

# Optimization

## Assignment 3

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Q1

$$\frac{\partial s(p)}{\partial p} = \frac{\partial}{\partial p} \int_0^p \|\dot{\gamma}(\tau)\| d\tau = \|\dot{\gamma}(p)\|$$

Let's define the inverse function:

$$p(s) = f^{-1}(s)$$

$$\frac{\partial}{\partial s} \gamma(p(s)) = \dot{\gamma}(p) \frac{\partial p(s)}{\partial s} \Rightarrow \dot{\gamma}(s) = \frac{\frac{\partial}{\partial p} \gamma(p)}{\frac{\partial s(p)}{\partial p}} = \frac{\dot{\gamma}(p)}{\|\dot{\gamma}(p)\|}$$

Therefore:

$$\|\dot{\gamma}(s)\| = \frac{\|\dot{\gamma}(p)\|}{\|\dot{\gamma}(p)\|} = 1$$

Q2

$$\langle \mathbf{T}, \mathbf{T} \rangle = 1$$

$$\frac{\partial}{\partial s} \langle \mathbf{T}, \mathbf{T} \rangle = 0$$

$$\frac{\partial}{\partial s} \langle \mathbf{T}, \mathbf{T} \rangle = 2 \langle \dot{\mathbf{T}}, \mathbf{T} \rangle = 0 \Rightarrow \mathbf{N} \perp \mathbf{T} \perp \dot{\mathbf{T}} \Rightarrow \dot{\mathbf{T}} \parallel \mathbf{N}$$

$$\Rightarrow \dot{\mathbf{T}} = k\mathbf{N}$$

Now to find  $k$ :

$$\langle \dot{\mathbf{T}}, \mathbf{N} \rangle = \langle k\mathbf{N}, \mathbf{N} \rangle = k \underbrace{\langle \mathbf{N}, \mathbf{N} \rangle}_{=1} = k$$

$$\Rightarrow k = \langle \dot{\mathbf{T}}, \mathbf{N} \rangle = [\ddot{x}(s) \quad \ddot{y}(s)] \begin{bmatrix} \dot{y}(s) \\ -\dot{x}(s) \end{bmatrix} = \ddot{x}(s)\dot{y}(s) - \ddot{y}(s)\dot{x}(s)$$

Q3

The normal to surface  $F(x, y, z) = 0$  is defined as,  $\mathbf{N} = \nabla F(x, y, z)$

In our case  $F(x, y, z) = z - z(x, y) \Rightarrow \mathbf{N} = (-z_x(x, y), -z_y(x, y), 1)$

We can also see that:

$$\begin{aligned} [1, 0, z_x]^T \times [0, 1, z_y]^T &= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 1 & 0 & z_x \\ 0 & 1 & z_y \end{vmatrix} = \hat{x}(-z_x) + \hat{y}(-z_y) + \hat{z}(1) \\ &= (-z_x(x, y), -z_y(x, y), 1) \end{aligned}$$

A Surface integral is defined as

$$A = \iint_T dA = \iint_T \left\| \frac{\partial \mathbf{r}}{\partial x} \times \frac{\partial \mathbf{r}}{\partial y} \right\| dx dy =$$

Where  $\mathbf{r} = (x, y, z(x, y))$

$$\iint_T \|[1, 0, z_x]^T \times [0, 1, z_y]^T\| dx dy = \iint_T \|(-z_x(x, y), -z_y(x, y), 1)\| dx dy =$$

$$\iint_T \sqrt{(z_x)^2 + (z_y)^2 + 1} dx dy$$

So  $dA = \sqrt{(z_x)^2 + (z_y)^2 + 1} dx dy$

Q4

The Euler Lagrange equation for

$$\min_{z(x,y)} \int_{\Omega} d\Omega = \min_{z(x,y)} \int_{\Omega} \sqrt{(z_x)^2 + (z_y)^2 + 1} dx dy$$

$$\int_{\Omega} F(x, y, z_x, z_y) dx dy$$

The Euler Lagrange equations:

