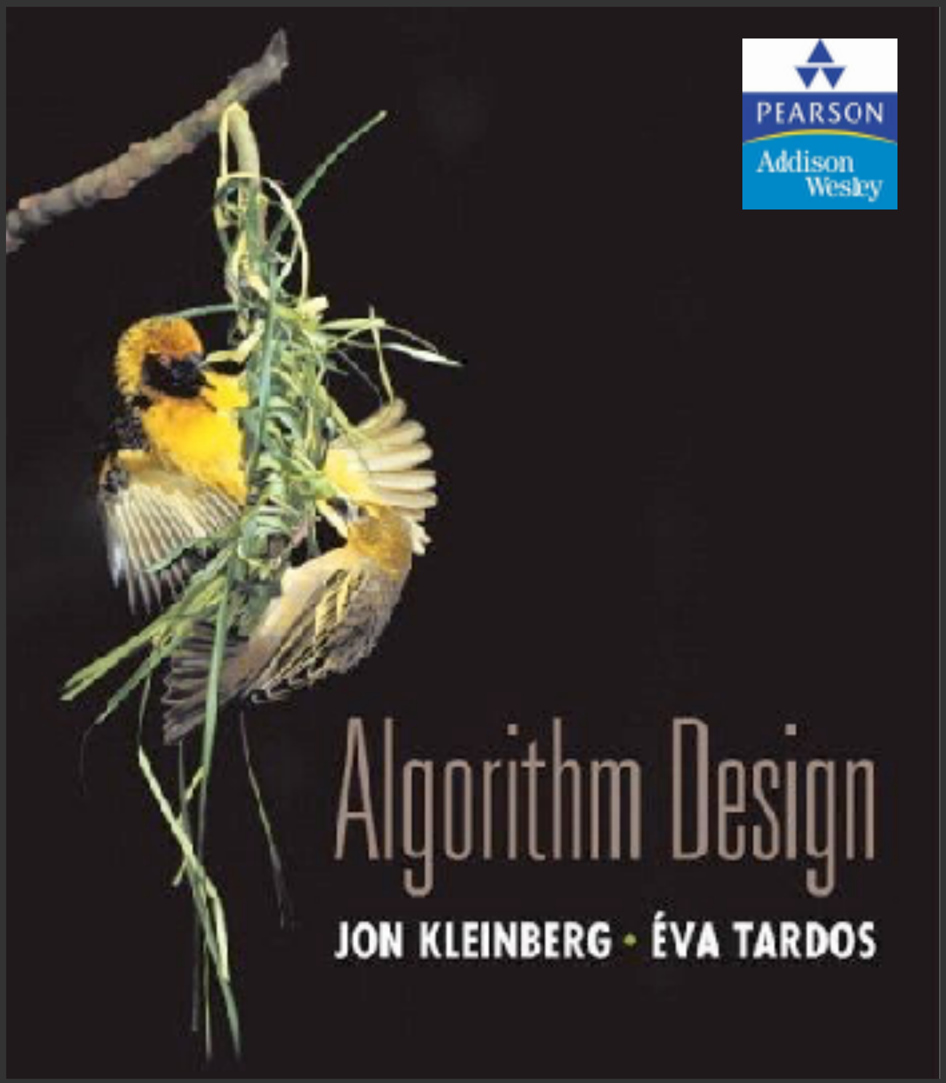

4. GREEDY ALGORITHMS I

- *coin changing*
- *interval scheduling and partitioning*
- *scheduling to minimize lateness*
- *optimal caching*

Lecture slides by Kevin Wayne
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http://www.cs.princeton.edu/~wayne/kleinberg-tardos

