QFT HW1

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Path Integral Basics

Solution

(a)

First we note that:

$$\begin{aligned} &q_H(t)\,|q,t\rangle = q\,|q,t\rangle \\ &\Rightarrow e^{Ht/\hbar}\hat{q}e^{-Ht/\hbar}\,|q,t\rangle = q\,|q,t\rangle \end{aligned}$$

where $\hat{q}$ is the Schrodinger picture operator

$$\begin{aligned} &\Rightarrow \hat{q}e^{-Ht/\hbar}\,|q,t\rangle = qe^{-Ht/\hbar}\,|q,t\rangle \\ &\Rightarrow e^{-Ht/\hbar}\,|q,t\rangle = |q\rangle \end{aligned}$$

$$\Rightarrow |q,t\rangle = e^{Ht/\hbar}\,|q\rangle \tag{1}$$

Now we look at the RHS:

$$\begin{aligned} \left\langle q^F, t_F\right| q_H(t)\left|q^I, t_I\right\rangle &\overbrace{=}^{(1)} \left\langle q^F\right| e^{-Ht_F/\hbar}q_H(t)e^{Ht_I/\hbar}\left|q^I\right\rangle \\ &= \left\langle q^F\right| e^{-H(t_F-t)/\hbar}\hat{q}e^{-H(t-t_I)/\hbar}\left|q^I\right\rangle \end{aligned}$$

We can write $\hat{q} = \int dq\, q\,|q\rangle\langle q|$, and plug this into the above to get:

$$\begin{aligned} &= \int dq\, q\left\langle q^F\right| e^{-H(t_F-t)/\hbar}\,|q\rangle\,\langle q|\, e^{-H(t-t_I)/\hbar}\left|q^I\right\rangle \\ &= \int dq \int [Dq]e^{\frac{i}{\hbar}\int_t^{t^F} dt' L}\int [Dq]e^{\frac{i}{\hbar}\int_{t^I}^{t} dt' L}q \\ &= \int [Dq]e^{\frac{i}{\hbar}\int_{t^I}^{t^F} dt' L}q(t) \end{aligned}$$

(b)

For $t > \tilde{t}$:

$$\begin{aligned} \left\langle q^F, t_F\right| q_H(t)q_H(\tilde{t})\left|q^I, t_I\right\rangle &\overbrace{=}^{(1)} \left\langle q^F\right| e^{-Ht_F/\hbar}q_H(t)q_H(\tilde{t})e^{Ht_I/\hbar}\left|q^I\right\rangle \\ &= \left\langle q^F\right| e^{-H(t_F-t)/\hbar}\hat{q}e^{-H(t-\tilde{t})/\hbar}\hat{q}e^{-H(\tilde{t}-t_I)/\hbar}\left|q^I\right\rangle \\ &= \iint dqdq'\, qq'\left\langle q^F\right| e^{-H(t_F-t)/\hbar}\,|q\rangle\,\langle q|\, e^{-H(t-\tilde{t})/\hbar}\,|q'\rangle\,\langle q'|\, e^{-H(\tilde{t}-t_I)/\hbar}\left|q^I\right\rangle \\ &= \iint dqdq' \int [Dq]e^{\frac{i}{\hbar}\int_t^{t^F} dt' L}\int [Dq]e^{\frac{i}{\hbar}\int_{\tilde{t}}^{t} dt' L}\int [Dq]e^{\frac{i}{\hbar}\int_{t^I}^{\tilde{t}} dt' L}qq' \\ &= \int [Dq]e^{\frac{i}{\hbar}\int_{t^I}^{t_F} dt' L}q(t)q(\tilde{t}) \end{aligned}$$

Therefore, if $t < \tilde{t}$, then:

