# Envy-Free Cake-Cutting in Two Dimensions


## Abstract

We consider the problem of fairly dividing a two-dimensional heterogeneous good, such as land or ad space in print and electronic media, among several agents. Classic cake-cutting procedures either consider a one-dimensional resource, or allocate each agent a collection of disconnected pieces. In practice, however, the two-dimensional shape of the allotted piece is of crucial importance in many applications. For example, when building houses or designing advertisements, in order to be useful, the allotments should be squares or rectangles with bounded aspect-ratio. We thus introduce the problem of fair two-dimensional division wherein the allocated piece must have a pre-specified geometric shape. Most classic cake-cutting procedures cannot handle such constraints. We present constructive cake-cutting procedures that satisfy the two most prominent fairness criteria, namely *envy-freeness* and *proportionality*. In scenarios where proportionality cannot be achieved due to the geometric constraints, our procedures provide a *partially-proportional* division, guaranteeing that the fraction allocated to each agent be at least a certain positive constant. We prove that in many natural scenarios the envy-freeness requirement is compatible with the best attainable partial-proportionality.



**JEL classification**: D63

*Keywords:* Cake Cutting, Envy Free, Two Dimensional Cake, Land Division, Geometry, Fairness


## 1. Introduction

Fair cake-cutting is an active field of research with various applications. A prominent application is is division of land (e.g. Berliant and Raa [1], Berliant et al. [2], Legut et al. [3], Chambers [4], Dall'Aglio and Maccheroni [5], Hüsseinov [6], Nicolò et al. [7]). The basic setting considers a heterogeneous good, such as a land-estate, to be divided among several agents. The agents may have different preferences over the possible pieces of the good, e.g. one agent prefers the forests while the other prefers the sea shore. The goal is to divide the good among the agents in a way deemed "fair". The common fairness criterion in economics is *Envy-freeness*, which means that no agent prefers getting a piece allotted to another agent. An important positive result has been proved by Weller [8]: when the agents' preferenecs are represented by non-atomic measures, there always exists a

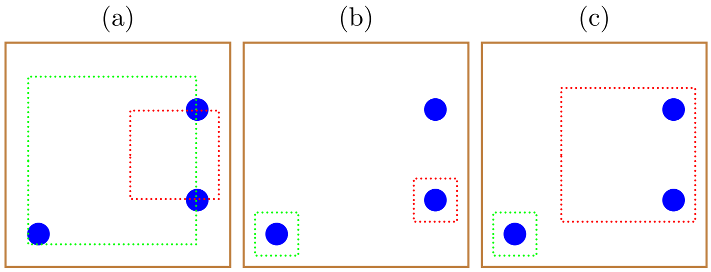


Figure 1: A square land-estate has to be divided between two people. The land-estate is mostly barren, except for three water-pools (discs). The agents have the same preferences: each agent wants a square land-plot with as much water as possible. The squares must not overlap. Hence:
(a) It is impossible to give both agents more than 1/3 of the water. Hence:
(b) An envy-free division must give each agent at most 1/3 of the water.
(c) But such a division cannot be Pareto-efficient since it is dominated by a division which gives one agent 1/3 and the other 2/3 of the water.

competitive-equilibrium with equal incomes, and the equilibrium allocation is both Pareto-efficient and envy-free. Hence, it is possible to simultaneously satisfy the two economic ideals of efficiency and equity. However, Weller's equilibrium allocation gives no guarantees about the geometric shape of the allotted pieces. A "piece" might even be a union of a countable number of disconnected cake-bits. So, Weller's positive result is valid only when the agents' preferences ignore the geometry of their allotted pieces. While such preferences may make sense when dividing an actual edible cake, they are not so sensible when dividing land.

Many authors have noted the importance of imposing some geometric constraints on the pieces. The most common constraint is *connectivity* - each agent should receive a single connected piece rather than a possibly infinite collection thereof. The cake is usually assumed to be the one-dimensional interval $[0, 1]$ and the allotted pieces are sub-intervals (e.g. Stromquist [9], Su [10], Nicolò and Yu [11], Azrieli and Shmaya [12]). This is usually justified by the reasoning that higher dimensional settings can always be projected onto one dimension, and hence fairness in one dimension implies fairness in higher dimensions. However, projecting back from the one dimension, the resulting two-dimensional plots are thin rectangular slivers, of little use in most practical applications; it is hard to build a house on a $10 \times 1,000$ meter plot even though its area is a full hectare, and a thin 0.1-inch wide advertisement space would ill-serve most advertises regardless of its height.

A multi-dimensional cake model has been studied by Berliant and Dunz [13]. Their results are mostly negative: when general value measures are combined with geometric preferences, a competitive-equilibrium might not exist. In Figure 1 we extend their negative result and show that, with geometric preferences, a Pareto-efficient-envy-free allocation might not exist, even without regard to competitive-equilibrium.

A possible solution is to go back to the first fairness criterion studied in the context of cake-cutting [14]: every agent should receive at least $1/n$ of the total cake value. This criterion is now termed *proportionality* and it is very common in the cake-cutting literature [15, 16]. With geometric preferences, a proportional division does not always exist (see Figure 1 again). We therefore relax the proportionality requirement and consider *partial proportionality*. Partial proportionality means that each agent receives a piece with a certain positive value, which is at least a certain fraction of the total cake value. Obviously we would like this fraction to be as large as possible.

In a previous paper [17], we studied the partial-proportionality criterion in isolation. We proved upper and lower bounds on the fraction that can be guaranteed to each agent, depending on the geometric shape of the cake and the geometric preferences of the agents. A typical result is: "when the cake is a square and there are $n$ agents who want square pieces, it is always possible to give each agent at least $1/(4n-4)$ of the total cake value, and it may be impossible to guarantee all agents more than $1/(2n)$ of the cake value". These results raise the questions of whether partial-proportionality is compatible with *Pareto-efficiency* and with *envy-freeness*.

The first question is easy: partial-proportionality is always compatible with Pareto-efficiency (under some natural compactness constraints), since partial-proportionality is preserved under Pareto-improvements. Hence, the results in our previous paper imply that, in all scenarios studied there, Pareto-efficient and partially-proportional allocations exist, where the level of partial-proportionality depends on the geometric shape of the cake and the desired shape of the pieces.

The second question is the topic of the present paper:

> When each agent wants a piece with a given geometric shape, does there always exist an allocation which is both envy-free and partially-proportional?
>
> If so, what is the largest proportionality compatible with envy-freeness, i.e, what is the largest fraction of the cake that can be guaranteed to every agent in an envy-free allocation?

Existing cake-cutting procedures are insufficient for answering the above questions, as the following example illustrates.

**Example 1.** You and a partner are going to divide a square land-estate. It is 100-by-100 square meters and its western side is adjacent to the sea. Your desire is to build a house near the sea-shore. You decide to use the classic procedure for envy-free division: “You cut, I choose”. You let your partner divide the land to two plots, knowing that you have the right to choose the plot that is more valuable according to your personal preferences. Your partner makes a cut parallel to the shoreline at a distance of only 1 meter

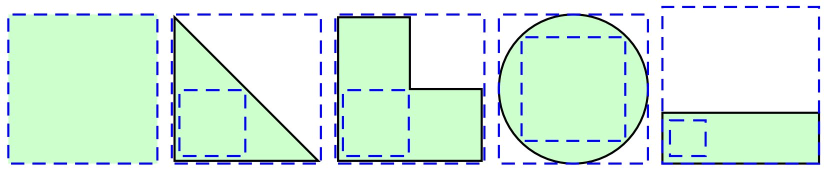


Figure 2: Fatness of several geometric shapes.

from the sea. [1] Which of the two plots would you choose? The western plot contains a lot of sea shore, but it is so narrow that it has no room for building anything. On the other hand, the eastern plot is large but does not contain any shore land. Whichever plot you choose, the division is not proportional for you, because your utility is far less than half the utility of the original land estate.

Of course the cake *could* be cut in a more sensible way (e.g. by a line perpendicular to the sea), but the current division procedures say nothing about how exactly the cake should be cut in each situation in order to guarantee that the division is fair in a way that respects the geometric preferences. While the cut-and-choose procedure still guarantees envy-freeness, it does not guarantee partial-proportionality since it does not guarantee any positive utility to agents who want square pieces.

The present paper shows cake-cutting procedures that guarantee both envy-freeness and partial-proportionality. Our procedures focus on agents who want *fat pieces* - pieces with a bounded length/width ratio, such as squares. The rationale is that a fat shape is more convenient to work with, build on, cultivate, etc.

### *1.1. Fatness*

We use the following formal definition of fatness, which is adapted from the computational geometry literature, e.g. Agarwal et al. [18], Katz [19]:

**Definition 1.1.** Let $R \geq 1$ be a real number. A $d$-dimensional piece is called *$R$-fat*, if it contains a $d$-dimensional cube $B^-$ and is contained in a parallel $d$-dimensional cube $B^+$, such that the ratio between the side-lengths of the cubes is at most $R$: $\text{len}(B^+)/\text{len}(B^-) \leq R$.

A 2-dimensional cube is a square. So, for example, the only 2-dimensional 1-fat shape is a square (it is also 2-fat, 3-fat etc.). An $L \times 1$ rectangle is

[1] The reason why he decided to cut this way is irrelevant since a fair division procedure is expected to guarantee that the division is fair for every agent playing by the rules, regardless of what the other agents do.

$L$-fat; a right-angled isosceles triangle is 2-fat (but not 1-fat) and a circle is $\sqrt{2}$-fat (see Figure 2).

Note that the fatness requirement is inherently multi-dimensional and cannot be reduced to a 1-dimensional requirement. Hence it cannot be satisfied by methods developed for a 1-dimensional cake.[2]

### 1.2. Results

We prove that envy-freeness and partial-proportionality are *compatible* in progressively more general geometric scenarios. Our proofs are constructive: in every geometric scenario (geometric shape of the cake and preferred shape of the pieces), we present a procedure that divides the cake with the following guarantees:

- *Envy-freeness*: every agent weakly prefers his/her allotted piece over the piece given to any other agent.
- *Partial-proportionality*: every agent receives a piece with a value of at least $\alpha$, which is a positive constant that depends on the geometric requirements.

In the following theorems, the partial-proportionality constant $\alpha$ is given in parentheses.

**Theorem 1.** *When dividing a cake to two agents, there is a procedure for finding an envy-free and partially-proportional allocation in the following cases:*

*(a) The cake is square and the usable pieces are squares* ($1/4$)*.*

*(b) The cake is an $R$-fat rectangle and the usable pieces are $R$-fat rectangles, where $R \geq 2$* ($1/3$)*.*

*(c) The cake is an arbitrary $R$-fat object and the pieces are $2R$-fat, where $R \geq 1$* ($1/2$)*.*

**Value-shape trade-off**: Theorem 1 illustrates a multiple-way trade-off between value and shape. Consider two agents who want to divide a square cake in an envy-free way. They have the following options:

- By projecting a 1-dimensional division obtained by any classic cake-cutting procedure, they can achieve a proportional allocation (a value of at least 1/2) with rectangular pieces but with no bound on the aspect ratio - the pieces might be arbitrarily thin.
- By (a), they can achieve an allocation with square pieces but only partial proportionality - the proportionality might be as low as 1/4.

[2]In contrast, the simpler requirement that the pieces be rectangles with an arbitrary length/width ratio can easily be reduced to a 1-dimensional requirement that the pieces are connected intervals. Such reduction is also possible for the requirements that the pieces be simplexes [20] or polytopes [5] with an unbounded aspect ratio.

- By (b), they can achieve a proportionality of 1/3 with 2-fat rectangles, which is a compromise between the previous two options.
- By (c), they can achieve an allocation that is *both* proportional *and* with 2-fat pieces, but the pieces might be non rectangular.

The proportionality constants in Theorem 1 are tight in the following sense: it is not possible to guarantee an allocation with a larger proportionality, even if envy is allowed. This means that envy-freeness is compatible with the largest possible proportionality - we don't have to compromise on proportionality to prevent envy.