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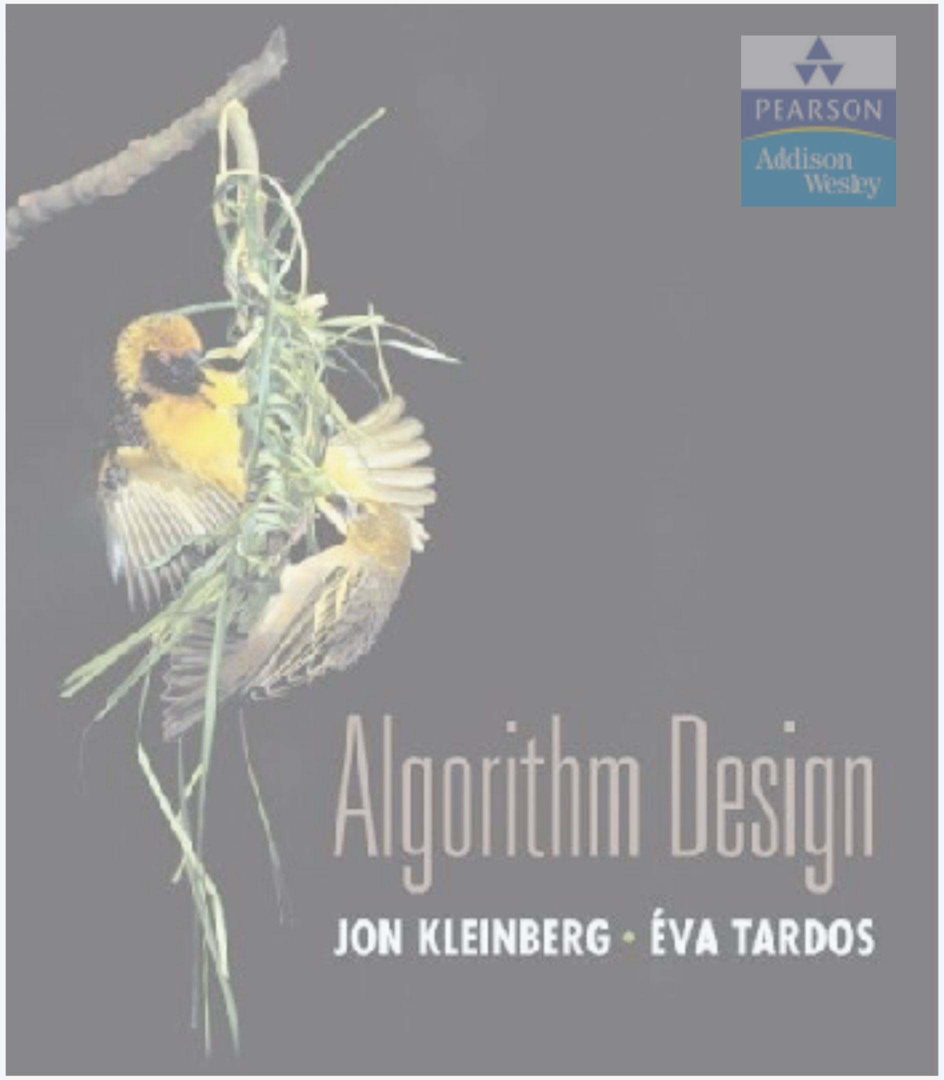
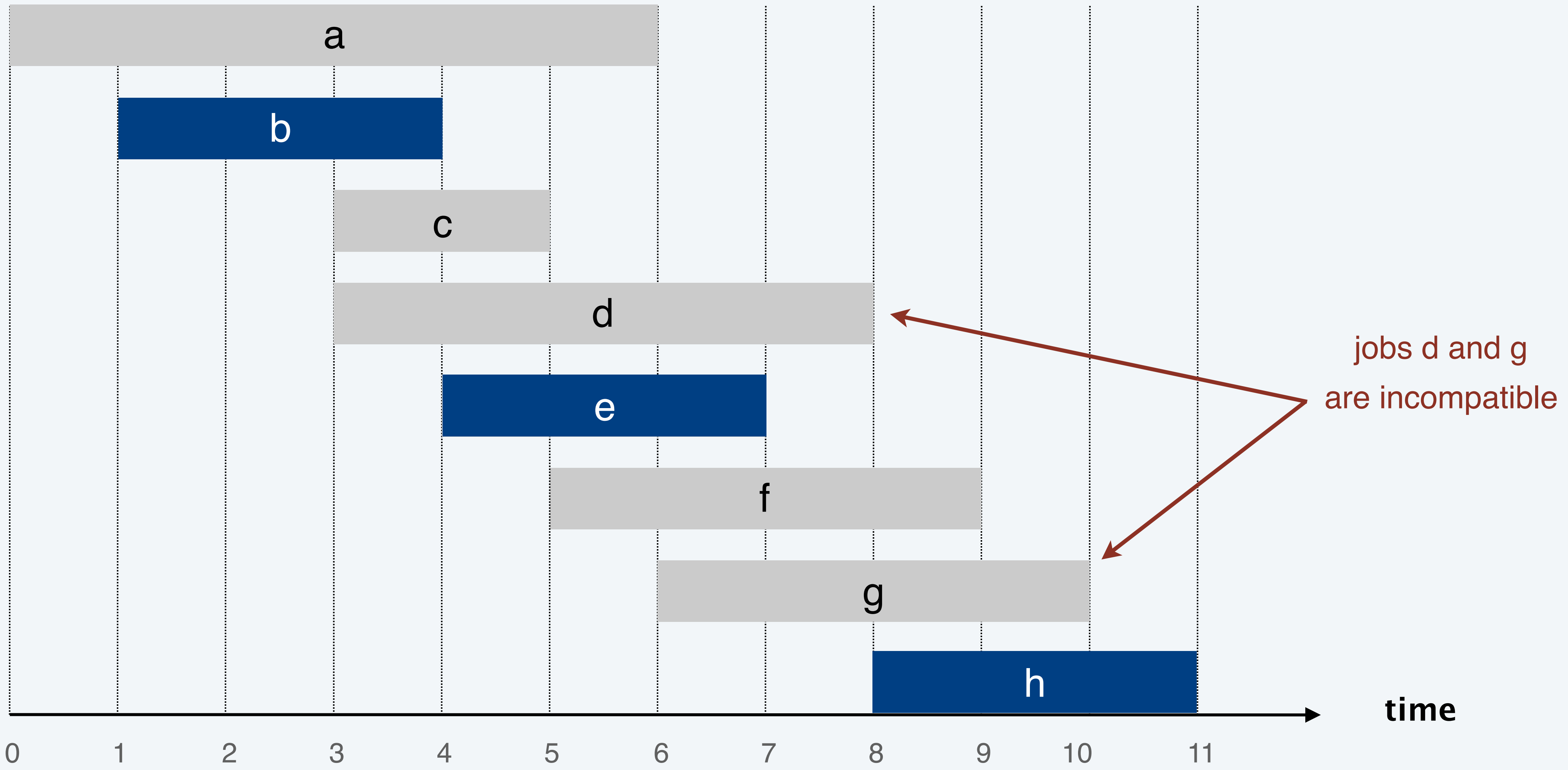


