Optimization

Assignment 3

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Q1

$$\frac{\partial s(p)}{\partial p} = \frac{\partial}{\partial p}\int_0^p \|\dot{\gamma}(\tau)\|\, d\tau = \|\dot{\gamma}(p)\|$$

Let's define the inverse function:

$$p(s) = f^{-1}(s)$$

$$\frac{\partial}{\partial s}\gamma\big(p(s)\big) = \dot{\gamma}(p)\frac{\partial p(s)}{\partial s} \Longrightarrow \dot{\gamma}(s) = \frac{\frac{\partial}{\partial p}\gamma(p)}{\frac{\partial s(p)}{\partial p}} = \frac{\dot{\gamma}(p)}{\|\dot{\gamma}(p)\|}$$

Therefore:

$$\|\dot{\gamma}(s)\| = \frac{\|\dot{\gamma}(p)\|}{\|\dot{\gamma}(p)\|} = 1$$

Q2

$$\langle \boldsymbol{T}, \boldsymbol{T}\rangle = 1$$

$$\frac{\partial}{\partial s}\langle \boldsymbol{T}, \boldsymbol{T}\rangle = 0$$

$$\frac{\partial}{\partial s}\langle \boldsymbol{T}, \boldsymbol{T}\rangle = 2\langle \dot{\boldsymbol{T}}, \boldsymbol{T}\rangle = 0 \Longrightarrow \boldsymbol{N} \perp \boldsymbol{T} \perp \dot{\boldsymbol{T}} \Longrightarrow \dot{\boldsymbol{T}} \parallel \boldsymbol{N}$$

$$\Longrightarrow \dot{\boldsymbol{T}} = k\boldsymbol{N}$$

Now to find k:

$$\langle \dot{\boldsymbol{T}}, \boldsymbol{N}\rangle = \langle k\boldsymbol{N}, \boldsymbol{N}\rangle = k\underbrace{\langle \boldsymbol{N}, \boldsymbol{N}\rangle}_{=1} = k$$

$$\Longrightarrow k = \langle \dot{\boldsymbol{T}}, \boldsymbol{N}\rangle = [\ddot{x}(s) \quad \ddot{y}(s)]\begin{bmatrix} \dot{y}(s) \\ -\dot{x}(s)\end{bmatrix} = \ddot{x}(s)\dot{y}(s) - \ddot{y}(s)\dot{x}(s)$$

Q3

The normal to surface $F(x, y, z) = 0$ is defined as, $\boldsymbol{N} = \nabla F(x, y, z)$

In our case $F(x, y, z) = z - z(x, y) \Longrightarrow \boldsymbol{N} = \big(-z_x(x, y), -z_y(x, y), 1\big)$

We can also see that:

$$[1,0, z_x]^T \times [0,1, z_y] = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = \hat{x}(-z_x) + \hat{y}(-z_y) + \hat{z}(1)$$
$$= \big(-z_x(x, y), -z_y(x, y), 1\big)$$

A Surface integral is defined as

$$A = \iint_T dA = \iint_T \left\|\frac{\partial \boldsymbol{r}}{\partial x} \times \frac{\partial \boldsymbol{r}}{\partial y}\right\| dxdy =$$

Where $\boldsymbol{r} = \big(x, y, z(x, y)\big)$

$$\iint_T \|[1,0, z_x]^T \times [0,1, z_y]\| dxdy = \iint_T \|(-z_x(x, y), -z_y(x, y), 1)\| dxdy =$$

$$\iint_T \sqrt{(z_x)^2 + (z_y)^2 + 1} dxdy$$

So $dA = \sqrt{(z_x)^2 + (z_y)^2 + 1} dxdy$

Q4

The Euler Lagrange equation for

$$\min_{z(x,y)} \int_\Omega d\Omega = \min_{z(x,y)} \int_\Omega \sqrt{(z_x)^2 + (z_y)^2 + 1} dxdy$$

$$\int_\Omega F(x, y, z_x, z_y) dxdy$$

The Euler Lagrange equations:

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\frac{\partial F}{\partial z_x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial z_y} = 0$$

$$F(x, y, z_x, z_y) = \sqrt{(z_x)^2 + (z_y)^2 + 1}$$

Let's deal with

$$\frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial z_x} = \frac{\frac{1}{2} 2z_x}{\sqrt{(z_x)^2 + (z_y)^2 + 1}} = \frac{z_x}{\sqrt{(z_x)^2 + (z_y)^2 + 1}}$$