$$\begin{aligned}
\int [Dq] e^{\frac{i}{\hbar}\int_{t^I}^{t^F} dt' L} q(t)q(\tilde{t}) &= \iint dq dq' \int [Dq] e^{\frac{i}{\hbar}\int_{\tilde{t}}^{t^F} dt' L} \int [Dq] e^{\frac{i}{\hbar}\int_{t}^{\tilde{t}} dt' L} \int [Dq] e^{\frac{i}{\hbar}\int_{t^I}^{t} dt' L} qq' \\
&= \iint dq dq' \, qq' \left\langle q^F \right| e^{-H(t_F-\tilde{t})/\hbar} \left| q \right\rangle \left\langle q \right| e^{-H(\tilde{t}-t)/\hbar} \left| q' \right\rangle \left\langle q' \right| e^{-H(t-t_I)/\hbar} \left| q^I \right\rangle \\
&= \left\langle q^F \right| e^{-H(t_F-\tilde{t})/\hbar} \hat{q} e^{-H(\tilde{t}-t)/\hbar} \hat{q} e^{-H(t-t_I)/\hbar} \left| q^I \right\rangle \\
&= \left\langle q^F \right| e^{-Ht_F/\hbar} q_H(\tilde{t}) q_H(t) e^{Ht_I/\hbar} \left| q^I \right\rangle \\
&= \left\langle q^F, t_F \right| q_H(\tilde{t}) q_H(t) \left| q^I, t_I \right\rangle
\end{aligned}$$

(c)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{dt}} q_H(t) &= \frac{\mathrm{d}}{\mathrm{dt}} \left(e^{\frac{i}{\hbar}Ht} \right) \hat{q} e^{-\frac{i}{\hbar}Ht} + e^{\frac{i}{\hbar}Ht} \hat{q} \frac{\mathrm{d}}{\mathrm{dt}} \left(e^{-\frac{i}{\hbar}Ht} \right) \\
&= \frac{i}{\hbar} H e^{\frac{i}{\hbar}Ht} \hat{q} e^{-\frac{i}{\hbar}Ht} - e^{\frac{i}{\hbar}Ht} \hat{q} \frac{i}{\hbar} H e^{-\frac{i}{\hbar}Ht}
\end{aligned}$$

because $[H, e^{Ht}] = 0$:

$$\begin{aligned}
&= \frac{i}{\hbar} e^{\frac{i}{\hbar}Ht} H \hat{q} e^{-\frac{i}{\hbar}Ht} - e^{\frac{i}{\hbar}Ht} \hat{q} \frac{i}{\hbar} H e^{-\frac{i}{\hbar}Ht} \\
&= \frac{i}{\hbar} e^{\frac{i}{\hbar}Ht} [H, \hat{q}] e^{-\frac{i}{\hbar}Ht} \\
&= \frac{i}{\hbar} e^{\frac{i}{\hbar}Ht} [\frac{\hat{p}^2}{2m} + V(q), \hat{q}] e^{-\frac{i}{\hbar}Ht} \\
&= \frac{i}{2m\hbar} e^{\frac{i}{\hbar}Ht} [\hat{p}^2, \hat{q}] e^{-\frac{i}{\hbar}Ht} \\
&= \frac{i}{2m\hbar} e^{\frac{i}{\hbar}Ht} (\hat{p}[\hat{p}, \hat{q}] + [\hat{p}, \hat{q}]\hat{p}) e^{-\frac{i}{\hbar}Ht} \\
&= \frac{i}{2m\hbar} e^{\frac{i}{\hbar}Ht} (-2i\hbar\hat{p}) e^{-\frac{i}{\hbar}Ht} \\
&= \frac{1}{m} p_H(t) \\
\Rightarrow & p_H(t) = m\dot{q}_H(t)
\end{aligned}$$

(d)

$$\int [Dq] e^{\frac{i}{\hbar}\int_{t^I}^{t^F} dt' L} q(t) \frac{\delta S}{\delta q(t)} \tag{2}$$

We write the action explicitly:

$$S[q(t)] = \int_{t^I}^{t^F} dt' \frac{1}{2} m \dot{q}(t')^2 - V(q(t')) \tag{3}$$

We choose a simple regularization where time is divided into even intervals $\Delta t = \Delta$: Then:

$$t_n = \Delta n q_n = q(t_n) \dot{q}(t_n) \to \frac{q_{n+1} - q_n}{\Delta}$$

$$S[\{q_n\}] = \sum_n \frac{1}{2} m \left(\frac{q_{n+1} - q_n}{\Delta} \right)^2 - V(q_n) \tag{4}$$