4. Greedy Algorithms I

- *coin changing*
- *interval scheduling and partitioning*
- *scheduling to minimize lateness*
- *optimal caching*

Interval scheduling

- Job j starts at s_j and finishes at f_j.
- Two jobs compatible if they don't overlap.
- Goal: find maximum subset of mutually compatible jobs.



Greedy algorithms

What makes an algorithm greedy?

Think of what describes the most simple algorithms you know.

Interval scheduling: greedy algorithms

Greedy template. Consider jobs in some natural order.
Take each job provided it's compatible with the ones already taken.

- [Earliest start time] Consider jobs in ascending order of s_j.
- [Earliest finish time] Consider jobs in ascending order of f_j.
- [Shortest interval] Consider jobs in ascending order of $f_j - s_j$.
- [Fewest conflicts] For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.

Greedy algorithms

What makes an algorithm greedy?

- simple and fast computations
- decision is based on local/few information
- once you make a decision you cannot go back to change it

Interval scheduling: greedy algorithms

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Interval scheduling: greedy algorithms

Greedy template. Consider jobs in some natural order. Take each job provided it's compatible with the ones already taken.

counterexample for earliest start time

counterexample for shortest interval

counterexample for fewest conflicts

Interval scheduling: earliest-finish-time-first algorithm

EARLIEST-FINISH-TIME-FIRST $(n, s_1, s_2, \ldots, s_n, f_1, f_2, \ldots, f_n)$

SORT jobs by finish time so that $f_1 \leq f_2 \leq \ldots \leq f_n$

$A \leftarrow \phi$ ⟵ set of jobs selected

FOR $j = 1$ TO n

    IF *job j is compatible with A*

        $A \leftarrow A \cup \{ j \}$

RETURN A

Proposition. Can implement earliest-finish-time first in $O(n \log n)$ time.

- Keep track of job j^* that was added last to A.
- Job j is compatible with A iff $s_j \geq f_{j^*}$.
- Sorting by finish time takes $O(n \log n)$ time.

Interval scheduling: analysis of earliest-finish-time-first algorithm

Theorem. The earliest-finish-time-first algorithm is optimal.

Pf. What do we need to prove?

What is the output of the greedy algorithm?

What does it mean that it is optimal?

Pf.

- our solution S may or may not be optimal, so let's pick a solution S* that we know is optimal. (what does that mean again?)
- there may be multiple optimal solutions, so let's choose S* to be the one most similar to S. (what makes it most similar?)
- by swapping two jobs we can make S* even more similar to S in contradiction with the choice of S* being most similar.