**Theorem 2.** *When dividing a cake to $n$ agents, there is a procedure for finding an envy-free and partially-proportional allocation in the following cases:*

*(a) The cake is square and the usable pieces are squares ($1/O(n^2)$).*

*(b) The cake is an $R$-fat rectangle and the usable pieces are $R$-fat rectangles, where $R \geq 1$ ($1/O(n^2)$).*

*(c) The cake is a $d$-dimensional $R$-fat object and the pieces are $\lceil n^{1/d} \rceil R$-fat,*[3] *where $d \geq 2$ and $R \geq 1$ ($1/n$).*

**Value-shape trade-off**: Part (a) and part (c) are duals in the following sense:

- Part (a) guarantees an envy-free division with perfect pieces (squares) but compromises on the proportionality level;
- Part (c) guarantees an envy-free division with perfect proportionality ($1/n$) but compromises on the fatness of the pieces.

The constant of the first compromise, $1/O(n^2)$, is not tight. It is possible to attain a division with square pieces and a proportionality of $1/O(n)$ which is not necessarily envy-free. We do not know if envy-freeness is compatible with a proportionality of $1/O(n)$. The constant of the second compromise, $\lceil n^{1/d} \rceil R$, *is* asymptotically tight: in order to guarantee a proportional division of an $R$-fat cake, with or without envy, we must allow the pieces to be $\Omega(n^{1/d} R)$-fat.

**Strategy considerations**: For the sake of simplicity, we present our division procedures as if all agents act according to their true value functions. However, the guarantees of the procedures are valid for any *single* agent who acts according to his own value function. E.g, the procedure of Theorem 2(c) guarantees that every agent acting according to his true value function receives a piece with a utility of at least $1/n$ and at least as good as the other pieces, regardless of what the other agents do. This is the common practice in the cake-cutting world.

The paper proceeds as follows. The introduction section is concluded by reviewing some related research. The formal definitions and model are

[3] $\lceil x \rceil$ denotes the *ceiling* of $x$ - the smallest integer which is larger than $x$.

provided in Section 2. Section 3 introduces the core geometric concepts and techniques. These geometric techniques are then applied in the construction of the envy-free division procedures for two agents (Section 4) and $n$ agents (Section 5).

## 1.3. Related work

### 1.3.1. Non-additive utilities

Utility functions that depend on geometric shape are non-additive. For example, consider an agent who wants to build a square house the utility of which is determined by its area. The utility of this agent from a $20 \times 20$ plot is 400, but if this plot is divided to two $20 \times 10$ plots, the utility from each plot is 100 and the sum of utilities is only 200. Thus they do not fit into the common cake-cutting model which assumes additivity. Previous work on cake-cutting with non-additive utilities can be classified to three types.

Berliant et al. [2], Maccheroni and Marinacci [21] focus on sub-additive, or concave, utility functions, in which the sum of the utilities of the parts is *more* than the utility of the whole. These utility functions are inapplicable in our scenario because, as illustrated above, utility functions that consider geometry are not necessarily sub-additive - the sum of the utilities of the parts might be less than the utility of the whole.

Sagara and Vlach [22], Dall'Aglio and Maccheroni [5], Hüsseinov and Sagara [23] consider general non-additive utility functions but provide only non-constructive existence proofs.

Su [10], Caragiannis et al. [24], Mirchandani [25] provide practical division procedures but they crucially assume that the cake is a 1-dimensional interval and cannot handle two-dimensional constraints.

### 1.3.2. Envy-free division

The classic cut-and-choose procedure guarantees an envy-free division but does not make any shape-related guarantees. When applied to agents with non-additive utility functions, the division is still envy-free but the utility per agent might be arbitrarily small. The same is true for all other procedures for envy-free division that we are aware of (Stromquist [9], Brams and Taylor [26], Reijnierse and Potters [27], Su [10], Barbanel and Brams [28], Manabe and Okamoto [29], Cohler et al. [30], Deng et al. [31], Kurokawa et al. [32], Chen et al. [33]).

### 1.3.3. Geometric constraints

Several authors diverge from the interval model by assuming a circular cake (e.g. Thomson [34], Brams et al. [35], Barbanel et al. [36]), but they still work in one dimension so the pieces are one-dimensional arcs corresponding to thin wedge-like slivers.

The importance of the multi-dimensional geometric shape of the plots was noted by several authors.

Hill [37], Beck [38], Webb [39], Berliant et al. [2] study the problem of dividing a disputed territory between several bordering countries, with the constraint that each country should get a piece that is adjacent to its border.

Ichiishi and Idzik [20], Dall'Aglio and Maccheroni [5] require the plots to be convex shapes such as multi-dimensional simplexes. However, there are no restrictions on the allocated simplexes and apparently these can be arbitrarily thin. Additionally, the proofs are purely existential.

Iyer and Huhns [40] describe a procedure for giving each agent a rectangular plot with an aspect ratio determined by the agent. Their procedure asks each of the $n$ agents to draw $n$ disjoint rectangles on the map of the two-dimensional cake. These rectangles are supposed to represent the "desired areas" of the agent. The procedure tries to give each agent one of his $n$ desired areas. However, it does not succeed unless each rectangle proposed by an individual intersects at most one other rectangle drawn by any other agent. If even a single rectangle of Alice intersects two rectangles of George (for example), then the procedure fails and no agent gets any piece.

In a previous paper [17] we considered the problem of partially-proportional division in two dimensions without envy considerations. Some of the upper bounds (impossibility results) on partial-proportionality are replicated from that paper to make the current paper stand alone.

## 2. Model and Terminology

The *cake* $C$ is a Borel subset of a Euclidean space $\mathbb{R}^d$. In most of the paper $d = 2$. *Pieces* are Borel subsets of $\mathbb{R}^d$. *Pieces of* $C$ are Borel subsets of $C$.

There is a family $S$ of pieces that are considered usable. An $S$*-piece* is an element of $S$.

There are $n \geq 1$ *agents*. Each agent $i \in \{1, ..., n\}$ has a value-density function $v_i$, which is an integrable, non-negative and bounded function on $C$. In the context of land division, the value-density function represents the quality of each land-spot in the eyes of the agent. It may depend upon factors such as the fertility of soil, the probability of finding oil, the existence of trees, etc.

The *value* of a piece $X$ to agent $i$ is marked by $V_i(X)$ and it is the integral of the value-density over the piece:

$$V_i(X) = \int_{x \in X} v_i(x)dx$$

We assume that for all agents $i$, $V_i(C) < \infty$. Hence each $V_i$ is a finite measure and it is absolutely-continuous with respect to the Lebesgue measure on $C$.

In the standard cake-cutting model [8, 4, 41, 33], the *utility function* of an agent is identical to his/her value measure. The present paper diverges from this model by considering agents whose utility functions depend both on value and on geometric shape. We assume that an agent can derive utility only from an $S$-piece; when his allotted land-plot is not an $S$-piece, he selects the most valuable $S$-piece contained therein and utilizes it. For each agent $i$, we define the *$S$-value function*, which assigns to each piece $X$ the value of the most valuable usable piece contained therein:

$$V_i^S(X) = \sup_{s \in S \ , \ s \subseteq X} V_i(s)$$

We assume that the utility of agent $i$ is equal to his $S$-value function $V_i^S$. In general, $V_i^S$ is not a measure since it is not additive (it is not even sub-additive). Hence, cake-cutting procedures that require additivity are not applicable. Note that the two most common cake-cutting models are special cases of our model:

- The model in which each agent may receive an arbitrary Borel subset [8] is a special case in which $S$ is the set of all Borel subsets of $C$.

- The model in which each agent must receive a connected piece [9] is a special case in which $C$ is an interval and $S$ is the set of intervals.

When the utilities of all agents are determined by $S$-value functions, we can restrict our attention to allocations in which each agent receives an $S$-piece. An *$S$-allocation* is a vector of $n$ $S$-pieces $X = (X_1, ..., X_n)$, one piece per agent, such that $\forall i : X_i \subseteq C$ and the $X_i$ are pairwise-disjoint. Some parts of the cake may remain unallocated (free disposal is assumed).

An $S$-allocation $X$ is called *envy-free* if the utility of an agent from his allocated $S$-piece is at least as large as his utility from every piece allocated to any other agent:

$$\forall i, j \in \{1, ..., n\} : \ V_i^S(X_i) \geq V_i^S(X_j)$$

In addition to envy-freeness, an allocation is assessed by the the fraction of the total cake value that is given to each agent. An allocation is called *proportional* if every agent receives a piece worth for him at least $1/n$ of the total cake value. Since a proportional $S$-allocation does not always exist (see e.g. Figure 1), we define:

**Definition 2.1.** (a) For a cake $C$, a family of usable pieces $S$ and an integer $n \geq 1$, the **envy-free proportionality** of $C$, $S$ and $n$, marked $\text{PropEF}(C, S, n)$, is the largest fraction $\alpha \in [0, 1]$ such that, for every set of $n$ value measures $(V_i, ..., V_n)$, there exists an envy-free $S$-allocation $(X_1, ..., X_n)$ for which $\forall i : \ V_i(X_i)/V_i(C) \geq \alpha$.[4]

(b) The **proportionality** of $C$, $S$ and $n$, marked $\text{Prop}(C, S, n)$, in defined exactly the same way, with the only difference being that all allocations are considered, rather than just the envy-free ones.

[4]Shortly: $\text{PropEF}(C, S, n) = \inf_V \sup_X \min_i V_i(X_i)/V_i(C)$, where the infimum is on all combinations of $n$ value measures $(V_1, ..., V_n)$, the supremum is on all envy-free $S$-allocations $(X_1, ..., X_n)$ and the minimum is on all agents $i \in \{1, ..., n\}$.

Obviously, because the supremum in $\text{PropEF}(C, S, n)$ is taken over a smaller set:

$$\forall C, n, S : \text{PropEF}(C, S, n) \leq \text{Prop}(C, S, n)$$

This means that, in theory, if we want to guarantee that there is no envy, we may have to "pay" in terms of proportionality. One of the goals of the current paper is to study if and how much we may have to pay.

Classic cake-cutting results imply that for every cake $C$:

$$\text{Prop}(C,\ All,\ n) = \text{PropEF}(C,\ All,\ n) = \frac{1}{n}$$

where "All" is the collection of all Borel subsets of $C$. That is: when all pieces are usable, every cake can be divided among every group of $n$ agents in an envy-free allocation in which the utility of each agent is at least $1/n$.

Our challenge in the rest of this paper will be to establish bounds on $\text{PropEF}(C, S, n)$ for various combinations of $C$ and $S$. All our possibility results (lower bounds) are on $\text{PropEF}(C, S, n)$ and therefore are also valid for $\text{Prop}(C, S, n)$; similarly, all our impossibility results (upper bounds) are for $\text{Prop}(C, S, n)$ and therefore are also valid for $\text{PropEF}(C, S, n)$.

## 3. Geometric Preliminaries

### *3.1. Geometric loss*

A key geometric concept in our analysis is the *geometric loss* - the maximum factor by which the utility of an agent can be reduced by his insistence on using pieces only from family $S$.

**Definition 3.1.** For a piece $C$ and family of usable pieces $S$, the *geometric loss* factor of $C$ relative to $S$ is defined as:

$$\text{Loss}(C, S) := \sup_V \frac{V(C)}{V^S(C)}$$

where the supremum is over all finite absolutely-continuous value measures $V$ having $V^S(C) > 0$. If there is no supremum, then we write $\text{Loss}(C, S) = \infty$.

When $C \in S$ the loss is 1, that is no loss, since in this case $V^S(C) = V(C)$. When $C \notin S$, the loss is generally larger than 1. For example, if $C$ is a 30-by-20 rectangle. The largest square contained in $C$ is 20-by-20. Hence, if the value density is uniform over $C$ (as in Figure 3/a), then $\frac{V(C)}{V^S(C)} = \frac{600}{400} = \frac{3}{2}$, implying that $\text{Loss}(C,\ Squares) \geq 3/2$. But the loss may be larger: suppose $V$ is uniform over the right and left sides of $C$ (as in Figure 3/b). In this case $\frac{V(C)}{V^S(C)} = 2$, implying that $\text{Loss}(C,\ Squares) \geq 2$. As we will see in Subsection 3.3, the loss in this case is exactly 2, and in general the loss of a rectangle with a length/width ratio of $L$ is $\lceil L \rceil$; a thinner rectangle has a larger loss.