$$\frac{\partial F}{\partial z} - \frac{\partial}{\partial x} \frac{\partial F}{\partial z_x} - \frac{\partial}{\partial y} \frac{\partial F}{\partial z_y} = 0$$

$$F(x, y, z_x, z_y) = \sqrt{(z_x)^2 + (z_y)^2 + 1}$$

Let's deal with

$$\frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial z_x} = \frac{\frac{1}{2} 2z_x}{\sqrt{(z_x)^2 + (z_y)^2 + 1}} = \frac{z_x}{\sqrt{(z_x)^2 + (z_y)^2 + 1}}$$

$$\frac{\partial}{\partial x} \frac{\partial F}{\partial z_x} = \frac{z_{xx} \sqrt{(z_x)^2 + (z_y)^2 + 1} - \frac{z_x}{2} (2z_x z_{xx} + 2z_y z_{xy})}{((z_x)^2 + (z_y)^2 + 1)^{\frac{3}{2}}}$$

$$= \frac{z_{xx} ((z_x)^2 + (z_y)^2 + 1) - ((z_x)^2 z_{xx} + z_x z_y z_{xy})}{((z_x)^2 + (z_y)^2 + 1)^{\frac{3}{2}}}$$

Likewise for  $z_y$ :

$$\begin{aligned}\frac{\partial}{\partial y} \frac{\partial F}{\partial z_y} &= \frac{z_{yy} \left( (z_x)^2 + (z_y)^2 + 1 \right) - \left( (z_y)^2 z_{yy} + z_y z_x z_{xy} \right)}{\left( (z_x)^2 + (z_y)^2 + 1 \right)^{\frac{3}{2}}} \\ \frac{\partial}{\partial x} \frac{\partial F}{\partial z_x} + \frac{\partial}{\partial y} \frac{\partial F}{\partial z_y} &= 0 \\ \Rightarrow z_{yy} \left( (z_x)^2 + (z_y)^2 + 1 \right) - \left( (z_x)^2 z_{xx} + z_x z_y z_{xy} \right) + z_{xx} \left( (z_x)^2 + (z_y)^2 + 1 \right) \\ &\quad - \left( (z_y)^2 z_{yy} + z_y z_x z_{xy} \right) = 0 \\ z_{yy} \left( (z_x)^2 + (z_y)^2 + 1 \right) - \left( (z_x)^2 z_{xx} + z_x z_y z_{xy} \right) + z_{xx} \left( (z_x)^2 + (z_y)^2 + 1 \right) \\ &\quad - \left( (z_y)^2 z_{yy} + z_y z_x z_{xy} \right) = 0 \\ \Rightarrow K &= z_{xx} \left( (z_y)^2 + 1 \right) - 2z_x z_y z_{xy} + z_{yy} \left( (z_x)^2 + 1 \right) = 0\end{aligned}$$

Q5

$$\begin{aligned}t^* &= \min_{\gamma(p)} \int_{p_0}^{p_1} \frac{n(\gamma(\tau))}{C} \|\dot{\gamma}(\tau)\| d\tau = \min_{\gamma(p)} \int_{p_0}^{p_1} F(\tau, \gamma(\tau), \dot{\gamma}(\tau)) d\tau \\ &\Rightarrow \frac{\partial F}{\partial \gamma} - \frac{d}{dp} \left( \frac{\partial F}{\partial \dot{\gamma}} \right) = 0\end{aligned}$$

Now find each part separately:

$$\frac{\partial F}{\partial \gamma} = \nabla_\gamma n \|\dot{\gamma}(p)\|$$

$$\begin{aligned}\frac{\partial F}{\partial \dot{\gamma}} &= n \frac{\dot{\gamma}}{\|\dot{\gamma}\|} \\ \frac{d}{dp} \left( \frac{\partial F}{\partial \dot{\gamma}} \right) &= \frac{d}{dp} \left( n \frac{\dot{\gamma}}{\|\dot{\gamma}\|} \right) = \frac{d}{dp} (n \dot{\gamma}(s)) = \frac{ds}{dp} \frac{d}{ds} (n \dot{\gamma}(s)) = \dot{\gamma}(p) \frac{d}{ds} (n \dot{\gamma}(s)) \\ &= \|\dot{\gamma}(p)\| \frac{\dot{\gamma}(p)}{\|\dot{\gamma}(p)\|} \frac{d}{ds} (n \dot{\gamma}(s))\end{aligned}$$