Interval scheduling: analysis of earliest-finish-time-first algorithm

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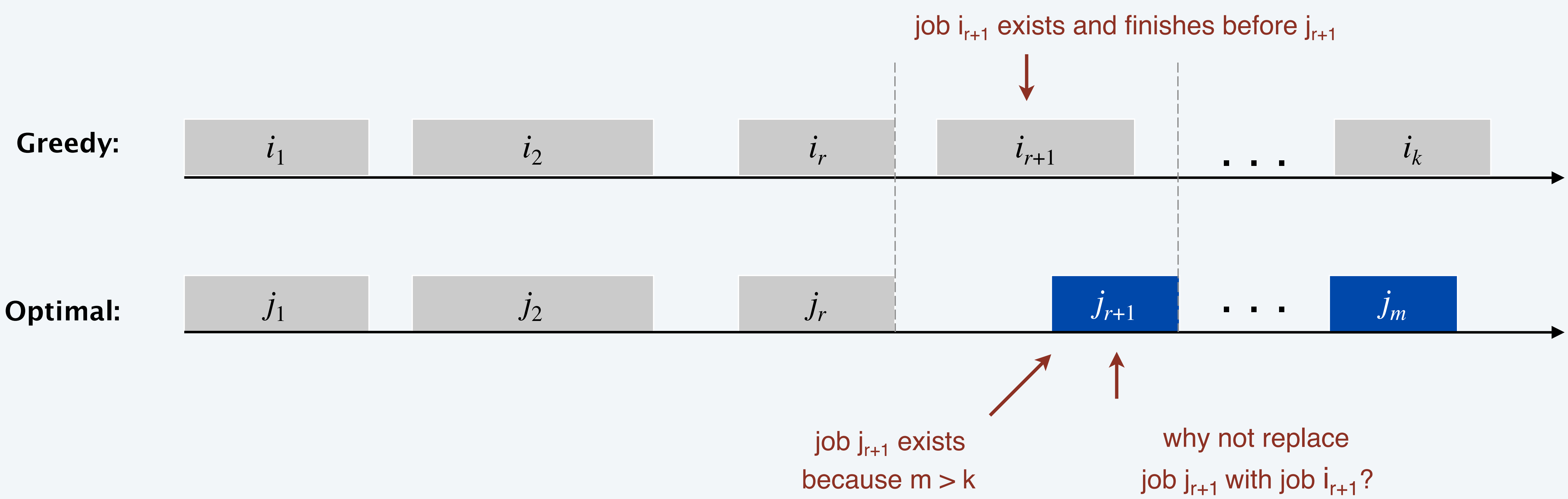

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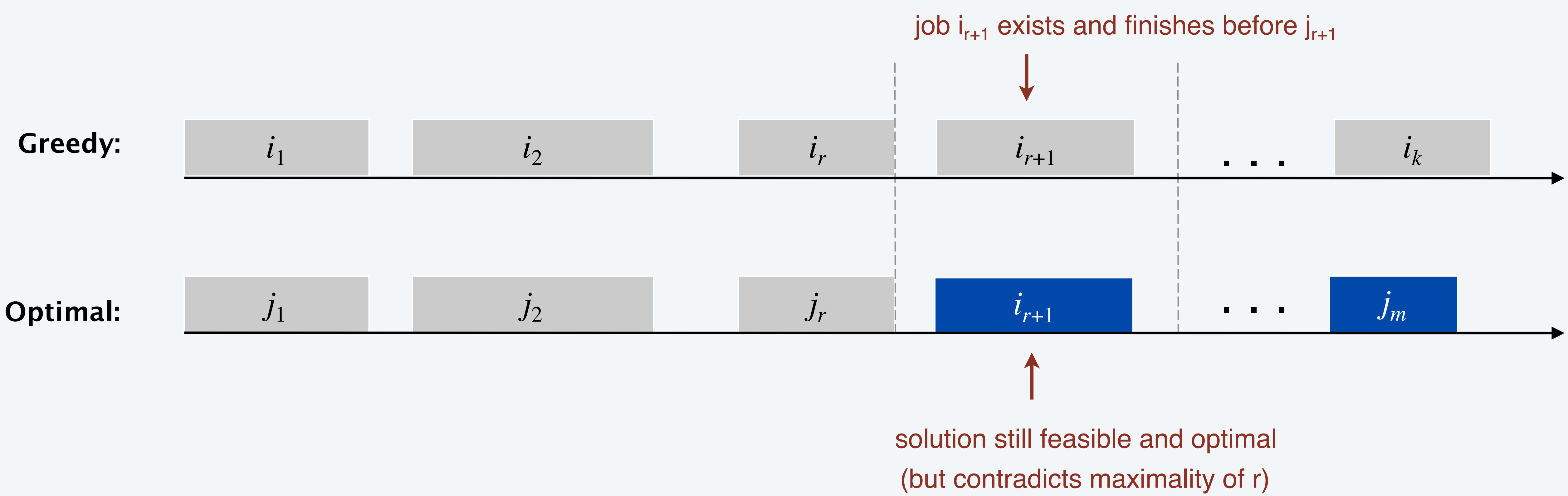
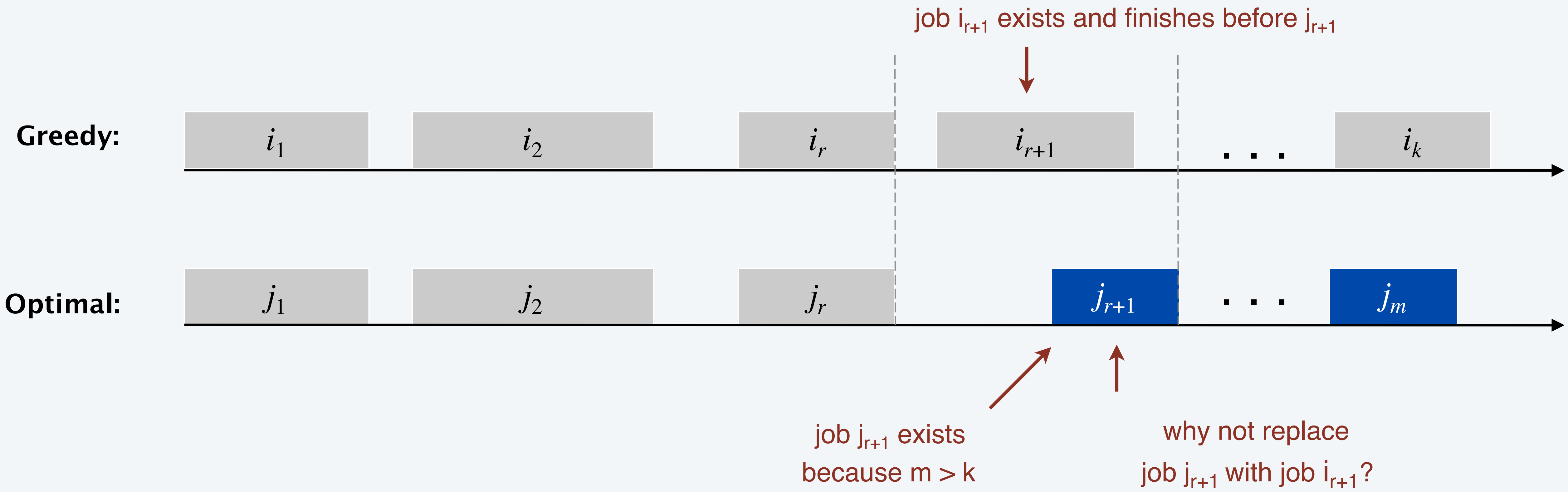


