Parcia 2 - Señales y Sistemas

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2.1 Expresión del espectro de fourier (exponencial, trigonométrica) de la señal $x(t) = |6\,sen(3t+\pi/4)|^2$, $t \in [-\pi, \pi]$

Exponencial

La señal $x(t) = |6\,sen(3t+\pi/4)|^2$ Se puede reescribir Como $x(t) = 18 + 18\,sen(6t)$, Así:

Con: $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp sen(\alpha)\,sen(\beta)$,

$sen^2(\alpha) = 1/2 - 1/2\cos(2\alpha)$

Se tiene: $x(t) = 36\,sen^2(3t+\pi/4) = 36(1/2 - 1/2\cos(6t+\pi/2))$

$= 36/2 \cancel{- 36/2(\cos(6t)\cos(\pi/2)} + \frac{36}{2}sen(6t)\,sen(\pi/2))$

$= 18 + 18\,sen(6t)$

Ahora, para la serie exponencial se tiene que:

Como $T = T/2 - (-T/2)$, entonces $\pi - (-\pi) = 2\pi = T$

y $w_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$ [rad/s]

Con $C_n = \frac{1}{T}\int_T x(t)e^{-jnw_0t}dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}(18+18\,sen(6t))e^{-jnw_0t}dt$

Resolviendo: $C_n = \frac{18}{2\pi}\int_{-\pi}^{\pi}e^{-jnw_0t} + \frac{18}{2\pi}\int_{-\pi}^{\pi}sen(6t)e^{-jnw_0t}dt$

Con: $e^{-jnw_0t} = \cos(nw_0t) - j\,sen(nw_0t)$

$C_n = -\frac{9}{jn\pi}\left[e^{-jnw_0(\pi)} - e^{-jnw_0(-\pi)}\right] + \frac{9}{\pi}\int_{-\pi}^{\pi}sen(6t)(\cos(nw_0t) - j\,sen(nw_0t))\,dt$

$C_n = \frac{9}{n\pi}\left[-e^{-jnw_0\pi} + e^{jnw_0\pi}\right] \cancel{+\frac{9}{\pi}\int_{-\pi}^{\pi}\underbrace{sen(6t)}_{impar}\underbrace{\cos(nw_0t)}_{par}dt} - \frac{9j}{\pi}\int_{-\pi}^{\pi}sen(6t)\,sen(nw_0t)\,dt$

Con: $sen(\alpha)\,sen(\beta) = \frac{1}{2}\cos(\alpha-\beta) - \cos(\alpha+\beta)$,

$e^{jnw_0t} - e^{-jnw_0t} = 2j\,sen(nw_0t)$

$$C_n = \frac{9}{n\pi}\left[\cancel{2j\,\mathrm{sen}(n(1)\pi)}^{\,0} - \frac{9}{\pi}j\int_{-\pi}^{\pi}\frac{1}{2}\left[\cos((6-n)t) - \cos((6+n)t)\right]dt\right.$$

$$C_n = -\frac{9j}{2\pi}\int_{-\pi}^{\pi}\cos((6-n)t)\,dt + \frac{9j}{2\pi}\int_{-\pi}^{\pi}\cos((6+n)t)\,dt$$

$$C_n = -\frac{9j}{2\pi(6-n)}\,\mathrm{sen}((6-n)t)\Big|_{-\pi}^{\pi} + \frac{9j}{2\pi(6+n)}\,\mathrm{sen}(((6+n)t)\Big|_{-\pi}^{\pi}$$

$$C_n = -\frac{9j}{2\pi(6-n)}\left[\mathrm{sen}((6-n)\pi - \mathrm{sen}((6-n)(-\pi)\right] + \frac{9j}{2\pi(6+n)}\left[\mathrm{sen}((6+n)\pi) - \mathrm{sen}(6+n)(-\pi)\right]$$

Como en todos los terminos esta sen $(n\pi)$ entonces el numerador siempre es cero, pero si el valor de n es 6, entonses se presenta una indeterminación, por esto se calcula C_6 y C_{-6} utilizando limites y L'hopital

$$C_6 = -9j\lim_{n\to 6}\frac{\frac{d}{dn}\left[\mathrm{sen}((6-n)\pi) - \mathrm{sen}((6-n)(-\pi))\right]}{\frac{d}{dn}\left[2\pi(6-n)\right]} + 0$$

$$C_6 = -9j\lim_{n\to 6}\frac{\cos((6-n)\pi)(-\pi) - \cos((6-n)\pi)\pi}{-2\pi}$$

$$C_6 = -9j\frac{\cos(0)(-\pi) - \cos(0)(\pi)}{-2\pi} = \frac{-9(-2\pi)}{-2\pi} = \boxed{-9j}$$

$$C_{-6} = 0 + 9j\lim_{n\to -6}\frac{\frac{d}{dn}\left[\mathrm{sen}((6+n)\pi) - \mathrm{sen}((6+n)(-\pi)\right)}{\frac{d}{dn}\left[2\pi(6+n)\right]}$$

$$C_{-6} = 9j\lim_{n\to -6}\frac{\cos((6+n)\pi)\pi - \cos((6+n)\pi)(-\pi)}{2\pi}$$

$$C_{-6} = 9j\frac{\cos(0)(\pi) - \cos(0)(-\pi)}{2\pi} = 9j\frac{(\pi+\pi)}{2\pi} = \boxed{9j}$$