$$
\begin{aligned}
\frac{\delta S}{\delta q(t)} \to \frac{\partial S[\{q_n\}]}{\partial q_m} &= \frac{\partial}{\partial q_m}\sum_n \frac{1}{2}m\left(\frac{q_{n+1}-q_n}{\Delta}\right)^2 - V(q_n)\\
&= \sum_n \frac{1}{2}m\frac{\partial}{\partial q_m}\left(\frac{q_{n+1}^2-2q_{n+1}q_n+q_n^2}{\Delta^2}\right) - \frac{\partial V(q_n)}{\partial q_m}\\
&= \sum_n \frac{1}{2}m\left(\frac{2\frac{\partial q_{n+1}}{\partial q_m} - 2\frac{\partial q_{n+1}}{\partial q_m}q_n - 2q_{n+1}\frac{\partial q_n}{\partial q_m} + 2\frac{\partial q_n}{\partial q_m}}{\Delta^2}\right) - \frac{\partial V(q_n)}{\partial q_n}\frac{\partial q_n}{\partial q_m}\\
&= \sum_n m\left(\frac{q_{n+1}-q_n}{\Delta}\right)\left(\frac{\delta_{n+1,m}-\delta_{nm}}{\Delta}\right) - \frac{\partial V(q_n)}{\partial q_n}\frac{\partial q_n}{\partial q_m}\\
&= \int dt' m\dot{q}(t')\frac{\partial}{\partial t'}\delta(t-t') - \frac{\partial V(q)}{\partial q}\delta(t'-t)
\end{aligned}
$$

To deal with the derivative of the delta function, we integrate by parts and then ignore the boudary term. We get:

$$
\frac{\delta S}{\delta q(t)} = -m\ddot{q} - \frac{\partial}{\partial q}V(q)
$$

We put this into (2):

$$
-\int [Dq]e^{\frac{i}{\hbar}\int_{t^I}^{t^F} dt' L} q(t)(m\ddot{q} + \frac{\partial}{\partial q}V) = (*) \tag{5}
$$

Once again we move to our regularization:

$$
(*) = \frac{1}{\Delta}\langle
$$

Now we calculate the same thing with integration by parts:

$$
\int [Dq]e^{\frac{i}{\hbar}\int_{t^I}^{t^F} dt' L} q(t)\frac{\delta S}{\delta q(t)} = -i\hbar\int [Dq]\frac{\delta e^{\frac{i}{\hbar}\int_{t^I}^{t^F} dt' L}}{\delta q(t)}q(t) = +i\hbar\int [Dq]e^{\frac{i}{\hbar}\int_{t^I}^{t^F} dt' L}\frac{\delta q(t)}{\delta q(t)} \tag{6}
$$

Harmonic Oscillator

Solution

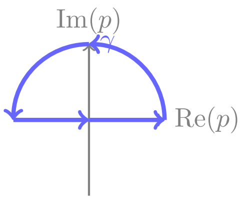
For an harmonic oscillator,

$$
q_S = \sqrt{\frac{\hbar}{2\omega}}(a^\dagger + a) \tag{7}
$$

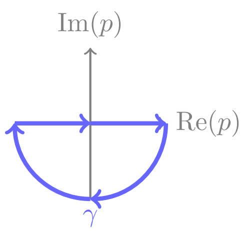
$$
\begin{aligned}
a^\dagger |n\rangle &= \sqrt{n+1}\,|n+1\rangle\\
a|n\rangle &= \sqrt{n}\,|n-1\rangle\\
e^{iHt/\hbar} &= \sum_n e^{i\omega n t}|n\rangle\langle n|
\end{aligned}
$$

Using these relations, we could calculate:
$\langle 0| q(t)q(0)|0\rangle = \langle 0| e^{\frac{i}{\hbar}Ht}q_S e^{-\frac{i}{\hbar}Ht}q_S|0\rangle =$
$= \langle 0|(\sum_k e^{i\omega k t}|k\rangle\langle k|)\sqrt{\frac{\hbar}{2\omega}}(a^\dagger + a)(\sum_n e^{-i\omega n t}|n\rangle\langle n|)\sqrt{\frac{\hbar}{2\omega}}(a^\dagger + a)|0\rangle$
$= (\sqrt{\frac{\hbar}{2\omega}})^2\sum_{k,n} e^{i\omega(k-n)t}\delta_{0k}\langle k|(a^\dagger + a)|n\rangle\delta_{n1}$
$= (\sqrt{\frac{\hbar}{2\omega}})^2 e^{-i\omega t}\langle 0|(a^\dagger + a)|1\rangle = (\sqrt{\frac{\hbar}{2\omega}})^2 e^{-i\omega t}\langle 0|0\rangle$
$= \boxed{\frac{\hbar}{2\omega}e^{-i\omega t}}$
If we do a Wick rotation by $t = -i\tau$, we get:
$\langle 0| q(\tau)q(0)|0\rangle = \frac{\hbar}{2\omega}e^{-\omega\tau}$
which is what we saw in class.

