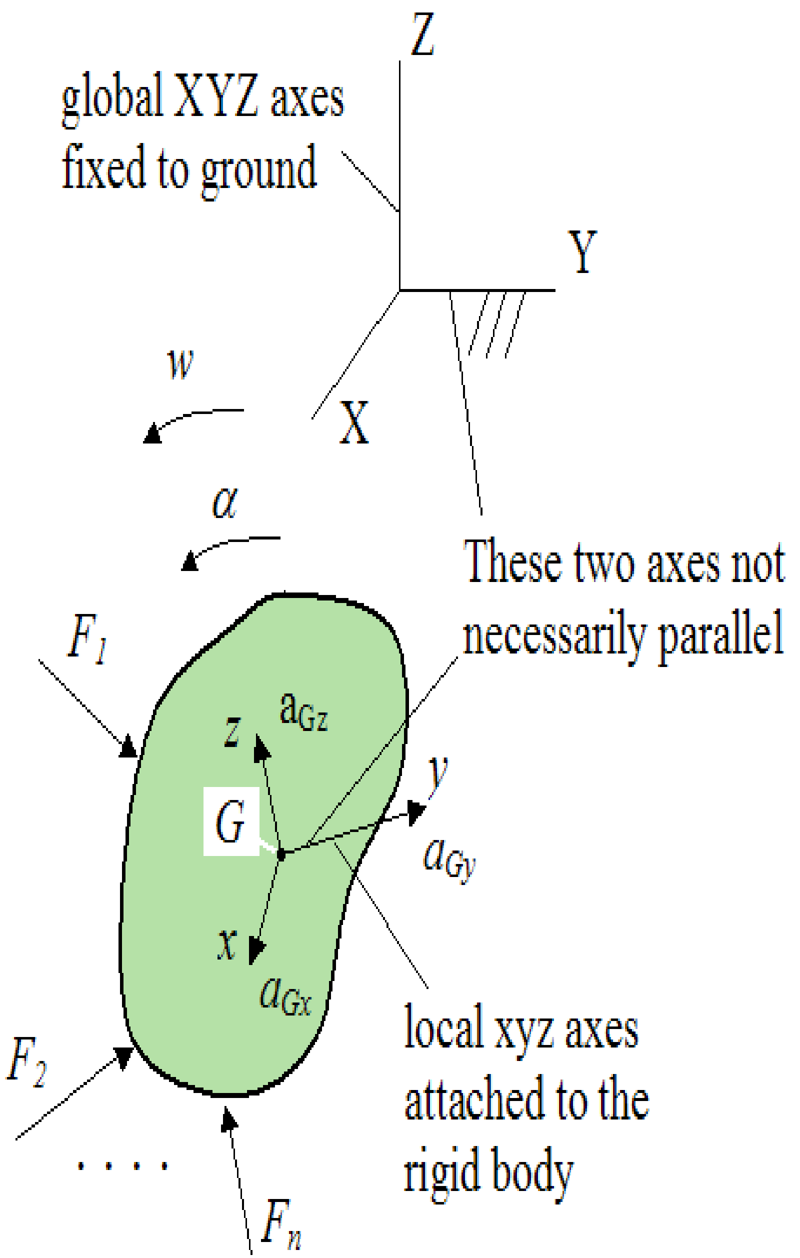
PHYS 216: ANALYTICAL MECHANICS I

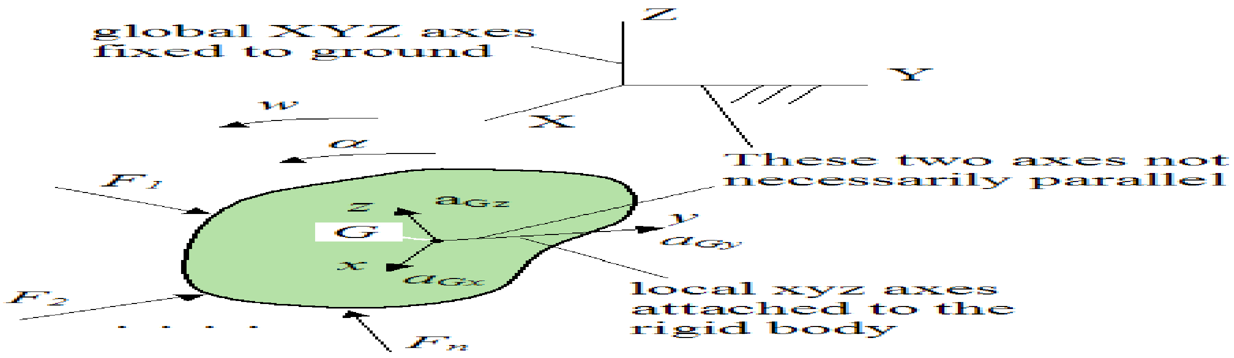
TODAY'S TOPIC: RIGID BODY DYNAMICS

OUTLINE:

1) **Review on Rigid Body Geometry: Definition of particles and rigid body.**
2) **Brief on Degrees of Freedom of a rigid body.**
3) **Basic kinematics of rigid bodies: classifications of general motion of RB.**



Review on Rigid Body Geometry

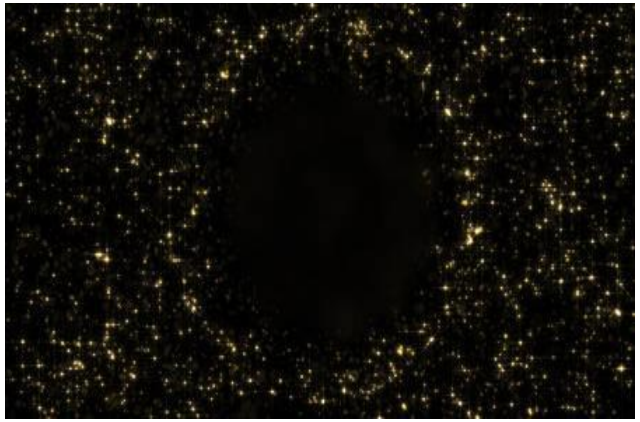


The term **rigid** is in reality a *mathematical idealization*, because all bodies deform by a certain amount under the application of loads.

If the deformation is small compared to the overall dimensions of the body, and energy dissipation due to elastic effects is negligible, the rigid body assumption can be safely used.

