x(1) = 0, y(0) = s_1, y(1) = s_2$$

is any differential line with start point is_1 and end point is_2. One has

$$\begin{aligned} d(\gamma) &= \int_0^1 \frac{\sqrt{x'(t)^2 + y'(t)^2}}{|y(t)|} dt \ge \int_0^1 \left|\frac{y'(t)}{y(t)}\right| dt \\ &\ge \left|\int_0^1 \frac{y'(t)}{y(t)} dt\right| \\ &= \left|\log \frac{s_2}{s_1}\right|. \end{aligned}$$

Next, since $g \in SL_2(\mathbb{R})$ acts on $\mathfrak{h}$ as conformal and isometry translations, it maps geodesics into geodesics. So we have the following proposition

Proposition 2.1. *The geodesics on $\mathfrak{h}$ are those lines and semi-circles orthogonal to x-axis. For any points $z, w \in \mathfrak{h}$,*

$$d(z, w) = \log \frac{|z - \overline{w}| + |z - w|}{|z - \overline{w}| - |z - w|}.$$

Proof. We have obtained the distance for points on y-axis. To establish the distance function on general points, we need only to find some $g \in SL_2(\mathbb{R})$ so that

$$z = g.i, \qquad w = g.(is)$$

so that $d(z, w) = d(i, is) = |\log s|$.

Assume that $z = x + iy$ and $w = u + iv$. The idea is the following. On taking $g_z = \begin{pmatrix} y^{1/2} & y^{-1/2}x \\ & y^{-1/2} \end{pmatrix}$ and $g_w = \begin{pmatrix} v^{1/2} & v^{-1/2}u \\ & v^{-1/2} \end{pmatrix}$, one has

$$z = g_z.i, \quad w = g_w.i = g_z.(g_z^{-1} g_w.i).$$

We can use KAK decomposition to write

$$g_z^{-1}g_w = \kappa_\varphi . \begin{pmatrix} s^{\frac{1}{2}} & \\ & s^{-\frac{1}{2}} \end{pmatrix} \kappa_\theta$$

so that

$$\begin{aligned} z &= g_z.i \\ &= g_z\kappa_\varphi.i, \\ w &= g_z\kappa_\varphi.\left(\begin{pmatrix} s^{\frac{1}{2}} & \\ & s^{-\frac{1}{2}} \end{pmatrix} \kappa_\theta.i\right) \\ &= g_z\kappa_\varphi.(is) \end{aligned}$$

and thus

$$d(z,w) = d(i,is) = |\log s|.$$

□

Remark 3. We define

$$u(z,w) := \frac{\cosh(d(z,w))-1}{2} = \frac{|z-w|^2}{4\mathrm{Im}z\mathrm{Im}w} \tag{2.4}$$

which is also a conformal invariance. It is used in the invariant automorphic kernel.

2.3. **The invariant measure and the Laplacian operator.** The invariant measure and the Laplacian-Beltrami operator are obtained via group theory,

$$d\mu(z) := \frac{dxdy}{y^2}, \qquad \Delta := -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Consider $L^2(\mathfrak{h})$ with the inner product

$$\langle f_1, f_2\rangle := \int_{\mathfrak{h}} f_1(z)\overline{f_2(z)}d\mu(z).$$

We are interested in the spectrum of $L^2(\mathfrak{h})$. The Laplacian-Beltrami operator is important for the decomposition of $L^2(\mathfrak{h})$. In fact, $C_c^\infty(\mathfrak{h})$ is a dense subspace in $L^2(\mathfrak{h})$ with respect to Fréchet topology. Roughly speaking, elements in $L^2(\mathfrak{h})$ can be expressed as 'limit' of a sequence of functions in $C_c^\infty(\mathfrak{h})$. The action of Δ on such sequence is also a sequence. So we can extend Δ to be an operator on $L^2(\mathfrak{h})$.

Proposition 2.2. *Δ is defined on $C_c^\infty(\mathfrak{h})$, which is a dense subspace of $L^2(\mathfrak{h})$. It is extended to a positive definite, unbounded, self-adjoint operators on $L^2(\mathfrak{h})$ and satisfies*

$$\langle \Delta f, f\rangle \geq \frac{1}{4}\langle f, f\rangle.$$

Proof. Set $\Delta^e = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Let d be the exterior derivative, which takes 1-forms to 2-forms. We have

$$\begin{aligned} dh &= \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy, \\ d\left(h_1 dx + h_2 dy\right) &= d(h_1)\wedge dx + d(h_2)\wedge dy \\ &= \frac{\partial h_1}{\partial y}dy\wedge dx + \frac{\partial h_2}{\partial x}dx\wedge dy \\ &= \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)dx\wedge dy. \end{aligned}$$

The Green's identity assert that

$$\int_\Omega \left(\frac{\partial h_2}{\partial x} - \frac{\partial h_1}{\partial y}\right)dx\wedge dy = \int_\Omega d\left(h_1 dx + h_2 dy\right) = \int_{\partial\Omega} h_1 dx + h_2 dy$$

Let f and g be two smooth functions defined in n.b.d. of a bounded region Ω, whose boundary is a smooth curve (or union of smooth curve) $\partial\Omega$. Note that

$$\begin{aligned} (\overline{g}\Delta^e f)dx\wedge dy &= \left(\overline{g}\frac{\partial^2 f}{\partial x^2} + \overline{g}\frac{\partial^2 f}{\partial y^2}\right)dx\wedge dy \\ &= \overline{g}d\left(\frac{\partial f}{\partial x}\right)\wedge dy - \overline{g}\left(\frac{\partial f}{\partial y}\right)\wedge dx \\ &= d\left(\overline{g}\frac{\partial f}{\partial x}\right)\wedge dy - d\left(\overline{g}\frac{\partial f}{\partial y}\right)\wedge dx - \left(\frac{\partial f}{\partial x}d(\overline{g})\wedge dy - \frac{\partial f}{\partial y}d(\overline{g})\wedge dx\right). \end{aligned}$$

Thus

$$\begin{aligned} -\langle \Delta f, g\rangle &= \int_{\mathfrak{h}} \overline{g}\Delta^e f dx\wedge dy \\ &= \int_\Omega d\left(\overline{g}\frac{\partial f}{\partial x}\right)\wedge dy - d\left(\overline{g}\frac{\partial f}{\partial y}\right)\wedge dx - \left(\int_\Omega \frac{\partial f}{\partial x}\frac{\partial \overline{g}}{\partial x}dx\wedge dy + \frac{\partial f}{\partial y}\frac{\partial \overline{g}}{\partial y}dx\wedge dy\right) \\ &= \int_{\partial\Omega} \left(\overline{g}\frac{\partial f}{\partial x}\right)dy - \left(\overline{g}\frac{\partial f}{\partial y}\right)dx - \left(\int_\Omega \frac{\partial f}{\partial x}\frac{\partial \overline{g}}{\partial x}dx\wedge dy + \frac{\partial f}{\partial y}\frac{\partial \overline{g}}{\partial y}dx\wedge dy\right) \end{aligned}$$

Note that $\partial\Omega$ is the boundary enclosed the support of f and g and thus the integrand vanishes, which gives

$$-\langle \Delta f, g\rangle = -\int_\Omega \left(\frac{\partial f}{\partial x}\frac{\partial \overline{g}}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial \overline{g}}{\partial y}\right)dx\wedge dy = -\int_\Omega \nabla f\cdot\nabla\overline{g}dx\wedge dy.$$

Thus one should has $\langle \Delta f, g\rangle = \langle f, \Delta g\rangle$, i.e. Δ is self-adjoint, and is positive definite.

Moreover, for $f \in C_c^\infty(\mathfrak{h})$,

$$\begin{aligned} \langle \Delta f, f\rangle &= \int_{-\infty}^{\infty}\int_0^{\infty} \nabla f\cdot\overline{\nabla f}dydx \\ &= \int_{-\infty}^{\infty}\int_0^{\infty} \left(\left|\frac{\partial f}{\partial x}\right|^2 + \left|\frac{\partial f}{\partial y}\right|^2\right)dydx \\ &\geq \int_{-\infty}^{\infty}\int_0^{\infty} \left|\frac{\partial f}{\partial y}\right|^2 dydx. \end{aligned}$$

Viewing f as functions in y, $f(y)=u(y)+iv(y)$ and

$$|f|^2=u^2+v^2,\quad \frac{\partial f}{\partial y}=u'+iv',\quad \left|\frac{\partial f}{\partial y}\right|^2=u'^2+v'^2$$

and one needs to show that

$$\int_0^\infty (u'^2+v'^2)dy \geq \frac{1}{4}\int_0^\infty \frac{u^2+v^2}{y^2}dy.$$

Note that

$$\int_0^\infty \frac{u^2}{y^2}dy=-\int_0^\infty u^2 d\frac{1}{y}=2\int_0^\infty \frac{u}{y}u'dy \leq 2\left(\int_0^\infty \frac{u^2}{y^2}dy\right)^{1/2}\left(\int_0^\infty u'^2dy\right)^{1/2}$$

which gives that

$$\int_0^\infty \frac{u^2}{y^2}dy \leq 4\int_0^\infty u'^2dy.$$

This finishes the proof. □

2.4. **Remark on spectral decomposition.** We have showed that Δ is a 'good' enough operators on $L^2(\mathfrak{h})$. We hope to build 'Fourier analysis' on $L^2(\mathfrak{h})$ as eigenfunctions of Δ. We recall the spectral decomposition in the simplest cases as follows.

Consider $L^2(\mathbb{R})$. Eigenfunctions of $\frac{d^2}{dx^2}$ are of the form

$$e^{2\pi iy},\quad y\in\mathbb{R}$$

and each $\phi\in L^2(\mathbb{R})$ has spectral decomposition

$$\phi(x)=\int_{\mathbb{R}}\widehat{\phi}(r)e^{2\pi irx}dr$$

where

$$\widehat{\phi}(r)=\langle\phi,e^{2\pi ir*}\rangle=\int_{\mathbb{R}}\phi(y)e^{-2\pi iry}dy$$

is the Fourier transform of ϕ. Here $e^{2\pi iry}$ are eigenfunctions of $\frac{d^2}{dx^2}$ which is not in $L^2(\mathbb{R})$.

Consider $L^2(\mathbb{Z}\backslash\mathbb{R})$, the space of functions on $\mathbb{R}$ with period in $\mathbb{Z}$. It admits discrete spectrum

$$e^{2\pi imx},\quad m\in\mathbb{Z}.$$

which are also eigenfunctions of Δ. They can be also obtained via the following theory. Let

$$(\hat{\mathbb{R}/\mathbb{Z}})=\{\psi:\mathbb{R}/\mathbb{Z}\to S^1,\quad \psi(a+b)=\psi(a)\psi(b)\}$$

be the group of complex continuous characters of $\mathbb{R}/\mathbb{Z}$ (all continuous group homomorphisms from $\mathbb{R}$ to S^1 with period in $\mathbb{Z}$). Then ψ is of the form $\psi(x)=e(mx)$ for some $m\in\mathbb{Z}$ and one has

$$(\hat{\mathbb{R}/\mathbb{Z}})\simeq\mathbb{Z},\qquad e(m*)\mapsto m.$$

Then we can build Fourier analysis as that $\phi\in L^2(\mathbb{Z}\backslash\mathbb{R})$,

$$\phi(x)=\sum_{\substack{m\\ \text{spectral parameter}}}\frac{\langle\phi,e(m*)\rangle_{\mathbb{R}/\mathbb{Z}}}{\langle e(m*),e(m*)\rangle_{\mathbb{R}/\mathbb{Z}}}e(mx)$$

where

$$\langle \phi, e(m*) \rangle_{\mathbb{R}/\mathbb{Z}} = \int_{\mathbb{R}/\mathbb{Z}} \phi(y)\overline{e(my)}dy$$

is the m-th Fourier coefficient of ϕ. We refer to GTM186 - Fourier analysis on number field (Ramakrishnan-Valenza) for much information.

2.5. **Eigenfunctions of Δ.** To build the spectral decomposition of $L^2(\mathfrak{h})$, we need to find eigenfunctions of Δ firstly.

Assume that f is an eigenfunction of Δ with $\Delta f = \lambda f$. We can assume that $f(z) = v(x)w(y)$ by separating parameters. One has

$$-y^2 \left(v''(x)w(y) + v(x)w''(y)\right) = \lambda v(x)w(y)$$

dividing $v(x)w(y)$ on both sides, one has

$$\frac{w''(y)}{w(y)} + \frac{\lambda}{y^2} = k = -\frac{v''(x)}{v(x)}$$

where k is the separable parameter.

Consider the differential equation

$$k = -\frac{v''(x)}{v(x)}, \quad \text{or equivalently,} \quad v'' + kv = 0.$$

It is independent of the eigenvalue λ. The characteristic function of the differential equation is

$$r^2 + k = 0, \quad r = \pm\sqrt{-k}$$

and thus we have solutions

$$e^{\sqrt{-k}x}, \quad e^{-\sqrt{-k}x}.$$

The growth condition implies that $k \geq 0$. Let $k = 4\pi^2 a^2$. The solutions are

$$e^{2\pi ai} \quad \text{and} \quad e^{-2\pi ai}.$$

Consider another differential equation,

$$\frac{w''(y)}{w(y)} + \frac{\lambda}{y^2} = 4\pi^2 a^2.$$

Assume that $w = y^{1/2}u(y)$ one has

$$\frac{dw}{dy} = \frac{u(y)}{2\sqrt{y}} + y^{1/2}u'(y), \quad \frac{d^2w}{dy^2} = y^{1/2}u''(y) + \frac{1}{\sqrt{y}}u' - \frac{1}{4y^{3/2}}u(y)$$

and thus the differential equation becomes

$$y^2u'' + yu' + \left((\lambda - \frac{1}{4}) - 4\pi a^2 y^2\right) u = 0. \tag{2.5}$$

Lemma 2.3. *Let $\lambda = s(1-s) = \frac{1}{4} + t^2$ with $s = \frac{1}{2} + it$. The solutions of (2.5) are as follows.*

- *If $a = 0$,*

$$\frac{y^s + y^{1-s}}{2}, \quad \frac{y^s - y^{1-s}}{2s-1}$$

- *If $a \neq 0$, solutions of (2.5) are*

$$K_{s-\frac{1}{2}}(2\pi|a|y), \quad I_{s-\frac{1}{2}}(2\pi|a|y).$$

with asymptotic formulas as $|z| \to \infty$

$$K_{s-\frac{1}{2}}(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z}, \quad I_{s-\frac{1}{2}}(z) \sim \sqrt{\frac{1}{2\pi z}} e^{z}$$

Remark 4. For the case $a = 0$ and $s \neq 1/2$, $\{y^s, y^{1-s}\}$ are linear dependent solutions in this case; For $a = 0$ and $s = 1/2$,

$$\{y^{1/2}, \quad y^{1/2}\log y\}$$

are the independent solutions.

Proof. Let $u(y) = F(2\pi|a|y)$ and $z = 2\pi|a|y$. One has

$$u' = 2\pi|a|F'(2\pi|a|y), \quad u'' = 4\pi^2 a^2 = F''(2\pi|a|y),$$

and thus the differential equation (2.5) is of the form

$$(2\pi|a|y)^2 F''(2\pi|a|y) + (2\pi|a|y)F'(2\pi|a|y) + \left(\lambda - \frac{1}{4} - (2\pi|a|y)^2\right) F(2\pi|a|y) = 0.$$

Note that $z = 2\pi|a|y$. It is

$$z^2 F''(z) + zF'(z) + \left(\lambda - \frac{1}{4} - z^2\right) F(z) = 0$$

or equivalently,

$$F''(z) + \frac{1}{z}F'(z) - \left(1 + \frac{(it)^2}{z^2}\right) F(z) = 0 \tag{2.6}$$

which is a Bessel equation. The linear independent solutions (2.6) are $K_{it}(z)$ and $I_{it}(z)$ with the growth condition as above. We refer to appendix A for detail. ☐

Proposition 2.4. *Eigenfunctions of Δ with eigenvalues $\lambda = s(1-s) = \frac{1}{4} + t^2$ satisfying moderate growth conditions are*

$$y^{1/2+it} = y^s \quad , \quad y^{1/2-it} = y^{1-s};$$

and

$$\sqrt{y}K_{s-\frac{1}{2}}(2\pi|a|y)e(ax), \qquad 0 \neq a \in \mathbb{R}.$$

2.6. **Relation with the spherical function in the Whittaker model of the principle series.** We have proved that most of the eigenfunctions are of the form (in the case $a \neq 0$)

$$W_s(az) = 2\sqrt{|a|y}K_{s-\frac{1}{2}}(2\pi|a|y)e(ax).$$

Here

$$W_s(z) := 2\sqrt{y}K_{s-\frac{1}{2}}(2\pi y)e(x) \tag{2.7}$$

is called the Whittaker function.

The expression (2.7) is good for explicit calculation and estimation. However, we need another way to construct the sphercial Whittaker function associated to the spectral parameter s, which is simple and good for generalization (Not good for explicit calculation and estimation).

Let $s \in \mathbb{C}$ be a spectral parameter. We define

$$I_s(z) = (\mathrm{Im} z)^s$$

which is eigenfunction of Δ with eigenvalue $s(1-s)$. We let

$$\psi : N(\mathbb{R}) \to \mathbb{C}, \quad \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \mapsto e^{2\pi i u}$$

be a fixed non-trivial additive character on the unipotent group. The sphercial Whittaker function associated to ψ is defined by

$$\tilde{W}_s(z) = \int_{-\infty}^{\infty} I_s\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} .z \right) \psi(-u) du \tag{2.8}$$

for $\mathrm{Im} z > 0$, and is generalized to the lower half plane via (see (1.27) in Iwaniec's book)

$$\tilde{W}_s(\overline{z}) = \tilde{W}_s(z).$$

This definition of $\tilde{W}_s(z)$ is easy to be generalized to the case SL_n (See formula (5.5.1) in Goldfeld's book, *automorphic forms and L-functions for the group* $GL_n(\mathbb{R})$.) However, for the explicit calculation and estimation, one needs much precisely information on the behaviour of $W_s(z)$ in terms of the so called generalized Bessel functions. These have been worked out recently for the case GL_3 (V. Blomer, Buttcane) and for the case $GL_2(\mathbb{C})$ (Qi Zhi.)

Proposition 2.5. *For* $\tilde{W}_s(z)$*, and* $a \neq 0$*, we have*

$$\begin{aligned} \Delta \tilde{W}_s(z) &= s(1-s)\tilde{W}_s(z), \\ \tilde{W}_s(az) &= e(ax)W_s(i|a|y) = \frac{\pi^s}{\Gamma(s)} 2\sqrt{|a|y} K_{s-\frac{1}{2}}(2\pi|a|y) e(ax). \end{aligned}$$

Remark 5. Note that the K-Bessel function satisfies $K_s(y) = K_{-s}(y)$ and $\overline{K_s(y)} = K_{\overline{s}}(y)$ by the integral representation in (1.25). For a general definition of the spherical Whittaker function in the Whittaker model of the unramified principle series of $PGL_2(\mathbb{R})$, see appendix B.

Proof. The first and the second result follows from the fact

$$z = x + iy = \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} .i$$

and that Δ commutes with the action of $g \in SL_2(\mathbb{R})$,

$$\Delta \left(f(g.z) \right) = \left(\Delta f \right)(g.z).$$

Note that

$$I_s(z) = (\mathrm{Im} z)^s, \quad I_s(g.z) = \left(\tfrac{y}{|cz+d|^2} \right)^s,$$
$$I_s\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} .z \right) = \left(\tfrac{y}{(x+u)^2+y^2} \right)^s$$

Thus

$$\tilde{W}_s(z) = \int_{-\infty}^{\infty} \left(\frac{y}{(x+u)^2+y^2} \right)^s e(-u) du = e(x) \int_{-\infty}^{\infty} \left(\frac{y}{u^2+y^2} \right)^s e(-u) du. \tag{2.9}$$

By the above and the definition on the lower half plane, obviously one has

$$\tilde{W}_s(az) = e(ax)W_s(i|a|y).$$

Moreover, by the definition,

$$\overline{\widetilde{W}_s(x+iy)} = e(-x)\int_{-\infty}^{\infty}\left(\frac{y}{u^2+y^2}\right)^{\overline{s}} e(u)du = \tilde{W}_{\overline{s}}(-x+iy).$$

The final step follows from the following proposition immediately. $\square$

Lemma 2.6. *Let $y \in (0,+\infty)$. For $\mathrm{Re}(s) > 0$ we have*

$$\int_{-\infty}^{\infty}\left(\frac{y}{u^2+y^2}\right)^{s} e(-au)du = \frac{\pi^s}{\Gamma(s)}\begin{cases}\pi^{-s+\frac{1}{2}}\Gamma(s-\frac{1}{2})y^{1-s}, & a=0\\ 2|a|^{s-\frac{1}{2}}\sqrt{|a|}K_{s-\frac{1}{2}}(2\pi|a|y), & a\neq 0\end{cases}$$

Proof. Recall that

$$\Gamma(s) = \int_0^{\infty} e^{-t}t^s\frac{dt}{t}$$

i.e. $\Gamma(s)$ is the Mellin transform of e^{-t} with transform kernel t^s. The integral involves $\left(\frac{y}{u^2+y^2}\right)^s$ which can be combined with t^s to get new kernel and then change variable.

By multiplying $\Gamma(s)$ and exchanging the integration,

$$\begin{aligned}
&\Gamma(s)\int_{-\infty}^{\infty}\left(\frac{y}{u^2+y^2}\right)^{s} e(-au)du = \int_0^{\infty}e^{-t}t^s\frac{dt}{t}\int_{-\infty}^{\infty}\left(\frac{y}{u^2+y^2}\right)^{s} e(-au)du\\
=\ & \int_{-\infty}^{\infty}e(-au)\left\{\int_0^{\infty}e^{-t}\left(\frac{ty}{u^2+y^2}\right)^s\frac{dt}{t}\right\}du = \int_{-\infty}^{\infty}e(-au)\left\{\int_0^{\infty}e^{-t\frac{u^2+y^2}{y}}t^s\frac{dt}{t}\right\}du\\
=\ & \int_0^{\infty}e^{-ty}t^s\left\{\int_{-\infty}^{\infty}e^{-\frac{t}{y}u^2-2\pi iau}du\right\}\frac{dt}{t}
\end{aligned}$$

Now we can calculate the inner integral and apply the fact that Bessel function is expressed as Mellin transform, namely

$$\int_0^{\infty}e^{-\frac{y}{2}(t+\frac{1}{t})}t^s\frac{dt}{t} = 2K_s(y).$$

$\square$

2.7. **Spectral decomposition of $L^2(\mathfrak{h})$.** We have obtained eigenfunctions of Δ. Now we give the spectral decomposition of $L^2(\mathfrak{h})$ as follows.

Theorem 2.7. *Denote by*

$$e_{a,s}(z) = \begin{cases} y^s, & a=0\\ \sqrt{y}K_{s-\frac{1}{2}}(2\pi|a|y)e(ax), & a\neq 0\end{cases}$$

For $f\in C_c^{\infty}(\mathfrak{h})$, denote by

$$\widehat{f}(a,s) = \int_{\mathfrak{h}} f(z)\overline{e_{a,s}(z)}\frac{dxdy}{y^2}.$$

One has

$$f(z) = \int_{a\in\mathbb{R}}\int_{\mathrm{Re}(s)=\frac{1}{2}}\widehat{f}(a,s)e_{a,s}(z)\frac{t\sinh(\pi t)}{\pi^2}dtda.$$

Remark 6. Note that $W_s(az) = 2\sqrt{\pi|a|}e_{s,a}(z)$. The above formula can be expressed as

$$\begin{aligned} f(z) &= \int_{a\in\mathbb{R}} \int_{\mathrm{Re}(s)=\frac{1}{2}} \langle f, e_{a,s}\rangle e_{a,s}(z) \frac{t\sinh(\pi t)}{\pi^2} dtda \\ &= \frac{1}{2\pi i} \int_{a\in\mathbb{R}} \int_{\mathrm{Re}(s)=\frac{1}{2}} \langle f, W_s(a*)\rangle W_s(az) \frac{t\sinh(\pi t)}{2\pi^2|a|} dads \end{aligned}$$

which is the same as the formula in Iwaniec's book.

Proof. To prove the identity, it is sufficient to prove it for $f(z) = h(x)g(y)$ with special values $x = 0$ and $y = 1$. Note

$$\hat{f}(a, \frac{1}{2} + it) = \int_{-\infty}^{\infty} h(u)e(-au)du \left(\int_0^\infty g(v)\sqrt{v}K_{it}(2\pi|a|v)\frac{dv}{v^2} \right).$$

Here note that $\overline{K_{s-\frac{1}{2}}}(2\pi|a|y) = K_{\bar{s}-\frac{1}{2}}(2\pi|a|y)$ and $K_{-s}(y) = K_s(y)$, $s = \frac{1}{2} + it$.