Now we will use residue calculus and calculate:
$\int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{e^{ip\tau}}{p^2+\omega^2} = \oint_\gamma \frac{dp}{2\pi}\frac{e^{ip\tau}}{p^2+\omega^2}$
where γ is the path in complex space which looks like:



The contour is at the limit as the radius of the semicircle goes to infinity. We chose the semicircle with $\mathrm{Im}[p] > 0$ so that the contribution from the arc goes to zero leaving us only with the contribution from the real axis. This works only for $\tau > 0$ There are poles at $p = \pm i\omega$. The residue from our contour is:
$Res(\frac{1}{2\pi}\frac{e^{ip\tau}}{p^2+\omega^2}) = \lim_{p\to i\omega}(p-i\omega)\frac{1}{2\pi}\frac{e^{ip\tau}}{p^2+\omega^2} = \lim_{p\to i\omega}\frac{1}{2\pi}\frac{e^{ip\tau}}{p+i\omega} = \frac{1}{4\pi}\frac{e^{-\omega\tau}}{i\omega}$
Now we plug this into the residue theorem and get:
$\int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{e^{ip\tau}}{p^2+\omega^2} = \oint_\gamma \frac{dp}{2\pi}\frac{e^{ip\tau}}{p^2+\omega^2} = 2\pi i \sum Res(\frac{1}{2\pi}\frac{e^{ip\tau}}{p^2+\omega^2}) = 2\pi i \frac{1}{2\pi}\frac{e^{-\omega\tau}}{2i\omega} = \frac{e^{-\omega\tau}}{2\omega}$
Note that here we assumed $\tau > 0$. When $\tau < 0$, we must choose the contour which looks like:



this way the contribution of the arc goes to zero. In this case the pole is at $-i\omega$, which changes the residue to:
$\frac{-1}{4\pi}\frac{e^{\omega\tau}}{i\omega}$. In addiction we notice that our contour is clockwise, which gives us a minus in front of the integral. In total our integral is: $\int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{e^{ip\tau}}{p^2+\omega^2} = \frac{e^{\omega\tau}}{2\omega}$
Putting these answers together for all τ, we get:

$$\boxed{\int_{-\infty}^{\infty} \frac{dp}{2\pi}\frac{e^{ip\tau}}{p^2+\omega^2} = \frac{e^{\omega|\tau|}}{2\omega}}$$

Wick's contractions basics

Show that:

$$\mathcal{Z}[J] = \int d^N q e^{-\frac{1}{2}q^T M q + J\cdot q} = (2\pi)^{\frac{N}{2}}(\det M)^{-\frac{1}{2}} e^{\frac{1}{2}J^T M^{-1} J} \tag{8}$$

Solution

M is diagonal, so it can be written as $M_{ij} = \delta_{ij}\lambda_i$ (no sum on i). We can rewrite the LHS of equation (8) as:
$\int d^N q e^{-\frac{1}{2}q^T M q + J\cdot q} = \int d^N q e^{-\frac{1}{2}\sum_i \lambda_i q_i^2 + J_i q_i} = \Pi_i \int d^N q e^{-\frac{1}{2}\lambda_i q_i^2 + J_i q_i}$
By completing the square, this becomes:
$= \Pi_i e^{\frac{1}{2}J_i^2/\lambda_i} \int d^N q e^{-\frac{\lambda_i}{2}(q_i - J_i/\lambda_i)^2}$
we compute this as a Gaussian:
$= \Pi_i e^{\frac{1}{2}J_i^2/\lambda_i}(2\pi/\lambda_i)^{1/2} = (2\pi)^{\frac{N}{2}}(\Pi_i \frac{1}{\lambda_i})(\Pi_i e^{\frac{1}{2}J_i^2/\lambda_i}) = (*)$.
We note that $M_{ij}^{-1} = \delta_{ij}\frac{1}{\lambda_i}$ (no sum on i), and the determinant is the product of the eigenvalues: $\det M = \Pi_i \frac{1}{\lambda_i}$. Plugging this into $(*)$:
$= (2\pi)^{\frac{N}{2}}(\det(M^{-1}))^{\frac{1}{2}} e^{\sum_{i,j}\frac{1}{2}J_i\delta_{ij}(\frac{1}{\lambda_i})J_j} = (2\pi)^{\frac{N}{2}}(\det M)^{-\frac{1}{2}} e^{\frac{1}{2}J^T(M^{-1})J}$.
which is what we wanted to prove.
Now we wish to show:

$$\frac{1}{Z[0]}\int d^N q e^{-\frac{1}{2}q^T M q} F(q) = \frac{1}{Z[0]} F\left(\frac{\partial}{\partial J}\right) Z[J]\bigg|_{J=0}. \tag{9}$$