Review on Rigid Body Geometry

Question: Distinguish between particles and rigid bodies



A

B

C

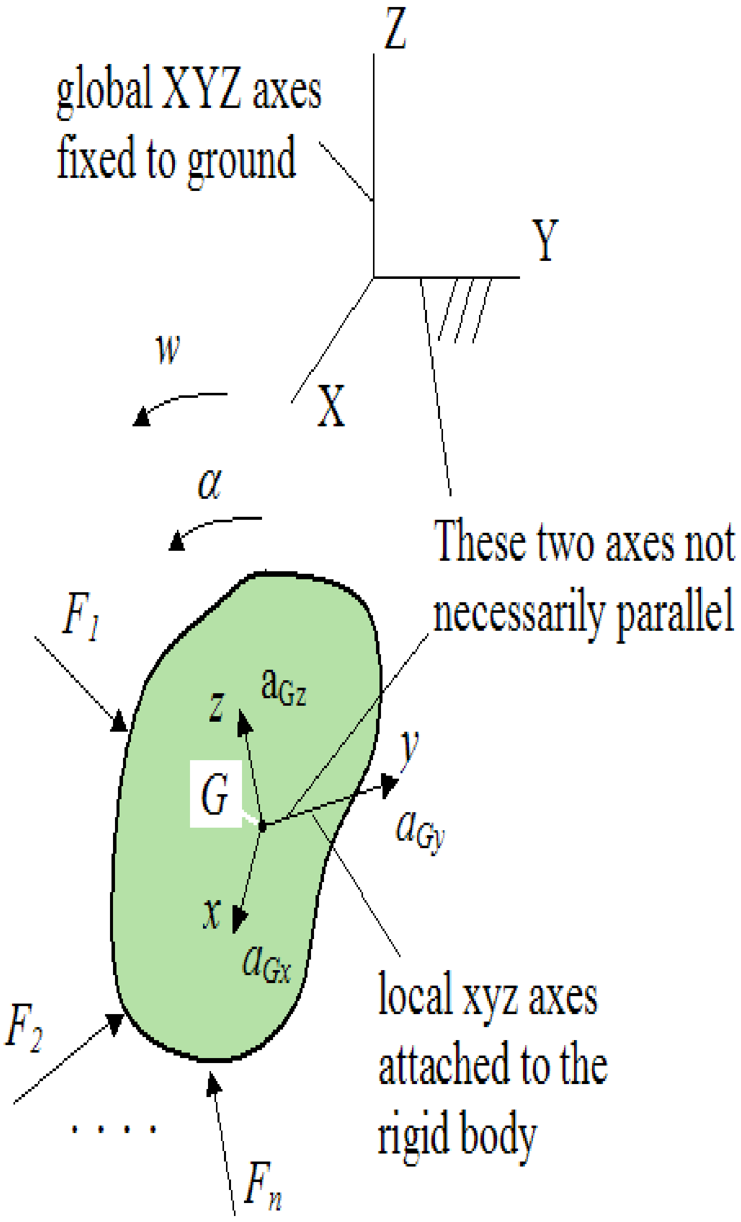
Review on Rigid Body Geometry

What is a particle?: A particle is defined as a body with no physical dimensions.

Note: This definition is also an idealization, as all bodies have physical dimensions.

If the dimensions of the body are much smaller than the path followed by the body, it becomes possible to neglect the physical dimensions of the body.

Hence, we do not consider any rotational motion and only three translational degrees of freedom are sufficient to describe the motion.



Review on Rigid Body Geometry

<u>What is a rigid body?</u> : A rigid body is defined as a body with physical dimensions where the distances between the particles that constitute the body remain unchanged ***<u>(deformation is zero or so small that it can be neglected).</u>***

In considering the rotational motion of a rigid body, the following are required to completely describe its motion:

1) six degrees of freedom

2) three translational motion

3) three rotational motion

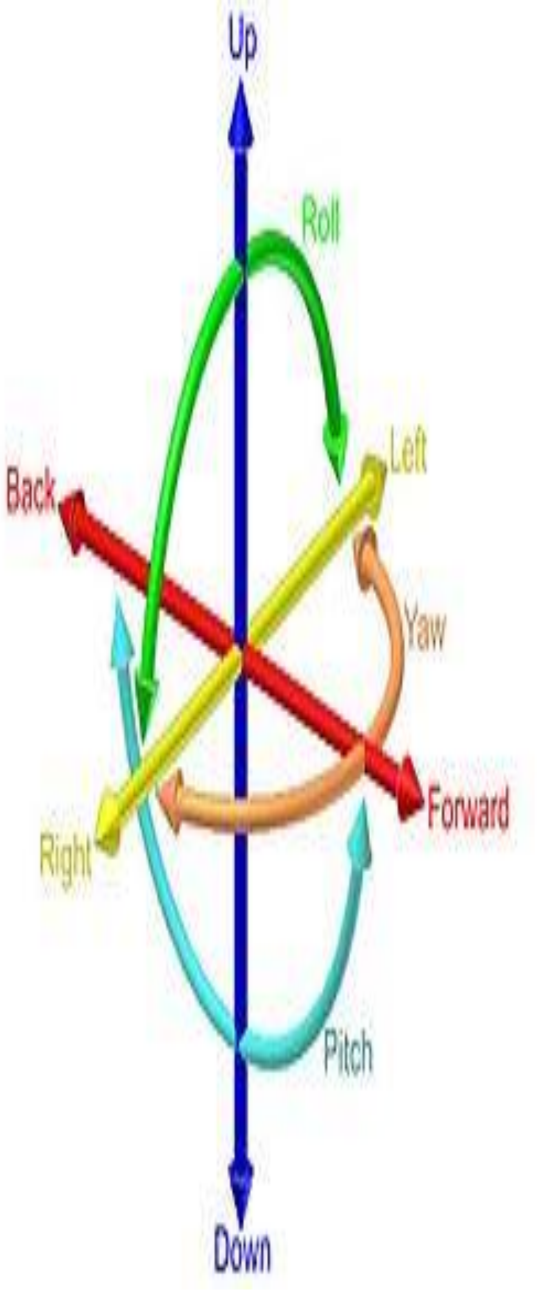
Brief on Degrees of Freedom of a rigid body

<u>So, what is a degree of freedom?:</u> The number of degrees of freedom **[*f*]** of a system represents the number of coordinates that are necessary to describe the motion of the particles of the system.

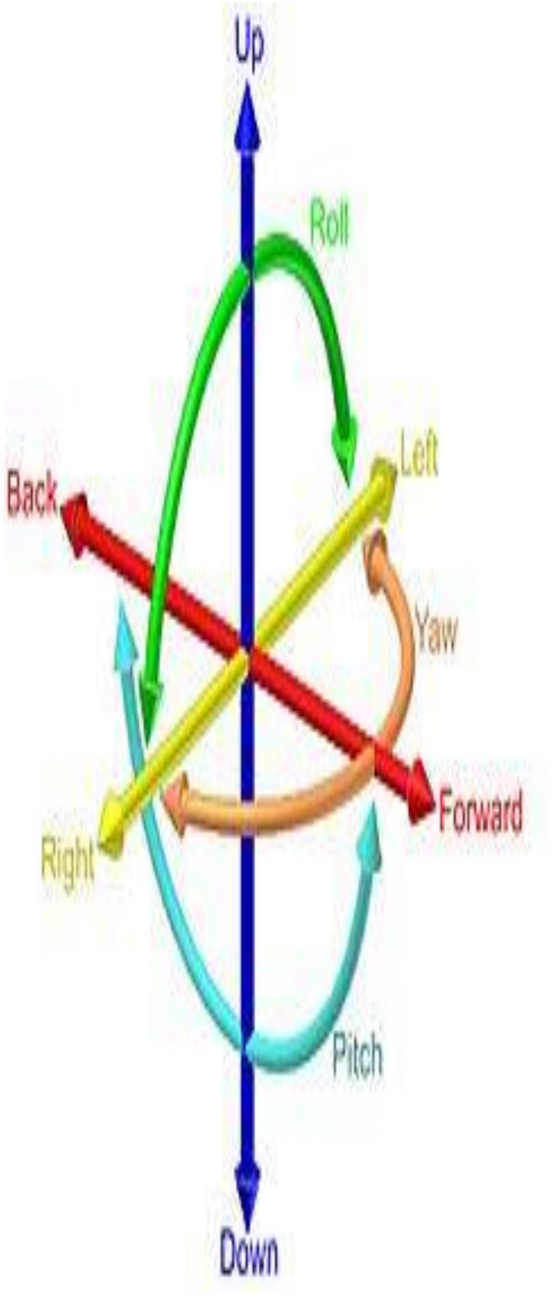
A mass point (particle) that can freely move in space has 3 translational degrees of freedom: (x,y,z).

If there are *n* mass points freely movable in space, this system has *3n* degrees of freedom:

$$(x_i, y_i, z_i),\ i = 1,...,n.$$



Brief on Degrees of Freedom of a rigid body



<u>For a Rigid Body:</u>

We look for the number of degrees of freedom of a rigid body that can freely move.

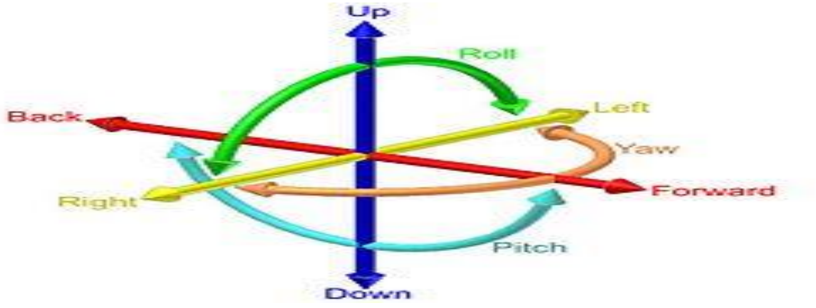
To describe a rigid body in space, one must know 3 non-collinear points of it. Hence,

one has 9 coordinates:

$r_1 = (x_1,y_1,z_1)$, $r_2 = (x_2,y_2,z_2)$, $r_3 = (x_3,y_3,z_3)$.