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial z_x} = \frac{z_{xx}\sqrt{(z_x)^2 + (z_y)^2 + 1} - \frac{z_x \frac{1}{2}(2z_x z_{xx} + 2z_y z_{xy})}{\sqrt{(z_x)^2 + (z_y)^2 + 1}}}{(z_x)^2 + (z_y)^2 + 1}$$

$$= \frac{z_{xx}\left((z_x)^2 + (z_y)^2 + 1\right) - \left((z_x)^2 z_{xx} + z_x z_y z_{xy}\right)}{\left((z_x)^2 + (z_y)^2 + 1\right)^{\frac{3}{2}}}$$

Likewise for z_y:

$$\frac{\partial}{\partial y}\frac{\partial F}{\partial z_y} = \frac{z_{yy}\left((z_x)^2 + \left(z_y\right)^2 + 1\right) - \left(\left(z_y\right)^2 z_{yy} + z_y z_x z_{xy}\right)}{\left((z_x)^2 + \left(z_y\right)^2 + 1\right)^{\frac{3}{2}}}$$

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial z_x} + \frac{\partial}{\partial y}\frac{\partial F}{\partial z_y} = 0$$

$$\Longrightarrow z_{yy}\left((z_x)^2 + \left(z_y\right)^2 + 1\right) - \left((z_x)^2 z_{xx} + z_x z_y z_{xy}\right) + z_{xx}\left((z_x)^2 + \left(z_y\right)^2 + 1\right) - \left(\left(z_y\right)^2 z_{yy} + z_y z_x z_{xy}\right) = 0$$

$$z_{yy}\left((z_x)^2 + \left(z_y\right)^2 + 1\right) - \left((z_x)^2 z_{xx} + z_x z_y z_{xy}\right) + z_{xx}\left((z_x)^2 + \left(z_y\right)^2 + 1\right) - \left(\left(z_y\right)^2 z_{yy} + z_y z_x z_{xy}\right) = 0$$

$$\Longrightarrow K = z_{xx}\left(\left(z_y\right)^2 + 1\right) - 2 z_x z_y z_{xy} + z_{yy}((z_x)^2 + 1) = 0$$

Q5

$$t^* = \min_{\gamma(p)} \int_{p_0}^{p_1} \frac{n\big(\gamma(\tau)\big)}{C} \|\dot{\gamma}(\tau)\| d\tau = \min_{\gamma(p)} \int_{p_0}^{p_1} F\big(\tau, \gamma(\tau), \dot{\gamma}(\tau)\big) d\tau$$

$$\Longrightarrow \frac{\partial F}{\partial \gamma} - \frac{d}{dp}\left(\frac{\partial F}{\partial \dot{\gamma}}\right) = 0$$

Now find each part separately:

$$\frac{\partial F}{\partial \gamma} = \nabla_\gamma n \|\dot{\gamma}(p)\|$$

$$\frac{\partial F}{\partial \dot{\gamma}} = n \frac{\dot{\gamma}}{\|\dot{\gamma}\|}$$

$$\frac{d}{dp}\left(\frac{\partial F}{\partial \dot{\gamma}}\right) = \frac{d}{dp}\left(n \frac{\dot{\gamma}}{\|\dot{\gamma}\|}\right) = \frac{d}{dp}\big(n\dot{\gamma}(s)\big) = \frac{ds}{dp}\frac{d}{ds}\big(n\dot{\gamma}(s)\big) = \dot{\gamma}(p)\frac{d}{ds}\big(n\dot{\gamma}(s)\big)$$
$$= \|\dot{\gamma}(p)\| \frac{\dot{\gamma}(p)}{\|\dot{\gamma}(p)\|}\frac{d}{ds}\big(n\dot{\gamma}(s)\big)$$

To summarize:

$$\nabla n \|\dot{\gamma}(p)\| - \|\dot{\gamma}(p)\| \frac{\dot{\gamma}(p)}{\|\dot{\gamma}(p)\|}\frac{d}{ds}\big(n\dot{\gamma}(s)\big) = 0$$

$$\Longrightarrow \|\dot{\gamma}(p)\| \left(\nabla n - \frac{\dot{\gamma}(p)}{\|\dot{\gamma}(p)\|}\frac{d}{ds}\big(n\dot{\gamma}(s)\big)\right) = 0 \; (*)$$