We want to show that

$$\begin{aligned} h(x)g(y) &= \int_{a\in\mathbb{R}} \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(u)e(-au)du \left(\int_0^\infty g(v)\sqrt{v}K_{it}(2\pi|a|v)\frac{dv}{v^2} \right) \right) \\ &\quad \sqrt{y}K_{it}(2\pi|a|y)e(ax)\frac{t\sinh(\pi t)}{\pi^2} dtda \\ &= \int_{a\in\mathbb{R}} \int_{-\infty}^{\infty} h(u)e(-au)du \left(\int_{-\infty}^{\infty} \int_0^\infty g(v)\sqrt{v}K_{it}(2\pi|a|v)\frac{dv}{v^2}\sqrt{y}K_{it}(2\pi|a|y)\frac{t\sinh(\pi t)}{\pi^2} dt \right) \\ &\quad e(ax)da. \end{aligned}$$

It is sufficient to consider

$$\begin{aligned} I(a,y) &= \int_{-\infty}^{\infty} \int_0^\infty g(v)\sqrt{v}K_{it}(2\pi|a|v)\frac{dv}{v^2}\sqrt{y}K_{it}(2\pi|a|y)\frac{t\sinh(\pi t)}{\pi^2} dt \\ &= \int_0^\infty \left(\int_0^\infty g(v)\frac{K_{it}(2\pi|a|v)}{\sqrt{v}}\frac{dv}{v} \right) \sqrt{y}K_{it}(2\pi|a|y)\frac{2t\sinh(\pi t)}{\pi^2} dt \\ &= 2\pi|a|y \int_0^\infty \left(\int_0^\infty \frac{g(\frac{v}{2\pi|a|})}{v}\frac{K_{it}(v)}{\sqrt{v}} dv \right) \frac{K_{it}(2\pi|a|y)}{\sqrt{2\pi|a|y}}\frac{2t\sinh(\pi t)}{\pi^2} dt. \end{aligned}$$

Applying Kontorovitch-Lebedev transform,

$$I(a,y) = 2\pi|a|y \times \left. \frac{g\left(\frac{v}{2\pi|a|}\right)}{v} \right|_{v=2\pi|a|y} = g(y)$$

We finish the proof. ◻

2.8. **Kontorovitch-Lebedev trasform.**

Proposition 2.8 (Kontorovitch-Lebedev)**.** *Let $h(y)$ with $y > 0$ be a function, one has*

$$\begin{aligned} g(y) &= \int_0^\infty \left(\int_0^\infty g(v)\frac{K_{it}(v)}{\sqrt{v}} dv \right) \frac{K_{it}(y)}{\sqrt{y}}\frac{2t\sinh(\pi t)}{\pi^2} dt \\ f(t) &= \frac{2t\sinh(\pi t)}{\pi^2} \int_0^\infty \frac{K_{it}(y)}{y} \left\{ \int_0^\infty f(u)K_{iu}(y)du \right\} dy. \end{aligned}$$

Proof. Recall the integral representation of K-Bessel function, for $s = it$, by viewing y as a parameter,

$$\begin{aligned} K_{it}(y) &= \frac{1}{2}\int_0^\infty e^{-\frac{y}{2}(t_0+\frac{1}{t_0})} t_0^{it}\frac{dt_0}{t_0}, \quad t_0 = e^x \\ &= \frac{1}{2}\int_{-\infty}^\infty e^{-y\cosh x} e^{2\pi i \frac{x}{2\pi} t} dx = \pi \int_{-\infty}^\infty e^{-y\cosh(2\pi x)} e^{2\pi i x t} dx \end{aligned}$$

and we know that $t \mapsto K_{it}(y)$ is the Fourier inverse transform of

$$h_y(x) := \pi e^{-y\cosh 2\pi x} \tag{2.10}$$

Recall the multiplication formula

$$\int f\hat{g} = \int \hat{f} g.$$

For f defined on $\mathbb{R}_+$, we extend it to $\mathbb{R}$ via $f(-t) = f(t)$, then

$$\begin{aligned} A &= \frac{2}{\pi^2} t\sinh(\pi t)\int_{y=0}^\infty K_{it}(y)\frac{1}{y}\left(\int_{u=0}^\infty f(u)K_{iu}(y)du\right)dy \\ &= \frac{1}{\pi^2} t\sinh(\pi t)\int_{y=0}^\infty K_{it}(y)\frac{1}{y}\left(\int_{u=-\infty}^\infty f(u)K_{iu}(y)du\right)dy, \\ &= \frac{1}{\pi^2} t\sinh(\pi t)\int_{y=0}^\infty K_{it}(y)\frac{1}{y}\left(\int_{u=-\infty}^\infty \widehat{f}(u)\pi e^{-y\cosh 2\pi u}du\right)dy \\ &= \frac{1}{2\pi^2} t\sinh(\pi t)\int_{y=0}^\infty K_{it}(y)\frac{1}{y}\left(\int_{u=-\infty}^\infty \widehat{f}(\frac{u}{2\pi})e^{-y\cosh u}du\right)dy \\ &= \frac{1}{2\pi^2} t\sinh(\pi t)\int_{u=-\infty}^\infty \widehat{f}(\frac{u}{2\pi})\left(\int_{y=0}^\infty K_{it}(y)\frac{1}{y}e^{-y\cosh u}dy\right)du. \end{aligned}$$

Next we applying the formula

$$\int_{y=0}^\infty e^{-y\cosh u} K_{it}(y)\frac{1}{y}dy = \pi\frac{\cos(tu)}{t\sinh(\pi t)}, \tag{2.11}$$

whose proof is in Page 177 in *Harmonic analysis on symmetric space*, one has

$$A = \int_{u=-\infty}^\infty \widehat{f}(\frac{u}{2\pi})\frac{\cos(tu)}{2\pi}du = f(t)$$

since f is even. We finish the proof. ☐

Remark 7. By (2.10),

$$\begin{aligned} & \pi e^{-y\cosh 2\pi x} = \int_{-\infty}^\infty K_{it}(y)e^{-2\pi i t x}dt \\ \Leftrightarrow\ & \pi e^{-y\cosh x} = \int_{-\infty}^\infty K_{it}(y)e^{-itx}dt = \pi e^{-y\cosh(-x)} = \int_{-\infty}^\infty K_{it}(y)e^{itx}dt \\ \Leftrightarrow\ & \pi e^{-y\cosh x} = \int_{-\infty}^\infty K_{it}(y)\frac{e^{-itx}+e^{itx}}{2}dt \\ \Leftrightarrow\ & \frac{\pi}{2}\exp(-y\cosh x) = \int_0^\infty K_{it}(y)\cos(tx)dt, \quad \mathrm{Re}(y) > 0. \end{aligned}$$

Via the integral representation, we can also prove that

$$\int_0^\infty y^{r-1}K_s(y)dy = 2^{r-2}\Gamma\left(\frac{r+s}{2}\right)\Gamma\left(\frac{r-s}{2}\right).$$

Proof. We give another proof as follows. It is sufficient to prove that the kernel function

$$W_R(x,y) = \frac{1}{\pi^2}\int_{-R}^{R} t\sinh(\pi t)\frac{K_{it}(x)K_{it}(y)}{\sqrt{xy}}dt$$

approaches $\delta(x-y)$, as $R\to\infty$. Since the problem is invariant under $SL_2(\mathbb{R})$, it is sufficient to consider the problem as $x,y\sim 0$.

Note

$$K_{it}(y)\sim 2^{it-1}\Gamma(it)y^{-it}+2^{-it-1}\Gamma(-it)y^{it},\quad y\to 0^+$$

which follows from the relation with I-Bessel and the power series for I-Bessel. Moreover,

$$\Gamma(it)\Gamma(-it) = \pi(t\sinh(\pi t))^{-1}$$

implies that

$$W_R(x,y)\sim \frac{1}{2\pi}\int_{-R}^{R} y^{-\frac{1}{2}-it}x^{-\frac{1}{2}+it}dt,\quad x,y\to 0^+.$$

Thus spectral measure is chosen to cancel the Gamma-factors. □

2.9. **Fourier expansion of functions in $L^2(\Gamma_\infty\backslash\mathfrak{h})$.** Let

$$\Gamma_\infty = \left\{\begin{pmatrix}1 & n\\ & 1\end{pmatrix}, n\in\mathbb{Z}\right\}$$

be a discrete subgroup. We know that elements in Γ_∞ are parabolic, and

$$\begin{pmatrix}1 & n\\ & 1\end{pmatrix}.z = z+n$$

has only one fixed point $z=\infty$.

Let $L^2(\Gamma_\infty\backslash\mathfrak{h})$ be the functions $f:\mathfrak{h}\to\mathbb{C}$ with

$$f\left(\begin{pmatrix}1 & n\\ & 1\end{pmatrix}.z\right) = f(z+n) = f(z),\quad \forall n\in\mathbb{Z}$$

$$\int_{\Gamma_\infty\backslash\mathfrak{h}} f(z)\overline{f(z)}\frac{dxdy}{y^2}<\infty.$$

Here a fundamental mesh is

$$\Gamma_\infty\backslash\mathfrak{h} = \{z = x+iy, 0\le x<1, y>0\}.$$

Proposition 2.9. *Let $f(z)$ be eigenfunctions of Δ with eigenvalue $\lambda = s(1-s)$ which satisfies*

- $f(z+m) = f(z)$, $\forall m\in\mathbb{Z}$
- $f(z)$ *is of moderate growth, i.e.*

$$f(z) = o\left(e^{2\pi y}\right),\quad y\to\infty$$

Then $f(z)$ has expansion

$$f(z) = a_{f,0}(y) + \sum_{n \neq 0} a_f(n) W_s(nz)$$

where $a_{f,0}(y)$ is a linear combination of y^s and y^{1-s} if $\lambda \neq \frac{1}{4}$, and $y^{1/2}$ and $y^{1/2} \log y$ if $\lambda = \frac{1}{4}$; and $a_f(n)$ are some coefficients (depending on f, called the Fourier coefficients of f).

3. The Hyperbolic Geometry - the Geodesic Polar coordinates

Recall that we have defined the distance function. Consider the radius with center i, i.e.

$$d(z, i) = r$$

It has hyperbolic area $4\pi(\sinh(r/2))^2$ and circumference $2\pi \sinh r$. On the other hand, its Euclidean center is $i \cosh r$ and radius $\sinh r$.

3.1. Cartan decomposition and polar coordinates.

Cartan decomposition and polar coordinates. Recall Cartan decomposition, $G = KAK$. For $g \in PSL_2(\mathbb{R})$, $g = \kappa(\varphi) a(e^r) \kappa(\theta)$ with

$$\begin{aligned} \kappa(\varpi) &= \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}, \quad 0 \leq \varphi < \pi \quad , \\ a(e^{-r}) &= \begin{pmatrix} e^{-\frac{r}{2}} & \\ & e^{r/2} \end{pmatrix}, \quad r \geq 0 \\ \kappa(\theta) &\in SO(2) \quad . \end{aligned}$$

It gives the geodesic polar coordinates

$$z = x + iy = \kappa(\varphi) a(e^{-r}).i = \kappa(\varphi) e^{-r}.i$$

Here $a(e^{-r})$ selects the point on y-axis with distance r with i on the geodesic, and $\kappa(\varphi)$ gives rotation of angle 2φ. With respect to the geodesic polar coordinates,

$$\begin{gathered} ds^2 = dr^2 + (\sinh r)^2 du^2, \quad d\mu(z) = \sinh r dr d\varphi \\ \Delta = -\tfrac{1}{\sinh r} \tfrac{\partial}{\partial r} \left(\sinh r \tfrac{\partial}{\partial r}\right) - \tfrac{1}{\sinh^2 r} \tfrac{\partial^2}{\partial \varphi^2} \end{gathered}$$

3.2. Spherical functions and spectral decomposition.

Spherical functions and spectral decomposition. We still want to obtain eigenfunctions of Δ with eigenvalue $\lambda = s(1-s)$. By separable parameters, eigenfunctions should be of the form

$$f(\kappa_\theta z) = \chi(\kappa_\theta) f(z), \quad \forall \kappa_\theta \in K$$

where

$$\chi : \kappa_\theta \mapsto e^{2im\theta}, \quad 0 \leq \theta < \pi.$$

By simlar argument, we start from the function

$$I_s(z) \quad = (\mathrm{Im} z)^s = (\mathrm{Im} \kappa_\varphi e^{-r}.i)^d$$

and thus to form such $f(z)$ as

$$\begin{aligned} f(z) &= \frac{1}{\pi}\int_0^{\pi} \operatorname{Im}(\kappa(-\theta)\kappa(\varphi)e^{-r}.i)^s \chi(\kappa_\theta)d\theta \\ &= \frac{1}{\pi}\int_0^{\pi} (\cosh r + \sinh r\cos 2\theta)^{-s} e^{2im(\theta+\varphi)}d\theta \\ &= \frac{\Gamma(1-s)}{\Gamma(1-s+m)} P_{-s}^m(\cosh r)e^{2im\varphi} \end{aligned}$$

where P_{-s}^m is the Legendre function defined by

$$\begin{aligned} P_\nu(z) &= F(-\nu, \nu+1, 1, \frac{1-z}{2}) \\ P_\nu^m(z) &= (z-1)^{m/2}\frac{d^m}{dz^m}P_\nu(z) \\ &= \frac{\Gamma(\nu+m+1)}{\pi\Gamma(\nu+1)}\int_0^{\pi}(z+\sqrt{z^2-1}\cos\alpha)^\nu \cos(m\alpha)d\alpha \\ &= \frac{\Gamma(\nu+m+1)}{2\pi\Gamma(\nu+1)}\int_0^{2\pi}(z+\sqrt{z^2-1}\cos\alpha)^\nu e^{im\alpha}d\alpha, \quad \operatorname{Re}(z)>0 \end{aligned}$$

Proposition 3.1. *The spherical function is defined by*

$$U_s^m(z) := P_{-s}^m(\cosh r)e^{2im\varphi},$$

Then for any $f \in C_c^\infty(\mathfrak{h})$, we have

$$\widehat{f}(m,s) := \langle f, U_s^m\rangle = \int_{\mathfrak{h}} f(z)U_s^m(z)d\mu(z)$$

and

$$f(z) = \sum_{m\in\mathbb{Z}} \frac{(-1)^m}{2\pi i}\int_{\operatorname{Re}(s)=1/2} \widehat{f}(m,s)U_s^m(z)t\tanh(\pi t)ds.$$

Remark 8. Spherical functions of order $m = 0$ depends only on the hyperbolic distance.

Proof. It follows from Fourier expansion of periodic function with the following inversion formula

$$g(u) = \int_0^\infty P_{-1/2+it}(u)\left(\int_1^\infty P_{-1/2+it}(v)g(v)dv\right)t\tanh(\pi t)dt$$

with $P_s(u) := P_s^0(u)$.

It is sufficient to show the kerne

$$V_R(x,y) = \int_0^R t\tanh(\pi t)P_{-\frac{1}{2}+it}(x)P_{-\frac{1}{2}+it}(y)dt$$

approaches $\delta(x-y)$ as $R\to\infty$.

Note

$$P_{-1/2+it}(x) \sim \frac{\Gamma(it)}{\sqrt{\pi}\Gamma(\frac{1}{2}+it)}(2x)^{-\frac{1}{2}+it} + \frac{\Gamma(-it)}{\sqrt{\pi}\Gamma(\frac{1}{2}-it)}(2x)^{-\frac{1}{2}+it}, \quad x\to\infty$$

for fixed real t, and

$$\frac{\Gamma(it)\Gamma(-it)}{\pi\Gamma(\frac{1}{2}+it)\Gamma(\frac{1}{2}-it)} = \frac{1}{\pi t\tanh(\pi t)}r,$$

thus

$$V_R(x,y) \sim \frac{1}{\pi}\int_0^R x^{-\frac{1}{2}+it}y^{-\frac{1}{2}-it}dt, \quad x,y\sim\infty$$

on right side of which is a Dirac delta family by Mellin inversion formula. $\square$

4. Helgason trasform on $\mathfrak{h}$

Set $B = K/M$ with $M = \{I, I\}$, where B is called the 'boundary' of $\mathfrak{h}$.

Let $f \in C_c^\infty(\mathfrak{h})$. For $s \in \mathbb{C}, k \in SO(2)$, we define

$$\mathcal{H}f(s,k) = \int_{\mathfrak{h}} f(z)\overline{\mathrm{Im}(kz)^s}\frac{dxdy}{y^2}.$$

Proposition 4.1. *One has*

$$f(z) = \frac{1}{4\pi}\int_{t\in\ R}\frac{1}{2\pi}\int_{\theta=0}^{2\pi}\mathcal{H}f(\frac{1}{2}+it,k_\theta)\mathrm{Im}(k_\theta z)^{\frac{1}{2}+it}t\tanh(\pi t)d\theta dt,$$

where $k_\theta \in SO(2)$.

The map $f \mapsto \mathcal{H}f$ takes $C_c^\infty(\mathfrak{h})$ one-to-one, onto the space of C^∞ functions $G(s,k)$ on $\mathbb{C}\times SO(2)$ which are holomorphic in s. It extends to an isometry mapping $L^2(\mathfrak{h}, \frac{dxdy}{y^2})$ onto $L^2(\mathbb{R}\times K, \frac{1}{8\pi^2}t\tanh(\pi t)dtd\theta)$ where $K = SO(2)$ is identified with $(0, 2\pi)$

Proposition 4.2. *The Helgason transform of K-invariant functions is a composition of Harish-Chandra and Mellin transforms. For $f_0 \in C_c^\infty(GL_2(\mathbb{R})^+, Z_\infty K_\infty)$, the action $\pi_{\epsilon_\pi,it_\pi}(f_0)$ on ϕ_0 is a scalar given by*

$$\mathcal{S}(f_0)(it_\pi) := \int_0^\infty\left[y^{-1/2}\int_{-\infty}^\infty f_0\left(\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & \\ & y^{-1/2}\end{pmatrix}\right)dx\right]y^{it_\pi}\frac{dy}{y},$$

where $\mathcal{S}(f_0)$ is called the spherical transform of f_0. Moreover, the spherical transform $\mathcal{S}$ defines a map

$$\mathcal{S} : C_c^\infty(GL_2(\mathbb{R})^+, Z_\infty K_\infty) \to PW^\infty(\mathbb{C})^{\mathrm{even}}, \quad f_0 \mapsto \mathcal{S}(f_0)$$

which is an isomorphism to the Paley-Wiener space of even functions.

5. Automorphic forms for $SL_2(\mathbb{Z})$

Proposition 5.1. *$SL_2(\mathbb{Z})$ is a disconnected subgroup of $SL_2(\mathbb{R})$. It has two generators, namely*

$$T = \begin{pmatrix}1 & 1\\ & 1\end{pmatrix}, \quad S = \begin{pmatrix} & -1\\ 1 & \end{pmatrix}$$

Proof. Note that $T^m = \begin{pmatrix}1 & m\\ & 1\end{pmatrix}$ and $S^2 = \begin{pmatrix}-1 & \\ & -1\end{pmatrix}$. Thus if $\begin{pmatrix}a & b\\ 0 & d\end{pmatrix} \in SL_2(\mathbb{Z})$, then $a = d \in \{\pm 1\}$ and $b \in \mathbb{Z}$ can be expressed by product of T and S. This shows that

$$\Gamma_\infty = \left\{\begin{pmatrix}\pm 1 & m\\ & \pm 1\end{pmatrix}, m\in\mathbb{Z}\right\} = \{\gamma\in SL_2(\mathbb{Z}), \gamma.\infty = \infty\}$$

can be expressed by T and S^2.

Assume $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ with $c \neq 0$. Note that $\det\gamma = ad - bc = 1$ which implies that $(a,c) = 1$, otherwise

$$ax + cy = 1$$

has no integral solution $(x,y) \in \mathbb{Z}^2$. Multiplying $\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$ on left one has

$$\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+mc & b+md \\ c & d \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ c & d \end{pmatrix}$$

by choosing suitable m we can assume that $0 \leq a_1 = a + mc < |c|$. Next, we multiplying S to obtain

$$S\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ c & d \end{pmatrix} = \begin{pmatrix} -c & -d \\ a_1 & b_1 \end{pmatrix}.$$

Note that $0 \leq a_1 < |c|$. repeat the above steps we will finally obtain a matrix with

$$\begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, \quad c_n = 0$$

which is an element in Γ_∞. □

5.1. **Fundamental domain.** we have

$$\Gamma\backslash\mathfrak{h} = \{z = x + iy, \quad -\frac{1}{2} \leq x \leq \frac{1}{2}, \sqrt{1-x^2} < y < \infty\}$$

This is non-compact and of finite volume, and has only one cusp ∞.

5.2. **Fourier expansion of $L^2(\Gamma\backslash\mathfrak{h})$.** We have shown that the Fourier expansion of functions $L^2(\Gamma_\infty\backslash\mathfrak{h})$ should be

$$f(z) = a_f(0,y) + \sum_{n\neq 0} a_f(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e(nx)$$

where

$$a_f(0,y) = \begin{cases} a_{f,1}(0)y^s + a_{f,2}(0)y^{1-s}, & s =\neq 1/2 \\ a_{f,1}(0)y^{1/2} + a_{f,2}(0)y^{1/2}\log y, & s = 1/2 \end{cases}$$

5.3. **Cusp forms.** Note that we add an compact region to the fundamental domain to consider

$$\begin{aligned} \int_{\Gamma\backslash\mathfrak{h}} |f(z)|^2 dz &\leq \int_{-1/2}^{1/2}\int_{\frac{\sqrt{3}}{2}}^{\infty} |f(z)|^2 dz \\ &= \int_{\sqrt{3}/2}^{\infty} |a_f(0,y)|^2 \frac{dy}{y^2} + \sum_{n\neq 0} |a_f(n)|^2 \int_{\sqrt{3}/2}^{\infty} |y| \left|K_{s-1/2}(2\pi|n|y)\right|^2 \frac{dy}{y^2} \end{aligned}$$

The asymptotic formula

$$K_{s-1/2}(z) \sim \sqrt{\frac{\pi}{2z}}e^{-z}$$

implies the sum over $n \neq 0$ is absolutely convergent.

For the constant term,

$$\int_{\sqrt{3}/2}^{\infty} |y^{2\mathrm{Re}(s)}| \frac{dy}{y^2} = \int_{\sqrt{3}/2}^{\infty} |y^{2(\mathrm{Re}(s)-1)}| dy = \infty$$

unless $\mathrm{Re}(s) < \frac{1}{2}$.

Proposition 5.2. *A function f which admits no constant term, namely*

$$\int_0^1 f(z)dx \neq 0$$

for all y, is called cusp form.

Via the Fourier expansion, and the property of K-Bessel function, if f is a cusp form, then it vanishes at $y \to \infty$.

If we realized $z = x + iy$ as matrix $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} \kappa_\theta$ and lifting maass forms as functions on

$$f : SL_2(\mathbb{R}) \to \mathbb{C}$$

The cuspidal condition is equivalently to

$$\int_{\mathbb{Z}\backslash\mathbb{R}} f\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx = 0$$

i.e. f admits no-'trivial' property at $N(\mathbb{Z}\backslash\mathbb{R})$.

6. L-function and functional equation of even and odd Maass cusp forms

6.1. **Even and odd Maass cusp forms.** We define

$$\iota : \mathfrak{h} \to \mathfrak{h}, \quad z = x + iy \mapsto -\overline{z} = -x + iy$$

Extend it to be an operator

$$f(z) \mapsto f(\iota z)$$

One has $\iota^2 = id$. Thus if f is eigen function of ι, then the eigenvalues should be ± 1.

- We call f is even Maass cusp form, if $\iota f = f$. In this case, by the Fourier expansion of $f(z)$, we have

$$a_f(-n) = a_f(n), \quad n \in \mathbb{Z}$$

- We call f is odd Maass cusp form, if $\iota f = -f$. In this case $a_f(-n) = -a_f(n)$.

Remark 9. The operator ι commutes with Δ and Hecke operators defined later.

6.2. **L-function asscoaited to even Maass cusp f.** Let f be an even Maass cusp form with spectral parameter $\frac{1}{2}+it$. We consider

$$I(s,f) := \int_0^\infty f(iy)|y|^{s-\frac{1}{2}}\frac{dy}{y}.$$

Note that

$$f(iy) = f(S.iy) = f\left(-\frac{1}{iy}\right) = f(i\frac{1}{y}).$$

Thus

$$\begin{aligned} I(s,f): &= \int_1^\infty f(iy)|y|^{s-\frac{1}{2}}\frac{dy}{y} + \int_0^1 f(i\frac{1}{y})|y|^{s-\frac{1}{2}}\frac{dy}{y} \\ &= \int_1^\infty f(iy)|y|^{s-\frac{1}{2}}\frac{dy}{y} + \int_1^\infty f(iy)|y|^{1-s-\frac{1}{2}}\frac{dy}{y} \end{aligned}$$

and one know that $I(s,f)$ is an entire function for all $s \in \mathbb{C}$ and satisfies the functional equation

$$I(s,f) = I(1-s,f).$$

On the other hand, for $\mathrm{Re}(s)$ large, by the Fourier expansion and f is even Maass cusp form.

$$f(iy) = \sum_{n\neq 0} a_f(n)\sqrt{y}K_{it}(2\pi|n|y) = 2\sum_{n\geq 1} a_f(n)\sqrt{y}K_{it}(2\pi ny),$$

we have

$$\begin{aligned} I(s,f) &= 2\sum_{n\geq 1} a_f(n)\int_0^\infty K_{it}(2\pi ny)y^s\frac{dy}{y} \\ &= 2\sum_{n\geq 1} a_f(n)\frac{1}{(2\pi n)^s}\int_0^\infty K_{it}(y)y^s\frac{dy}{y} \\ &= 2(2\pi)^{-s}\sum_{n\geq 1}\frac{a_f(n)}{n^s}\int_0^\infty K_{it}(y)y^s\frac{dy}{y} \end{aligned}$$

Lemma 6.1. *One has*

$$\int_0^\infty K_\nu(y)y^s\frac{dy}{y} = 2^{s-2}\Gamma\left(\frac{s+\nu}{2}\right)\Gamma\left(\frac{s-\nu}{2}\right)$$

which is absolutely convergent if $\mathrm{Re}(s) > \mathrm{Re}(\nu)$.