However, these coordinates are <u>mutually perpendicular</u>.

Brief on Degrees of Freedom of a rigid body



Since by definition we are dealing with a rigid body, the distances between any two points are constant. One obtains:

Recall: [$r_1 = (x_1,y_1,z_1)$, $r_2 = (x_2,y_2,z_2)$, $r_3 = (x_3,y_3,z_3)$]

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2 = C_1^2 = \text{constant},$$
$$(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 = C_2^2 = \text{constant},$$
$$(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2 = C_3^1 = \text{constant}.$$

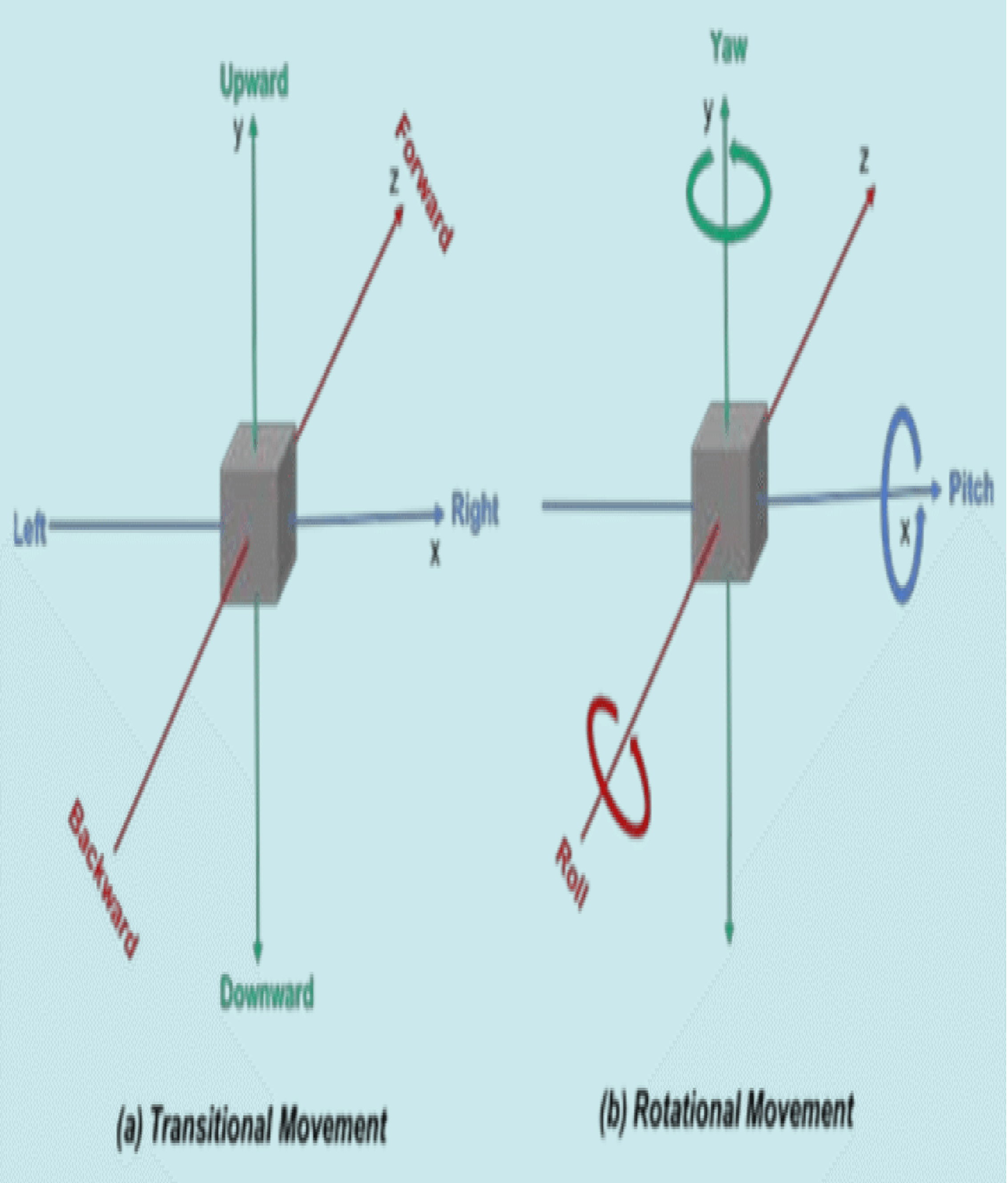
Three coordinates can be eliminated by means of these 3 equations.

The remaining 6 coordinates represent the 6 degrees of freedom.

These are the 3degrees of freedom of translation and the 3degrees of freedom of rotation

Brief on Degrees of Freedom of a rigid body

The motion of a rigid body can always be understood as a translation of any of its points relative to an inertial system and a rotation of the body about this point (Chasles' theorem)



Spin-It: Optimizing Moment of Inertia for Spinnable Objects

Moritz Bächer — Disney Research Zurich
Emily Whiting — ETH Zurich
Bernd Bickel — Disney Research Zurich
Olga Sorkine-Hornung — ETH Zurich

Basic kinematics of rigid bodies

Points to note:

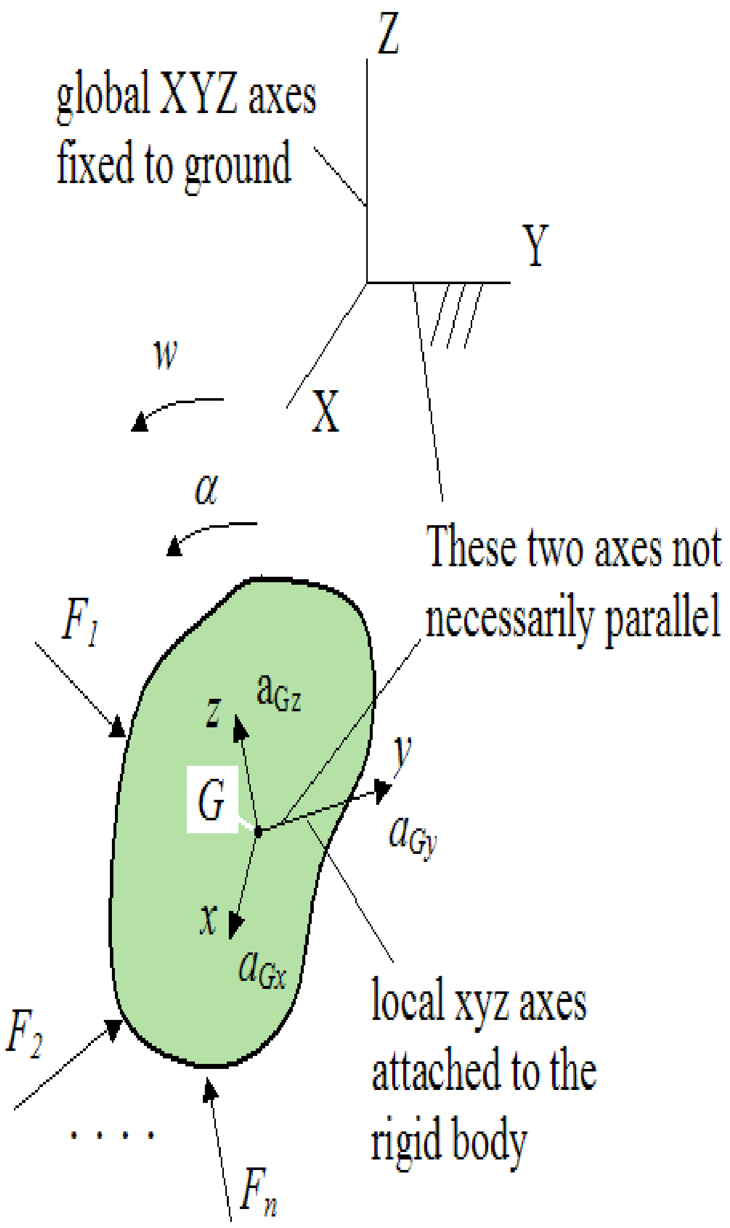
1) We will consider kinematical relations that describe the motion of rigid bodies.