We still need to calculate:

$$\frac{d}{ds}\big(n\dot{\gamma}(s)\big)=\left(\nabla n\cdot\frac{d\gamma}{ds}\right)\dot{\gamma}(s)+n\ddot{\gamma}(s)=\langle\nabla n,\boldsymbol{T}\rangle\boldsymbol{T}+n(k\boldsymbol{N})$$

Substitute in (*) and we get

$$\nabla n=\langle\nabla n,\boldsymbol{T}\rangle\boldsymbol{T}+kn\boldsymbol{N}\ (**)$$

Now take the equation we are trying to prove:

$$(\langle\nabla n,\boldsymbol{N}\rangle-kn)\boldsymbol{N}=\boldsymbol{0}$$

Substitute (**) into it:

$$(\langle\langle\nabla n,\boldsymbol{T}\rangle\boldsymbol{T}+kn\boldsymbol{N}\ ,\boldsymbol{N}\rangle-kn)\boldsymbol{N}\stackrel{?}{=}\boldsymbol{0}$$

$$\Longrightarrow\langle\langle\nabla n,\boldsymbol{T}\rangle\boldsymbol{T}+kn\boldsymbol{N}\ ,\boldsymbol{N}\rangle-kn=\langle\nabla n,\boldsymbol{T}\rangle\underbrace{\langle\boldsymbol{T},\boldsymbol{N}\rangle}_{=0}+kn\underbrace{\langle\boldsymbol{N},\boldsymbol{N}\rangle}_{=1}-kn=0$$

Q.E.D

Q6

From the previous question:

$$t^*=\min_{\gamma(p)}\int_{p_0}^{p_1}\frac{n\big(\gamma(\tau)\big)}{C}\|\dot{\gamma}(\tau)\|d\tau=\min_{y}\int_{x_0}^{x_1}\frac{n(y)}{C}\|\dot{\gamma}(\tau)\|d\tau$$
$$=\min_{y}\int_{x_0}^{x_1}\frac{n(y)}{C}\sqrt{\dot{x}(s)+\dot{y}(s)}ds=\min_{y}\int_{x_0}^{x_1}n(y)\sqrt{1+\dot{y}(x)}dx$$

In this case $F(\gamma,\dot{\gamma})=n(y)\sqrt{\dot{x}(p)+\dot{y}(p)}\ ,\gamma(x)=ax+l$ so we get:

$$\frac{\partial F}{\partial\gamma}-\frac{d}{dp}\left(\frac{\partial F}{\partial\dot{\gamma}}\right)=0$$

$$\frac{\partial F}{\partial\gamma}=\frac{\partial}{\partial\gamma}\left(n(y)\sqrt{\dot{x}(p)+\dot{y}(p)}\right)=\dot{n}(y)=0\left(n(y)=\begin{cases}n_1, & y\le l\\ n_2, & y>l\end{cases}\right)$$

Where $\begin{aligned}x(p)&=\sin\theta\,p\\ y(p)&=\cos\theta\,p+l\end{aligned}$

$$\frac{\partial F}{\partial\dot{\gamma}}=\frac{\partial}{\partial\dot{x}}\left(n(y)\sqrt{\dot{x}(p)+\dot{y}(p)}\right)=n(y)\frac{\dot{x}}{\sqrt{\dot{x}(p)+\dot{y}(p)}}$$

$$\frac{d}{dx}\left(n(y)\frac{\dot{x}}{\sqrt{\dot{x}(p)+\dot{y}(p)}}\right)=0\Longrightarrow n(y)\frac{\dot{x}}{\sqrt{\dot{x}(p)+\dot{y}(p)}}=C$$

$$n(y)\sin\theta=C$$

From the boundary conditions: $n(y=l)\Longrightarrow n_1\sin\theta_1=n_2\sin\theta_2$

Q7

Using the conservation of energy principal:

$$mgy_0 = mgy + \frac{1}{2}mv^2$$

The function we wish to calculate is:

$$\min_{\gamma} \int_{t_0}^{t_1} dt = \min_{\gamma} \int_{A}^{B} \frac{ds}{v}$$

Where $ds = \sqrt{\dot{x}(p)^2 + \dot{y}(p)^2}dp$ or alternatively, $\sqrt{\dot{x}(p)^2 + \dot{y}(p)^2}dp = \sqrt{1 + y_x^2}dx$

From the principle of energy conservation, $v = \sqrt{2g(y_0 - y)}$

$$\min_{\gamma} \int_{A}^{B} \frac{ds}{v} = \min_{\gamma} \int_{A}^{B} \frac{1}{\sqrt{2g(y_0 - y)}} \sqrt{1 + y_x^2}dx$$