To summarize:

$$\begin{aligned}\nabla n \|\dot{\gamma}(p)\| - \|\dot{\gamma}(p)\| \frac{\dot{\gamma}(p)}{\|\dot{\gamma}(p)\|} \frac{d}{ds} (n \dot{\gamma}(s)) &= 0 \\ \Rightarrow \|\dot{\gamma}(p)\| \left( \nabla n - \frac{\dot{\gamma}(p)}{\|\dot{\gamma}(p)\|} \frac{d}{ds} (n \dot{\gamma}(s)) \right) &= 0 \quad (*)\end{aligned}$$

We still need to calculate:

$$\frac{d}{ds} (n \dot{\gamma}(s)) = \left( \nabla n \cdot \frac{d\gamma}{ds} \right) \dot{\gamma}(s) + n \ddot{\gamma}(s) = \langle \nabla n, \mathbf{T} \rangle \mathbf{T} + n(k\mathbf{N})$$

Substitute in (\*) and we get

$$\nabla n = \langle \nabla n, \mathbf{T} \rangle \mathbf{T} + kn \mathbf{N} (**)$$

Now take the equation we are trying to prove:

$$(\langle \nabla n, \mathbf{N} \rangle - kn) \mathbf{N} = \mathbf{0}$$

Substitute (\*\*) into it:

$$\begin{aligned} & (\langle \langle \nabla n, \mathbf{T} \rangle \mathbf{T} + kn \mathbf{N}, \mathbf{N} \rangle - kn) \mathbf{N} \stackrel{?}{=} \mathbf{0} \\ \Rightarrow & \langle \langle \nabla n, \mathbf{T} \rangle \mathbf{T} + kn \mathbf{N}, \mathbf{N} \rangle - kn = \underbrace{\langle \nabla n, \mathbf{T} \rangle \langle \mathbf{T}, \mathbf{N} \rangle}_{=0} + kn \underbrace{\langle \mathbf{N}, \mathbf{N} \rangle}_{=1} - kn = 0 \end{aligned}$$

Q.E.D

Q6

From the previous question:

$$\begin{aligned} t^* &= \min_{\gamma(p)} \int_{p_0}^{p_1} \frac{n(\gamma(\tau))}{C} \|\dot{\gamma}(\tau)\| d\tau = \min_y \int_{x_0}^{x_1} \frac{n(y)}{C} \|\dot{\gamma}(\tau)\| d\tau \\ &= \min_y \int_{x_0}^{x_1} \frac{n(y)}{C} \sqrt{\dot{x}(s) + \dot{y}(s)} ds = \min_y \int_{x_0}^{x_1} n(y) \sqrt{1 + \dot{y}(x)} dx \end{aligned}$$

In this case  $F(\gamma, \dot{\gamma}) = n(y) \sqrt{\dot{x}(p) + \dot{y}(p)}$  so we get:

$$\begin{aligned} & \frac{\partial F}{\partial \gamma} - \frac{d}{dp} \left( \frac{\partial F}{\partial \dot{\gamma}} \right) = 0 \\ \Rightarrow & \begin{cases} \frac{\partial F}{\partial x} - \frac{d}{dp} \left( \frac{\partial F}{\partial \dot{x}} \right) = 0 \\ \frac{\partial F}{\partial y} - \frac{d}{dp} \left( \frac{\partial F}{\partial \dot{y}} \right) = 0 \end{cases} \end{aligned}$$

$$\frac{\partial F}{\partial \gamma} = \frac{\partial}{\partial y} (n(y) \sqrt{\dot{x}(p) + \dot{y}(p)}) = \dot{n}(y) = 0 \left( n(y) = \begin{cases} n_1, & y \leq l \\ n_2, & y > l \end{cases} \right)$$