2a) The field of kinematics is conveniently divided into two major components:

1] analysis

2] synthesis

2b) The focus here will be on kinematic analysis as kinematic synthesis is usually needed when designing interconnected bodies and mechanisms.



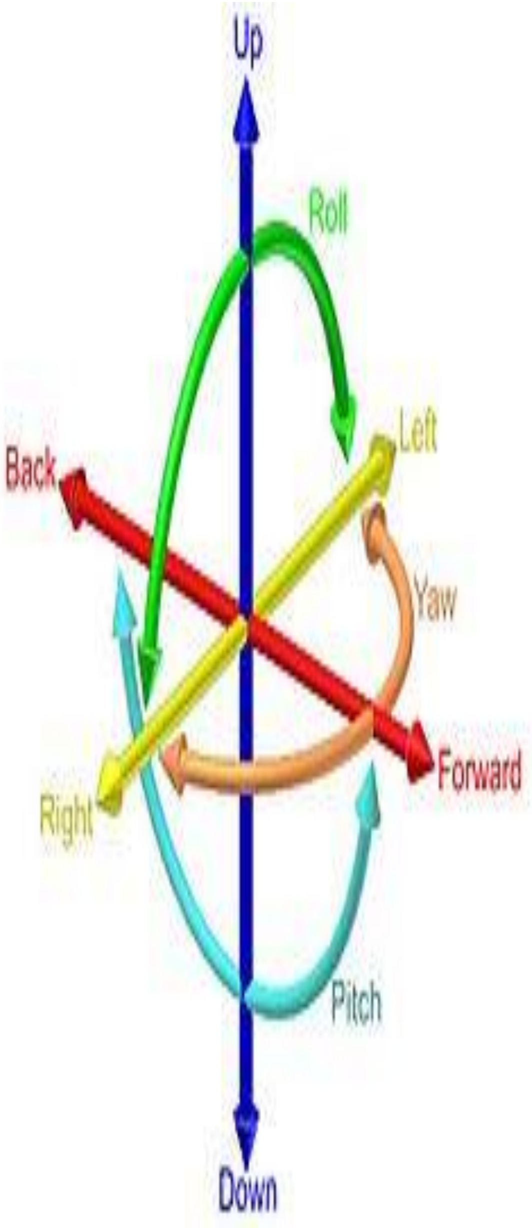
Basic kinematics of rigid bodies

The general motion of a rigid body can be classified into three categories.

a] Pure translation

b] Pure rotation

c] Combined translation and rotation



Basic kinematics of rigid bodies

To describe the kinematics, we will make use of the relative motion equations in (*Recall: equation 1 under moving coordinates frame*).

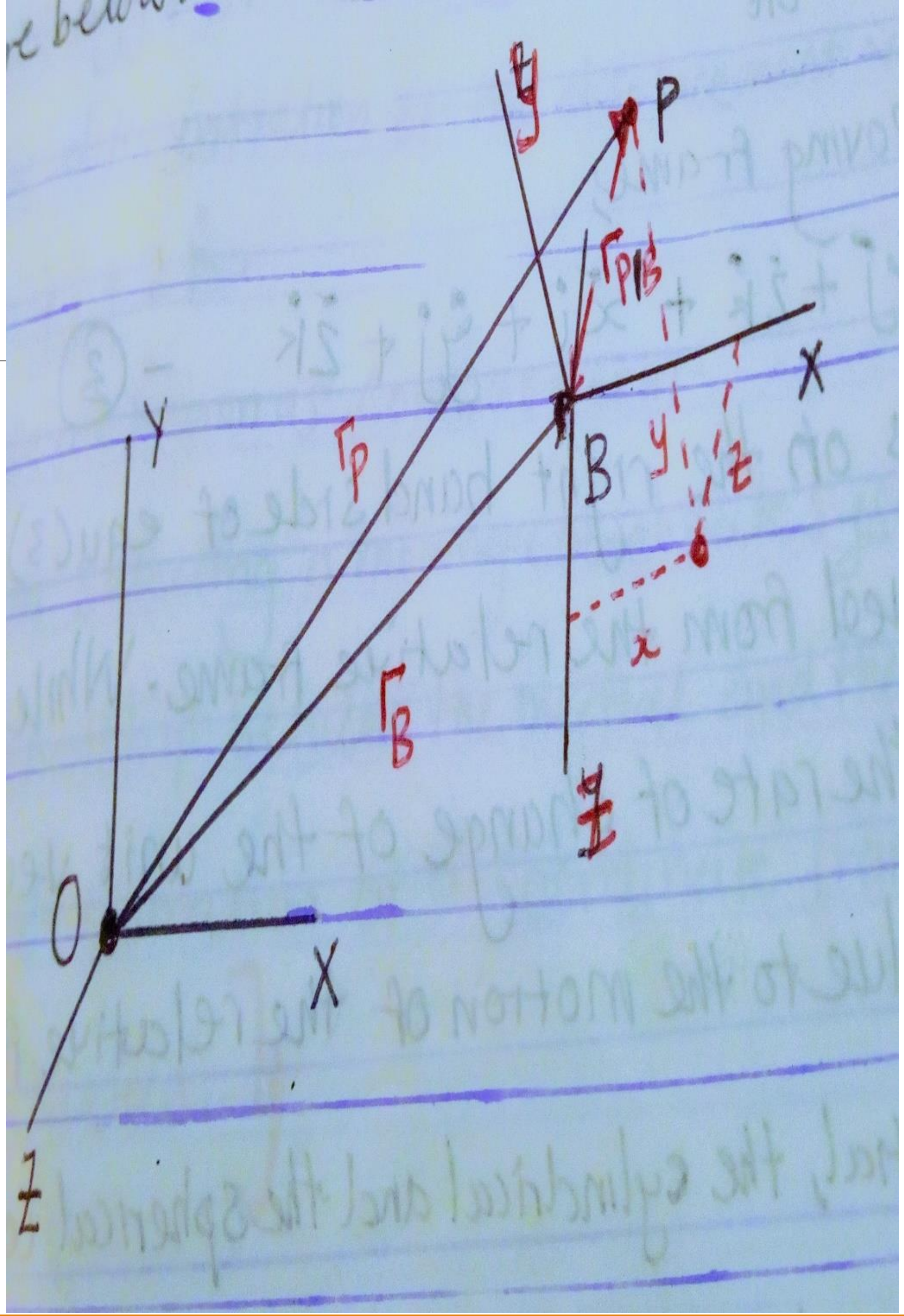
The velocity of a point ***P***, whose motion is observed from a rotating coordinate system with origin at B is expressed as:

$$v_p = v_B + v_{P/B} = V_B + \omega \times r_{P/B} + v_{rel} \quad \text{---[equ. 1]}$$

Where; V_B is the velocity of the origin of the reference frame.

ω is the angular velocity of the reference frame.

v_{rel} is the velocity of the point ***P*** as observed from the moving reference frame.