Assuming $y_0 = 0$ we get:

$$\min_{\gamma} \int_{t_0}^{t_1} dt = \min_{\gamma} \int_{A}^{B} \frac{1}{\sqrt{2gy}} \sqrt{1 + y_x^2}dx$$

Q8

The Euler Lagrange equation is:

$$F(x, y, y_x) = \sqrt{\frac{1 + y_x^2}{2gy}}$$

Either we use the Hamiltonian of Euler Lagrange equation which is:

$H = y_x \frac{\partial F}{\partial y_x} - F = Const$

It is constant because of F is not a function of x (because $\frac{d}{dx}\left(F - y_x \frac{\partial F}{\partial y_x}\right) = \frac{\partial F}{\partial x} = 0)$

So $y_x \frac{\partial F}{\partial y_x} = \frac{y_x^2}{\sqrt{2gy(1+y_x^2)}}$

Which gives us the Hamiltonian of:

$$\sqrt{\frac{1 + y_x^2}{2gy}} - \frac{y_x^2}{\sqrt{2gy(1 + y_x^2)}} = C$$

$$(1 + y_x^2) - y_x^2 = C\sqrt{2gy(1 + y_x^2)}$$

$$1 = C^2 2gy(1 + y_x^2)$$

Where $k = \frac{1}{2C^2 g}$

$$y_x = \sqrt{\frac{k^2 - y}{y}} \ (*)$$

The cycloid equations are:

$$x = \frac{1}{2}k^2(\theta - \sin\theta)$$

$$y = \frac{1}{2}k^2(1 - \cos\theta)$$

By substituting the cycloid equation in the (*) we get:

$$y_x = \sqrt{\frac{k^2 - y}{y}} \Longrightarrow \left(\frac{dy}{dx}\right)^2 = \frac{k^2 - y}{y}$$

$$dy = \frac{1}{2}k^2 \sin\theta d\theta$$

$$dx = \frac{1}{2}k^2(1 - \cos\theta)d\theta$$

So this gives us:

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{\frac{1}{2}k^2 \sin\theta d\theta}{\frac{1}{2}k^2(1 - \cos\theta)d\theta}\right)^2 = \left(\frac{\sin\theta}{1 - \cos\theta}\right)^2$$

$$\begin{aligned}\frac{k^2 - y}{y} &= \frac{k^2 - \frac{1}{2}k^2(1 - \cos\theta)}{\frac{1}{2}k^2(1 - \cos\theta)} = \frac{k^2 - \frac{1}{2}k^2(1 - \cos\theta)}{\frac{1}{2}k^2(1 - \cos\theta)} = \frac{1 + \cos\theta}{1 - \cos\theta} \\ &= \frac{(1 + \cos\theta)(1 - \cos\theta)}{(1 - \cos\theta)^2} = \frac{1 - \cos^2\theta}{(1 - \cos\theta)^2} = \frac{\sin^2\theta}{(1 - \cos\theta)^2} = \left(\frac{dy}{dx}\right)^2 \blacksquare\end{aligned}$$

Q9

From the previous question (Q6) we showed that:

$$t^* = \min_{\gamma(p)} \int_{p_0}^{p_1} \frac{n\big(\gamma(\tau)\big)}{C} \|\dot{\gamma}(\tau)\| d\tau = \min_{y} \int_{x_0}^{x_1} n(y)\sqrt{1 + \dot{y}^2(x)}dx$$

So it can be seen that

$$n(y) = \frac{1}{\sqrt{2gy}}$$

Q10

We have been given that $\nabla S = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}}$

We would like to show that: $\nabla S = n\boldsymbol{T}$

$$\nabla S = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}} = \frac{\partial}{\partial \dot{\gamma}}(n\|\gamma(s)\|) = n\frac{\partial}{\partial \dot{\gamma}}(\|\gamma(s)\|) + \underbrace{\frac{\partial}{\partial \dot{\gamma}}(n)}_{=0} \|\gamma(s)\| = n\frac{\gamma(s)}{\|\gamma(s)\|} \underbrace{=}_{\|\gamma(s)\|=1} n\boldsymbol{T}$$

$\blacksquare$

Q11

$$S^{AB} = \int_A^B \frac{n\big(\gamma(\tau)\big)}{C} \|\dot{\gamma}(\tau)\| d\tau = S^{AC} + S^{CB}$$
$$= \int_A^C \frac{n\big(\gamma(\tau)\big)}{C} \|\dot{\gamma}(\tau)\| d\tau + \int_C^B \frac{n\big(\gamma(\tau)\big)}{C} \|\dot{\gamma}(\tau)\| d\tau$$