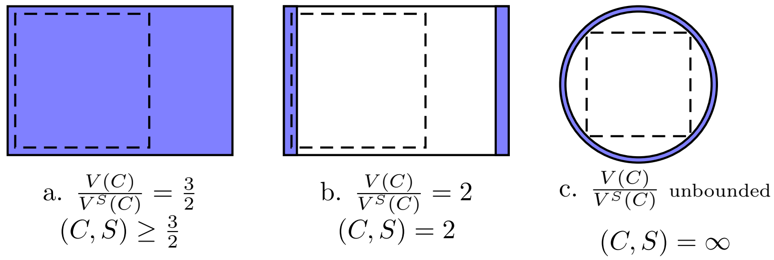


Figure 3: Geometric loss factors relative to the family of squares.

For some combinations of $C$ and $S$, the geometric loss factor might be infinite. For example, if $C$ is a circle and that $V$ is nonzero only in a very narrow strip near the perimeter (as in Figure 3/c), any square contained in $C$ intersects the valuable strip only in the corners. and the intersection might be arbitrarily small. Hence, $V^S(C)$ might be arbitrarily small and $\text{Loss}(C, \textit{Squares}) = \infty$.

### *3.2. Chooser Lemma*

The following lemma relates the geometric loss factor to cake partitions.

**Lemma 3.1.** *For every cake $C$, partition $X = (X_1, \ldots, X_m)$ of $C$, family $S$ and value measure $V$:*

$$\max_{i \in \{1,\ldots,m\}} V^S(X_i) \geq \frac{V(C)}{\sum_{i=1}^m Loss(X_i, S)}$$

*Proof.* Denote the denominator in the right-hand side by:

$$\text{Loss}(X, S) := \sum_{i=1}^m \text{Loss}(X_i, S)$$

By additivity of $V$:

$$\sum_{i=1}^m V(X_i) = V(C)$$

Multiply both sides by the $\text{Loss}(X, S) = \sum_{i=1}^m \text{Loss}(X_i, S)$:

$$\sum_{i=1}^m V(X_i) \cdot \text{Loss}(X, S) = \sum_{i=1}^m \text{Loss}(X_i, S) \cdot V(C)$$

By the pigeonhole principle, at least one of the $m$ summands in the left-hand side must be greater than or equal to the corresponding summand in the right-hand side. I.e., there exists an $i$ for which:

$$V(X_i) \cdot \text{Loss}(X, S) \geq \text{Loss}(X_i, S) \cdot V(C)$$

By Definition 3.1 and the definition of supremum, for every value measure $V$:

$$\text{Loss}(X_i, S) \geq \frac{V(X_i)}{V^S(X_i)}$$

Combining the above two inequalities yields:

$$V(X_i) \cdot \text{Loss}(X, S) \geq \frac{V(X_i) \cdot V(C)}{V^S(X_i)}$$

which is equivalent to:

$$V^S(X_i) \geq \frac{V(C)}{\text{Loss}(X, S)}$$

□

Motivated by the Chooser Lemma and its proof, we define the expression $\text{Loss}(X, S) := \sum_{i=1}^{m} \text{Loss}(X_i, S)$ as the *geometric loss of the partition* $X$. The Chooser Lemma implies that *smaller* geometric-loss is *better* for the chooser. This is easy to see in Example 1, involving a partition of a 100-by-100 land-estate:

- A partition to 100-by-1 and 100-by-99 rectangles has a geometric loss of 102 (the loss of the 100-by-1 sliver is 100 and the loss of the 100-by-99 rectangle is 2). Hence, the utility guarantee for a chooser who wants square pieces is only 1/102.[5]
- In contrast, a partition to two 100-by-50 rectangles has a geometric loss of 4 (2+2) and the chooser can always get a square with a utility of at least 1/4.

### *3.3. Cover Numbers and Cover Lemma*

Since smaller geometric loss is better, it is useful to have an upper bound on the geometric loss.

**Definition 3.2.** For a cake $C$ and family $S$, $CoverNum(C, S)$ is the smallest number of $S$-pieces whose union is exactly $C$.

Some examples are depicted in Figure 4.

**Lemma 3.2.** *For every cake $C$ and family $S$:*

$$Loss(C, S) \ \leq \ CoverNum(C, S)$$

[5]The upper bound of 1/102 is tight. This can be shown by describing a specific value measure and showing that a chooser with this value measure can get a utility of at most 1/102 from either piece.

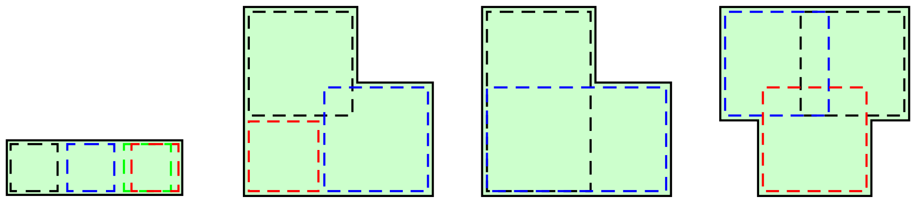


Figure 4: Cover numbers of several geometric shapes.

*Proof.* Let $m = \text{CoverNum}(C, S)$. By definition of CoverNum, there are $m$ $S$-pieces $X_1, ..., X_m$, possibly overlapping, that cover the cake $C$:

$$C = X_1 \cup X_2 \cup ... \cup X_m$$

Let $V$ be any value measure. A measure is additive, so:

$$V(X_1) + V(X_2) + ... + V(X_m) \geq V(C)$$

By the pigeonhole principle, there is at least one piece $X_i \in S$ with:

$$V(X_i) \geq \frac{V(C)}{m}$$

On the other hand, since $X_i$ is an $S$-piece and it is contained in $C$, its value is bounded by the supremum $V^S$:

$$V^S(C) \geq V(X_i)$$

Combining the above two inequalities yields:

$$V^S(C) \geq V(C)/m$$

Combining this into the definition $\text{Loss}(C, S) = \sup_V \frac{V(C)}{V^S(C)}$, yields:

$$\text{Loss}(C, S) \leq \sup_V \frac{V(C)}{V(C)/m} = \sup_V m = m$$

$\square$

By Definition 3.1, for every value measure $V$: $V^S(C) \geq \frac{V(C)}{\text{Loss}(C,S)}$. By Lemma 3.2, this implies $V^S(C) \geq \frac{V(C)}{\text{CoverNum}(C,S)}$. Thus, for example, in the 30×20 rectangle of Figure 3, $\text{CoverNum}(C, \mathit{Squares}) = 2$ so $\text{Loss}(C, \mathit{Squares}) \leq 2$ so $V^S(C) \geq V(C)/2$. This means that every agent, with any value measure, can get from $C$ a utility of at least half its total value.

*3.4. Knife functions*

Moving knives have been used to cut cakes ever since the seminal paper of Dubins and Spanier [42]. We generalize the concept of a moving knife to handle geometric shape constraints.

**Definition 3.3.** A function $K$, from the real interval $[0, 1]$ to Borel subsets of $\mathbb{R}^d$, is called a *knife function* if it has the following properties:

1. *Monotonicity*: for every $t' \geq t$: $K(t') \supseteq K(t)$;
2. *Lebesgue-continuity*: The Lebesgue measure of $K(t)$ is a continuous function of $t$.

If $K$ is defined over the interval $[0, 1]$ with $K(0) = A$ and $K(1) = B$, then we say that $K$ is *a knife function from $A$ to $B$*.

Given two bounded Borel subsets of $\mathbb{R}^d$, $A$ and $B$, does there always exist a knife function from $A$ to $B$? By definition, a necessary condition is that $A \subset B$. By the following lemma, this condition is also sufficient.

**Lemma 3.3.** *(Existence of Knife Function) Let $A$ and $B$ be two bounded Borel subsets of $\mathbb{R}^d$ with $A \subseteq B$. There exists a knife function from $A$ to $B$.*

*Proof (based on Fish [43]).* Pick a point $O \in B$. For every $r \geq 0$ let $D(r)$ be the open $d$-ball of radius $r$ around $O$. Since $B$ is bounded, there is a certain radius $r_{max}$ such that $B \subseteq D(r_{max})$. For every $t \in [0, 1]$, define $D^*(t) = D(t \cdot r_{max})$, so $D^*(t)$ is an open ball whose radius grows continuously from 0 to $r_{max}$. Define: $G(t) = [A \cup D^*(t)] \cap B$. Clearly, $G(0) = A$, $G(1) = B$ and $G$ is (weakly) monotonically increasing. The continuity of $Lebesgue(G(t))$ follows from the fact that $Lebesgue(D^*(t))$ is continuous and for every $\Delta t$: $Lebesgue(G(t + \Delta t)) - Lebesgue(G(\Delta t)) \leq Lebesgue(D^*(t + \Delta t)) - Lebesgue(D^*(\Delta t))$. □

*3.5. Geometric loss of knife functions*

Let $C$ be a cake and $K_C$ a knife function from $[0, 1]$ to Borel subsets of $C$ (we call such function "a knife function on $C$"). The *complement* of $K_C$ with respect to $C$, marked $\overline{K_C}$, is defined by:

$$\overline{K_C}(t) = C \setminus K_C(t)$$

By definition 3.3, the function $\overline{K_C}$ is monotonically decreasing and $Lebesgue(\overline{K_C}(t))$ is a continuous function of $t$.

When a knife function $K_C$ is "stopped" at a certain time $t \in [0, 1]$, it induces a partition of the cake $C$ to the part which was already covered by the knife, $K_C(t)$, and the part not covered, $\overline{K_C}(t)$. Based on this partition, the geometric loss of the knife can be defined:

**Definition 3.4.** Let $C$ be a cake, $K_C$ a knife function on $C$ and $S$ a family of pieces. Define the *geometric loss* of $K_C$ as:

$$\text{Loss}(K_C, S) = \sup_{t \in [0,1]} \{\text{Loss}(K_C(t), S) + \text{Loss}(\overline{K_C}(t), S)\}$$

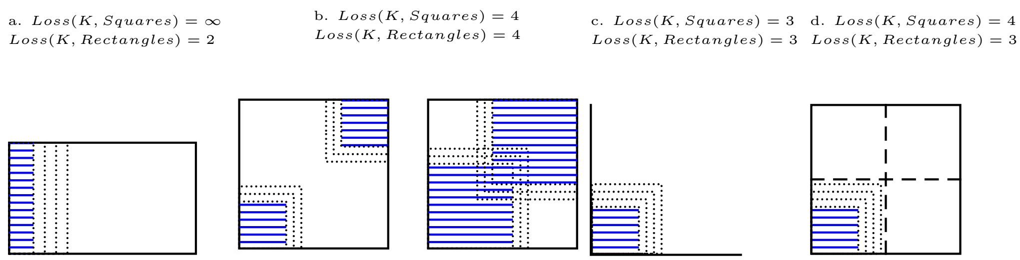


Figure 5: Several knife functions. The area filled with horizontal lines marks $K(t)$ in a certain intermediate time $t \in (0, 1)$. Dotted lines mark future knife locations.

Whenever a knife is stopped, the resulting partition has a geometric loss of at most $Loss(K_C, S)$. This means that such a knife is useful for fairly dividing a cake among agents who want $S$-pieces.

Recall that the smallest possible Loss of a single piece is 1 (which means "no loss"); hence the smallest possible loss of a knife function is 1+1=2. Some examples are illustrated in Figure 5, from left to right:

(a) Let $C = [0, L] \times [0, 1]$ and $K_C(t) = [0, L] \times [0, t]$. Both $K_C(t)$ and its complement are rectangles so their geometric loss relative to the family of rectangles is 1. Hence Loss($K_C$, *Rectangles*) $= 1 + 1 = 2$. In contrast, the geometric loss of these rectangles relative to the family of squares is unbounded, so: Loss($K_C$, *Squares*) $= \infty$.

(b) Let $C = [0, 1] \times [0, 1]$ and $K_C(t) = [0, t] \times [0, t] \cup [1-t, 1] \times [1-t, 1]$. For every $t$, $K_C(t)$ is a union of two squares and its complement is also a union of two squares. By the Cover Lemma, each such union has a geometric loss of 2 (relative to the family of squares). Hence, Loss($K_C$, *Squares*) $= 2+2 = 4$.

(c) Let $C$ be the top-right quarter-plane and $S$ the family of squares and quarter-planes (we consider a quarter-plane to be a square with infinite side-length). Define the following knife function: $K_C(t) = [0, x/(1-x)] \times [0, x/(1-x)]$. $K_C(t)$ is a square and its complement can be covered by two quarter-planes, so the geometric loss of $K_C$ is 1+2=3.