Proof. Recall that

$$K_\nu(y) = \frac{1}{2}\int_0^\infty e^{-y(t+\frac{1}{t})/2}t^s\frac{dt}{t}$$

for all values of ν. Thus

$$L.H.S. = \frac{1}{2}\int_0^\infty\int_0^\infty e^{-\frac{yt}{2}-\frac{y}{2t}}t^\nu y^s\frac{dy}{y}\frac{dt}{t}.$$

We hope to separable parameters and thus take $u=\frac{ty}{2}$ $v=\frac{y}{2t}$ so that

$$\frac{du}{u}\wedge\frac{dv}{v} = 2\frac{dt}{t}\wedge\frac{dy}{y}$$

and thus

$$L.H.S. = 2^{s-2} \int_0^\infty \int_0^\infty e^{-u-v} u^{(s+\nu)/2} v^{(s-\nu)/2} \frac{du}{u} \frac{dv}{v} = R.H.S.$$

$\square$

Theorem 6.2. *Let f be an even maass cusp form for $SL_2(\mathbb{Z})$ with eigenvalue $\lambda = \frac{1}{4} + t^2$ (spectral parameter $\frac{1}{2} + it$). Then we have*

$$f(z) = \sum_{n \neq 0} a_f(n) \sqrt{y} K_{it}(2\pi |n| y) e(nx)$$

with $a_f(-n) = a_f(n)$. The L-function

$$L(s, f) := \sum_{n \geq 1} \frac{a_f(n)}{n^s}$$

is defined for $\mathrm{Re}(s)$ *large and has analytic continuation to all $s \in \mathbb{C}$. Denote by*

$$\Lambda(s, f) = \pi^{-s} \Gamma(\frac{s+it}{2}) \Gamma(\frac{s-it}{2}) L(s, f)$$

one has the functional equation

$$\Lambda(s, f) = \Lambda(1-s, f).$$

6.3. **L-function associated to odd maass cusp form.** Assume that f is odd maass cusp form. It has Fourier expansion with $a_f(-n) = -a_f(n)$ and we can define

$$L(s, f) = \sum_{n \geq 1} \frac{a_f(n)}{n^s}$$

for $\mathrm{Re}(s)$ large. Since f is odd, the integral $I(s, f)$ vanishes.

To establish the functional equation, on taking

$$g(z) := \frac{1}{4\pi i} \frac{\partial f}{\partial x}(z),$$

one needs to consider

$$I(s, g) := \int_0^\infty g(iy) y^{s+\frac{1}{2}} \frac{dy}{y}$$

and establishes

$$\begin{aligned} \Lambda(s, f) &= \pi^{-s} \Gamma\left(\frac{s+it-1}{2}\right) \Gamma\left(\frac{s-it-1}{2}\right) L(s, f) \\ &= (-1)\Lambda(1-s, f). \end{aligned}$$

For more information, we refer to page 107 in Bump's book.

7. The theory of Hecke operators -Does the L-function admits Euler product

Let $\Gamma = SL_2(\mathbb{Z})$. For any subgroups $G \subset \Gamma$, we have

$$f(\gamma.z) = f(z), \quad \gamma \in G \subset \Gamma.$$

To obtain much more information, we consider the action of a much bigger discontinuous subgroup.

Consider

$$M_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{Z} \right\}$$

It is the biggest in some sense. We decompose $M_2(\mathbb{Z})$ as

$$M_2(\mathbb{Z}) = \sum_n G_n$$

where

$$G_n = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \det g = ad - bc = n \right\}.$$

One has

$$\Gamma G_n = G_n \Gamma$$

Remark 10. We are interested in those G_n with $n \geq 1$.

7.1. Slash operator.

For $g \in GL_2(\mathbb{R})$ and $f \in L^2(\Gamma\backslash\mathfrak{h})$, we define the operator

$$f \mapsto f|_g, \quad f|_g(z) = f(g.z)$$

Note that f is automorphic for Γ then

$$f|_\gamma(z) = f(\gamma z) = f(z).$$

Remark 11. The slash operator is defined by left translation, which commutes with the action of Δ naturally. Thus f is automorphic for Γ if and only if

$$f|_\gamma = f, \quad \forall \gamma \in \Gamma.$$

7.2. The right cosets $\Gamma\backslash G_n$.

Start from elements in $g \in G_n$, we hope to construct new operators T_n which map automorphic forms to automorphic forms.

Note that for Γg,

$$f|_{\Gamma g}(z) = f(\Gamma g.z) = f(g.z) = f|_g(z)$$

So we need only consider the right cosets $\Gamma\backslash G$.

Lemma 7.1. *For G_n, we have the right coset decomposition*

$$G_n = \bigcup_{g \in \Delta_n} \Gamma g,$$

where Δ_n is the set of representative elements given by

$$\Delta_n = \{ \begin{pmatrix} a & b \\ & d \end{pmatrix}, \quad ad = n, 0 \leq b < d\}$$

Proof. For any $\rho = \begin{pmatrix} a & c \\ & * \end{pmatrix} \in G_n$, and $\gamma = \begin{pmatrix} * & * \\ \tau & \delta \end{pmatrix} \in \Gamma$, i.e. $(\tau, \delta) = 1$

$$\gamma\rho = \begin{pmatrix} a & * \\ \tau a + \delta c & * \end{pmatrix}$$

Note that $ax + cy = 0$ always have solutions $(x_0, y_0) \in \mathbb{Z}^2 - \{0, 0\}$. By dividing the greatest common divisor (x_0, y_0), we can assume that they are coprime. Thus on taking $\tau = x_0$ and $\delta = y_0$, we have $(\tau, \delta) = 1$. So there exists such $\gamma = \begin{pmatrix} * & * \\ \tau & \delta \end{pmatrix}$ so that

$$\gamma\rho = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

So we consider the representative elements in $\{\begin{pmatrix} a & * \\ & d \end{pmatrix}, \quad ad = n\}$ By multiplying $\pm I \in \Gamma$, we can assume that $a > 0$ and $d > 0$.

By multiplying $\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$, with $m \in \mathbb{Z}$

$$\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ & d \end{pmatrix} = \begin{pmatrix} a & b + md \\ & d \end{pmatrix}$$

and we can assume that $0 \leq b_1 = b_m d \leq d - 1$. This shows the final result.

$\square$

Lemma 7.2. *There exists an $1 - 1$ correspondence between*

$$\Delta_n \times \Gamma = \Gamma \times \Delta_n$$

i.e. for any $\rho, \gamma \in \Gamma$, there exists unique ρ' and γ' so that

$$\rho.\gamma = \gamma'.\rho$$

Remark 12. Although elements $\rho \in \Delta_n$ is not in the normalizer of Γ,

$$g.\Gamma = \Gamma.g$$

But all the set should be. This suggests us to define

$$T_n f(z) := \sum_{g \in \Delta_n} f|_g(z)$$

Then for any $\gamma \in \Gamma$,

$$\begin{aligned} (T_n f)(\gamma.z) &= \sum_{g \in \Delta_n} f_g(\gamma z) = \sum_{g \in \Delta_n} f(g\gamma z) = \sum_{g \in \Delta_n} f(\gamma' g' z) \\ &= \sum_{g' \in \Delta_n} f(g' z) = (T_n f)(z). \end{aligned}$$

So T_n maps automorphic forms to be automorphic forms.

7.3. **Definition of the Hecke operators.**

Proposition 7.3. *For* $G_n = \{g \in M_2(\mathbb{Z}), \det g = n\}$, *we have*

$$\Gamma G_n = G_n \Gamma.$$

A set of representative elements of the right cosests $\Gamma \backslash G_n$ *is*

$$\Delta_n = \left\{ \begin{pmatrix} a & b \\ & d \end{pmatrix}, \quad ad = n, 0 \le b < d \right\}$$

We define

$$(T_n f)(z) = \sum_{g \in \Delta_n} f|_g(z)$$

- T_n *commutes to each other for all* $n \ge 1$, *and* T_n *commutes with* Δ *and* ι.
- T_n *maps automorphic forms to be automorphic forms.*

Remark 13. Thus we can assume an automorphic cuspidal forms are eigenfunction of Δ with eigenvalues $\frac{1}{4} + t^2$, even or odd, and is eigenfunctions for all Hecke operators.

7.4. **The action of Hecke operators.** . Let

$$f(z) = \sum_{n \neq 0} a_f(n) \sqrt{y} K_{it}(2\pi |n| y) e(nx)$$

be a Maass cusp form with spectral parameter $\frac{1}{2} + it$. Assume that $f(z)$ is an eigenfunction of T_m with eigenvalue $\lambda_f(m)$,

$$T_m f(z) = \lambda_f(m) f(z) = \sum_{n \neq 0} a_f(n) \lambda_f(m) \sqrt{y} K_{it}(2\pi |n| y) e(nx).$$

On the other hand,

$$T_m f(z) = \sum_{ad=m} \sum_{b \mod d} f|_{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}}(z)$$

Note that

$$f|_{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}}(z) = f\left(\frac{az+b}{d}\right) = \sum_{n \neq 0} a_f(n) \lambda_f(m) \sqrt{\frac{a}{d} y} K_{it}(2\pi |n| \frac{a}{d} y) e(n \frac{ax+b}{d}).$$

Thus $T_m f(z)$ has Fourier expansion

$$\begin{aligned} T_m f(z) &= \sum_{ad=m} \sum_{b \mod d} \sum_{n \neq 0} a_f(n) \sqrt{\frac{a}{d} y} K_{it}(2\pi |n| \frac{a}{d} y) e(n \frac{ax+b}{d}) \\ &= \sum_{ad=m} \sum_{n \neq 0} a_f(n) \sqrt{\frac{a}{d} y} K_{it}(2\pi |n| \frac{a}{d} y) e(n \frac{ax}{d}) \sum_{b \mod d} e\left(n \frac{b}{d}\right) \end{aligned}$$

Note that $\sum_{b \bmod d} e\left(n\frac{b}{d}\right) = d\delta_{d|n}$. One has

$$\begin{aligned} T_m f(z) &= \sum_{ad=m}\sum_{d|n} a_f(n)\sqrt{\frac{a}{d}y}K_{it}(2\pi|n|\frac{a}{d}y)e(n\frac{ax}{d})d \\ &= \sqrt{m}\sum_{ad=m}\sum_{d|n} a_f(n)\sqrt{y}K_{it}(2\pi|n|\frac{a}{d}y)e(n\frac{ax}{d}) \\ &= \sqrt{m}\sum_{ad=m}\sum_{\ell} a_f(d\ell)\sqrt{y}K_{it}(2\pi|d\ell|\frac{a}{d}y)e(d\ell\frac{ax}{d}) \\ &= \sqrt{m}\sum_{a|m}\sum_{\ell} a_f(\frac{m}{a}\ell)\sqrt{y}K_{it}(2\pi|\ell|ay)e(\ell ax) \end{aligned}$$

Let $n = \ell a$, then $n \geq 1$ and a satisfies the condition $a \mid n$ and $a \mid m$. Thus

$$T_m f(z) = \sqrt{m}\sum_{n}\sum_{a|(m,n)} a_f\left(\frac{mn}{a}\right)\sqrt{y}K_{it}(2\pi|n|y)e(nx)$$

Proposition 7.4. *Define*

$$T_m f(z) = \frac{1}{\sqrt{m}}\sum_{ad=m}\sum_{b \bmod d} f|_{\begin{pmatrix} a & b \\ & d\end{pmatrix}}(z)$$

Assume $f(z)$ has Fourier expansion

$$f(z) = \sum_n a_f(n)\sqrt{y}K_{it}(2\pi|n|y)e(nx)$$

and f is eigenfunction of T_m with eigenvalue $T_m f(z) = \lambda_f(m)f(z)$. Then

$$\lambda_f(m)a_f(n) = \sum_{d|(m,n)} a_f\left(\frac{mn}{d^2}\right) \tag{7.12}$$

Remark 14. Assume that f is eigenfunction of all Hecke operators. One has

$$\lambda_f(n)a_f(\pm 1) = \sum_{d|(1,n)} a_f\left(\frac{\pm 1n}{d}\right) = a_f(\pm n) \tag{7.13}$$

So we can write $a_f(n) = a_f(1)\lambda_f(n)$ and thus the Fourier expansion of $f(z)$ is

$$f(z) = \sum_{n\neq 0} a_f(\mathrm{sign}(n))\lambda_f(n)\sqrt{y}K_{it}(2\pi|n|y)e(nx).$$

7.5. **Properties of Hecke eigenvalues.** Assume that f is eigenfunction of all Hecke operators T_m with eigenvalues $\lambda_f(m)$. By (7.12) and (7.13), we have the following Hecke relation.

$$\lambda_f(m_1)\lambda_f(m_2) = \sum_{d|(m_1,m_2)} \lambda\left(\frac{m_1 m_2}{d^2}\right) \tag{7.14}$$

Thus $m \mapsto \lambda_f(m)$ is a multiplicative function with $\lambda_f(1) = 1$.

By the Hecke relation, we have the recurrent formula

$$\lambda_f(p^n)\lambda_f(p) = \lambda_f(p^{n+1}) + \lambda_f(p^{n-1})$$

and thus for $\lambda_f(1) = 1$ and $\lambda_f(p)$,

$$\begin{aligned}
\lambda_f(p^2) &= -\lambda_f(1) + \lambda_f(p)^2 = -1 + \lambda_f(p)^2, \\
\lambda_f(p^3) &= -\lambda_f(p) + \lambda_f(p)\lambda_f(p^2) = -2\lambda_f(p) + \lambda_f(p)^3 \\
\lambda_f(p^4) &= -\lambda_f(p^2) + \lambda_f(p)\lambda_f(p^3) = -\lambda_f(1) + \lambda_f(p)^2 + \lambda_f(p)\left(-2\lambda_f(p) + \lambda_f(p)^3\right) \\
&= -1 - \lambda_f(p)^2 + \lambda_f(p)^4 \\
\lambda_f(p^5) &= -\lambda_f(p^3) + \lambda_f(p)\lambda_f(p^4) = -(-2\lambda_f(p) + \lambda_f(p)^3) + \lambda_f(p)(-1 - \lambda_f(p)^2 + \lambda_f(p)^4) \\
&= -\lambda_f(p)
\end{aligned}$$

8. Eisenstein series

We recall the definition of the Maass forms as follows. For

$$f : \mathfrak{h} \to \mathbb{C}$$

1. f is eigenfunction of Δ with eigenvalue $\lambda = s(1-s) = \frac{1}{4} + t^2$, $s = \frac{1}{2} + it$.
2. $f(\gamma.z) = f(z)$ for $\gamma \in SL_2(\mathbb{Z})$.
3. $f \in L^2(\Gamma\backslash\mathfrak{h})$

3'. f is of moderate growth, i.e. $f(x+iy) = o(e^{2\pi y})$ for some N.

We call f is cusp form, if

$$\int_0^1 f(x+iy)dx = 0$$

for all but finite number of y, and we know that

$$\begin{aligned}
& f \text{ cusp form} \Leftrightarrow f \text{ vanishes at the cusp} \\
\Leftrightarrow\ & \text{As functions on } SL_2(\mathbb{R}) = N(\mathbb{R})A(\mathbb{R})SO(2),\ f \text{ vanishes on } N(\mathbb{Z}\backslash\mathbb{R}).
\end{aligned}$$

We will introduce the Eisenstein series which is related to the spectrum of

$$L^2(\Gamma\backslash\mathfrak{h}) - L^2_{\text{cusp}}(\Gamma\backslash\mathfrak{h})$$

8.1. **Definition.** We start from the function

$$I_s(z) := (\text{Im} z)^s$$

which is eigenfunction of Δ with eigenvalue $s(1-s)$ and is Γ_∞-invariant. To construct function which is invariant under Γ_∞, we define

$$E(z,s) := \sum_{\delta \in \Gamma_\infty\backslash\Gamma} I_s(\delta.z).$$

Formally, $E(z,s)$ is an automorphic forms in z with eigenvalue $s(1-s)$ of Δ. However, the sum over $\Gamma_\infty\backslash\Gamma$ is an infinite sum, and $E(z,s)$ may be divergent.

Lemma 8.1. *A set of the representative elements for $\Gamma_\infty\backslash\Gamma$ is*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} * & * \\ c & d \end{pmatrix}, c > 0, d \in \mathbb{Z}, (c,d) = 1.$$

Proof. Note that for given $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

$$I_s\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}.z\right) = \frac{y^s}{|cz+d|^{2s}}$$

so we need only to determine $\begin{pmatrix} * & * \\ c & d \end{pmatrix}$ in the representative element for $\Gamma_\infty \backslash \Gamma$.

Let $\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} \in \Gamma_\infty$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. We have

$$\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-cm & b-dm \\ c & d \end{pmatrix} = \begin{pmatrix} a^* & b^* \\ c & d \end{pmatrix}$$

For fixed $(c,d)=1$, as $m \in \mathbb{Z}$ varies, (a^*, b^*) varies over solutions of

$$xd - yc = 1.$$

So the representative elements in this equivalence class is uniquely characterized by (c,d).

By multiplying $\pm I$, we can assume either $c > 0$, or $c = 0$ and $d = 1$. Thus a representative elements for the coset $\Gamma_\infty \backslash \Gamma$ are

$$\begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} + \bigcup_{c>0} \bigcup_{\substack{d\in\mathbb{Z} \\ (d,c)=1}} \begin{pmatrix} * & * \\ c & d \end{pmatrix}$$

$\square$

By the above lemma, we have formually

$$\begin{aligned} E(z,s) &= \left(\sum_{c=0}\sum_{d=1} + \sum_{c>0}\sum_{\substack{d\in\mathbb{Z} \\ (d,c)=1}}\right)\frac{y^s}{|cz+d|^{2s}} \\ &= \frac{1}{2}\frac{1}{2}\sum_{\substack{(c,d)\in\mathbb{Z}^2-\{(0,0)\} \\ (c,d)=1}} \frac{y^s}{|cz+d|^{2s}}. \end{aligned}$$

Multiplying $\zeta(2s)$, we can get rid of the coprime condition to obtain

$$\begin{aligned} \zeta(2s)E(z,s) &= \sum_{m\geq 1}\frac{1}{m^{2s}}\left(\sum_{c=0}\sum_{d=1} + \sum_{c>0}\sum_{\substack{d\in\mathbb{Z} \\ (d,c)=1}}\right)\frac{y^s}{|cz+d|^{2s}} \\ &= \left(\sum_{c=0}\sum_{d\geq 1} + \sum_{c\geq 1}\sum_{d\in\mathbb{Z}}\right)\frac{y^s}{|cz+d|^{2s}} \\ &= \frac{1}{2}\sum_{(c,d)\neq(0,0)}\frac{y^s}{|cz+d|^{2s}} \end{aligned}$$

Thus for any fixed $z \in \mathfrak{h}$, the infinite sum over c and d are absolutely convergent if $\mathrm{Re}(s) > 1$.

Proposition 8.2. *Let $s \in \mathbb{C}$ be a spectral parameter. The Eisenstein series $E(z,s)$ is defined by*

$$E(z,s) = \sum_{\delta \in \Gamma_\infty \backslash \Gamma} I_s(\delta.z) = \frac{1}{2} \sum_{\substack{(c,d) \in \mathbb{Z}^2 - \{(0,0)\} \\ (c,d)=1}} \frac{y^s}{|cz+d|^{2s}}$$

for $\mathrm{Re}(s) > 1$*; it is an automorphic form for $SL_2(\mathbb{Z})$ in the sense*

$$E(\gamma.z, s) = E(z,s), \quad \gamma \in SL_2(\mathbb{Z})$$

and is an eigenfunction of Δ with eigenvalue

$$\lambda = s(1-s).$$

8.2. **Fourier expansion of Eisenstein series.** For $E(z,s)$ defined as above, we consider the Fourier expansion of $E(z,s)$,

$$E(z,s) = \sum_n a(y,n;s) e(nx)$$

where

$$\begin{aligned} a(y,n;s) : &= \int_0^1 E(x+iy,s) e(-nx) dx \\ &= \int_{x \in \mathbb{Z} \backslash \mathbb{R}} \sum_{\delta \in \Gamma_\infty \backslash \Gamma} I_s(\delta.z) \, e(-nx) dx \end{aligned}$$

We need the following proposition which will give the relation between the right coset $\Gamma_\infty \backslash \Gamma$ and the double cosets $\Gamma_\infty \backslash \Gamma / \Gamma_\infty$.

Lemma 8.3. *The double coset of* $\Gamma_\infty \backslash \Gamma / \Gamma_\infty$ *is*

$$\bigcup_{c \geq 0} \bigcup_{\substack{d \bmod c \\ (d,c)=1}} \begin{pmatrix} * & * \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bigcup \bigcup_{c>0} \bigcup_{\substack{d \bmod c \\ (d,c)=1}} \begin{pmatrix} * & * \\ c & d \end{pmatrix}$$

and thus the representative elements in the right coset $\Gamma_\infty \backslash \Gamma$ can be expressed as

$$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix} + \bigcup_{c>0} \bigcup_{\substack{d \in \mathbb{Z} \\ (c,d)=1}} \bigcup_{m \in \mathbb{Z}} \begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} \tag{8.15}$$

Remark 15. Formula (8.15) can be obtained directly from Lemma 8.1.

Proof. For $\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}$ in Γ_∞ (multiplying $\pm I$ if necessary), and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$,

$$\begin{pmatrix} 1 & m \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} = \begin{pmatrix} a+mc & b+md \\ c & d \end{pmatrix} \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix} = \begin{pmatrix} a+mc & b+md+n(a+mc) \\ c & d+nc \end{pmatrix}$$

Multiplying $\pm I$ if necessary, we can assume that $c \geq 0$ and $d \geq 0$.