So $t^{AB} = \min_{\gamma(p)} \int_A^B \frac{n(\gamma(\tau))}{c} \|\dot{\gamma}(\tau)\| d\tau = \min_{\gamma(p)} \left\{ \int_A^C \frac{n(\gamma(\tau))}{c} \|\dot{\gamma}(\tau)\| d\tau + \int_C^B \frac{n(\gamma(\tau))}{c} \|\dot{\gamma}(\tau)\| d\tau \right\} = \min_{\gamma(p)} \left\{ \int_A^C \frac{n(\gamma(\tau))}{c} \|\dot{\gamma}(\tau)\| d\tau \right\} + \min_{\gamma(p)} \left\{ \int_C^B \frac{n(\gamma(\tau))}{c} \|\dot{\gamma}(\tau)\| d\tau \right\} = t^{AC} + t^{CB}$ ■

Q12

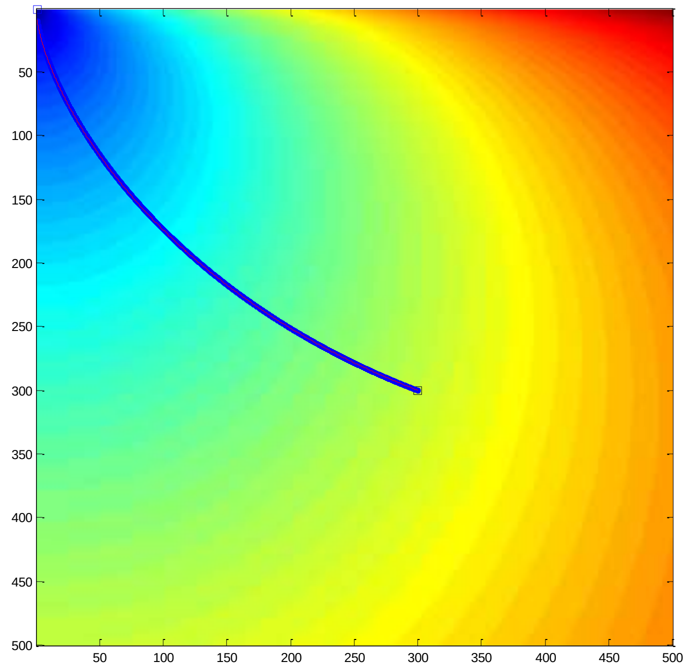
Matlab

Q13

Matlab

Q14

Matlab



Q15

From (9) we had:

$$\min_{z(x,y)} \int_{\Omega} d\Omega = \min_{z(x,y)} \int_{\Omega} \sqrt{(z_x)^2 + \left(z_y\right)^2 + 1} dxdy$$

With the taylor series for

$$(1+x)^{0.5} \approx 1 + \frac{1}{2}x$$

We can approximate this expression to

$$\min_{z(x,y)} \int_{\Omega} d\Omega \approx \min_{z(x,y)} \int_{\Omega} \left((z_x)^2 + \left(z_y\right)^2\right) dxdy$$

The E-L equations are:

$$\frac{\partial F}{\partial z} - \frac{d}{dx}\frac{\partial F}{\partial z_x} - \frac{d}{dy}\frac{\partial F}{\partial z_y} = 0$$

$$F\left(x, y, z_x, z_y\right) = (z_x)^2 + \left(z_y\right)^2$$

So taking each part separately:

$$\frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial z_x} = 2z_x$$

$$\frac{\partial F}{\partial z_y} = 2z_y$$

$$\frac{d}{dx}\frac{\partial F}{\partial z_x} = 2z_{xx}$$

$$\frac{d}{dy}\frac{\partial F}{\partial z_y} = 2z_{yy}$$

And the E-L equation is:

$$z_{xx} + z_{yy} = \nabla^2 z = 0$$

Q16

$$I = \lambda\langle \boldsymbol{l}, \boldsymbol{N}\rangle$$

Assuming the Albedo is the same and the light direction is $l = [0 \quad 0 \quad 1]^T$ we write

$$I = \langle \boldsymbol{l}, \boldsymbol{N}\rangle = \frac{[0 \quad 0 \quad 1]^T[-z_x(x,y) \quad -z_y(x,y) \quad 1]}{\|\boldsymbol{N}\|} = \frac{1}{\sqrt{1 + (z_x)^2 + \left(z_y\right)^2}} = I(x,y)$$

$$\Rightarrow \frac{1}{\sqrt{1 + \|\nabla z\|^2}} = I \Rightarrow \|\nabla z\| = \sqrt{\frac{1 - I^2}{I^2}} = F(x,y)$$