Interval scheduling: analysis of earliest-finish-time-first algorithm

Theorem. The earliest-finish-time-first algorithm is optimal.

Pf. [by contradiction]

- Assume greedy is not optimal, and let's see what happens.
- Let $i_1, i_2, \ldots\ i_k$ denote set of jobs selected by greedy.
- Let $j_1, j_2, \ldots\ j_m$ denote set of jobs in an optimal solution with $i_1 = j_1, i_2 = j_2, \ldots, i_r = j_r$ for the largest possible value of r.



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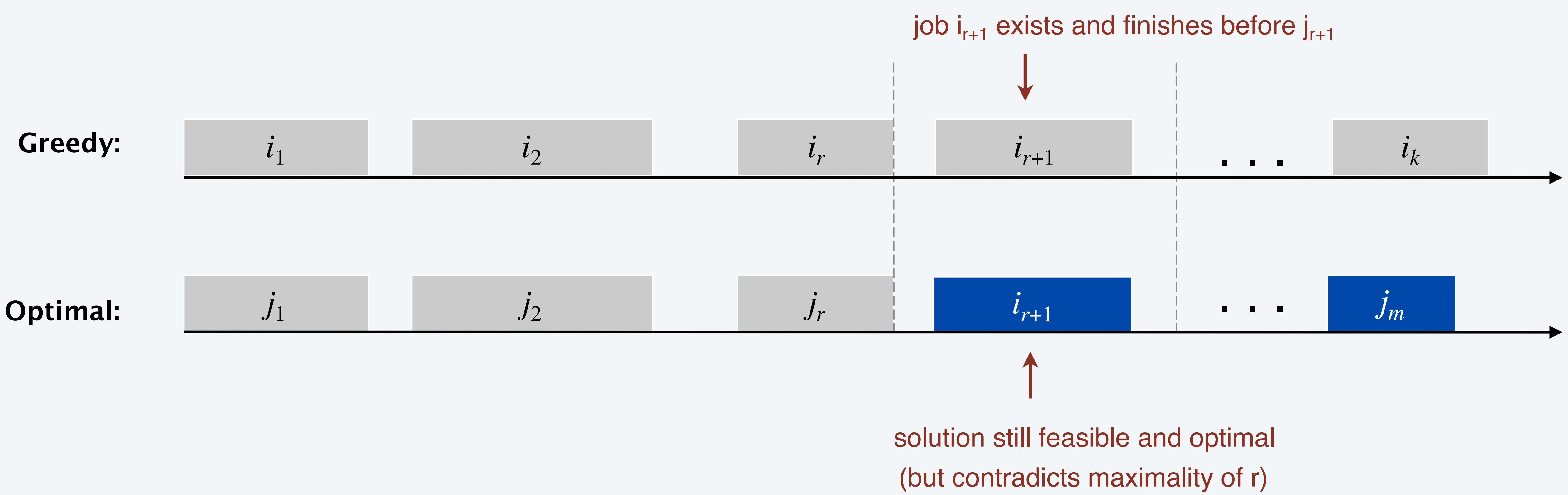
- Let j_1, j_2, ... j_m denote set of jobs in an optimal solution with $i_1 = j_1$, $i_2 = j_2$, ..., $i_r = j_r$ for the largest possible value of r.