Note that $c \geq 0$, and d is restricted in the reduced class modulo c. elements in the first row are determined by the same reason in the right coset $\Gamma_\infty \backslash \Gamma$. □

By the above lemma, for $\mathrm{Re}(s) > 1$,

$$\begin{aligned}
a(y,n;s) &= \int_0^1 I_s(\begin{pmatrix}1 & \\ & 1\end{pmatrix}.z)e(-nx)dx \\
&\quad + \sum_{c\geq 1}\sum_{\substack{d \bmod c \\ (d,c)=1}}\sum_{m\in\mathbb{Z}}\int_0^1 I_s\left(\begin{pmatrix}* & * \\ c & d\end{pmatrix}\begin{pmatrix}1 & m \\ & 1\end{pmatrix}.z\right)e(-nx)dn \\
&= y^s\int_0^1 e(-nz)dx + \sum_{c\geq 1}\sum_{\substack{d \bmod c \\ (d,c)=1}}\sum_{m\in\mathbb{Z}}\int_0^1 I_s\left(\begin{pmatrix}* & * \\ c & d\end{pmatrix}.(x+m+iy)\right)e(-n(x+m))dx \\
&= y^s\delta_{n,0} + \sum_{c\geq 1}\sum_{\substack{d \bmod c \\ (c,d)=1}}\int_{-\infty}^{\infty}\frac{y^s}{((cx+d)^2+c^2y^2)^s}e(-nx)dx \\
&= y^s\delta_{n,0} + \sum_{c\geq 1}\frac{1}{c^{2s}}\sum_{\substack{d \bmod c \\ ((c,d)=1)}} y^s\int_{-\infty}^{\infty}\frac{1}{\left((x+\frac{d}{c})^2+y^2\right)^s}e(-nx)dx \\
&= y^s\delta_{n,0} + \sum_{c\geq 1}\frac{1}{c^{2s}}\sum_{\substack{d \bmod c \\ ((c,d)=1)}} e\left(\frac{d}{c}n\right)y^s\int_{-\infty}^{\infty}\frac{1}{(x^2+y^2)^s}e(-nx)dx
\end{aligned}$$

Lemma 8.4 (Lemma 2.6)**.** *We have*

$$\pi^{-s}\Gamma(s)y^s\int_{-\infty}^{\infty}\frac{1}{(x^2+y^2)^s}e(-nx)dx = \begin{cases}\pi^{-s+\frac{1}{2}}\Gamma(s-\frac{1}{2})y^{1-s}, & n=0\\ 2|n|^{s-\frac{1}{2}}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y), & n\neq 0.\end{cases}$$

Proof. Note that

$$\Gamma(s) = \int_0^\infty e^{-t}t^s\frac{dt}{t}.$$

Thus

$$\begin{aligned}
&\pi^{-s}y^s\int_0^\infty e^{-t}t^s\frac{dt}{t}\int_{-\infty}^{\infty}\frac{1}{(x^2+y^2)^s}e(-nx)dx \\
=&\int_{-\infty}^{\infty}\left(\int_0^\infty\left(\frac{y}{\pi(x^2+y^2)}t\right)^s e^{-t}\frac{dt}{t}\right)dx = \int_{-\infty}^{\infty}\left(\int_0^\infty t^s e^{-t\frac{\pi(x^2+y^2)}{y}}\frac{dt}{t}\right)dx \\
=&\int_0^\infty t^s e^{-t\pi y}\left(\int_{-\infty}^{\infty}e^{-\frac{\pi t}{y}x^2}e^{-2\pi inx}dx\right)\frac{dt}{t}
\end{aligned}$$

Since

$$\int_{-\infty}^{\infty}e^{-\frac{\pi t}{y}x^2}e^{-2\pi inx}dx = \begin{cases}\sqrt{\frac{y}{t}}, & n=0\\ \sqrt{\frac{y}{t}}e^{-\frac{\pi yn^2}{t}}, & n\neq 0\end{cases}$$

The result follows immediately by the expression of Γ-function and K-Bessel function. $\square$

8.2.1. *The constant term.* By the above lemma, the constant term of the normalized Eisenstein series $E^*(z,s)$ is

$$\begin{aligned} a_0^*(y,s) := \pi^{-s}\Gamma(s)\zeta(2s)a(y,0;s) &= \pi^{-s}\Gamma(s)\zeta(2s)y^s + \zeta(2s)\sum_{c\geq 1}\frac{\varphi(c)}{c^{2s}}\pi^{-s+\frac{1}{2}}\Gamma(s-1/2)y^{1-s} \\ &= \pi^{-s}\Gamma(s)\zeta(2s)y^s + \zeta(2s)\sum_{c\geq 1}\frac{\sum_{d|c}\mu(d)\frac{c}{d}}{c^{2s}}\pi^{-s+\frac{1}{2}}\Gamma(s-1/2)y^{1-s} \end{aligned}$$

Note that

$$\sum_{c\geq 1}\frac{\sum_{d|c}\mu(d)\frac{c}{d}}{c^{2s}} = \sum_{c\geq 1}\frac{\mu(d)}{c^{2s}}\sum_{c\geq 1}\frac{c}{c^{2s}},$$

thus

$$a_0^*(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{-s+\frac{1}{2}}\Gamma(s-\frac{1}{2})\zeta(2s-1)y^{1-s}$$

and

$$\zeta(2s-1) = \frac{\pi^{-\frac{1-(2s-1)}{2}}\Gamma\left(\frac{1-(2s-1)}{2}\right)\zeta(1-(2s-1))}{\pi^{-\frac{2s-1}{2}}\Gamma(\frac{2s-1}{2})}$$

Thus finally, the constant of the Eisenstein series is

$$a_0^*(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{-(1-s)}\Gamma(1-s)\zeta(2-2s)y^{1-s}.$$

Here $s=\frac{1}{2}$ is a possible pole of $a_0^*(y,s)$, with

$$\mathrm{Res}_{s=\frac{1}{2}}a_0^*(y,s) = \pi^{-\frac{1}{2}}\Gamma(1/2)y^{1/2}\frac{1}{2} + \pi^{-\frac{1}{2}}\Gamma(1/2)y^{1/2}\frac{-1}{2} = 0,$$

i.e. $s=\frac{1}{2}$ is not a pole; $s=1$ is a simple pole with residue

$$\mathrm{Res}_{s=1}a_0^*(y,s) = \frac{1}{2}, \quad \mathrm{Res}_{s=0}a_0^*(y,s) = -\frac{1}{2}$$

Proposition 8.5. *The constant term of $E^*(z,s)$ is*

$$a_0^*(y,s) = \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{-(1-s)}\Gamma(1-s)\zeta(2-2s)y^{1-s},$$

it has meromorphic continuation for $s\in\mathbb{C}$ with simple poles at $s=1$ and $s=0$ with the residue

$$\mathrm{Res}_{s=1}a_0^*(y,s) = \frac{1}{2}, \quad \mathrm{Res}_{s=0}a_0^*(y,s) = -\frac{1}{2}.$$

Moreover,

$$a_0^*(y,s) = a_0^*(y,1-s).$$

8.2.2. *The non-constant term.* Next, we consider the non-constant term. For $n \neq 0$,

$$\begin{aligned} a_n^*(y,s) : &= \pi^{-s}\Gamma(s)\zeta(2s)\int_0^1 E(z,s)e(-nx)dx \\ &= \zeta(2s)\sum_{c\geq 1}\frac{1}{c^{2s}}\sum_{\substack{d \bmod c \\ (d,c)=1}} e\left(\frac{d}{c}n\right)2|n|^{s-\frac{1}{2}}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y). \end{aligned}$$

Note that

$$\sum_{\substack{d \bmod c \\ (c,d)=1}} e\left(\frac{dn}{c}\right) = S(0,n;c) = \sum_{\delta|(c,n)}\mu(\frac{c}{\delta})\delta,$$

is the Ramanujan sum, and

$$\sum_{c\geq 1}\frac{1}{c^{2s}}S(0,n;c) = \sum_{\delta|n}\delta\sum_{c\geq 1}\mu(c)\frac{1}{(c\delta)^{2s}} = \zeta(2s)^{-1}\sigma_{1-2s}(n).$$

Thus

$$a_n^*(y;s) = 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y).$$

One has

$$K_{s-\frac{1}{2}} = K_{\frac{1}{2}-s} = K_{1-s-\frac{1}{2}}$$

and

$$\begin{aligned} |n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|) &= \sum_{d||n|}\left(\frac{|n|}{d^2}\right)^{s-\frac{1}{2}} = \sum_{d_1d_2=|n|} d_1^{s-\frac{1}{2}}d_2^{\frac{1}{2}-s} \\ &= |n|^{1-s-\frac{1}{2}}\sigma_{1-2(1-s)}(|n|). \end{aligned}$$

This gives the following result.

Proposition 8.6. *The non-constant term*

$$a_n^*(y;s) = 2|n|^{s-\frac{1}{2}}\sigma_{1-2s}(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)$$

has analytic continuation for $s \in \mathbb{C}$ and satisfies

$$a_n^*(y;s) = a_n^*(y;1-s).$$

8.2.3. *Conclusion.*

Theorem 8.7. *For $E^*(z,s) = \pi^{-s}\Gamma(s)\zeta(2s)E(z,s)$, we have*

$$\begin{aligned} E^*(z,s) &= \pi^{-s}\Gamma(s)\zeta(2s)y^s + \pi^{-(1-s)}\Gamma(1-s)\zeta(2-2s)y^{1-s} \\ &\quad +2\sum_{n\neq 0}|n|^{s-\frac{1}{2}}\sigma_{1-2s}(|n|)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e(nx). \end{aligned}$$

$E^*(z,s)$ is defined for $\mathrm{Re}(s) > 1$ and has meromorphic continuation to $s \in \mathbb{C}$ and satisfies the functional equation

$$E^*(z,s) = E^*(z,1-s).$$

Moreover, $s=1$ and $s=0$ are two simple pole of $E^*(z,s)$, and the resiue at $s=1$ is the constant function (in z)

$$\mathrm{Res}_{s=1}E^*(z,s)=\frac{1}{2},$$

and

$$E^*(x+iy,s)=O\left(y^{\max\mathrm{Re}(s),1-\mathrm{Re}(s)}\right),\quad y\to\infty.$$

8.3. **Orthogonal relation with cusp forms.** For $E(z,s)$, we know that

$$\overline{E(z,s)}=\sum_{\delta\in\Gamma_\infty\backslash\Gamma}I_{\overline{s}}(\delta.z)=E(z,\overline{s}).$$

For $f\in L^2_{cusp}$,

$$\begin{aligned}\langle f,E(,s)\rangle &= \int_{\Gamma\backslash\mathfrak{h}}f(z)\overline{E(z,s)}d\mu(z)\overset{\text{unfold}}{=}\int_{\Gamma\backslash\mathfrak{h}}\sum_{\delta\in\Gamma_\infty\backslash\Gamma}f(\delta.z)I_s(\delta.z)d\mu(\delta.z)\\ &= \int_{\Gamma_\infty\backslash\mathfrak{h}}f(z)I_s(z)d\mu(z)=\int_0^\infty y^s\left(\int_0^1 f(x+iy)dx\right)\frac{dy}{y^2}\\ &= 0.\end{aligned}$$

Remark 16. We refer to section F for an overview of Eisenstein series in representation language.

8.4. **Inner product with Eisenstein series.** For any $h\in L^2(\Gamma\backslash\mathfrak{h})$ with the Fourier expansion

$$h(z)=a_{h,0}(y)+\sum_{n\neq0}a_{n,h}(y)e(nx),$$

and thus

$$\begin{aligned}\langle h,E^*(\ ,\overline{s})\rangle &= \int_{\Gamma\backslash\mathfrak{h}}h(z)E(z,s)d\mu(z)\overset{unfold}{=}\int_{\Gamma_\infty\backslash\mathfrak{h}}h(z)I_s(z)d\mu(z)\\ &= \int_0^\infty\left(\int_0^1 h(x+iy)dx\right)y^s\frac{dy}{y^2}\\ &= \int_0^\infty a_{h,0}(y)y^{s-1}\frac{dy}{y}\end{aligned}$$

Thus the inner product of $h\in L^2(\Gamma\backslash\mathfrak{h})$ with Eisenstein series is just the Mellin transform of the constant term $a_{h,0}(y)$ of $h(z)$. In the next section, we use this fact to derive the analytic property of the Rankin-Selberg L-function.

8.5. **Application of Eisenstein series -Rankin-Selberg integrals.** We start from two cusp forms, f and g with

$$\begin{aligned}f(z) &= \sum_{n\neq0}a_f(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e(nx),\\ g(z) &= \sum_{n\neq0}a_g(n)\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y)e(nx),\end{aligned}$$

Note that $f(z)E(z,s)$ and $g(z)$ are both vanishes at the cusp $i\infty$ and thus

$$I(s,f,g) = \langle fE(*,s), g\rangle = \int_{\Gamma\backslash\mathfrak{h}} f(z)\overline{g(z)}E(z,s)d\mu(z)$$

are well defined.

For $\mathrm{Re}(s) > 1$, by unfolding Eisenstein series we have

$$\begin{aligned} I(s,f,g) &= \int_{\Gamma\backslash\mathfrak{h}} \sum_{\delta\in\Gamma_\infty\backslash\Gamma} f(\delta.z)\overline{g(\delta.z)}I_s(\delta.z)d\mu(\delta.z) \\ &= \int_0^1\int_0^\infty f(z)\overline{g(z)}y^s dx\frac{dy}{y^2}. \end{aligned}$$

Applying the Fourier expansion of $f(z)$ and $g(z)$,

$$\begin{aligned} I(s,f,g) &= \sum_{m\neq 0}\sum_{n\neq 0} a_g(m)\overline{a_f(n)}\left\{\int_0^\infty K_{it_f}(2\pi|m|y)K_{it_g}(2\pi|n|y)y^{s+1}\frac{dy}{y^2}\right\}\int_0^1 e((m-n)x)dx \\ &= \left(a_f(1)\overline{a_g(1)}+a_f(-1)\overline{a_g(-1)}\right)\sum_{m>0}\lambda_f(m)\overline{\lambda_g(m)}\int_0^\infty K_{it_f}(2\pi|m|y)K_{it_g}(2\pi|n|y)y^s\frac{dy}{y}. \end{aligned}$$

Lemma 8.8. *We have*

$$\int_0^\infty K_\mu(y)K_\nu(y)y^s\frac{dy}{y} = 2^{s-3}\frac{\Gamma\left(\frac{s-\mu-\nu}{2}\right)\Gamma\left(\frac{s-\mu+\nu}{2}\right)\Gamma\left(\frac{s+\mu-\nu}{2}\right)\Gamma\left(\frac{s+\mu+\nu}{2}\right)}{\Gamma(s)}$$

Proof. Recall (1.25),

$$K_s(y) = \frac{1}{2}\int_0^\infty e^{-\frac{y}{2}(t+\frac{1}{t})}t^s\frac{dt}{t}.$$

Thus

$$I = 2^{-2}\int_0^\infty\int_0^\infty\int_0^\infty e^{-\frac{y}{2}(t_1+\frac{1}{t_1})}e^{-\frac{y}{2}(t_2+\frac{1}{t_2})}t_1^\mu t_2^\nu\frac{dt_1}{t_1}\frac{dt_2}{t_2}y^s\frac{dy}{y}.$$

Changing variable in a suitable situation, we will obtain the result. ☐

Therefore,

$$\begin{aligned} I(s,f,g) &= \left(a_f(1)\overline{a_g(1)}+a_f(-1)\overline{a_g(-1)}\right)\sum_{m>0}\lambda_f(m)\overline{\lambda_g(m)}(2\pi m)^{-s}\int_0^\infty K_{it_f}(y)K_{it_g}(y)y^s\frac{dy}{y} \\ &= \left(a_f(1)\overline{a_g(1)}+a_f(-1)\overline{a_g(-1)}\right)(2\pi)^{-s}\sum_{m>0}\frac{\lambda_f(m)\overline{\lambda_g(m)}}{m^s}2^{s-3}\prod_{\epsilon_f,\epsilon_g\in\{\pm1\}}\frac{\Gamma\left(\frac{s+\epsilon_f it_f+\epsilon_g it_g}{2}\right)}{\Gamma(s)} \end{aligned}$$

and

$$\begin{aligned} I^*(s,f,g) &= \pi^{-s}\Gamma(s)\zeta(2s)I(s,f,g) \\ &= \frac{\left(a_f(1)\overline{a_g(1)}+a_f(-1)\overline{a_g(-1)}\right)}{8}\pi^{-2s}\prod_{\epsilon_f,\epsilon_g\in\{\pm1\}}\Gamma\left(\frac{s+\epsilon_f it_f+\epsilon_g it_g}{2}\right)L(s,f\otimes g) \end{aligned}$$

where

$$L(s, f \otimes g) = \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n)\overline{\lambda_g(n)}}{n^s}$$

is called the Rankin-Selberg convolution L-functions. We also denote by

$$\Lambda(s, f \otimes g) = \pi^{-2s} \prod_{\pm,\pm} \Gamma\left(\frac{s \pm it_f \pm it_g}{2}\right) L(s, f \otimes g)$$

as the complete L-function and thus

$$I(s, f, g) = \frac{a_f(1)\overline{a_g(1)}}{4} \Lambda(s, f \otimes g).$$

for f and g both even or odd.

On the other hand, Recall that

$$E^*(z, s) = E^*(z, 1 - s),$$

and $s = 1$ is a simple pole of $E^*(z, s)$ with residue $\frac{1}{2}$, we have

$$I^*(s, f, g) = I^*(1 - s, f, g),$$

which gives the functional equation

$$\Lambda(s, f \otimes g) = \Lambda(1 - s, f \otimes g),$$

and

$$\begin{aligned}
\mathrm{Res}_{s=1} I^*(z, f, g) &= \frac{a_f(1)\overline{a_g(1)}}{4} \mathrm{Res}_{s=1} \Lambda(s, f \otimes g) \\
&= \frac{1}{2}\langle f, g\rangle.
\end{aligned}$$

Lemma 8.9. *For f and g be two normalized Maass cusp form,*

$$\langle f, g\rangle = \begin{cases} \|f\|^2, & f = g \\ 0, & f \neq g \end{cases}$$

Proof. Note that we choose f and g be orthogonal basis of L^2_{cusp}, it is obviously. $\square$

Proposition 8.10. *Let $f, g \in \mathcal{B}_{cusp}$ be two even or odd maass cusp forms. We have*

$$I^*(s, f, g) := \int_{\Gamma\backslash\mathfrak{h}} f(z)\overline{g(z)} E(z, s) d\mu(z) = \frac{a_f(1)\overline{a_g(1)}}{4} \Lambda(s, f \otimes g),$$

where

$$\Lambda(s, f \otimes g) = \pi^{-2s} \prod_{\pm} \prod_{\pm} \Gamma\left(\frac{s \pm it_f \pm it_g}{2}\right) L(s, f \otimes g)$$

with

$$L(s, f \otimes g) = \zeta(2s) \sum_{n \geq 1} \frac{\lambda_f(n)\overline{\lambda_g(n)}}{n^s}.$$

Then $\Lambda(s, f\otimes g)$ has analytic continuation for $s\in\mathbb{C}$ except for a possible simple pole at $s=1$ and $s=0$ if $f=g$, in which case

$$\mathrm{Res}_{s=1}\Lambda(s,f\otimes f)=\frac{2\langle f,f\rangle}{|a_f(1)|^2}=\begin{cases}\dfrac{2}{|a_f(1)|^2}, & \textit{if we normaliz } f \textit{ to be orthornormal basis, i.e. } \langle f,f\rangle=1\\ 2\langle f,f\rangle & \textit{if we normaliz } f \textit{ to be } a_f(1)=1.\end{cases}$$

Remark 17 (Real coefficients)**.** Note that $\lambda_f(n)$ are eigen values of the Hecke operators, and the Hecke operators are self-dual and thus $\lambda_f(n)$ are real!

8.6. **Euler products of Rankin-Selberg L-functions.** Note that

$$\begin{aligned} L(s,f) &= \sum_{n\geq 1}\frac{\lambda_f(n)}{n^s}=\prod_p(1-\alpha_1(p)p^{-s})^{-1}(1-\alpha_2(p)p^{-s})^{-1}\\ L(s,g) &= \sum_{n\geq 1}\frac{\lambda_g(n)}{n^s}=\prod_p(1-\beta_1(p)p^{-s})^{-1}(1-\beta_2(p)p^{-s})^{-1}\end{aligned}$$

Lemma 8.11 (Lemma 1.6.1 in Bump)**.** *If*

$$\begin{aligned}\sum_{r=0}^{\infty}A(r)x^r &= (1-\alpha_1x)^{-1}(1-\alpha_2x)^{-1}\\ \sum_{r=0}^{\infty}B(r)x^r &= (1-\beta_1x)^{-1}(1-\beta_2x)^{-1}\end{aligned}$$

then

$$\sum_{r=0}^{\infty}A(r)B(r)x^r=\left(1-\alpha_1\alpha_2\beta_1\beta_2x^2\right)\prod_{i,j=1}^{2}(1-\alpha_i\beta_jx)^{-1}.$$

By the above lemma, we have

$$L(s,f\otimes g)=\prod_p\prod_{i=1}^{2}\prod_{j=1}^{2}\left(1-\alpha_i(p)\beta_j(p)p^{-s}\right)^{-1}$$

Specially, if $f=g$,

$$\begin{aligned} L(s,f\otimes f) &= \prod_p\prod_{i,j=1}^{2}\left(1-\alpha_i(p)\overline{\alpha}_j(p)p^{-s}\right)\\ &= \zeta(s)\prod_p(1-\alpha_1^2(p)p^{-s})^{-1}(1-p^{-s})^{-1}(1-\alpha_2^2p^{-s})^{-1}\\ &= \zeta(s)L(s,sym^2f).\end{aligned}$$

and

$$\mathrm{Res}_{s=1}L(s,f\otimes f)=L(1,sym^2f).$$

The symmetric square L-function is another story.

9. Poincare series 1 - General definition

The Eisenstein series is constructed as follows. We start from the function

$$\tilde{h}_0(z) := I_s(z)$$

which is an eigenfunction of Δ, invariant under Γ_∞, and then construct

$$E(z,s) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f_0(\gamma.z)$$

The most important property is that the inner product of Eisenstein series and automorphic forms involves the constant term in the Fourier expansion of f,

$$\langle f, E(z,\overline{s})\rangle = \int_0^1 \int_0^\infty f(z) h_0(z) \frac{dxdy}{y^2} = \int_0^\infty \left(\int_0^1 f(x+iy) e_n(x) dx \right) y^s \frac{dy}{y^2}$$

9.1. **Poincare series - definition.**

Following the idea above, we can construct a lot of automorphic forms as follows.

- We consider Γ_∞-functions. Set

  $$\tilde{h}(z) := h(\mathrm{Im} z) e(m \mathrm{Re} z)$$

  with $m \in \mathbb{Z}$ and $f \in C_c^\infty((0,\infty))$. It is naturally a function which is invariant under Γ_∞. As $m \in \mathbb{Z}$ and $h \in C_c^\infty((0,\infty))$ varies, it varies over allmost all these functions.
- We construct

  $$\sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(\mathrm{Im}\delta.z) e(m \mathrm{Re}\delta z)$$

- Note that $e(mz) = e(mx)e^{-2\pi y}$. Instead of the above defintion, Kuznetsov us

  $$P_m(z,h) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} h(\mathrm{Im}\delta.z) e(mz)$$

  which is called the Poincare series. It is well-defined by the following lemma.

Lemma 9.1. *Let $T > 0$. Let z be in the fundamental domain. The number*

$$\{\gamma \in \Gamma_\infty \backslash \Gamma, \quad \mathrm{Im}\delta.z > T\}$$

is finite.

Proof. Recall that

$$\Gamma_\infty \backslash \Gamma = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix} \bigcup \{ \begin{pmatrix} * & * \\ c & d \end{pmatrix} \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}, \quad c > 0, b \bmod c, (b,c) = 1, m \in \mathbb{Z} \}$$

Note that for fixed $z = x_0 + iy_0 \in \mathcal{F}$ and $\delta = \begin{pmatrix} & \\ c & d \end{pmatrix} \begin{pmatrix} 1 & m \\ & 1 \end{pmatrix}$ be a representative element in the coset $\Gamma_\infty \backslash \Gamma$,

$$\mathrm{Im}\delta.z = \frac{y_0}{|c(x_0 + m + iy_0) + d|^2} > T \leftrightarrow (cx_0 + m + d)^2 + c^2 y_0^2 < \frac{y_0}{T}$$

Obviously there are only finite number choice of such pair $c > 0$, and hence $d \bmod c$ with $(d,c) = 1$, and hence m.

$\square$

9.2. **Poncare series - Inner product with automorphic forms.** Now,

$$\begin{aligned}
\langle f, P_m(z,h)\rangle &= \int_{\Gamma_\infty\backslash\Gamma} f(z)\overline{P_m(z,h)}\frac{dxdy}{y^2} \\
&= \int_0^\infty \overline{h(y)}e^{-2\pi my}\left(\int_0^\infty f(x+iy)e(-mx)dx\right)\frac{dy}{y^2} \\
&= \int_0^\infty \overline{h(y)}e^{-2m\pi y}a_f(m,y)\frac{dy}{y^2} \qquad (9.16)
\end{aligned}$$

where $a_f(m,y)$ is the m-th Fourier coefficients of f.

Note that for $f\in L^2(\Gamma\backslash\mathfrak{h})$, $f(z)=\sum_n a_f(n,y)e(nx)$, we know $f\equiv 0$ iff $a_f(n,y)=0$ for all n. The inner product of f with $P_m(z,h)$ implies that

$$\{P_m(z,h),\quad m\in\mathbb{Z}, h\in C_c^\infty(\mathfrak{h})\}$$

spans $L^2(\Gamma\backslash\mathfrak{h})$. Especially, for the case $m=0$, i.e.

$$P(z,h):=P_0(z,h)=\sum_{\delta\in\Gamma_\infty\backslash\Gamma}h(\mathrm{Im}(\delta.z)),$$

called Pseudo-Eisenstein series, which are orthogonal to L^2_{cusp} and spans $L^2-L^2_{cusp}$.

Proposition 9.2. *Poincare series $P_m(z,h)$ span L^2, and Pseudo-Eisenstein series $P(z,h)$ span $L^2-L^2_{cusp}$.*

Remark 18. To study L^2_{cusp}, we need to express $P(z,h)$ in terms of sum (integral) over eigenfunctions of Δ.

9.3. **Poincare series - Fourier expansion.** We consider the n-th Fourier coefficients of $P_m(z,h)$, namely

$$a_{m,h}(n)=\int_0^1 P_m(z,h)e(-nx)dx$$

By double coset decomposition in Lemma 8.3, we have

$$\begin{aligned}
a_{m,h}(n) &= \int_0^1 h(\mathrm{Im}(z))e(mz)e(-nx)dx \\
&\quad+\int_0^1\sum_{c\geq 1}\sideset{}{^*}\sum_{d \bmod c}\sum_{\ell\in\mathbb{Z}} h\left(\mathrm{Im}\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&\ell\\&1\end{pmatrix}.z\right)\right)e\left(m\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}1&\ell\\&1\end{pmatrix}.z\right)e(-nx)dx \\
&= h(y)e^{-2\pi my}\delta_{m,n}+\sum_{c\geq 1}\sideset{}{^*}\sum_{d \bmod c}\int_{-\infty}^\infty h\left(\frac{y}{(cx+d)^2+c^2y^2}\right)e\left(m\frac{ax+b+iay}{cx+d+icy}\right)e(-nx)dx \\
&\overset{x+\frac{d}{c}\mapsto x}{=} h(y)e^{-2\pi my}\delta_{m,n}+\sum_{c\geq 1}\sideset{}{^*}\sum_{d \bmod c}\int_{-\infty}^\infty h\left(\frac{y}{c^2x^2+c^2y^2}\right)e\left(m\frac{a(x-\frac{d}{c})+b+iay}{c(x-\frac{d}{c})+d+icy}\right)e(-n(x-\frac{d}{c}))dx \\
&= h(y)e^{-2\pi my}\delta_{m,n}+\sum_{c\geq 1}\sum_{ad\equiv 1 \bmod c}e\left(\frac{am+dn}{c}\right)\int_{-\infty}^\infty h\left(\frac{y}{c^2x^2+c^2y^2}\right)e\left(-\frac{m}{c^2x+ic^2y}-nx\right)dx
\end{aligned}$$

9.4. **A Remark on Mellin transform.** We use the following notation,

$$H(s) := \int_0^\infty h(t)t^{-s}\frac{dt}{t}, \quad h(y) = \frac{1}{2\pi i}H(s)y^s ds$$

which coincides with the notation in Arthur's notes, and is different with the original Mellin transfrom.