By definition:

$$F(\frac{\partial}{\partial J}) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{\partial^{(n)} F}{\partial \mathbf{q}^n}|_{\mathbf{q}=0}(\frac{\partial}{\partial \mathbf{J}})^n$$

We look at the RHS of (9) and plug in the definition of $Z[J]$ and $F(\frac{\partial}{\partial J})$ above we get:

$$\begin{aligned}
\frac{1}{Z[0]} F\left(\frac{\partial}{\partial J}\right) Z[J]|_{J=0} &= \frac{1}{Z[0]}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^{(n)} F}{\partial \mathbf{q}^n}|_{\mathbf{q}=0}(\frac{\partial}{\partial \mathbf{J}})^n \int d^N q e^{-\frac{1}{2}q^T M q + J\cdot q}|_{J=0} \\
&= \frac{1}{Z[0]}\int d^N q e^{-\frac{1}{2}q^T M q}\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^{(n)} F}{\partial \mathbf{q}^n}|_{\mathbf{q}=0}\underbrace{(\frac{\partial}{\partial \mathbf{J}})^n e^{J\cdot q}}_{\equiv A}|_{J=0} = (*)
\end{aligned}$$

We look at A:

$$A = (\frac{\partial}{\partial \mathbf{J}})^n e^{J\cdot q} = (\frac{\partial}{\partial \mathbf{J}})^{n-1} e^{J\cdot q}\mathbf{q} = (\frac{\partial}{\partial \mathbf{J}})^{n-2} e^{J\cdot q}\mathbf{q}^2 = \cdots = e^{J\cdot q}\mathbf{q}^n \tag{10}$$

We plug A into $(*)$:

$$\begin{aligned}
\frac{1}{Z[0]} F\left(\frac{\partial}{\partial J}\right) Z[J]|_{J=0} &= \frac{1}{Z[0]}\int d^N q e^{-\frac{1}{2}q^T M q + J\cdot q}\overbrace{\sum_{n=0}^{\infty}\frac{1}{n!}\frac{\partial^{(n)} F}{\partial \mathbf{q}^n}|_{\mathbf{q}=0}\mathbf{q}^n}^{\text{Maclauren Series of } F(q)}|_{J=0} \\
&= \frac{1}{Z[0]}\int d^N q e^{-\frac{1}{2}q^T M q + J\cdot q} F(q)|_{J=0} \\
&= \frac{1}{Z[0]}\int d^N q e^{-\frac{1}{2}q^T M q} F(q)
\end{aligned}$$

We proved (9).

Anharmonic Oscilator

$$H = \frac{1}{2}p^2 + \frac{1}{2}q^2 + \frac{1}{4!}gq^4$$

(a)

Solution

We want to compute the propagator:

$$\langle q(\tau_2) q(\tau_1)\rangle = \frac{\int [Dq] e^{\int_{\tau_1}^{\tau_2} d\tau' L_E} q(\tau_1) q(\tau_2)}{\int [Dq] e^{\int_{\tau_1}^{\tau_2} d\tau' L_E}} \tag{11}$$

In our case $L_E = \frac{1}{2}(\partial_\tau q)^2 + \frac{1}{2}q^2 + \frac{1}{4!}gq^4$. We label $L'_E = -(\frac{1}{2}(\partial_\tau q)^2 + \frac{1}{2}q^2)$, and we can rewrite (11) as:

$$\begin{aligned}
\langle q(\tau_2) q(\tau_1)\rangle &= \frac{\int [Dq] e^{-\int_{\tau_1}^{\tau_2} d\tau' L'_E} e^{\int_{\tau_1}^{\tau_2} d\tau'(-\frac{1}{4!}gq^4)} q(\tau_1) q(\tau_2)}{\int [Dq] e^{-\int_{\tau_1}^{\tau_2} d\tau' L'_E} e^{\int_{\tau_1}^{\tau_2} d\tau'(-\frac{1}{4!}gq^4)}} \\
&= \frac{1}{\mathcal{Z}_g}\int [Dq] e^{-\int_{\tau_1}^{\tau_2} d\tau' L'_E} e^{\int_{\tau_1}^{\tau_2} d\tau'(-\frac{1}{4!}gq^4)} q(\tau_1) q(\tau_2)
\end{aligned} \tag{12}$$