Basic kinematics of rigid bodies

The expression for the acceleration of ***P*** is:

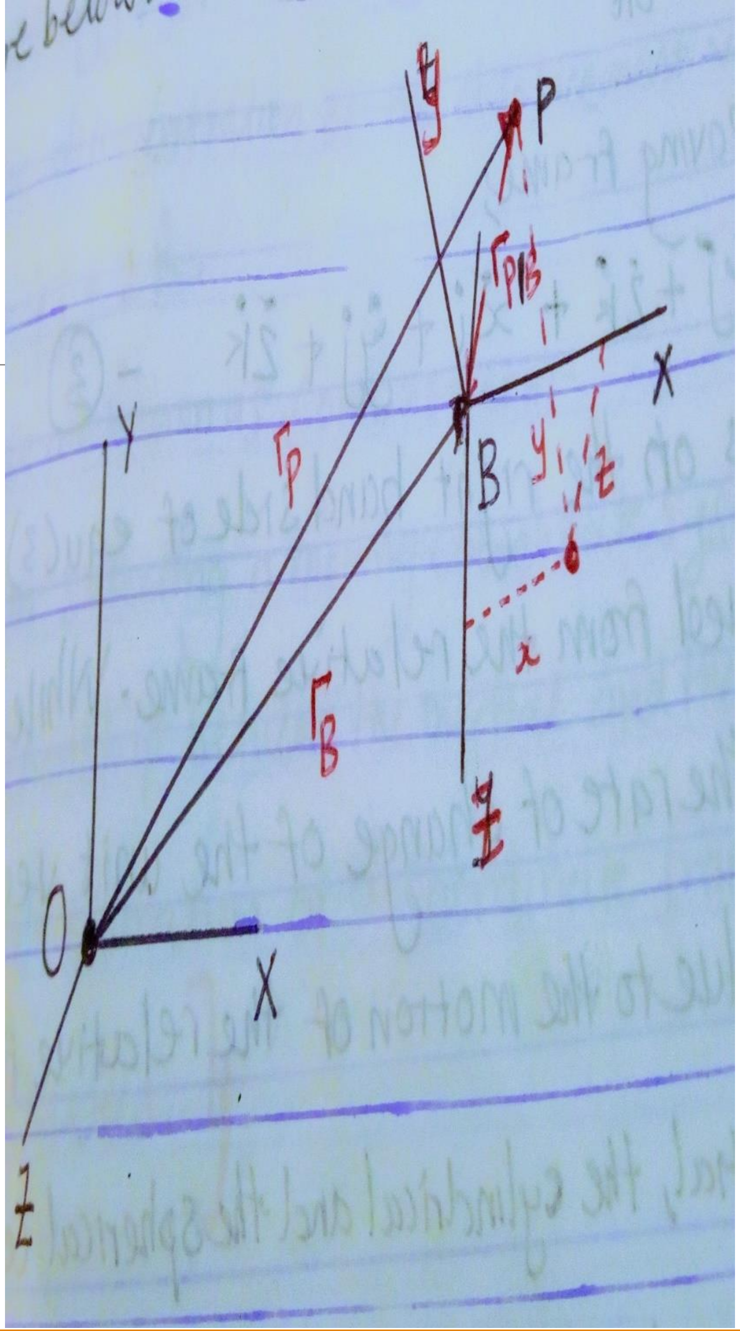
$$a_P = a_B + \propto \times r_{P/B} + \omega \times \left(\omega \times r_{P/B}\right) + 2\omega \times V_{rel} + a_{rel} \quad \text{---[equ. 2]}$$

The two equations just described above have a significant application for rigid bodies.

The moving frame can be attached to a point on the body and it moves with the body.

With this configuration, the relative axes are called <u>**body-fixed axes**</u> or the <u>**body axes.**</u>

The origin of the reference frame is usually selected as the center of the mass of (if it exists), or the center of rotation.



Basic kinematics of rigid bodies

The angular velocity and angular acceleration of the body are then the angular velocity and angular acceleration of the reference frame respectively.

V_{rel} and a_{rel} become the velocity and acceleration of point $\boldsymbol{P}$ with respect to the body.

If $\boldsymbol{P}$ is a point fixed on the rigid body, then $V_{rel} = \mathbf{0}$ and $a_{rel} = \mathbf{0}$.

Note: In the majority of dynamics problems involving three dimensional motion, one attaches the relative axes to the body (the study of axisymmetric bodies primarily makes use of this special case, which will be discussed in Analytical Mechanics II)

Classification of the general motion of rigid body

1] <u>PURE TRANSLATION</u>

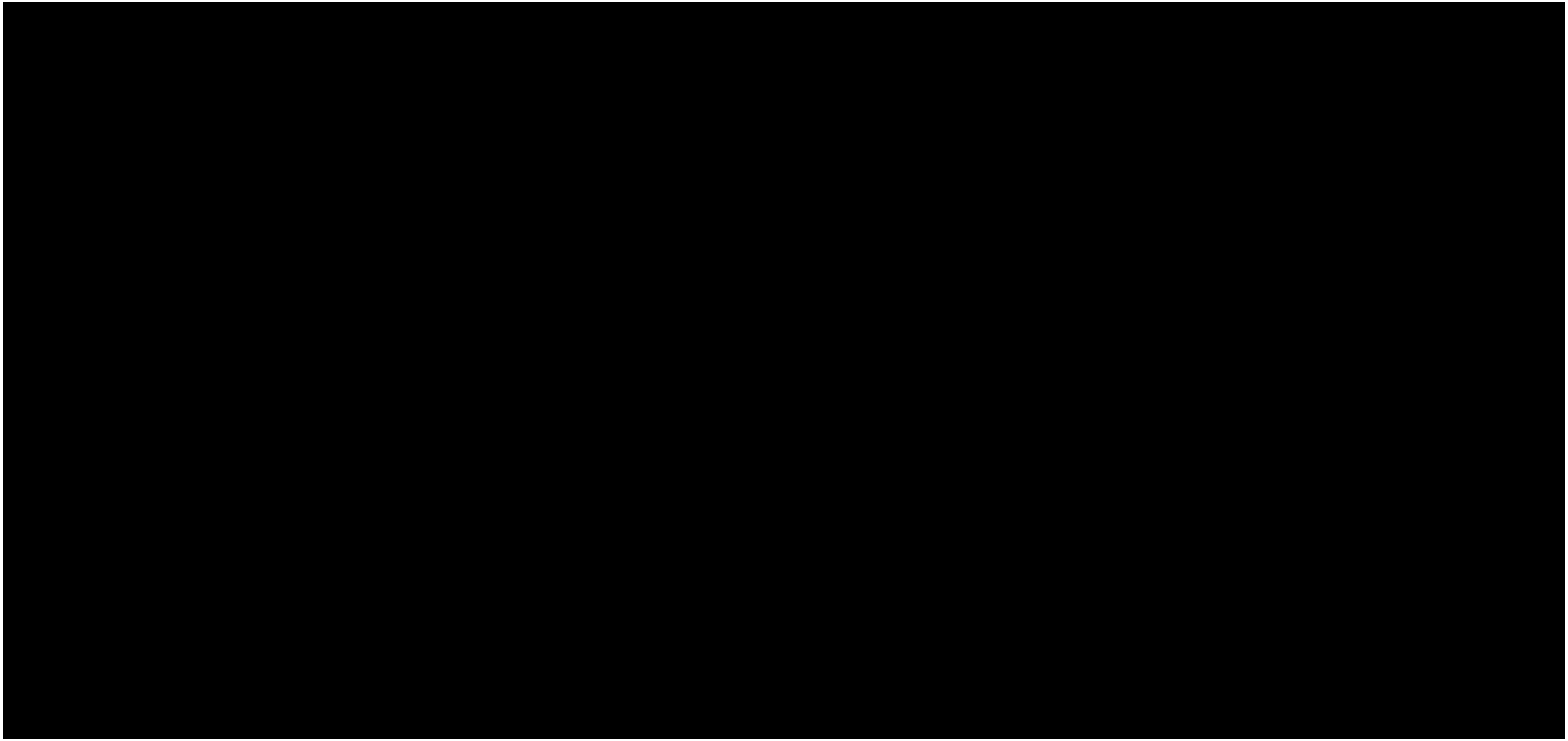
In this case, the rigid body moves with no angular velocity and no angular acceleration.

i.e: $\omega = 0, \alpha = 0.$

Every point on the body has the same translational velocity and acceleration, so that three translational parameters are sufficient to describe the motion.