Where  $\begin{matrix} x(p) = \sin \theta p \\ y(p) = \cos \theta p + l \end{matrix}$

$$\begin{aligned} \frac{\partial F}{\partial \dot{x}} &= \frac{\partial}{\partial \dot{x}} (n(y) \sqrt{\dot{x}(p) + \dot{y}(p)}) = n(y) \frac{\dot{x}}{\sqrt{\dot{x}(p) + \dot{y}(p)}} \\ \frac{d}{dx} \left( n(y) \frac{\dot{x}}{\sqrt{\dot{x}(p) + \dot{y}(p)}} \right) &= 0 \Rightarrow n(y) \frac{\dot{x}}{\sqrt{\dot{x}(p) + \dot{y}(p)}} = \text{Const} \\ n(y) \sin \theta &= \text{Const} \end{aligned}$$

From the boundary conditions:  $n(y = l) \Rightarrow n_1 \sin \theta_1 = n_2 \sin \theta_2$

Q7

Using the conservation of energy principal:

$$-mgy_0 = -mgy + \frac{1}{2}mv^2$$

The function we wish to calculate is:

$$\min_{\gamma} \int_{t_0}^{t_1} dt = \min_{\gamma} \int_A^B \frac{ds}{v}$$

Where  $ds = \sqrt{\dot{x}(p)^2 + \dot{y}(p)^2} dp$  or alternatively,  $\sqrt{\dot{x}(p)^2 + \dot{y}(p)^2} dp = \sqrt{1 + y_x^2} dx$

From the principle of energy conservation,  $v = \sqrt{2g(-y_0 + y)}$

$$\min_{\gamma} \int_A^B \frac{ds}{v} = \min_{\gamma} \int_A^B \frac{1}{\sqrt{2g(-y_0 + y)}} \sqrt{1 + y_x^2} dx$$

Assuming  $y_0 = 0$  we get:

$$\min_{\gamma} \int_{t_0}^{t_1} dt = \min_{\gamma} \int_A^B \frac{1}{\sqrt{2gy}} \sqrt{1 + y_x^2} dx$$

Q8

$$F(x, y, y_x) = \sqrt{\frac{1 + y_x^2}{2gy}}$$

Because  $F(x, y, y_x) = F(y, y_x) \Rightarrow \frac{dH(x)}{dx} = -\frac{\partial F}{\partial x} = 0$ , it is easier to use the Hamiltonian which is:

$$H = y_x \frac{\partial F}{\partial y_x} - F = \text{Const}$$

It is constant because of  $F$  is not a function of  $x$ . So

$$y_x \frac{\partial F}{\partial y_x} = \frac{y_x^2}{\sqrt{2gy(1 + y_x^2)}}$$

Which gives us the Hamiltonian of:

$$\begin{aligned} \sqrt{\frac{1 + y_x^2}{2gy}} - \frac{y_x^2}{\sqrt{2gy(1 + y_x^2)}} &= C \\ (1 + y_x^2) - y_x^2 &= C \sqrt{2gy(1 + y_x^2)} \\ 1 &= C^2 2gy(1 + y_x^2) \end{aligned}$$

Where  $k = \frac{1}{2C^2g}$

$$y_x = \sqrt{\frac{k^2 - y}{y}} (*)$$

The cycloid equations are:

$$x = \frac{1}{2}k^2(\theta - \sin \theta)$$

$$y = \frac{1}{2}k^2(1 - \cos \theta)$$

By substituting the cycloid equation in the (\*) we get:

$$y_x = \sqrt{\frac{k^2 - y}{y}} \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{k^2 - y}{y}$$

$$dy = \frac{1}{2}k^2 \sin \theta d\theta$$

$$dx = \frac{1}{2}k^2(1 - \cos \theta)d\theta$$

So this gives us:

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{\frac{1}{2}k^2 \sin \theta d\theta}{\frac{1}{2}k^2(1 - \cos \theta)d\theta}\right)^2 = \left(\frac{\sin \theta}{1 - \cos \theta}\right)^2$$

$$\begin{aligned} \frac{k^2 - y}{y} &= \frac{k^2 - \frac{1}{2}k^2(1 - \cos \theta)}{\frac{1}{2}k^2(1 - \cos \theta)} = \frac{k^2 - \frac{1}{2}k^2(1 - \cos \theta)}{\frac{1}{2}k^2(1 - \cos \theta)} = \frac{1 + \cos \theta}{1 - \cos \theta} \\ &= \frac{(1 + \cos \theta)(1 - \cos \theta)}{(1 - \cos \theta)^2} = \frac{1 - \cos^2 \theta}{(1 - \cos \theta)^2} = \frac{\sin^2 \theta}{(1 - \cos \theta)^2} = \left(\frac{dy}{dx}\right)^2 \blacksquare \end{aligned}$$

Q9

From the previous question (Q6) we showed that:

$$t^* = \min_{\gamma(p)} \int_{p_0}^{p_1} \frac{n(\gamma(\tau))}{C} \|\dot{\gamma}(\tau)\| d\tau = \min_y \int_{x_0}^{x_1} n(y) \sqrt{1 + \dot{y}^2(x)} dx$$

So it can be seen that

$$n(y) = \frac{1}{\sqrt{2gy}}$$

Q10

We have been given that  $\nabla S = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}}$

We would like to show that:  $\nabla S = n\mathbf{T}$

$$\nabla S = \frac{\partial \mathcal{L}}{\partial \dot{\gamma}} = \frac{\partial}{\partial \dot{\gamma}} (n \|\dot{\gamma}(s)\|) = n \frac{\partial}{\partial \dot{\gamma}} (\|\dot{\gamma}(s)\|) + \underbrace{\frac{\partial}{\partial \dot{\gamma}} (n)}_{=0} \|\dot{\gamma}(s)\| = n \frac{\dot{\gamma}(s)}{\|\dot{\gamma}(s)\|} \Big|_{\|\dot{\gamma}(s)\|=1} \equiv n\mathbf{T}$$

■

Q11

$$S^{AB} = \int_A^B \frac{n(\gamma(\tau))}{C} \|\dot{\gamma}(\tau)\| d\tau = S^{AC} + S^{CB}$$

$$= \int_A^C \frac{n(\gamma(\tau))}{C} \|\dot{\gamma}(\tau)\| d\tau + \int_C^B \frac{n(\gamma(\tau))}{C} \|\dot{\gamma}(\tau)\| d\tau$$

$$\text{So } t^{AB} = \min_{\gamma(p)} \int_A^B \frac{n(\gamma(\tau))}{c} \|\dot{\gamma}(\tau)\| d\tau = \min_{\gamma(p)} \left\{ \int_A^C \frac{n(\gamma(\tau))}{c} \|\dot{\gamma}(\tau)\| d\tau + \int_C^B \frac{n(\gamma(\tau))}{c} \|\dot{\gamma}(\tau)\| d\tau \right\} =$$

$$\min_{\gamma(p)} \left\{ \int_A^C \frac{n(\gamma(\tau))}{c} \|\dot{\gamma}(\tau)\| d\tau \right\} + \min_{\gamma(p)} \left\{ \int_C^B \frac{n(\gamma(\tau))}{c} \|\dot{\gamma}(\tau)\| d\tau \right\} = t^{AC} + t^{CB} \blacksquare$$