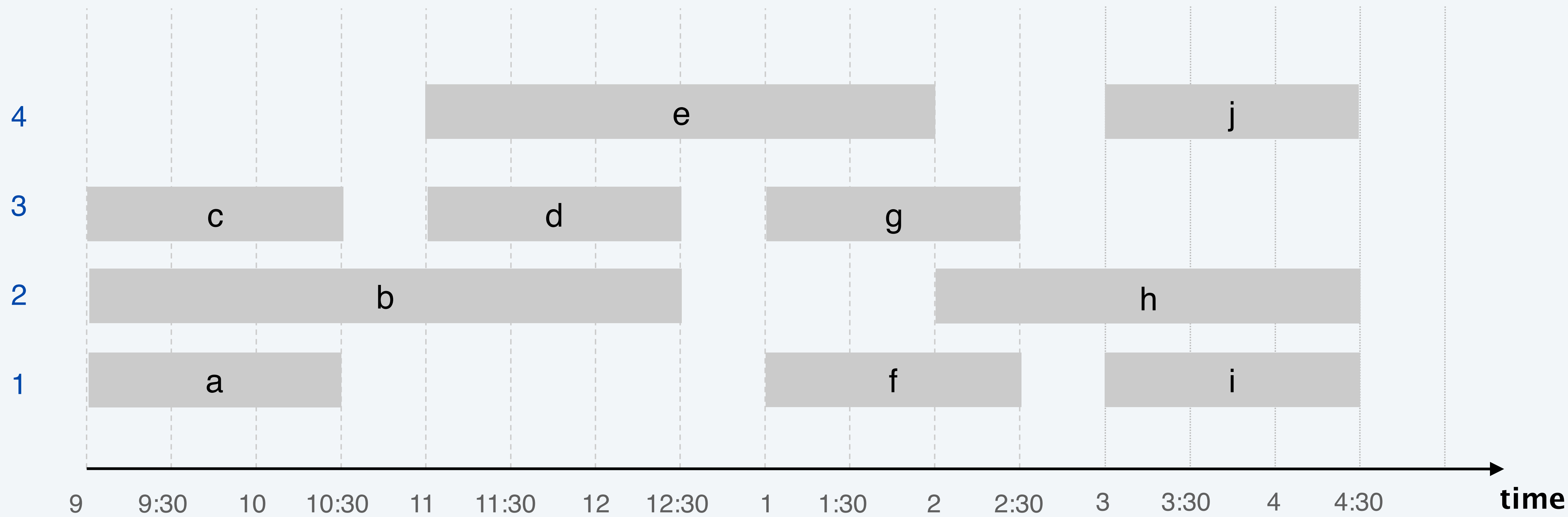


Interval partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two lectures occur at the same time in the same room.

Ex. This schedule uses 4 classrooms to schedule 10 lectures.