(d) Let $C = [0, 2] \times [0, 2]$ and $K_C(t) = [0, t] \times [0, t]$. Note that $K_C(0) = \emptyset$ and $K_C(1) = [0, 1] \times [0, 1] =$ the bottom-left quarter of $C$. For every $t$, $K_C(t)$ is a square and its complement is an L-shape, similar to the L-shapes in Figure 4, which can be covered by 3 squares. Hence, Loss($K_C$, *Squares*) $= 3 + 1 = 4$.

### *3.6. Generic Knife Procedure*

A knife function $K_C$ on a cake $C$ can be used to attain an envy-free division of $C$ between two agents. The idea is simple:

**Generic Knife Procedure:**

Each agent $i \in \{A, B\}$ selects a time $t_i \in [0, 1]$ such that:

$$V_i^S(K_C(t_i)) = V_i^S(\overline{K_C}(t_i))$$

Rename the agents, if needed, such that $t_A \leq t_B$.
Select any time $t^* \in [t_A, t_B]$.
Give $K_C(t^*)$ to agent A and $\overline{K_C}(t^*)$ to agent B.

This procedure obviously generates an envy-free division. Moreover, the geometric-loss of the partition is at most $\text{Loss}(K_C, S)$, so by the Chooser Lemma (3.1), the utility of each agent is at least $1/\text{Loss}(K_C, S)$.

The challenge is in the first step: we must be sure that each agent $i$ can, indeed, select a time $t_i$ such that the utilities on both sides of the knife are equal. This requires that both the utility covered by the knife $V_i^S(K_C(t))$ and the utility not covered by the knife $V_i^S(\overline{K_C}(t))$ change continuously as a function of $t$. Hence, we define:

**Definition 3.5.** Given a cake $C$ and a family $S$, a knife function $K_C$ is called *$S$-good* if for every absolutely-continuous value-measure $V$, both $V^S(K_C(t))$ and $V^S(\overline{K_C}(t))$ are continuous functions of $t$.

The knife functions in Figure 5 are all Square-good. This is easy to explain intuitively. The idea is that all squares in $K_C(t)$ grow continuously and all squares in $\overline{K_C}(t)$ shrink continuously; no square with a positive area is created abruptly in $K_C(t)$ and no square with a positive area is destroyed abruptly in $\overline{K_C}(t)$. Hence, $V^S$, which is the value of the most valuable square, also increases/decreases continuously. A formal proof that these functions are square-good requires some elaborate technical geometric definitions which we defer to Appendix A.

Armed with these $S$-good knives, we go back to cake-cutting.

### *3.7. Generic Knife Procedure Proof*

**Lemma 3.4.** *Let $C$ be a cake and $C_0, C_1$ pieces such that: $C_0 \subseteq C_1 \subseteq C$. Let $K_C$ be an $S$-good knife-function from $C_0$ to $C_1$. Assume that an agent has a value function $V$ such that:*

- $V^S(C_0) \leq V^S(C \setminus C_0)$
- $V^S(C_1) \geq V^S(C \setminus C_1)$

*Then there exists a time $t_i \in [0, 1]$ in which the utilities on both sides of the knife are equal:*

$$V^S(K_C(t_i)) = V^S(\overline{K_C(t_i)})$$

*Proof.* When $t = 0$:

$$V^S(K_C(t)) = V^S(C_0) \leq V^S(C \setminus C_0) = V^S(\overline{K_C(t)})$$

and when $t = 1$:

$$V^S(K_C(t)) = V^S(C_1) \geq V^S(C \setminus C_1) = V^S(\overline{K_C(t)})$$

Since $K_C$ is $S$-good, by Definition 3.5 both $V^S(K_C(t))$ and $V^S(\overline{K_C(t)})$ are continuous functions of $t$. Hence the lemma follows from the intermediate value theorem. □

**Lemma 3.5.** *(Knife Lemma) Let $C$ be a cake and $C_0, C_1$ pieces such that: $C_0 \subseteq C_1 \subseteq C$. Let $K_C$ be an $S$-good knife-function from $C_0$ to $C_1$. If there are two agents and for every agent $i$:*
- $V_i^S(C_0) \leq V_i^S(C \setminus C_0)$ *and*
- $V_i^S(C_1) \geq V_i^S(C \setminus C_1)$,

*then $C$ can be divided using the Generic Knife Procedure (see beginning of Subsection 3.6) and every agent playing by the rules is guaranteed an envy-free share with a utility of at least:*

$$\max\left(V_i^S(C_0),\, V_i^S(C \setminus C_1),\, \frac{1}{Loss(K_C,\, S)}\right)$$

*Proof.* Consider an agent $i$ who plays by the rules and declares a time $t_i$ for which $V_i^S(K_C(t_i)) = V_i^S(\overline{K_C(t_i)})$. Denote this equal utility by $U$. By declaring time $t_i$, the agent is guaranteed to receive, either a piece that contains $K_C(t_i)$ or a piece that contains $\overline{K_C(t_i)}$. In both cases the agent feels no envy and receives a utility of at least $U$. This utility is bounded from below in three ways:

- $U \geq V^S(C_0)$, because the piece $K_C(t_i)$ contains $C_0$.
- $U \geq V^S(C \setminus C_1)$, because the complement piece $\overline{K_C(t_i)}$ contains $C \setminus C_1$.
- $U \geq 1/\mathrm{Loss}(K_C,\, S)$ by the Chooser Lemma, because the loss of the partition at any time is bounded by $\mathrm{Loss}(K_C,\, S)$.

□

Note that the Generic Knife Procedure is discrete and finite: we don't need to actually move the knife until one of the agents shouts "stop"; we ask the agents in advance in what time they would like to "stop the knife".

The Chooser Lemma and the Knife Lemma are the main tools we use to construct division procedures.

## 4. Envy-Free Division For Two agents

### *4.1. Squares and rectangles*

Our first generic envy-free division procedure is based on a single knife function.

**Lemma 4.1.** (Single Knife Procedure). *Let $C$ be a cake, $S$ a family of pieces and $M \geq 2$ an integer. If there exists an $S$-good knife-function $K_C$ from $\emptyset$ to $C$ having*

$$Loss(K_C,\, S) \leq M,$$

*then*

$$Prop(C, S, 2) \geq PropEF(C, S, 2) \geq \frac{1}{M}.$$

*Proof.* The cake can be divided using the Generic Knife Procedure, taking $C_0 = \emptyset$ and $C_1 = C$. The assumptions of the Knife Lemma (3.5) hold trivially because $C_0 = C \setminus C_1 = \emptyset$. Hence each agent playing by the rules receives an envy-free piece with a utility of at least $1/M$. □

The knife function in Figure 5/b is Square-good and its Square-loss is 4. Applying Lemma 4.1 to that knife function yields our first sub-theorem:

**Theorem 1(a).** PropEF($Square$, $Squares$, 2) $\geq 1/4$

The generality of Lemma 4.1 allows us to get more results with no additional effort. For example:

- By the knife function of Figure 5/b: PropEF($Square, Square\,pairs$, 2) $\geq 1/2$. I.e., if each agent has to receive a union of two squares (as it is common when dividing land to settlers, e.g. one land-plot for building and another one for agriculture, etc.), then a proportional division is possible since the knife function in example (b) has a geometric loss of 2 relative to the family of square pairs.
- By Figure 5/c: PropEF($Quarter\,Plane$, $Generalized\,Squares$, 2) $\geq 1/3$.

All bounds presented above are tight in the strong sense described in the introduction, i.e., it is not possible to guarantee both agents a larger utility *even if envy is allowed.* This is obvious for the $\geq 1/2$ results, since a proportionality of $1/n$ is the best that can be guaranteed to $n$ agents. For the other results, the matching upper bound is proved in Appendix B.

### *4.2. Cubes and archipelagos*

In some cases it may be difficult to find a single knife function that covers the entire cake. This is so, for example, when the cakes are multi-dimensional cubes or unions of disjoint squares. To handle such cases, the following lemma suggests a generalized division procedure employing several knife functions.

**Lemma 4.2.** (Single Partition Procedure). *Let $C$ be a cake, $S$ a family of pieces and $M \geq 2$ an integer such that:*

*(a) $C$ has a partition with a geometric loss of at most $M$, i.e. $C = \sqcup_{j=1}^{m} C_j$ and:*

$$\sum_{j=1}^{m} Loss(C_j, S) \leq M$$

*(b) For every part $C_j$, there is an $S$-good knife function from $\emptyset$ to $C_j$ with a geometric loss of at most $M$, i.e.:*

$$\forall j : \exists K_{C_j} : \ Loss(K_{C_j}, S) \leq M$$

*Then:*

$$Prop(C, S, 2) \geq PropEF(C, S, 2) \geq \frac{1}{M}$$

*Proof.* $C$ can be divided using the following procedure.

(1) Each agent chooses the part $C_j$ that gives him maximum utility. If the choices are different, then each agent receives his chosen piece and the rest of $C$ is discarded. This allocation is obviously envy-free, and by the Chooser Lemma the utility of each agent is at least $1/M$.

(2) If both agents chose the same part $C_j$, then ask each agent to choose either $C_j$ or $\overline{C_j}$ (where $\overline{C_j} := C \setminus C_j$). If the choices are different then each agent receives his chosen piece; this allocation is again envy-free and by the Chooser Lemma the utility of each agent is at least $1/M$. If the choices are identical then there are two cases:

(3-a) Both agents chose $C_j$. By assumption b, there exists a knife function $K_{C_j}$ from $\emptyset$ to $C_j$ with a geometric loss of at most $M$. Apply the Generic Knife Procedure with that knife function. The requirements of the Knife Lemma (3.5) are met since for both agents, $V_i^S(\emptyset) \leq V^S(C \setminus \emptyset)$ (trivially) and $V^S(C_j) \geq V^S(C \setminus C_j)$ (both agents preferred $C_j$ over $\overline{C_j}$). Hence, the Knife Lemma guarantees each agent an envy-free share with a utility of at least $1/\text{Loss}(K_{C_j}, S) \geq 1/M$.

(3-b) Both agents chose $\overline{C_j}$. By Lemma 3.3, there exists a knife function from $\emptyset$ to $\overline{C_j}$. Apply the Generic Knife Procedure with any such knife function. The requirements of the Knife Lemma (3.5) are met since for both agents, $V^S(\emptyset) \leq V^S(C \setminus \emptyset)$ (trivially) and $V^S(\overline{C_j}) \geq V^S(C \setminus \overline{C_j})$ (both agents preferred $\overline{C_j}$ over $C_j$). Additionally, the fact that in step (2) both agents chose $C_j$ implies, by the Chooser Lemma, that $V^S(C_j) \geq 1/M$. The Knife Lemma guarantees each agent an envy-free share with a utility of at least $V^S(\overline{C_j}) \geq V^S(C_j) \geq 1/M$. □

Several applications of Lemma 4.2 are presented below.

(a) PropEF$(Square, Squares, 2) \geq 1/4$. **Proof**: A square cake can be partitioned to a 2-by-2 grid of squares. The loss of the partition relative to the family of squares is 4, satisfying requirement a. Each quarter has a knife function with a loss of 4 (see Figure 5/d), satisfying requirement b. The

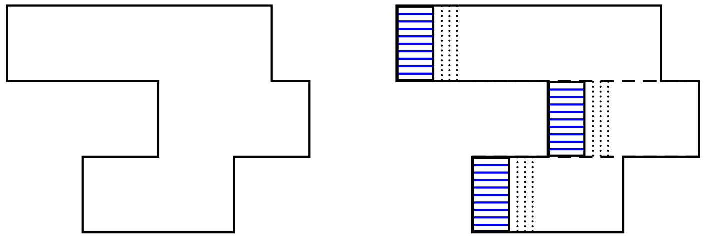

Figure 6: (a) A cake made of a union of 3 disjoint rectangles.
(b) Three knife functions, each having a geometric loss of at most 4, proving that PropEF($C$, *rectangles*, 2) $\geq 1/4$.

advantage of this result over the identical result presented in the previous subsection is that it can be easily extended to higher dimensions:

(b) **Multi-dimensional cakes**: Let $C$ be a $d$-dimensional cube with a side-length of 2. $C$ can be partitioned to $2^d$ cubes with a side-length of 1. For each smaller cube there is a knife function, analogous to Figure 5/d, with a geometric loss of $2^d$. Hence PropEF($d\,cube$, $d\,cubes$, 2) $\geq 1/2^d$.