Q12

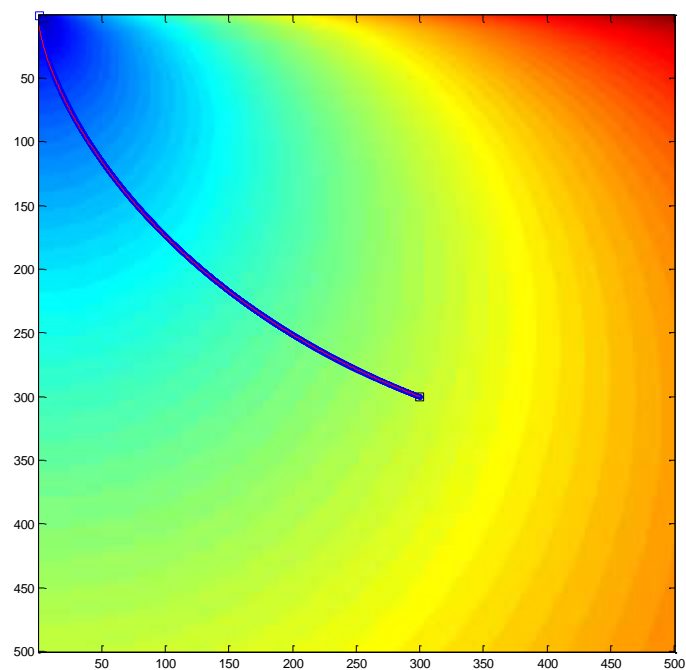
Matlab

Q13

Matlab

Q14

Matlab





### Q15

From (9) we had:

$$\min_{z(x,y)} \int_{\Omega} d\Omega = \min_{z(x,y)} \int_{\Omega} \sqrt{(z_x)^2 + (z_y)^2 + 1} dx dy$$

With the Taylor series for

$$(1+x)^{0.5} \approx 1 + \frac{1}{2}x$$

We can approximate this expression to

$$\min_{z(x,y)} \int_{\Omega} d\Omega \approx \min_{z(x,y)} \int_{\Omega} ((z_x)^2 + (z_y)^2) dx dy$$

The E-L equations are:

$$\frac{\partial F}{\partial z} - \frac{d}{dx} \frac{\partial F}{\partial z_x} - \frac{d}{dy} \frac{\partial F}{\partial z_y} = 0$$

$$F(x, y, z_x, z_y) = (z_x)^2 + (z_y)^2$$

So taking each part separately:

$$\frac{\partial F}{\partial z} = 0$$

$$\frac{\partial F}{\partial z_x} = 2z_x = 2p$$

$$\frac{\partial F}{\partial z_y} = 2z_y = 2q$$

$$\frac{d}{dx} \frac{\partial F}{\partial z_x} = 2z_{xx}$$

$$\frac{d}{dy} \frac{\partial F}{\partial z_y} = 2z_{yy}$$

And the E-L equation is:

$$z_{xx} + z_{yy} = \nabla^2 z = p_x + q_x = 0$$

### Q16

$$I = \lambda \langle \mathbf{l}, \mathbf{N} \rangle$$

Assuming the Albedo is the same and the light direction is  $\mathbf{l} = [0 \ 0 \ 1]^T$  we write

$$I = \langle \mathbf{l}, \mathbf{N} \rangle = \frac{[0 \ 0 \ 1]^T [-z_x(x, y) \ -z_y(x, y) \ 1]}{\|\mathbf{N}\|} = \frac{1}{\sqrt{1 + (z_x)^2 + (z_y)^2}} = I(x, y)$$

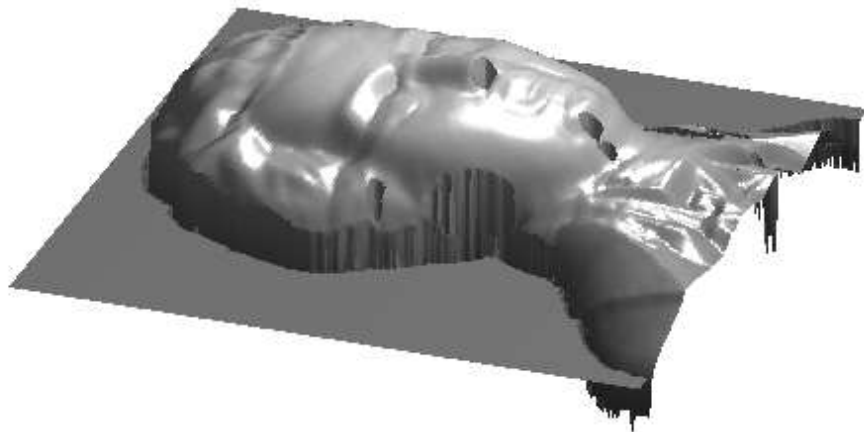
$$\Rightarrow \frac{1}{\sqrt{1 + \|\nabla z\|^2}} = I \Rightarrow \|\nabla z\| = \sqrt{\frac{1 - I^2}{I^2}} = F(x, y)$$

$F(x, y) = 0 \Rightarrow I(x, y) = 1$  which mean a saturated pixel. This happens when we have specularity instead of an albedo surface. This violates the basic assumption of our model.