10. Pseudo Eisenstein series and the spectrum decomposition

We are interested in $L^2 - L^2_{cusp}$. Note that for $m = 0$,

$$P(z,h) = P_0(z,h) = \sum_{\delta\in\Gamma_\infty\backslash\Gamma} h(\delta.z)$$

which is orthogonal to L^2_{cusp}, called Psudo-Eisenstein series.

10.1. **Relation with Eisenstein series.** Note that $h \in C_c^\infty((0,\infty))$. We need the spectral decomposition of $L^2(0,\infty)$, which is related to Mellin transform, namely

$$H(s) := \langle f, *^s\rangle = \int_0^\infty f(y)y^{-s}\frac{dy}{y}, \quad h(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma} H(s)y^s ds$$

Applying this one has

$$\begin{aligned} P(z,h) &= \sum_{\delta\in\Gamma_\infty\backslash\Gamma} h(\delta.z) = \sum_{\delta\in\Gamma_\infty\backslash\Gamma} \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma} H(s)I_s(\delta.z)ds \\ &= \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\sigma} H(s)E(z,s)ds \end{aligned}$$

10.2. **Constant term.** Psudo Eisenstein series is orthogonal to cusp forms. By the double coset decomposition, its constant term is

$$\begin{aligned} \int_0^1 P(z,h)dx &= h(y) + \sum_{c\geq 1 d} \sum_{\mathrm{mod}\ c}^{*} \int_{-\infty}^{\infty} h\left(\frac{y}{(cx+d)^2 + c^2y^2}\right)dx \\ &= h(y) + \sum_{c\geq 1}\varphi(c)\int_{-\infty}^{\infty} h\left(\frac{y}{c^2(x^2+y^2)}\right)dx \end{aligned}$$

For $h \in C_c^\infty(0,\infty)$, we decompose $h(y)$ as integral (sum) with the power function y^s in y. Recall that for $h(y)$, we have

$$H(s) = \int_0^\infty h(t)t^{-s}\frac{dt}{t}, \quad h(y) = \frac{1}{2\pi i}\int_{(\sigma)} H(s)y^s ds.$$

Thus

$$\int_0^1 P(z,h)dx = h(y) + \int_{(\sigma)} \frac{1}{2\pi i}H(s)\sum_{c\geq 1}\varphi(c)\frac{y^s}{c^{2s}}\left\{\int_{-\infty}^{\infty}\left(\frac{1}{x^2+y^2}\right)^s dx\right\}$$

By lemma 8.4, we have

$$\int_{-\infty}^{\infty}\frac{1}{(x^2+y^2)^s}dx = \pi^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}y^{1-2s}.$$

and hus

$$\sum_{c\geq 1}\varphi(c)\frac{1}{c^{2s}}y^s\int_{-\infty}^{\infty}\frac{1}{(x^2+y^2)^s}dx = y^{1-s}\pi^{1/2}\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{c\geq 1}\frac{\varphi(c)}{c^{2s}}$$
$$= y^{1-s}\frac{\pi^{-(s-\frac{1}{2})}\Gamma(s-\frac{1}{2})\zeta(2s-1)}{\pi^{-s}\Gamma(s)\zeta(2s)} = y^{1-s}\frac{\xi(2s-1)}{\xi(2s)}$$

This gives that

$$\int_0^1 P(z,h)dx = h(y) + \frac{1}{2\pi i}\int_\sigma H(s)y^{1-s}\frac{\xi(2s-1)}{\xi(2s)}ds. \tag{10.17}$$

10.3. **Inner products of Pseudo-Eisenstein series.** Now, we consider

$$\begin{aligned}
\langle P(,h_1),P(,h_2)\rangle &= \int_0^\infty \overline{h_2(y)}\left(h_1(y)+\sum_{c\geq 1}\varphi(c)\int_{-\infty}^{\infty}h_1\left(\frac{y}{c(x^2+y^2)}\right)dx\right)\frac{dy}{y^2}\\
&= \int_0^\infty h_1(y)\overline{h_2(y)}\frac{dy}{y^2} + \int_0^\infty \overline{h_2(y)}\left(\sum_{c\geq 1}\varphi(c)\int_{-\infty}^{\infty}h_1\left(\frac{y}{c(x^2+y^2)}\right)dx\right)\frac{dy}{y^2}.
\end{aligned}$$

As a function in y,

$$y\mapsto \sum_{c\geq 1}\varphi(c)\int_{-\infty}^{\infty}h_1\left(\frac{y}{c(x^2+y^2)}\right)dx$$

is not compactly supported on $(0,\infty)$, To tackle this problem, we apply the Mellin transform (valid for $\mathrm{Re}(s)=\sigma$ large), see (10.17), one has

$$\begin{aligned}
\langle P(,h_1),P(,h_2)\rangle &= \int_0^\infty \overline{h_2(y)}\left(h_1(y)+\frac{1}{2\pi i}\int_\sigma H_1(s)y^{1-s}\frac{\xi(2s-1)}{\xi(2s)}ds\right)\frac{dy}{y^2}\\
&= \int_0^\infty h_1(y)\overline{h_2(y)}\frac{dy}{y^2} + \int_0^\infty \overline{h_2(y)}\frac{1}{2\pi i}\int_\sigma H_1(s)y^{1-s}\frac{\xi(2s-1)}{\xi(2s)}ds\frac{dy}{y^2}\\
&= \int_0^\infty h_1(y)\overline{h_2(y)}\frac{dy}{y^2} + \frac{1}{2\pi i}\int_{(\sigma)}\frac{\xi(2s-1)}{\xi(2s)}H_1(s)\left\{\int_0^\infty \overline{h_2(y)}y^{1-s}\frac{dy}{y^2}\right\}ds\\
&= \int_0^\infty h_1(y)\overline{h_2(y)}\frac{dy}{y^2} + \frac{1}{2\pi i}\int_{(\sigma)}\frac{\xi(2s-1)}{\xi(2s)}H_1(s)\left\{\overline{\int_0^\infty h_2(y)y^{-\overline{s}}\frac{dy}{y}}\right\}ds\\
&= \int_0^\infty h_1(y)\overline{h_2(y)}\frac{dy}{y^2} + \frac{1}{2\pi i}\int_{(\sigma)}\frac{\xi(2s-1)}{\xi(2s)}H_1(s)\overline{H_2(\overline{s})}ds
\end{aligned}$$

Remark 19. Note that the convergence problem are now

$$\frac{\xi(2s-1)}{\xi(2s)}$$

which has meromoprhic continuation now. So we can move the integral line from $\mathrm{Re}(s)=\sigma>1$ to $\mathrm{Re}(s)=1/2$, passing simple pole at $s=1$.

The first term has no convergence problem, and one has

$$\begin{aligned}
\int_0^\infty h_1(y)\overline{h_2(y)}\frac{dy}{y^2} &= \int_0^\infty \left\{\frac{1}{2\pi i}\int_{(\sigma)} H_1(s)y^s ds\right\}\overline{h_2(y)}\frac{dy}{y^2} \\
&= \frac{1}{2\pi i}\int_{(\sigma)} H_1(s)\left\{\overline{\int_0^\infty h_2(y)y^{-(1-\overline{s})}\frac{dy}{y}}\right\}ds \\
&= \frac{1}{2\pi i}\int_{\sigma} H_1(s)\overline{H_2(1-\overline{s})}ds \\
&\overset{\sigma=1/2}{=} \frac{1}{2\pi}\int_{-\infty}^\infty H_1(\frac{1}{2}+it)\overline{H_2(\frac{1}{2}+it)}dt,
\end{aligned}$$

and the second term, we move the line of integration to $\mathrm{Re}(s) = 1/2$, passing a simple pole at $s = 1$ coming from $\zeta(2s-1)$, one has

$$\begin{aligned}
&\frac{1}{2\pi i}\int_{(\sigma)}\frac{\xi(2s-1)}{\xi(2s)}H_1(s)\overline{H_2(\overline{s})}ds \\
=\ &\frac{1}{2\pi}\int_{-\infty}^\infty \frac{\xi(2it)}{\xi(1+2it)}H_1(\frac{1}{2}+it)\overline{H_2(\frac{1}{2}-it)}dt + H_1(1)\overline{H_2(1)}\frac{1}{2}\frac{\pi^{-1/2}\Gamma(1/2)}{\pi^{-1}\Gamma(1)\zeta(2)} \\
=\ &\frac{3}{\pi}H_1(1)\overline{H_2(1)} + \frac{1}{2\pi}\int_{-\infty}^\infty H_1(1/2+it)\overline{H_2(1/2-it)}\frac{\xi(1-2it)}{\xi(1+2it)}dt
\end{aligned}$$

and thus we have

$$\langle P(,h_1), P(,h_2)\rangle = \frac{3}{\pi}H_1(1)\overline{H_2(1)} + \frac{1}{2\pi}\int_{-\infty}^\infty H_1(\frac{1}{2}+it)\overline{\left\{H_2(\frac{1}{2}+it) + \frac{\xi(1+2it)}{\xi(1-2it)}H_2(\frac{1}{2}-it)\right\}}dt.$$

Moreover, note that

$$\begin{aligned}
I &= \frac{1}{2\pi}\int_{-\infty}^\infty H_1(\frac{1}{2}+it)\overline{\left\{H_2(\frac{1}{2}+it) + \frac{\xi(1+2it)}{\xi(1-2it)}H_2(\frac{1}{2}-it)\right\}}dt \\
&\overset{-t\mapsto t}{=} \frac{1}{2\pi}\int_{-\infty}^\infty H_1(\frac{1}{2}-it)\overline{\left\{H_2(\frac{1}{2}-it) + \frac{\xi(1-2it)}{\xi(1+2it)}H_2(\frac{1}{2}+it)\right\}}dt \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty H_1(\frac{1}{2}-it)\overline{\frac{\xi(1-2it)}{\xi(1+2it)}\left\{\frac{\xi(1+2it)}{\xi(1-2it)}H_2(\frac{1}{2}-it) + H_2(\frac{1}{2}+it)\right\}}dt \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty H_1(\frac{1}{2}-it)\frac{\xi(1+2it)}{\xi(1-2it)}\overline{\left\{\frac{\xi(1+2it)}{\xi(1-2it)}H_2(\frac{1}{2}-it) + H_2(\frac{1}{2}+it)\right\}}dt
\end{aligned}$$

and thus

$$I = \frac{1}{4\pi}\int_0^\infty \left(H_1(\frac{1}{2}+it) + \frac{\xi(1+2it)}{\xi(1-2it)}H_1(\frac{1}{2}-it)\right)\overline{\left(H_2(\frac{1}{2}+it) + \frac{\xi(1+2it)}{\xi(1-2it)}H_2(\frac{1}{2}-it)\right)}dt$$

Therefore, finally we have the following proposition.

Proposition 10.1. *We have*

$$\begin{aligned}\langle P(,h_1), P(,h_2)\rangle &= \frac{3}{\pi}H_1(1)\overline{H_2(1)}\\ &+\frac{1}{4\pi}\int_0^\infty \left(H_1(\frac{1}{2}+it)+\frac{\xi(1+2it)}{\xi(1-2it)}H_1(\frac{1}{2}-it)\right)\overline{\left(H_2(\frac{1}{2}+it)+\frac{\xi(1+2it)}{\xi(1-2it)}H_2(\frac{1}{2}-it)\right)}dt.\end{aligned}$$

Recall

$$P(z,h)=\frac{1}{2\pi i}\int_\sigma H_1(s)E(z,s)ds.$$

We have the following.

- Firstly,

$$\begin{aligned}\langle P(z,h),1\rangle &= \int_{\Gamma\backslash\mathfrak{h}} P(z,h)\overline{1}d\mu(z) \overset{unfold}{=} \int_0^\infty h(y)\frac{dy}{y^2}\\ &= \int_0^\infty h(y)y^{-1}\frac{dy}{y} = H(1),\end{aligned}$$

- For $s=\frac{1}{2}+it$,

$$\begin{aligned}\langle P(z,h),E(z,1/2+it)\rangle &= \int_{\Gamma\backslash\mathfrak{h}} P(z,h)E(z,\overline{s})d\mu(z)\\ &\overset{\text{unfold } P(z,h)}{=} \int_0^\infty h(y)\left\{\int_0^1 E(z,\overline{s})dx\right\}\frac{dy}{y^2}\\ &= \int_0^\infty h(y)\left(y^{\overline{s}}+\frac{\xi(2\overline{s}-1)}{\xi(2\overline{s})}y^{1-\overline{s}}\right)\frac{dy}{y^2}\\ &= \int_0^\infty h(y)y^{-(\frac{1}{2}+it)}\frac{dy}{y}+\frac{\xi(-2it)}{\xi(1-2it)}\int_0^\infty h(y)y^{-(\frac{1}{2}-it)}\frac{dy}{y}\\ &= H(\frac{1}{2}+it)+\frac{\xi(1+2it)}{\xi(1-2it)}H(\frac{1}{2}-it)\end{aligned}$$

By the above argument, the main proposition in prop 10.1 can be expressed as the following.

Theorem 10.2. *We have*

$$\begin{aligned}\langle P(,h_1),P(,h_2)\rangle &= \frac{3}{\pi}\langle P(,h_1),1\rangle\overline{\langle P(,h_2),1\rangle}\\ &+\frac{1}{4\pi}\int_{-\infty}^\infty\langle P(,h_1),E(,1/2+it)\rangle\overline{\langle P(,h_2),E(,1/2+it)\rangle}dt.\end{aligned}$$

It gives that

$$P(z,h)=\frac{3}{\pi}\langle P(,h),1\rangle+\frac{1}{4\pi}\int_{-\infty}^\infty\langle P(z,h),E(z,1/2+it)\rangle E(z,1/2+it)dt.$$

10.4. **Main theorem on the spectral decomposition.**

Theorem 10.3. *We have $L^2=L^2_{cusp}+L^2_{res}+L^2_{cont}$, where L^2_{cusp} is the space of cusp forms, L^2_{cont} is the space of continuous spectrum, consisting of $E(z,\frac{1}{2}+it)$, and L^2_{res} is the residue spectrum coming* *from the residue of Eisenstein series. Moreover, $L^2_{cusp}+L^2_{res}=L^2_{disc}$ is the space of discrete spectrum. Given $f\in L^2$, we have*

$$f(z) = \sum_{\varphi\in\mathfrak{B}_{cusp}} \frac{\langle f,\varphi\rangle}{\langle\varphi,\varphi\rangle}+\frac{3}{\pi}\int_{\Gamma\backslash\mathfrak{h}} f(z)d\mu(z)+\frac{1}{4\pi}\int_{-\infty}^{\infty}\langle f,E(*,\frac{1}{2}+it)\rangle E(z,\frac{1}{2}+it)dt.$$

Theorem 10.4. *We have the Parseval identity and the Parseval identity*

$$\begin{aligned}\langle f,g\rangle &= \frac{3}{\pi}\langle f,1\rangle\overline{\langle g,1\rangle}\\ &+\frac{1}{4\pi}+\sum_{\varphi\in\mathcal{B}_{cusp}}\frac{\langle f,\varphi\rangle\overline{\langle g,\varphi\rangle}}{\langle\varphi,\varphi\rangle}\int_{-\infty}^{\infty}\langle f,E(,1/2+it)\rangle\overline{\langle g,E(,1/2+it)\rangle}dt.\end{aligned}\tag{10.18}$$

10.5. **Another way.** Instead of the constant term of the Eisenstein series, we can obtain the spectral decomposition (formally) via the relation between P-series an Eisenstein series an the global propery of $E(z,s)$ as follows.

For $\mathrm{Re}(s)>1$, by the definition of the Eisenstein series, we have

$$P(z,h_1)=\frac{1}{2\pi i}\int_{(\sigma)}H_1(s)E(z,s)ds.$$

Thus

$$\langle P(,h_1),P(,h_2)\rangle=\int_{\Gamma\backslash\mathfrak{h}}\left\{\frac{1}{2\pi i}\int_{\sigma}H_1(s)E(z,s)ds\right\}\overline{P(z,h_2)}dz$$

Moving the line of the integration to $\mathrm{Re}(s)=1/2$, passing a simple pole at $s=1$ with the residue

$$\begin{aligned}&\frac{1}{2\pi^{-1}\Gamma(1)\zeta(2)}\int_{\Gamma\backslash\mathfrak{h}}H_1(1)\overline{P(z,h_2)}d\mu(z)\\ =\ &\frac{3}{\pi}H_1(1)\int_{\Gamma\backslash\mathfrak{h}}\overline{P(z,h_2)}d\mu(z),\end{aligned}$$

one has

$$\langle P(,h_1),P(,h_2)\rangle = H_1(1)\int_{\Gamma\backslash\mathfrak{h}}\frac{3}{\pi}\overline{P(z,h_2)}d\mu(z)+\frac{1}{2\pi}\int_{-\infty}^{\infty}H_1(\frac{1}{2}+it)\int_{\Gamma\backslash\mathfrak{h}}E(z,\frac{1}{2}+it)\overline{P(z,h_2)}d\mu(z)dt.$$

Next, by the functional equation of the Eisenstein series,

$$E(z,s)=\frac{\xi(2s-1)}{\xi(2s)}E(z,1-s),$$

we have

$$
\begin{aligned}
\langle P(,h_1), P(,h_2)\rangle &= H_1(1)\int_{\Gamma\backslash\mathfrak{h}}\frac{3}{\pi}\overline{P(z,h_2)}d\mu(z)\\
&+\frac{1}{4\pi}\int_{-\infty}^{\infty}H_1(\frac{1}{2}+it)\int_{\Gamma\backslash\mathfrak{h}}E(z,\frac{1}{2}+it)\overline{P(z,h_2)}d\mu(z)dt.\\
&+\frac{1}{4\pi}\int_{-\infty}^{\infty}H_1(\frac{1}{2}+it)\frac{\xi(2it)}{\xi(1+2it)}\int_{\Gamma\backslash\mathfrak{h}}E(z,\frac{1}{2}-it)\overline{P(z,h_2)}d\mu(z)dt\\
&= H_1(1)\int_{\Gamma\backslash\mathfrak{h}}\frac{3}{\pi}\overline{P(z,h_2)}d\mu(z)\\
&+\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(H_1(\frac{1}{2}+it)+\frac{\xi(2it)}{\xi(1+2it)}H_1(\frac{1}{2}-it)\right)\int_{\Gamma\backslash\mathfrak{h}}E(z,\frac{1}{2}+it)\overline{P(z,h_2)}d\mu(z)dt.
\end{aligned}
$$

This gives

$$P(z,h)=H_1(1)\frac{3}{\pi}+\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(H_1(\frac{1}{2}+it)+\frac{\xi(2it)}{\xi(1+2it)}H_1(\frac{1}{2}-it)\right)E(z,\frac{1}{2}+it)dt.$$

11. Poincare series and the Kuznetsov's trace formfula

We know that

$$L^2(\Gamma\backslash\mathfrak{h})=L^2_{cusp}\oplus L^2_{res}\oplus L^2_{cont}$$

The space $L^2_{res}\oplus L^2_{cont}$ is clearly. The problem is, how about the space L^2_{cusp}? How to study the basis of L^2_{cusp}?

11.1. Redefine the Poincare series.

Recall that Poincare series

$$P_m(z,h):=\sum_{\delta\in\Gamma_\infty\backslash\Gamma}h(\mathrm{Im}\delta.z)e(m\delta.z)$$

spans $L^2(\Gamma\backslash\mathfrak{h})$. For $m>0$,

$$y\mapsto e^{-2\pi my}$$

is exponential decay as $y\to\infty$, even if we replace $h(y)$ by y^s for any s.

Definition 11.1. We redefine the Poincare series as

$$U_m(z,s):=\sum_{\delta\in\Gamma_\infty\backslash\Gamma}I_s(\delta.z)e(mz)$$

Note that for the case $m=0$, it is Eisenstein series.

Remark 20. At least, the series is absolutely convergent for $\mathrm{Re}(s)>1$ just like the argument as the Eisenstein series. Moreover, by Mellin transform, $h(y)=\frac{1}{2\pi i}\int_{(\sigma)}H(s)y^s\frac{dy}{y}$ and thus

$$P_m(z,h)=\frac{1}{2\pi i}\int_{(\sigma)}H(s)\sum_{\delta\in\Gamma_\infty\backslash\Gamma}\mathrm{Im}(\delta.z)^se(m\delta.z)ds=\frac{1}{2\pi i}\int_{(\sigma)}H(s)U_m(z,s)ds.$$

Following [DeIw1982], we study $U_m(z,s)$.

Firstly, note that for $m \geq 1$, $U_m(z,s) \in L^2(\Gamma\backslash\mathfrak{h})$. It is due to the fact that

$$I_s(z)e(mz) = y^s e^{-2\pi my} e(2\pi imx)$$

which is exponential decay as $y \to \infty$.

11.2. **Fourier coefficients of Poincare series.** We consider the n-th Fourier coefficients of Poincare series,

$$a_{m,s}(n,y) = \int_0^1 U_m(x+iy,s)e(-nx)dx = \int_0^1 \sum_{\delta\in\Gamma_\infty\backslash\Gamma} I_s(\delta.z)e(m\delta.z)e(-nx)dx$$

By the double coset decomposition, we have

$$\begin{aligned}
a_{m,s}(n,y) \quad &= \quad \delta_{m,n}y^s e^{-2\pi ny} + \sum_{c\geq 1}\sum_{d \bmod c}^{*} \int_{-\infty}^{\infty} \left(\frac{y}{(cx+d)^2+c^2y^2}\right)^s e\left(m\frac{az+b}{cz+d}\right) e(-nx)dx \\
&\overset{x+\frac{d}{c}\mapsto x}{=} \quad \delta_{m,n}y^s e^{-2\pi ny} + \sum_{c\geq 1}\sum_{d \bmod c}^{*} \int_{-\infty}^{\infty} \left(\frac{y}{(cx)^2+c^2y^2}\right)^s e\left(m\frac{a(x-\frac{d}{c})+b+iay}{c(x-\frac{d}{c})+d+icy}\right) e(-n(x-\frac{d}{c}))dx \\
&= \quad \delta_{m,n}y^s e^{-2\pi ny} + \sum_{c\geq 1}\frac{y^s}{c^{2s}}\sum_{d \bmod c}^{*} e\left(\frac{ma+nd}{c}\right)\int_{-\infty}^{\infty} \left(\frac{1}{x^2+y^2}\right)^s e\left(m\frac{-\frac{ad-bc}{c})}{cx+icy}\right) e(-nx)dx \\
&= \quad \delta_{m,n}y^s e^{-2\pi ny} + y^s\sum_{c\geq 1}\frac{1}{c^{2s}}S(m,n;c)\int_{-\infty}^{\infty} \left(\frac{1}{x^2+y^2}\right)^s e\left(\frac{-m}{c^2(x+iy)} - nx\right)dx.
\end{aligned}$$

11.3. **Poincare series spans $L^2(\Gamma\backslash\mathfrak{h})$.** Note that

$$\overline{U(z,s)} = \sum_{\delta\in\Gamma_\infty\backslash\mathfrak{h}} \overline{I_s(\delta.z)e(mz)} = \sum_{\delta\in\Gamma_\infty\backslash\mathfrak{h}} I_{\overline{s}}(\delta.z)e(-m\overline{z}).$$

Let $\mathfrak{S}$ be the space spanned by all Poincare series $U_m(z,s)$. For $f \in L^2(\Gamma\backslash\mathfrak{h})$, we have the inner product

$$\langle f, U_m(z,\overline{s})\rangle = \int_0^{\infty} y^{s-1}e^{-2m\pi y}\left\{\int_0^1 f(x+iy)e(-mx)dx\right\}\frac{dy}{y}. \tag{11.19}$$

Thus if f is orthogonal to the space $\mathfrak{S}$, then the Mellin transform of all the Fourier coefficients of f should be zero, and hence f is zero. This shows that the Poincare sereis spans $L^2(\Gamma\backslash\mathfrak{h})$.