(c) **Archipelagos**: Let $C$ be an archipelago which is a union of $m$ disjoint rectangular islands. Then PropEF($C$, *Rectangles*, 2) $\geq \frac{1}{m+1}$. **Proof**: The geometric loss of the partition of $C$ to $m$ rectangles is obviously $m < m+1$, satisfying requirement a. For each part $C_j$, define a knife function $K_{C_j}$ based on a line sweeping from one side of the rectangle to the other side, similar to Figure 5/a. $K_{C_j}(t)$ is always a rectangle. Its complement can be covered by $m$ rectangles: one rectangle to cover $C_j \setminus K_{C_j}(t)$ and additional $m-1$ rectangles to cover $C \setminus C_j$. Hence the geometric loss of every $K_{C_j}$ is $1+1+m-1 = m+1$, satisfying requirement b (see Figure 6).

(d) Let $C$ be an archipelago which is a union of $m$ disjoint square islands. Then PropEF($C$, *Squares*, 2) $\geq \frac{1}{m+3}$. The proof is the same as in (c), the only difference being that each of the knife functions on the $C_j$ is a union of two squares, similar to Figure 5/b.

All bounds proved above are tight. The tightness of (a) was already proved in Figure B.14/a and the tightness of (b) can be proved by an analogous $d$-dimensional cake. (c) is tight in the following sense: for every $m$ there is a cake $C$, which is a union of $m$ disjoint rectangles, having Prop($C$, *Rectangles*, 2) $\leq \frac{1}{m+1}$. See Appendix B. (d) is tight in a similar sense by a similar proof.

### *4.3. Fat rectangles*

More types of cakes can be handled by adding partition steps.

**Lemma 4.3.** (Multiple Partition Procedure). *Let $C$ be a cake, $S$ a family of pieces and $M \geq 2$ an integer such that:*

*(a) $C$ has a partition with a geometric loss of at most $M$, i.e. $C =$*

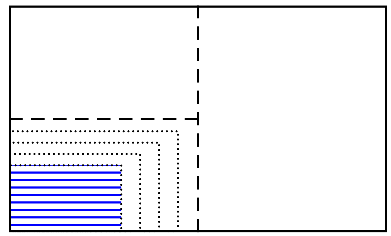

Figure 7: A knife function with a geometric loss of 3, proving that PropEF($C$, $2\,fat\,rectangles$, 2) $\geq 1/3$.

$\sqcup_{j=1}^{m} C_j$ *and:*

$$\sum_{j=1}^{m} Loss(C_j, S) \leq M$$

*(b) Every part $C_k$ can be further partitioned such that, if $C_k$ is replaced with its partition, then the geometric loss of the resulting partition of $C$ is at most $M$, i.e. for every $k$, $C_k = \sqcup_{j'=1}^{m_k} C_k^{j'}$ and:*

$$\sum_{j \neq k} Loss(C_j, S) + \sum_{j'=1}^{m_k} Loss(C_k^{j'}, S) \leq M$$

*(c) In the sub-partition of every $C_k$, for every part $C_k^{j'}$ there is a knife function with a geometric loss of at most $M$:*

$$\forall k, j' : \exists K_{C_k^{j'}} : \; Loss(K_{C_k^{j'}}, S) \leq M$$

*Then:*

$$Prop(C, S, 2) \geq PropEF(C, S, 2) \geq \frac{1}{M}$$

The proof is a straightforward refinement of the procedure used to prove Lemma 4.2.

As a corollary of Lemma 4.3, we get the second part of our Theorem 1:

**Theorem 1(b).** *For every $R \geq 2$:*

$$PropEF(R\,fat\,rectangle,\; R\,fat\,rectangles,\; 2) \geq 1/3$$

*Proof.* The proof relies on the following geometric fact: for every $R \geq 2$, an $R$-fat rectangle can be bisected to two $R$-fat rectangles using a straight line through the center of its longer sides.

Apply Lemma 4.3 in the following way. Let $C$ be an $R$-fat rectangle. Partition $C$ in the middle of its longer side. The two halves are $R$-fat so the geometric loss of the partition is $1 + 1 < 3$. Each half can be further

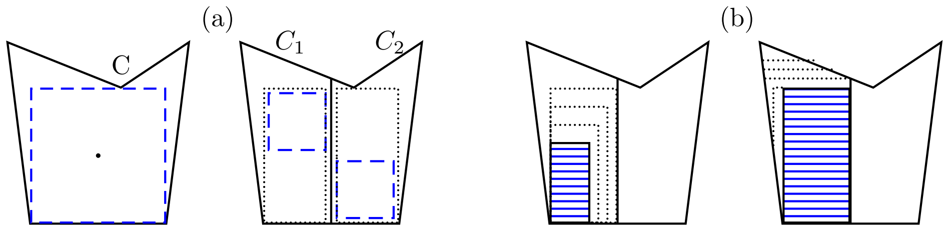


Figure 8: Dividing a general $R$-fat cake to two people.
(a) The $R$-fat cake $C$ (left) and its two pieces $C_1$ and $C_2$ (right). Dashed lines indicate the enclosed square/s. Dotted lines indicate $B_1$ and $B_2$.
(b) The knife function on $C_1$ in $t \in [0, \frac{1}{2}]$ (left) and in $t \in [\frac{1}{2}, 1]$ (right).

partitioned along its longer side to two rectangles, which are also R-fat (each of these is exactly one quarter of $C$). When a part is replaced by its sub-partition, the geometric loss of the resulting partition is thus $2 + 1 = 3$. For each quarter-rectangle, there is a knife function (growing from the corner towards the center, analogous to Figure 5/d) with a geometric loss of 3. See Figure 7. □

The bound of 1/3 is tight; see Appendix B, Figure B.15.

Lemma 4.3 can be further refined by adding more sub-partition steps. For example, by adding a third sub-partition step we can prove that if $C$ is an archipelago of $m$ disjoint R-fat rectangles (with $R \geq 2$) then:

$$\text{PropEF}(C,\ R\text{-fat rectangles},\ 2) \geq \frac{1}{m+2}$$

and this bound is tight. The proof is a analogous to examples (c) and (d) after Lemma 4.2.

Note that the impossibility results are valid when the pieces are $R$-fat rectangles for every finite $R$, while the impossibility result of the square in Figure B.14/b is valid for every $R < 2$. This implies that 2-fat rectangles are a good practical compromise between fatness and fairness: if we require fatter pieces ($R < 2$) then the proportionality drops from 1/3 to 1/4, while if we allow thinner pieces ($R > 2$) the proportionality remains 1/3 for all $R < \infty$.

### *4.4. Fat cakes of arbitrary shape*

Our most general result involves cakes which are arbitrary Borel sets. The result is valid for cakes of any dimensionality, but for simplicity the proofs are presented for subsets of $\mathbb{R}^2$.

**Theorem 1(c).** *For every $R \geq 1$, If $C$ is $R$-fat and $S$ is the family of $2R$-fat pieces then:*

$$PropEF(C, S, 2) = Prop(C, S, 2) = \frac{1}{2}$$

*Proof.* The proof uses Lemma 4.2 (the Single Partition Procedure). We show a partition of $C$ to two pieces and a knife-function on each piece.

Scale, rotate and translate the cake $C$ such that the largest square contained in $C$ is $B^- = [-1,1] \times [-1,1]$. By definition of fatness (see Subsection 1.1), $C$ is now contained in a square $B^+$ of side-length at most $2R$.

Using a straight line through the origin, divide the square $B^-$ to two 2-by-1 rectangles $B_1$ and $B_2$. Continue this line to a curve that divides $C$ to two parts, $C_1$ and $C_2$ (see Figure 8/a). Every $C_i$ contains $B_i$ which contains a square with a side-length of 1. Every $C_i$ is of course still contained in $B^+$ which is square of size $2R$. Hence every $C_i$ is $2R$-fat. Hence the geometric loss of the partition $C = C_1 \sqcup C_2$, relative to the family or $2R$-fat shapes, is 2, satisfying requirement (a) of Lemma 4.2.

For every $i \in \{1, 2\}$, define the following knife function $K_{C_i}$ on $C_i$:

- For $t \in [0, \frac{1}{2}]$, $K_{C_i}(t) = (B_i)^{2t}$, i.e., the piece $B_i$ dilated by a factor of $2t$. Hence $K_{C_i}(0) = \emptyset$ and $K_{C_i}(\frac{1}{2}) = B_i$.

- For $t \in [\frac{1}{2}, 1]$, $K_{C_i}(t)$ is any knife function from $B_i$ to $C_i$.

An illustration is provided in Figure 8/b. $K_{C_i}(t)$ is always $2R$-fat, since in $[0, \frac{1}{2}]$ it is a scaled-down version of the 2-by-1 rectangle $B_i$ and in $[\frac{1}{2}, 1]$ it contains $B_i$ and is contained in the square $B^+$. $C \setminus K_{C_i}(t)$ is also $2R$-fat, since it contains $B_{3-i}$ and is contained in $B^+$. Hence, by Appendix A, $K_{C_i}$ is an $S$-good knife function. The geometric loss of $K_{C_i}$, relative to the family of $2R$-fat shapes, is 2, satisfying requirement (b) of Lemma 4.2.

Both requirements of Lemma 4.2 are satisfied, and its conclusion is exactly the claimed theorem. □

Theorem 1(c) implies that we can satisfy the two main fairness requirements: proportionality and envy-freeness, while keeping the allocated pieces sufficiently fat. The fatness guarantee means that each allotted piece: (a) contains a sufficiently *large* square, (b) is contained in a sufficiently *small* square. In the context of land division, these guarantees can be interpreted as follows: (a) Each land-plot has sufficient room for building a large house in a convenient shape (square); (b) The parts of the land that are valuable to the agent are close together, since they are bounded in a sufficiently small square.

Finally we note that a different technique leads to a version of Theorem 1(c) which guarantee that the pieces are not only $2R$-fat but also *convex* (if the original cake is convex); hence an agent can walk in a straight line from his square house to his valuable spots without having to enter or circumvent the neighbor's fields. See Appendix C for details.

### *4.5. Between envy-freeness end proportionality*

For all cakes $C$ and families of usable pieces $S$ studied in this section, we constructively proved the existence of an envy-free partition with a proportionality of at least $\text{Prop}(C, S, 2)$. Using our notation, this fact can be expressed succinctly as:

$$\mathrm{PropEF}(C,S,2) = \mathrm{Prop}(C,S,2)$$

In other words, in these cases, envy-freeness is compatible with the best possible partial-proportionality.

It is an open question whether this equality holds for *every* combination of cakes $C$ and families $S$.

What *can* we say about the relation between proportionality and envy-freeness for arbitrary $C$ and $S$? In addition to the trivial upper bound $\mathrm{PropEF}(C,S,2) \leq \mathrm{Prop}(C,S,2)$, we have the following lower bound:

**Lemma 4.4.** *For every cake $C$ and family $S$:*

$$\mathit{PropEF}(C,S,2) \geq \mathit{Prop}(C,S,2) \cdot \inf_{s\in S} \mathit{PropEF}(s,S,2)$$

*Proof.* Let $p = \mathrm{Prop}(C,S,2)$ and $e = \inf_{s\in S} \mathrm{PropEF}(s,S,2)$. The following meta-procedure provides an envy-free partition of $C$ in which the utility of each agent is at least $p \cdot e$.

Divide $C$ between the two agents such that each agent $i$ receives a piece $X_i$ with $V_i^S(X_i) \geq p$. Ask each agent whether he envies the other agent and proceed accordingly:

(a) If no agent envies the other agent, then the partition is envy-free and the utility of each agent is at least $p \geq p \cdot e$.

(b) If both agents envy each other, then just switch the pieces. The resulting partition is envy-free and the utility of each agent is more than $p \geq p \cdot e$.