In this case, we have three degrees of freedom.

Note: A rigid body is capable of moving along a curved trajectory without any rigid body rotation. [example is that of the landing of an aircraft].



Classification of the general motion of rigid body

2] <u>PURE ROTATION</u>

In this case, the motion of the rigid body is described using rotational parameters alone.

The velocity and acceleration of any point on the body can be expressed in terms of the angular velocity, angular acceleration, and the distance of the point from the rotation center.

This motion is separated into two categories.

a] Rotation about a fixed axis.

b] Rotation about a fixed point.

<u>Note</u>: Rotation about a fixed axis is a special case of rotation about a fixed point. For 2D motion, the above two categories coincide.

Classification of the general motion of rigid body: Pure Rotation

2a] **Rotation About a Fixed Axis**

Consider a body rotating about a fixed axis h. The direction of the angular velocity vector ω is along the fixed axis (as shown in Fig. A).

Recall that angular velocity denoted by $\boldsymbol{e}_h$, the unit vector along the fixed axis can then be written as:

$$\boldsymbol{\omega} = \omega \boldsymbol{e}_h \qquad \boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \dot{\omega} \boldsymbol{e}_h = \alpha \boldsymbol{e}_h \qquad \text{---[equ. 3]}$$

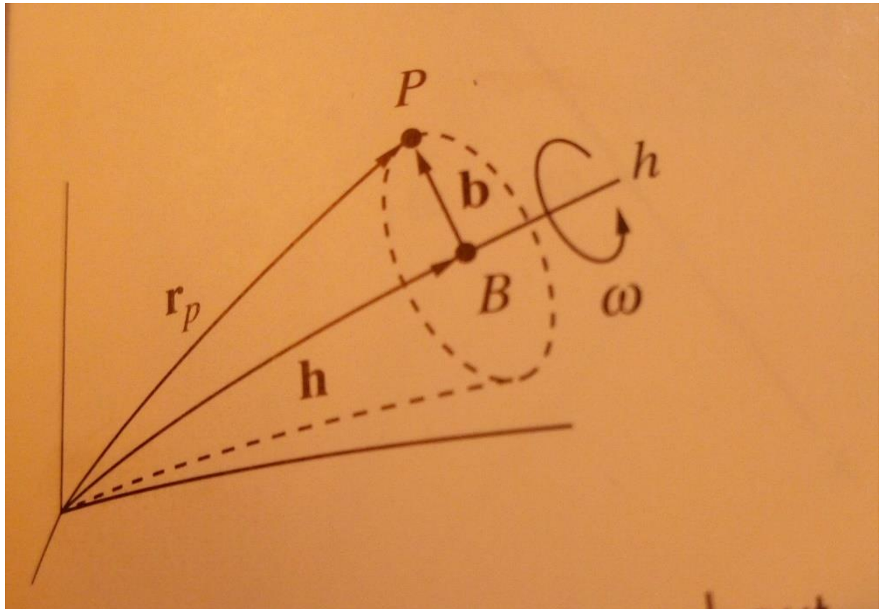
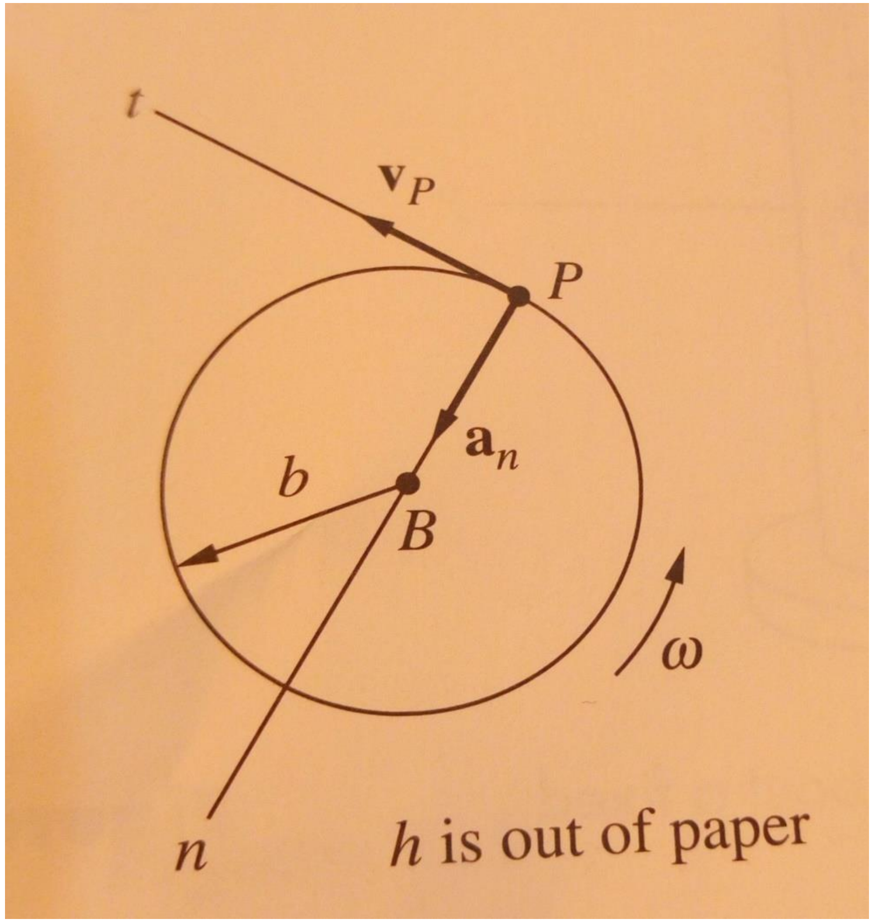


Figure A Rotation about a fixed axis

Classification of the general motion of rigid body: Pure Rotation

Note: The unit vector $\boldsymbol{e}_h$ is similar to the bi-normal vector: used in conjunction with the normal and tangential coordinates.

The only difference here is that the direction of $\boldsymbol{e}_h$ is fixed.



Classification of the general motion of rigid body: Pure Rotation

Let’s consider a point P on the body and express its position in terms of its components along the h axis and the plane perpendicular to the h axis.

Hence, we write $\boldsymbol{r}_p$ as:

$$\boldsymbol{r}_p = \boldsymbol{h} + \boldsymbol{b} \qquad \text{---[equ. 4]}$$

Where $\boldsymbol{h} = h\boldsymbol{e}_h$

$\mathbf{b} = -b\boldsymbol{e}_n$ (perpendicular distance from point $\boldsymbol{P}$ to the axis of rotation)

$\boldsymbol{e}_n$ is the associated unit vector and also in the normal direction.

When $\boldsymbol{P}$ is fixed on the body, its velocity is:

$$\boldsymbol{V}_p = \boldsymbol{\omega} \times \boldsymbol{r}_p = \omega\boldsymbol{e}_h \times (\boldsymbol{h} + \boldsymbol{p}) = \omega\boldsymbol{e}_h \times (h\boldsymbol{e}_h - b\boldsymbol{e}_n) = \omega b\boldsymbol{e}_t \quad \text{---[equ. 5]}$$

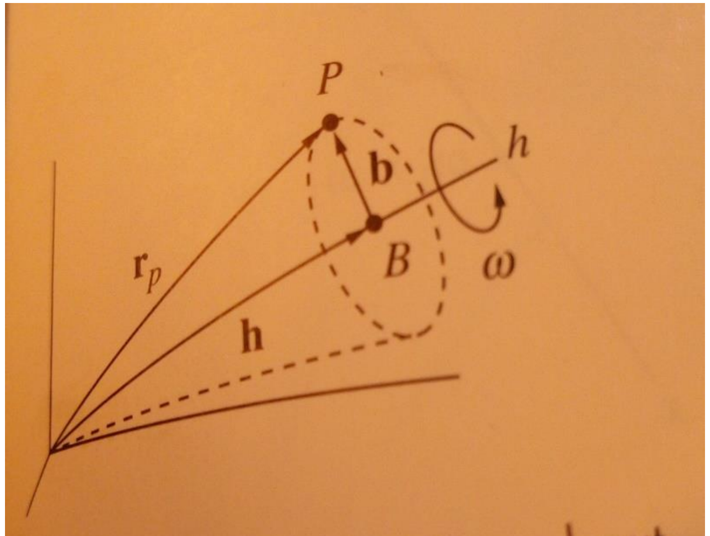


Figure A Rotation about a fixed axis

Classification of the general motion of rigid body: Pure Rotation

The magnitude of the velocity is bω, which leads to the conclusion that the velocity of a point on the body is dependent only on the perpendicular distance between that point and the axis of rotation.