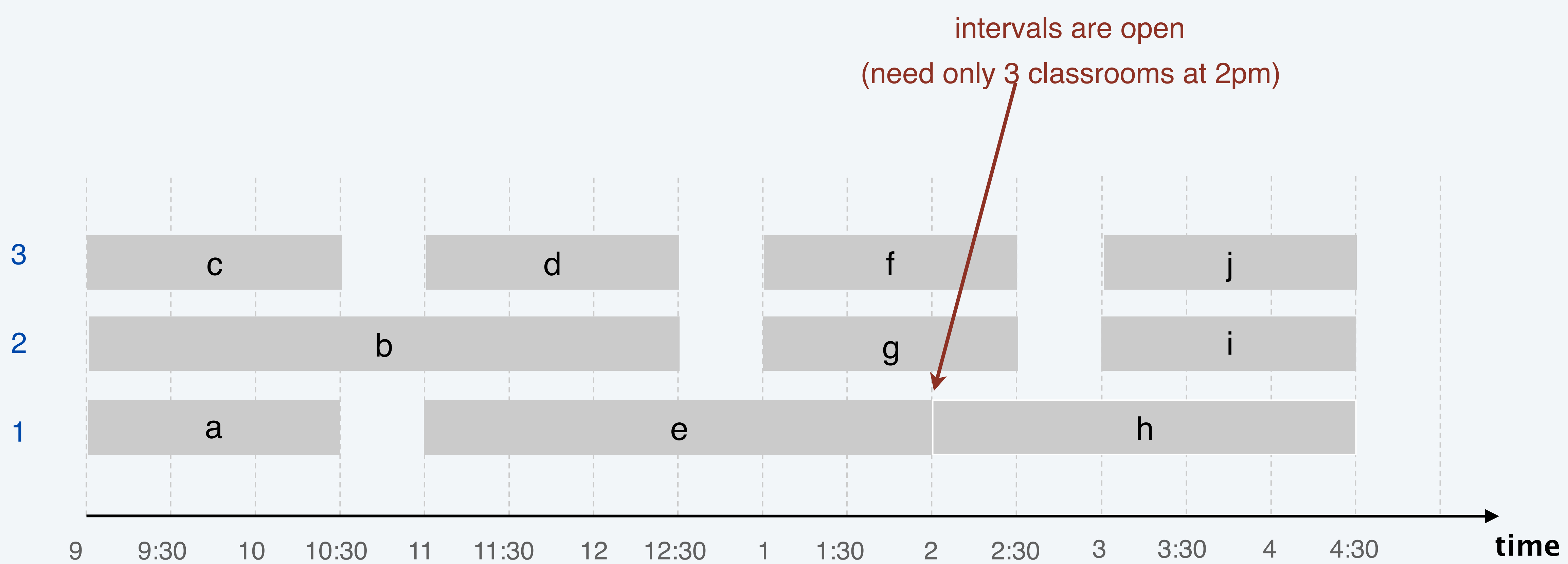


Interval partitioning

Interval partitioning.

- Lecture j starts at s_j and finishes at f_j.
- Goal: find minimum number of classrooms to schedule all lectures so that no two lectures occur at the same time in the same room.

Ex. This schedule uses 3 classrooms to schedule 10 lectures.



Greedy algorithms

What makes an algorithm greedy?

- simple and fast computations
- decision is based on local/few information
- once you make a decision you cannot go back to change it

Interval partitioning: greedy algorithms

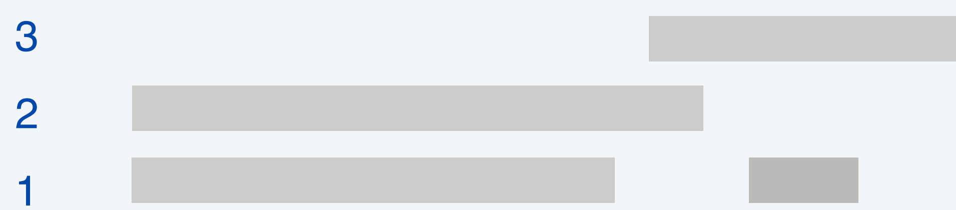
Greedy template. Consider lectures in some natural order. Assign each lecture to an available classroom (which one?); allocate a new classroom if none are available.

- [Earliest start time] Consider lectures in ascending order of s_j.
- [Earliest finish time] Consider lectures in ascending order of f_j.
- [Shortest interval] Consider lectures in ascending order of $f_j - s_j$.
- [Fewest conflicts] For each lecture j, count the number of conflicting lectures c_j. Schedule in ascending order of c_j.

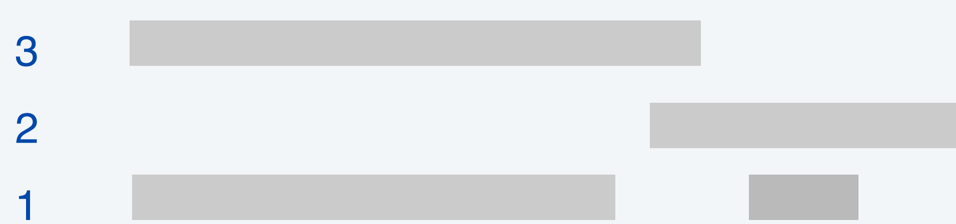
Interval partitioning: greedy algorithms

Greedy template. Consider lectures in some natural order. Assign each lecture to an available classroom (which one?); allocate a new classroom if none are available.

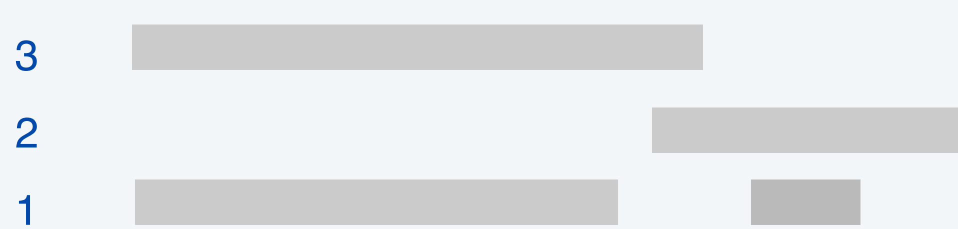
counterexample for earliest finish time



counterexample for shortest interval



counterexample for fewest conflicts



Interval partitioning: earliest-start-time-first algorithm

EARLIEST-START-TIME-FIRST $(n, s_1, s_2, \ldots, s_n, f_1, f_2, \ldots, f_n)$

SORT lectures by start time so that $s_1 \leq s_2 \leq \ldots \leq s_n$.