(c) If only agent $i$ envies agent $1-i$, then the piece $X_{1-i}$ has a value of at least $p$ to both agents. Divide $X_{1-i}$ between the two agents in an envy-free manner. The utility of each agent is at least $p \cdot e$. □

So by previous results we have the following partial-compatibility results for *every* cake $C$:

- $\mathrm{Prop}(C,\ \mathit{Squares},\ 2) \geq \mathrm{PropEF}(C,\ \mathit{Squares},\ 2) \geq \frac{1}{4}\mathrm{Prop}(C,\ \mathit{Squares},\ 2)$
- $\mathrm{Prop}(C,\ R\,\mathit{fat\,rects},\ 2) \geq \mathrm{PropEF}(C,\ R\,\mathit{fat\,rects},\ 2) \geq \frac{1}{3}\mathrm{Prop}(C,\ R\,\mathit{fat\,rects},\ 2)$ (for $R \geq 2$)
- $\mathrm{Prop}(C,\ \mathit{Rectangles},\ 2) \geq \mathrm{PropEF}(C,\ \mathit{Rectangles},\ 2) \geq \frac{1}{2}\mathrm{Prop}(C,\ \mathit{Rectangles},\ 2)$

## 5. Envy-Free Division For $n$ agents

### *5.1. The Simmons–Su procedure*

Our procedure for $n$ agents is a generalization of a procedure for envy-free division of a 1-dimensional cake. The procedure was developed by Simmons and described by Su [10]. We briefly describe the 1-dimensional procedure below (see Figure 9). The cake $C$ is the 1-dimensional interval $[0,1]$ and $S$

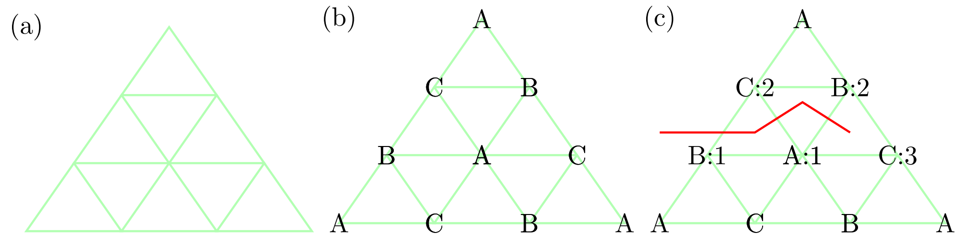


Figure 9: An illustration of the Simmons-Su procedure for $n = 3$ agents.
(a) A triangulation of the simplex of partitions.
(b) Assigning an agent to each vertex of the triangulation.
(c) Touring the triangulation until a fully-labeled simplex is found.

is the family of intervals. A partition of $C$ to $n$ intervals can be described by a vector of length $n$ whose elements are the lengths of the intervals. The sum of all lengths in a partition is 1, so the set of all partitions is an $(n-1)$-dimensional simplex in $\mathbb{R}^n$. Triangulate the simplex of partitions to a collection of $(n-1)$-dimensional sub-simplexes. Assign each vertex of the triangulation to one of the $n$ agents, such that in each sub-simplex, all $n$ agents are represented. Start at an arbitrary sub-simplex. For each vertex of the sub-simplex (which represents a partition of $C$), ask the owner of that vertex: "if $C$ is partitioned according to this vertex, which piece would you prefer?". The answer is an integer between 1 and $n$; label that vertex with that integer. If each vertex of the sub-simplex has a different label, we say it is "fully labeled". If the current simplex is not fully-labeled, move to another simplex and repeat. It is possible to prove, based on Sperner's lemma, that this "tour" eventually ends in a fully-labeled sub-simplex. Triangulate this sub-simplex to even smaller sub-simplices and repeat the entire procedure. This process creates an infinite sequence of sub-simplices. It is possible to prove that, under certain weak conditions, this sequence converges to a single point. This point represents a partition in which each of the $n$ agents prefers a different piece. By definition, this partition is envy-free.

Note that the above procedure is infinite - the envy-free partition is found only at the limit of an infinite sequence. In fact, Stromquist [44] proved that when $n \geq 3$, an envy-free partition to $n$ agents with connected pieces cannot be found by a finite procedure. Therefore, Simmons' infinite procedure is the best that can be hoped for.

The above procedure can be used for finding an approximately-envy-free division. For example, suppose that an interval is divided among several agents and they all agree that a 1 centimeter movement of the border between their plots is irrelevant. Then the simplex of partitions can be divided to sub-simplices of side-length 1 cm. A fully-labeled simplex can be found in finite time. All points in that simplex correspond to a division which is approximately-envy-free for all practical purposes.

*5.2. Knife tuples*

To generalize Simmons's procedure, we need not a single knife function but an $(n-1)$-tuple of knife functions, $\{k_i\}_{i=1}^{n-1}$, such that each function runs on the remainder of the previous functions. I.e:

- $k_1$ is a knife-function from $\emptyset$ to $C$.
- For every time $t_1 \in [0, 1]$, $k_2(t_1)$ is a knife-function from $\emptyset$ to $C \setminus k_1(t_1)$.
- For every pair of times $t_1, t_2 \in [0, 1]$, $k_3(t_1, t_2)$ is a knife-function from $\emptyset$ to $C \setminus k_1(t_1) \setminus k_2(t_2)$. And so on.

An example with $n - 1 = 2$ knife functions is shown in Figure 10. There, $k_1$ is a growing pair-of-squares (like the one shown in Figure 5/b). For every $t_1$, $k_2(t_1)$ is a growing union-of-four-squares. It starts at an empty set and grows until it covers all of $C \setminus k_1(t_1)$.

The *geometric loss* of such a tuple is defined as the maximum geometric loss of the resulting partitions, over all possible combinations of $t_1, ..., t_{n-1} \in [0, 1]$.

**Lemma 5.1.** *Let $C$ be a cake and $S$ a family of pieces. If there is an $(n-1)$-tuple of $S$-good knife-functions on $C$ with a geometric loss of at most $M$, then:*

$$Prop(C, S, n) \geq PropEF(C, S, n) \geq \frac{1}{M}$$

*Proof.* Consider the vector $X = [x_1, ..., x_n]$, where $\forall i : x_i \in [0, 1]$ and $\sum_{i=1}^{n} x_i = 1$. Each such vector corresponds to a partition of the 1-dimensional interval $[0, 1]$ to $n$ connected intervals: each $x_i$ is the length of interval $i$. The collection of all such vectors is an $(n - 1)$-dimensional simplex in $\mathbb{R}^n$.

The vector $X$ also corresponds to a partition of a cake $C$ to $n$ pieces using $(n - 1)$ knife functions: $[k_1(t_1), k_2(t_2), ..., k_{n-1}(t_{n-1}), C \setminus \cup k_{i=1}^{n-1}(t_i)]$, such that:

- $t_1 = x_1$;
- If $x_2 \geq 0$ (which implies $x_1 < 1$), then $t_2 = \frac{x_2}{1-x_1}$; otherwise $t_2 = 0$.
- If $x_3 \geq 0$ (which implies $x_1 + x_2 < 1$), then $t_3 = \frac{x_3}{1-x_1-x_2}$; otherwise $t_3 = 0$. And so on.

Using Simmons's procedure [10], it is possible to find a sequence of divisions that converges to an envy-free division, based on the $n - 1$ knife functions: triangulate the simplex, traverse the vertexes of the triangulation, let the agents select which piece they prefer in each vertex and proceed until a sub-simplex is found where the answers are all different. If the geometric loss of this tuple of knife functions is $M$, then by the Chooser Lemma the proportionality of the resulting division is at least $1/M$. [6] □

[6]When $n = 3$, the three-knives procedure of Stromquist [9] can be used instead of Simmons' procedure.

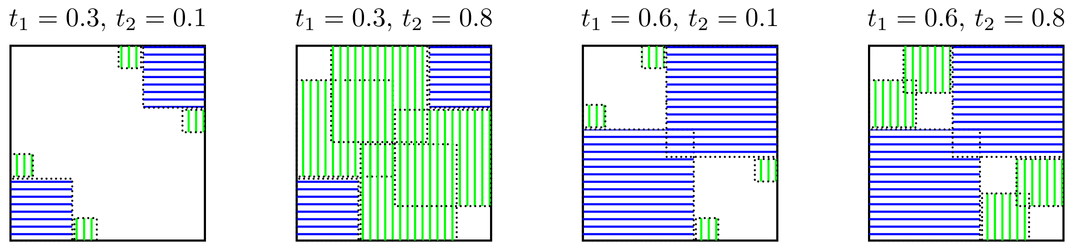


Figure 10: A pair of knife functions. $k_1(t_1)$ is filled with horizontal blue lines, $k_2(t_1)(t_2)$ is filled with vertical green lines and the remainder ($C \setminus k_1(t_1) \setminus k_2(t_1)(t_2)$) is white.

The knife functions $k_i$ don't have to be defined sequentially from $i = 1$ to $i = n - 1$; they can also be defined similarly to a binary-tree. For example, if $n = 4$ then we can define the following triple of functions:

- $k_2$ is a knife-function from $\emptyset$ to $C$.
- For every time $t_2 \in [0, 1]$, $k_1(t_2)$ is a knife-function from $\emptyset$ to $k_2(t_2)$.
- For every time $t_2 \in [0, 1]$, $k_3(t_2)$ is a knife-function from $\emptyset$ to $C \setminus k_2(t_2)$.

We now apply Lemma 5.1 in its two variants to prove our Theorem 2.

*5.3. Square cake and square pieces*

**Theorem 2(a).** *For every $n \geq 1$:*

$$\text{PropEF}(Square, Squares, n) \geq \frac{1}{2^{2\lceil log_2 n \rceil}} = \frac{1}{\Theta(n^2)}$$

*Proof.* We first describe the case $n = 3$. The knife-function $k_1$ is the union of two corner-squares growing towards the center, as in Figure 5/b. For every time $t_1$, $k_2(t_1)$ is the union of *four* corner-squares as illustrated in Figure 10. Since the squares meet only at their corners, no square is created or destroyed abruptly, and the knife-function is square-good. By counting the number of covering squares in different time combinations, it is possible to show that the geometric loss of the remainder ($C \setminus k_1(t_1) \setminus k_2(t_2)$) is at most 4. This is also obviously true for $k_2(t_2)$. For $k_1(t_1)$ the geometric loss is obviously 2. Hence the total geometric loss of the knife-tuple is at most 10 and $\text{PropEF}(Square, squares, n = 3) \geq 1/10$.[7]

Next, consider the case $n = 4$. Use the binary-tree variant of Lemma 5.1. Define three knife functions as follows:

[7] In [45] we proved the same result using a variation on Stromquist's 3-knives procedure [9].

- $k_2$ is a union of two corner-squares growing towards the center, as in Figure 5/b.

- For every $t_2$, $k_1(t_2)$ is a union of four corner-squares growing inside $k_2(t_2)$, as in Figure 10.

- For every $t_2$, $k_3(t_2)$ is a union of four corner-squares growing inside $C \setminus k_2(t_2)$, as in Figure 10.

The function $k_2$ induces a partition of $C$ that can be covered by 4 squares. The function $k_1$ induces, on two of these squares, partitions that can be covered by 4 squares. Similarly, the function $k_3$ induces, on the other two squares, partitions that can be covered by 4 squares. All in all, the partition of $(k_1, k_2, k_3)$ can be covered by $4 \cdot 4 = 16$ squares. Hence, the geometric loss of the knife-tuple $(k_1, k_2, k_3)$ is 16.

For every $n$ which is a power of 2, an $(n-1)$-tuple of knife functions can be constructed recursively in a similar manner:

*Claim.* For every $n$ which is a power of 2, there exists an $(n-1)$-tuple of knife functions with geometric loss (relative to the squares) of at most $n^2$.

*Proof.* By induction on $\log_2(n)$.

The base is $n = 2$. The "1-tuple" is the knife-function in Figure 5/b.

Assume the claim is true for $n$. So there is an $(n-1)$-tuple of knife-functions on a square, with geometric loss $n^2$.

We prove that the claim is true for $2n$. We construct a $(2n-1)$-tuple of knife-functions as follows:

- The function $k_n$ is the knife-function in Figure 5/b - a union of two squares growing from two opposite corners.

- The functions $k_1, \ldots, k_{n-1}$ are the $(n-1)$-tuple of knife-functions whose existence is assumed by induction. They run on each of the two squares of $k_n(t_n)$. This means that each of these two squares is now partitioned such that the partition can be covered by $n^2$ squares.