Q17

Matlab

Q18

Show that:

$$\nabla^2 z = \frac{\partial^2 z}{\partial^2 x} + \frac{\partial^2 z}{\partial^2 y} \approx -4z[x, y] + z[x+, y] + z[x - 1, y] + z[x, y + 1] + z[x, y - 1]$$

$$\frac{\partial z}{\partial x} \approx z[x + 1, y] - z[x, y]$$

$$\frac{\partial z}{\partial y} \approx z[x, y + 1] - z[x, y]$$

$$\frac{\partial^2 z}{\partial^2 x} \approx z[x + 1, y] - z[x, y] - (z[x, y] - z[x - 1, y]) = z[x + 1, y] - 2z[x, y] + z[x - 1, y]$$

$$\begin{aligned} \frac{\partial^2 z}{\partial^2 y} &\approx z[x, y + 1] - z[x, y] - (z[x, y] - z[x, y - 1]) \\ &= z[x, y + 1] - 2z[x, y] + z[x, y - 1] \end{aligned}$$

$$\Longrightarrow \nabla^2 z = \frac{\partial^2 z}{\partial^2 x} + \frac{\partial^2 z}{\partial^2 y} \approx -4z[x, y] + z[x+, y] + z[x - 1, y] + z[x, y + 1] + z[x, y - 1]$$

Q19

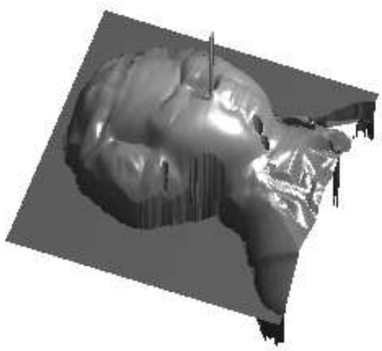
Matlab

Estimated Depth image

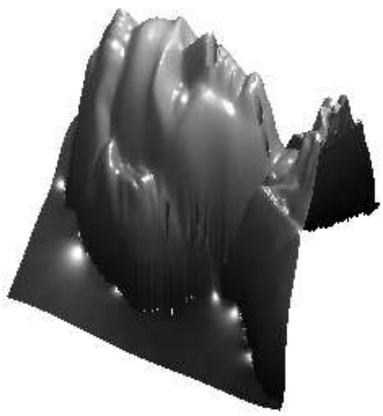
Q20

A qualitative comparison between both methods:

$|\nabla z|=F(x,y)$

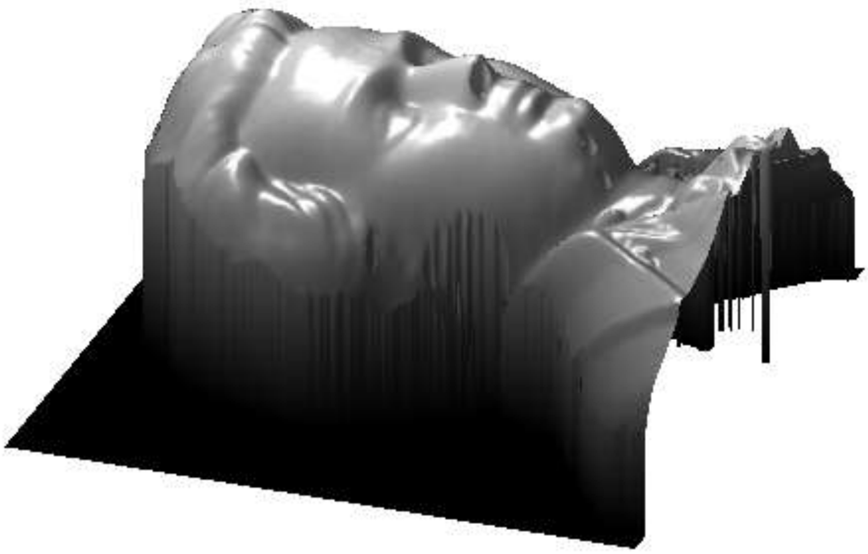


$|\nabla^2 z|=p_x + q_y$



The original bust looks like this:

Ground Truth



In conclusion the linear method is much better that the method based on a single light source.