Queuing Theory - Optimal Traffic Light Duration

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Problem Context

- Traffic congestion is a persistent and growing issue in Ho Chi Minh City.
- Current traffic light systems may not adapt efficiently to dynamic vehicle flows.
- Optimizing traffic light durations can significantly reduce waiting times and improve overall traffic flow.

Figure: A HCMC's crossroad in peak hour

Markov Processes in Traffic Modeling

- Traffic patterns can be modeled as stochastic systems due to their random and dynamic nature.
- Markov processes are a powerful tool to describe the probabilistic transitions of vehicles through intersections.
- By modeling each direction as a queue and traffic signals as service phases, we can analyze system behavior and optimize light durations.

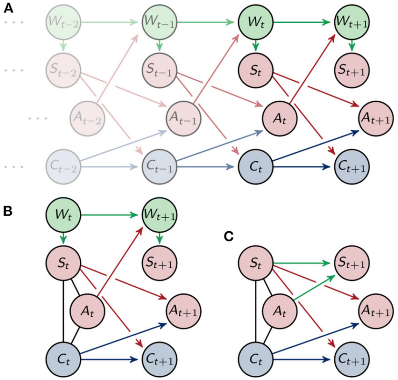


Figure: Sample processes

Relevance & Impact: Why This Project Matters

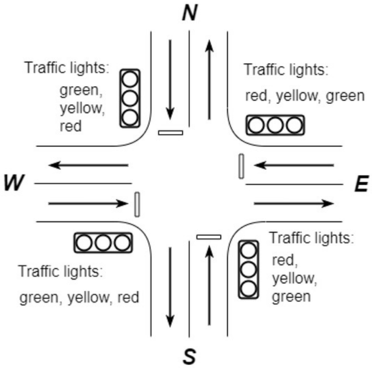


Figure: A model of traffic lights

- Course relevance: Applying continuous-time Markov chains and stochastic modeling.
- Real-world value: Quantitative tool for traffic signal optimization in congested cities.
- Class interest: Relatable, visual, and shows how math can drive real change.

Continuous-Time Markov Chain

Overview:

Continuous-time Markov chain models

- Transitions between discrete states
- Duration of time in each state (exponentially distributed) $(*)$

Each model is specified by

- Transition matrix $\tilde{\mathbf{P}}$
- Initial distribution α
- Exponential time parameters λ_{S_i},

Continuous-time Markov chain

Example. 3-state weather chain

- State space $\mathcal{S} = \{\text{rain, snow, clear}\}$
- Exponential time parameters $(\lambda_r, \lambda_s, \lambda_c) = (1/3, 1/6, 1/12)$
- Transition matrix $P = \begin{array}{cc} & \begin{array}{ccc} \text{rain} & \text{snow} & \text{clear} \end{array} \\ \begin{array}{c} \text{rain} \\ \text{snow} \\ \text{clear} \end{array} & \begin{pmatrix} 0 & 1/2 & 1/2 \\ 3/4 & 0 & 1/4 \\ 1/4 & 3/4 & 0 \end{pmatrix} \end{array}$

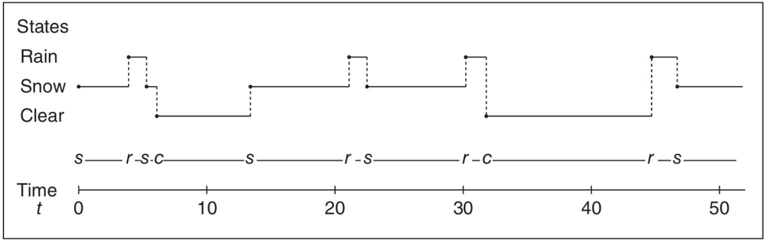


Figure: Realization of a continuous-time weather chain.

Continuous-Time Markov Chain

Markov Property

A continuous-time stochastic process $(X_t)_{t\geq 0}$ with discrete state space $\mathcal{S}$ is a continuous-time Markov chain if:

$$P(X_{t+s} = j \mid X_s = i, X_u = x_u, 0 \leq u < s) = P(X_{t+s} = j \mid X_s = i)$$

for all $s, t \geq 0$, $i, j, x_u \in \mathcal{S}$, and $0 \leq u < s$.

Time-Homogeneous Property

The process is said to be time-homogeneous if this probability does not depend on s. That is,

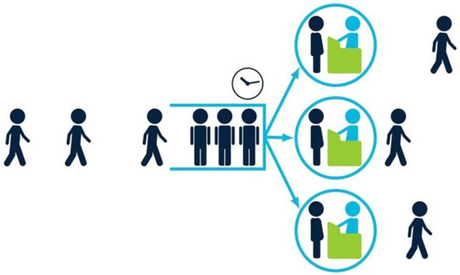
$$P(X_{t+s} = j \mid X_s = i) = P(X_t = j \mid X_0 = i), \quad \text{for all } s \geq 0$$

Queueing Theory

Queuing theory is the study of waiting lines, or queues.

General framework:

1. *Customers* arrive at a facility for *service*.
2. They wait for service, creating a *queue*.
3. When the service is completed, they leave the system.



Applications:

- Traffic light control
- Call centers
- Cloud computing
- Emergency services

Queuing Models

Standard notation

A queuing model is described by the standard notation of the form

$$A/B/n,$$

in which

- A: arrival time distribution
- B: service time distribution
- n: number of servers.