We can expand the exponent containing the anharmonic term:

$$e^{\int_{\tau_1}^{\tau_2} d\tau'(-\frac{1}{4!}gq^4)} = \sum_{n=0}^{\infty}\frac{1}{n!}(-\frac{1}{4!}g)^n \overbrace{\int\cdots\int}^{n} d\tau'_1 \ldots d\tau'_n q(\tau'_1)^4 \ldots q(\tau'_n)^4$$

We are only interested in the n=2 terms:

$$e^{\int_{\tau_1}^{\tau_2} d\tau' (-\frac{1}{4!} g q^4)} = \sum_{n=0}^{\infty} \frac{1}{n!} (-\frac{1}{4!} g)^n \overbrace{\int \cdots \int}^{n} d\tau_1' \ldots d\tau_n' q(\tau_1')^4 \ldots q(\tau_n')^4$$

We plug this into the numerator of (**??**) and get:

$$\langle q(\tau_2) q(\tau_1) \rangle_{\text{anharmonic}} = \frac{\mathcal{Z}_0}{\mathcal{Z}_g} \sum_{n=0}^{\infty} \frac{1}{n!} (-\frac{1}{4!} g)^n \overbrace{\int \cdots \int}^{n} d\tau_1' \ldots d\tau_n' \langle q(\tau_1) q(\tau_2) q(\tau_1')^4 \ldots q(\tau_n')^4 \rangle_{\text{harmonic}}$$

where

$$\frac{\mathcal{Z}_g}{\mathcal{Z}_0} = \sum_{n=0}^{\infty} \frac{1}{n!} (-\frac{1}{4!} g)^n \overbrace{\int \cdots \int}^{n} d\tau_1' \ldots d\tau_n' \langle q(\tau_1')^4 \ldots q(\tau_n')^4 \rangle_{\text{harmonic}}$$

We can see that the Feynman diagrams of g^n have n, $q(\tau')^n$ terms to be contracted together with $q(\tau_1)$ and $q(\tau_2)$. Notice, however, that the the Wick contractions that don't include $q(\tau_1)$ and $q(\tau_2)$ can be factored out and cancel out exactly with the $\frac{\mathcal{Z}_0}{\mathcal{Z}_\}}$ term. The Feynman diagrams for g^2 have 2 four legged nodes and 2 one legged nodes. The different diagrams appear in Figure. We only look at part of the charts with $q(\tau_1)$ and $q(\tau_2)$. From diagrams (a), (b), and (c):

$$(a) = 8 \times 4 \times 3 \times 3 \times \frac{1}{2!} (\frac{-g}{4!})^2 \int d\tau_1' \int d\tau_2' G(\tau_1, \tau_1') G(\tau_1', \tau_1') G(\tau_1', \tau_2') G(\tau_2', \tau_2') G(\tau_2', \tau_2)$$

$$(b) = 8 \times 4 \times 3 \times 3 \times 2 \times 2 \times \frac{1}{2!} (\frac{-g}{4!})^2 \int d\tau_1' \int d\tau_2' G(\tau_1, \tau_1') [G(\tau_1', \tau_2')]^3 G(\tau_2', \tau_2)$$

$$(c) = 8 \times 3 \times 2 \times 4 \times 3 \times \frac{1}{2!} (\frac{-g}{4!})^2 \int d\tau_1' \int d\tau_2' G(\tau_1, \tau_1') [G(\tau_1', \tau_2')]^2 G(\tau_2', \tau_2') G(\tau_2', \tau_2)$$