11.4. **Inner product of Poincare series 1 - the geometric side.** Note that

$$\overline{U_m(z;s)} = U_m(-\overline{z};\overline{s}).$$

and

$$\overline{U_m(z,\overline{s})} = \sum_{\delta\in\Gamma_\infty\backslash\Gamma} I_s(\delta.z)e(-mx)e^{-2\pi my}.$$

We consider

$$\begin{aligned}
& \langle U_n(z,s_1), U_m(z,\overline{s}_2)\rangle \\
= & \int_0^\infty y^{s_2-1}e^{-2\pi my}\int_0^1 U_n(z,s_1)e(-mx)dx\frac{dy}{y} \\
= & \int_0^\infty y^{s_2-1}e^{-2\pi my}\delta_{m,n}y^{s_1}e^{-2\pi ny}\frac{dy}{y} \\
& + \int_0^\infty y^{s_2-1}e^{-2\pi my}y^{s_1}\sum_{c\geq 1}\frac{1}{c^{2s_1}}S(n,m;c)\int_{-\infty}^\infty\left(\frac{1}{x^2+y^2}\right)^{s_1}e\left(-\frac{n}{c^2(x+iy)}-mx\right)dx\frac{dy}{y} \\
= & \delta_{m,n}\int_0^\infty e^{-2\pi(m+n)y}y^{s_1+s_2-1}\frac{dy}{y}+\sum_{c\geq 1}\frac{1}{c^{2s_1}}S(m,n;c)I(s_1,s_2,m,n,c)
\end{aligned}$$

where we have

$$\int_0^\infty e^{-2\pi(m+n)y}y^{s_1+s_2-1}\frac{dy}{y}=\frac{1}{(2\pi(m+n))^{s_1+s_2-1}}\Gamma(s_1+s_2-1)$$

and (Lemma 4.1 in [DeIw1982])

$$\begin{aligned}
I(s_1,s_2,m,n,c) &= \int_0^\infty y^{s_1+s_2-1}e^{-2\pi my}\int_{-\infty}^\infty\frac{1}{(x^2+y^2)^{s_1}}e\left(-\frac{n}{c^2(x+iy)}-mx\right)dx\frac{dy}{y} \\
&= -i2^{3-s_1-s_2}c^{s_1-s_2}\left(\frac{m}{n}\right)^{\frac{s_1-s_1}{2}}\int_{-i}^{i}K_{s_1-s_2}\left(\frac{4\pi\sqrt{mn}}{c}\theta\right)\left(\theta+\frac{1}{\theta}\right)^{s_1+s_2-2}\frac{d\theta}{\theta}.
\end{aligned}$$

where the integral is performed though the half circle $|z|=1$, $\mathrm{Re}(z)>0$ starting from the point $-i$.

The above argument gives the analytic continuation of the inner product of Poincare series for $\mathrm{Re}(s_1)>\frac{3}{4}$ and $\mathrm{Re}(s_2)>\frac{3}{4}$. On taking $s_1=1+it$ and $s_2=1-it$, we have the following proposition (Lemma 4.2 in [DeIw1982]).

Proposition 11.2. *We have*

$$\langle U_n(z,1+it), U_m(z,1+it)\rangle=\frac{\delta_{m,n}}{4\pi\sqrt{mn}}-2i\left(\frac{m}{n}\right)^{it}\sum_{c\geq 1}\frac{S(m,n;c)}{c^2}\int_{-i}^{i}K_{2it}\left(\frac{4\pi\sqrt{mn}}{c}\theta\right)\frac{d\theta}{\theta}.$$

11.5. **Inner products of Poincare series 1 - the spectral side.** By Parseval identity, for $f\in\mathcal{B}$ a basis of $L^2(\Gamma\backslash\mathfrak{h})$,

$$\langle U_n(z,s_1), U_m(z,\overline{s_2})\rangle=\int_{f\in\mathcal{B}}\langle U_n(z,s_1),f\rangle\overline{\langle U_m(z,\overline{s}_2),f\rangle}df.$$

So we need the inner product of Poincare series with respect to the basis.

11.5.1. Since $m\geq 1$, obviously one has

$$\langle 1, U_m(z,\overline{s})\rangle=0 \tag{11.20}$$

11.5.2. If $f \in \mathfrak{B}_{cusp}$ with eigenvalues $\lambda = \frac{1}{4} + \nu_f^2$ and

$$f(z) = \sum_{n \neq 0} a_f(n) \sqrt{y} K_{i\nu_f}(2\pi|n|y) e(nx)$$

with $a_f(n) = \lambda_f(|n|) a_f(\text{sign}(n))$. We have

$$\begin{aligned}
\langle f, U_m(z, \overline{s}) \rangle &= a_f(m) \int_0^\infty y^{s-1} e^{-2\pi m y} \sqrt{y} K_{i\nu_f}(2\pi|m|y) \frac{dy}{y} \\
&= a_f(n) \int_0^\infty y^{s-\frac{1}{2}} e^{-2\pi m y} K_{i\nu_f}(2\pi|m|y) \frac{dy}{y} \\
&= \frac{a_f(m)}{(2\pi m)^{s-\frac{1}{2}}} \pi^{1/2} 2^{\frac{1}{2}-s} \frac{\Gamma(s-\frac{1}{2}-i\nu_f)\Gamma(s-\frac{1}{2}+i\nu_f)}{\Gamma(s)} \\
&= \frac{a_f(m)}{(4\pi m)^{s-\frac{1}{2}}} \pi^{1/2} \frac{\Gamma(s-\frac{1}{2}-i\nu_f)\Gamma(s-\frac{1}{2}+i\nu_f)}{\Gamma(s)}. \qquad (11.21)
\end{aligned}$$

where we have use the following formula (see Lemma G.2)

$$\int_0^\infty e^{-y} y^{s-\frac{1}{2}} K_\nu(y) \frac{dy}{y} = \pi^{1/2} 2^{\frac{1}{2}-s} \frac{\Gamma(s-\frac{1}{2}-\nu)\Gamma(s-\frac{1}{2}+\nu)}{\Gamma(s)}. \qquad (11.22)$$

Thus the cuspidal spectrum contributes to $\langle U_n(z, s_1), U_m(z, \overline{s_2}) \rangle$ is

$$\begin{aligned}
& \sum_{f \in \mathfrak{B}_{cusp}} \frac{\overline{\langle f, U_n(z, s_1) \rangle} \langle f, U_m(z, \overline{s_2}) \rangle}{\langle f, f \rangle} \\
= & \sum_{f \in \mathfrak{B}_{cusp}} \frac{1}{\langle f, f \rangle} \overline{\frac{a_f(n)}{(4\pi n)^{\overline{s}_1-\frac{1}{2}}} \pi^{1/2} \frac{\Gamma(\overline{s}_1 - 1/2 - i\nu_f)\Gamma(\overline{s}_1 - 1/2 + i\nu_f)}{\Gamma(\overline{s}_1)}} \\
& \frac{a_f(m)}{(4\pi m)^{s_2-\frac{1}{2}}} \pi^{1/2} \frac{\Gamma(s_2 - 1/2 - i\nu_f)\Gamma(s_2 - 1/2 + i\nu_f)}{\Gamma(s_2)} \\
= & \frac{\pi n^{-s_1+\frac{1}{2}} m^{-s_2+\frac{1}{2}}}{(4\pi)^{s_1+s_2-1}} \frac{1}{\Gamma(s_1)\Gamma(s_2)} \sum_{f \in \mathfrak{B}_{cusp}} \frac{a_f(m)\overline{a_f(n)}}{\langle f, f \rangle} \prod_{i=1,2} \prod_{\pm} \Gamma(s_i - \frac{1}{2} \pm i\nu_f)
\end{aligned}$$

11.5.3. Recall

$$\begin{aligned}
E(z, \frac{1}{2} + i\nu) &= y^{\frac{1}{2}+i\nu} + \frac{\xi(2i\nu)}{\xi(1+2i\nu)} y^{\frac{1}{2}-i\nu} \\
&\quad + 2\frac{1}{\xi(1+2i\nu)} \sum_{n \neq 0} |n|^{i\nu} \sigma_{-2i\nu}(|n|) \sqrt{y} K_{i\nu}(2\pi|n|y) e(nx).
\end{aligned}$$

Thus

$$\begin{aligned}
\langle E(z, \frac{1}{2} + i\nu), U_m(z, \overline{s}) \rangle &= \int_0^\infty y^{s-1} e^{-2\pi m y} 2 \frac{1}{\xi(1+2i\nu)} m^{i\nu} \sigma_{-2i\nu}(m) \sqrt{y} K_{i\nu}(2\pi m y) \frac{dy}{y} \\
&= 2\frac{1}{\xi(1+2i\nu)} m^{i\nu} \sigma_{-2i\nu}(m) \int_0^\infty y^{s-\frac{1}{2}} e^{-2\pi m y} K_{i\nu}(2\pi m y) \frac{dy}{y}.
\end{aligned}$$

By (11.22) again, we have

$$\begin{aligned}\langle E(z,\frac{1}{2}+i\nu),U_m(z,\overline{s})\rangle &= 2\frac{1}{\xi(1+2i\nu)}m^{i\nu}\sigma_{-2i\nu}(m)\frac{1}{(2\pi m)^{s-\frac{1}{2}}}\pi^{1/2}2^{\frac{1}{2}-s}\frac{\Gamma(s-\frac{1}{2}-i\nu)\Gamma(s-\frac{1}{2}+i\nu)}{\Gamma(s)}\\ &= 2\frac{m^{i\nu}\sigma_{-2i\nu}(m)}{\xi(1+2i\nu)}\frac{1}{(4\pi m)^{s-\frac{1}{2}}}\pi^{1/2}\frac{\Gamma(s-\frac{1}{2}-i\nu)\Gamma(s-\frac{1}{2}+i\nu)}{\Gamma(s)} \qquad (11.23)\end{aligned}$$

and thus this part contributes

$$\begin{aligned}&\frac{1}{4\pi}\int_{-\infty}^{\infty}\langle U_n(z,s_1),E(z,1/2+i\nu)\rangle\overline{\langle U_m(z,\overline{s}_2),E(z,1/2+i\nu)\rangle}d\nu\\ =\ &\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{\pi}{(4\pi)^{s_1+s_2-1}n^{s_1-\frac{1}{2}}m^{s_2-\frac{1}{2}}}\frac{1}{\Gamma(s_1)\Gamma(s_2)}\left\{\overline{\frac{2n^{i\nu}\sigma_{-2i\nu}(n)}{\xi(1+2i\nu)}}\frac{2m^{i\nu}\sigma_{-2i\nu}(m)}{\xi(1+2i\nu)}\right\}\prod_{i=1,2}\prod_{\pm}\Gamma(s_i-\frac{1}{2}\pm i\nu)d\nu\\ =\ &\frac{\pi n^{-s_1+\frac{1}{2}}m^{-s_2+\frac{1}{2}}}{(4\pi)^{s_1+s_2-1}}\frac{1}{\Gamma(s_1)\Gamma(s_2)}\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{m}{n}\right)^{i\nu}\frac{\sigma_{2i\nu}(n)}{\xi(1-2i\nu)}\frac{\sigma_{-2i\nu}(m)}{\xi(1+2i\nu)}\prod_{i=1,2}\prod_{\pm}\Gamma(s_i-\frac{1}{2}\pm i\nu)d\nu.\end{aligned}$$

Note that

$$\xi(1-2i\nu)\xi(1+2i\nu)=\pi^{-1}\Gamma(\frac{1}{2}-i\nu)\Gamma(\frac{1}{2}+i\nu)\zeta(1-2i\nu)\zeta(1+2i\nu),$$

and thus

$$\begin{aligned}&\frac{1}{4\pi}\int_{-\infty}^{\infty}\langle U_n(z,s_1),E(z,1/2+i\nu)\rangle\overline{\langle U_m(z,\overline{s}_2),E(z,1/2+i\nu)\rangle}d\nu\\ =\ &\frac{\pi n^{-s_1+\frac{1}{2}}m^{-s_2+\frac{1}{2}}}{(4\pi)^{s_1+s_2-1}}\frac{1}{\Gamma(s_1)\Gamma(s_2)}\int_{-\infty}^{\infty}\left(\frac{m}{n}\right)^{i\nu}\frac{\sigma_{2i\nu}(n)}{\zeta(1-2i\nu)}\frac{\sigma_{-2i\nu}(m)}{\zeta(1+2i\nu)}\frac{\prod_{i=1,2}\prod_{\pm}\Gamma(s_i-\frac{1}{2}\pm i\nu)}{\Gamma(\frac{1}{2}+i\nu)\Gamma(\frac{1}{2}-i\nu)}d\nu.\end{aligned}$$

Therefore, we have the following result.

Lemma 11.3. *We have*

$$\begin{aligned}\langle U_n(z,s_1),U_m(z,\overline{s}_2)\rangle &= \frac{\pi n^{-s_1+\frac{1}{2}}m^{-s_2+\frac{1}{2}}}{(4\pi)^{s_1+s_2-1}}\frac{1}{\Gamma(s_1)\Gamma(s_2)}\\ &\left\{\sum_{f\in\mathfrak{B}_{cusp}}\frac{a_f(m)\overline{a_f(n)}}{\langle f,f\rangle}\prod_{i=1,2}\prod_{\pm}\Gamma(s_i-\frac{1}{2}\pm i\nu_f)\right.\\ &\left.\quad+\int_{-\infty}^{\infty}\left(\frac{m}{n}\right)^{i\nu}\frac{\sigma_{2i\nu}(n)}{\zeta(1-2i\nu)}\frac{\sigma_{-2i\nu}(m)}{\zeta(1+2i\nu)}\frac{\prod_{i=1,2}\prod_{\pm}\Gamma(s_i-\frac{1}{2}\pm i\nu)}{\Gamma(\frac{1}{2}+i\nu)\Gamma(\frac{1}{2}-i\nu)}d\nu.\right\}\end{aligned}$$

On taking $s_1=1+it$ and $s_2=1-it$, by the facts

$$|\Gamma(\frac{1}{2}+ib)|^2=\frac{\pi}{\cosh(\pi b)},\quad |\Gamma(ib)|^2=\frac{\pi}{b\sinh(\pi b)},\quad \Gamma(s+1)=s\Gamma(s),$$

we have

$$
\begin{aligned}
\Gamma(1/2+i\nu)\Gamma(1/2-i\nu) &= \frac{\pi}{\sin\pi(1/2-i\nu)} = \frac{\pi}{\cos(i\nu\pi)} = \frac{\pi}{\cosh(\pi\nu)} \\
\prod_{\pm}\prod_{\pm}\Gamma(1/2\pm i\nu\pm it)\Gamma(1/2\pm i\nu\pm it) &= \frac{\pi^2}{\cosh(\pi(t+\nu))\cosh(\pi(t-\nu))} \\
\Gamma(1+2it)\Gamma(1-2it) &= 2it\Gamma(2it)(-2it)\Gamma(-2it) = 4t^2\frac{\pi}{2t\sinh(2\pi t)}.
\end{aligned}
$$

we have the following results on the spectral side.

Proposition 11.4 (Lemma 4.5 in [DeIw1982])**.** *We have*

$$
\begin{aligned}
\langle U_n(z,1+it), U_m(z,1+it)\rangle &= \frac{1}{4\sqrt{mn}}\left(\frac{m}{n}\right)^{it}\frac{\sinh(\pi t)}{t}\left\{\pi\sum_{f\in\mathfrak{B}_{cusp}}\frac{|a_f(1)|^2}{\langle f,f\rangle}\frac{\lambda_f(m)\lambda_f(n)}{\cosh(\pi(\nu_f-t))\cosh(\pi(\nu_f+t))}\right. \\
&\quad \left.+\int_{-\infty}^{\infty}\left(\frac{m}{n}\right)^{i\nu}\frac{\sigma_{-2i\nu}(m)\sigma_{2i\nu}(n)}{|\zeta(1+2i\nu)|^2}\frac{\cosh\pi\nu}{\cosh\pi(\nu-t)\cosh\pi(\nu+t)}d\nu\right\}
\end{aligned}
$$

11.6. **Inner products of Poincare series 1 - the pre trace formula.** By propositions 11.4 and 11.2, we take

$$
\begin{aligned}
H(\nu,t) :&= \frac{\cosh(\pi\nu)}{\cosh(\pi(\nu-t))\cosh(\pi(\nu+t))} \\
D_{2it}(x) &= -\frac{2it}{\sinh(\pi t)}\int_{-i}^{i}K_{2it}(x\theta)\frac{d\theta}{\theta} = \frac{t}{\sinh(2\pi t)}\int_{x}^{\infty}(J_{2it}(\theta)+J_{-2it}(\theta))\frac{d\theta}{\theta},
\end{aligned}
$$

we have the following result.

Proposition 11.5. *We have*

$$
\begin{aligned}
&\frac{t}{\pi\sinh(\pi t)}\delta_{m,n} - \sum_{c\geq 1}\frac{4\pi\sqrt{mn}}{c^2}S(m,n;c)D_{2it}\left(\frac{4\pi\sqrt{mn}}{c}\right) \\
= \ &\pi\sum_{f\in\mathfrak{B}_{cusp}}\frac{|a_f(1)|^2}{\langle f,f\rangle}\frac{H(\nu_f,t)}{\cosh(\pi\nu_f)}\lambda_f(m)\lambda_f(n) + \int_{-\infty}^{\infty}\left(\frac{m}{n}\right)^{i\nu}\frac{\sigma_{-2i\nu}(m)\sigma_{2i\nu}(n)}{|\zeta(1+2i\nu)|^2}H(\nu,t)d\nu
\end{aligned}
$$

11.7. **Inner products of Poincare series 2.** If one consider

$$\langle U_n(z,1+it), \overline{U_m(z,1-it)}\rangle$$

(see lemma 4.4 in [DeIw1982] for the calculation of the geometric side), one has another pre-trace formula as follows.

Proposition 11.6 (Lemma 4.8 in [DeIw1982])**.** *We have*

$$
\begin{aligned}
&\pi\sum_{f\in\mathfrak{B}_{cusp}}\frac{a_f(m)a_f(n)}{\langle f,f\rangle}\frac{H(\nu_f,t)}{\cosh(\pi\nu_f)} + \int_{-\infty}^{\infty}(mn)^{i\nu}\frac{\sigma_{-2i\nu}(m)\sigma_{-2i\nu}(n)}{\xi(1+2i\nu)^2}H(\nu,t)d\nu \\
= \ &\sum_{c\geq 1}\frac{4\pi\sqrt{mn}}{c^2}S(m,n;c)K_{2it}\left(\frac{4\pi\sqrt{mn}}{c}\right).
\end{aligned}
$$

Remark 21. In this formula, since $m,n\geq 1$, $\int_0^1 e((m+n)x)dx = 0$, i.e. the diagonal term vanishes.

11.8. **Test function and integral transform.** We refer to this part in page 228 in [DeIw1982].
Let $\varphi \in C_c^3(0,\infty)$. For real t, set

$$\begin{aligned}
g(t) &= -\int_0^\infty (J_{2it}(x)+J_{-2it}(x))\left(\frac{\varphi(x)}{x}\right)' dx\\
\hat{\varphi}(r) &= \frac{\pi}{\sinh(\pi r)}\int_0^\infty \frac{J_{2ir}(x)-J_{-2ir}(x)}{2i}\varphi(x)\frac{dx}{x}\\
\check{\varphi}(r) &= \frac{4}{\pi}\cosh(\pi r)\int_0^\infty K_{2ir}(x)\varphi(x)\frac{dx}{x}\\
\tilde{\varphi}(\ell) &= \int_0^\infty J_\ell(y)\varphi(y)\frac{dy}{y}\\
\varphi_B(x) &= \sum_{\substack{k>0\\ k \text{ odd}}} 2k\tilde{\varphi}(k)J_k(x)\\
\varphi_H &= \varphi-\varphi_B
\end{aligned}$$

Applying the following integral transform

$$\begin{aligned}
-\int_{-\infty}^\infty t\sinh(2\pi t)H(r,t)K_{2it}(\theta)dt &= \theta K_{2ir}(\theta)\\
\int_{-\infty}^\infty \frac{tg(t)}{\sinh(\pi t)}dt &= \int_0^\infty J_0(x)\varphi(x)dx\\
x\int_x^\infty\int_{-\infty}^\infty \frac{tg(t)}{\sinh(\pi t)}(J_{2it}(u)+J_{-2it}(u))dt\frac{du}{u} &= \varphi_H(x)\\
\int_{-\infty}^\infty H(r,t)g(t)dt &= \frac{2}{\pi}\hat{\varphi}(t).
\end{aligned}$$

One has the following

Theorem 11.7 (Prop 2 in [DeIw1982])**.** *For $\varphi \in C_c^3(0,\infty)$, we have*

$$\begin{aligned}
&\sum_{f\in\mathcal{B}_{cusp}} \frac{|a_f(1)|^2}{\langle f,f\rangle}\frac{\lambda_f(m)\lambda_f(n)}{\cosh(\pi\nu_f)}\widehat{\varphi}(\nu_f)+\int_{-\infty}^\infty\left(\frac{m}{n}\right)^{-i\nu}\frac{\sigma_{-2i\nu}(n)\sigma_{2i\nu}(m)}{|\zeta(1+2i\nu)|^2}\hat{\varphi}(\nu)d\nu\\
=\;&\frac{\delta_{m,n}}{2\pi}\int_0^\infty J_0(x)\varphi(x)dx+\sum_{c\geq 1}\frac{S(m,n;c)}{c}\varphi_H\left(\frac{4\pi\sqrt{mn}}{c}\right).
\end{aligned}$$

11.9. **Kunzetsov's trace formula - Final version.** This is a formula in Li Xiaoqing's paper.

Proposition 11.8. *Assume $u_j \in \mathfrak{B}_{cusp}$ with $\langle u_j,u_j\rangle = 1$ with $\lambda_j = \frac{1}{4}+t_j^2$ the following*

$$u_j(z) = \sum_{n\neq 0}\rho_j(n)\sqrt{y}K_{it_j}(2\pi|n|y)e(nx).$$

Let $h(z)$ be a test function satisfying the following.

- *$h(z)$ is holomorphic in $|\mathrm{Im}(z)| \leq \sigma$*
- *$h(z) \ll (1+|z|)^{-\theta}$ in the strip with $\sigma > 1/2$ and $\theta > 2$*

One has

$$\sum_{j\geq 1} h(t_j)\frac{\rho_j(n)\overline{\rho_j(m)}}{\cosh(\pi t_j)} + \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{m}{n}\right)^{ir}\sigma_{2ir}(n)\sigma_{-2ir}(m)\frac{h(r)}{|\zeta(1+2ir)|^2}dr$$
$$= \frac{\delta_{m,n}}{\pi^2}\int_{-\infty}^{\infty} r\tanh(\pi r)h(r)dr + \sum_{c\geq 1}\frac{S(n,m;c)}{c}h^+\left(\frac{4\pi\sqrt{mn}}{c}\right),$$

with

$$\sigma_\nu(n) = \sum_{d|n} d^\nu, \qquad h^+(x) = \frac{2i}{\pi}\int_{-\infty}^{\infty} J_{2ir}(x)\frac{h(r)r}{\cosh \pi r}dr.$$

Remark 22. Note that the L.H.S. in the cuspidal parameter should be

$$\sum_{j\geq 1}\frac{|\rho_j(1)|^2}{\langle u_j, u_j\rangle}\lambda_j(m)\lambda_j(n)\frac{\phi(t_j)}{\cosh(\pi t_j)}.$$

Remark 23. For the proof of the Kuznetsov's trace formula via the relative trace formula, we refer to V. Blomer, *The relative trace formula in analytic number theory*, arXiv:1912.08137.

12. Truncated Eisenstein series and Maass-Selberg Relation

Now, Let T be a sufficiently large parameter, we define the truncated spectral function

$$I_s^T(z) = \Lambda^T I_s(z) = \begin{cases} y^s, & y = \mathrm{Im} z \leq T \\ 0, & y > T \end{cases}$$

Then we set

$$\Lambda^T E(z,s) = E^T(z,s) := \sum_{\delta\in\Gamma_\infty\backslash\Gamma} I_s^T(\delta.z)$$

which is called Truncated Eisenstein series.

Appendix A. Bessel equations

This part is a note on Chapter 7 in *(Te Shu Han Shu Gai Lun, Chinese verion, Wang Zhuxi and Guo dunren)*.

We change variables in (2.6) as follows. Let $F(z) = G(iz)$. One has

$$F(z) = G(iz), \quad F'(z) = iG'(iz), \quad F''(z) = -G''(iz)$$

and thus (2.6) is

$$-G''(iz) + \frac{1}{z} iG'(i\zeta) - \left(1 + \frac{(it)^2}{z^2}\right) G(iz) = 0$$

on taking $iz = \zeta$, one has

$$G''(\zeta) + \frac{1}{\zeta} G'(\zeta) + \left(1 - \frac{(it)^2}{\zeta^2}\right) G(\zeta) = 0 \tag{1.24}$$

Here $\nu = it$ is called the order of the Bessel function.

A.1. **Basic solutions.**

A.1.1. Assume $2\nu \notin \mathbb{Z}$. We rewrite (1.24) as

$$\zeta^2 G'' + \zeta G' + (\zeta^2 - \nu^2) G = 0.$$

So we assume that $G = \sum_{k \geq 0} c_k z^{k+\rho}$ with $c_0 \neq 0$ to obtain the solutions

$$J_{\pm\nu}(\zeta) = \sum_{k \geq 0} \frac{(-1)^k}{k!} \frac{1}{\Gamma(\pm\nu + k + 1)} \left(\frac{\zeta}{2}\right)^{2k \pm \nu}$$

which are linear independent since $2\nu \notin \mathbb{Z}$. They are <u>Bessel function of the first kind</u>.