Queuing Models: The $M/D/1$ Queue

A queuing model of the form

$$M/D/1$$

consists of

- **M**: Markovian (Poisson) arrival time distribution
- **D**: Deterministic service time
- **1**: One server

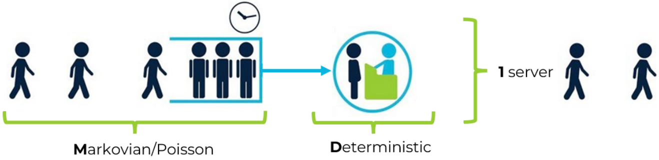


Figure: An $M/D/1$ queue

Queuing Models: The $M/D/1$ Queue

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Key Properties

- Arrival rate: λ
- Service time: $1/\mu$ (fixed)
- Utilization: $\rho = \lambda/\mu$

Little's Law

Theorem (Little's Formula)

In a queuing system, let

- L: average number of customers in the system
- λ: arrival rate
- W: average time a customer spends in the system

then the system is stable when

$$L = \lambda W.$$

Applications

- System design and capacity planning
- Estimating unknown parameters

Problem Formulations

- Vehicles arrive at intersections as a Poisson process with rate λ.
- Service rate during green is deterministic: μ cars/s.
- Two scenarios modeled:
  (a) Paired directions (NS and EW alternate).
  (b) Independent directions (N, E, S, W take turns).
- Modeled as $M/D/1$ queues with interruptions.
- Yellow time is fixed: 8s for (a), 16s for (b).
- Objective: Minimize average waiting time W by optimizing green durations.

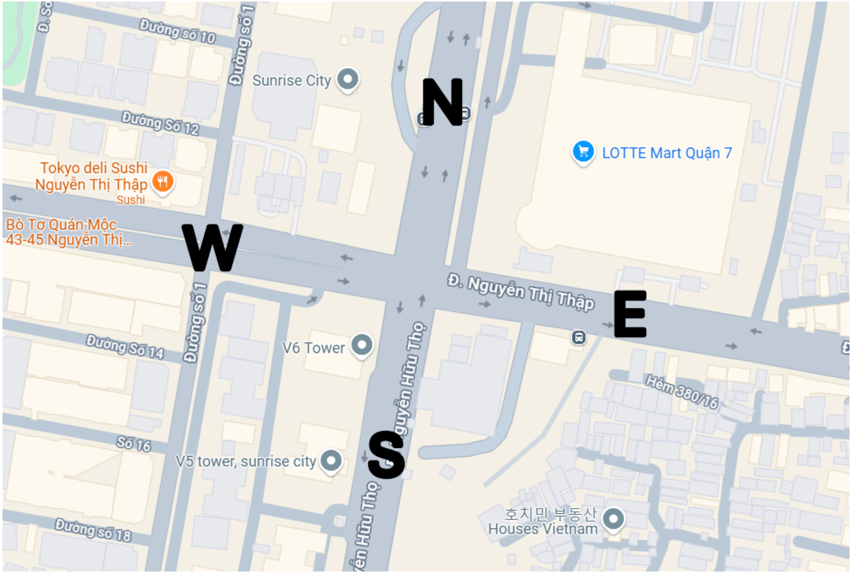
Passenger Car Unit (PCU) Conversion Table

Conversion Coefficient from Vehicle Types to PCUs

Vehicle Type	**Motorcycle**	**Car ($\leq$12 seats)**	**Other Vehicles**	
			Bus (12–30 seats)	**Truck ($<$2 tons)**
PCU Value	0.30	1.00	1.25	1.50

Source: JICA (2004), Urban Transport Master Plan – Ho Chi Minh City, Chungsuk Engineering & Bach Khoa (2006, p.49)

Nguyen Huu Tho - Nguyen Thi Thap Crossroad Layout



Traffic Count Data

Traffic Count Data with PCU Conversion Values

Location	Motorbike	Car	Bus	Truck (<2 tons)	PCU/7min	PCU/s
Nguyen Huu Tho (N)	358	53	2	14	183.90	0.437857
Nguyen Huu Tho (S)	315	35	2	19	160.50	0.382143
Nguyen Thi Thap (E)	446	65	1	40	260.05	0.619167
Nguyen Thi Thap (W)	422	46	1	30	218.85	0.521071

Data collected from 7:30–8:00 AM, May 12, 2025

Simulations

- Numerical optimization:
    - **Scenario (a)**: Used `minimize_scalar` to find optimal G_{NS}.
    - **Scenario (b)**: Solved using Lagrange multipliers and `root_scalar`.
- Animation Simulations:
    - Visualized queue lengths over time using `matplotlib.animation`.
    - Green/red phases alternate per scenario logic.
- Inputs:
    - $\lambda_N = 0.438$, $\lambda_S = 0.382$, $\lambda_E = 0.619$, $\lambda_W = 0.521$
    - Service rate $\mu = 5.5$
    - Cycle time $C = 110$s

Optimized Green Time Allocation

Comparison of Green Durations

Approach	**Real Green (s)**	**Opt. Green a (s)**	**Opt. Green b (s)**
Nguyen Huu Tho (N)	60	49.24	22.36
Nguyen Huu Tho (S)	60	49.24	21.12
Nguyen Thi Thap (E)	42	52.76	26.34
Nguyen Thi Thap (W)	42	52.76	24.18

Total cycle time = 110 s (includes 4s Yellow per phase).

Discussion

- Queueing model offers analytical insight into signal optimization.
- Trade-off:
  - Simpler set-up in Scenario (a), but lower risk of accidents in Scenario (b)
- Simulations confirm theoretical predictions and effectiveness of optimization
- Practical considerations:
  - Traffic variability
  - Real-time adaptation
  - Data collection over a period of time
- Framework can be extended to adaptive traffic light control.