Now we calculate each integral:

$$\begin{aligned}
(a) &= 8 \times 4 \times 3 \times 3 \times \frac{1}{2!} (\frac{g}{4!})^2 \int d\tau_1' \int d\tau_2' G(\tau_1, \tau_1') G(\tau_1', \tau_1') G(\tau_1', \tau_2') G(\tau_2', \tau_2') G(\tau_2', \tau_2) \\
&= \frac{g^2}{4} \int d\tau_1' \int d\tau_2' G(\tau_1, \tau_1') G(\tau_1', \tau_1') G(\tau_1', \tau_2') G(\tau_2', \tau_2') G(\tau_2', \tau_2) \\
&= \frac{g^2}{4} \int d\tau_1' \int d\tau_2' G(\tau_1 - \tau_1') G(\tau_1' - \tau_1') G(\tau_1' - \tau_2') G(\tau_2' - \tau_2') G(\tau_2' - \tau_2)
\end{aligned}$$

we use the fact that for a harmonic oscillator:

$$G(\tau) = \frac{1}{2\omega} e^{-\omega|\tau|} = \int \frac{dp}{2\pi} \frac{e^{p\tau}}{p^2 + \omega^2} \tag{13}$$

plugging this into (a):

$$(a) = \frac{g^2}{16} \int d\tau_1' \int d\tau_2' \int \frac{dp}{2\pi} \frac{e^{p(\tau_1 - \tau_1')}}{p^2 + \omega^2} \int \frac{dp'}{2\pi} \frac{e^{p'(\tau_1' - \tau_2')}}{p'^2 + \omega^2} \int \frac{dp''}{2\pi} \frac{e^{p''(\tau_2' - \tau_2)}}{p''^2 + \omega^2}$$

Now we calculate (b)

$$(b) = g^2 \int d\tau_1' \int d\tau_2' G(\tau_1, \tau_1') [G(\tau_1', \tau_2')]^3 G(\tau_2', \tau_2)$$

We plug in (13):

$$(b) = g^2 \int d\tau_1' \int d\tau_2' \int \frac{dp}{2\pi} \frac{e^{p(\tau_1' - \tau_1)}}{p^2 + \omega^2} \left(\int \frac{dp'}{2\pi} \frac{e^{p'(\tau_2' - \tau_1')}}{p'^2 + \omega^2} \right)^3 \int \frac{dp''}{2\pi} \frac{e^{p''(\tau_2' - \tau_1')}}{p''^2 + \omega^2}$$

We do the same thing for (c):

$$(c) = \frac{g^2}{4} \int d\tau_1' \int d\tau_2' \int \frac{dp}{2\pi} \frac{e^{p(\tau_1' - \tau_1)}}{p^2 + \omega^2} \left(\int \frac{dp'}{2\pi} \frac{e^{p'(\tau_2' - \tau_1')}}{p'^2 + \omega^2} \right)^2 \int \frac{dp''}{2\pi} \frac{e^{p''(\tau_2' - \tau_1')}}{p''^2 + \omega^2}$$

(b) Comute in standard perturbation theory the correction to the ground state energy in first order in g and compare to the expression we obtained in class.

Solution

In standard perturbation theory the first order correction is given bey:

$$\Delta E_n = \langle n| H_{pert} |n\rangle$$

where $|n\rangle$ are the energy eigenstates of the free hamiltonian and H_{pert} is the pertubation part of the hamiltonian. The free energy hamiltonian eigenstates are $|n\rangle = (a^\dagger)^n |0\rangle$. In class we only looked at the ground state energy $|0\rangle$. We will also do that here. Therefore:

$$\begin{aligned}
\Delta E_0 &= \langle 0| H_{pert} |0\rangle \\
&= \frac{g}{4!} \langle 0| \hat{q}^4 |0\rangle = (*)
\end{aligned}$$

We plug in (7):

$$\begin{aligned}
(*) &= \frac{g}{4!} \langle 0| \left(\sqrt{\frac{\hbar}{2\omega}} (a^\dagger + a) \right)^4 |0\rangle \\
&= \frac{g\hbar^2}{4\omega^2 4!} \langle 0| \left(a^\dagger + a\right)^4 |0\rangle \\
&= \frac{g}{4!} \frac{\hbar^2}{4\omega^2} \langle 0| \left(a^\dagger + a\right) \left(a^\dagger + a\right) \left(a^\dagger + a\right) \left(a^\dagger + a\right) |0\rangle \\
&= \frac{g}{4!} \frac{\hbar^2}{4\omega^2} \langle 0| \left((a^\dagger)^2 + a^\dagger a + a a^\dagger + a^2\right) \left((a^\dagger)^2 + a^\dagger a + a a^\dagger + a^2\right) |0\rangle \\
&= \frac{g}{4!} \frac{\hbar^2}{4\omega^2} \langle 0| \left((a^\dagger)^2 + a^\dagger a + a a^\dagger + a^2\right) \left((a^\dagger)^2 + a^\dagger a + a a^\dagger + a^2\right) |0\rangle
\end{aligned}$$