Note: Rotation about a fixed axis is a single degree of freedom problem.

If we differentiate Eq. [5] to derive the acceleration of point P:

$$\boldsymbol{a}_p = \frac{d}{dt}\left(\boldsymbol{\omega} \times \boldsymbol{r}_p\right) = \boldsymbol{\alpha} \times \boldsymbol{r}_p + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_p) \quad \text{---[equ. 6]}$$

This is recognized as the sum of the tangential plus normal components. Hence we have:

$$\boldsymbol{a}_p = \boldsymbol{a}_t + \boldsymbol{a}_n \quad \text{---[equ. 7]}$$

Classification of the general motion of rigid body: Pure Rotation

From Eq. [7],

$$\boldsymbol{a}_t = \alpha \times r_p \qquad \boldsymbol{a}_n = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r}_p) \qquad \text{---[equ. 8]}$$

The magnitude of the tangential component of the acceleration is $\boldsymbol{a}_t = \alpha$b.

The magnitude of the normal component of the acceleration is $\boldsymbol{a}_n = \omega^2 b$

Hence Eq. [8] can be expressed as:

$\boldsymbol{a}_t = b\alpha e_t$

$\boldsymbol{a}_n = -\omega^2 b = b\omega^2 e_n$ ---[equ. 9]

Classification of the general motion of rigid body: Pure Rotation

2b] <u>**Rotation About a Fixed Point**</u>

In this case, the angular velocity vector $\boldsymbol{\omega}$ does not lie on a fixed axis.

The rate of change of the angular velocity depends on a change in direction as well as a change in magnitude.

A body rotating about a fixed point has three degrees of freedom, and the angular velocity is usually a combination of two or more rotation components.

Classification of the general motion of rigid body: Pure Rotation

Let's consider a cylinder in the Fig. B

The shaft is rotating about the fixed Z axis with angular velocity ω_1.

The cylinder is spinning with angular velocity ω_3 about axis h, which lies on the yz plane and makes an angle of β with the rod.

Hence the angular velocity of the cylinder can be expressed as:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \boldsymbol{\omega}_3 = \boldsymbol{\omega}_1 \boldsymbol{K} + \dot{\beta}\boldsymbol{i} + \omega_3 \boldsymbol{h}$$

$$= \omega_1 \boldsymbol{K} + \dot{\beta}\boldsymbol{i} + \omega_3(\cos\beta \boldsymbol{k} - \sin\beta \boldsymbol{j}) \qquad \text{---[equ. 10]}$$

Where $\boldsymbol{\omega}_3$ - spin rate $\quad \boldsymbol{\omega}_1$ - precession rate $\quad \beta$ – nutation angle.

The rate of change of β is called the nutation rate.

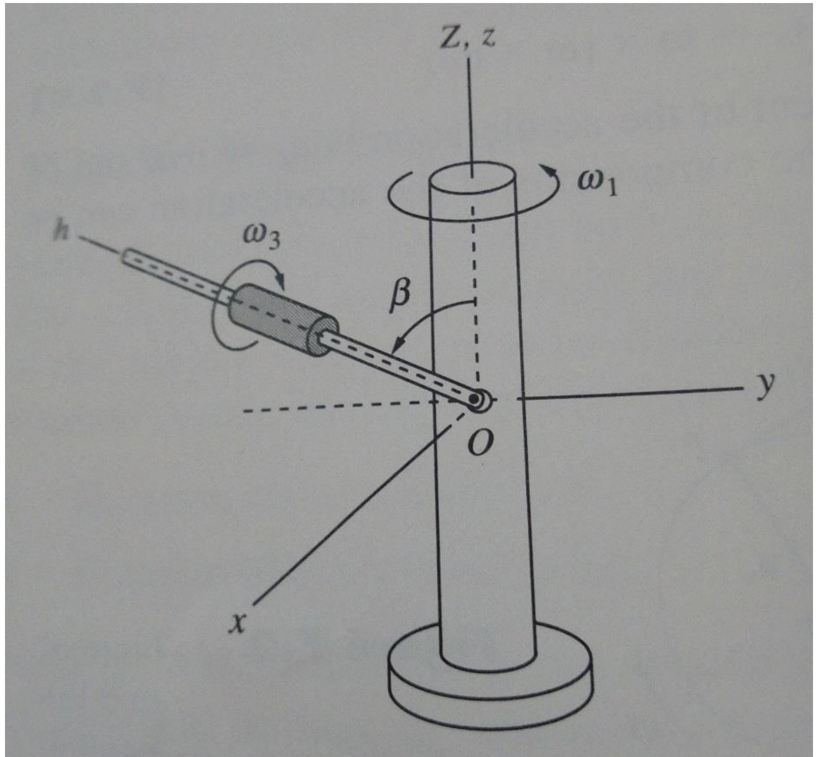


Figure B Rotation about a fixed point

Classification of the general motion of rigid body: Pure Rotation

Note: If the spin rate and precession rate are constant

A) The components of the angular velocity are constant in magnitude

B) The angular acceleration of the body is not zero, because the direction of the angular velocity vector is changing.

If we apply the transport theorem, we obtain:

$\boldsymbol{\alpha} = \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_2 + \dot{\boldsymbol{\omega}}_3 + \omega_1 \dot{\times} (\omega_2 + \omega_3) + \omega_2 + \omega_3$ ---[equ. 11]

The last two terms in this equation are also known as gyroscopic effects.

Fig. C shows this effect for $\omega_2 = 0$.

The term $\dot{\boldsymbol{\omega}}_1 + \dot{\boldsymbol{\omega}}_3$ describes the change in the magnitude of the angular vector

The term $\omega_1 \times \omega_3$ describes the change in direction.

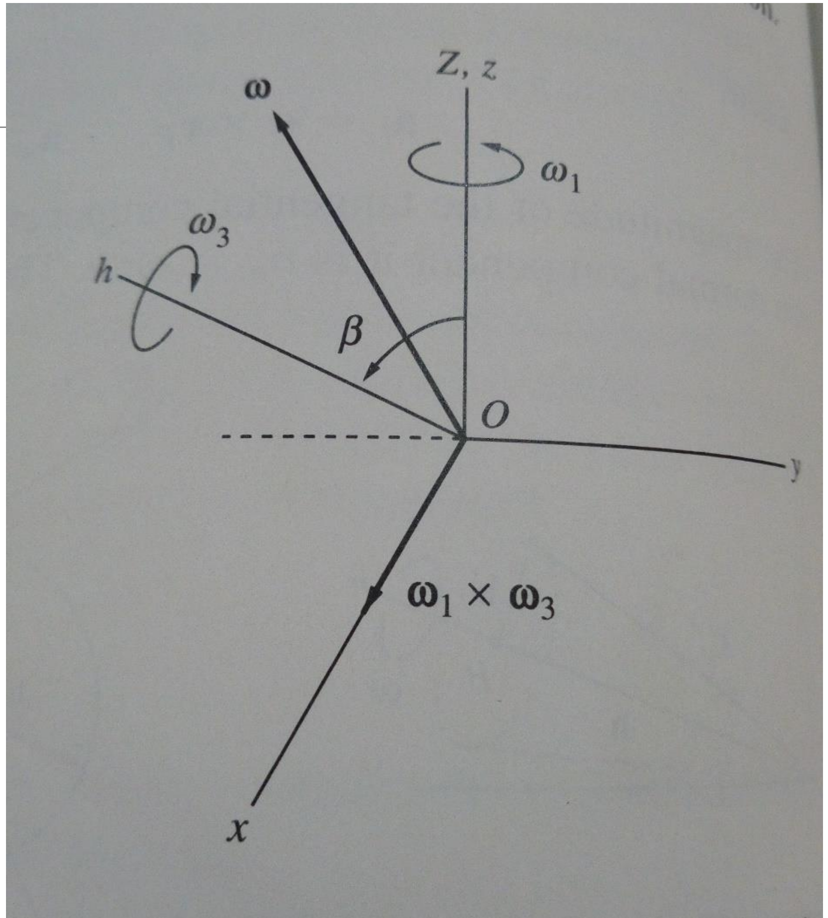


Figure C Angular velocities and gyroscopic effect

Classification of the general motion of rigid body: Pure Rotation

The line specifying the direction of the angular velocity vector $\boldsymbol{\omega}$ is known as the instantaneous axis of acceleration, or the instantaneous axis of rotation.