- The functions $k_{n+1}, \ldots, k_{2n-1}$ are the same $(n-1)$-tuple of knife-functions, but they run on each of the two squares of $C \setminus k_n(t_n)$. Again, each of these two squares is now partitioned such that the partition can be covered by $n^2$ squares.

All in all, the partition of this $(2n-1)$-tuple of knife functions can be covered by $4n^2$ squares. □

When $n$ is not a power of two, it is possible to round it to the next power of two - $2^{\lceil log_2 n \rceil}$. The geometric loss is at most $2^{2\lceil log_2 n \rceil}$, which is between $n^2$ and $4n^2$. □

**Theorem 2(b).** *If $C$ is an $R$-fat rectangle and $S$ the family of $R$-fat rectangles then:*

$$PropEF(C,S,n) \geq \frac{1}{2^{2\lceil log_2 n\rceil}} = \frac{1}{\Theta(n^2)}$$

*Proof.* Scale the coordinate system such that $C$ becomes a square. Use Theorem 2(a) and get a division with square pieces. Scale the coordinate system back. Now the pieces are $R$-fat rectangles. □

The $1/O(n^2)$ is not tight. The tightest impossibility result currently known [17] is $\mathrm{Prop}(Square, squares, n) \leq 1/(2n)$. Moreover, there is a procedure for non-envy-free division that proves $\mathrm{Prop}(Square, squares, n) \geq 1/(4n-4)$. We do not know if it is possible to attain an envy-free division with a proportionality of $1/O(n)$.

In the following subsection we show that it is possible to attain an envy-free *and proportional* division for every $n$, by making a certain compromise on the family of usable pieces.

*5.4. Fat cakes of arbitrary shape*

**Theorem 2(c).** *Let $C$ be a $d$-dimensional $R$-fat cake and $n \geq 2$ an integer. Let $S$ be the family of $mR$-fat pieces, where $m$ be the smallest integer such that $n \leq m^d$ (i.e. $m = \lceil n^{1/d}\rceil$). Then:*

$$PropEF(C,S,n) = \frac{1}{n}$$

*Proof.* For simplicity we present the proof for $d = 2$. The adaptations required for arbitrary $d$ are straightforward.

Let $C$ be an $R$-fat 2-dimensional cake. By definition of fatness it contains a square $B^-$ of side-length $x$ and it is contained in a parallel square $B^+$ of side-length $R\cdot x$, for some $x > 0$. Partition the square $B^-$ to a grid of $m\times m$ smaller squares, $B_1, ..., B_{m^2}$, each of side-length $\frac{x}{m}$. For every $i$, denote by $\overline{B_i}$ the union of all $m^2-1$ squares different than $B_i$ (i.e. $B^- \setminus B_i$). Denote by $\overline{B^-}$ the cake outside the enclosed square (i.e. $C \setminus B^-$).

Define the following knife function $k_1$ from $\emptyset$ to $C$ (see Figure 11):

- For $t \in [0, \frac{1}{3}]$: $k_1(t) = (B_1)^{3t}$, i.e., the square $B_1$ dilated by a factor of $3t$. Hence $k_1(0) = \emptyset$ and $k_1(\frac{1}{3}) = B_1$.

- For $t \in [\frac{1}{3}, \frac{2}{3}]$: $k_1(t)$ is any knife function from $B_1$ to $B_1 \cup \overline{B^-}$. Note that at time $\frac{2}{3}$, the cake not yet covered by the knife, $C \setminus k_1(\frac{2}{3})$, exactly equals $\overline{B_1}$.

- For $t \in [\frac{2}{3}, 1]$: $k_1(t)$ is $C \setminus [(\overline{B_1})^{3(1-t)}]$, i.e., the cake not yet covered by the knife is $\overline{B_1}$ dilated by a factor proportional to the remaining time. Hence $k_1(1) = C$.

$k_1(t)$ is always $mR$-fat because:

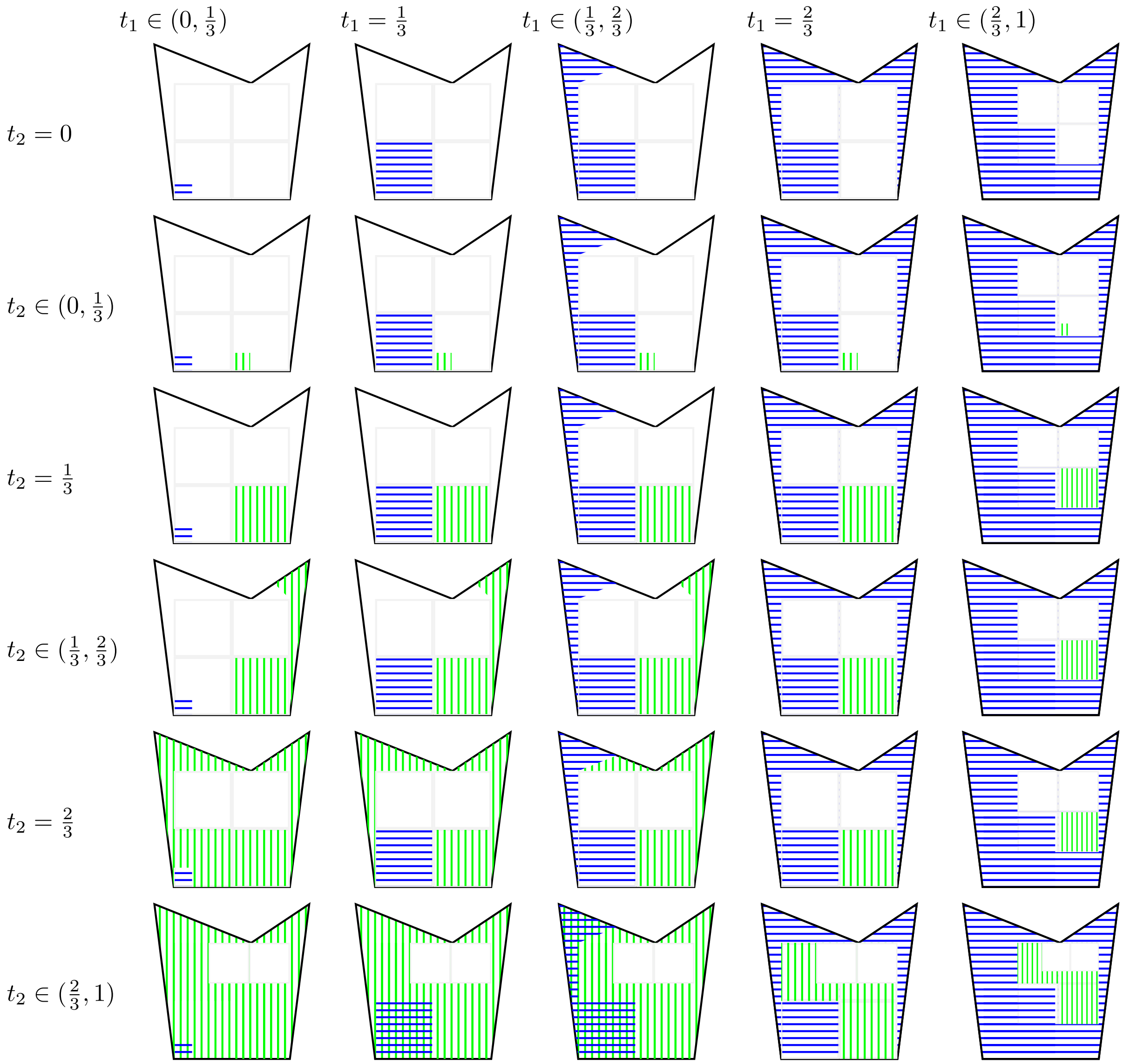


Figure 11: Dividing a general $R$-fat cake to $n = 3$ people. $k_1(t_1)$ is filled with horizontal lines, $k_2(t_2)$ is filled with vertical lines and the remainder $(C \setminus k_1(t) \setminus k_2(t))$ is white. Note that each of these three pieces is $2R$-fat, where $R$ is the fatness of the original cake.

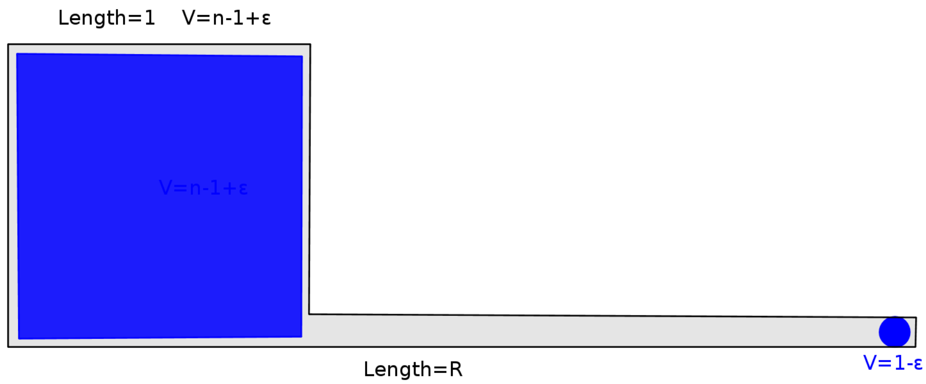


Figure 12: A fat cake in which every proportional division must use slim pieces. See Lemma 5.2.

- In $[0, \frac{1}{3}]$, $k_1(t)$ it is a square, which is 1-fat;
- In $[\frac{1}{3}, 1]$, $k_1(t)$ contains the square $B_1$ and is contained in the larger square $B^+$.

For every $t_1 \in [0, 1]$, we have to define a second knife function $k_2$ from $\emptyset$ to $\overline{k_1(t_1)}$. This can be done analogously to $k_1$ but using the square $B_2$. This is possible because:

- In $[0, \frac{2}{3}]$, $\overline{k_1(t_1)}$ contains the square $B_2$ itself;
- In $(\frac{2}{3}, 1]$, $\overline{k_1(t_1)}$ contains a dilated version of $B_2$ which is contained in a dilated version of the larger square $B^+$.

To define an $(n-1)$-tuple of knife functions, select arbitrary $n$ pieces out of the $m^2$ pieces and proceed in the same way. $\square$

Figure 11 shows an example of the construction for $n = 3$ agents. Here $m = \lceil\sqrt{3}\rceil = 2$ so each agent receives an envy-free $2R$-fat land-plot with a utility of at least 1/3.

Theorem 2(c) implies that we can guarantee proportionality by compromising on the fatness of the pieces - allowing the pieces to be thinner than the cake by a factor of $\lceil n^{1/d}\rceil$. This factor is asymptotically optimal:

**Lemma 5.2.** *For every $R \geq 1$, there is an $(R+1)$-fat cake $C$ for which, for every $m' \leq (n-1)^{1/d}$:*

$$Prop(C,\, m'R\,fat\,shapes,\, n) < \frac{1}{n}$$

*Proof.* For simplicity we present the proof for $d = 2$. Let $C$ be an $(R+1)$-fat cake in the following shape: a $1 \times 1$ square at the left and an $R \times \delta$ rectangle at the right , where $\delta > 0$ is an arbitrarily small constant (in particular, $\delta < \sqrt{\frac{1}{n-1+\epsilon}}$). Assume that $C$ is a desert with the following water sources: (1) The left square, containing $n - 1 + \epsilon$ water units, where $\epsilon > 0$ is an arbitrarily small constant; (2) A small circle at the end of the right rectangle, containing $1 - \epsilon$ water units (see Figure 12).

Assume that $C$ has to be divided among $n$ agents whose value functions are proportional to the amount of water. To get a proportional division, each agent must receive exactly 1 unit of water. This means that at least one agent, say Alice, has to receive some of the right pool and some of the left pool. Alice can take from the left pool at most 1 unit of water. This means that the largest square contained in Alice's allotment has a side-length of at most $\sqrt{\frac{1}{n-1+\epsilon}}$. The smallest square containing Alice's allotment has a side-length of at least $R$. Hence, Alice's piece is not $m'R$-fat for every $m' \leq \sqrt{n-1}$. $\square$

## 6. Conclusion and Future Work

We presented the problem of fairly dividing a cake to agents whose utility functions depend on geometric shape. Our main contributions are several generic, symmetric, anonymous division procedures for envy-free division. For two agents, our procedures have the best possible partial-proportionality guarantees in various geometric scenarios. For $n$ agents, our procedures guarantee a positive partial proportionality.