Q17

Matlab

$$|\nabla z| = F(x, y)$$



Q18

Show that:

$$\nabla^2 z = \frac{\partial^2 z}{\partial^2 x} + \frac{\partial^2 z}{\partial^2 y} \approx -4z[x, y] + z[x+, y] + z[x - 1, y] + z[x, y + 1] + z[x, y - 1]$$

$$\frac{\partial z}{\partial x} \approx z[x + 1, y] - z[x, y]$$

$$\frac{\partial z}{\partial y} \approx z[x, y + 1] - z[x, y]$$

$$\frac{\partial^2 z}{\partial^2 x} \approx z[x + 1, y] - z[x, y] - (z[x, y] - z[x - 1, y]) = z[x + 1, y] - 2z[x, y] + z[x - 1, y]$$

$$\begin{aligned} \frac{\partial^2 z}{\partial^2 y} &\approx z[x, y + 1] - z[x, y] - (z[x, y] - z[x, y - 1]) \\ &= z[x, y + 1] - 2z[x, y] + z[x, y - 1] \end{aligned}$$

$$\Rightarrow \nabla^2 z = \frac{\partial^2 z}{\partial^2 x} + \frac{\partial^2 z}{\partial^2 y} \approx -4z[x, y] + z[x+, y] + z[x - 1, y] + z[x, y + 1] + z[x, y - 1]$$

Q19  
Matlab

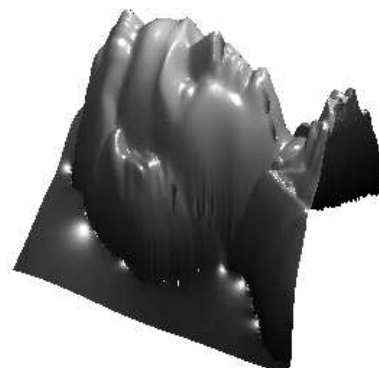
Estimated Depth image



Q20  
A qualitative comparison between both methods:

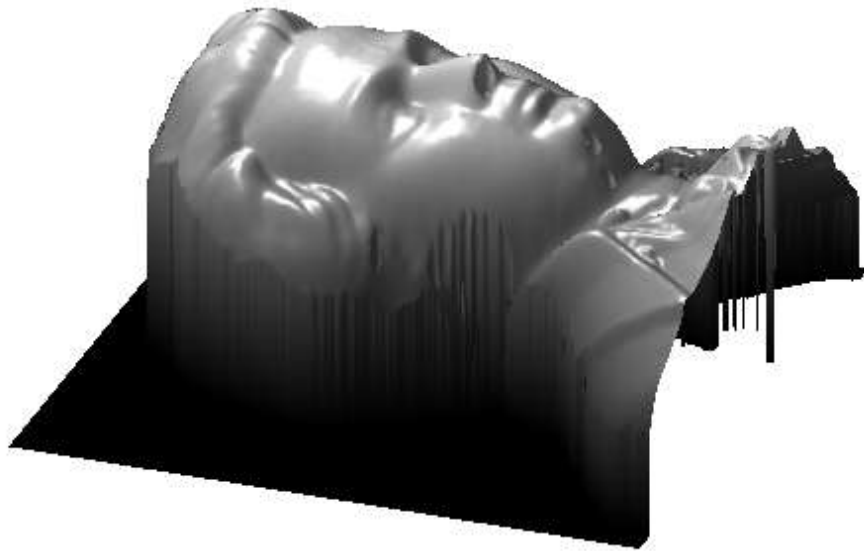
$$|\nabla z| = F(x, y)$$

$$|\nabla^2 z| = p_x + q_y$$



The original bust looks like this:

Ground Truth



In conclusion the linear method is much better than the method based on a single light source.