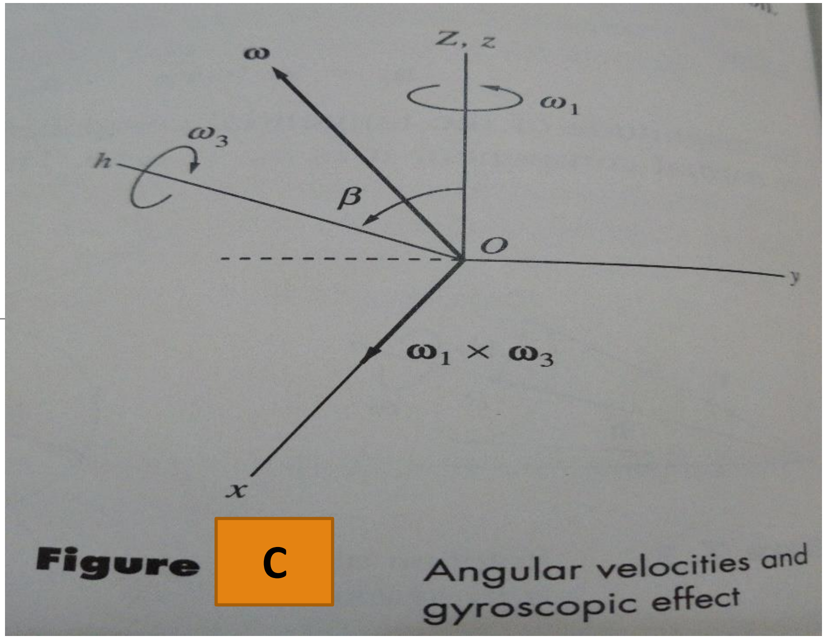
The unit vector along this axis is defined as:

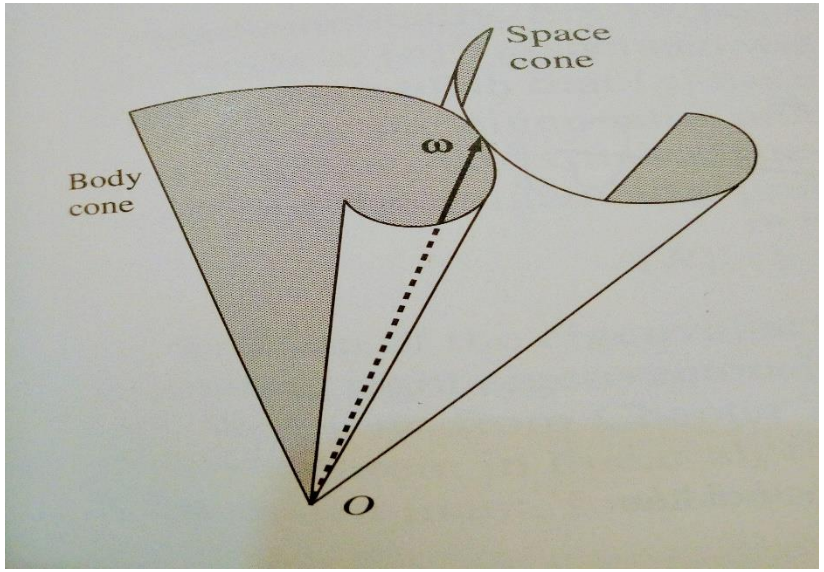
$$n = \frac{\boldsymbol{\omega}}{\omega} \quad \text{---[equ. 12]}$$

Where $\omega = |\boldsymbol{\omega}|$ is the magnitude of the angular velocity.

The rigid body can be viewed as rotating about the axis defined by $\boldsymbol{\omega}(t)$ at a particular instant.

The trajectory of the instantaneous axis of rotation defines the *body and space cones*. Body and space cones are helpful in visualizing the motion of rigid bodies.





Classification of the general motion of rigid body

3] **Combined Translation and Rotation**

A body undergoing combined translation and rotation requires both translational and angular parameters to describe its motion.

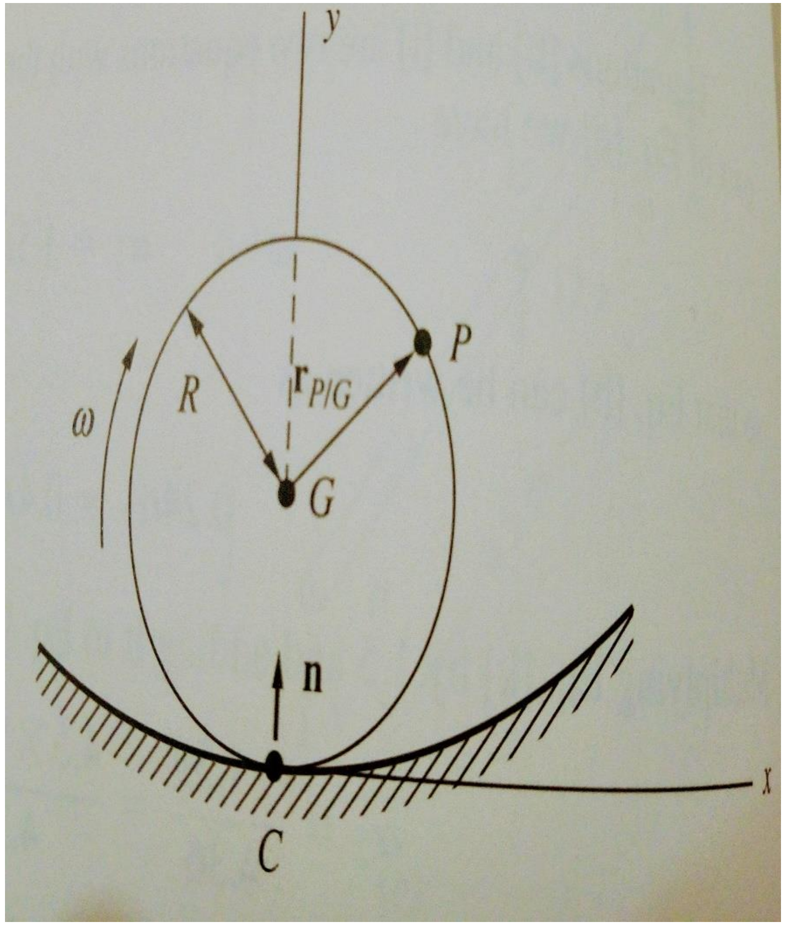
The unrestricted three-dimensional motion of a rigid body is a six degree of freedom problem.

Combined translation and rotation of an arbitrary rigid body is too general to be described in broad terms.

From the diagram, once the instant center is located, the velocity of a point P on the body can be found from the relation:

$v_p = \omega \times q$ ---[equ. 13]

Where q = $r_{P/C}$ *and* C *is the instant center*.



About Lecturer:

Opadele A.E is a physics enthusiast with special interest in Medical Physics. He loves to present the complex theories in physics in seemingly simple approach for effectual understanding.

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If you have any questions, let me know