Se calcula el nivel DC de la señal

$$C_0 = \frac{1}{T}\int_T |x(t)|^2\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}(18 + 18\,\mathrm{sen}(6t))\,dt = \frac{18}{2\pi}t\Big|_{-\pi}^{\pi} + \frac{18}{2\pi}\left[-\cos(6t)\right]_{-\pi}^{\pi}$$

$$C_0 = \frac{18}{2\pi}\left[\pi + \pi\right] + \frac{18}{2\pi}\left[-\cancel{\cos}(6\pi) + \cancel{\cos}(-6\pi)\right] = 18 + \frac{18}{2\pi}\left[\cancel{1-1}\right] = \boxed{18}$$

finalmente

$$C_n = \begin{cases} 18 & n=0 \\ 9j & n=-6 \\ -9j & n=6 \\ 0 & n \neq \{0,-6,6\} \end{cases}$$

Sabiendo que:

$$\hat{x}(t) = \sum_{n=-N}^{N} C_n e^{jnw_0 t}$$

Remplazando:

$$\hat{x}(t) = (C_{-6} + C_0 + C_6) e^{jnw_0 t}$$

$$\hat{x}(t) = 9j e^{j6t} + 18 e^{0} - 9j e^{-j6t} = 18 + 9j(e^{j6t} - e^{-j6t})$$

Como $e^{jwt} - e^{-jwt} = 2j\,sen(wt)$ y $j^2 = 1$

$$\hat{x}(t) = 18 + (2j)(9j)\,sen(6t) = \boxed{18 + 18\,sen(6t)}$$

Para la trigonometrica se tiene:

$a_n = 2Re\{C_n\}$, $b_n = 2Im\{C_n\}$, $C_0 = a_0$

$a_n = 2(0) = 0$, $b_n = 2(9) = 18$, $a_0 = 18$

Reemplazando en $\hat{x}(t) = a_0 + \sum_{0} a_n \cos(nw_0 t) + b_n\,sen(nw_0 t)$

$$\hat{x}(t) = 18 + (0)\cos(nw_0 t) + 18\,sen(6w_0 t) \quad , \quad \boxed{\hat{x}(t) = 18 + 18\,sen(6t)}$$

La potencia de $x(t) = P_x = \frac{1}{T}\int_T |x(t)|^2 dt$

$$P_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} |18 + 18\,sen(6t)|^2 dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} 18^2 dt + \frac{2}{2\pi}\int_{-\pi}^{\pi} (18)(18\,sen(6t))\,dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} 18^2 sen^2(6t)\,dt$$

$$P_x = \frac{18^2}{2\pi} t\Big|_{-\pi}^{\pi} + \frac{\cancel{2}(18)(18)}{\cancel{2}\pi 6}\, -\cos(6t)\Big|_{-\pi}^{\pi} + \frac{1 \cdot 18^2}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2} - \frac{1}{2}\cos(12t)\,dt$$

$$P_x = \frac{18^2}{2\pi}[\pi - (-\pi)] + \frac{18^2}{6\pi}[\underbrace{\cancel{-\cos(6\pi) + \cos(-6\pi)}}_{0}] + \frac{18^2}{4\pi} t\Big|_{-\pi}^{\pi} - \frac{18^2}{4\pi(12)}\, sen(12t)\Big|_{-\pi}^{\pi}$$

$$P_x = 18^2 + 0 + \frac{18^2}{2} - \frac{18^2}{4\pi(12)}\left[\cancel{sen(12\pi) - sen(12(-\pi))}\right]$$

$$P_x = 18^2 + \frac{18^2}{2} = \boxed{486}\ \text{W}$$

2.2 Señal portadora $C(t) = A_c \cos(2\pi F_c t)$, $A_c, F_c \in \mathbb{R}$

Señal mensaje $m(t) \in \mathbb{R}$

Encontrar el espectro en frecuencia de la señal modulada AM

$$Y(t) = \left(1 + \frac{m(t)}{A_c}\right) c(t)$$

El espectro en frecuencia se calcula como:

$$Y(w) = F\{y(t)\} = F\{(1 + \frac{m(t)}{A_c}) c(t)\}$$

Por propiedades de linealidad, se puede reescribir como:

$$Y(w) = F\{C(t)\} + \frac{1}{A_c} F\{m(t)\ c(t)\}$$

Teniendo en cuenta las tablas de fourier conocidas

$$F\{c(t)\} = F\{A_c \cos 2\pi F_c t\} = A_c F\{\cos(2\pi F_c t)\}$$

con: $\cos(x) = \frac{e^{jx} + e^{-jx}}{2}$, $F\{e^{\pm j w_0 t}\} = 2\pi\, \delta(w \mp w_0)$

Se tiene: $C(w) = F\{c(t)\} = \frac{A_c}{2} F\{e^{j2\pi F_c t} + e^{-j2\pi F_c t}\}$

$$C(w) = A_c \pi \left(\delta(w - 2\pi F_c) + \delta(w + 2\pi F_c)\right)$$

Para $\frac{1}{A_c} F\{m(t)\ c(t)\} = \frac{1}{A_c} F\{m(t)\ A_c \cos(2\pi F_c t)\}$

$$= \frac{A_c}{2A_c} F\{m(t)\, e^{j2\pi F_c t} + m(t)\, e^{-j2\pi F_c t}\}$$

De las tablas de transformadas se tiene:

$$F\{x(t)\, e^{\pm j w_0 t}\} = X(w \mp w_0)$$

así: $\frac{1}{2}\left(M(w - 2\pi F_c) + M(w + 2\pi F_c)\right)$

Finalmente el espectro $X(w)$ queda:

$$Y(w) = F\{c(t)\} + \frac{1}{A_c} F\{m(t)\ c(t)\}$$

$$\boxed{Y(w) = A_c \pi \left(\delta(w - 2\pi F_c) + \delta(w + 2\pi F_c)\right) + \frac{1}{2}\left(M(w - 2\pi F_c) + M(w + 2\pi F_c)\right)}$$