A.1.2. If $\nu = n \in \mathbb{N} \cup \{0\}$, we know that $\Gamma(-n + k + 1) \to \infty$ $(k < n)$. Thus

$$\begin{aligned} J_{-n}(\zeta) &= \sum_{k \geq n} \frac{(-1)^k}{k!} \frac{1}{\Gamma(-n + k + 1)} \left(\frac{\zeta}{2}\right)^{2k - n} = \sum_{k \geq 0} \frac{(-1)^{k+n}}{(k+n)!} \frac{1}{\Gamma(k+1)} \left(\frac{\zeta}{2}\right)^{2k+n} \\ &= (-1)^n J_n(\zeta), \end{aligned}$$

which are linear dependent. In this case, <u>$J_n(\zeta)$ is entire function</u>. So we need another linear independent solution, which is $Y_n(\zeta)$ with

$$Y_\nu(\zeta) := \frac{\cos(\pi\nu) J_\nu(\zeta) - J_{-\nu}(\zeta)}{\sin(\pi\nu)}$$

which are <u>Bessel function of the second kind</u>.

A.1.3. If $2\nu \in \mathbb{Z} - 2\mathbb{Z}$, i.e. $\nu = n + \frac{1}{2}$ with $n \in \mathbb{Z}$, the $J_{n+\frac{1}{2}}(\zeta)$ can be expressed via elementary functions as follows.

$$J_{1/2}(\zeta) = \sum_{k\geq 0} \frac{(-1)^k}{k!} \frac{1}{\Gamma(k+\frac{3}{2})} \left(\frac{\zeta}{2}\right)^{2k+\frac{1}{2}}$$

Note that

$$\Gamma(k+\frac{3}{2}) = \Gamma(2k+2)2^{-2k-1}\sqrt{\pi}/\Gamma(k+1),$$

one has

$$J_{1/2}(\zeta) = \sqrt{\frac{2}{\pi\zeta}} \sum_{k\geq 0} \frac{(-1)^k}{(2k+1)!} \zeta^{2k+1} = \sqrt{\frac{2}{\pi\zeta}} \sin\zeta.$$

and similarly,

$$J_{-1/2}(\zeta) = \sqrt{\frac{2}{\pi\zeta}} \cos\zeta.$$

Remark 24. Another basis solutions are expressed as

$$H^1_\nu(z) := J_\nu(z) + iY_\nu(z), \qquad H^2_\nu(z) := J_\nu(z) - iY_\nu(z)$$

which are called Bessel function of the third kind.

A.2. **Solutions of** (2.6)**.** Consider (2.6). We know $\nu = it \notin \mathbb{Z}$ and $F(z) = G(iz) = G(\zeta)$. Solutions for (2.6) are

$$J_{it}(iz), \quad J_{-it}(iz)$$

which are linear independent.

We introduce new functions defined by

$$\begin{aligned} I_\nu(z) &= \begin{cases} e^{-\nu\frac{\pi i}{2}} J_\nu(ze^{\frac{\pi i}{2}}), & (-\pi < \arg z < \frac{\pi}{2}) \\ e^{\nu\frac{3\pi i}{2}} J_\nu(z^{-\frac{3\pi i}{2}}), & (\frac{\pi}{2} < \arg z < \pi) \end{cases} \\ &= \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{\Gamma(\nu+k+1)} \left(\frac{z}{2}\right)^{2k} \end{aligned}$$

Then we know $I_{it}(z)$ are one of the solution of (2.6).

- If $\nu \notin \mathbb{Z}$, $I_{\pm\nu}(z)$ are two linear independent solutions. One define

$$K_\nu(z) = \frac{\pi}{2\sin(\pi\nu)}(I_{-\nu}(z) - I_\nu(z))$$

  and use $K_\nu(z)$ and $I_\nu(z)$ as linear independent solutions of (2.6).
- If $\nu = n \in \mathbb{Z}$, $I_{-n}(z) = I_n(z)$. In this case, $K_\nu(z)$ and $I_\nu(z)$ are still linear independent.

Therefore, we have the following result.

Proposition A.1. *Two linear independent solutions of* (2.6) *are*

$$K_{it}(z) \quad \text{and} \quad I_{it}(z).$$

A.3. **Important property of J-Bessel function.** We have

$$J_{\pm\nu}(z) := \sum_{k\geq 0} \frac{(-1)^k}{k!} \frac{1}{\Gamma(\pm\nu + k + 1)} \left(\frac{z}{2}\right)^{2k\pm\nu}, \qquad \mathrm{Re}(\nu) \geq 0$$

Special values are given as

$$J_{-n}(z) = (-1)^n J_n(z), \quad J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z, \quad J_{-1/2}(z) = \sqrt{\frac{2}{\pi z}} \cos z$$

and the fact that $J_n(z)$ is entire function in z.

The recurrent differential formula are

$$\frac{d}{dz}(z^\nu J_\nu) = z^\nu J_{\nu-1}, \quad \frac{d}{dz}(z^{-\nu} J_\nu) = -z^{-\nu} J_{\nu+1}$$
$$\left(\frac{d}{zdz}\right)^m (z^\nu J_\nu) = z^{\nu-m} J_{\nu-m}, \quad \left(\frac{d}{zdz}\right)^m (z^{-\nu} J_\nu) = (-1)^m z^{-\nu-m} J_{\nu+m}$$

and some recurrent formulas are

$$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{z} J_\nu, \quad J_{\nu-1} - J_{\nu+1} = 2J_v'$$
$$J_{\pm\nu}(ze^{\pi i}) = e^{\pm\nu\pi i} J_{\pm\nu}(z)$$

Integral representations of J-Bessel function are

$$\begin{aligned}
J_\nu(z) &= \frac{(z/2)^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_{-1}^1 e^{izt}(1-t^2)^{\nu-\frac{1}{2}} dt, \quad \mathrm{Re}(\nu) > -1/2, \arg(1-t^2) = 0 \\
&= \frac{(\frac{z}{2})^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\pi e^{iz\cos\theta} \sin^{2\nu}\theta d\theta, \quad \text{by taking } t = \cos\theta \text{ as above} \\
&= \frac{(\frac{z}{2})^\nu}{\sqrt{\pi}\Gamma(\nu + \frac{1}{2})} \int_0^\pi \cos(z\cos\theta) \sin^{2\nu}\theta d\theta \\
&= \frac{(z/2)^\nu}{2\pi i} \int_{-\infty}^{0+} e^{t - \frac{z^2}{4t}} t^{-\nu-1} dt, \quad |\arg t| < \pi \\
&= \frac{1}{2\pi i} \int_{-\infty}^{0+} e^{\frac{z}{2}(t - \frac{1}{t})} t^{-\nu-1} dt, \quad |\arg z| < \frac{\pi}{2}, |\arg t| < \pi
\end{aligned}$$

We also has the following important integral representation for the K-Bessel function, namely

$$\int_0^\infty e^{-\frac{y}{2}(t+\frac{1}{t})} t^s \frac{dt}{t} = 2K_s(y). \tag{1.25}$$

which implies

$$K_s(y) = K_{-s}(y)$$

by $t \mapsto t^{-1}$ and $s \mapsto -s$.

A.4. **Asymptotic formulas as $|z| \to \infty$.** Let

$$\begin{aligned}
(\nu, 0) &= 1 \\
(\nu, p) &= \frac{\Gamma(\frac{1}{2} + \nu + p)}{p!\Gamma(\frac{1}{2} + \nu - p)} = (-\nu, p).
\end{aligned}$$

Some asymptotic formulas as $|z| \to \infty$ are

$$J_\nu(z) \sim \sqrt{\frac{2}{\pi z}} \Bigg[\cos(z - \frac{\pi}{2}\nu - \frac{\pi}{4}) \sum_{m\geq 0} (-1)^m \frac{(\nu, 2m)}{(2z)^{2m}} - \sin(z - \frac{\pi}{2}\nu - \frac{\pi}{4}) \sum_{m\geq 0} \frac{(-1)^m(\nu, 2m+1)}{(2z)^{2m+1}} \Bigg], \quad -\pi < \arg z < \pi$$

Other range of $\arg z$, for example, $0 < \arg z < 2\pi$ can be obtained by the recurrent relation

$$J_\nu(z) = e^{\nu\pi i} J_\nu(ze^{-\pi i}) \sim e^{(\nu+\frac{1}{2})\pi i} \sim \sqrt{\frac{2}{\pi z}} \Bigg[\cos(z + \frac{\pi}{2}\nu + \frac{\pi}{4}) \sum_{m\geq 0} (-1)^m \frac{(\nu, 2m)}{(2z)^{2m}} - \sin(z + \frac{\pi}{2}\nu + \frac{\pi}{4}) \sum_{m\geq 0} \frac{(-1)^m(\nu, 2m+1)}{(2z)^{2m+1}} \Bigg]$$

One has also

$$K_\nu(z) \sim \sqrt{\frac{\pi}{2z}} e^{-z} \Bigg[1 + \sum_{n\geq 1} \frac{(\nu, n)}{(2z)^n} \Bigg], \quad |\arg z| < \frac{3}{2}\pi$$

$$I_\nu(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sum_{n\geq 0} \frac{(-1)^n(\nu, n)}{(2z)^n} + \frac{e^{-z+(\nu+\frac{1}{2})\pi i}}{\sqrt{2\pi z}} \sum_{n\geq 0} \frac{(\nu, n)}{(2z)^n}, \quad -\frac{\pi}{2} < \arg z < \frac{3}{2}\pi.$$

A.5. **Asymptotic formulas as $|\nu| \to \infty$.** For fixed z, as $|\nu|$ large,

$$J_\nu(z) \sim \exp\left(\nu + \nu \log \frac{z}{2} - (\nu - \frac{1}{2}) \log \nu \right) \left[c_0 + \frac{c_1}{\nu} + \frac{c_2}{\nu^2} + \cdots \right]$$

with $c_0 = \frac{1}{\sqrt{2\pi}}$.

A.6. **Asymptotic formulas as both $|\nu|$ and $|z|$ large.** We refer to sections 7.11-7.12 in the Book on special functions by Wang Zhuxi-Guo Dunren.

Appendix B. Spherical Whittaker function in Whittaker model

Some notations are as follows.

- For compact subgroup,

$$O(2) = SO(2) \bigcup \begin{pmatrix} -1 & \\ & 1 \end{pmatrix} SO(2), \quad PSO(2) = SO(2)/\{\pm I\}.$$

  Note that $GL_2(\mathbb{R}) = GL_2^+(\mathbb{R}) \bigcup GL_2^-(\mathbb{R})$. We have

$$\mathfrak{h} = SL_2(\mathbb{R})/SO(2) = GL_2(\mathbb{R})/Z(\mathbb{R})O(2)$$

- Next, $PGL_2(\mathbb{R}) = GL_2(\mathbb{R})/Z(\mathbb{R})$ and

$$PGL_2(\mathbb{R})/PSO(2) = GL_2(\mathbb{R})/Z(\mathbb{R})SO(2).$$

We are dealing with spherical representations for $PGL_2 = GL_2/Z$ over $\mathbb{R}$. Note that for $g \in GL_2(\mathbb{R})$,

$$g = z\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\pm 1 & \\ & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & \\ & y^{-1/2}\end{pmatrix}\kappa_\theta, \quad \kappa_\theta \in SO(2), z \in Z(\mathbb{R}).$$

Let π_∞ be an irreducible unramified unitary infinite dimensional representation of G_∞ with trivial central character, which can be realized as the normalized induced representation

$$\pi(\epsilon_\pi, it_\pi) = \mathrm{Ind}_{B(\mathbb{R})}^{G(\mathbb{R})}\chi_{\epsilon_\pi, it_\pi},$$

where B is the standard parabolic subgroup of G and $\chi_{\epsilon_\pi,it_\pi}$ is a character of $B(\mathbb{R})$ given by

$$\chi_{\epsilon_\pi,it_\pi}\begin{pmatrix}a & b\\ 0 & d\end{pmatrix} = \mathrm{sgn}(a)^{\epsilon_\pi}\mathrm{sgn}(d)^{\epsilon_\pi}\left|\frac{a}{d}\right|^{it_\pi}, \quad \begin{pmatrix}a & b\\ 0 & d\end{pmatrix} \in B(\mathbb{R}).$$

Here $\epsilon_\pi \in \{0,1\}$ and $\{\pm t_\pi\}$ is the set of spectral parameters of π such that

- either $t_\pi \in \mathbb{R}$, in which case $\pi_\infty = \pi(\epsilon_\pi, it_\pi)$ is a principal series,
- or $t_\pi \in i\mathbb{R}$ with $0 < |t_\pi| < \frac{1}{2}$, in which case $\pi_\infty = \pi(\epsilon_\pi, it_\pi)$ is a complementary series.

B.1. **spherical Whittaker function.** The spherical vector in $\pi_\infty = \pi(\epsilon_\pi, it_\pi)$ is the function

$$f_0 : GL_2(\mathbb{R}) \to \mathbb{C}$$

such that

$$f_0\left(z\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\pm 1 & \\ & 1\end{pmatrix}\begin{pmatrix}y & \\ & 1\end{pmatrix}\kappa_\theta\right) = (\pm 1)^{\epsilon_\pi}y^{\frac{1}{2}+it_\pi}.$$

Via the theory of intertwining operator and Whittaker function, for given a non-degenerate character on $N(\mathbb{R})$ (i.e. $a \neq 0$),

$$\psi_a : N(\mathbb{R}) \to \mathbb{C}, \quad \psi_a(n(x)) = e^{2\pi i a x} = e(ax),$$

the associated spherical Whittaker function (in the Whittaker model) is defined by

$$\begin{aligned} W_0(g,\psi_a) :&= \int_{N_P(\mathbb{R})\cap w_s N_P(\mathbb{R})w_s^{-1}\backslash N_P(\mathbb{R})} f_0(w_s^{-1}ng)\overline{\psi_a(n)}dn \\ &= \int_{-\infty}^{\infty} f_0\left(\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)e(-ax)dx \end{aligned}$$

Obviously one has

$$\begin{aligned} & W_0\left(z\begin{pmatrix}1 & x_0\\ & 1\end{pmatrix}\begin{pmatrix}\pm y_0 & \\ & 1\end{pmatrix}g\kappa_\theta, \psi_a\right) \\ =\ & e(ax_0)\int_{-\infty}^{\infty} f_0\left(\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}\pm y_0 & \\ & 1\end{pmatrix}g\right)e(-ax)dx \\ =\ & e(ax_0)\int_{-\infty}^{\infty} f_0\left(\begin{pmatrix}\pm y_0 & \\ & \pm y_0\end{pmatrix}\begin{pmatrix}\pm y_0^{-1} & \\ & 1\end{pmatrix}\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\begin{pmatrix}1 & \pm y_0^{-1}x\\ & 1\end{pmatrix}g\right)e(-ax)dx \\ =\ & (\pm 1)^{\epsilon_\pi}y_0^{\frac{1}{2}-it}e(ax_0)\int_{-\infty}^{\infty} f_0\left(\begin{pmatrix} & -1\\ 1 & \end{pmatrix}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}g\right)e(-(\pm y_0 a)x)dx \\ =\ & (\pm 1)^{\epsilon_\pi}y_0^{\frac{1}{2}-it}e(ax_0)W_0(g,\psi_{\pm y_0 a}). \end{aligned}$$

Based on the argument above, by the Iwasawa decomposition, we need only to determine the values

$$\begin{aligned} W_0\left(\begin{pmatrix} y & \\ & 1\end{pmatrix}, \psi_1\right) &= \int_{-\infty}^{\infty} f_0\left(\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\begin{pmatrix} y & \\ & 1\end{pmatrix}\right) e(-x)dx \\ &= \int_{-\infty}^{\infty} f_0\begin{pmatrix} & -1 \\ y & x\end{pmatrix} e(-x)dx = \int_{-\infty}^{\infty}\left(\frac{y}{x^2+y^2}\right)^{\frac{1}{2}+it} e(-x)dx \end{aligned}$$

where we have used the following explicit Iwasawa decomposition. The expression above coincides with (2.9) and the calculation are in Lemma 2.6.

Lemma B.1 (Explicit Iwasawa decomposition)**.** *For v real, then $\forall g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in GL_2(\mathbb{R})^+$, it has unique decomposition*

$$g = \begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} z & \\ & z\end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2}\end{pmatrix}\kappa_\theta \tag{2.26}$$

with

$$z = z(g) = \sqrt{ad-bc}, \quad x = x(g) = \frac{ac+bd}{c^2+d^2}, \quad y = y(g) = \frac{ad-bc}{c^2+d^2}$$

and

$$\theta = \theta(g) = \arctan(-c/d), \quad \textit{of period } \pi.$$

Appendix C. Mellin transform - Harmonic analysis on $L^2(\mathbb{R}^+, \frac{dy}{y})$

C.1. Mellin transform.

Proposition C.1. *For $f \in C_c^\infty(\mathbb{R}^+, \frac{dy}{y})$, the Mellin transform is defined by*

$$(\mathcal{M}(f)(s)) = \varphi(s) = \int_0^\infty f(t)t^s\frac{dt}{t}$$

and the Mellin inversion formula is

$$f(y) = (\mathcal{M}^{-1}\varphi)(y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\varphi(s)y^{-s}ds.$$

Remark 25. For $f \in L^2(\mathbb{R}^+, \frac{dy}{y})$, at the discontinuous point,

$$\frac{1}{2}\left(f(y+)+f(y-)\right) = \frac{1}{2\pi i}\lim_{r\to\infty}\int_{c-ir}^{c+ir}(\mathcal{M}f)(s)y^{-s}ds.$$

C.2. **Harmonic analysis.** Now we build the harmonic analysis as follows. For $L^2(\mathbb{R}^+, \frac{dy}{y})$, the Laplacian operator on $\mathbb{R}^+$ is $\Delta = -y^2\frac{d^2}{dy^2}$. Since $\widehat{\mathbb{R}}^+$ is commutative multiplicative group with invariant measure $\frac{dy}{y}$, The spectrum of $L^2(\mathbb{R}^+)$ (eigenfunctions of Δ) can also obtained via duality theory.

Let

$$\widehat{\mathbb{R}^+} = \{\chi : \mathbb{R}^+ \to S^1, \quad \text{continuous}, \chi(ab) = \chi(a)\chi(b)\}.$$

be the group of the continuous characters of $\mathbb{R}^+$. Via the differentiable map $\log : \mathbb{R}^+ \to \mathbb{R}$, we know that the multiplicative character χ is of the form

$$\chi : \mathbb{R}^+ \xrightarrow{\log} \mathbb{R} \xrightarrow{e()} S^1, \quad \chi(y) = e^{i\theta\log y}$$

for some $s = i\theta \in \mathbb{R}$. One has

$$\widehat{\mathbb{R}^+} \simeq i\mathbb{R}, \qquad \chi_{i\theta} \mapsto i\theta.$$

Remark 26. Obviously $\chi_s(y) = y^s$ are eigenfunctions of $\Delta = -y^2 \frac{d^2}{y^2}$ with eigenvalue

$$s(1-s).$$

Note that $\chi_s(y)$ are not in $L^2(\mathbb{R}^+)$. For $f \in L^2(\mathbb{R}^+)$ and $\chi_\theta \in \widehat{R^+}$, we have the inner product

$$\varphi(i\theta) \quad = \quad \langle f, \chi_{i\theta} \rangle = \int_{\mathbb{R}^+} f(y) e^{-i\theta \log y} \frac{dy}{y} = \int_0^\infty f(y) y^{-i\theta} \frac{dy}{y} = \mathcal{M}(f)(-i\theta)$$

which gives a function in $s = i\theta \in i\mathbb{R}$.

If the spectrum are all discrete, we expect to establish

$$f(y) = \sum_{\substack{\theta \\ \text{spectral}}} \frac{\langle f, \chi_{i\theta} \rangle}{\langle \chi_{i\theta}, \chi_{i\theta} \rangle} \chi_{i\theta};$$

If the spectrum are all continuous, one expect that

$$f(y) = \int_{\substack{s=i\theta \\ \text{spectrum}}} \langle f, \chi_{i\theta} \rangle \chi_{i\theta}(y) \frac{d\theta}{\langle \chi_{i\theta}, \chi_{i\theta} \rangle}$$

But we know $\chi_{i\theta}$ is not in $L^2(\mathbb{R}^+)$ and $\frac{1}{\langle \chi_{i\theta}, \chi_{i\theta} \rangle}$ has no means. However, one can replace it by a function $\mu(\theta)$, which is known as 'spectral measure' (Plancheral measure), and establish

$$\begin{aligned} f(y) \quad &= \quad \int_{-\infty}^{\infty} \mathcal{M}(f)(-i\theta) \chi_{i\theta}(y) \mu(\theta) d\theta \\ &= \quad \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \mathcal{M}(f)(i\theta) y^{-i\theta} \left(2\pi \mu(-\theta)\right) d(i\theta) \\ &= \quad \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} M(f)(s) y^{-s} \left(2\pi \mu(is)\right) ds. \end{aligned}$$

So if we can prove that the spectral measure $\mu(i\theta) = \frac{1}{2\pi}$, then the theory of the spectral decomposition on $L^2(\mathbb{R}^+)$ is just the theory of Mellin and Mellin inverse transform, by some tricks on analytic continuation for the convergence.

Lemma C.2. *The spectral measure is* $\mu(i\theta) = \frac{1}{2\pi}$.

Proof. choose suitable test function f and consider the value of f at a special point, for example, at $y = 1$.

$\square$

C.3. **Proof via Fourier transform.** Recall the argument above,

$$\varphi(i\theta) = \int_{\mathbb{R}^+} f(y) e^{-i\theta \log y} \frac{dy}{y}.$$

On taking $y = e^{2\pi x}$,

$$\varphi(i\theta) = (\mathcal{M} f)(i\theta) = 2\pi \int_{-\infty}^{\infty} f(e^{2\pi x}) e^{-2\pi i x \theta} dx = \int_{-\infty}^{\infty} F(x) e^{-2\pi i x \theta} dx, \qquad F(x) = 2\pi f(e^{2\pi x}).$$

and thus

$$\begin{aligned} F(x) &= 2\pi f(e^{2\pi x}) = 2\pi f(y) \\ &= \int_{-\infty}^{\infty} \varphi(i\theta) e^{2\pi i\theta x} d\theta \overset{x=\frac{1}{2\pi}\log y}{=} \int_{-\infty}^{\infty} \varphi(i\theta) y^{i\theta} d\theta \end{aligned}$$

which is

$$\begin{aligned} f(y) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathcal{M}f)(-i\theta) y^{i\theta} d\theta \\ &= \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=0} (\mathcal{M}f)(s) y^{-s} ds. \end{aligned}$$

We finish the proof of the Mellin transform.

C.4. **Useful table for Mellin and Mellin inversion formula.** We list the following useful table for Mellin transforms.

$f(y), y > 0$	$\mathcal{M}(f)(s)$
$\exp(-ay), y > 0$	$a^{-s}\Gamma(s)$
$\exp\left[-\frac{a}{2}(y + \frac{1}{y})\right], a > 0$	$2K_s(a)$

An application of the Mellin transform can be found in Lemma 2.6.

Appendix D. Gauss Hypergeometric function

The Gauss hypergeometric function is defined by

$$\begin{aligned} F(\alpha, \beta, \gamma, z) &= \sum_{n \geq 0} \frac{(\alpha)_n (\beta)_n}{n!(\gamma_n)} z^n, \quad |z| < 1 \\ &= \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \sum_{n \geq 0} \frac{1}{n!} \frac{\Gamma(\alpha+n)\Gamma(\beta+n)}{\Gamma(\gamma+n)} z^n, \quad |z| < 1. \end{aligned} \tag{4.27}$$

where

$$(\alpha)_0 = 1, \quad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

Here (4.27) is defined for $|z| < 1$. $F(\alpha, \beta, \gamma, z)$ is analytic continuated to $z \in \mathbb{C}$ except for $z = 1$ and $z = \infty$, which may be branch points of $F(\alpha, \beta, \gamma, z)$.

D.1. **Integral representations.** We have

$$F(\alpha, \beta, \gamma, z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt, \quad \mathrm{Re}(\gamma) > \mathrm{Re}(\beta) > 0, \quad |\arg(1-z)| > \pi.$$

Here $\frac{1}{z}$ is the singular pint of $\frac{1}{(1-zt)^{-\alpha}}$(except that α is negative integer), and thus $z = 1$ and $z = \infty$ are branches points.