The tools developed in this paper are generic and can work for cakes and pieces of other geometric shapes. In fact, our tools reduce the envy-free division problem to a geometric problem - the problem of finding appropriate knife functions.

One way to generalize our model is to consider a utility function which takes into account both the value contained in the best usable piece and the total value of the piece, e.g.: $U(P) = WV^S(P) + (1 - W)V(P)$, where $W$ is an agent-dependent constant.

Other topics not covered in the current paper are:

- Absolute size constraints on the usable pieces instead of the relative fatness constraints studied here, e.g. let $S$ be the family of all rectangles with length and width of at least 10 meters.
- Personal geometric preferences - letting each agent $i$ specify a different family $S_i$ of usable pieces.

## A. Geometric conditions for $S$-good knife functions

Recall Definition 3.5:

> Given a cake $C$ and a family $S$, a knife function $K_C$ is called $S$-*good* if for every absolutely-continuous value-measure $V$, both $V^S(K_C(t))$ and $V^S(\overline{K_C}(t))$ are continuous functions of $t$.

This section presents two different geometric properties of a knife function $K_C$, each of which guarantees that a knife function is $S$-good.

### *A.1. S-smoothness*

The first property is simple: both the knife function and its complement should always return $S$-pieces.

**Lemma A.1.** *Let $V$ be a continuous measure and $K$ a knife function on a given cake $C$. If for all $t$: $K(t) \in S$ and $\overline{K}(t) \in S$, then the real functions $V^S \circ K$ and $V^S \circ \overline{K}$ are continuous functions of $t$.*

*Proof.* By definition of a knife function, $Lebesgue(K(t))$ is a continuous function of $t$. Since the measure $V$ is absolutely continuous with respect to Lebesgue measure, $V(K(t))$ is also a continuous function of $t$. The assumption $K(t) \in S$ implies that $\forall t \in [0,1] : V^S(K(t)) = V(K(t))$, so $V^S(K(t))$ is also a continuous function of $t$. An analogous proof applies to $V^S(\overline{K}(t))$. $\square$

If a knife function satisfies the conditions of Lemma A.1, we say that it is *S-smooth*. The knife function in Figure 5/a is Rectangle-smooth but not Square-smooth. The other knife functions in that figure are neither Rectangle-smooth nor Square-smooth (e.g in c, $K(t)$ is a square but $\overline{K}(t)$ is not).

### *A.2. S-continuity*

The second property is more involved. The knife function may return pieces that are not from $S$. However, it must change in a way that $S$-pieces are not created or destroyed abruptly, but rather grow or shrink in a continuous manner.

**Definition A.1.** A piece $s$ is called a *$\epsilon$-predecessor* of a piece $s'$ if $s \subseteq s'$ and $Area(s' \setminus s) < \epsilon$.

**Definition A.2.** Let $S$ be a family of pieces. A knife function $K(t)$ is called *S-continuous* if for every $\epsilon > 0$ there exists $\delta > 0$ such that, for all $t$ and $t'$ having $|t' - t| < \delta$:

(a) Every $S$-piece $s' \subseteq K(t')$ has an $\epsilon$-predecessor $S$-piece $s \subseteq K(t)$.

(b) Every $S$-piece $s' \subseteq \overline{K}(t')$ has an $\epsilon$-predecessor $S$-piece $s \subseteq \overline{K}(t)$.

**Lemma A.2.** *Let $V$ be a continuous measure and $K$ a knife function. If $K(t)$ is $S$-continuous, then the real functions $V^S \circ K$ and $V^S \circ \overline{K}$ are uniformly-continuous functions of $t$.*

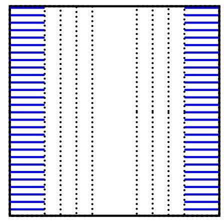

Figure A.13: Knife functions that are not $S$-continuous.

*Proof.* Mark $v := V^S \circ K$. Given $\epsilon' > 0$, we show the existence of $\delta > 0$ such that, for every $t, t'$, if $|t' - t| < \delta$ then $|v(t') - v(t)| < \epsilon'$.

Given $\epsilon'$, by the continuity of $V$, there is an $\epsilon > 0$ such that every piece $s$ having $Area(s) < \epsilon$ has $V(s) < \epsilon'$.

Given that $\epsilon$, by the $S$-continuity of $K$ there is a $\delta > 0$ such that, if $|t' - t| < \delta$, then every $S$-piece $s' \subseteq K(t')$ has an $\epsilon$-predecessor $S$-piece $s \subseteq K(t)$. This means that $s \subseteq s'$ and $Area(s' \setminus s) < \epsilon$, hence $V(s' \setminus s) < \epsilon'$, hence $V(s) > V(s') - \epsilon$.

Since $v(t) = V^S(K(t))$, by definition $v(t) = \sup_{s \in S, s \subseteq K(t)} V(s)$. Similarly, $v(t') = \sup_{s' \in S, s' \subseteq K(t')} V(s')$. Hence, $v(t) > v(t') - \epsilon$.

By symmetric arguments (replacing the roles of $t$ and $t'$), $v(t') > v(t) - \epsilon$. Hence $|v(t') - v(t)| < \epsilon$.

An analogous proof applies to the function $V^S \circ \overline{K_C}$. □

Lemmas A.1 and A.2 give:

**Corollary A.1.** *If a knife function is either $S$-smooth or $S$-continuous (or both), then it is $S$-good.*

*A.3. Examples*

Consider for example the knife-function in Figure 5/a, $K_C(t) = [0, L] \times [0, t]$. Intuitively, it is easy to see that it is Square-continuous, no squares with positive area are created abruptly. Formally, given $\epsilon > 0$, select a $\delta$ such that $2\delta + \delta^2 < \epsilon$. For every $t, t'$ with $|t' - t| < \delta$, for every axis-parallel square in $K_C(t')$ with side-length $a + \delta$, there is a contained square in $K_C(t)$ with side-length at least $a$. The difference between these squares has an area of at most $2\delta + \delta^2 < \epsilon$.

As a negative example, consider the knife function $K_C(t) = [0, t] \times [0, 1] \cup [1 - t, 1] \times [0, 1]$ defined on the cake $C = [0, 1] \times [0, 1]$. This function describes two rectangles that approach each other from two opposite sides of the cake (see Figure A.13). It is not Square-continuous. Intuitively, a square of side-length 1 is created at time $t = 0.5$, when the two components of $K_C(t)$ meet. Formally, let $\epsilon = 0.75$. For every $\delta > 0$, select $t = 0.5 - \frac{\delta}{3}$ and $t' = 0.5 + \frac{\delta}{3}$. Then $K_C(t')$ contains the square $s' = [0, 1] \times [0, 1]$, but all squares $s \subseteq K_C(t)$ have a side-length of less than 0.5, hence $Area(s' \setminus s) > 0.75 = \epsilon$.

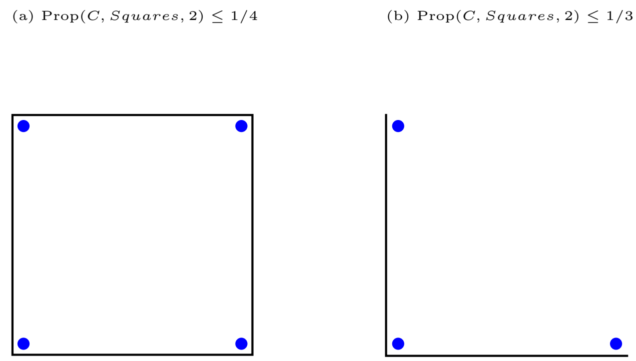


Figure B.14: Impossibility results (upper bounds) on proportionality. At least one agent receives at most one pool.

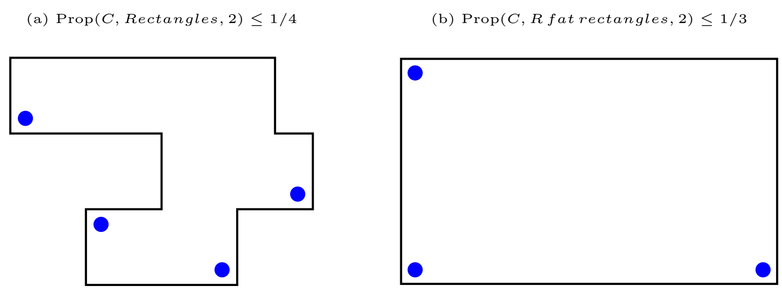


Figure B.15: More upper bounds on proportionality.

Figure 5 shows knife functions that are $S$-continuous but not $S$-smooth. Thus one may think that $S$-continuity is more permissive than $S$-smoothness. But this is not the case: $S$-continuity and $S$-smoothness are two independent properties. To see this, let $S'$ be the family of *rectangle-pairs* (defined as unions of two rectangles). The function $K_C$ defined in the previous paragraph (and Figure A.13) is $S'$-smooth, because both $K_C(t)$ and $\overline{K_C}(t)$ are rectangle-pairs. However, $K_C$ is not $S'$-continuous because some rectangle-pairs (e.g. $[0,1] \times [0,0.2] \cup [0,1] \times [0.8,1]$) are created abruptly at time $t = 0.5$.

## B. Upper bounds on proportionality

To complement the positive results presented in Section 4, we present here some negative results - upper bounds on the attainable proportionality level.

The following generic technique is used in all impossibility results. Assume that $C$ is a desert with $M$ water-pools. Consider two agents whose

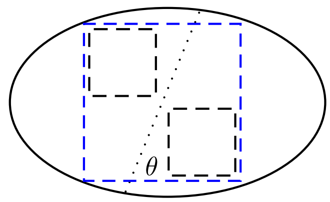


Figure C.16: Convex variant of Figure 8. A convex R-fat cake (the ellipse) is divided by a rotating knife (dotted line) to two 2R-fat convex pieces.

value measure is proportional to the amount of water in their land-plot. Suppose it is possible to give each agent an $S$-piece containing a single water pool, but impossible to give both agents more than one pool since there is room for at most a single $S$-piece touching two pools. Therefore, at least one agent has at most one pool and a utility of at most $1/M$. The arrangements of pools and the corresponding upper bounds are presented in Figures B.14 and B.15.

## C. Convex version of Subsection 4.4

The following theorem is a variant of Theorem 1(c) in which the cake must be convex and the pieces are guaranteed to be convex. The convexity requirement, while seemingly simple, implies that we cannot use the usual knife functions anymore. For example, if $C$ is a circle then every knife function (which must be a straight line to keep the pieces convex) must start with an infinitely slim piece. Hence we must use another technique which can be called a *rotating-knife*.

**Theorem.** *For every $R \geq 1$, If $C$ is convex and $R$-fat and $S$ is the family of convex $2R$-fat pieces then:*

$$PropEF(C, S, 2) = Prop(C, S, 2) = \frac{1}{2}$$

*Proof.* Scale, rotate and translate the cake $C$ such that the largest square contained in $C$ is $B^- = [-1, 1] \times [-1, 1]$. By definition of fatness, $C$ is now contained in a square $B^+$ of side-length at most $2R$.

Consider a line passing through the origin at angle $\theta \in [0°, 360°]$ from the x axis (see Figure C.16). This line cuts the contained square $B^-$ into two quadrangles, each of which contains a square with side-length 1. Because $C$ is convex, this line also cuts the boundary of $C$ at exactly two points, splitting $C$ to two convex pieces. Each of these two pieces is $2R$-fat since it contains a square with side-length 1 and it is contained in $B^+$ whose side-length is $2R$.

Let $W(\theta)$ be the value of the piece for agent #1 at the left-hand side of the line when facing at angle $\theta$. Because the value measure is continuous, $W$

is continuous. When $\theta$ rotates by 180°, the piece that was at the left-hand side is now at the right-hand side and vice versa (e.g. when $\theta = 0°$ the left-hand side is above the line and when $\theta = 180°$ the right-hand side is above the line). Hence if $W(\theta) > 1/2$ then $W(180° + \theta) = 1 - W(\theta) < 1/2$ and vice versa. Hence by the continuity of $W$ there must be a $\theta$ for which $W(\theta) = 1/2$. Cut the cake at the line in angle $\theta$. Let agent #2 choose a piece and give the other piece to agent #1. Now both agents have a piece which is convex and $2R$-fat and their value is at least 1/2. $\square$

So far we have not managed to generalize the rotating-knife technique to more than two agents.

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