$d \leftarrow 0$ ← number of allocated classrooms

FOR $j = 1$ TO n

IF *lecture j is compatible with some classroom*

Schedule lecture j in any such classroom k.

ELSE

Allocate a new classroom d + 1.

Schedule lecture j in classroom d + 1.

$d \leftarrow d + 1$

RETURN *schedule*.

Interval partitioning: earliest-start-time-first algorithm

Proposition. The earliest-start-time-first algorithm can be implemented in $O(n \log n)$ time.

Pf. Store classrooms in a priority queue (key = finish time of its last lecture).

- To determine whether lecture j is compatible with some classroom, compare s_j to key of min classroom k in priority queue.
- To add lecture j to classroom k, increase key of classroom k to f_j.
- Total number of priority queue operations is $O(n)$.
- Sorting by start time takes $O(n \log n)$ time. ▪

Remark. This implementation chooses a classroom k whose finish time of its last lecture is the earliest.

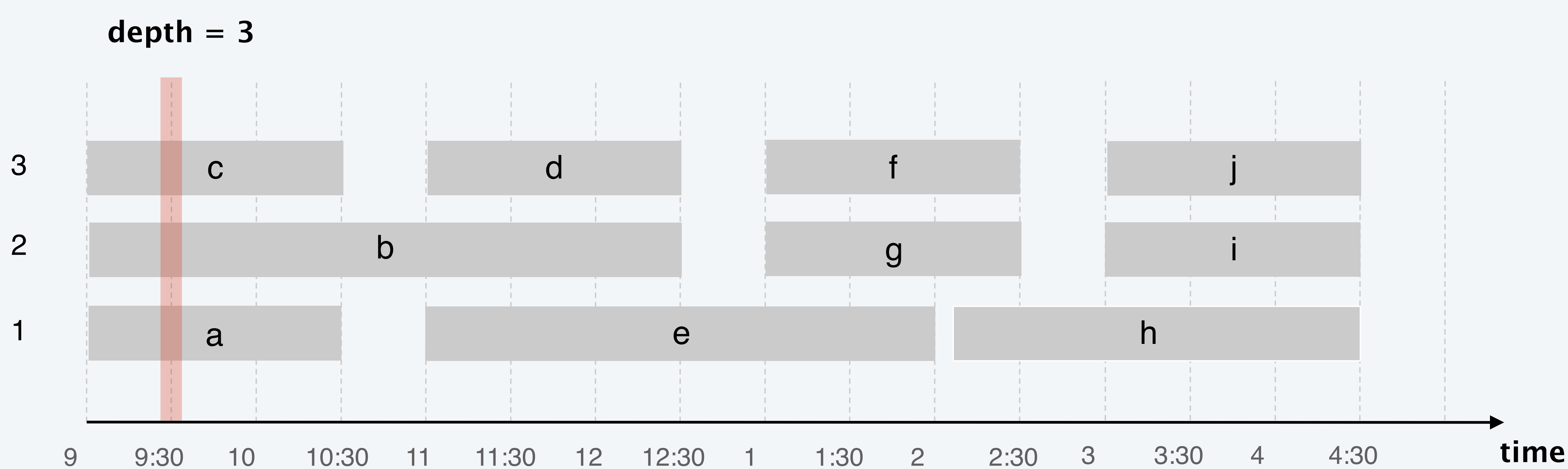
Interval partitioning: lower bound on optimal solution

Def. The depth of a set of open intervals is the maximum number of intervals that contain any given time.

Key observation. Number of classrooms needed ≥ depth.

Q. Does minimum number of classrooms needed always equal depth?

A. Yes! Moreover, earliest-start-time-first algorithm finds a schedule whose number of classrooms equals the depth.



Interval partitioning: analysis of earliest-start-time-first algorithm

Observation. The earliest-start-time first algorithm never schedules two incompatible lectures in the same classroom.

Theorem. Earliest-start-time-first algorithm is optimal.

Pf.

- Let d = number of classrooms that the algorithm allocates.
- Classroom d is opened because we needed to schedule a lecture, say j, that is incompatible with all $d-1$ other classrooms.
- These d lectures each end after s_j.
- Since we sorted by start time, all these incompatibilities are caused by lectures that start no later than s_j.
- Thus, we have d lectures overlapping at time $s_j + \varepsilon$.
- Key observation $\Rightarrow$ all schedules use ≥ d classrooms. ▪