D.2. **Barnes integral representations.** We have

$$F(\alpha,\beta,\gamma,z)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\Gamma(\alpha+s)\Gamma(\beta+s)}{\Gamma(\gamma+s)}\Gamma(-s)(-z)^s ds,$$

where one need that $\arg(-z)<\pi$ and $\mathrm{Re}(s)$ is in the left of poles of $\Gamma(-s)$, and right of poles of $\Gamma(\alpha+s)\Gamma(\beta+s)$. This implies that

$$\alpha \text{ and } \beta \text{ are not zero or negative integers.}$$

Lemma D.1 (Barnes lemma)**.** *For $\gamma-\alpha-\beta\in\mathbb{Z}$,*

$$\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\Gamma(\alpha+s)\Gamma(\beta+s)\Gamma(\gamma-s)\Gamma(\delta-s)ds=\frac{\Gamma(\alpha+\gamma)\Gamma(\alpha+\delta)\Gamma(\beta+\gamma)\Gamma(\beta+\delta)}{\Gamma(\alpha+\beta+\gamma+\delta)}$$

D.3. **Special values.** One has

$$F(\alpha,\beta,\gamma,1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)},\quad \gamma\notin\{0\}\cup\mathbb{Z}^-,\quad \gamma-\alpha-\beta\notin\mathbb{Z}$$

Other cases are the following.

1. If $\alpha-\beta=m$ with $m\in\mathbb{N}$, see formula (2) in page 120
2. If $\gamma-\alpha-\beta\in\mathbb{Z}$, see (8) in page 123
3. If $\gamma-\alpha-\beta\in\mathbb{N}$, see (9) in page 123
4. Moreover, if α or β is $-n$, in this case, $F(\alpha,\beta,\gamma,z)$ is a polynomial, called Jacobi polynomial (see (1) page). In this case,

$$F(-n,\beta,\gamma,1)=\frac{\Gamma(\gamma)\Gamma(\gamma-\beta+n)}{\Gamma(\gamma+n)\Gamma(\gamma-\beta)}.$$

D.4. **Relation with Chebechy polynomial.**

D.5. **Kummer's formula.**

$$F(\alpha,\beta,1+\alpha-\beta,-1)=\frac{\Gamma(1+\alpha-\beta)\Gamma(1+\frac{\alpha}{2})}{\Gamma(1+\alpha)\Gamma(1+\frac{\alpha}{2}-\beta)}$$

D.6. **asymptotic formulas.** As $z\to\infty$ and the parameter $\gamma\to\infty$, see section 4.14.

Appendix E. Legendre functions

Legendre functions are solutions of the Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+\nu(\nu+1)y=0, \tag{5.28}$$

which is obtained via the separable method from the Laplacian differential equation under the polar coordinate.

E.1. **Legendre polynomial.** Case $\nu = n \in \mathbb{N}$, (5.28) has solutions, which are called Legendre polynomials,

$$\begin{aligned} P_n(x) &= \frac{1}{2^n}\sum_{r=0}^{[\frac{n}{2}]}(-1)^r\frac{(2n-2r)!}{r!(n-r)!(n-2r)!}x^{n-2r} \\ &= \frac{(2n)!}{2^n(n!)^2}x^nF\left(-\frac{n}{2},\frac{1-n}{2},\frac{1}{2}-n,x^{-2}\right) \\ &= F(n+1,-n,1,\frac{1-x}{2}) \end{aligned}$$

and $P_n(1) = 1$.

One has integral representation of

$$P_n(x) = \frac{1}{\pi}\int_0^\pi (x+\sqrt{x^2-1}\cos\theta)d\theta$$

and the recurrent formulas

$$P_1(x) - xP_0(x) = 0$$
$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0, \quad n \geq 1.$$

E.1.1. *Orthogonal property of Legendre polynomials.*

Proposition E.1. *Let $f(x)$ be a polynomial with $\deg f = k$. If $k < n$, one has*

$$\int_{-1}^1 f_k(x)P_n(x)dx = 0$$

Moreover,

$$\int_{-1}^1 P_m(x)P_n(x) = \frac{2}{2n+1}\delta_{m,n}.$$

If $(1-x^2)^{-1/4}f(x)$ is integrable over $[-1,1]$, then

$$\frac{1}{2}(f(x+0)+f(x-0)) = \sum_{k\geq 0}\frac{2k+1}{2}P_k(x)\int_{-1}^1 f(t)P_k(t)dt$$

E.1.2. *Zeroes of $P_n(x)$.* $P_n(x)$ has n number of simple pole appears in $[-1,1]$.

E.2. **Lengedre function.** The solutions of

$$(1-z^2)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + [\nu(\nu+1) - \frac{\mu^2}{1-z^2}]u = 0$$

are the associated Legendre function defined by

$$P_\nu^\mu(z) := \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2}F(-\nu,\nu+1,1-\mu,\frac{1-z}{2}), \quad |\arg(z\pm 1)| < \pi.$$

One has integral representation

$$P_s^a(z) = \frac{\Gamma(s+a+1)}{2\pi\Gamma(s+1)}\int_0^{2\pi}(z+\sqrt{z^2-1}\cos\alpha)^s e^{ia\alpha}d\alpha, \quad \mathrm{Re}(z) > 0$$

For the order $a=0$,

$$\begin{aligned} P_s(z) &\sim \frac{1}{\sqrt{\pi}}\frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}(2z)^s, \quad z\to\infty \\ P_{-s}(z) &= P_{s-1}(z), \quad \mathrm{Re}(z)>0 \\ P_s(z) &= F(-s,s+1,1,\frac{1-z}{2}), \quad |z-1|<2. \end{aligned}$$

Appendix F. Philosophy of Eisenstein series

This part is based on Casselman's paper at Casselman's homepage.

F.1. The lift of classical modular and maass forms to GL_2.

Let Γ be a principal congruence subgroup of level N in $SL(2,\mathbb{Z})$, $\mathfrak{h}^2$ the Poincare upper half plane. By strong approximate theorem, we know

$$\Gamma\backslash\mathfrak{h}^2 \longrightarrow GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A})/K_{\mathbb{R}}K_f \tag{6.29}$$

is bijective, where $K_{\mathbb{R}}=SO(2)$, and K_f is compact open subgroup of $GL_2(\mathbb{A}_f)$ defined in the following.

- For $p\nmid N$, define $K_p = GL_(\mathbb{Z}_p)$.
- For $p\mid N$, consider the diagonal embedding

  $$SL(2,\mathbb{Z})\hookrightarrow \prod_{p\mid N} GL_2(\mathbb{Z}_p).$$

  Let K_N be open subgroup of $\prod_{p\mid N} GL_2(\mathbb{Z}_p)$ such that
  - the preimage of K_N in $SL(2,\mathbb{Z})$ is Γ
  - $\det(K_N)=\prod_{p\mid N}\mathbb{Z}_p^\times$.

  Eg,

  $$K_N=\left\{k, k\equiv\begin{pmatrix}1 & \\ & *\end{pmatrix} \bmod N\right\}$$
- Let $K_f = K_N\prod_{p\nmid N} K_p$.

Remark 27. By (6.29), we can lift the maass cusp form on $\Gamma\backslash\mathfrak{h}^2$ to be a function on $GL_2(\mathbb{Q})\backslash G(\mathbb{A})$ which is fixed by right action under $K_{\mathbb{R}}K_f$, and lift holomorphic modular form to be a function on $GL_2(\mathbb{Q})\backslash G(\mathbb{A})$ fixed by K_f and transforming in a certain way under $K_{\mathbb{R}}$.

F.2. Hecke operators.

For any $g\in G(\mathbb{A}_f)$, we can define Hecke operator T_g in the following.

- Consider the double cosets $K_f g K_f$, we have right coset decomposition

  $$K_f g K_f = \bigcup_i g_i K_f.$$

  Thus for $f\in GL(2,\mathbb{Q})\backslash GL(2,\mathbb{A})/K_{\mathbb{R}}K_f$, define the action of T_g on f via

  $$T_g f(x) = \sum_i f(xg_i)$$
- Adele scheme allows us to separable global problem into local ones. For $p\nmid N$, the Hecke operator T_p and $T_{p,p}$ corresponds to

  $$K_p\begin{pmatrix}1 & \\ & p\end{pmatrix}K_p, \qquad K_p\begin{pmatrix}p & \\ & p\end{pmatrix}K_p,$$

and the action of T_p and $T_{p,p}$ can be expressed as convolution with characteristic function of the above double cosets.

F.3. **Automorphic forms.** Let G be a reductive group defined over a number field F. Automorphic forms on G is just functions $f : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ satisfying

- a condition of moderate growth on adele version of Siegel set,
- smooth at real primes, and is $(Z(\mathfrak{g}_\infty^{\mathbb{C}}), K_\infty)$-finite,
- fixed with respect to the right translation of open subgroup K_f of $G(\mathbb{A}_f)$.

Here G is reductive over F. We know that G is unramified over F_v for almost all v. That is to say, G is unramified outside a finite set of primes D_G, or equivalently

- G arises by base extension from smooth reductive group over $O_F[1/N]$ for some integer N,
- For $p \mid N$, G/F_p arise by base extension from smooth reductive group scheme over O_p, i.e.
$$G(F_p) = G(O_p) \otimes_{O_p} F_p.$$

Remark 28. In general, we dealing with automorphic forms on connected reductive groups G over F with center character defined by a Hecke character ω.

Hecke operators are determined through convolution by functions on $K_f\backslash G(\mathbb{A}_f)/K_f$ with K_f as above. One can express $K_f = K_S \prod_{p\notin S} K_p$ where $S \supset D_G$ such that $K_p = G(O_F)$ for $p \notin S$. So by local global principle, the Hecke operators involving convolution of functions (generated by characteristic functions of double cosets) on $K_p\backslash G(F_p)/K_p$, i.e. functions in $\mathcal{H}(G(F_p), K_p)$ for $p \notin S$. We will show that $\mathcal{H}(G(F_p), K_p)$ is commutative algebra, and the action on spherical vectors gives a scalar, by prove the Satake isomorphism.

F.4. **Eisenstein series on $\mathfrak{h}$.** Let $\Gamma = SL(2,\mathbb{Z})$ and $z = x + iy \in \mathfrak{h}$. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1/2$,
$$E_s(z) := \sum_{\substack{c\geq 0\\ (c,d)=1}} \frac{y^{s+\frac{1}{2}}}{|cz+d|^{2s+1}} = \sum_{\Gamma_P\backslash\Gamma} Im(\gamma.z)^{s+\frac{1}{2}},$$
where
$$\Gamma_P = \{\gamma.(i\infty) = (i\infty), \gamma \in \Gamma\} = P \cap \Gamma = \left\{\pm I, \begin{pmatrix} 1 & n \\ & 1 \end{pmatrix}, n \in \mathbb{Z}\right\}$$
It is well defined since $Im(\gamma.z)$ is Γ_P invariant and is absolutely convergent for $\mathrm{Re}(s) > 1/2$. Moreover,

- It is eigenfunction of Laplacian operator with eigenvalue
$$\Delta E_s = (s^2 - \frac{1}{4})E_s.$$
- $$E_s(z) \sim y^{1/2+s} + c(s)y^{\frac{1}{2}-s}, \quad y \to \infty.$$
i.e., the constant term along parabolic subgroup P is
$$\int_0^1 E_s(x+iy)dx = y^{1/2+s} + c(s)y^{\frac{1}{2}-s}$$
- It satisfies the functional equation
$$E_s(z) = c(s)E_{1-s}(z)$$
which implies that
$$c(s)c(1-s) = 1.$$

We can calculate the constant term directly by

$$\begin{aligned}\int_0^1 E_s(x+iy)dx &= y^{s+\frac{1}{2}} + y^{s+\frac{1}{2}}\left(\int_{-\infty}^{\infty}\frac{dw}{|w^2+y^2|^{s+\frac{1}{2}}}\right)\sum_{c>0}\frac{\varphi(c)}{c^{2s+1}} \\ &= y^{s+\frac{1}{2}} + y^{s+\frac{1}{2}}\frac{\zeta_{\mathbb{R}}(2s)}{\zeta_{\mathbb{R}}(2s+1)}\sum_{c>0}\frac{\varphi(c)}{c^{2s+1}}\end{aligned}$$

where $\zeta_{\mathbb{R}}(s) = \pi^{-\frac{s}{2}}\Gamma(s/2)$, and $\varphi(c) = \sum_{\substack{n\leq c\\(n,c)=1}} 1$. Moreover,

$$\sum_{c>0}\frac{\varphi(c)}{c^{2s+1}} = \prod_p\sum_{n\geq 0}\frac{\varphi(p^n)}{p^{n(2s+1)}} = \frac{\zeta(2s)}{\zeta(2s+1)}$$

It implies that

$$c(s) = \frac{\xi(2s)}{\xi(2s+1)}.$$

F.5. **The lift of classical Eisenstein series.** We can lift the Eisenstein series to $SL(2,\mathbb{R})$. Via the decomposition $SL(2,\mathbb{R}) = BK$, or

$$SL(2,\mathbb{R}) \to \mathfrak{h}^2, \quad \begin{pmatrix} a & b \\ c & d\end{pmatrix} \mapsto \frac{ai+b}{ci+d}.$$

we can define

$$\tilde{E}_s(g_{\mathbb{R}}) = E_s(g.i).$$

Moreover, since

$$SL(2,\mathbb{A}) = SL(2,\mathbb{Q})(SL(2,\mathbb{R})K_f)$$

with $K_f = \prod_p K_p$, we can lift the Eisenstein series to be functions on $SL(2,\mathbb{A})$ via

$$\mathcal{E}_s(\gamma g_{\mathbb{R}} k_f) = \tilde{E}_s(g_{\mathbb{R}}) = E_s(g_{\mathbb{R}}.i), \quad g = \gamma g_{\mathbb{R}} k_f.$$

F.6. **Philosophy of cusp form.** Recall Parabolic induction in representation of Lie groups. Consider $G = SL(2,\mathbb{R})$ and $B(\mathbb{R}) = M(\mathbb{R}) \ltimes N(\mathbb{R})$. Let σ be a representation of the levi-component $M(\mathbb{R})$. We inflate it to be a representation of $P(\mathbb{R})$ by letting $N(\mathbb{R})$ acting trivially, and normalized parabolic induced to be a representation of $G(\mathbb{R})$ via

$$i^G_{M,P}\sigma = ind^G_P(\sigma\otimes\delta^{1/2})$$

Let Γ be a principal congruence subgroup of $SL(2,\mathbb{Z})$. The question is, how to obtain an automorphic forms on $G(\mathbb{R})$ w.r.t. Γ from automomrphic forms on $M(\mathbb{R})$ via the above philosophy?

- Assume we have an automorphic form on $M(\mathbb{R})$ with respect to $M_\Gamma = M(\mathbb{R})\cap\Gamma$, i.e. a function
$$I_s : M_\Gamma\backslash M(\mathbb{R}) \to \mathbb{C}, \quad I_s\left(\begin{pmatrix}\pm y^{\frac{1}{2}} & \\ & \pm y^{-1/2}\end{pmatrix}\right) = |y|^s$$
- Inflate it to be a function on $P(\mathbb{R})$ via letting $N(\mathbb{R})$ acts trivially, we have an automorphic forms on P, namely
$$I_s : P_\Gamma N(\mathbb{R})\backslash P(\mathbb{R}) \to \mathbb{C}, \quad I_s\left(\begin{pmatrix}\pm y^{\frac{1}{2}} & x \\ & \pm y^{-1/2}\end{pmatrix}\right) = |y|^s$$

- Finally, we use the normalized induction to obtain a function on $G(\mathbb{R})$, i.e. to obtain a function
$$\varphi_s : P_\Gamma N(\mathbb{R})\backslash G(\mathbb{R}) \to \mathbb{C}, \quad \varphi_s\left(\begin{pmatrix} \pm y^{1/2} & x \\ & \pm y^{-1/2} \end{pmatrix} g\right) = |y|^{s+\frac{1}{2}}\varphi_s(g).$$
Note that we also need φ_s is right $K_{\mathbb{R}}$-finite.
- The above function φ_s is defined on $G(\mathbb{R})$, but only with automorphism with respect to P_Γ. To obtain a real automorphic form on $G(\mathbb{Z})\backslash G(\mathbb{R})$, we finally define
$$\mathcal{E}(g, \varphi_s) := \sum_{\gamma \in P_\Gamma \backslash \Gamma} \varphi_s(\gamma g)$$
It is an automorphic forms on $\Gamma\backslash G(\mathbb{R})$.

F.7. **Constant term of Eisenstein series.** Now we consider the constant term of Eisenstein series. Recall that if Φ is an automorphic form on $\Gamma\backslash G(\mathbb{R})$, the constant term is defined by
$$\hat{\Phi}(g) = \int_{N(\mathbb{Z})\backslash N(\mathbb{R})} \Phi(ng)dn$$
Since for $\gamma \in P_\Gamma$, $n\gamma g = \gamma n' g$. It is easy to see that $\hat{\Phi}(g)$ is a function on $P_\Gamma N(\mathbb{R})\backslash G(\mathbb{R})$.

Remark 29. The constant term along parabolic subgroup $P = M^P \ltimes N^P$ is defined by
$$\hat{\Phi}_P(g) := \int_{N^P(\mathbb{Z})\backslash N^P(\mathbb{R})} \Phi(ng)dn$$
maps automorphic form to be function on $P_\Gamma N(\mathbb{R})\backslash G(\mathbb{R})$.

F.8. **Adele version.** Recall that we have
$$\mathbb{A}^\times = \mathbb{Q}^\times(\mathbb{R}_+^\times \prod_p \mathbb{Z}_p^\times).$$
For $G = SL_2$, it gives
$$G(\mathbb{A}) = G(\mathbb{Q})\left(G(\mathbb{R})(\prod_p G(Z_p))\right), \quad M(\mathbb{A}) = M(\mathbb{Q})\left(M(\mathbb{R})(\prod_p M(Z_p))\right).$$
Let Γ be a principal congruent subgroup of level N in $SL(2,\mathbb{Z})$. Recall $K_f = K_N \prod_p K_p$ is defined in section F.1.

Question: How to obtain an Adele analogue of function φ_s on $P_\Gamma N(\mathbb{R})\backslash G(\mathbb{R})$?

- By Langlands decomposition, we have
$$G(\mathbb{A}) = P(\mathbb{A})K_{\mathbb{R}}K_f$$
Moreover, the decomposition
$$P(\mathbb{A}) = M(\mathbb{A})N(\mathbb{A}), \quad M(\mathbb{A}) = M(\mathbb{Q})(M(\mathbb{R})(\prod_p M(Z_p))$$
implies that
$$P(\mathbb{Q})N(\mathbb{A})\backslash G(\mathbb{A})/K_f \simeq P_\Gamma N(\mathbb{R})\backslash G(\mathbb{R}) \tag{6.30}$$

- Let φ_s be the unique function on $P(\mathbb{Q})N(\mathbb{A})\backslash G(\mathbb{A})/K_{\mathbb{R}}K_f$ such that
$$\varphi_s(pg) = \delta_P(p)^{1/2+s}\varphi_s(g), \qquad \varphi_s(1) = 1.$$
Here δ_P is the modulus function on $P(\mathbb{A})$ defined by the product of local ones.
- The Eisenstein series $\mathcal{E}(*, \varphi_s)$ is defined by
$$\mathcal{E}(g, \varphi_s) := \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi_s(\gamma g).$$

Question: How about the constant term of the Eisenstein series?
$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \mathcal{E}_s(ng)dn = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \sum_{\gamma \in P(\mathbb{Q})\backslash G(\mathbb{Q})} \varphi_s(\gamma ng)dn.$$

- The idea is to use the Bruhat decomposition, i.e.
$$G(\mathbb{Q}) = P(\mathbb{Q}) \cup \bigcup_{w \neq 1} P(\mathbb{Q})wN'_w(\mathbb{Q}), \quad P(\mathbb{Q})\backslash G(\mathbb{Q}) = \{1\} \cup \bigcup_{w \neq 1} wN'_w(\mathbb{Q}),$$
where $N = N_w \cdot N'_w = N'_w \cdot N_w$ with
$$N_w = wNw^{-1} \cap N, \quad N'_w = w\overline{N}w^{-1} \cap N.$$
- Thus we can express the constant term as
$$E_P(g, \varphi_s) = \varphi_s(g) + \sum_{w \neq 1} \int_{N'_w(\mathbb{A})} \varphi_s(wng)dn.$$
In the case $G = SL_2$, we have
$$E_P(g, \varphi_s) = \varphi_s(g) + \int_{N(\mathbb{A})} \varphi_s(wng)dn.$$
- So we need to calculate the local integral
$$\int_{N(\mathbb{Q}_v)} \varphi_{s,v}(wn)dn,$$
where $\int_{N(\mathbb{Q}_v)} \varphi_{s,v}(wn)dn$ is a constant given by intertwining operator acts on spherical vector of spherical representations, for almost all v.

Appendix G. Some integral transform

We review the proof of Lemma 8.4 (Lemma 2.6).

Lemma G.1. *We have*
$$\pi^{-s}\Gamma(s)y^s \int_{-\infty}^{\infty} \frac{1}{(x^2+y^2)^s} e(-nx)dx = \begin{cases} \pi^{-s+\frac{1}{2}}\Gamma(s-\frac{1}{2})y^{1-s}, & n = 0 \\ 2|n|^{s-\frac{1}{2}}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|n|y), & n \neq 0. \end{cases}$$

Proof. Note that
$$\Gamma(s) = \int_0^{\infty} e^{-t}t^s \frac{dt}{t}.$$

Thus

$$\begin{aligned}
& \pi^{-s} y^s \int_0^\infty e^{-t} t^s \frac{dt}{t} \int_{-\infty}^\infty \frac{1}{(x^2+y^2)^s} e(-nx) dx \\
= & \int_{-\infty}^\infty \left(\int_0^\infty \left(\frac{y}{\pi(x^2+y^2)} t \right)^s e^{-t} \frac{dt}{t} \right) dx = \int_{-\infty}^\infty \left(\int_0^\infty t^s e^{-t \frac{\pi(x^2+y^2)}{y}} \frac{dt}{t} \right) dx \\
= & \int_0^\infty t^s e^{-t\pi y} \left(\int_{-\infty}^\infty e^{-\frac{\pi t}{y} x^2} e^{-2\pi i n x} dx \right) \frac{dt}{t}
\end{aligned}$$

Since

$$\int_{-\infty}^\infty e^{-\frac{\pi t}{y} x^2} e^{-2\pi i n x} dx = \begin{cases} \sqrt{\frac{y}{t}}, & n = 0 \\ \sqrt{\frac{y}{t}} e^{-\frac{\pi y n^2}{t}}, & n \neq 0 \end{cases}$$

The result follows immediately by the expression of Γ-function and K-Bessel function. □

Next, we consider (11.22).

Lemma G.2. *One has*

$$\int_0^\infty e^{-y} y^{s-\frac{1}{2}} K_\nu(y) \frac{dy}{y} = \pi^{1/2} 2^{\frac{1}{2}-s} \frac{\Gamma(s-\frac{1}{2}-\nu)\Gamma(s-\frac{1}{2}+\nu)}{\Gamma(s)}$$

Proof. Note that K-Bessel function is Mellin transform of $\frac{1}{2} e^{a(y+\frac{1}{y})}$, (see (1.25)),

$$\int_0^\infty e^{-\frac{y}{2}(t+\frac{1}{t})} t^s \frac{dt}{t} = 2K_s(y).$$

Applying this one has

$$\begin{aligned}
L.H.S. &= \int_0^\infty e^{-y} y^{s-\frac{1}{2}} \frac{1}{2} \int_0^\infty t^\nu e^{-\frac{y}{2}(t+\frac{1}{t})} \frac{dt}{t} \frac{dy}{y} \\
&= \frac{1}{2} \int_0^\infty \int_0^\infty t^\nu y^{s-\frac{1}{2}} t^\nu e^{-\frac{y}{2}(t+\frac{1}{t}+2)} \frac{dt}{t} \frac{dy}{y}
\end{aligned}$$

Let $yt = u_1$, $\frac{y}{t} = u_2$ and thus

$$y = \sqrt{u_1 u_2}, \quad t = \sqrt{\frac{u_1}{u_2}}$$

and thus

$$\begin{aligned}
dy &= \frac{1}{2}\sqrt{\frac{u_2}{u_1}} du_1 + \frac{1}{2}\sqrt{\frac{u_1}{u_2}} du_2 \\
dt &= \frac{1}{2}\sqrt{\frac{1}{u_1 u_2}} du_1 - \frac{1}{2}\sqrt{\frac{u_1}{u_2^3}} du_2
\end{aligned}$$

and thus

$$ty = u_1, \quad dydy = -\frac{1}{2}\frac{1}{u_2} du_1 du_2.$$

It gives

$$
\begin{aligned}
L.H.S. &= \frac{1}{2}\int_0^\infty\int_0^\infty e^{-\sqrt{u_1u_2}}(u_1u_2)^{\frac{s}{2}-\frac{1}{4}}\left(\frac{u_1}{u_2}\right)^{\frac{\nu}{2}}\frac{1}{2}\frac{du_1du_2}{u_1u_2}\\
&= \frac{1}{4}\int_0^\infty\int_0^\infty e^{-\sqrt{u_1u_2}}u_1^{\frac{s}{2}+\frac{\nu}{2}-\frac{1}{4}}u_2^{\frac{s}{2}-\frac{\nu}{2}-\frac{1}{4}}e^{-\frac{(\sqrt{u_1}+\sqrt{u_2})^2}{2}}\frac{du_1du_2}{u_1u_2}
\end{aligned}
$$

□

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